Frobenius Kernel

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Abstract

Let G be a finite group and H be a non-trivial proper subgroup of G. The group G is called a Frobenius group if $H \cap H^g = 1$ for all $g \in G - H$. The set

$$K = (G - \bigcup_{g \in G} H^g) \cup \{1\}$$

is called the Frobenius kenel and H is called the Frobenius complement. Using character Theory it is proved that K is a normal subgroup of G. In this paper we present some group theoretical proofs that K is a subgroup of G under certain conditions.

keywords: Frobenius group, Frobenius complement, Frobenius kernel.

Mathematics Subject classification: 20H20, 20F50.

Introduction

Let G be a finite group acting on a set Ω, $|\Omega| > 1$. Then G is called a Frobenius group if

(a) G acts transitively on Ω,

(b) $G_\alpha \neq 1$ for any $\alpha \in \Omega$,

(c) $G_\alpha \cap G_\beta = 1$ for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.

Introduction

Let $H = G_\alpha$ for some $\alpha \in \Omega$, then for any $\beta \in \Omega$, the group G_β is conjugate to G_α, i. e.

$$G_\beta = G_\alpha^g = H^g$$

for some $g \in G$.

Introduction

Therefore $F = G \setminus \bigcup_{g \in G} H^g$ is the set of elements of G that don't fix any element of Ω. We set

$$K = F \cup \{1\} = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}.$$

The subgroup H is called a Frobenius complement and the set K is called the Frobenius kernel of G.

Introduction

It is easy to prove that $N_G(H) = H$ and that

$$|K| = |G| + 1 - (|H|)[G : H] - 1 = [G : H] = n$$

Therefore $|G| = |K||H|$ and $H \cap K = 1$.

Introduction

An equivalent definition of a Frobenius group is the following: G is called a Frobenius group with complement H if

$$1 \neq H \lneqq G$$

and

$$H \cap H^g = 1$$

for all

$$g \in G \setminus H.$$

Introduction

It is proved by G. Frobenius in 1901.

★ G. Frobenius, Über auflősbare Gruppen , IV., Berl. Ber., 1901 (1901), PP. 1216-1230.

That the Frobenius kernel K is a normal subgroup of G.

Introduction

The proof by Frobenius uses the theory of character. But since 1901 many attempts have been made to prove the normality of K without using character theory. Ofcourse K contains the unit element 1 and it is a normal subset of G, but the difficulty is to prove that K is closed under multiplication.

Character Theory

Forethere proofs of the normality of K in G can be found in the following refrences where character theory is used:

★ L. Dornhoff, Group Representation Theory, Part A: ordinary representation theory, Vol 7, Pure and Appl. Math, Marcel Dekker, Inc., New York (97)

★ W. Feit, On a conjecture of Frobenius, Proc. Amer. Math. Soc. 7(1956)177-187.

★ L. C. Grove, Groups and Characters, Pure and Appl. Math John Wiley and sons Inc. New York 1997.

★ M. Hall Jr, The theory of groups, The Macmillan company, New York, 1959.

★ B. Huppert, Endlichp Gruppen, Springer-Verlag, 1967.

★ W. Koapp and P. Schmid, A note on Frobenius groups, J. Group Theory, 12(2009) 393-400.

Character Theory

Assume that G is a Frobenius group with complement H and kernel K. We will present a character theoretic proof that K is a normal subgroup of G. This proof is a modification of the proof in:

★ W. Knapp and P. Schmid, A note on Frobenius groups, Journal of group theory 12(2009) 393-400.

Character Theory

Define the function $\psi : G \longrightarrow \mathbb{C}$ by

$$\psi(g) = \begin{cases} |H|, & \text{if} \quad g \in K \\ \\ 0, & \text{otherwise} \end{cases}$$

Since K is a normal subset of G, ψ is a class function on G. We will prove ψ is a character of G with $ker\psi = K$, proving $K \trianglelefteq G$.

Character Theory

Let $\chi \in Irr(G)$. We will show that

$$C_\chi = \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{x \in G} \chi(x)\psi(x) = \frac{1}{|G|} \sum_{x \in K} \chi(x)|H| = \frac{1}{n} \sum_{x \in K} \chi(x)$$

is a non-negative integer.

Character Theory

If $\chi = 1_G$ the trivial character of G, then $(\chi, \psi) = 1$. So assume $\chi \neq 1_G$.
$G - K = \bigcup_{g \in G} (H-1)^g$ is a disjoint union of n conjugates of $H - 1$, hence

$$(\chi, 1_G) = \frac{1}{|G|}\sum_{g \in G} \chi(g) = \frac{1}{|H|}\sum_{h \in H-1} \chi(h) + \frac{1}{n|H|}\sum_{x \in K} \chi(x) =$$

$$(\chi_H, 1_H) - \frac{\chi(1)}{|H|} + \frac{C_\chi}{|H|} \Longrightarrow C_\chi = \chi(1) - |H|(\chi_H, 1_H)$$

is an integer and $|H| \mid C_\chi - \chi(1)$.

Character Theory

Therefore ψ equals a linear combinators of irreducible characters of G with integer coefficients.

Character Theory

Next we compute

$$1 = (\chi, \chi) = \frac{1}{|H|} \sum_{h \in H-1} |\chi(h)|^2 + \frac{1}{|G|} \sum_{x \in K} |\chi(x)|^2$$

$$\frac{1}{|H|} \sum_{h \in H-1} |\chi(h)|^2 = (\chi_H, 1_H) - \frac{\chi(1)^2}{|H|}$$

is a non-negative rational number.

Character Theory

By Cauchy-Schwartz inequality

$$\sum_{x \in K} |\chi(x)|^2 \geq \frac{1}{n} (\sum_{x \in K} |\chi(x)|)^2$$

with equality iff

$$|\chi(x)| = \chi(1)$$

for all x.

Character Theory

Moreover

$$\frac{1}{n}\sum_{x\in K}|\chi(x)|^2 \geq C_\chi^2$$

with equality iff $\chi(x) \in \mathbb{R}$, with the same signe. Therefore

$$1 \geq ((\chi_H, \chi_H) - \frac{\chi(1)^2}{|H|}) + \frac{C_\chi^2}{|H|} \qquad (*)$$

Character Theory

But

$$C_\chi^2 - \chi(1)^2 = (C_\chi - \chi(1))(C_\chi + \chi(1))$$

is divisible by $|H| \Longrightarrow$ the right-hand side of $(*)$ is a positive integer and consequently we must have equality. Thus

$$C_\chi = \frac{1}{n} \sum_{x \in K} |\chi(x)| = \chi(1)$$

is the degree of χ and this proves that $ker\psi = K \trianglelefteq G$.

Character Theory

In the paper entiiled " Some properties of the finite Frobenius groups " published in Aut J. Math. and comput. 4(1)(2023)57-61, which was dedicated to Prof. J. Moori we obtained some character theoretic properties of the finite Frobenius groups as follows.

Character Theory

Recal that G is a Frobenius group with complement H and kernel K, $G = KH$, $K \cap H = 1$, if K is a normal subgroup of G, $n = [G : H]$.
G acts on the set of cosets of H as a transitive permutation group of degree n and the number of orbits of H on this set is called the rank of G and is denoted by s.

Character Theory

Proposition

Let χ be the permutation character of H acting on K by conjugation. Then $\chi = s\rho_H + 1_H$ where ρ_H and 1_H are the regular and the identity character of H.

Proposition

Let G be a Frobenius group with kernel K as a subset. If all elements of K commute, then K is a normal subgroup of G.

proof. K is a normal subset of G with identity and G acts on it by conjugation. Let η be the permutation character associated with this action. For $g \in G$, $\eta(g)$ is the number of $k \in K$ such that $k^g = g^{-1}kg = k$. We have $\eta(1) = |K|$ and if $g \neq 1$, and $k^g = k$, then $k \in C_G(g) \cap K = C_K(g)$. Since $G = K \underset{g \in G}{\cup} H^g$ we distinguish the following cases.

Character Theory

Since $G = K \underset{g\in G}{\cup} H^g$ we distinguish the following cases.

(I) $1 \in K, k^g = k \Longrightarrow k \in C_K(g) \Longrightarrow \eta(g) = |C_K(g)|$

(II) $1 \neq g \in \bigcup_{g\in G} H^g \Longrightarrow g$ belongs to some conjugate of H

Some we may take

$g \in H \Longrightarrow k^g = k \Longrightarrow g = g^k \in H \cap H^k \Longrightarrow k \in H \cap K = 1$
$\Longrightarrow \eta(g) = 1.$

Character Theory

Therefore

$$\eta(g) = \begin{cases} |K|, & \text{if } \ g = 1, \\ 1, & \text{if } 1 \neq g \in \{H^x \mid x \in G\}, \\ |C_K(g)|, & \text{if } \ 1 \neq g \in K. \end{cases}$$

By assumption all elements of K cummute, hence

$$\eta(g) = \begin{cases} |K|, & \text{if } \ g \in K, \\ 1, & \text{if } 1 \neq g \in \{H^x \mid x \in G\}. \end{cases}$$

Now we see that $ker\eta = K \trianglelefteq G$.

Group Theory

As we mentioned earlier no group theory proof exists for the Frobenius kernel to be a subgroup, but in some special cases there is a proof that we will present here.

Group Theory

If G is a Frobenius group with complement H and kernel K, then

$$N_G(H) = H$$

and $|K| = [G : H]$.

Lemma

If $N \trianglelefteq G$, $G = NH$, $N \cap H = 1$, then $N \leq K$.

Lemma (Burnside)

If G is a finite group and P is a Sylow p-subgroup of G such that $N_G(P) = C_G(P)$, then P has a normal complement in G, i. e. , there is $N \trianglelefteq G$ such that $G = NP$, $N \cap P = 1$.

Theorem

Let G be a Frobenius group with complement H and kernel K. Assume that H is an abelian p-group. Then K is a normal subgroup of G.

proof. From the fact that $N_G(H) = H$ and the fact that H is abelian we obtain

$$H = N_G(H) \geq C_G(H) \geq H$$

Therefore $N_G(H) = C_G(H)$. But $(|H| : [G : H]) = 1$ from which it follows that H is a Sylow p-subgroup of G. Now by Burnside's theorem H has a normal complement N in G, i. e. $G = NH$, $N \cap H = 1$ and $N \trianglelefteq G$.

Corollary

Let G be a finite Frobenius group with complement H and kernel K. Suppose H is centralized by a Sylow p-subgroup of G. Then $K \trianglelefteq G$.

proof. By assumption $H \leq C_G(P)$ where P is a Sylow p-subgroup of G. But:

$$1 \neq x \in H \Longrightarrow C_G(x) \leq H$$

Therefore $C_G(P) \leq H \Longrightarrow C_G(P) = H = N_G(P)$. Now by Burnside's theorem: $\exists N \trianglelefteq G, G = NP. \Longrightarrow |N| = [G : P]$.

By Lemma 1,

$$N \leq K \Longrightarrow |N| = [G : P] \leq |K| = [G : H]. \Longrightarrow |P| \geq |H| \Longrightarrow P = H$$

By Theorem 3 $\Longrightarrow K \leq G$.

Another group theoretical proof under different conditions exists that we mention below. If $2 \mid |H|$, there is an elementary proof that $K \leq G$ due to Bender:

Group Theory

Let t be an element of order 2 in H and $g \in G \backslash H$. Then either

$$a = t \cdot g^{-1} t g = t t^{g} = [t, g]$$

is in K or $\exists x \in G$ such that $1 \neq a \in H^{x}$.

If $a \in H^x \Longrightarrow a \in H^x \cap H^{xt} \cap H^{xt^g}$
Because $a^t = a^{-1} = a^{tg} \Longrightarrow H^x = H^{xt} = H^{xt^g}, t, t^g \in H^x$
If $H^x = H$ contradicts $t \in H$ and $t^g \notin H$.
Therefore: $tt^g \in K$ if $g \in G \backslash H$.

Let $\{g_1, \cdots, g_n\}$ be a transversal of H in G, $n = [G : H]$.
$tt^{g_i} = tt^{g_j} \iff t^{g_i} = t^{g_j} \iff t^{g_i g_j^{-1}} = t \iff g_i g_j^{-1} \in H$
The elements $tt^{g_1}, \cdots, tt^{g_n}$ are pairwise distinct.
$\implies K = \{t^{g_1}t, \cdots, t^{g_n}t\}$

Now we show that $K \leq G$. For $t^{g_i t}$ there exists g_s such that $t^{g_i t} = tt^{g_s}$.
$\Longrightarrow (tt^{g_i})(tt^{g_j}) = t(t^{g_i}t)t^{g_j} = t(tt^{g_s})t^{g_j} = t^{g_s}t^{g_j} = (tt^{g_i g_s^{-1}})g_s \in K^{g_s} = K$
$tt^g \in K$ for $g \in G \backslash H$.

If the complement H is solvable, then K is a subgroup of G.
★ R. H. Shaw, Remarks on a theorem of Frobenius, Proc. Amer. Math. Soc., 3(1952) 970-972.

H acts on $K - \{1\}$ by conjugation without a fixed point, orbits of size $|H|$. Therefore $|H| \mid |K| - 1 \Longrightarrow (|H|, |K|) = 1$. If $K \leq G$, then K is a Hall-subgroup of G. Also H is a $Hall-subgroup$ of G.

Group Theory

Looking at the Frobenius group G as a transitive permutation group on the set Ω, $|\Omega| = n$, $H = G_\alpha$, $\alpha \in \Omega$. Then $|\Omega| = [G : H]$. The number of orbits of H on Ω is called the rank of G, $r = rank(G)$.

Each nontrivial H-orbit has six $|H|$ and there are $s = \dfrac{n-1}{|H|}$ such orbits, $rank(G) = 1 + s = 1 + \dfrac{n-1}{|H|}$.

If $rank(G) \leq 3$, then $K \leq G$ by using elementary group theory.
★ W. Knapp and P. Schmiott, Frobenius groups of low rank, Acch Math. 117(2021) 121-127.

The proof uses the fact that H is a Hall-subgroup of G, and for every prime p dividing $|K| = n = [G : H]$, the Sylow p-subgroup of G are contained in K. Thus for small rank, consequence of the Sylow theorem implies $K \leq G$. In particular if $[G : H]$ is n prime power then K is a Sylow subgroup of G.

Properties of the Frobenius Kernel

Suppose G is a Frobenius group with complement H and kernel K. Assume $K \trianglelefteq G$, $G = HK$.
G has a unique kernel. If K is solvable, then H is nilpotent.

Properties of the Frobenius Kernel

Thompson showed that K is always nilpotent. Any subgroup of H of order p^2 or pq is cyclic p, q primes. If $P \in Syl_p(H)$, $p \neq 2$, then P is cyclic and $p = 2$, P is cyclic or generalized quaternion.

Properties of the Frobenius Kernel

K has an automorphism without fixed points, if $|H|$ is even, then K is abelian.
Example that K is non-abelian.
D_{2n}, n odd is Frobenius with kernel of order 2.
Praglandan Perumal: Msc. Thesis
$V_2(5) : SL_2(5)$
$GF(q)^*$ acts by right multiplication on $(GF(q), +)$. The corresponding semidirect product $GF(q)^*(q)$ is Frobenius group.

Thank You for Your Attention