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On the existence results for (p, q)-Kirchhoff type systems involving nonlocal operator and Caffarelli-Kohn-Nirenberg exponents

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Abstract

Using the method of sub-super solutions, we study the existence of positive solutions for a class of kirchhoff type systems with combined nonlinear effects involving nonlocal operator.

1. introduction

The study of positive solutions of singular partial differential equations or systems has been an extremely active research topic during the past few years. Such singular nonlinear problems arise naturally and they occupy a central role in the interdisciplinary research between analysis, geometry, biology, elasticity, mathematical physics, etc. The paper deal with the existence of positive solution for the nonlinear system

$$\begin{cases} -M_1\Big(\displaystyle\int_\Omega |\nabla u|^p dx\Big) div\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-(a+1)p+c_1}(\alpha_1 f(v) + \beta_1 h(u)), & x \in \Omega, \\ -M_2\Big(\displaystyle\int_\Omega |\nabla v|^q dx\Big) div\left(|x|^{-bq}|\nabla v|^{q-2}\nabla v\right) = |x|^{-(b+1)q+c_2}(\alpha_2 g(u) + \beta_2 k(v)), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $0 \in \Omega$, $1 < p, q < N$, $0 < a < \frac{N-p}{p}$, $0 < b < \frac{N-q}{q}$ and $c_1, c_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ are positive parameters. Here M_1, M_2 satisfy the following condition:
(H1) $M_i : \mathbb{R}_0^+ \to \mathbb{R}^+$, $i = 1, 2$, are two continuous and increasing functions and $0 < m_i \leq M_i(t) \leq m_{i,\infty}$ for all $t \in \mathbb{R}_0^+$, where $\mathbb{R}_0^+ := [0, +\infty)$.
Moreover $f, g, h, k : [0, \infty) \to [0, \infty)$ are nondecreasing continuous. System (1) is related to the stationary problem of a model introduced by Kirchhoff [14].In recent years, problems involving Kirchhoff type operators have been studied

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in many papers, we refer to [1, 2, 6, 9, 18–20] in which the authors have used variational method and topological method to get the existence of solutions for (1).These problems are interesting in applications and raise many difficult mathematical problems.
A crucial milestone in the understanding of the elliptic problems involving the singular quasilinear elliptic operator $-\text{div}\,(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$ is the paper by Caffarelli, Kohn and Nirenberg [7] (see also [11]). The study of this type of problem is motivated by its various applications,for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [4], [8]). The above problem is nonlocal because of the presence of the term involving the function M, which makes the equation in (1) a pointwise identity no longer.
We are inspired by the ideas in the interesting paper [17], in which the authors considered (1) in the case $M_1(t) = M_2(t) \equiv 1$. Using the sub-supersolution method combining a comparison principle introduced in [3], the authors established the existence of a positive solution for (1). We make the following assumptions:
(H2) $f, g, h, k : [0, \infty) \to [0, \infty)$ are C^1 nondecreasing functions such that

$$\lim_{s\to\infty} f(s) = \lim_{s\to\infty} g(s) = \lim_{s\to\infty} h(s) = \lim_{s\to\infty} k(s) = \infty.$$

(H3) For all $A > 0$,

$$\lim_{s\to\infty} \frac{f\left(Ag(s)^{\frac{1}{q-1}}\right)}{s^{p-1}} = 0.$$

(H4)

$$\lim_{s\to\infty} \frac{h(s)}{s^{p-1}} = \lim_{s\to\infty} \frac{k(s)}{s^{q-1}} = 0.$$

Now we are ready to state our existence result.

Theorem 1.1. *Assume (H1)-(H4) hold. Then there exists a positive large solution of system (1) when $\alpha_1 + \beta_1$ and $\alpha_2 + \beta_2$ are large.*

2. Preliminaries

In this paper, we denote $W_0^{1,p}(\Omega, ||x||^{-ap})$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $||u|| = \left(\int_\Omega ||x||^{-ap}|\nabla u|^p dx\right)^{\frac{1}{p}}$. To precisely state our existence result, we consider the eigenvalue problem

$$\begin{cases} -div\left(|x|^{-sr}|\nabla\phi|^{r-2}\nabla\phi\right) = \lambda|x|^{-(s+1)r+t}|\phi|^{r-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

For $r = p, s = a$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2) such that $\phi_{1,p}(x) > 0$ in Ω, and $||\phi_{1,p}||_\infty = 1$ and for $r = q, s = b$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of (2) such that $\phi_{1,q}(x) > 0$ in Ω, and $||\phi_{1,q}||_\infty = 1$ (see [15, 22]). It can be shown that $\frac{\partial\phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$. Here n is the outward normal. This result is well known and hence, depending on Ω, there exist positive constants $m, \delta, \sigma_p, \sigma_q$ such that

$$\lambda_{1,r}|x|^{-(s+1)r+t}\phi_{1,r}^r - |x|^{-sr}|\nabla\phi_{1,r}|^r \leq -m, \quad x \in \bar{\Omega}_\delta, \tag{3}$$

$$\phi_{1,r} \geq \sigma_r, \quad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\delta, \tag{4}$$

with $r = p, q; s = a, b; t = c_1, c_2$ and $\bar{\Omega}_\delta = \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$ (see [15]). We will also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0^{1,p}(\Omega, ||x||^{-ap}) \times W_0^{1,q}(\Omega, ||x||^{-bq})$ for the system

$$\begin{cases} -div\left(|x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p\right) = |x|^{-(a+1)p+c_1}, & x \in \Omega, \\ -div\left(|x|^{-bq}|\nabla\zeta_q|^{q-2}\nabla\zeta_q\right) = |x|^{-(b+1)q+c_2}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

to discuss our existence result. It is known that $\zeta_r(x) > 0$ in Ω and $\frac{\partial\zeta_r(x)}{\partial n} < 0$ on $\partial\Omega$, for $r = p, q$ (see [15]).

We will prove our results by using the method of sub- and supersolutions, we refer the readers to recent papers [1, 5, 12, 13] on the topic.
A pair of nonnegative functions $(\psi_1,\psi_2),(z_1,z_2)$ are called subsolution and supersolution of (1) if they satisfy $(\psi_1,\psi_2)=(0,0)=(z_1,z_2)$ on $\partial\Omega$ and

$$\begin{aligned}
&M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx \le \int_\Omega |x|^{-(a+1)p+c_1}(\alpha_1 f(\psi_2)+\beta_1 h(\psi_1))w dx,\\
&M_2\left(\int_\Omega |\nabla\psi_2|^q\,dx\right)\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx \le \int_\Omega |x|^{-(b+1)q+c_2}(\alpha_2 g(\psi_1)+\beta_2 k(\psi_2))w dx,\\
&M_1\left(\int_\Omega |\nabla z_1|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w dx \ge \int_\Omega |x|^{-(a+1)p+c_1}(\alpha_1 f(z_2)+\beta_1 h(z_1))w dx,\\
&M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w dx \ge \int_\Omega |x|^{-(b+1)q+c_2}(\alpha_2 g(z_1)+\beta_2 k(z_2))w dx,
\end{aligned} \tag{6}$$

for all $w\in W=\{w\in C_0^\infty(\Omega)|w\ge 0, x\in\Omega\}$. A key role in our arguments will be played by the following auxiliary result. Its proof is similar to those presented in [12], the reader can consult further the papers [1, 13].

Lemma 2.1. *Assume that $M:\mathbb{R}_0^+\to\mathbb{R}^+$ is continuous and increasing, and there exists $m_0>0$ such that $M(t)\ge m_0$ for all $t\in\mathbb{R}_0^+$. If the functions $u,v\in W_0^{1,p}(\Omega,|x|^{-ap})$ satisfy*

$$\begin{aligned}
&M\left(\int_\Omega |\nabla u|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx\\
\le\ & M\left(\int_\Omega |\nabla v|^p\,dx\right)\int_\Omega |x|^{-ap}|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx
\end{aligned} \tag{7}$$

for all $\varphi\in W_0^{1,p}(\Omega,|x|^{-ap})$, $\varphi\ge 0$, then $u\le v$ in Ω.

From Lemma 3.1 we can establish the basic principle of the sub- and supersolutions method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases}
-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)=|x|^{-(a+1)p+c_1}h(x,u,v) \text{ in } \Omega,\\
-M_2\left(\int_\Omega |\nabla v|^q\,dx\right)\operatorname{div}(|x|^{-bq}|\nabla v|^{q-2}\nabla v)=|x|^{-(b+1)q+c_2}k(x,u,v) \text{ in } \Omega,\\
u=v=0 \text{ on } x\in\partial\Omega,
\end{cases} \tag{8}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h,k:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions

(HK1) $h(x,s,t)$ and $k(x,s,t)$ are Carathéodory functions and they are bounded if s,t belong to bounded sets.

(KH2) There exists a function $g:\mathbb{R}\to\mathbb{R}$ being continuous, nondecreasing, with $g(0)=0$, $0\le g(s)\le C(1+|s|^{\min\{p,q\}-1})$ for some $C>0$, and applications $s\mapsto h(x,s,t)+g(s)$ and $t\mapsto k(x,s,t)+g(t)$ are nondecreasing, for a.e. $x\in\Omega$.

If $u,v\in L^\infty(\Omega)$, with $u(x)\le v(x)$ for a.e. $x\in\Omega$, we denote by $[u,v]$ the set $\{w\in L^\infty(\Omega):\ u(x)\le w(x)\le v(x)$ for a.e. $x\in\Omega\}$. Using 2.1 and the method as in the proof of Theorem 2.4 of [15] (see also Section 4 of [10]), we can establish a version of the abstract lower and upper-solution method for our class of the operators as follows.

Proposition 2.2. *Let $M_1,M_2:\mathbb{R}_0^+\to\mathbb{R}^+$ be two functions satisfying the condition **(H1)**. Assume that the functions h,k satisfy the conditions (HK1) and (HK2). Assume that $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$, are respectively, a weak subsolution and a weak supersolution of system (8) with $\underline{u}(x)\le\overline{u}(x)$ and $\underline{v}(x)\le\overline{v}(x)$ for a.e. $x\in\Omega$. Then there exists a minimal (u_*,v_*) (and, respectively, a maximal (u^*,v^*)) weak solution for system (8) in the set $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. In particular, every weak solution $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ of system (8) satisfies $u_*(x)\le u(x)\le u^*(x)$ and $v_*(x)\le v(x)\le v^*(x)$ for a.e. $x\in\Omega$.*

3. Proof of the main results

Proof. Sincef, g, h, k are continuous and nondecreasing, we have $f(x), g(x), h(x), k(x) \geq k_0$ for all $x \geq 0$ and for some $k_0 > 0$. Choose $r > 0$ such that

$$r \leq \min\{|x|^{-(a+1)p+c_1}, |x|^{-(b+1)q+c_2}\},$$

in $\bar{\Omega}_\delta$. We shall verify that

$$(\psi_1, \psi_2) = \left(\left[\frac{(\alpha_1 + \beta_1)k_0 r}{m m_{1,\infty}} \right]^{\frac{1}{p-1}} \left(\frac{p-1}{p} \right) \phi_{1,p}^{\frac{p}{p-1}}, \left[\frac{(\alpha_2 + \beta_2)k_0 r}{m m_{2,\infty}} \right]^{\frac{1}{q-1}} \left(\frac{q-1}{q} \right) \phi_{1,q}^{\frac{q}{q-1}} \right)$$

is a sub-solution of (1). Let $w \in W$, Then a calculation shows that

$$\begin{aligned} &M_1 \left(\int_\Omega |\nabla \psi_1|^p \, dx \right) \int_\Omega |x|^{-ap} |\nabla \psi_1|^{p-2} \nabla \psi_1 \nabla w dx \\ &\qquad = M_1 \left(\int_\Omega |\nabla \psi_1|^p \, dx \right) \left(\frac{(\alpha_1 + \beta_1)k_0 r}{m m_{1,\infty}} \right) \int_\Omega |x|^{-ap} \phi_{1,p} |\nabla \phi_{1,p}|^{p-2} \nabla \phi_{1,p} \nabla w dx \\ &\qquad \leq m_{1,\infty} \left(\frac{(\alpha_1 + \beta_1)k_0 r}{m m_{1,\infty}} \right) \int_\Omega |x|^{-ap} |\nabla \phi_{1,p}|^{p-2} \nabla \phi_{1,p} \left[\nabla(\phi_{1,p} w) - |\nabla \phi_{1,p}|^p w \right] dx \\ &\qquad \leq \left(\frac{(\alpha_1 + \beta_1)k_0 r}{m} \right) \int_\Omega [\lambda_{1,p} |x|^{-(a+1)p+c_1} \phi_{1,p}^p - |x|^{-ap} |\nabla \phi_{1,p}|^p] w dx. \end{aligned}$$

Similarly

$$\begin{aligned} &M_2 \left(\int_\Omega |\nabla \psi_2|^q \, dx \right) \int_\Omega |x|^{-bq} |\nabla \psi_2|^{q-2} \nabla \psi_2 \nabla w dx \\ &\qquad \leq \left(\frac{(\alpha_2 + \beta_2)k_0 r}{m} \right) \int_\Omega [\lambda_{1,q} |x|^{-(b+1)q+c_2} \phi_{1,q}^q - |x|^{-bq} |\nabla \phi_{1,q}|^q] w dx. \end{aligned}$$

First we consider the case when $x \in \bar{\Omega}_\delta$. We have $\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla \phi_{1,p}|^p \leq -m$ on $\bar{\Omega}_\delta$. Since $\psi_1(x), \psi_2(x) \geq 0$ in Ω, it follows that

$$-k_0 r \leq \min\{|x|^{-(a+1)p+c_1} f(\psi_2), |x|^{-(a+1)p+c_1} h(\psi_1)\},$$

in Ω. Hence, we have

$$\begin{aligned} &\left(\frac{(\alpha_1 + \beta_1)k_0 r}{m} \right) \int_{\bar{\Omega}_\delta} [\lambda_{1,p} |x|^{-(a+1)p+c_1} \phi_{1,p}^p - |x|^{-ap} |\nabla \phi_{1,p}|^p] w dx \\ &\leq -(\alpha_1 + \beta_1)k_0 r \int_{\Omega_\delta} w dx \leq \int_{\bar{\Omega}_\delta} |x|^{-(a+1)p+c_1} (\alpha_1 f(\psi_2) + \beta_1 h(\psi_1)) w dx. \end{aligned}$$

A similar argument shows that

$$\begin{aligned} &M_2 \left(\int_\Omega |\nabla \psi_2|^q \, dx \right) \left(\frac{(\alpha_2 + \beta_2)k_0 r}{m m_{2,\infty}} \right) \int_{\bar{\Omega}_\delta} [\lambda_{1,q} |x|^{-(b+1)q+c_2} \phi_{1,q}^q - |x|^{-bq} |\nabla \phi_{1,q}|^q] w dx \\ &\leq \int_{\bar{\Omega}_\delta} |x|^{-(b+1)q+c_2} (\alpha_2 g(\psi_1) + \beta_2 k(\psi_2)) w dx. \end{aligned}$$

On the other hand, on $\Omega \setminus \bar{\Omega}_\delta$, since $\phi_{1,p} \geq \sigma_p, \phi_{1,q} \geq \sigma_q$ for some $0 < \sigma_p, \sigma_q < 1$,
if $\alpha_1 + \beta_1$ and $\alpha_2 + \beta_2$ are enough large, then by (H2) we have

$$f(\psi_2), h(\psi_1), g(\psi_1), k(\psi_2) \geq \frac{k_0 r}{m} \max\{\lambda_{1,p}, \lambda_{1,q}\}.$$

Hence

$$\begin{aligned}
&\left(\frac{(\alpha_1+\beta_1)k_0 r}{m}\right)\int_{\Omega\setminus\bar{\Omega}_\delta}[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}|^p-|x|^{-ap}|\nabla\phi_{1,p}|^p]wdx\\
&\left(\frac{(\alpha_1+\beta_1)k_0 r}{m}\right)\int_{\Omega\setminus\bar{\Omega}_\delta}|x|^{-(a+1)p+c_1}\lambda_{1,p}wdx\\
&\leq\int_{\Omega\setminus\bar{\Omega}_\delta}|x|^{-(a+1)p+c_1}(\alpha_1 f(\psi_2)+\beta_1 h(\psi_1))wdx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(\frac{(\alpha_2+\beta_2)k_0 r}{m}\right)\int_{\Omega\setminus\bar{\Omega}_\delta}[\lambda_{1,q}|x|^{-(b+1)q+c_2}\phi_{1,q}|^q-|x|^{-bq}|\nabla\phi_{1,q}|^q]wdx\\
&\leq\int_{\Omega\setminus\bar{\Omega}_\delta}|x|^{-(b+1)q+c_2}(\alpha_2 g(\psi_1)+\beta_2 k(\psi_2))wdx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega |x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla wdx\\
&\leq\int_\Omega |x|^{-(a+1)p+c_1}(\alpha_1 f(\psi_2)+\beta_1 h(\psi_1))wdx,\\
&\int_\Omega |x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla wdx\\
&\leq\int_\Omega |x|^{-(b+1)q+c_2}(\alpha_2 g(\psi_1)+\beta_2 k(\psi_2))wdx,
\end{aligned}$$

i.e., (ψ_1,ψ_2) is a sub-solution of (1).
Now, we will prove there exists a M large enough so that

$$(z_1,z_2)=\left(M\zeta_p(x),(\frac{\alpha_2+\beta_2}{m_2})^{\frac{1}{q-1}}g(M||\zeta_p||_\infty)^{\frac{1}{q-1}}\zeta_q(x)\right),$$

is a super-solution of (1).
A calculation shows that:

$$\begin{aligned}
M_1\left(\int_\Omega|\nabla z_1|^p\,dx\right)\int_\Omega|x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1\nabla wdx &=M^{p-1}M_1\left(\int_\Omega|\nabla z_1|^p\,dx\right)\int_\Omega|x|^{-ap}|\nabla\zeta_p|^{p-2}\nabla\zeta_p\nabla wdx\\
&\geq m_1M^{p-1}\int_\Omega|x|^{-(a+1)p+c_1}wdx.
\end{aligned}$$

By (H3)-(H4) we can choose M large enough so that

$$\begin{aligned}
m_1M^{p-1}&\geq\alpha_1 f\left((\frac{\alpha_2+\beta_2}{m_2})^{\frac{1}{q-1}}||\zeta_q||_\infty g(M||\zeta_p||_\infty)^{\frac{1}{q-1}}\right)+\beta_1 h(M||\zeta_p||_\infty)\\
&\geq\alpha_1 f\left((\frac{\alpha_2+\beta_2}{m_2})^{\frac{1}{q-1}}\zeta_q(x)g(M||\zeta_p||_\infty)^{\frac{1}{q-1}}\right)+\beta_1 h(M\zeta_p(x))\\
&=\alpha_1 f(z_2)+\beta_1 h(z_1).
\end{aligned}$$

Hence

$$\begin{aligned}
&M_1\left(\int_\Omega|\nabla z_1|^p\,dx\right)\int_\Omega|x|^{-ap}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla wdx\\
&\geq\int_\Omega|x|^{-(a+1)p+c_1}(\alpha_1 f(z_2)+\beta_1 h(z_1))wdx.
\end{aligned}$$

Again, by (H4) for M large enough we have

$$g(M||\zeta_p||_\infty) \geq k\left((\frac{\alpha_2+\beta_2}{m_2})^{\frac{1}{q-1}} g(M||\zeta_p||_\infty)^{\frac{1}{q-1}} ||\zeta_q||_\infty \right).$$

Hence

$$\begin{aligned}
&M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |x|^{-bq}|\nabla z_2|^{q-2}\nabla w dx\\
=&(\frac{\alpha_2+\beta_2}{m_2})\int_\Omega |x|^{-(b+1)q+c_2} g(M||\zeta_p||_\infty) w dx\\
\geq&\int_\Omega |x|^{-(b+1)q+c_2}[\alpha_2 g(z_1)+\beta_2 g(M||\zeta_p||_\infty)] w dx\\
\geq&\int_\Omega |x|^{-(b+1)q+c_2}\left[\alpha_2 g(z_1)+\beta_2 k\left((\frac{\alpha_2+\beta_2}{m_2}) g(M||\zeta_p||_\infty)^{\frac{1}{q-1}}||\zeta_q||_\infty\right)\right] w dx\\
\geq&\int_\Omega |x|^{-(b+1)q+c_2}\left[\alpha_2 g(z_1)+\beta_2 k\left(z_2\right)\right] w dx,
\end{aligned}$$

i.e., (z_1, z_2) is a supersolution of (1) with $z_i \geq \psi_{i,\lambda}$, $i = 1, 2$ for a M large enough. Thus, by ([16]) there exists a positive solution (u, v) of (1) such that $(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2)$. This completes the proof of Theorem 1.1.

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Existence and stability results for a class of nonlinear fractional q-integro-differential equation

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Abstract

This paper deals with the stability results for solution of a fractional q–integro-differential problem with integral conditions. Using the Krasnoselskii's, Banach fixed point theorems, we proof the existence and uniqueness results. Based on the results obtained, conditions are provided that ensure the generalized Ulam stability of the original system on time scale. The results are illustrated by the examples under numerical technique.

1. Introduction and formulation of the problem

It is interesting to study solution to fractional q-integro-differential problem with integral conditions, which will allow a generalized stability. The authors in [1], considered the problem for the system (1) and we generalized the system in the $q\mathbb{FDE}$ which it is not explicitly presented, and therefore it makes sense to consider for $\mathfrak{t} \in \bar{\mathbb{I}}$, $\sigma, \nu \in \mathbb{I}$, the problem for the system

$$^{C}\mathbb{D}_q^{\sigma+\nu}[y](\mathfrak{t}) = h_1(\mathfrak{t}, y(\mathfrak{t})) + \mathbb{I}_q^{\sigma}[h_2](\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}, \xi, y(\xi))\, \mathrm{d}\xi, \tag{1}$$

under boundary condition $y(0) = \eta \int_0^{\tau^*} y(\xi)\, \mathrm{d}\xi$, for $\tau^* \in \mathbb{I}$, where η is a real constant, $^{C}\mathbb{D}_q^{\sigma+\nu}$ is the Caputo fractional q-derivative of order $\sigma+\nu$, $\mathbb{I}_q^{\sigma}$ denotes the left sided Riemann–Liouville fractional q-integral of order σ and $h_i : \bar{\mathbb{I}} \times \mathfrak{H} \to \mathfrak{H}$ $(i = 1, 2)$, $\Theta : \bar{\mathbb{I}}^2 \times \mathfrak{H} \to \mathfrak{H}$, are an appropriate functions satisfying some conditions which will be stated later. $\mathfrak{H}$ here is a Banach space equipped with the norm $\|.\|$.

Here we focused our study on the question of existence and uniqueness in Sec. 3. And Sec. 4 is devoted to show a generalized stability. Note that this representation also allows us to generalize the results obtained recently in the literature. The paper is ended by two examples illustrating our results.

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2. Notations and notions preliminaries

We recall some essential preliminaries that are used for the results of the subsequent sections. Let $\mathfrak{t}_0 \in \mathbb{R}$ and $q \in \mathbb{I}$. The time scale $\mathbb{T}_{t_0}$ is defined by $\mathbb{T}_{t_0} = \{0\} \cup \{\mathfrak{t} : \mathfrak{t} = \mathfrak{t}_0 q^n, \forall n \in \mathbb{N}\}$. If there is no confusion concerning $\mathfrak{t}_0$ we shall denote $\mathbb{T}_{\mathfrak{t}_0}$ by $\mathbb{T}$. Let $s \in \mathbb{R}$. Define $[s]_q = (1-q^s)/(1-q)$ [7]. The q-factorial function $(y-z)_q^{(n)}$ is defined by

$$(y-z)_q^{(n)} = \prod_{k=0}^{n-1} (y - zq^k), \qquad n \in \mathbb{N}_0, \tag{2}$$

and $(y-z)_q^{(0)} = 1$, where $y, z \in \mathbb{R}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ ([2]). Also, we have

$$(y-z)_q^{(\sigma)} = y^\sigma \prod_{k=0}^{\infty} \frac{y - zq^k}{y - zq^{\sigma+k}}, \qquad \sigma \in \mathbb{R}, s \neq 0. \tag{3}$$

In the paper [4], the authors proved $(y-z)_q^{(\sigma+\nu)} = (y-z)_q^{(\sigma)}(y-q^\sigma z)_q^{(\nu)}$ and $(sy-sz)_q^{(\sigma)} = s^\sigma (y-z)_q^{(\sigma)}$. If $z=0$, then it is clear that $y^{(\sigma)} = y^\sigma$. The q-Gamma function is given by [7]

$$\Gamma_q(y) = (1-q)^{1-y}(1-q)_q^{(y-1)}, \qquad (y \in \mathbb{R} \backslash \{\cdots, -2, -1, 0\}).$$

In fact, by using (3), we have

$$\Gamma_q(y) = (1-q)^{1-y} \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{y+k-1}}. \tag{4}$$

Algorithm 1 shows the MATLAB lines for calculation of $\Gamma_q(y)$ which we tend n to infinity in it.

Algorithm 1: MATLAB lines for calculation $\Gamma_q(x)$.

```
function p = qGamma(q,x,n)
s=1;
for k=0:n
        s=s*(1-q^(k+1))/(1-q^(x+k-1));
end;
p = s*(1-q)^(1-x);
end
```

Note that, $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$ [4, Lemma 1]. For any positive numbers σ and ν, the q-beta function define by

$$B_q(\sigma,\nu) = \int_0^1 (1-\xi)_q^{(\sigma-1)} \xi^{\nu-1}\, \mathrm{d}_q\xi = \frac{\Gamma_q(\sigma)\Gamma_q(\nu)}{\Gamma_q(\sigma+\nu)}. \tag{5}$$

For a function $w : \mathbb{T} \to \mathbb{R}$, the q-derivative of w, is

$$\mathbb{D}_q[y](\mathfrak{t}) = \left(\frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)_q y(\mathfrak{t}) = \frac{y(q\mathfrak{t}) - y(\mathfrak{t})}{\mathfrak{t}(1-q)}, \tag{6}$$

for all $\mathfrak{t} \in \mathbb{T} \backslash \{0\}$, and $\mathbb{D}_q[y](0) = \lim_{\mathfrak{t}\to 0} \mathbb{D}_q[y](\mathfrak{t})$ [2]. Also, the higher order q-derivative of the function y is defined by $\mathbb{D}_q^n[y](\mathfrak{t}) = \mathbb{D}_q\left[\mathbb{D}_q^{n-1}[y]\right](\mathfrak{t})$, for all $n \geq 1$, where $\mathbb{D}_q^0[y](\mathfrak{t}) = y(\mathfrak{t})$ [2]. In fact

$$\mathbb{D}_q^n[y](\mathfrak{t}) = \frac{1}{\mathfrak{t}^n(1-q)^n} \sum_{k=0}^{n} \frac{(1-q^{-n})_q^{(k)}}{(1-q)_q^{(k)}} q^k y(\mathfrak{t}q^k), \tag{7}$$

for $\mathfrak{t} \in \mathbb{T} \setminus \{0\}$ [3].

Remark 2.1. *By using Eq.* (2)*, we can change Eq.* (7) *as follows:*

$$\mathbb{D}_q^n[y](\mathfrak{t}) = \frac{1}{\mathfrak{t}^n(1-q)^n} \sum_{k=0}^{n} \prod_{i=0}^{k-1} \frac{1-q^{i-n}}{1-q^{i+1}}\, q^k y(\mathfrak{t}q^k). \tag{8}$$

The q-integral of the function y is defined by

$$\mathbb{I}_q[y](\mathfrak{t}) = \int_0^{\mathfrak{t}} y(\xi)\, \mathrm{d}_q\xi = \mathfrak{t}(1-q)\sum_{k=0}^{\infty} q^k y(\mathfrak{t}q^k), \tag{9}$$

for $0 \leq \mathfrak{t} \leq b$, provided the series is absolutely converges [2]. By using the Algorithm 2, we can obtain the numerical results of $\mathbb{I}_q[y](\mathfrak{t})$ when $n \to \infty$.

Algorithm 2: MATLAB lines for calculation $\mathcal{I}_q[w](t)$.

```
function p = Iq(q,x,n,fun)
s=1;
for k=0:n
        s=s+q^k*eval(subs(fun,x*q^k));
end;
p=x*(1-q)*s;
end
```

If s in $[0, b]$, then

$$\int_s^b y(\xi)\, \mathrm{d}_q\xi = \mathbb{I}_q[y](b) - \mathbb{I}_q[y](s) = (1-q)\sum_{k=0}^{\infty} q^k \left[by(bq^k) - sy(sq^k)\right],$$

whenever the series exists. The operator $\mathbb{I}_q^n$ is given by $\mathbb{I}_q^0[y](\mathfrak{t}) = y(\mathfrak{t})$ and

$$\mathbb{I}_q^n[y](\mathfrak{t}) = \mathbb{I}_q\left[\mathbb{I}_q^{n-1}[y]\right](\mathfrak{t}),$$

for $n \geq 1$ and $y \in C([0, b])$ [2]. It has been proved that

$$\mathbb{D}_q\left[\mathbb{I}_q[y]\right](\mathfrak{t}) = y(\mathfrak{t}), \qquad \mathbb{I}_q\left[\mathbb{D}_q[y]\right](\mathfrak{t}) = y(\mathfrak{t}) - y(0),$$

whenever the function y is continuous at $\mathfrak{t} = 0$ [2]. The fractional Riemann–Liouville type q-integral of the function y is defined by

$$\mathbb{I}_q^{\sigma}[y](\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{(\sigma-1)} \frac{y(\xi)}{\Gamma_q(\sigma)}\, \mathrm{d}_q\xi, \qquad \mathbb{I}_q^0[y](\mathfrak{t}) = y(\mathfrak{t}), \tag{10}$$

for $\mathfrak{t} \in [0, 1]$ and $\sigma > 0$ [3, 6].

Remark 2.2. *By using Eqs.* (3)*,* (4) *and* (9)*, we obtain*

$$\begin{aligned}\int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{(\sigma-1)} \frac{y(\xi)}{\Gamma_q(\sigma)}\, \mathrm{d}_q\xi &= \frac{1}{\Gamma_q(\sigma)} \int_0^{\mathfrak{t}} \mathfrak{t}^{\sigma-1} \prod_{i=0}^{\infty} \frac{\mathfrak{t}-\xi q^i}{\mathfrak{t}-\xi q^{\sigma+i-1}} y(\xi)\, \mathrm{d}_q\xi \\ &= \mathfrak{t}^{\sigma}(1-q)^{\sigma} \prod_{i=0}^{\infty} \frac{1-q^{\sigma+i-1}}{1-q^{i+1}} \sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty} \frac{1-q^{k+i}}{1-q^{\sigma+k+i-1}} y(\mathfrak{t}q^k).\end{aligned}$$

Therefore,

$$\mathbb{I}_q^{\sigma}[y](\mathfrak{t}) = \mathfrak{t}^{\sigma}(1-q)^{\sigma} \lim_{n\to\infty} \sum_{k=0}^{n} q^k \prod_{i=0}^{n} \frac{\left(1-q^{\sigma+i-1}\right)\left(1-q^{k+i}\right)}{\left(1-q^{i+1}\right)\left(1-q^{\sigma+k+i-1}\right)} y(\mathfrak{t}q^k), \tag{11}$$

The Caputo fractional q-derivative of the function y is defined by

$$ {}^{C}\mathbb{D}_q^{\sigma}[y](\mathfrak{t}) = \mathbb{I}_q^{[\sigma]-\sigma}\left[\mathbb{D}_q^{[\sigma]}[y]\right](\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{([\sigma]-\sigma-1)} \frac{\mathbb{D}_q^{[\sigma]}[y](\xi)}{\Gamma_q([\sigma]-\sigma)}\, \mathrm{d}_q\xi \tag{12} $$

for $\mathfrak{t} \in \{0,1]$ and $\sigma > 0$ [6, 9]. It has been proved that $\mathbb{I}_q^{\nu}\left[\mathbb{I}_q^{\sigma}[y]\right](\mathfrak{t}) = \mathbb{I}_q^{\sigma+\nu}[y](\mathfrak{t})$, and ${}^{C}\mathbb{D}_q^{\sigma}\left[\mathbb{I}_q^{\sigma}[y]\right](\mathfrak{t}) = y(\mathfrak{t})$, where $\sigma, \nu \geq 0$ [6]. Also, [6]

$$ \mathbb{I}_q^{\sigma}\left[\mathbb{D}_q^{n}[y]\right](\mathfrak{t}) = \mathbb{D}_q^{n}\left[\mathbb{I}_q^{\sigma}[y]\right](\mathfrak{t}) - \sum_{k=0}^{n-1} \frac{\mathfrak{t}^{\sigma+k-n}\mathbb{D}_q^{k}[y](0)}{\Gamma_q(\sigma+k-n+1)}, \qquad \sigma > 0,\ n \geq 1. $$

Remark 2.3. *From Eq.(4), Remark 2.1 and Eq. (11) in Remark 2.2, we obtain*

$$ \begin{aligned} &\int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{([\sigma]-\sigma-1)} \frac{\mathbb{D}_q^{[\sigma]}[y](\xi)}{\Gamma_q([\sigma]-\sigma)}\, \mathrm{d}_q\xi \\ &= \int_0^{\mathfrak{t}} \frac{\mathfrak{t}^{[\sigma]-\sigma-1}}{\Gamma_q([\sigma]-\sigma)} \left[\prod_{i=0}^{\infty} \frac{\mathfrak{t}-\xi q^{i}}{\mathfrak{t}-\xi q^{[\sigma]-\sigma-1+i}}\right] \\ &\quad \times \left(\frac{1}{\mathfrak{t}^{[\sigma]}(1-q)^{[\sigma]}} \sum_{k=0}^{[\sigma]} \left[\prod_{i=0}^{k-1} \frac{(1-q^{i-[\sigma]})}{(1-q^{i+1})}\right] q^{k} y(\mathfrak{t}q^{k}) \right) \mathrm{d}_q\xi \\ &= \frac{1}{\mathfrak{t}^{\sigma}(1-q)^{\sigma-[\sigma]}} \sum_{k=0}^{\infty} \left(\left[\prod_{i=0}^{\infty} \frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\right] \right. \\ &\quad \times \left.\left(\sum_{m=0}^{[\sigma]} \left[\prod_{i=0}^{m-1} \frac{(1-q^{i-[\sigma]})}{(1-q^{i+1})}\right] q^{m} y\left(\mathfrak{t}q^{k+m}\right) \right)\right). \end{aligned} $$

Thus, we have

$$ \begin{aligned} {}^{c}\mathbb{D}_q^{\sigma}[y](\mathfrak{t}) &= \frac{1}{\mathfrak{t}^{\sigma}(1-q)^{\sigma-[\sigma]}} \lim_{n\to\infty} \sum_{k=0}^{n} \left(\left[\prod_{i=0}^{n} \frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\right] \right. \\ &\quad \times \left.\left(\sum_{m=0}^{[\sigma]} \left[\prod_{i=0}^{m-1} \frac{(1-q^{i-[\sigma]})}{(1-q^{i+1})}\right] q^{m} y\left(\mathfrak{t}q^{k+m}\right) \right)\right). \end{aligned} \tag{13} $$

Now, we introduce some basic definitions, lemmas and theorems which are used in the subsequent sections.

Lemma 2.4. *[8] Let $y \in AC^{n}[\mathfrak{t}_1, \mathfrak{t}_2]$. Then, one has $\mathbb{I}^{\sigma}[{}^{C}\mathbb{D}_q^{\sigma}[y]](\mathfrak{t}) = y(\mathfrak{t}) + \sum_{i=0}^{n-1} c_i(\mathfrak{t}-\mathfrak{t}_1)^{i}$, $(c_0, c_1, \ldots, c_{n-1} \in \mathbb{R})$, for $n-1 < \sigma \leq n$, $n \in \mathbb{N}$.*

Lemma 2.5. *[8] Let $n-1 < \sigma \leq n$, $n \in \mathbb{N}$ and $y \in C[\mathfrak{t}_1, \mathfrak{t}_1]$. Then for all $\mathfrak{t} \in [\mathfrak{t}_1, \mathfrak{t}_2]$, we have ${}^{C}\mathbb{D}_{\mathfrak{t}_1}^{\sigma}[\mathbb{I}_{\mathfrak{t}_1}^{\sigma}[y]](\mathfrak{t}) = y(\mathfrak{t})$.*

Lemma 2.6. *[8] Let $\sigma \in (0,1)$. Then for each $y \in AC[0,1]$, $\mathbb{I}^{\sigma}[\mathbb{D}^{\sigma}[y]](\mathfrak{t}) = y(\mathfrak{t})$ for a.e. $\mathfrak{t} \in [0,1]$, where*

$$ \mathbb{D}^{\sigma}[y](\mathfrak{t}) = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}} \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)^{-\sigma} \frac{y(\xi)}{\Gamma(1-\sigma)}\, \mathrm{d}\xi. $$

Lemma 2.7. *(Banach fixed point theorem, [5]) Let $\mathfrak{B}$ be a non-empty complete metric space and $\mathcal{T}: \mathfrak{B} \to \mathfrak{B}$ is contraction mapping. Then, there exists a unique point $y \in \mathfrak{B}$ such that $\mathcal{T}(y) = y$.*

Lemma 2.8. *([5], Krasnoselskii fixed point theorem) Let $\mathfrak{E}$ be bounded, closed and convex subset in a Banach space $\mathfrak{B}$. If $\mathcal{T}_1, \mathcal{T}_2 : \mathfrak{E} \to \mathfrak{E}$ are two applications satisfying the following conditions: (A1) $\mathcal{T}_1(y) + \mathcal{T}_2(z) \in \mathfrak{E}$ for every $y, z \in \mathfrak{E}$; (A2) $\mathcal{T}_1$ is a contraction; (A3) $\mathcal{T}_2$ is compact and continuous. Then there exists $\mathfrak{v}^* \in \mathfrak{B}$ such that $\mathcal{T}_1(\mathfrak{v}^*) + \mathcal{T}_2(\mathfrak{v}^*) = \mathfrak{v}^*$.*

3. Existence results

Before presenting our main results, we need the following auxiliary lemma.

Lemma 3.1. *Let $\sigma+\nu \in \mathbb{I}$ and $\eta\tau^* \neq 1$. Assume that h_1, h_2 and Θ are three continuous functions. If $y \in C(\overline{\mathbb{I}}, \mathfrak{H})$, then y is solution of* (1) *iff y satisfies the IE*

$$\begin{aligned} y(\mathrm{t}) = &\int_0^{\mathrm{t}} \frac{(\mathrm{t}-\xi)_q^{(\sigma+\nu-1)}}{\Gamma_q(\sigma+\nu)} \Bigg[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s \\ &+ \int_0^{\xi} \frac{(\xi-a)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\Bigg]\,\mathrm{d}_q\xi \\ &+ \frac{\eta}{1-\eta\tau^*} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \Bigg[h_1(s, y(s)) + \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r \\ &+ \int_0^{s} \frac{(s-r)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\Bigg]\,\mathrm{d}_q s. \end{aligned} \tag{14}$$

Proof. Let $y \in C(\overline{\mathbb{I}}, \mathfrak{H})$ be a solution of (1). Firstly, we show that y is solution of integral equation (14). By Lemma 2.4, we obtain

$$\mathbb{I}_q^{\sigma+\nu}\left[{}^{C}\mathbb{D}_q^{\sigma+\nu}[y](\mathrm{t})\right] = y(\mathrm{t}) - y(0). \tag{15}$$

From equation (1) we have

$$\begin{aligned} \mathbb{I}_q^{\sigma+\nu}\left[{}^{C}\mathbb{D}_q^{\sigma+\nu}[y](t)\right] &= \mathbb{I}_q^{\sigma+\nu}\left[h_1(\mathrm{t}, y(\mathrm{t})) + +\mathbb{I}_q^{\sigma}[h_2](\mathrm{t}, y(\mathrm{t})) \int_0^{\mathrm{t}} \Theta(\mathrm{t}, \xi, y(\xi))\,\mathrm{d}\xi\right] \\ &= \int_0^{\mathrm{t}} \frac{(\mathrm{t}-\xi)_q^{(\sigma+\nu-1)}}{\Gamma_q(\sigma+\nu)} \Bigg[h_1(\mathrm{t}, y(\mathrm{t})) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s \\ &\quad + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\Bigg]\,\mathrm{d}_q\xi \end{aligned} \tag{16}$$

By substituting 16 in 15 with nonlocal condition in problem 14, we get

$$\begin{aligned} y(\mathrm{t}) = &\int_0^{\mathrm{t}} \frac{(\mathrm{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \Bigg[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s \\ &+ \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\Bigg]\,\mathrm{d}_q\xi + y(0). \end{aligned} \tag{17}$$

From integral boundary condition of our problem with using Fubini's theorem and after some computations, we get

$$
\begin{aligned}
y(0) &= \eta \int_0^{\tau^*} y(\xi)\,\mathrm{d}\xi \\
&= \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left(h_1(s, y(s)) + \int_0^s \varTheta(s, r, y(r))\,\mathrm{d}r \right.\right.\\
&\quad \left.\left. + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r \right) \mathrm{d}_q s \right] \mathrm{d}_q \xi + \eta \tau^* y(0) \\
&= \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} h_1(s, y(s))\,\mathrm{d}_q s \right] \mathrm{d}_q \xi \\
&\quad + \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \int_0^s \varTheta(s, r, y(r))\,\mathrm{d}r\,\mathrm{d}_q s \right] \mathrm{d}_q \xi \\
&\quad + \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\,\mathrm{d}_q s \right] \mathrm{d}_q \xi + \eta \tau^* y(0) \\
&= \eta \int_0^{\tau^*} \left(\int_s^{\tau^*} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\,\mathrm{d}_q \xi \right) h_1(s, y(s))\,\mathrm{d}_q s \\
&\quad + \eta \int_0^{\tau^*} \left(\int_s^{\tau^*} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\,\mathrm{d}_q \xi \right) \left(\int_0^s \varTheta(s, r, y(r))\,\mathrm{d}r \right) \mathrm{d}_q s \\
&\quad + \eta \int_0^{\tau^*} \left(\int_s^{\tau^*} \frac{(\xi - s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\,\mathrm{d}_q \xi \right) \\
&\quad \times \left(\int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r \right) \mathrm{d}_q s + \eta \tau^* y(0),
\end{aligned}
$$

that is

$$
\begin{aligned}
y(0) = \frac{\eta}{1 - \eta\tau^*} \int_0^{\tau^*} \frac{(\tau^* - s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu)} \Bigg[& h_1(s, y(s)) + \int_0^s \varTheta(s, r, y(r))\,\mathrm{d}r \\
& + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r \Bigg] \mathrm{d}_q s.
\end{aligned} \tag{18}
$$

Finally, by substituting (18) in (17), we find (14). Conversely, from Lemma 14 and by applying the operator ${}^{C}\mathbb{D}_q^{\sigma+\nu}$ on both sides of (14), we find

$$
\begin{aligned}
{}^{C}\mathbb{D}_q^{\sigma+\nu}[y](\mathfrak{t}) &= {}^{C}\mathbb{D}_q^{\sigma+\nu} \left[\mathbb{I}_q^{\sigma+\nu} \left[h_1(\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\mathfrak{t}} \varTheta(\mathfrak{t}, \xi, y(\xi))\,\mathrm{d}s + \mathbb{I}_q^{\sigma} h_2(\mathfrak{t}, y(\mathfrak{t})) \right] \right] + {}^{C}\mathbb{D}_q^{\sigma+\nu} y(0) \\
&= h_q(\mathfrak{t}, y(\mathfrak{t})) + \mathbb{I}_q^{\sigma} h_2(\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\mathfrak{t}} \varTheta(\mathfrak{t}, \xi, y(\xi))\,\mathrm{d}\xi.
\end{aligned} \tag{19}
$$

This means that y satisfies the equation in problem (1). Furthermore, by substituting $\mathfrak{t}$ by 0 in integral equation (14), we have clearly that the integral boundary condition in (1) holds. Therefore, y is solution of problem (1), which completes the proof. □

In order to prove the existence and uniqueness of solution for the problem (1) in $C(\bar{\mathbb{I}}, \mathfrak{H})$, we use two fixed point theorems. Firstly, we transform the system (1) into fixed point problem as $y = \mathfrak{U}y$, where $\mathfrak{U} : (\bar{\mathbb{I}}, \mathfrak{H}) \to (\bar{\mathbb{I}}, \mathfrak{H})$ is an operator defined by following

$$\begin{aligned}\mathfrak{U}y(\mathfrak{t}) = &\int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu)}\left[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\right] \mathrm{d}_q\xi \\ &+ \frac{\eta}{1-\eta\tau^*}\int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\left[h_1(s, y(s)) + \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r\right. \\ &\left.+ \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\right] \mathrm{d}_q s. \end{aligned} \tag{20}$$

3.1. Existence result by Krasnoselskii's fixed point

Theorem 3.2. *Consider continuous functions $h_1, h_2 : \overline{\mathbb{I}} \times \mathfrak{H} \to \mathfrak{H}$ and $\Theta : \overline{\mathbb{I}}^2 \times \mathfrak{H} \to \mathfrak{H}$ such that satisfying: (H_1) The inequalities $\|h_j(\mathfrak{t}, y(\mathfrak{t})) - h_j(\mathfrak{t}, z(\mathfrak{t}))\| \leq \mu_j \|y(\mathfrak{t}) - z(\mathfrak{t})\|,\ j = 1, 2$ and*

$$\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s})) - \Theta(\mathfrak{t}, \mathfrak{s}, z(\mathfrak{s}))\| \leq \mu^* \|y(\mathfrak{s}) - z(\mathfrak{s})\|,$$

where $\mu^*, \mu_j \geq 0,\ (j = 1, 2)$ with $\mu = \max\{\mu_1, \mu_2, \mu^*\}$; (H_2) There exist three functions $\varrho^*, \varrho_j \in L^\infty(\overline{\mathbb{I}}, \mathbb{R}^+)$, $(j = 1, 2)$, such that

$$\|h_j(\mathfrak{t}, y(\mathfrak{t}))\| \leq \varrho_j(\mathfrak{t})\|y(\mathfrak{t})\|, \qquad j = 1, 2,$$

and

$$\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s}))\| \leq \varrho^*(\mathfrak{t})\|y(\mathfrak{s})\|,$$

$\forall \mathfrak{t} \in \overline{\mathbb{I}}$, $y, z \in \mathfrak{H}$ and

$$(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G} := \Big\{(\mathfrak{t}, \mathfrak{s}) : 0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1\Big\}.$$

If $\lambda \leq 1$ and $\mu\lambda^* \leq 1$, then the problem (1) has at least one solution on $\overline{\mathbb{I}}$, where

$$\begin{aligned}\lambda = &\frac{\|\varrho_1\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty} B_q(\sigma+1, \sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu)} \\ &+ \frac{|\eta|\|\varrho_1\|_{L^\infty}{\tau^*}^{\sigma+\nu+1} + |\eta|\|\varrho^*\|_{L^\infty}{\tau^*}^{\sigma+\nu+1}}{|1-\eta\tau^*|\Gamma_q(\sigma+\nu+2)} \\ &+ \frac{|\eta|\|\varrho_2\|_{L^\infty}{\tau^*}^{2\sigma+\nu+1} B_q(\sigma+1, \sigma+\nu+1)}{|1-\nu\tau^*\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}, \end{aligned} \tag{21}$$

and

$$\lambda^* = \frac{|\eta|}{|1-\eta\tau^*|}\left[\frac{2{\tau^*}^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu+2)} + \frac{{\tau^*}^{2\sigma+\nu+1} B_q(\sigma+1, \sigma+\nu+1)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}\right]. \tag{22}$$

Proof. For any function $y \in C(\overline{\mathbb{I}}, \mathfrak{H})$, we define the norm

$$\|y\|_* := \max\Big\{e^{-\mathfrak{t}}\|y(\mathfrak{t})\| \ : \ \mathfrak{t} \in \overline{\mathbb{I}}\Big\},$$

and consider the closed ball $\mathbb{B}_\ell := \{y \in C(\overline{\mathbb{I}}, \mathfrak{H}) \ : \ \|y\|_* \leq \ell\}$. Next, let us define the operators $\mathfrak{U}_1, \mathfrak{U}_2$ on $\mathbb{B}_\ell$ as follows

$$\begin{aligned}\mathfrak{U}_1 y(\mathfrak{t}) = &\int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu)}\left[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s\right. \\ &\left.+ \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\right] \mathrm{d}_q\xi. \end{aligned} \tag{23}$$

and

$$\mathfrak{U}_2 y(\mathfrak{t}) = \frac{\eta}{1-\eta\tau^*}\int_0^{\tau} \frac{(\tau - s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\bigg[h_1(s, y(s)) + \int_0^s \Theta(s, r, y(r))\,\mathrm{d}r + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\bigg]\,\mathrm{d}_q s. \tag{24}$$

For $y, z \in \mathbb{B}_\ell$, $\mathfrak{t} \in \overline{\mathbb{I}}$ and by the assumption (H_2), we find

$$\begin{aligned}
\|\mathfrak{U}_1 y(\mathfrak{t}) + \mathfrak{U}_2 z(\mathfrak{t})\| &\le \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\bigg[\|h_1(\xi, y(\xi))\| + \int_0^{\xi} \|\Theta(\xi, s, y(s))\|\,\mathrm{d}s \\
&\quad + \int_0^{\xi} \frac{(\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(\xi, y(\xi))\|\,\mathrm{d}_q s\bigg]\,\mathrm{d}_q\xi \\
&\quad + \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^* - s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\bigg[\|h_1(s, z(s))\| + \int_0^s \|\Theta(s, r, z(r))\|\,\mathrm{d}r \\
&\quad + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(r, z(r))\|\,\mathrm{d}_q r\bigg]\,\mathrm{d}_q s \\
&\le \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu)}\bigg[\varrho_1(\xi)\|y(\xi)\| + \int_0^{\xi} \varrho^*(\xi)\|y(s)\|\,\mathrm{d}s \\
&\quad + \int_0^{\xi} \frac{(\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\varrho^*(s)\|y(s)\|\,\mathrm{d}_q s\bigg]\,\mathrm{d}_q\xi \\
&\quad + \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^* - s)^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\bigg[\varrho_1(s)\|z(s)\| + \int_0^s \varrho^*(s)\|z(r)\|\,\mathrm{d}r \\
&\quad + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\varrho_2(r)\|z(r)\|\,\mathrm{d}_q r\bigg]\,\mathrm{d}_q s \\
&\le \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\bigg[\|\varrho_1\|_{\mu^\infty}\|y\|_* e^{\xi} + \|\varrho^*\|_{L^\infty}\|y\|_*(e^{\xi}-1) \\
&\quad + \|\varrho_2\|_{L^\infty}\|y\|_* \int_0^{\xi} \frac{(\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)} e^s\,\mathrm{d}_q s\bigg]\,\mathrm{d}_q\xi \\
&\quad + \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^* - s)^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\bigg[\|\varrho_1\|_{L^\infty}\|z\|_* e^s + \|\varrho^*\|_{L^\infty}\|z\|_*(e^{s-1}) \\
&\quad + \|\varrho_2\|_{L^\infty}\|z\|_* \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} e^r\,\mathrm{d}r\bigg]\,\mathrm{d}_q s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathfrak{U}_1 y + \mathfrak{U}_2 z\|_* \leq & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \bigg[\|\varrho_1\|_{L^\infty} \|y\|_* \frac{e^{\xi}}{e^{\mathfrak{t}}} + \|\varrho^*\|_{L^\infty} \|y\|_* \frac{(e^{\xi}-1)}{e^{\mathfrak{t}}} \\
& + \|\varrho_2\|_{L^\infty} \|y\|_* \int_0^{\xi} \frac{(\xi-s)_q^{\sigma\sigma-1}}{\Gamma_q(\sigma)} \frac{e^s}{e^{\mathfrak{t}}} \, \mathrm{d}_q s \bigg] d_q \xi \\
& + \frac{|\eta|}{|1-\eta\tau^*|} \int_0^{\tau^*} \frac{(\tau^*-s)^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \bigg[\|\varrho_1\|_{L^\infty} \|z\|_* \frac{e^s}{e^{\mathfrak{t}}} + \|\varrho^*\|_{L^\infty} \|z\|_* \frac{(e^s-1)}{e^{\mathfrak{t}}} \\
& + \|\varrho_2\|_{L^\infty} \|z\|_* \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} \frac{e^r}{e^{\mathfrak{t}}} \, \mathrm{d}_q r \bigg] \, \mathrm{d}_q s \\
\leq & \, \ell \bigg[\frac{\|\varrho_1\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty}}{\Gamma_q(\sigma+1)\Gamma_q(\nu+1)} \int_0^1 (1-\xi)_q^{\sigma+\nu+1} \xi^{\sigma} \, \mathrm{d}_q \xi \\
& + \frac{|\eta| \|\varrho_1\|_{L^\infty} {\tau^*}^{\sigma+\nu+1} + |\eta| \|\varrho^*\|_{L^\infty} {\tau^*}^{\sigma+\nu+1}}{|1-\eta\tau^*| \Gamma_q(\sigma+\nu+1)} \\
& + \frac{|\eta| \|\varrho_2\|_{L^\infty}}{|1-\eta\tau^*| \Gamma_q(\sigma\sigma+1) \Gamma_q(\sigma+\nu+1)} \int_0^{\tau^*} (\tau^*-s)^{\sigma+\nu} s^{\sigma} \, \mathrm{d}_q s \bigg] \\
= & \, \ell \bigg[\frac{\|\varrho_1\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty} \nu(\sigma+1, \sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\nu+1)} \\
& + \frac{|\eta|}{|1-\eta\tau^*|} \bigg(\frac{\|\varrho_1\|_{L^\infty} {\tau^*}^{\sigma+\nu+1} + \|\varrho^*\|_{L^\infty} {\tau^*}^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu+2)} \\
& + \frac{\|\varrho_2\|_{L^\infty} {\tau^*}^{2\sigma+\nu+1} \nu(\sigma+1, \sigma+\nu+1)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)} \bigg) \bigg] = \ell\lambda \leq \ell.
\end{aligned} \tag{25}$$

This implies that $(\mathfrak{U}_1 y + \mathfrak{U}_2 z) \in \mathbb{B}_{\ell}$. Here we used the computations

$$\begin{aligned}
\int_0^1 (1-\xi)_q^{\sigma+\nu} \xi^{\sigma} \, \mathrm{d}_q \xi &= \beta_q(\sigma+1, \sigma+\nu), \\
\int_0^{\tau^*} (\tau^*-s)_q^{\sigma+\nu} s^{\sigma} \, \mathrm{d}_q \xi &= {\tau^*}^{2\sigma+\nu+1} \nu(\sigma+1, \sigma+\nu+1),
\end{aligned}$$

and the estimations:

$$\frac{e^{\xi}}{e^{\mathfrak{t}}} \leq 1, \quad \frac{e^{s}}{e^{\mathfrak{t}}} \leq 1, \quad \frac{e^{r}}{e^{\mathfrak{t}}} \leq 1.$$

In this step, we show that $\mathfrak{U}_2$ is a contraction mapping. Let $y, z \in \mathfrak{H},\ \mathfrak{t} \in \bar{\mathbb{I}}$. We have

$$\begin{aligned}\|\mathfrak{U}_2 y(\mathfrak{t}) - \mathfrak{U}_2 z(\mathfrak{t})\| \leq{}& \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\|h_1(s,y(s)) - h_1(s,\nu(s))\| \\ &+ \int_0^s \|\Theta(s,r,y(r)) - \Theta(s,r,z(r))\|\,\mathrm{d}r \\ &+ \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(r,y(r)) - h_2(r,z(r))\|\,\mathrm{d}_q r\Big]\,\mathrm{d}_q s \\ \leq{}& \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\mu_1\|y-z\|_* e^s + \int_0^s \mu^*\|y-z\|_* e^r\,\mathrm{d}r \\ &+ \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\mu_2\|y-z\|_* e^r\,\mathrm{d}r\Big]\,\mathrm{d}_q s \\ \leq{}& \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\mu\|y-z\|_* e^s + \mu\|y-z\|_*(e^{s-1}) \\ &+ \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\mu\|y-z\|_* e^r\,\mathrm{d}_q r\Big]\,\mathrm{d}_q s\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathfrak{U}_2 y - \mathfrak{U}_2 z\|_* \leq{}& \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\mu\|y-z\|_*\frac{e^s}{e^{\mathfrak{t}}} - \mu\|y-z\|_*\frac{(e^s-1)}{e^{\mathfrak{t}}} \\ &+ \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\mu\|y-z\|_*\frac{e^r}{e^{\mathfrak{t}}}\,\mathrm{d}_q r\Big]\,\mathrm{d}_q s \\ \leq{}& \frac{|\eta|\mu}{|1-\eta\tau^*|}\left[\frac{2\tau^{*\sigma+\nu+1}}{\Gamma_q(\sigma+\nu+2)} + \frac{\tau^{*2\sigma+\nu+1}\nu(\sigma+1,\sigma+\nu+1)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}\right]\|y-z\|_*.\end{aligned}$$

Then since $\mu\lambda^* \leq 1$, $\mathfrak{U}_2$ is a contraction mapping. The continuity of the functions h_1, h_2 and Θ implies that $\mathfrak{U}_1$ is continuous and $\mathfrak{U}_1\mathbb{B}_\ell \subset \mathbb{B}_\ell$, for each $y \in \mathbb{B}_\ell$, i.e., $\mathfrak{U}_1$ is uniformly bounded on $\mathbb{B}_\ell$ as

$$\begin{aligned}\|(\mathfrak{U}_1 y)(\mathfrak{t})\| \leq{}& \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\Big[\|h_1(\xi,y(\xi))\| + \int_0^\xi \|\Theta(\xi,s,y(s))\|\,\mathrm{d}s \\ &+ \int_0^\xi \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(s,y(s))\|\,\mathrm{d}_q s\Big]\,\mathrm{d}_q\xi,\end{aligned}$$

which implies that

$$\begin{aligned}\|\mathfrak{U}_1 y\|_* \leq{}& \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\Big[\|\varrho_1\|_{L^\infty}\|y\|_*\frac{e^\xi}{e^{\mathfrak{t}}} + \varrho^*\|_{L^\infty}\|y\|_*\frac{(e^\xi-1)}{e^{\mathfrak{t}}} \\ &+ \varrho_2\|_{L^\infty}\|y\|_*\int_0^\xi \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\frac{e^s}{e^{\mathfrak{t}}}\,\mathrm{d}_q s\Big]\,\mathrm{d}_q\xi \\ \leq{}& \ell\left[\frac{\|\varrho_*\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty}\nu(\sigma+1,\sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\nu+1)}\right] \leq \ell\lambda \leq \ell.\end{aligned} \tag{26}$$

Finally, we will show that $(\mathfrak{U}_1\mathbb{B}_\ell)$ is equi-continuous. For this end, we put

$$\overline{h}_j = \sup_{(\mathfrak{t},y(\mathfrak{t}))\in\bar{\mathbb{I}}\times\mathbb{B}_\ell}\|h_j(\mathfrak{t},y(\mathfrak{t}))\|, \qquad j=1,2$$

$$\overline{\Theta} = \sup_{(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))\in\mathbb{G}\times\mathbb{B}_\ell}\int_0^\xi \|\Theta(\mathfrak{t},\xi,y(\xi))\|\,\mathrm{d}\xi.$$

Let for any $y \in \mathbb{B}_\ell$ and for each $\mathfrak{t}_1, \mathfrak{t}_2 \in \overline{\mathbb{I}}$ with $\mathfrak{t}_1 \leq \mathfrak{t}_2$, we have

$$\begin{aligned}
\|(\mathfrak{U}_1 y)(\mathfrak{t}_2) - (\mathfrak{U}_1 y)(\mathfrak{t}_1)\| \leq & \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \frac{(\mathfrak{t}_2 - \xi)^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \Bigg[\|h_1(\xi, y(\xi))\| \\
& + \int_0^\xi \|\Theta(\xi, s, y(s))\| \, \mathrm{d}_q s \\
& + \int_0^\xi \frac{(\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, y(s))\| \, \mathrm{d}s \Bigg] \mathrm{d}_q \xi \\
& + \frac{1}{\Gamma_q(\sigma)} \int_0^{\mathfrak{t}_1} \Big[(\mathfrak{t}_1 - \xi)_q^{\sigma+\nu-1} - (\mathfrak{t}_2 - \xi)_q^{\sigma+\nu-1} \Big] \\
& \times \Bigg[\|h_1(\xi, y(\xi))\| + \int_0^\xi \|\Theta(\xi, s, y(s))\| \, \mathrm{d}s \\
& + \int_0^\xi \frac{(\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, y(s))\| \, \mathrm{d}s \Bigg] \mathrm{d}_q \xi \\
\leq & \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \frac{(\mathfrak{t}_2 - \xi)_q^{\sigma+\nu-1}}{\Gamma(\sigma+\nu)} \Bigg[\overline{h}_1 + \overline{\Theta} + \int_0^\xi \frac{\overline{h}_2 (\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \, \mathrm{d}s \Bigg] \mathrm{d}_q \xi \\
& + \frac{1}{\Gamma_q(\sigma+\nu)} \int_0^{\mathfrak{t}_1} \Big[(\mathfrak{t}_1 - \xi)_q^{\sigma+\nu-1} - (\mathfrak{t}_2 - \xi)_q^{\sigma+\nu-1} \Big] \\
& \times \Bigg[\overline{h}_1 + \overline{\Theta} + \int_0^\xi \frac{\overline{h}_2 (\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \, \mathrm{d}_q s \Bigg] \mathrm{d}_q \xi \\
\leq & \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \frac{(\mathfrak{t}_2 - \xi)^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \Bigg[\overline{h}_1 + \overline{\Theta} + \frac{\overline{h}_2}{\Gamma_q(\sigma+1)} \Bigg] \mathrm{d}_q \xi \\
& + \frac{1}{\Gamma_q(\sigma+\nu)} \int_0^{\mathfrak{t}_1} \Big[(\mathfrak{t}_1 - \xi)_q^{\sigma+\nu-1} - (\mathfrak{t}_2 - \xi)_q^{\sigma+\nu-1} \Big] \\
& \times \Bigg[\overline{h}_1 + \overline{\Theta} + \frac{\overline{h}_1}{\Gamma_q(\sigma+1)} \Bigg] \mathrm{d}_q \xi + \frac{1}{\Gamma_q(\sigma+\nu+1)} \Bigg[\overline{h}_1 + \overline{\Theta} \\
& + \frac{\overline{h}_2}{\Gamma_q(\sigma+1)} \Bigg] \Big[2(\mathfrak{t}_2 - \mathfrak{t}_1)_q^{\sigma+\nu} + \mathfrak{t}_1^{\sigma+\nu} - \mathfrak{t}_2^{\sigma+\nu} \Big].
\end{aligned}$$

The RHS of the last inequality is independent of y and tends to zero when $|\mathfrak{t}_2 - \mathfrak{t}_1| \to 0$, this means that

$$|\mathfrak{U}_1 y(\mathfrak{t}_2) - \mathfrak{U}_1 y,$$

which implies that $\mathfrak{U}_1 \mathbb{B}_\ell$ is equi-continuous, then $\mathfrak{U}_1$ is relatively compact on $\mathbb{B}_\ell$. Hence by Arzelá-Ascoli theorem, $\mathfrak{U}_1$ is compact on $\mathbb{B}_\ell$. Now, all hypothesis of Theorem 3.2 hold, therefore the operator $\mathfrak{U}_1 + \mathfrak{U}_2$ has a fixed point on $\mathbb{B}_\ell$. So the problem (1) has at least one solution on $\overline{\mathbb{I}}$. This proves the theorem. □

3.2. Existence and uniqueness result

Theorem 3.3. *Assume that* (H_1) *holds. If* $\mu\lambda < 1$*, then the BVP* (1) *has a unique solution on* $\overline{\mathbb{I}}$.

Proof. Define $m = \max\{m_1, m_2, m^*\}$, where m_j and m^* are positive numbers such that

$$m_j = \sup_{\mathfrak{t} \in \overline{\mathbb{I}}} \|h_j(\mathfrak{t}, 0)\|, (j = 1, 2), \qquad m^* = \sup_{(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G}} \|\Theta(\mathfrak{t}, \mathfrak{s}, 0)\|.$$

We fix $\ell \geq \frac{m^*\lambda}{1-\mu\lambda}$ and we consider $\mathbb{N}_\ell = \{y \in C(\bar{\mathbb{I}}, \mathfrak{H}) \,:\, \|y\|_* \leq \ell\}$. Then, in view of the assumption (H_1), we have

$$\begin{aligned}
\|h_q(\mathfrak{t}, y(\mathfrak{t}))\| &= \|h_1(\mathfrak{t}, y(\mathfrak{t})) - h_1(\mathfrak{t}, 0) + h_1(\mathfrak{t}, 0)\| \\
&\leq \|h_q(\mathfrak{t}, y(\mathfrak{t})) - h_q(\mathfrak{t}, 0)\| + \|h_1(\mathfrak{t}, 0)\| \\
&\leq \mu_1\|y\|_* + m_1,
\end{aligned}$$

$$\|h_2(\mathfrak{t}, y(\mathfrak{t}))\| \leq \mu_2\|y\|_* + m_2,$$

and $\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s}))\| \leq \mu^*\|y\|_* + m^*$. In the first step, we show that $\mathfrak{U}\mathbb{N}_\ell \subset \mathbb{N}_\ell$. For each $\mathfrak{t} \in \bar{\mathbb{I}}$ and for any $y \in \mathbb{N}_\ell$,

$$\begin{aligned}
\|\mathfrak{U}y(\mathfrak{t})\| \leq& \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\|h_1(\xi, y(\xi))\| + \int_0^{\xi} \|\Theta(\xi, s, y(s))\| \,\mathrm{d}s \right.\\
&\left. + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, y(s))\| \,\mathrm{d}_q s\right] \mathrm{d}_q\xi \\
&+ \frac{|\eta|}{|1-\eta\tau^*|} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \left[\|h_1(s, z(s))\| \right.\\
&\left. + \int_0^{\xi} \|\Theta(s, r, z(r))\| \,\mathrm{d}r + \int_0^{\xi} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(r, z(r))\| \,\mathrm{d}r\right] \mathrm{d}_q s \\
\leq& (\mu\ell + m)\lambda \leq \ell.
\end{aligned}$$

Hence, $\mathfrak{U}\mathbb{N}_\ell \subset \mathbb{N}_\ell$. Now, in the second step, we shall show that $\mathfrak{U} : \mathbb{N}_\ell \to \mathbb{N}_\ell$ is a contraction. From the assumption (H_1) we have for any $y, z \in \mathbb{N}_\ell$ and for each $\mathfrak{t} \in \bar{\mathbb{I}}$

$$\begin{aligned}
\|\mathfrak{U}y(\mathfrak{t}) - \mathfrak{U}z(\mathfrak{t})\| \leq& \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\|h_1(\xi, y(\xi)) - h_1(\xi, z(\xi))\| \right.\\
&+ \int_0^{\xi} \|\Theta(\xi, s, y(s)) - \Theta(\xi, s, z(s))\| \,\mathrm{d}s \\
&\left. + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, y(s)) - h_2(s, z(s))\| \,\mathrm{d}_q s\right] \mathrm{d}_q\xi \\
&+ \frac{|\eta|}{|1-\eta\tau^*|} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma - q(\sigma+\nu+1)} \left[\|h_1(s, y(s)) - h_1(s, z(s))\| \right.\\
&+ \int_0^{s} \|\Theta(s, r, y(r)) - \Theta(s, r, z(r))\| \,\mathrm{d}r \\
&\left. + \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(r, y(r)) - h_2(r, z(r))\| \,\mathrm{d}_q r\right] \mathrm{d}_q s \\
\leq& \mu\lambda\|y-z\|_*.
\end{aligned} \tag{27}$$

Since $\mu\lambda < 1$, it follows that $\mathfrak{U}$ is a contraction. All assumptions of Lemma 2.2 are satisfied, then there exists $y \in C(\bar{\mathbb{I}}, \mathfrak{H})$ such that $\mathfrak{U}y = y$, which is the unique solution of the problem (1.1) in $C(\bar{\mathbb{I}}, \mathfrak{H})$. $\square$

4. Generalized Ulam stabilitiestle

The aim is to discuss the Ulam stability for problem (1), by using the integration

$$
\begin{aligned}
z(\mathrm{t}) = & \int_0^{\mathrm{t}} \frac{(\mathrm{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\|h_1(\xi, z(\xi))\| + \int_0^{\xi} \|\Theta(\xi, s, z(s))\| \, \mathrm{d}s \right. \\
& \left. + \int_0^{\xi} \frac{(\xi - s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, z(s))\| \, \mathrm{d}_q s \right] \mathrm{d}_q \xi \\
& + \frac{\eta}{1-\eta\tau^*} \int_0^{\tau^*} \frac{(\tau^* - s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \left[h_1(s, z(s)) + \int_0^s \Theta(s, r, z(r)) \, \mathrm{d}r \right. \\
& \left. + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, z(r)) \, \mathrm{d}_q r \right] \mathrm{d}_q s.
\end{aligned}
$$

Here $z \in C(\overline{\mathbb{I}}, \mathfrak{H})$ possess a fractional derivative of order $\sigma+\nu$, where $0 < \sigma+\nu < 1$ and $h_j : \overline{\mathbb{I}} \times \mathfrak{H} \to \mathfrak{H}$ and $\Theta : \overline{\mathbb{I}}^2 \times \mathfrak{H} \to \mathfrak{H}$, are continuous functions. Then we define the nonlinear continuous operator $\mathfrak{P} : C(\overline{\mathbb{I}}, \mathfrak{H}) \to C(\overline{\mathbb{I}}, \mathfrak{H})$, as follows

$$
\mathfrak{P}z(\mathrm{t}) = {}^{C}\mathbb{D}_q^{\sigma+\nu} z(\mathrm{t}) - h_1(\mathrm{t}, v(\mathrm{t})) - \mathbb{I}_q^{\sigma} h_2(\mathrm{t}, v(\mathrm{t})) - \int_0^{\mathrm{t}} \Theta(\mathrm{t}, \xi, z(\xi)) \, \mathrm{d}\xi.
$$

For each $\epsilon > 0$ and for each solution z of problem (1), such that

$$
\|\mathfrak{P}z\|_* \le \epsilon, \tag{28}
$$

the problem (1), is said to be Ulam–Hyers stable if we can find a solution $y \in C(\overline{\mathbb{I}}, \mathfrak{H}\mathfrak{U})$ of problem (1) and $\gamma \in \mathbb{R}^{\ge 0}$, satisfying the inequality $\|y - z\|_* \le \gamma\epsilon^*$, is a positive real number depending on ϵ. Consider function $\wp$ in $C(\mathbb{R}^+, \mathbb{R}^+)$ such that for each solution z of problem (1), we can find a solution $u \in C(\overline{\mathbb{I}}, \mathfrak{H})$ of the problem (1) such that $\|y(\mathrm{t}) - z(\mathrm{t})\|_* \le \wp(\epsilon)$, $\mathrm{t} \in \overline{\mathbb{I}}$. Then the problem (1), is said to be generalized Ulam–Hyers stable. For each $\epsilon > 0$ and for each solution z of problem (1), the problem (1) is called Ulam-Hyers-Rassias stable with respect to $\varrho \in C(\overline{\mathbb{I}}, \mathbb{R}^+)$ if

$$
\|\mathfrak{P}z(\mathrm{t})\|_* \le \epsilon\varrho(t), \qquad \mathrm{t} \in \overline{\mathbb{I}}, \tag{29}
$$

and there exist a real number $\gamma > 0$ and a solution $z \in C(\overline{\mathbb{I}}, \mathfrak{H})$ of problem (1) such that

$$
\|y(\mathrm{t}) - z(\mathrm{t})\| \le \gamma\epsilon_*\varrho(\mathrm{t}), \qquad \forall \mathrm{t} \in \overline{\mathbb{I}},
$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. *Under assumption (H_1) in Theorem 3.1, with $\mu\lambda < 1$. The problem (1.1) is both Ulam–Hyers and generalized Ulam–Hyers stable.*

Proof. Let $y \in C(\overline{\mathbb{I}}, \mathfrak{H})$ be a solution of problem (1), satisfying (14) in the sense of Theorem 3.2. Let z be any solution satisfying (28). Lemma 2.4 implies the equivalence between the operators $\mathfrak{P}$ and $\mathcal{T} - \mathfrak{I}_d$ (where $\mathfrak{I}_d$ is the identity operator) for every solution $z \in C(\overline{\mathbb{I}}, \mathfrak{H})$ of problem (1) satisfying $\mu\lambda < 1$. Therefore, we deduce by the fixed-point property of the operator $\mathcal{T}$ that

$$
\begin{aligned}
\|z(\mathrm{t}) - y(\mathrm{t})\|_* &= \|z(\mathrm{t}) - \mathcal{T}z(\mathrm{t}) + \mathcal{T}z(\mathrm{t}) - y(\mathrm{t})\|_* \\
&= \|z(\mathrm{t}) - \mathcal{T}z(\mathrm{t}) + \mathcal{T}z(\mathrm{t}) - \mathcal{T}y(\mathrm{t})\|_* \\
&\le \|\mathcal{T}z(\mathrm{t}) - \mathcal{T}y(\mathrm{t})\| + \|\mathcal{T}z(\mathrm{t}) - z(\mathrm{t})\|_* \\
&\le \mu\lambda\|y - z\|_* + \epsilon,
\end{aligned}
$$

because $\mu\lambda < 1$ and $\epsilon > 0$, we find

$$
\|u - v\|_* \le \frac{\epsilon}{1-\mu\lambda}.
$$

Fixing $\epsilon_* = \frac{\epsilon}{1-\mu\lambda}$ and $\gamma = 1$, we obtain the Ulam–Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $\wp(\epsilon) = \frac{\epsilon}{1-\mu\lambda}$. □

Theorem 4.2. *Assume that* (H_1) *holds with* $\mu < \lambda{-}1$, *and there exists a function* $\varrho \in C(\bar{\mathbb{I}}, \mathbb{R}^+)$ *satisfying the condition 29. Then the problem (1) is Ulam-Hyers-Rassias stable with respect to* ϱ.

Proof. We have from the proof of Theorem 4.1, $\|y(\mathfrak{t}) - z(\mathfrak{t})\|_* \leq \epsilon_* \varrho(\mathfrak{t})$, $\forall \mathfrak{t} \in \bar{\mathbb{I}}$, where $\epsilon_* = \frac{\epsilon}{1-\mu\lambda}$, and so the proof is cimpleted. □

5. Illustrative of our outcome

First we present Example 5.1, for illustrative our main result.

Example 5.1. *Consider the following fractional integro-differential problem*

$$ {}^{C}\mathbb{D}_q^{\frac{68}{77}}[y](\mathfrak{t}) = \frac{(15-2\mathfrak{t})y(\mathfrak{t})}{25} + \mathbb{I}_q^{\frac{5}{11}}\left[\frac{(5-\mathfrak{t})\sin(y(\mathfrak{t}))}{43}\right] + \int_0^{\mathfrak{t}} \frac{y(\xi)\exp(-(\mathfrak{t}+\xi))}{20}\, \mathrm{d}\xi, \tag{30}$$

with boundary condition

$$y(0) = -\frac{15}{2}\int_0^{0.6} y(\xi)\,\mathrm{d}\xi, \qquad \forall \mathfrak{t} \in \mathbb{I}.$$

Clearly $\sigma + \nu = \frac{68}{77}$, $\sigma = \frac{5}{11}$, $\tau^* = 0.6$ *and* $\eta = -\frac{15}{2}$. *To illustrate our results in Theorem 3.2 and Theorem 4.1, we take for* $y, z \in \mathfrak{H} = \mathbb{R}^+$ *and* $\mathfrak{t} \in [0,1]$ *the following continuous functions:*

$$h_1(\mathfrak{t}, y(\mathfrak{t})) = \frac{(15-2\mathfrak{t})y(\mathfrak{t})}{25}, \qquad h_2(\mathfrak{t}, y(\mathfrak{t})) = \frac{(5-\mathfrak{t})\sin(y(\mathfrak{t}))}{43},$$

$\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s})) = \frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}$. *Now, for* $y, z \in \mathfrak{H}$, *we have*

$$\|h_1(\mathfrak{t}, y(\mathfrak{t})) - h_1(\mathfrak{t}, z(\mathfrak{t}))\| \leq \frac{3}{5}\|y(\mathfrak{t}) - z(\mathfrak{t})\|,$$

$$\|h_2(\mathfrak{t}, y(\mathfrak{t})) - h_2(\mathfrak{t}, z(\mathfrak{t}))\| \leq \frac{5}{43}\|y(\mathfrak{t}) - z(\mathfrak{t})\|,$$

and

$$\begin{aligned}\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s})) - \Theta(\mathfrak{t}, \mathfrak{s}, z(\mathfrak{s}))\| &= \left\|\frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20} - \frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}\right\| \\ &\leq \frac{1}{20}\|y(\mathfrak{s}) - z(\mathfrak{s})\|,\end{aligned}$$

for each $\mathfrak{t}, \mathfrak{s} \in \mathbb{I}$ *and* $(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G}$. *Hence,* $\mu_1 = \frac{17}{25}$, $\mu_2 = \frac{7}{43}$, $\mu^* = \frac{1}{20}$ *and so*

$$\mu = \max\left\{\mu_1, \mu_2, \mu^*\right\} = \frac{17}{25}.$$

Also, we obtain

$$\begin{aligned}\|h_1(\mathfrak{t}, y(\mathfrak{t}))\| &= \left\|\frac{(15-2\mathfrak{t})y(\mathfrak{t})}{25}\right\| \leq \left|\frac{15-2\mathfrak{t}}{25}\right|\|y(\mathfrak{t})\|, \\ \|h_2(\mathfrak{t}, y(\mathfrak{t}))\| &= \left\|\frac{(5-2\mathfrak{t})\sin(y(\mathfrak{t}))}{43}\right\| \leq \left|\frac{5-2\mathfrak{t}}{43}\right|\|y(\mathfrak{t})\|, \\ \|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s}))\| &\leq \left\|\frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}\right\| \leq \left\|\frac{\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}\right\|\|y(\mathfrak{s})\|,\end{aligned}$$

for each $\mathfrak{t}, \mathfrak{s} \in \mathbb{I}$. *Hence,*

$$\varrho_1(\mathfrak{t}) = \frac{15-2\mathfrak{t}}{25}, \quad \varrho_2(\mathfrak{t}) = \frac{5-2\mathfrak{t}}{43}, \quad \varrho^*(\mathfrak{t}) = \frac{\exp(-\mathfrak{t})}{20},$$

Table 1: Numerical results of λ and λ^* for $q = \frac{3}{8}, \frac{1}{2}, \frac{8}{9}$ in Example 5.1.

n	$q=\frac{3}{8}$		$q=\frac{1}{2}$		$q=\frac{8}{9}$	
	λ	λ^*	λ	λ^*	λ	λ^*
1	0.93177	1.34571	0.71630	0.99360	0.11402	0.07701
2	0.94654	1.39205	0.73885	1.06376	0.11895	0.09638
3	0.95212	1.40943	0.75025	1.09885	0.12377	0.11354
4	0.95422	1.41595	0.75598	1.11640	0.12828	0.12878
5	0.95500	1.41840	0.75885	1.12518	0.13242	0.14232
6	0.95530	1.41931	0.76029	1.12957	0.13618	0.15436
7	0.95541	1.41966	0.76101	1.13176	0.13957	0.16506
8	0.95545	1.41978	0.76137	1.13286	0.14262	0.17458
9	0.95546	1.41983	0.76155	1.13341	0.14536	0.18304
10	<u>0.95547</u>	1.41985	0.76164	1.13368	0.14781	0.19057
11	0.95547	<u>1.41986</u>	0.76168	1.13382	0.15001	0.19727
12	0.95547	1.41986	0.76170	1.13389	0.15197	0.20322
13	0.95547	1.41986	<u>0.76172</u>	1.13392	0.15372	0.20852
14	0.95547	1.41986	0.76172	1.13394	0.15528	0.21323
15	0.95547	1.41986	0.76172	<u>1.13395</u>	0.15667	0.21741
16	0.95547	1.41986	0.76173	1.13395	0.15791	0.22114
17	0.95547	1.41986	0.76173	1.13396	0.15901	0.22445
18	0.95547	1.41986	0.76173	1.13396	0.16000	0.22739
⋮	⋮	⋮	⋮	⋮	⋮	⋮
76	0.95547	1.41986	0.76173	1.13396	0.16792	0.25095
77	0.95547	1.41986	0.76173	1.13396	0.16792	<u>0.25096</u>
78	0.95547	1.41986	0.76173	1.13396	<u>0.16793</u>	0.25096
79	0.95547	1.41986	0.76173	1.13396	0.16793	0.25096
80	0.95547	1.41986	0.76173	1.13396	0.16793	0.25096

for all $\mathfrak{t} \in \bar{\mathbb{I}}$, $y, z \in \mathfrak{H}$ *and* $(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G}$. *By the above, we find that*

$$\begin{aligned}\lambda = &\frac{\frac{3}{5} + \frac{1}{20}}{\Gamma_q\left(\frac{5}{11} + \frac{3}{7} + 1\right)} + \frac{\frac{5}{43} B_q\left(\frac{5}{11} + 1, \frac{5}{11} + \frac{3}{7}\right)}{\Gamma_q\left(\frac{5}{11} + 1\right)\Gamma_q\left(\frac{5}{11} + \frac{3}{7}\right)}\\ &+ \frac{\left|-\frac{15}{2}\right| \times \frac{3}{5} \times 0.6^{\frac{5}{11}+\frac{3}{7}+1} + \left|-\frac{15}{2}\right| \times \frac{1}{20} \times 0.6^{\frac{5}{11}+\frac{3}{7}+1}}{\left|1 - 0.6\left(-\frac{15}{2}\right)\right| \Gamma_q\left(\frac{5}{11} + \frac{3}{7} + 2\right)}\\ &+ \frac{\left|-\frac{15}{2}\right| \times \frac{5}{43} \times 0.6^{\frac{10}{11}+\frac{3}{7}+1} B_q\left(\frac{5}{11} + 1, \frac{5}{11} + \frac{3}{7} + 1\right)}{\left|1 - \frac{5}{11} \times 0.6\right| \Gamma_q\left(\frac{5}{11} + 1\right)\Gamma_q\left(\frac{5}{11} + \frac{3}{7} + 1\right)},\end{aligned} \tag{31}$$

and

$$\lambda^* = \frac{\left|-\frac{15}{2}\right|}{\left|1 - 0.6\left(-\frac{15}{2}\right)\right|}\left[\frac{2 \times 0.6^{\frac{5}{11}+\frac{3}{7}+1}}{\Gamma_q\left(\frac{5}{11} + \frac{3}{7} + 2\right)} + \frac{0.6^{2\frac{5}{11}+\frac{3}{7}+1} B_q\left(\frac{5}{11} + 1, \frac{5}{11} + \frac{3}{7} + 1\right)}{\Gamma_q\left(\frac{5}{11} + 1\right)\Gamma_q\left(\frac{5}{11} + \frac{3}{7} + 1\right)}\right]. \tag{32}$$

With consider $q = \frac{3}{8}, \frac{1}{2}, \frac{8}{9}$, *we can see the results of* λ *and* λ^* *in Table 1. These results are plotted in Fig. 1. Then, we get*

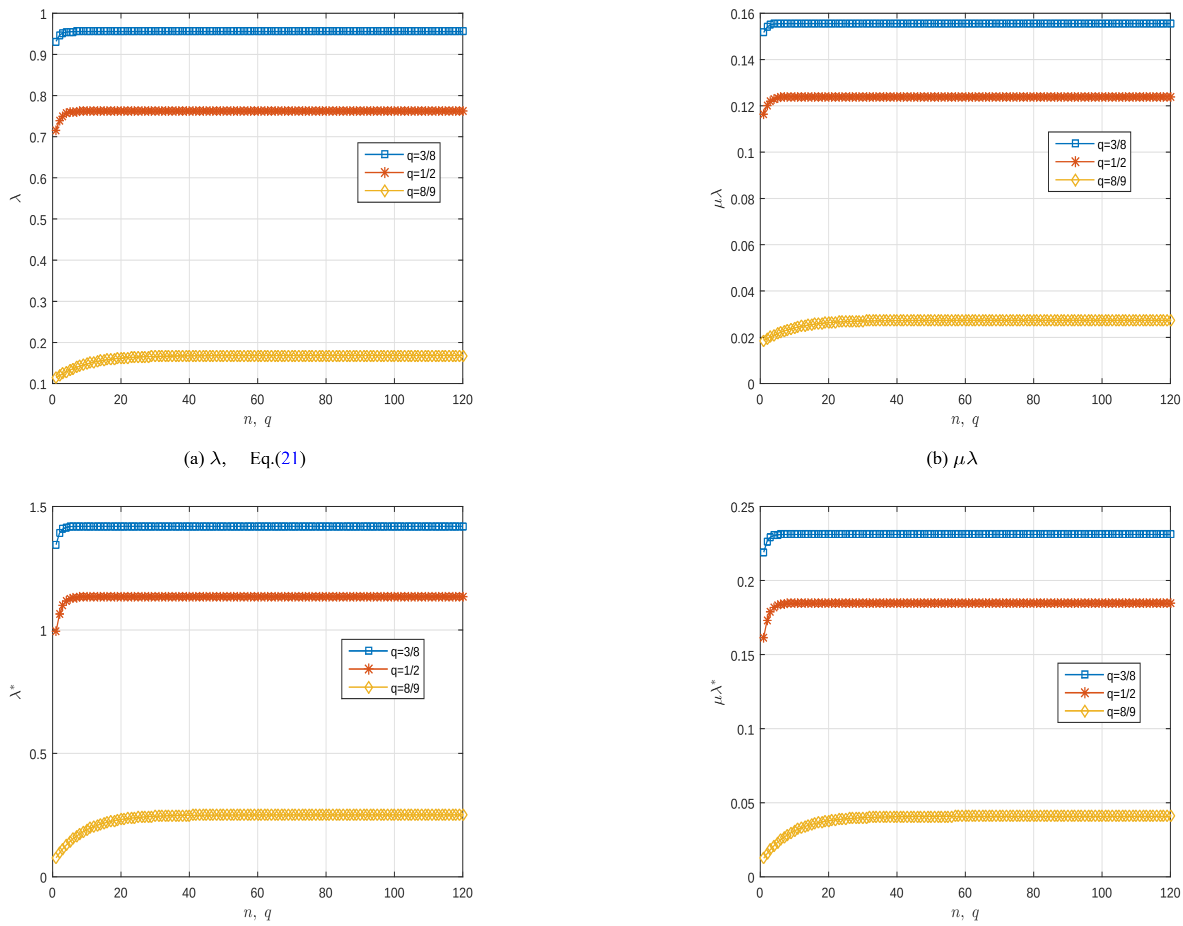
$$\begin{aligned}\lambda_j &= 0.95547 < 1, \quad 0.76172 < 1, \quad 0.16793 < 1,\\ \lambda_j^* &= 1.41986, \quad 1.13395, \quad 0.25096,\\ \mu\lambda_j^* &= 0.9655 < 1, \quad 0.7711 < 1, \quad 0.1707 < 1,\end{aligned}$$

for $q_j = \frac{3}{8}, \frac{1}{2}, \frac{8}{9}$ *respectively. All assumptions of Theorem 3.2 are satisfied. Hence, there exists at least one solution for the problem (30) on* $\bar{\mathbb{I}}$. *By take the same functions, we result the assumption*

$$\mu\lambda_j = 0.6497 < 1, \quad 0.5180 < 1, \quad 0.1142 < 1,$$

then the system (30) is Ulam–Hyers stable, then it is generalized Ulam–Hyers stable. It is Ulam-Hyers-Rassias stable if there exists a continuous and positive function $\varrho_j \in C(\bar{\mathbb{I}}, \mathbb{R}^+)$ *such that*

$$\|y(\mathfrak{t}) - z(\mathfrak{t})\| \leq \epsilon_{j*}\varrho(\mathfrak{t}) = \frac{\epsilon_j \varrho(\mathfrak{t})}{1 - \mu\lambda_j},$$



(a) λ, Eq.(21)

(b) $\mu\lambda$

(c) λ^*, Eq.(22)

(d) $\mu\lambda^*$

Fig. 1: Graphical representation of λ, λ^* and $\mu\lambda$, $\mu\lambda^*$ for $q = \frac{3}{8}, \frac{1}{2}, \frac{8}{9}$ in Example 5.1.

which it satisfies in assumption of the Theorem 4.2.

In the next example, we review and check Theorem 3.3 numerically.

Example 5.2. *Consider the following fractional integro-differential problem*

$$
\begin{aligned}
{}^{C}\mathbb{D}_q^{\frac{29}{45}}[y](\mathrm{t}) = \frac{(16-\sqrt{\mathrm{t}})\tan^{-1}(y(\mathrm{t}))}{75} + \mathbb{I}_q^{\frac{4}{9}}\left[\frac{2\mathrm{t}\sin^{-1}(y(\mathrm{t}))}{21}\right] \\
+ \int_0^{\mathrm{t}} \frac{y(\xi)\exp(-(3\mathrm{t}+\xi))}{10}\, \mathrm{d}\xi,
\end{aligned}
\tag{33}
$$

with boundary condition

$$
y(0) = -\frac{5}{2}\int_0^{0.95} y(\xi)\, \mathrm{d}\xi, \qquad \mathrm{t} \in \mathbb{I}.
$$

Clearly $\sigma+\nu = \frac{29}{45}$, $\sigma = \frac{4}{9}$, $\tau^* = 0.95$ *and* $\eta = \frac{5}{2}$. *To illustrate our results in Theorem 3.3, we take for* $y, z \in \mathfrak{H} = \mathbb{R}^+$ *and* $\mathrm{t} \in \mathfrak{I}$ *the following continuous functions:*

$$
h_1(\mathrm{t}, y(\mathrm{t})) = \frac{(16-\sqrt{\mathrm{t}})\tan^{-1}(y(\mathrm{t}))}{75}, \qquad h_2(\mathrm{t}, y(\mathrm{t})) = \frac{2\mathrm{t}\sin^{-1}(y(\mathrm{t}))}{21},
$$

$\Theta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))=\frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10}$. *Now, for* $y,z\in\mathfrak{H}$*, we have*

$$\begin{aligned}\|h_1(\mathfrak{t},y(\mathfrak{t}))-h_1(\mathfrak{t},z(\mathfrak{t}))\|&=\left\|\frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(y(\mathfrak{t}))}{75}-\frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(z(\mathfrak{t}))}{75}\right\|\\&\leq\frac{17}{75}\left\|y(\mathfrak{t})-z(\mathfrak{t})\right\|,\end{aligned}$$

$$\begin{aligned}\|h_2(\mathfrak{t},y(\mathfrak{t}))-h_2(\mathfrak{t},z(\mathfrak{t}))\|&=\left\|\frac{2\mathfrak{t}\sin^{-1}(y(\mathfrak{t}))}{21}-\frac{2\mathfrak{t}\sin^{-1}(z(\mathfrak{t}))}{43}\right\|\\&\leq\frac{2}{21}\left\|y(\mathfrak{t})-z(\mathfrak{t})\right\|,\end{aligned}$$

and

$$\begin{aligned}\|\Theta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))-\Theta(\mathfrak{t},\mathfrak{s},z(\mathfrak{s}))\|&=\left\|\frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10}-\frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10}\right\|\\&\leq\frac{1}{10}\left\|y(\mathfrak{s})-z(\mathfrak{s})\right\|,\end{aligned}$$

for each $\mathfrak{t},\mathfrak{s}\in\mathbb{I}$ *and* $(\mathfrak{t},\mathfrak{s})\in\mathbb{G}$*. Hence,* $\mu_1=\frac{17}{75}$, $\mu_2=\frac{2}{21}$, $\mu^*=\frac{1}{10}$ *and so* $\mu=\max\{\mu_1,\mu_2,\mu^*\}=\frac{17}{25}$*. Also, we obtain*

$$\begin{aligned}\|h_1(\mathfrak{t},y(\mathfrak{t}))\|&=\left\|\frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(y(\mathfrak{t}))}{75}\right\|\leq\left|\frac{16-\sqrt{\mathfrak{t}}}{75}\right|\|y(\mathfrak{t})\|,\\\|h_2(\mathfrak{t},y(\mathfrak{t}))\|&=\left\|\frac{2\mathfrak{t}\sin^{-1}(y(\mathfrak{t}))}{21}\right\|\leq\left|\frac{2\mathfrak{t}}{21}\right|\|y(\mathfrak{t})\|,\\\|\Theta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))\|&\leq\left\|\frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10}\right\|\leq\left\|\frac{\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10}\right\|\|y(\mathfrak{s})\|,\end{aligned}$$

for each $\mathfrak{t},\mathfrak{s}\in\mathbb{I}$*. Hence,* $\varrho_1(\mathfrak{t})=\frac{16-\sqrt{\mathfrak{t}}}{75}$, $\varrho_2(\mathfrak{t})=\frac{2\mathfrak{t}}{21}$ *and* $\varrho^*(\mathfrak{t})=\frac{\exp(-3\mathfrak{t})}{10}$ *for all* $\mathfrak{t}\in\overline{\mathbb{I}}$, $y,z\in\mathfrak{H}$ *and* $(\mathfrak{t},\mathfrak{s})\in\mathbb{G}$*. By the above, we find that*

$$\begin{aligned}\lambda&=\frac{\|\varrho_1\|_{L^\infty}+\|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)}+\frac{\|\varrho_2\|_{L^\infty}B_q(\sigma+1,\sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu)}\\&\quad+\frac{|\eta|\|\varrho_1\|_{L^\infty}\tau^{*\sigma+\nu+1}+|\eta|\|\varrho^*\|_{L^\infty}\tau^{*\sigma+\nu+1}}{|1-\eta\tau^*|\Gamma_q(\sigma+\nu+2)}\\&\quad+\frac{|\eta|\|\varrho_2\|_{L^\infty}\tau^{*2\sigma+\nu+1}B_q(\sigma+1,\sigma+\nu+1)}{|1-\eta\tau^*|\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}\\&=\frac{\frac{16}{75}+\frac{1}{10}}{\Gamma_q(\frac{4}{9}+\frac{1}{5}+1)}+\frac{\frac{2}{21}B_q(\frac{4}{9}+1,\frac{4}{9}+\frac{1}{5})}{\Gamma_q(\frac{4}{9}+1)\Gamma_q(\frac{4}{9}+\frac{1}{5})}\\&\quad+\frac{|2.5|\times\frac{16}{75}0.95^{\frac{4}{9}+\frac{1}{5}+1}+|2.5|\frac{1}{10}0.95^{\frac{4}{9}+\frac{1}{5}+1}}{|1-2.5\times0.95|\Gamma_q(\frac{4}{9}+\frac{1}{5}+2)}\\&\quad+\frac{|2.5|\frac{4}{9}\times\frac{2}{21}\times0.95^{\frac{8}{9}+\frac{1}{5}+1}B_q(\frac{4}{9}+1,\frac{4}{9}+\frac{1}{5}+1)}{|1-2.5\times0.95|\Gamma_q(\frac{4}{9}+1)\Gamma_q(\frac{4}{9}+\frac{1}{5}+1)}.\end{aligned}\tag{34}$$

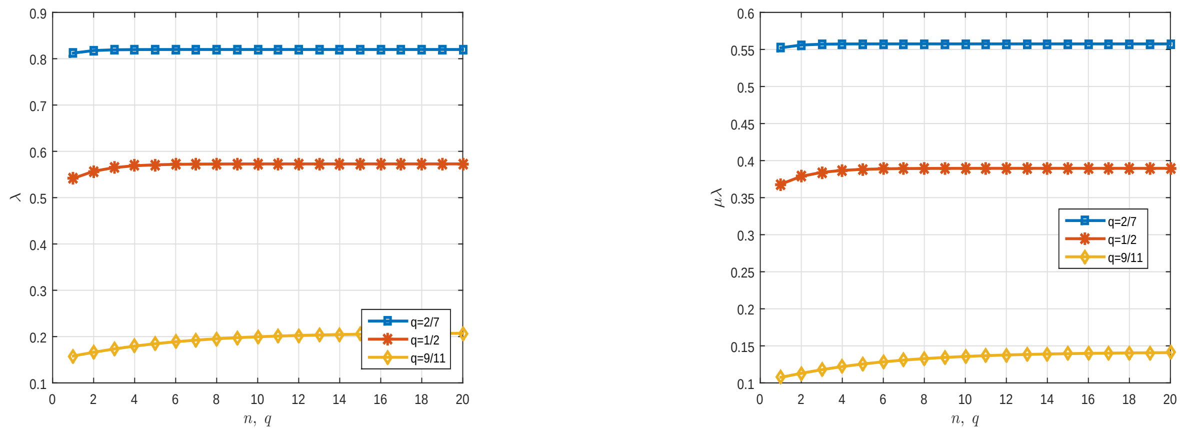
With consider $q=\frac{2}{7},\frac{1}{2},\frac{9}{11}$*, we can see the results of* λ *and* λ^* *in Table 2. These results are plotted in Fig. 2. Then, we get*

$$\begin{aligned}\lambda_j&=0.81987,\quad 0.57290,\quad 0.20831,\\\mu\lambda_j&=0.55751<1,\quad 0.38957<1,\quad 0.14165,\end{aligned}$$

for $q_j=\frac{2}{7},\frac{1}{2},\frac{9}{11}$ *respectively. All assumptions of Theorem 3.3 are satisfied. Hence, there exists at least one solution for the problem (33) on* $\overline{\mathbb{I}}$*.*

Table 2: Numerical results of λ and $\mu\lambda$ for $q = \frac{2}{7}, \frac{1}{2}, \frac{9}{11}$ in Example 5.2.

n	$q = \frac{2}{7}$		$q = \frac{1}{2}$		$q = \frac{9}{11}$	
	λ	$\mu\lambda$	λ	$\mu\lambda$	λ	$\mu\lambda$
1	0.81214	0.55225	0.54150	0.36822	0.15811	0.10752
2	0.81764	0.55600	0.55700	0.37876	0.16610	0.11295
3	0.81923	0.55708	0.56491	0.38414	0.17332	0.11785
4	0.81969	0.55739	0.56890	0.38685	0.17947	0.12204
5	0.81982	0.55748	0.57090	0.38821	0.18462	0.12554
6	0.81986	0.55750	0.57190	0.38889	0.18887	0.12843
7	$\underline{0.81987}$	$\underline{0.55751}$	0.57240	0.38923	0.19238	0.13082
8	0.81987	0.55751	0.57265	0.38940	0.19526	0.13278
9	0.81987	0.55751	0.57278	0.38949	0.19763	0.13439
10	0.81987	0.55751	0.57284	0.38953	0.19956	0.13570
11	0.81987	0.55751	0.57287	0.38955	0.20115	0.13678
12	0.81987	0.55751	0.57289	0.38956	0.20245	0.13767
13	0.81987	0.55751	$\underline{0.57290}$	$\underline{0.38957}$	0.20352	0.13839
14	0.81987	0.55751	0.57290	0.38957	0.20439	0.13898
15	0.81987	0.55751	0.57290	0.38957	0.20510	0.13947
⋮	⋮	⋮	⋮	⋮	⋮	⋮
43	0.81987	0.55751	0.57290	0.38957	0.20830	0.14164
44	0.81987	0.55751	0.57290	0.38957	0.20830	$\underline{0.14165}$
45	0.81987	0.55751	0.57290	0.38957	0.20830	0.14165
46	0.81987	0.55751	0.57290	0.38957	$\underline{0.20831}$	0.14165
47	0.81987	0.55751	0.57290	0.38957	0.20831	0.14165



(a) λ, Eq.(21) (b) $\mu\lambda$

Fig. 2: Graphical representation of λ and $\mu\lambda$ for $q = \frac{2}{7}, \frac{1}{2}, \frac{9}{11}$ in Example 5.2.

6. Conclusion

The q-integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional q-calculus and its applications in various areas of science and technology. q-integro-differential boundary value problems occur in the mathematical modeling of a variety of physical operations. Using the Krasnoselskii's, Banach fixed point theorems, we proof the existence and uniqueness results. Based on the results obtained, conditions are provided that ensure the generalized Ulam stability of the original system. The results for investigating Eq. (1) on a time scale, are illustrated by two examples.

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Existence of solutions for a (p, q)-Laplace equation with Steklov boundary conditions

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Abstract

Here, the existence of at least one nontrivial solution for the (p, q)-Laplacian problem

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) + div(|\nabla u|^{q-2}\nabla u) = f(x,u) & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} + |\nabla u|^{q-2}\frac{\partial u}{\partial n} = g(x,u) & x \in \partial\Omega \end{cases}$$

is done, where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$ and $q, p \geq 2$, via variational methods.

1. Introduction

Usually solutions to (p,q)-Laplacian problems are the steady state solutions of the reaction diffusion systems. Reaction–diffusion systems are mathematical models which correspond to several physical phenomena. This system has a wide range of applications in physics and related sciences like chemical reaction design, biophysics, plasma physics, geology, and ecology. This equations also arise in the study of soliton-like solutions of the nonlinear Schrödinger equation as a model for elementary particles for example waves in a discrete electrical lattice. These problems have been intensively studied in the last decates.
In this note, we investigate the existence of at least one weak solutions for a (p, q)-Laplacian problem with Steklov boundary conditions and our starting point is introducing some notations and recalling a basic result which compose the tools that are needed for proving our claim.

Notations and Preliminaries:

Throught this note Ω is a bounded domain in $\mathbb{R}^N, N \geq 3$, for $p > 1$, by $|.|_p$ we denote the norm on the Lebesgue space $L^p(\Omega)$, and $\|.\|_p$ denotes the norm of the Sobolev space $W_0^{1,p}(\Omega)$, i.e. $\|u\|_p = |\nabla u|_p$.

$f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that hold in the following coditions

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$(f1)$ The constants $a_1, a_2 \geq 0$ exist such that

$$|f(x,t)| \leq a_1 + a_2|t|^{\theta-1}, \qquad (x,t) \in \Omega \times \mathbb{R}$$

with $1 < \theta < p^*$, where

$$p^*(N) = \begin{cases} \frac{Np}{N-p} & p < N, \\ \infty & p \geq N; \end{cases}$$

$(f2)$ $f(x,t)t \geq 0$ for all $(x,t) \in \Omega \times \mathbb{R}$;

$(f3)$ $f(x,0) \neq 0$ for all $x \in \Omega$;

$(g1)$ The constant $b \geq 0$ exists such that

$$|g(x,t)| \leq b|t|^{\gamma-1}, \qquad (x,t) \in \partial\Omega \times \mathbb{R}$$

with $1 < \gamma < p_*^{\partial}(N)$, where

$$p_*^{\partial}(n) = \begin{cases} \frac{(N-1)p}{N-p} & p < N, \\ \infty & p \geq N; \end{cases}$$

$(g2)$ There exists constant $\mu > 0$ such that

$$\mu G(x,t) \leq tg(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}$$

where $G(x,t) = \int_0^t g(x,s)ds$ and $p, q, \mu, \gamma, \theta$ satisfy the following relations

$$2 \leq p < \gamma < q < \mu \quad \& \quad \theta < \mu. \tag{1}$$

Definition 1.1. ((PS) compactness condition) Let X be a reflexive Banach space. We say that $I \in C(X, \mathbb{R})$ satisfies the Palais-Smale (PS) compactness condition if any sequence $\{u_k\} \subset X$ such that

- $\{I(u_k)\}$ is bounded, and
- $I'(u_k) \to 0$ in X,

has a convergent subsequence in X.

The Mountain Pass Theorem (MPT) is an existence theorem from the calculus of variations and is as follows.

Theorem 1.2. *Let $(X, \|.\|_X)$ be a reflexive Banach space. Suppose that the functional $I : X \to (-\infty, +\infty]$ satisfies (PS) compactness condition and also the following assertions*

(i) $I(0) = 0$*;*

(ii) There exists $e \in V$ such that $I(e) \leq 0$;

(iii) There exists positive constant ρ such that $I(u) > 0$, if $\|u\|_X = \rho$;

Then I has a critical value $c \leq \rho$ which is characterized by

$$c = \inf_{h\in\Gamma} \sup_{t\in[0,1]} I(h(t)),$$

where $\Gamma = \{h \in C([0,1], V) : h(0) = 0\ ,\ h(1) = e\}$.

2. Main result

We state now the main result of the paper:

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 3$, constants $p, q, \gamma, \mu, \theta$ hold the relations (1) and the functions f, g satisfy the assumptions $(f1), (f_2)$ and $(g1), (g2)$, respectively. Then the Steklov problem*

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) + div(|\nabla u|^{q-2}\nabla u) = f(x,u) & x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} + |\nabla u|^{q-2}\frac{\partial u}{\partial n} = g(x,u) & x \in \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

admits at least one nontrivial (weak) solution.

We point out in [2–8] authors have probed some elliptic equations with different boundary conditions usually on the Heisenberg groups.

We continue by the definition of weak solution for the problem ($\mathcal{P}$).

Definition 2.2. (Weak solution) We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of ($\mathcal{P}$) if the following integral equality is true

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla v dx + \int_\Omega f(x,u)v dx = \int_{\partial\Omega} g(x,u)v d\sigma,$$

for any $v \in W_0^{1,p}(\Omega)$.

We consider the Euler-Lagrange energy functional corresponding to the problem ($\mathcal{P}$); i.e.,

$$I(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx + \int_\Omega F(x,u)dx - \int_{\partial\Omega} G(x,u)d\sigma,$$

that in which

$$F(x,t) = \int_0^t f(x,s)ds \quad \& \quad G(x,t) = \int_0^t g(x,s)ds.$$

Clearly, every critical point of I is a weak solution of the problem ($\mathcal{P}$). To prove that I has a critical point we apply MPT (Theorem 1.2):

Firstly, we verify that I satisfies MPT conditions:

Proof. It is clear that $I(0) = 0$. Since $p < q$, so $W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ and from $(g2)$, one has

$$C|t|^\mu \leq G(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}.$$

for some suitable $C > 0$. Take $e \in W^{1,q}(\Omega) \subset W^{1,p}(\Omega)$, then for appropriate $a_1', a_2' > 0$, we have

$$\begin{aligned} I(te) &= \tfrac{t^p}{p}\|e\|_p^p + \tfrac{t^q}{q}\|e\|_q^q + \textstyle\int_\Omega F(x,te)dx - \int_{\partial\Omega} G(x,te)d\sigma \\ &\leq \tfrac{t^p}{p}\|e\|_q^p + \tfrac{t^q}{q}\|e\|_q^q + a_1'\|u\|_q + a_2'|t|^\theta\|u\|_q^\theta - C|t|^\mu \textstyle\int_{\partial\Omega}|e|^\mu d\sigma. \end{aligned}$$

Now, since $\mu > q > \gamma > p > 1$ and $\mu > \theta$ for t large enough $I(te)$ is negative. We now prove condition (iii) of MPT. From $(f2)$ one gains that $F(x,t) \geq 0$ for $t \in \mathbb{R}$. Take u with $\|u\|_p = \rho > 0$. Using standard embedding it follows that

$$I(u) \geq \frac{1}{p}\rho^p + \frac{1}{q}\rho^q - b'\rho^\gamma > 0,$$

provided $\rho > 0$ is small enough. Therefore, MPT conditions are held for the functional I.
Now, we verify Palais-Smale compactness condition. In fact we show that any (PS)-sequence is bounded. To this end, suppose that $\{u_k\}$ is a sequence in $W_0^{1,p}(\Omega)$ such that

$$\{I(u_k)\} \text{ is bounded} \quad \& \quad I'(u_k) \to 0 \quad \text{in } W_0^{1,p}(\Omega).$$

Using the standard embedding, there exists $b' > 0$ such that

$$\begin{aligned} I'(u_k)u_k &= \|u_k\|_p^p + \|u_k\|_q^q + \int_\Omega f(x,u_k)u_k dx - \int_{\partial\Omega} g(x,u_k)u_k d\sigma, \\ &\geq \|u_k\|_p^p + \|u_k\|_p^q - b'\|u_k\|_p^\gamma \end{aligned}$$

Thus, for large enough k we have

$$\|u_k\|_p^q \leq \|u_k\|_p^p + \|u_k\|_p^q \leq b'\|u_k\|_p^\gamma.$$

Since $\gamma < q$, $\{u_k\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$ as desired. $\square$

Remark 2.3. *Mountain pass theorem (MPT) ensures that problem ($\mathcal{P}$) has at least one weak solution and by $(f3)$, it is a nontrivial solution.*

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Wendel's Theorem on Homogeneous Spaces

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Abstract

Here we extended the Wendel's theorem by a general result related to homogeneous space. Also, some results are given for $A^1(G/H)$ that is not true for $L^1(G/H)$ in general, such as existence of bounded approximate identity.

1. First Section

In this section, we provide a summary of the mathematical notion and definitions which will be used in the sequel. Let G be a locally compact group and H be a compact subgroup of G and λ_G, λ_H be Haar measures on G, H respectively. Let Δ_G, Δ_H be modular functions on G, H respectively and $q : G \longrightarrow G/H$ given by $x \longrightarrow q(x) := xH$ is canonical map. The quotient space G/H is considered as a homogeneous space that G acts on it by $x(yH) = (xy)H$. A rho-function for the pair (G, H) is a continuous function $\rho : G \longrightarrow (0, \infty)$ such that $\rho(x\xi) = \rho(x)\frac{\Delta_H(\xi)}{\Delta_G(\xi)}$ $(x \in G, \xi \in H)$. By [2, Proposition 2.54] the pair (G, H) always admits a rho-function, and each rho-function ρ induces a strongly quasi-invariant measure μ on G/H such that $\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}$ $(x, y \in G, \xi \in H)$. The map $T_\rho : L^1(G) \longrightarrow L^1((G/H), \mu)$ is defined by

$$T_\rho f(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)} d\lambda_H(\xi) \quad (xH \in G/H).$$

The map T_ρ is a surjective bounded linear map with $\|T_\rho\| \leq 1$ and the Weil's formula holds:

$$\int_{G/H} T_\rho f(xH) d\mu(xH) = \int_G f(x) d\lambda_G(x) \quad (f \in L^1(G)).$$

It has been shown that $L^1(G/H, \mu) = L^1(G/H)$ becomes a Banach algebra by multiplication $\varphi \star \psi = T_\rho(\rho(\varphi \circ q) \star \rho(\psi \circ q))$ $(\varphi, \psi \in L^1(G/H))$ and $L^1(G/H)$ is isomerically isomorphic to the closed subalgebra $L^1(G : H)=\{f \in L^1(G);\ f(x\xi) = f(x)\ , x \in G, \xi \in H\}$ of $L^1(G)$ via T_ρ. The left translation of $\varphi \in L^1(G/H)$ by $x \in G$ is defined by

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$$L_x\varphi(yH) = T_\rho(L_x(\varphi \circ q))(yH) = \tfrac{\rho(x^{-1}y)}{\rho(y)}\varphi(x^{-1}yH) \quad (yH \in G/H).$$

Also, there is a bounded surjective linear map $T_\infty : L^\infty(G) \longrightarrow L^\infty(G/H)$ defined by

$$T_\infty(f)(xH) = \textstyle\int_H f(x\xi)d\lambda_H(\xi) \quad (x \in G, f \in L^\infty(G)),$$

and $T_\infty(C_0(G)) = C_0(G/H)$. For $m \in M(G)$, $\tilde{T} : M(G) \longrightarrow M(G/H)$ by $\tilde{T}(m)(E) = m(q^{-1}(E))$ ($E \subseteq G/H$ is a Borel set) were introduced by Reiter and Stegeman in [5]. For $\nu \in M(G/H)$ there exists $m_\nu \in M(G)$ such that $\int_G f dm_\nu = \int_{G/H} T_\infty(f)d\nu$ ($f \in C_0(G)$). For $m \in M(G)$, and $\nu, \omega \in M(G/H), \varphi \in C_0(G/H)$, $m \star \nu$ and $\nu \star \omega$ are defined by

$$m \star \nu(\varphi) = \tilde{T}(m \star m_\nu)(\varphi) = \int_{G/H}\int_G \varphi(xyH)dm(x)d\nu(yH),$$

$$(\nu \star \omega)(\varphi) = \tilde{T}(m_\nu \star m_\omega)(\varphi) = \int_{G/H}\int_{G/H}\int_H \varphi(x\xi yH)d\lambda_H(\xi)d\upsilon(xH)d\omega(yH).$$

In other words $L^1(G/H)$ is an ideal of $M(G/H)$ and for $\varphi \in L^1(G/H)$, $\nu \in M(G/H)$ and $x \in G$, we have

$$(\varphi \star \nu)(xH) = \int_{\frac{G}{H}} \Delta_G(y^{-1})\int_H \varphi(x\xi y^{-1}H)\frac{\rho(x\xi y^{-1})}{\rho(x)}d\lambda_H(\xi)d\nu(yH),$$

$$(\nu \star \varphi)(xH) = \int_{\frac{G}{H}}\int_H \varphi(\xi y^{-1}xH))\frac{\rho(\xi y^{-1}x)}{\rho(x)}d\lambda_H(\xi)d\nu(yH).$$

For more details see [1, 6]. In [4] we showed that an analogue of wendel's result holds for $L^1(G/H)$ only when H is normal. In this paper we show that if H is closed subgroup of G the wendel's theorem holds for $A^1(G/H)$, as an ideal of $L^1(G/H)$.

2. Second Section

Let G be a compact group and H be a closed subgroup of G. Define $A^1(G/H)$, as follow

$$A^1(G/H) = \{\varphi \in L^1(G/H) : \varphi(hxH) = \varphi(xH), x \in G, h \in H\}.$$

For $\varphi, \psi \in A^1(G/H)$, the involution and convolution on $A^1(G/H)$ are defined by the relations

$$\varphi \star \psi(xH) \;\;=\;\; \int_G \varphi(yH)\psi(y^{-1}xH)d\lambda_G(y),$$

and

$$\begin{aligned}\varphi^*(xH) &= T_\rho((\varphi \circ q)^*)(xH)\\ &= \overline{(\varphi \circ q)}(x^{-1})\Delta_G(x^{-1}).\end{aligned}$$

So $A^1(G/H)$ is subalgebra of $L^1(G/H)$ and $A^1(G/H)$ with this involution is a Banach $*$- algebra. For $\varphi \in L^1(G/H)$ the map $J_1 : L^1(G/H) \longrightarrow L^1(G/H)$ is defined via

$$J_1(\varphi)(xH) \;\;= \textstyle\int_H \varphi(hxH)d\lambda_H(h).$$

J_1 is a linear operator, $J_1(C_c(G/H)) \subseteq C_c(G/H)$, $J_1(L^1(G/H)) = A^1(G/H)$ and $\|J_1(\varphi)\|_{L^1(G/H)} \leq \|\varphi\|_{L^1(G/H)}$. For more details see [3].

Theorem 2.1. *Let G be a compact group and H be a closed subgroup of G. The Banach algebra $A^1(G/H)$ has a right approximate identity.*

Proof. Let $(\gamma_\alpha)_{\alpha\in I}$ be a right approximate identity for $L^1(G/H)$, for all $\alpha \in I$ let $J_1(\gamma_\alpha) = \eta_\alpha$, then $(\eta_\alpha)_{\alpha\in I}$ is a right approximate identity for $A^1(G/H)$. Since $A^1(G/H)$ is an involution Banach algebra, it has a bounded left approximate identity. So $A^1(G/H)$ has a bounded approximate identity. □

Remark 2.2. *Let G be a compact group and H be a closed subgroup of G. Let*

$$L^1(G:H) := \{f \in L^1(G) : f(x\xi) = f(x)\},$$

and

$$A^1(G:H) := \{f \in L^1(G) : f(\xi x) = f(x)\}.$$

Then $A^1(G:H)$, $L^1(G:H)$ are Banach subalgebra of $L^1(G)$. For $\varphi \in A^1(G/H)$ take $(\varphi \check{\circ} q)(x) = (\varphi \circ q)(x^{-1})$, then $(\varphi \check{\circ} q) \in L^1(G:H) \cap A^1(G:H)$ and so $\check{\varphi} \in A^1(G/H)$, $\|\varphi\|_{L^1(G/H)} = \|\check{\varphi}\|_{L^1(G/H)}$.

Definition 2.3. Let G be a compact group and H be a closed subgroup of G. $M^1(G/H)$ is subalgebra of $M(G/H)$ given by

$$M^1(G/H) = \{\nu \in M(G/H) : \nu(hE) = \nu(E), \forall h \in H, E \in B_{G/H}\},$$

where $B_{G/H}$ is the σ-algebra of Borel sets.

Theorem 2.4. *Let G be a compact group and H be a closed subgroup of G. Then $M^1(G/H)$ has an identity.*

Theorem 2.5. *Let G be a compact group and H be a closed subgroup of G. If G/H has been attached a strongly G-invariant measure μ, then $A^1(G/H)$ is an ideal of $M^1(G/H)$.*

Proof. For $\nu \in M^1(G/H)$, $\varphi \in A^1(G/H)$ and $\psi \in C_0(G/H)$ we can write,

$$\begin{aligned}
\nu \star \varphi(\psi) &= (\nu \star \mu_\varphi)(\psi) \\
&= \int_{G/H}\int_{G/H}\int_H \psi(xhyH)\varphi(yH)d\nu(xH)d\mu(yH) \\
&= \int_{G/H}\int_{G/H}\int_H \psi(xyH)\varphi(h^{-1}yH)d\nu(xH)d\mu(yH) \\
&= \int_{G/H}\int_{G/H}\int_H \psi(yH)\varphi(h^{-1}x^{-1}yH)d\nu(xH)d\mu(yH).
\end{aligned}$$

Thus

$$\begin{aligned}
\nu \star \varphi(yH) &= \int_{G/H}\int_H \varphi(h^{-1}x^{-1}yH)d\nu(xH)d\lambda_H(h) \\
&= \int_{G/H} \varphi(x^{-1}yH)d\nu(xH).
\end{aligned}$$

For $\xi \in H$ we have

$$\begin{aligned}
\nu \star \varphi(\xi yH) &= \int_{G/H} \varphi(x^{-1}\xi yH)d\nu(xH) \\
&= \int_{G/H} \varphi(x^{-1}\xi^{-1}\xi yH)d\nu(\xi xH) \\
&= \int_{G/H} \varphi(x^{-1}yH)d\nu(xH) \\
&= \nu \star \varphi(yH).
\end{aligned}$$

Also

$$\varphi \star \nu(yH) = \int_{G/H} \int_H \varphi(yhx^{-1}H) d\lambda_H(h) d\nu(xH),$$

and

$$\begin{aligned} \varphi \star \nu(\xi yH) &= \int_{G/H} \int_H \varphi(\xi yhx^{-1}H) d\lambda_H(h) d\nu(xH) \\ &= \varphi \star \nu(yH). \end{aligned}$$

□

Definition 2.6. Let G be a compact group and H be a closed subgroup of G. Let $A(G/H)$, be the closed linear subspace of $C(G/H)$ given by

$$A(G/H) = \{f \in C(G/H) : f(\xi xH) = f(xH), \forall \xi \in H, xH \in G/H\}.$$

Theorem 2.7. *Let G be a compact group and H be a closed subgroup of G. If $T : A^1(G/H) \longrightarrow A^1(G/H)$ is a bounded linear operator such that $T(\varphi \star \psi) = \varphi \star T(\psi)$ (or $T(\varphi \star \psi) = T(\varphi) \star \psi$), then there exists $\mu \in M^1(G/H)$ such that $T(\varphi) = \varphi \star \mu$ ($T(\varphi) = \mu \star \varphi$).*

Proof. For $\varphi_1 \in A(G/H)$, $\varphi_2 \in A^1(G/H)$, $\nu \in M^1(G/H)$, we have

$$\begin{aligned} &< \varphi_1, \nu \star \varphi_2 > = < \varphi_1 \star \overline{\varphi_2^\star}, \nu >, \\ &< \varphi_1, \varphi_2 \star \nu > = < \overline{\varphi_2^\star} \star \varphi_1, \nu >, \end{aligned}$$

and it is easy to sea that $\varphi_1 \star \overline{\varphi_2^\star}, \overline{\varphi_2^\star} \star \varphi_1 \in A(G/H)$. Let $(\eta_\alpha)_\alpha$ be a bounded approximate identity for $A^1(G/H)$ and $\eta_0 \in M^1(G/H)$ be a $\omega^\star$- accumulation point of $(T(\eta_\alpha))_\alpha$ .Then we have

$$T(\varphi) = \lim_\alpha T(\varphi \star \eta_\alpha) = \lim_\alpha \varphi \star T(\eta_\alpha) = \varphi \star \eta_0.$$

□

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Mean Ergodicity of multiples of composition operators on some Banach spaces

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Abstract

In this paper we consider the multiples of composition operator λC_φ on some Banach spaces with special properties, such as Bergman and Hardy spaces and discuss about mean ergodicity and uniform mean ergodicity of λC_φ on X. In the case $\lambda \in \mathbb{C}$, $|\lambda| = 1$ and φ is an elliptic automorphism, mean ergodicity and uniform mean ergodicity of λC_φ are completely characterized.

1. Introduction

1.1. Ergodic Operators

Suppose T is a bounded linear operator on a locally convex Hausdorff space X, the Cesáro means of T is defined by

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad (n \in \mathbb{N}).$$

An operator T is uniformly mean ergodic if the sequence of Cesáro means of T converges in operator norm topology and it is mean ergodic if $\{T_{[n]}\}_n$ converges in the strong operator topology. Also it is called power bounded if $\sup_{n\in\mathbb{N}} ||T^n|| < \infty$.

An easy calculation shows that $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n}T_{[n-1]}$. So if T is uniformly mean ergodic or mean ergodic then for all $x \in X$, $\lim_{n\to\infty} \frac{||T^n x||}{n} = 0$.

The study of mean ergodicity of linear operators on Banach spaces goes back to 1931, when Von Numann proved that for a unitary operator T on a Hilbert space H, there is a projection P on H, such that $T_{[n]}$ converges to P in the strong operator topology. In 1939 Lorch demonstrated that for reflexive Banach spaces, power bounded operators are mean ergodic. Dunford in 1943 stated the connection between the spectral properties of an operator and its uniform mean ergodicity. [5] is a perfect survey on ergodic theory.

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Recall that by $\sigma(T)$ (spectrum of T) we mean the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible. The approximate point spectrum $\sigma_{ap}(T)$, is the set of all $\lambda \in \mathbb{C}$ for which there is $\{x_n\} \subseteq X$ with $||x_n|| = 1$ such that $\lim_{n\to\infty} ||T(x_n) - \lambda x_n|| = 0$. It is well known that $\sigma_{ap}(T) \subseteq \sigma(T)$.
The following theorem is due to Dunford and Lin, see [1].

Theorem 1.1. *If an operator T on a Banach space X is uniformly mean ergodic then both $(||T^n||/n)_n$ converges to 0 and either $1 \in \mathbb{C} \setminus \sigma(T)$ or 1 is a pole of order 1 of the resolvent $R_T : \mathbb{C} \setminus \sigma(T) \to L(X), R_T(\lambda) = (T - \lambda I)^{-1}$. Consequently if 1 is an accumulation point of $\sigma(T)$, then T is not uniformly mean ergodic.*

Proof. [1]. □

1.2. Self Maps of the Unit Disk

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$. The analytic self map of the unit disk are divided in two classes of elliptic and non-elliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. The non-elliptic one has a unique fixed point $p \in \overline{\mathbb{D}}$, such that $\{\varphi_n\}_n$ converges to p uniformly on compact subset of $\mathbb{D}$. This point is called Denjoy-Wolff point.
An elliptic automorphism φ, has an interior fixed point p with $|\varphi'(p)| = 1$. The holomorphic automorphism of $\mathbb{D}$ which is defined by $\Phi_p(z) = \frac{p-z}{1-\bar{p}z}$, interchanges 0 and p and $\Phi_p^{-1} = \Phi_p$. Let $\phi = \Phi_p \ o\ \varphi \ o\ \Phi_p$. $\phi(0) = 0$ and since φ is an elliptic automorphism, $\phi(z) = \alpha z$ for some $\alpha \in \partial\mathbb{D}$. If $\alpha^n = 1$ for some $n \in \mathbb{N}$, ϕ is called rational rotation and otherwise it is called irrational rotation. Also $C_\phi = C_{\varphi_p} \ o\ C_\varphi \ o\ C_{\varphi_p}$, C_φ and C_ϕ are similar so they have the same ergodic properties. Afterwards without loss of generality, we may assume that $\varphi(z) = \alpha z$, where $|\alpha| = 1$.
For further study on holomorphic self maps of the unit disk, see [3].
Each $\varphi \in H(\mathbb{D})$ induces a linear composition operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ by $C_\varphi(f)(z) = f(\varphi(z))$ for every $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.
Let X be a Banach space that continuously embedded in $H(\mathbb{D})$. $\varphi \in H(\mathbb{D})$ is called a symbol for X, if $C_\varphi(X) \subseteq X$. By Close Graph Theorem φ is symbol for X if and only if C_φ is bounded linear operator from X in to itself. Throughout this paper, suppose the space X satisfies the following conditions:

1. automorphisms are symbols for X.
2. Polynomials are dense in it.
3. For each symbol ψ of X with $\psi(0) = 0$, C_ψ is power bounded on X.
4. For $\alpha \in \partial\mathbb{D}$ there is $f \in X$ such that $\lim_{z\to\alpha} Ref(z) = +\infty$.
5. $B(0, r_e(C_\varphi)) \subseteq \sigma(C_\varphi)$ for each univalent symbol φ with Denjoy-Wolff point $0 \in \mathbb{D}$.

Bergman spaces $A^p(\mathbb{D})$ and the classical Hardy spaces $H^p(\mathbb{D})$ for all $p \geq 1$ are the examples of such spaces, for details see [4]. We recall that For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is defined by

$$H^p(\mathbb{D}) = \{f \in H(\mathbb{D}) : ||f||_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p < \infty\}.$$

When $p \geq 1$, $H^p(\mathbb{D})$ is a Banach space with norm $||.||_p$. Also the Bergman space is the space of all analytic functions that

$$||f||^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty.$$

The study of ergodic properties of composition operators has received a special attention from many authors and this topic was investigated on various spaces of holomorphic functions. In [2] and [4] the authors completely characterized power bounded, mean ergodic and uniformly mean ergodic composition operators on various Banach spaces of analytic functions. In this paper, we discuss about the following questions: when $\lambda C_\varphi : X \to X$, is mean ergodic or uniformly mean ergodic?

2. Main Results

For a positive integer n, the nth iterates of φ is denoted by φ_n.

Proposition 2.1. *Let $\varphi \in \mathbb{D}$ be a symbol for X and $\lambda \in \mathbb{C}$. If λC_φ is power bounded, mean ergodic or uniformly mean ergodic on X, then $|\lambda| \leq 1$.*

Proof. Since $||\lambda^n C_{\varphi_n} 1|| = |\lambda|^n \leq ||\lambda^n C_{\varphi_n}||$ and if λC_φ is mean ergodic or uniformly mean ergodic, $\frac{||\lambda^n C_{\varphi_n} 1||}{n} = \frac{|\lambda^n|}{n} \to 0$ as $n \to \infty$, in three cases $\{|\lambda|^n\}_n$ must be a bounded sequence. □

Proposition 2.2. *Let $\varphi \in \mathbb{D}$ be a symbol for X and $\lambda \in \mathbb{C}$. If $|\lambda| = 1$, then λC_φ is power bounded if and only if C_φ is power bounded if and only if φ has an interior fixed point.*

Proof. The Proposition follows from the fact that $||(\lambda C_\varphi)^n|| = ||(C_\varphi)^n||$ and the Proposition 2.2 of [4]. □

Let $\mathcal{K}(X)$ be the space of all bounded and compact operators on X. The essential norm of operator T is defined by $||T||_e = inf\{||T - K|| : K \in \mathcal{K}(X)\}$. Also the essential spectral radios of T is $r_e(T) = \lim_{n\to\infty} ||T^n||^{\frac{1}{n}}$. $T \in L(X)$ is quasicompact if $r_e(T) < 1$.

Theorem 2.3. *Let $\varphi \in \mathbb{D}$ be a univalent symbol for X with $z_0 \in \mathbb{D}$ as its Denjoy-Wolff point. If $|\lambda| = 1$, then λC_φ is uniformly mean ergodic if and only if λC_φ is quasicompact.*

Proof. By the hypothesis λC_φ is power bounded, so if it is quasicompact, the uniform mean ergodicity follows from Yosida-Kakutani mean ergodic Theorem, see [1]. Conversely, suppose λC_φ is uniformly mean ergodic. We show $r_e(\lambda C_\varphi) < 1$. Without loss of generality we may assume that $z_0 = 0$. Since λC_φ is power bounded, there exists $M > 0$ such that $||\lambda^n C_{\varphi_n}||_e \leq ||\lambda^n C_{\varphi_n}|| \leq M$. So $r_e(\lambda C_\varphi) = \lim_{n\to\infty} ||\lambda^n C_{\varphi_n}||^{\frac{1}{n}} \leq 1$. Moreover, $\overline{B(0, r_e(C_\varphi))} \subseteq \sigma(C_\varphi)$, so $\overline{B(0, r_e(\lambda C_\varphi))} \subseteq \sigma(\lambda C_\varphi)$. If $r_e(\lambda C_\varphi) = 1$, clearly, $\overline{B(0, 1)} \subseteq \sigma(\lambda C_\varphi)$. By Dunford-Lin Theorem λC_φ can not be uniformly mean ergodic, so we must have $r_e(\lambda C_\varphi) < 1$ and λC_φ is quasicompact. □

Theorem 2.4. *Suppose φ is an elliptic automorphism which is similar to an irrational rotation and $|\lambda| = 1$. Then λC_φ is mean ergodic operator on X.*

Proof. As we said in the introduction, we may assume $\varphi(z) = \alpha z$, where $|\alpha| = 1$. Let $m \in \mathbb{N}$. If for all $k \in \mathbb{N}$, $\lambda\alpha^k \neq 1$, then:

$$
\begin{aligned}
||(\lambda C_\varphi)_{[n]} z^m|| &= \frac{1}{n} ||\sum_{j=1}^{n} \lambda^j \alpha^{mj} z^m|| = \frac{1}{n} ||z^m||| \sum_{j=1}^{n} \lambda^j \alpha^{mj}| \\
&\leq \frac{2}{n|1 - \lambda\alpha^m|} ||z^m|| \to 0, \quad as\ n \to \infty.
\end{aligned}
$$

If $\lambda\alpha^k = 1$ for some $k \in \mathbb{N}$,

$$
\begin{aligned}
||(\lambda C_\varphi)_{[n]} z^m|| &= \frac{1}{n} ||\sum_{j=1}^{n} (\lambda\alpha^k)^j \alpha^{jm-kj} z^m|| = \frac{1}{n} ||z^m||| \sum_{j=1}^{n} \alpha^{j(m-k)} z^m| \\
&\leq \frac{2}{n|1 - \alpha^{m-k}|} ||z^m|| \to 0, \quad as\ n \to \infty.
\end{aligned}
$$

By linearity, for all polynomials P, $||(\lambda C_\varphi)_{[n]} P|| \to 0$, as $n \to \infty$. Since polynomials are dense in X and λC_φ is power bounded, the result follows. □

Theorem 2.5. *Suppose φ is an elliptic automorphism which is similar to an irrational rotation and $|\lambda| = 1$. Then λC_φ is not uniformly mean ergodic on X.*

Proof. As before, let $\varphi(z) = \alpha z$, where $|\alpha| = 1$. Since $\alpha^k \neq 1$ for all $k \in \mathbb{N}$, $\overline{\{\alpha^k : k \in \mathbb{N}\}} = \partial\mathbb{D}$ and one easily can show that $\overline{\{\lambda\alpha^k : k \in \mathbb{N}\}} = \partial\mathbb{D}$. Let $z_0 \in \partial\mathbb{D}$. There exists subsequence $\{n_k\} \subseteq \mathbb{N}$ such that $\lambda\alpha^{n_k} \to z_0$. Put $g_{n_k} = \frac{z^{n_k}}{||z^{n_k}||}$. Clearly, $g_{n_k} \in X$ and

$$||\lambda C_\varphi g_{n_k} - z_0 g_{n_k}|| = |\lambda\alpha^{n_k} - z_0| \to 0, \ as\ n \to \infty.$$

So $\partial\mathbb{D} \subseteq \sigma_{ap}(\lambda C_\varphi) \subseteq \sigma(\lambda C_\varphi)$, by Dunford theorem λC_φ is not uniformly mean ergodic. □

Theorem 2.6. *Suppose φ is an elliptic automorphism which is similar to a rational rotation and $|\lambda| = 1$. Then λC_φ is uniformly mean ergodic and consequently, mean ergodic on X.*

Proof. Let $\varphi(z) = \alpha z$, where $|\alpha| = 1$ and $\alpha^{k_1} = 1$, for some $k_1 \in \mathbb{N}$. First suppose that for some $k_2 \in \mathbb{N}$, $\lambda^{k_2} = 1$. Let k be the smallest common multiple of k_1, k_2. Fix $0 \leq r < k$. So for all $n = lk + r$ with $l \geq 0$,

$$(\lambda C_\varphi)_{[n]} = \frac{l}{lk+r}\sum_{m=1}^{k}\lambda^m C_{\varphi_m} + \frac{1}{lk+r}\sum_{m=1}^{r}\lambda^m C_{\varphi_m},$$

when $n \to \infty$, $\frac{l}{lk+r} \to \frac{1}{k}$ and $\frac{1}{lk+r} \to 0$, so $(C_\varphi)_{[n]} \to \frac{l}{k}\sum_{m=1}^{k}\lambda^m C_{\varphi_m}$ and λC_φ is uniformly mean ergodic. Now, suppose for all $k \in \mathbb{N}$, $\lambda^k \neq 1$. In this case for each r, that $0 \leq r < k_1$ and $n = lk + r$ with $l \geq 0$, we have:

$$(\lambda C_\varphi)_{[n]} = \frac{1}{n}\sum_{m=0}^{l-1}\lambda^{mk_1}\sum_{m=1}^{k_1}\lambda^m C_{\varphi_m} + \frac{1}{n}\lambda^{lk_1}\sum_{m=1}^{r}\lambda^m C_{\varphi_m}.$$

Let $M = \sum_{m=1}^{k_1}||C_{\varphi_m}||$, then

$$||(\lambda C_\varphi)_{[n]}|| \leq \frac{2M}{n|1-\lambda^{k_1}|} + \frac{M}{n} \to 0, \ as\ n \to \infty,$$

and λC_φ is uniformly mean ergodic.

□

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Tripled Fixed Point Theorems Under Pata-Type Conditions in Ordered Modular Metric Spaces

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Abstract

In this paper, we generalize and improve the results given in [V. Pata, A fixed point theorem in metric spaces, J. Fixed Point Theory Appl. 10 (2011) 299-305], for tripled fixed points in two cases for monotone and mixed-monotone mappings with three variables in ordered modular metric spaces.

1. Introduction

Modular metric spaces were introduced in [4, 5]. Behind this new notion, there exists a physical interpretation of the modular. A modular on a set bases on a nonnegative (possibly infinite valued) "field of (generalized) velocities": to each time $\lambda > 0$ (the absoulute value of) an averge velocity $\omega_\lambda(x, y)$ is associated in such that in order to cover the distance between points $x, y \in X$, it takes time λ to move from x to y with velocity $\omega_\lambda(x, y)$, while a metric on a set stands for non-negative finite distances between any two points of the set. The process of access to this notion of modular metric spaces is different. Actually we deal with these spaces as the nonlinear version of the classical modular spaces as introduced by Nakano [9] on vector spaces and modular function spaces introduced by Musielack [7].
Recently some authors have introduced and have established some notions and fixed point results in modular metric spaces [6]. Many authors investigated on the existence of the fixed points for contraction type mapping in partially ordered metric spaces [1].
In this paper, we generalize and improve the results of Pata [10] for tripled fixed points in two cases for monotone and mixed-monotone mappings with three variables in ordered modular metric spaces.

Definition 1.1. Let X be an arbitrary set. A function $\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ that will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $x, y \in X$ and for all $\lambda > 0$, is said to be a modular metric on X (or simply a modular if no ambiguity arises) if it satisfies the following three conditions:
(i) given $x, y \in X$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ iff $x = y$;

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(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$ and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.
If instead of (i), we have only the condition:
(i_1) $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudomodular on X and if ω satisfies (i_1) and
(i_2) given $x, y \in X$, if there exists $\lambda > 0$, possibly depending on x and y, such that $\omega_\lambda(x, y) = 0$ implies that $x = y$, then ω is called a *strict modular* on X.

Definition 1.2. [4] Given a modular ω on X, the sets

$$X_\omega \equiv X_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) \to 0 \ as \ \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_\circ) = \{x \in X : \omega_\lambda(x, x_\circ) < \infty \ \ for \ some \ \ \lambda > 0\}$$

are said to be modular spaces (around $x_\circ$). Also the modular spaces X_ω and X^*_ω can be equipped with metrics d_ω and d^*_ω, generated by ω and given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\}, \quad x, y \in X_\omega$$

and

$$d^*_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\}, \quad x, y \in X^*_\omega$$

If ω is a convex modular on X, then according to [4, Theorem 3.6] the two modular spaces coincide, $X_\omega = X^*_\omega$.

Definition 1.3. Given a modular metric space X_ω, a sequence of elements $\{x_n\}_{n=1}^\infty$ from X_ω is said to be modular convergent ($\omega-$convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x such that $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$. This will be written briefly as $x_n \overset{\omega}{\to} x$, as $n \to \infty$.

Definition 1.4. [6] A sequence $\{x_n\} \subset X_\omega$ is said to be ω-*Cauchy* if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{m,n\to\infty} \omega_\lambda(x_n, x_m) = 0$, i.e.,

$$\forall \varepsilon > 0 \ \exists \ \ n_\circ(\varepsilon) \in \mathbb{N} \ \ such \ \ that \ \ \forall n, m \geq n_\circ(\varepsilon) \quad : \quad \omega_\lambda(x_n, x_m) \leq \varepsilon.$$

Modular metric space X_ω is said to be ω-*complete* if each ω-Cauchy sequence from X_ω is modular convergent to an $x \in X_\omega$.

Remark 1.5. *A modular $\omega = \omega_\lambda$ on a set X, for given $x, y \in X$, is non-increasing on λ. Indeed if $0 < \lambda < \mu$, then we have*

$$\omega_\mu(x, y) \leq \omega_{\mu-\lambda}(x, x) + \omega_\lambda(x, y) = \omega_\lambda(x, y)$$

for all $x, y \in X$.

Lemma 1.6. *Suppose that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are three sequences in modular metric space X_ω. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are ω-convergent to x, y and z (respectively) iff the sequence $\{(x_n, y_n, z_n)\}$ is ω-convergent to (x, y, z).*

Throughout the paper, $(X; \omega; \preceq)$ always denotes a partially ordered metric space, i.e., a triple where $(X; \preceq)$ is a partially ordered set and $(X; \omega)$ is a metric space.
For $x; y \in X$, $x \asymp y$ will denote that x and y are comparable, i.e., either $x \preceq y$ or $y \preceq x$ holds.
Recall that the space $(X; \omega; \preceq))$ is said to be regular if it has the following properties:
(i) if for a non-decreasing sequence $\{x_n\}$, $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $x_n \preceq x$ for all n;
(ii) if for a non-increasing sequence $\{x_n\}$, $x_n \longrightarrow x$ as $n \longrightarrow \infty$, then $x_n \succeq x$ for all n.

2. Main results

Throughout the paper, $\psi : [0;1] \longrightarrow [0;1)$ will be a fixed increasing function, continuous at zero, satisfying $\psi(0) = 0$. We will use the following terminology.

Definition 2.1. Let $F : X_\omega^3 \longrightarrow X_\omega$ be a mapping.
(1) F is called non-decreasing if it is non-decreasing in all three variables.
(2) F is called mixed-monotone if it is non-decreasing in the first and third variables, and non-increasing in the second variable.
(3) A point $Y = (x, y, z) \in X_\omega^3$ is called a tripled fixed point of the first kind (or Borcut kind [2]) if

$$F(x, y, z) = z, \quad F(y, x, z) = y, \quad F(z, y, x) = z. \tag{1}$$

(4) A point $Y = (x, y, z) \in X_\omega^3$ is called a tripled fixed point of the second kind (or Berinde-Borcut kind [3]) if

$$F(x, y, z) = z, \quad F(y, x, y) = y, \quad F(z, y, x) = z. \tag{2}$$

Remark 2.2. *In what follows, tripled fixed point results of the first kind will be proved for monotone mappings, while those of the second type will be connected with mixed-monotone mappings. It will be clear in the sequel that part* (3) *of the previous definition can be modified in several ways. In fact, any three combinations of elements* x, y, z *can be taken instead of* $(x, y, z), (y, x, z)$ *and* (z, y, x) *in (1), with the only condition that the first entry of each triple matches the right-hand side. In particular, the "cyclic" case, i.e., the condition*

$$F(x, y, z) = x, \quad F(y, z, x) = and F(z, x, y) = z \tag{3}$$

can be considered. It will also be clear which modifications should be made to the results that follows, so we will not state them explicitly. Moreover, the same treatment can be applied in the case of arbitrary number of variables. It is important to notice that this considerably differs from the case of "mixed-monotone situation". Namely, as was shown in [7], in this case only some particular combinations are possible (in particular, the cyclic case cannot be treated in this way).

Lemma 2.3. *(i) If relations* $\sqsubseteq_1$ *and* $\sqsubseteq_2$ *are defined on* X_ω^3 *by*

$$Y \sqsubseteq_1 V \Longleftrightarrow x \preceq u \wedge y \preceq v \wedge z \preceq w, \quad Y = (x, y, z),\ V = (u, v, w) \in X_\omega^3$$

and

$$Y \sqsubseteq_2 V \Longleftrightarrow x \preceq u \wedge y \succeq v \wedge z \preceq w, \quad Y = (x, y, z),\ V = (u, v, w) \in X_\omega^3$$

and $\Omega : X_\omega^3 \times X_\omega^3 \to \mathbb{R}^+$ *is given by*

$$\Omega_\lambda(Y, V) = \omega_\lambda(x, u) + \omega_\lambda(y, v) + \omega_\lambda(z, w), \quad Y = (x, y, z),\ V = (u, v, w) \in X_\omega^3$$

then $(X_\omega^3, \Omega, \sqsubseteq_1)$, $i = 1, 2$ *are ordered modular metric space. The space* (X_ω^3, Ω) *is* ω*-complete if and only if* (X_ω, ω) *is* ω*-complete. Moreover, the spaces* $(X_\omega^3, \Omega, \sqsubseteq_1)$ *are regular if and only if* $(X_\omega, \Omega, \sqsubseteq_1)$ *is such.*
(ii) If $F : X_\omega^3 \to X_\omega$ *is non-decreasing (w.r.t.* $\preceq$*), then the mapping* $T_F^1 : X_\omega^3 \to X_\omega^3$ *given by*

$$T_F^1 Y = (F(x, y, z), F(y, x, z), F(z, y, x)) \quad Y = (x, y, z) \in X_\omega^3$$

is non-decreasing w.r.t. $\sqsubseteq_1$. *(iii) If* $F : X_\omega^3 \to X_\omega$ *is mixed-monotone, then the mapping* $T_F^2 : X_\omega^3 \to X_\omega^3$ *given by*

$$T_F^2 Y = (F(x, y, z), F(y, x, y), F(z, y, x)) \quad Y = (x, y, z) \in X_\omega^3$$

is non-decreasing w.r.t. $\sqsubseteq_2$.
(iv) The mappings $T_F^i, i = 1, 2$ *are* ω*-continuous if and only if* F *is* ω*-continuous.*
(v) The mapping F *has a tripled fixed point of the first (resp. of the second) kind if and only if the mapping* T_F^1 *(resp.* T_F^2 *) has a fixed point in* X_ω^3.

If what follows, $Y_0 = (x_0, y_0, z_0)$ will be a fixed element in X_ω^3 and for $Y = (x, y, z) \in X_\omega^3$, we will denote $\Omega_\lambda(Y, Y_0) = \omega_\lambda(x, x_0) + \omega_\lambda(y, y_0) + \omega_\lambda(z, z_0)$. It will be clear that the obtained results do not depend on the particular choice of the point Y_0. We will prove first some results for monotone mappings and tripled fixed points of the first (Borcut) kind.

Theorem 2.4. *Let $F : X_\omega^3 \to X_\omega$ be a non-decreasing mapping, and suppose that there exist $x_0, y_0, z_0 \in X_\omega$ such that $x_0 \preceq F(x_0, y_0, z_0),\ y_0 \preceq F(y_0, x_0, z_0),\ z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \geq 0, \alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}&\omega_\lambda(F(x,y,z),F(u,v,w)) + \omega_\lambda(F(y,x,z),F(v,u,w)) + \omega_\lambda(F(z,y,x),F(w,v,u))\\ &\quad\leq (1-\epsilon)\left(\omega_\lambda(x,u) + \omega_\lambda(y,v) + \omega_\lambda(z,w)\right) + \Lambda\epsilon^\alpha\psi(\epsilon)\left[1 + \Omega_\lambda(Y,Y_0) + \Omega_\lambda(V,V_0)\right]^\beta\end{aligned}$$

holds for all $\epsilon \in [0,1]$ and all $x, y, z, u, v, w \in X_\omega$ with $(x \preceq u,\ y \preceq v$ and $z \preceq w)$ or $(x \succeq u,\ y \succeq v$ and $z \succeq w)$. Finally, suppose that F is ω-continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in X_\omega^3$ of the first kind.

Corollary 2.5. *Let $F : X_\omega^3 \to X_\omega$ be a non-decreasing mapping, and suppose that there exist $x_0, y_0, z_0 \in X_\omega$ such that $x_0 \preceq F(x_0, y_0, z_0),\ y_0 \preceq F(y_0, x_0, z_0),\ z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \geq 0, \alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}&\omega_\lambda(F(x,y,z) + F(u,v,w))\\ &\quad\leq \frac{1-\epsilon}{3}\left(\omega_\lambda(x,u) + \omega_\lambda(y,v) + \omega_\lambda(z,w)\right) + \Lambda\epsilon^\alpha\psi(\epsilon)\left[1 + \Omega_\lambda(Y,Y_0) + \Omega_\lambda(V,V_0)\right]^\beta\end{aligned}$$

holds for all $\epsilon \in [0,1]$ and all $x, y, z, u, v, w \in X_\omega$ with $(x \preceq u,\ y \preceq v$ and $z \preceq w)$ or $(x \succeq u,\ y \succeq v$ and $z \succeq w)$. Finally, suppose that F is ω-continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in X_\omega^3$ of the first kind.

Consider now mixed-monotone mappings and tripled fixed points of the second (Berinde-Borcut) kind.

Theorem 2.6. *Let $F : X_\omega^3 \to X_\omega$ be a mixed-monotone mapping, and suppose that there exist $x_0, y_0, z_0 \in X_\omega$ such that $x_0 \preceq F(x_0, y_0, z_0),\ y_0 \succeq F(y_0, x_0, z_0),\ z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \geq 0, \alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}&\omega_\lambda(F(x,y,z),F(u,v,w)) + \omega_\lambda(F(y,x,z),F(v,u,w)) + \omega_\lambda(F(z,y,x),F(w,v,u))\\ &\quad\leq (1-\epsilon)\left(\omega_\lambda(x,u) + \omega_\lambda(y,v) + \omega_\lambda(z,w)\right) + \Lambda\epsilon^\alpha\psi(\epsilon)\left[1 + \Omega_\lambda(Y,Y_0) + \Omega_\lambda(V,V_0)\right]^\beta\end{aligned}$$

holds for all $\epsilon \in [0,1]$ and all $x, y, z, u, v, w \in X_\omega$ with $(x \preceq u,\ y \succeq v$ and $z \preceq w)$ or $(x \succeq u,\ y \preceq v$ and $z \succeq w)$. Finally, suppose that F is ω-continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in X_\omega^3$ of the second kind.

Corollary 2.7. *Let $F : X_\omega^3 \to X_\omega$ be a mixed-monotone mapping, and suppose that there exist $x_0, y_0, z_0 \in X_\omega$ such that $x_0 \preceq F(x_0, y_0, z_0),\ y_0 \succeq F(y_0, x_0, z_0),\ z_0 \preceq F(z_0, y_0, x_0)$. Let, for some fixed constants $\Lambda \geq 0, \alpha \geq 1$ and $\beta \in [0, \alpha]$, the inequality*

$$\begin{aligned}&\omega_\lambda(F(x,y,z),F(u,v,w))\\ &\quad\leq \frac{1-\epsilon}{3}\left(\omega_\lambda(x,u) + \omega_\lambda(y,v) + \omega_\lambda(z,w)\right) + \Lambda\epsilon^\alpha\psi(\epsilon)\left[1 + \Omega_\lambda(Y,Y_0) + \Omega_\lambda(V,V_0)\right]^\beta\end{aligned}$$

holds for all $\epsilon \in [0,1]$ and all $x, y, z, u, v, w \in X_\omega$ with $(x \preceq u,\ y \succeq v$ and $z \preceq w)$ or $(x \succeq u,\ y \preceq v$ and $z \succeq w)$. Finally, suppose that F is ω-continuous or that the space is regular. Then F has a tripled fixed point $Y^* = (x^*, y^*, z^*) \in X_\omega^3$ of the first kind.

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Some results about metrizability of a modular space

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Abstract

In this paper we prove a metrization theorem on modular spaces. Then we get a sufficient and necessary condition for a modular space to be metrizable.

1. Introduction and Preliminaries

A modular space is a pair (X, ρ) where X is a real linear space and ρ is a real valued functional on X which satisfies the conditions:

1. $\rho(x) = 0$ if and only if $x = 0$,
2. $\rho(-x) = \rho(x)$,
3. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for any nonnegative real numbers α, β with $\alpha + \beta = 1$.

The functional ρ is called a modular on X. There are many arguably important special instances of well known spaces in which these properties are fulfilled. Interestingly, it is shown that a modular induces a vector space $X_\rho = \{x \in X : \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}$ which is called a modular linear space. Furthermore, Musielak and Orlicz in [1] naturally provide the first definitions of the following key concepts in a modular space (X, ρ):

D1. A sequence $\{x_n\}$ in $B \subseteq X$ is said to be ρ-convergent to a point $x \in B$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

D2. A ρ-closed subset $B \subseteq X$ is meant that it contains the limit of all its ρ-convergent sequences.

D3. A sequence $\{x_n\}$ in $B \subseteq X$ is said to be ρ-Cauchy if $\rho(x_m - x_n) \to 0$ as $m, n \to \infty$.

D4. A subset B of X is said to be ρ-complete if each ρ-Cauchy sequence in B is ρ-convergent to a point of B.

D5. **ρ-bounded subsets:** A subset $B \subseteq X_\rho$ is called ρ-bounded if $\sup\limits_{x,y \in B} \rho(x - y) < \infty$.

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D6. ρ-compact subsets: A ρ-closed subset $B \subseteq X$ is called ρ-compact if any sequence $\{x_n\} \subset B$ has a ρ-convergent subsequence.

For a modular space (X, ρ), the function ω_ρ which is said growth function is defined on $[0, \infty)$ as follows:

$$\omega_\rho(t) = \inf\{\omega : \rho(tx) \leq \omega\rho(x) : x \in X, 0 < \rho(x)\}.$$

It is easy to show that when (X, ρ) satisfies $\omega_\rho(2) < \infty$, then every ρ-convergent sequence in (X, ρ) is ρ-Cauchy. Also, we note that in such cases every ρ-compact set is ρ-bounded and ρ-complete.
In 2000 Branciari [2] introduced the notion of a ν-generalized metric space. A 2-generalized metric space was also called a generalized metric space, or for short, g.m.s, or rectangular metric space. ν-generalized metric spaces were investigated by many authors and various fixed point theorems in such spaces were stated and references therein. There were also examples for ν-generalized metrics that are not metrics.

Definition 1.1. Let X be a nonempty set, $\nu \in \mathbb{N}$ and $d : X \times X \to [0, \infty)$ be a function such that for all $x, y \in X$,
1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + ... + d(u_\nu, y)$ for all distinct points $u_1, ..., u_\nu$ are not belong to $\{x, y\}$. Then d is called a ν-generalized metric and (X, d) is called a ν-generalized metric space, or for short, ν- g.m.s.

As in a generalization of a rectangular metric space, George et al. [5] introduced the notion of a rectangular b-metric space. This notion was also introduced independently by Roshan et al. [7].

Definition 1.2. Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function such that for some $s \geq 1$ and for all $x, y \in X$, all distinct points u, v do not belong to $\{x, y\}$,
1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$.
Then d is called a rectangular b-metricon X and (X, d, s) is called a rectangular b-metric space with coefficient s.

The convergence, Cauchy sequence and the completeness in rectangular b-metric spaces were defined similar as in metric spaces. Note that the topology on a rectangular b-metric space is explicitly understood the topology induced by its convergence.

Definition 1.3. Let (X, d, s) be a rectangular b-metric space.
1.A sequence $\{x_n\}$ is called convergent to x, written as $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$.
2.A sequence $\{x_n\}$ is called Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
3.(X, d, s) is called complete if each Cauchy sequence is a convergent sequence.

The metrizability of generalized metric spaces was also attracted by many authors. One of the most interesting quantitative metrization theorem of generalized metric spaces is Frink' s result. Note that Frink' s metrization technique has impacted many results. In 1998 Aimar et al.[1] improved Frink' s metrization technique to give a direct proof of Macías-Segovia theorem on the metrization of a b-metric space (X, d, s). In 2006 Schroeder showed the limit of Frink' s construction by constructing a counterexample that for a given b-metric space (X, d, s), the function defined by 1.2 is not a metric [[8], Example2]. In 2009 Paluszynski and Stempak [6] also improved Frink' s metrization technique to produce a metric d from a given b-metric space (X, D, K). Recently Dung et al. [3] constructed a simple.

2. Main Results

The following result is a metrization theorem of rectangular b-metric spaces.

Theorem 2.1. *Let (X, d, s) be a rectangular b-metric space such that the limit of a convergent sequence is unique.Then*
1. There exists a metric d on X such that $\lim_{n\to\infty} x_n = x$ in (X, d, s) if and only if $\lim_{n\to\infty} x_n = x$ in (X, d). In particular, (X, d, s) is metrizable by the metric d.
2. A sequence $\{x_n\}$ is Cauchy in (X, d, s) if and only if it is Cauchy in (X, d). In particular, (X, d, s) is complete if and only if (X, d) is complete.

From Theorem 2.1, we get the following results. For other results on metrization of ν-generalized metric spaces, see [4] for example.

Corollary 2.2. *([3], Theorem 5.3). Let (X, D) be a rectangular metric space such that the limit of a convergent sequence is unique.Then*
1.There exists a metric d on X such that $\lim_{n\to\infty} x_n = x$ in (X, D) if and only if $\lim_{n\to\infty} x_n = x$ in (X, d). In particular, (X, D) is metrizable by the metric d.
2.A sequence $\{x_n\}$ is Cauchy in (X, D) if and only if it is Cauchy in (X, d). In particular, (X, D) is complete if and only if (X, d) is complete.

Corollary 2.3. *Let (X, d, s) be a rectangular b-metric space. Then (X, d, s) is metrizable if and only if the limit of a convergent sequence in (X, d, s) is unique.*

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Estimates on coefficients of a general subclass of bi-univalent functions by means of Horadam polynomial expansions

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Abstract

In this paper, we introduce a new subclass $\mathcal{B}_{a,b,p,q}(\lambda,\mu,x)$ of bi-univalent functions by using Horadam polynomials. Furthermore, we obtain upper bounds for the general coefficients for functions in this subclass. Moreover, we obtain the upper bounds for the initial Taylor-Maclaurin coefficients and also, Fekete-Szegö inequalities for functions in this subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

1. Introduction

Let $\mathcal{A}$ be a class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit disk

$$\mathbb{U} = \{z : c \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$ (for more details see [2]). So every function $f \in \mathcal{S}$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, (z \in \mathbb{U})$ and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geqq \frac{1}{4}\right).$$

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In fact, the inverse function f^{-1} has a series expansion of the form:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n = w - a_2 w^2 + (2a_2^2 - a3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. For an interesting examples of subclasses of bi-univalent functions see [6, 8]. Horzum and Kocer [4] studied Horadam polynomials sequence $h_n(x, a, b; p, q)$, or briefly $h_n(x)$, which are given by the following recurrence relation

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \qquad (n \geq 2), \tag{3}$$

with $h_1 = a$, $h_2 = bx$ and $h_3 = pbx^2 + aq$ where (a, b, p, q are some real constants). The mentioned polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many of sciences, specially in the mathematics, statistics and physics (For more information see [3, 5]).

Remark 1.1. (see [3]) *Let $\Omega(x, z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then*

$$\Omega(x, z) = \frac{a + (b - ap)xz}{1 - pxz - qz^2} = \sum_{n=1}^{\infty} h_n(x) z^{n-1}. \tag{4}$$

In this paper, we introduce a new subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$ of bi-univalent functions by applying Horadam polynomials. We use the Faber polynomial expansion to find not only the estimates of the coefficients $|a_2|$ and $|a_3|$, but also the estimates of the coefficients $|a_n|$ for functions in this subclass. Consequently, we generalize and improve the works of Alamoush [1] and Bulut [7].

2. Subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$

Definition 2.1. A function $f \in \Sigma$, given by (1), is said to be in the subclass

$$\mathcal{B}_{a,b,p,q}(\lambda, \mu, x) \quad (\lambda \geqq 1; \mu \geqq 0)$$

of bi-univalent functions, if the following conditions are satisfied:

$$(1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu - 1} \prec 1 - a + \Omega(x, z)$$

and

$$(1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu - 1} \prec 1 - a + \Omega(x, w),$$

where the function g is the inverse of the function f, given by (2), the function $\Omega(x, z)$ is given by (4), and $z, w \in \mathbb{U}$.

Remark 2.2. *There are several choices of the parameters μ, λ and parameters a, b, p, q of the function $\Omega(x, z) = \frac{a+(b-ap)xz}{1-pxz-qz^2}$ which would provide interesting subclasses of bi-univalent functions. For example, we have the following special cases*

- *(A) By putting $\lambda = \mu = 1$, the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$ reduces to the subclass $\Sigma'(x)$ which was defined by Alamoush [1].*
- *(B) By putting $\mu = a = 1$, $b = p = 2$ and $q = -1$, the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$ reduces to the subclass $\mathcal{B}_{\Sigma}(\lambda, x)$ which was introduced by Bulut et al. [7].*
- *(C) By putting $\mu = \lambda = a = 1$, $b = p = 2$ and $q = -1$, the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$ reduces to the subclass $\mathcal{B}_{\Sigma}(x)$ which was introduced by Bulut et al. [7].*

(D) *By putting $\lambda = 1$, $\mu = 0$, $b = p = 2$ and $q = -1$; the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$ reduces to the subclass $\mathcal{S}^*_\Sigma(x)$ which was introduced by Bulut et al. [7].*

Now, we begin by obtaining the estimates on the general coefficients $|a_n|$ for functions in the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$.

Theorem 2.3. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$. If $a_2 = \cdots = a_{n-1} = 0$, then*

$$|a_n| \leqq \frac{|bx|}{\mu + (n-1)\lambda}, \quad (n \geqq 3).$$

Now, we find the initial coefficient bounds and Fekete-Szegö inequalities for functions in the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$.

Theorem 2.4. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{B}_{a,b,p,q}(\lambda, \mu, x)$. Then*

$$|a_2| \leqq \min\left\{\frac{|bx|}{\mu+\lambda}, |bx|\sqrt{\frac{2|bx|}{|(\mu+2\lambda)(\mu+1)b^2x^2 - 2(\mu+\lambda)^2(pbx^2+aq)|}}\right\},$$

$$|a_3| \leqq \min\left\{\frac{b^2x^2}{(\mu+\lambda)^2} + \frac{|bx|}{\mu+2\lambda}, \frac{2|bx|^3}{|(\mu+2\lambda)(\mu+1)b^2x^2 - 2(\mu+\lambda)^2(pbx^2+aq)|} + \frac{|bx|}{\mu+2\lambda}\right\}$$

and

$$|a_3 - \delta a_2^2| \leqq |bx| \begin{cases} \frac{1}{\mu+2\lambda}; & |T(\delta)| \leqq \frac{1}{2(\mu+2\lambda)} \\ 2|T(\delta)|; & |T(\delta)| \geq \frac{1}{2(\mu+2\lambda)}, \end{cases}$$

where

$$T(\delta) = \frac{b^2x^2(1-\delta)}{(\mu+2\lambda)(\mu+1)b^2x^2 - 2(\mu+\lambda)^2(pbx^2+aq)}.$$

3. Corollaries and Consequences

If we put $\lambda = \mu = 1$ in Theorem 2.3, we conclude the following result.

Corollary 3.1. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\Sigma'(x)$. If $a_2 = \cdots = a_{n-1} = 0$, then*

$$|a_n| \leqq \frac{|bx|}{n} \qquad (n \geqq 3).$$

If we put $\mu = a = 1$, $b = p = 2$ and $q = -1$ in Theorem 2.3, we conclude the following result.

Corollary 3.2. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{B}_\Sigma(\lambda, x)$. If $a_2 = \cdots = a_{n-1} = 0$, then*

$$|a_n| \leqq \frac{2x}{1 + (n-1)\lambda} \qquad (n \geqq 3).$$

If we put $\mu = \lambda = a = 1$, $b = p = 2$ and $q = -1$ in Theorem 2.3, we conclude the following result.

Corollary 3.3. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{B}_\Sigma(x)$. If $a_2 = \cdots = a_{n-1} = 0$, then*

$$|a_n| \leqq \frac{2x}{n} \qquad (n \geqq 3).$$

If we put $\mu = 0$, $\lambda = 1$, $b = p = 2$ and $q = -1$ in Theorem 2.3, we conclude the following result.

Corollary 3.4. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{S}^*_\Sigma(x)$. If $a_2 = \cdots = a_{n-1} = 0$, then*

$$|a_n| \leqq \frac{2x}{n-1} \qquad (n \geqq 3).$$

If we put $\lambda = \mu = 1$ in Theorem 2.3, we conclude the following result.

Corollary 3.5. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\Sigma^{'}(x)$. then*

$$|a_2| \leqq \min\left\{\frac{|bx|}{2}, |bx|\sqrt{\frac{|bx|}{|3b^2x^2 - 4(pbx^2 + aq)|}}\right\},$$

$$|a_3| \leqq \min\left\{\frac{b^2x^2}{4} + \frac{|bx|}{3}, \frac{|bx|^3}{|3b^2x^2 - 4(pbx^2 + aq)|} + \frac{|bx|}{3}\right\}$$

and

$$|a_3 - \delta a_2^2| \leqq |bx| \begin{cases} \frac{1}{3}; & \left|\frac{b^2x^2(1-\delta)}{6b^2x^2 - 8(pbx^2+aq)}\right| \leqq \frac{1}{6} \\ \\ 2|T(\delta)|; & \left|\frac{b^2x^2(1-\delta)}{6b^2x^2 - 8(pbx^2+aq)}\right| \geq \frac{1}{6}. \end{cases}$$

Remark 3.6. *Corollary 3.5 is an improvement of result obtained by Alamoush [1].*

If we put $\mu = a = 1$, $b = p = 2$ and $q = -1$ in Theorem 2.4, we conclude the following result.

Corollary 3.7. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{B}_\Sigma(\lambda, x)$. Then*

$$|a_2| \leqq \min\left\{\frac{2x}{1+\lambda}, 2x\sqrt{\frac{2x}{|(1+\lambda)^2 - 4\lambda^2x^2|}}\right\},$$

$$|a_3| \leqq \min\left\{\frac{4x^2}{(1+\lambda)^2} + \frac{2x}{1+2\lambda}, \frac{8x^3}{|(1+\lambda)^2 - 4\lambda^2x^2|} + \frac{2x}{1+2\lambda}\right\}$$

and

$$|a_3 - \delta a_2^2| \leqq 2x \begin{cases} \frac{1}{1+2\lambda}; & \left|\frac{2x^2(1-\delta)}{(1+\lambda)^2 - 4\lambda^2x^2}\right| \leqq \frac{1}{2(1+2\lambda)} \\ \\ 2|T(\delta)|; & \left|\frac{2x^2(1-\delta)}{(1+\lambda)^2 - 4\lambda^2x^2}\right| \geq \frac{1}{2(1+2\lambda)}. \end{cases}$$

Remark 3.8. *Corollary 3.7 is an improvement of result obtained by Bulut et al. [7, Corollary 1].*

If we put $\mu = \lambda = a = 1$, $b = p = 2$ and $q = -1$ in Theorem 2.4, we conclude the following result.

Corollary 3.9. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{B}_\Sigma(x)$. Then*

$$|a_2| \leqq x \quad , \quad |a_3| \leqq x^2 + \frac{2x}{3}$$

and

$$|a_3 - \delta a_2^2| \leqq \begin{cases} \frac{2x}{3}; & |\delta - 1| \leqq \frac{1-x^2}{3x^2} \\ \\ \frac{2|\delta-1|x^3}{1-x^2}; & |\delta - 1| \geq \frac{1-x^2}{3x^2}. \end{cases}$$

Remark 3.10. *Corollary 3.9 is a refinement for estimate of $|a_2|$ obtained by Bulut et al. [7, Corollary 3].*

If we put $\lambda = 1$, $\mu = 0$, $b = p = 2$ and $q = -1$ in Theorem 2.4, we conclude the following result.

Corollary 3.11. *Let the function $f \in \Sigma$, given by (1), be in the subclass $\mathcal{S}^*_\Sigma(x)$. Then*

$$|a_2| \leqq 2x \quad \text{and} \quad |a_3| \leqq 4x^2 + x$$

and

$$|a_3 - \delta a_2^2| \leqq \begin{cases} x; & |\delta - 1| \leqq \frac{1}{8x^2} \\ \\ 8|\delta - 1|x^3; & |\delta - 1| \geq \frac{1}{8x^2}. \end{cases}$$

Remark 3.12. *Corollary 3.11 is a refinement for estimate of $|a_2|$ obtained by Bulut et al. [7, Corollary 4].*

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An application of a Poisson distribution series on certain analytic functions

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Abstract

In the present paper, we investigate new connections between the Poisson distribution series and some subclasses of normalized analytic functions. Further, we consider an integral operator related to Poisson distribution series. Some interesting special cases of our main results are also considered.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1) which are also univalent in $\mathbb{U}$(see [2]).

The class $\mathcal{S}^*(\alpha)$ of starlike functions of order α, with $0 \le \alpha < 1$, defined by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{U} \right\},$$

and the class $C(\alpha)$ of convex functions of order α, with $0 \le \alpha < 1$, defined by

$$C(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{U} \right\} = \{ f \in \mathcal{A} : zf'(z) \in \mathcal{S}^*(\alpha)\},$$

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were introduced by Robertson in [7]. We also write $\mathcal{S}^*(\alpha) = \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Further, $C(0) = C$ is the well-known standard class of convex functions.
A function $f \in \mathcal{A}$ of the form (1) is said to be in the class $S_p(\alpha, \beta)$, if it satisfies the following condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \geq \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \ (\alpha \geq 0,\ 0 \leq \beta < 1,\ z \in \mathbb{U}).$$

A function $f \in \mathcal{A}$ of the form (1) is said to be in the class $UCV(\alpha, \beta)$, if it satisfies the following condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta \ (\alpha \geq 0,\ 0 \leq \beta < 1,\ z \in \mathbb{U}).$$

The classes $S_p(\alpha, \beta)$ and $UCV(\alpha, \beta)$ were introduced by Bharati et al. [6]. It is easy to verify that $f \in UCV(\alpha, \beta) \Leftrightarrow zf' \in S_p(\alpha, \beta)$.
Recently, many researchers investigated connections between various subclasses of univalent funcions and some power series that their coefficients were probabilities of the elementary distributions such as Poisson, Pascal, etc. Also, they obtained necessary and sufficient conditions for these distribution series on certain subclasses of univalent functions (see for example [1, 3]).
A variable x is said to have Poisson distribution, if it takes the values $0, 1, 2, 3, \cdots$ with probabilities e^{-m}, $\frac{me^{-m}}{1!}$, $\frac{m^2e^{-m}}{2!}$, $\frac{m^3e^{-m}}{3!}, \cdots$, respectively, where m is called the parameter. Thus

$$P(x = k) = \frac{e^{-m}m^k}{k!},\ \ k = 0, 1, 2, 3, \cdots,\ \ m > 0.$$

Recently, Porwal [4] introduce a power series whose coefficients are probabilities of Poisson distribution

$$\varphi_m(z) = z + \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}z^n, \tag{2}$$

where m > 0. It is easy to see that the radius of convergence of $\varphi_m(z)$ is infinite.
Porwal and Kumar [5] introduced a new linear operator

$$I_m : \mathcal{A} \longrightarrow \mathcal{A}$$

by using the convolution or Hadamard product, as below

$$I_m(f) = \varphi_m(z) * f(z) = z + \sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!}e^{-m}a_nz^n,\quad (z \in \mathbb{U}). \tag{3}$$

For convenience throughout in the sequel, we use the following notations.

$$\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-1)!} = e^m - 1,$$

$$\sum_{n=2}^{\infty}\frac{m^{n-1}}{(n-2)!} = me^m,$$

$$\sum_{n=3}^{\infty}\frac{m^{n-1}}{(n-3)!} = m^2e^m,$$

$$\sum_{n=2}^{\infty}\frac{m^{n-1}}{n!} = \frac{1}{m}(e^m - 1 - m).$$

Lemma 1.1. *[6] A function $f \in A$ belongs to the class $S_p(\alpha, \beta)$, if*

$$\sum_{k=2}^{\infty} \big(k(1+\alpha) - (\alpha+\beta)\big)|a_k| \leq 1 - \beta.$$

Lemma 1.2. *[6] A function $f \in A$ belongs to the class $UCV(\alpha, \beta)$, if*

$$\sum_{k=2}^{\infty} k\big(k(1+\alpha) - (\alpha+\beta)\big)|a_k| \leq 1 - \beta.$$

In this paper, we determine some sufficient conditions for power series that coefficients are probabilities of the Poisson distribution and other related series to be in the subclasses of analytic functions $S_p(\alpha, \beta)$ and $UCV(\alpha, \beta)$. Finally, we give conditions for the integral operator $T_m(z) = \int_0^z \frac{\varphi_m(t)}{t} dt$ belonging to the these classes.

2. Main results

Theorem 2.1. *A sufficient condition for the function φ_m given by (2) to be in the class $S_p(\alpha, \beta)$ is*

$$e^m(1+\alpha)m \leq 1 - \beta. \tag{4}$$

Theorem 2.2. *A sufficient condition for the function φ_m given by (2) to be in the class $UCV(\alpha, \beta)$ is*

$$e^m\big((1+\alpha)m^2 + (3+2\alpha-\beta)m\big) \leq 1 - \beta. \tag{5}$$

Theorem 2.3. *(i) If the condition*

$$e^m\big((1+\alpha)m^2 + (3+2\alpha-\beta)m\big) \leq 1 - \beta \tag{6}$$

holds, then the operator I_m defined by (3) maps the class $\mathcal{S}^*$ to the class $S_p(\alpha, \beta)$, that is $I_m(\mathcal{S}^*) \subset S_p(\alpha, \beta)$.

(ii) If the condition

$$e^m(1+\alpha)m \leq 1 - \beta \tag{7}$$

defined by (3) maps the class C to the class $S_p(\alpha, \beta)$, that is $I_m(C) \subset S_p(\alpha, \beta)$.

3. An integral operator

In this section, we will examine some inclusion properties of integral operator associated with the function $\varphi_m(z)$ as follow

$$T_m(z) = \int_0^z \frac{\varphi_m(t)}{t} dt, \tag{8}$$

where $\varphi_m(z)$ is given by (2).

Theorem 3.1. *A sufficient condition for the function T_m given by (8) to be in the class $S_p(\alpha, \beta)$ is*

$$(\alpha+\beta)\big((m-1)e^m + 1\big) \leq m(1-\beta). \tag{9}$$

Theorem 3.2. *A sufficient condition for the function T_m given by (8) to be in the class $UCV(\alpha, \beta)$ is*

$$e^m(1+\alpha)m \leq 1 - \beta. \tag{10}$$

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On a class of strongly starlike functions involving q-derivative operator

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Abstract

By applying the q-derivative, we introduce a new subclass of univalent function, so called q-strongly starlike functions and The class of functions defined by SL_q. In the present paper, using the definition of subordination, we investigation necessary and sufficient condition for function in this class. Furthermore, we obtain some some upper bounds on the coefficients.

1. Introduction

Let $\mathcal{H}$ be the class of all analytic functions in the unite disc $D = \{z : |z| < 1\}$ on the complex plan $\mathcal{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f^{'}(0) = 1$. Therefor, a function $f \in \mathcal{A}$ having the expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

The class of univalent functions in $\mathcal{A}$ denoted by S. We say that a function f is subordinate to a function g in D (and write $f \prec g$ or $f(z) \prec g(z)$) if there exist a Schwarz function w such that $f(z) = g(w(z))$, $z \in D$. If the function g is univalent in D, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subset g(D)$ (See [?]).
Jacson's q-derivative of function $f(z)$, introduced by:

$$D_{q,z}f(z) = \frac{f(z) - f(qz)}{z(1-q)}, z \in D, z \neq 0, |q| < 1.$$

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We note that $D_q f(z) \to f'(z)$ as $q \to 1^-$ and $D_q f(0) = f'(0)$, where f' is the ordinary derivative of f.
In the theory of q-calculus, we introduced q-real number $[n]_q$ by

$$[n]_q := \frac{1-q^n}{1-q}, n \in \mathbb{N}.$$

The q-derivative of function f , defined by eqnarray (1) is as follows:

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, (z \in D, 0 < q < 1).$$

For more details see [**?**].
In this paper, we introduced the class SL_q^* of strongly functions as follow

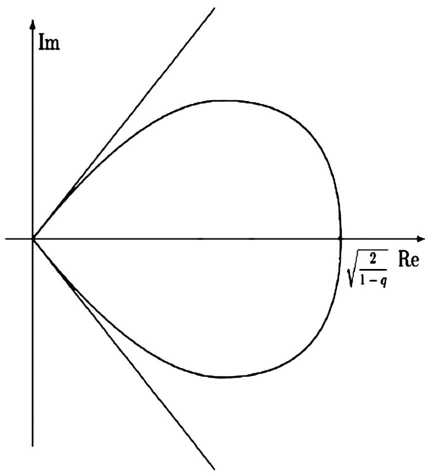
$$SL_q^* = \left\{ f \in \mathcal{A} : \left| \left(z\frac{D_q f(z)}{f(z)} \right)^2 - \frac{1}{1-q} \right| \leq \frac{1}{1-q} \right\}.$$

For $f \in SL_q^*$ the image of D under $z\frac{D_q f(z)}{f(z)}$ lies in the angular sector

$$\Omega = \left\{ w \in \mathbb{C} : \left| w^2 - \frac{1}{1-q} \right| = \frac{1}{1-q} \right\}.$$

Remark 1.1. *Notice that for fixed $q \in (0,1)$, the set Ω is the interior of the right half plan of the lemniscate*

$$\left| w^2 - \frac{1}{1-q} \right| = \frac{1}{1-q}.$$



2. Subordination Result

Theorem 2.1. *Let $f(z) \in \mathcal{A}$. Then $f(z) \in SL_q^*$ if and only if*

$$z\frac{D_q f(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\frac{1}{2}}.$$

Proof. Let $f(z) \in \mathcal{A}$, we have

$$\left| z\frac{D_q f(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q},$$

the above inequality is equivalent to

$$\left| \left(z\frac{D_q f(z)}{f(z)} \right)^2 - M \right| \leq M, M = \frac{1}{1-q} > 1.$$

Therefore, we define the function $\psi(z) = \frac{1}{M} z \frac{D_q f(z)}{f(z)} - 1$ in the unite disc D and so

$$\phi(z) = \frac{\psi(z) - \phi(0)}{1 - \overline{\psi(0)}\psi(z)} = \frac{\frac{1}{M} z \frac{D_q f(z)}{f(z)} - (\frac{1}{M} - 1)}{1 - (\frac{1}{M} - 1)(\frac{1}{M} z \frac{D_q f(z)}{f(z)} - 1)}. \tag{2}$$

and then $\phi(0) = 0, |\phi(z)| < 1$. Therefore by the Schwarz Lemma

$$|\phi(z)| \leq |z|. \tag{3}$$

From (2) and (3), we obtain

$$\left(z\frac{D_q f(z)}{f(z)} \right)^2 = \frac{1 + \phi(z)}{1 - (1 - \frac{1}{M})\phi(z)} = \frac{1 + \phi(z)}{1 - q\phi(z)}$$

Since $z\frac{D_q f(z)}{f(z)}$ having positive real part in the unite disc D, by applying Remark 1.1, we have

$$z\frac{D_q f(z)}{f(z)} = \left(\frac{1 + \phi(z)}{1 - q\phi(z)} \right)^{\frac{1}{2}} \tag{4}$$

The equality (2) shows that

$$z\frac{D_q f(z)}{f(z)} \prec \left(\frac{1+z}{1-qz} \right)^{\frac{1}{2}} ..$$

Conversely, if

$$z\frac{D_q f(z)}{f(z)} \prec \left(\frac{1+z}{1-qz} \right)^{\frac{1}{2}}.$$

then, we have

$$z\frac{D_q f(z)}{f(z)} = \left(\frac{1 + \phi(z)}{1 - q\phi(z)} \right)^{\frac{1}{2}}.$$

Therefore, we obtain

$$\left(z\frac{D_q f(z)}{f(z)} \right)^2 - M = M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)},$$

on the other hand, the function $\left(\frac{1-M}{M} + \phi(z)\right) / \left(1 + \frac{1-M}{M}\phi(z)\right)$ maps the unit circle onto itself, then we have

$$\left| \left(z\frac{D_q f(z)}{f(z)} \right)^2 - M \right| = \left| M\frac{\frac{1-M}{M} + \phi(z)}{1 + \frac{1-M}{M}\phi(z)} \right| \leq M.$$

After this step we can see the desired result. □

3. Coefficient bounds

Theorem 3.1. *If the function $f(z) \in \mathcal{A}$ belongs to the class SL_q^* then*

$$\sum_{k=2}^{\infty} |a_k|^2 \left((1-q) [k]_q^2 - 2 \right) \leq 1 + q. \tag{5}$$

Proof. If SL_q^*, ??? then $z\frac{D_q f(z)}{f(z)} \prec \left(\frac{1+z}{1-qz}\right)^{\frac{1}{2}}$. Hence

$$z\frac{D_q f(z)}{f(z)} = \left(\frac{1+\phi(z)}{1-q\phi(z)}\right)^{\frac{1}{2}},$$
$$z\frac{D_q f(z)}{f(z)} = \left(\frac{1+\phi(z)}{1-q\phi(z)}\right)^{\frac{1}{2}},$$

where $\phi(z)$ satisfies $\phi(0) = 0$ and $|\phi(z)| < 1$ for $|z| < 1$.Therefore

$$\left(f^2(z) + q\,(zDf(z))^2 \right) \phi(z) = (zD_q f(z))^2 - f^2(z) \tag{6}$$

and using this, we can obtain

$$\begin{aligned}
2\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k} + 2\pi q \sum_{k=1}^{\infty} |a_k|^2 [k]_q^2 r^{2k} &= \int_0^{2\pi} \left| f(re^{i\theta}) \right| d\theta + q \int_0^{2\pi} \left| re^{i\theta} D_q f(re^{i\theta}) \right|^2 d\theta \\
&\geqslant \int_0^{2\pi} \left| f^2(re^{i\theta}) + q \left(re^{i\theta} D_q f(re^{i\theta}) \right)^2 \right| d\theta \\
&\geqslant \int_0^{2\pi} \left| f^2(re^{i\theta}) + q \left(re^{i\theta} D_q f(re^{i\theta}) \right)^2 \right| \left| \phi(re^{i\theta}) \right| d\theta \\
&= \int_0^{2\pi} \left| \left(re^{i\theta} D_q f(re^{i\theta}) \right)^2 - f^2(re^{i\theta}) \right| d\theta
\end{aligned}$$

$$\geqslant \int_0^{2\pi} \left| re^{i\theta} D_q f(re^{i\theta}) \right|^2 d\theta - \int_0^{2\pi} \left| f^2(re^{i\theta}) \right|^2 d\theta$$

$$= 2\pi \sum_{k=1}^{\infty} |a_k|^2 [k]_q^2 r^{2k} - 2\pi \sum_{k=1}^{\infty} |a_k|^2 r^{2k}$$

For $0 \leqslant r < 1$. The extremes in this sequence of inequalities gives

$$2 \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \geqslant \sum_{k=1}^{\infty} (1-q) |a_k|^2 [k]_q^2 r^{2k}, 0 \leqslant r < 1.$$

Eventually, if we let $r \to 1^-$, then we obtain (5). □

Corollary 3.2. *If the function $f(z) \in \mathcal{A}$ belongs to the class SL_q^* then*

$$|a_k| \leqslant \sqrt{\frac{1+q}{(1-q)[k]_q^2}},$$

for $k \geq 2$.

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Some properties Of Generalized Composition Operators On Weighted Hilbert Spaces Of Analytic Functions

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Abstract

In this paper we consider generalized composition operators on wieghted Hilbert spaces of analytic functions and determine their spectra. Moreover we investigate some classic properties of these operators.

1. Introduction

In recent years, considerable attention has been given to the delineation of weighted composition operators and generalized composition operator with regard to basic properties like boundedness, compactness, essential norm and some others. There are many great papers on the investigation of generalized composition operator acting on the spaces of analytic functions. For instance, one can see [12]. Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{D} \ : \ |z| < 1\}$ in the complex plane. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Given a positive integrable function $\omega \in C^2[0\ ,\ 1]$, we extend ω on $\mathbb{D}$ by setting $\omega(z) = \omega(|z|)$ for each $z \in \mathbb{D}$. Let ω be the weighted function such that $\omega(z)dm(z)$ defines a finite measure on $\mathbb{D}$, that is $\omega \in L^1(\mathbb{D}\ ,\ dm)$. For such a weighted ω, the weighted space $\mathcal{H}_\omega$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{\mathcal{H}_\omega} = |f(0)|^2 + \|f^{'}\|^2_\omega$$

where

$$\|f^{'}\|^2_\omega = \int_{\mathbb{D}} |f^{'}(z)|^2\omega(z)dm(z) < \infty$$

Let φ be an analytic function maps $\mathbb{D}$ into itself and g be an analytic function on $\mathbb{D}$. The generalized composition operator induced by the map g and φ is defined by the integral operator

$$I_{(g,\varphi)}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta$$

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We refer the reader to the monographs ([1], [2], [3], [9], [10], [11]). Composition operators on weighted Hilbert space $\mathcal{H}_\omega$ have been studied by many authors, see for example [4], [8] and the related references therein.

2. Preliminaries

In the this section we gave some useful definitions and auxiliary that are crucial for the paper's main results. For $a, z \in \mathbb{D}$ let $\sigma_a(z)$ be the Mobius transformation of $\mathbb{D}$ which interchanges 0 and a, that is $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. Obviously $\sigma_a^{'}(z) = -\frac{1-|a|^2}{(1-\bar{a}z)^2}$ for every $z \in \mathbb{D}$.

For $a \in \mathbb{D}$ set $f_a(z) = \frac{1}{\sqrt{\omega(a)}}\frac{(1-|a|^2)^{1+\delta}}{(1-\bar{a}z)^{1+\delta}}$. Then by Lemma3,[12], $\|f_a\|_{\mathcal{H}_\omega} \approx 1$.

Lemma 2.1. *t1 If ω be a non-increasing and $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$, (We say $\omega$is admissible weight) then there exists $C > 0$ such that $\frac{1}{C}\omega(z) \leq \omega(\sigma_a(z)) \leq C\omega(z)$.*

Proof. See lemma 2.1[12] □

Definition 2.2. Let $\varphi \in Hol(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The generalized counting Nevanlinna function associated to ω and g is defined for every $z \in \mathbb{D}\backslash\varphi(0)$ by

$$N_{\varphi,\omega,g}(z) = \sum_{\varphi(a)=z \ a\in\mathbb{D}} \left|\frac{g(a)}{\varphi'(a)}\right|^2 \omega(a)$$

With definition above consider the function

$$\tau_{\varphi,\omega,g}(z) \quad = \quad \frac{N_{\varphi,\omega,g}(z)}{\omega(z)}$$

Definition 2.3. [12] Let μ be a positive Borel measure. We say μ is a ω- Carleson measure if there exists cinstant $C > 0$ such that for all $f \in \mathcal{H}_\omega$,

$$\int_{\mathbb{D}} |f'(z)|^2 d\mu(z0 \leq C\|f\|^2_{\mathcal{H}_\omega}$$

3. Main Results

In this section, we determine the spectrum of the generalized composition operators on the space $\mathcal{H}_\omega$ using the ω-Carleson measures.

Theorem 3.1. *Suppose $I_{(g,\varphi)}$ is a compact operator on the weighted Hilbert space of analytic functions.*

(a) If the composition map φ has only one fixed point a in the unit disk then

$$\sigma(I_{(g,\varphi)}) \subseteq \{0\ ,\ g(a)(\varphi^{'}(a))^{m-1}\ ;\ \ m \in \mathbb{N}\}$$

(b) If there exists a positive integer n such that $\varphi^n(z) = z$ for $z \in \mathbb{D}$ then

$$\sigma(I_{(g,\varphi)}) \subseteq \{\lambda \in \mathbb{C}\ ;\ \lambda^n = \prod_{i=0}^{n-1} g\circ\varphi^i(z)\}$$

(c) If $\varphi^n \neq Id$ for every n then $\sigma(I_{(g,\varphi)}) = \{0\}$.

Proof. **(a)** For any $\lambda \in \sigma(I_{(g,\varphi)})$ since $I_{(g,\varphi)}$ is a compact operator then λ is an eigenvalue and for the eigenvector f of λ, $\lambda f(z) = I_{(g,\varphi)}f(z)$. So we have that $\lambda f^{'}(z) = f^{'}(\varphi(z))g(z)$.
If $f^{'}(a) \neq 0$ then $\lambda f^{'}(a) = f^{'}(a)g(a)$ and so $\lambda = g(a)$.
If $f^{'}(a) = g(a) = 0$ then for $f^{'}(z) = \sum_{n=1}^{\infty} A_n(z-a)^n$, $f^{'}(\varphi(z)) = \sum_{n=1}^{\infty} B_n(z-a)^n$ and $\lambda = g(a)$ $g(z) = \sum_{n=1}^{\infty} C_n(z-a)^n$ can to shows that $\lambda = 0$.
If $f^{'}(a) = 0$ and $g(a) \neq 0$ then by differentiating we have $\lambda f^{''}(z) = \varphi^{'}(z)f^{''}(\varphi(z))g(z) + f^{'}(\varphi(z))g^{'}(z)$ and so

$\lambda = g(a)\varphi^{'}(a)$.
Let $f^{'}$ have a zero of order m at a. For $m > 1$ we have that

$$\lambda f^{(m)}(z) = \sum_{j=1}^{m-1} \alpha_j f^{(j)}(\varphi(z)) + g(z)(\varphi^{'}(z))^{m-1} f^{(m)}(\varphi(z))$$

where $f^{(k)}$ stand for k^{th} derivative of f and $\alpha_j^{'}$s are functions which consists of various products of derivatives of g and $\varphi(z)$. The exact values of these are not important to us. Now let $z = a$. Since f has a zero of order m at a all the terms in the sum $\sum_{j=1}^{m-1} \alpha_j f^{(j)}(\varphi(z))$ vanish, and this yield $\lambda = g(a)(\varphi^{'}(a))^{m-1}$. The above computation shows that only possible eigenvalues are of the form $g(a)(\varphi^{'}(a))^{m-1}$.
(b) It is clear that $\lambda^n f(z) = \int_0^z f^{'}(\varphi^n(\zeta))g(\varphi^{n-1}(\zeta))\dots g(\varphi(\zeta))g(\zeta)d\zeta$ then we have
$\lambda^n f^{'}(z) = f^{'}(z)g(\varphi^{n-1}(z))\dots g(\varphi(z))g(z)$ so $\lambda^n = g(\varphi^{n-1}(z))\dots g(\varphi(z))g(z)$.
(c) we know that $I^n_{(g,\varphi)}f(z) = I_{(\prod_{i=0}^{n-1} g\circ\varphi^i, \varphi^n)}f(z) = \int_0^z f^{'}(\varphi^n(\zeta))g(\varphi^{n-1}(\zeta))\dots g(\varphi(\zeta))g(\zeta)d\zeta$. So by lemma 4[12] , change of variable and Fubinis theorem we get

$$\begin{aligned}
\|I^n_{(g,\varphi)}f\|^2_{\mathcal{H}_\omega} &= \int_{\mathbb{D}} |(f^{'}\circ\varphi^n g\circ\varphi^{n-1}\dots g\circ\varphi g)(z)|^2\omega(z)dm(z) \\
&= \int_{\mathbb{D}} |(f^{'}\circ\varphi^{n-1} g\circ\varphi^{n-2}\dots g\circ\varphi g)(\varphi(z))|^2|g(z)|^2\omega(z)dm(z) \\
&= \int_{\mathbb{D}} |(f^{'}\circ\varphi^{n-1} g\circ\varphi^{n-2}\dots g\circ\varphi g)(z)|^2 d\mu_{\omega,g,\varphi}(z) \\
&\leq \int_{\mathbb{D}} |(f^{'}\circ\varphi^{n-1} g\circ\varphi^{n-2}\dots g\circ\varphi g)(\lambda)|^2\omega(\lambda)(\int_{E(z,r)} \frac{d\mu_{\omega,g,\varphi}(z)}{\omega(z)(1-|z|^2)^2})dm(\lambda)
\end{aligned}$$

Since $\chi_{E(z,r)} = \chi_{E(\lambda,r)}$ and $1-|\lambda|^2 \approx 1-|z|^2$ for all $z \in \mathbb{D}$, we get

$$\|I_{(g,\varphi)}f\|^2_{\mathcal{H}_\omega} = \int_{\mathbb{D}} |(f^{'}\circ\varphi^{n-1} g\circ\varphi^{n-2}\dots g\circ\varphi g)(\lambda)|^2\omega(\lambda)\frac{\mu_{\omega,g,\varphi}(E(\lambda,r))}{\omega(\lambda)(1-|\lambda|^2)^2})dm(\lambda)$$

Now consider the right hand side of the last inequality. Since $I_{(g,\varphi)}$ is compact then $\mu_{\omega,g,\varphi}$ is vanishing ω- Carleson measure. Then for a given $\epsilon > 0$ there exists $r \in (0\ ,\ 1)$ such that

$$\begin{aligned}
&\int_{|z|>r} |(f^{'}\circ\varphi^{n-1} g\circ\varphi^{n-2}\dots g\circ\varphi g)(z)|^2\omega(z)\frac{\mu_{\omega,g,\varphi}(E(z,r))}{\omega(z)(1-|z|^2)^2})dm(z) \\
\leq\ & C_1\epsilon\int_{|z|>r} |(f^{'}\circ\varphi^{n-1} g\circ\varphi^{n-2}\dots g\circ\varphi g)(z)|^2\omega(z)dm(z) \\
=\ & C_1\epsilon\int_{|z|>r} |(f^{'}\circ\varphi^{n-2} g\circ\varphi^{n-3}\dots g\circ\varphi g)(\varphi(z))|^2|g(z)|^2\omega(z)dm(z) \\
=\ & C_1\epsilon\int_{|z|>r} |(f^{'}\circ\varphi^{n-2} g\circ\varphi^{n-3}\dots g\circ\varphi g)(z)|^2 d\mu_{\omega,g,\varphi}(z) \\
\leq\ & \int_{\mathbb{D}} |(f^{'}\circ\varphi^{n-2} g\circ\varphi^{n-3}\dots g\circ\varphi g)(\lambda)|^2\omega(\lambda)(\int_{E(z,r)} \frac{d\mu_{\omega,g,\varphi}(z)}{\omega(z)(1-|z|^2)^2})dm(\lambda) \\
\leq\ & C_1C_2\epsilon^2\int_{|z|>r} |(f^{'}\circ\varphi^{n-3} g\circ\varphi^{n-4}\dots g\circ\varphi g)(z)|^2 d\mu_{\omega,g,\varphi}(z)
\end{aligned}$$

By induction we have that

$$\begin{aligned}\|I^n_{(g,\varphi)}f\|^2_{\mathcal{H}_\omega} &\leq C_1C_2\dots C_n\epsilon^n\int_{\mathbb{D}}|f^{'}(z)|\omega(z)dm(z)\\ &\leq \epsilon^{2n}M\end{aligned}$$

Choose $\epsilon > 0$ so that $|g(\zeta)| < \epsilon$, and let $U \subseteq \mathbb{D}$ be an open neighborhood of ζ such that $|g(z)| < \epsilon$. Now choose $r > 0$ so that $\{\zeta \ ; \ |\frac{\zeta-r}{1-\bar{\zeta}r}|\} \subseteq U$. Therefor, for $m \geq 0$ and $z \in \{\zeta \ ; \ |\frac{\zeta-r}{1-\bar{\zeta}r}|\}$, $|g(\varphi_m(z))| < \epsilon$. Thus, since g and f continues and for constant C we get

$$\begin{aligned}\int_{|z|\leq r} & |f^{'}(\varphi^{n-1}(z))g(\varphi^{n-2}(z))\dots g(\varphi(z))g(z)|^2\omega(z)\frac{\mu_{\omega,g,\varphi}(E(z,r))}{\omega(z)(1-|z|^2)^2})dm(z)\\ \leq & \ \epsilon^{2(n-m-1)}M_1^{2(m+1)}\int_{\mathbb{D}}\mu_{\omega,g,\varphi}(E(z,r))dm(z)\\ \leq & \ \epsilon^{2(n-m-1)}M_1^{2(m+1)}C\\ \leq & \ \epsilon^{2n}N\end{aligned}$$

Hence we have

$$\|I^n_{(g,\varphi)}f\|^{\frac{2}{n}}_{\mathcal{H}_\omega} \leq \epsilon^2(M+N)^{\frac{1}{n}}$$

Since ϵ is arbitrary, we get $\lim_{n\to\infty}(\|I^n_{(g,\varphi)}f\|^2_{\mathcal{H}_\omega})^{\frac{1}{n}} = 0$. □

Theorem 3.2. *Let $I_{(g,\varphi)} \in B(\mathcal{H}_\omega)$. Then $\sigma_p(I_{(g,\varphi)}) \subseteq \{\lambda \in \mathbb{C} \ ; \ |\lambda|^2 = 2|\lambda|\|I_{(g,\varphi)}\| + \|I_{(g,\varphi)}\|^2\} \cup \{0\}$.*

Proof. Let $0 \neq \lambda \in \mathbb{C}$ be such that $\lambda \in \sigma_p(I_{(g,\varphi)})$, then there exists a function $0 \neq f \in \mathcal{H}_\omega$ such that $(\lambda I - I_{(g,\varphi)})f = 0$. Since $I_{(g,\varphi)}f(0) = 0$ then we have

$$\begin{aligned}0 &= \|\lambda f - I_{(g,\varphi)}f\|^2_{\mathcal{H}_\omega}\\ &= |\lambda f(0) - I_{(g,\varphi)}f(0)|^2 + \int_{\mathbb{D}}|\lambda f^{'}(z) - I^{'}_{(g,\varphi)}f(z)|^2\omega(z)dm(z)\\ &\geq |\lambda f(0)|^2 + \int_{\mathbb{D}}(|\lambda f^{'}(z)| - |I^{'}_{(g,\varphi)}f(z)|)^2\omega(z)dm(z)\\ &\geq |\lambda f(0)|^2 + \int_{\mathbb{D}}(|\lambda f^{'}(z)|^2 - |I^{'}_{(g,\varphi)}f(z)|^2 - 2|\lambda f^{'}(z)||I^{'}_{(g,\varphi)}f(z)|)\omega(z)dm(z)\\ &= |\lambda f(0)|^2 + \int_{\mathbb{D}}|\lambda f^{'}(z)|^2\omega(z)dm(z) - \int_{\mathbb{D}}|I^{'}_{(g,\varphi)}f(z)|^2\omega(z)dm(z)\\ &- 2\int_{\mathbb{D}}|\lambda f^{'}(z)||I^{'}_{(g,\varphi)}f(z)|\omega(z)dm(z)\\ &\geq |\lambda|^2(|f(0)|^2 + \|f^{'}\|^2_\omega) - (|I_{(g,\varphi)}f(0)|^2 + \|I^{'}_{(g,\varphi)}f(z)\|^2_\omega) - 2|\lambda|\|f^{'}\|^2_\omega(|I_{(g,\varphi)}f(0)|^2 + \|I^{'}_{(g,\varphi)}f(z)\|^2_\omega)^{\frac{1}{2}}\\ &= |\lambda|^2\|f\|^2_{\mathcal{H}_\omega} - \|I_{(g,\varphi)}f\|^2_{\mathcal{H}_\omega} - 2|\lambda|\|f^{'}\|_\omega\|I_{(g,\varphi)}\|\|f\|_{\mathcal{H}_\omega}\\ &\geq |\lambda|^2\|f\|^2_{\mathcal{H}_\omega} - \|I_{(g,\varphi)}\|^2\|f\|^2_{\mathcal{H}_\omega} - 2|\lambda|\|f\|_{\mathcal{H}_\omega}\|I_{(g,\varphi)}\|\|f\|_{\mathcal{H}_\omega}\\ &= (|\lambda|^2 - 2|\lambda|\|I_{(g,\varphi)}\| - \|I_{(g,\varphi)}\|^2)\|f\|^2_{\mathcal{H}_\omega}\end{aligned}$$

and hence $\lambda \in \{\lambda \in \mathbb{C} \ ; \ |\lambda|^2 = 2|\lambda|\|I_{(g,\varphi)}\| + \|I_{(g,\varphi)}\|^2\}$.

□

Theorem 3.3. *Let ω be an admissible weight and $I_{(g,\varphi)}$ be abounded operator on $\mathcal{H}_\omega$. Then the following are equivalent.*

1. *The operator $I_{(g,\varphi)} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ has closed range.*
2. *There exists a constant $C > 0$ such that for all $f \in \mathcal{H}_\omega$*

$$\int_{\mathbb{D}} |f^{'}(z)|^2 d\mu_{\omega,g,\varphi}(z) \geq C \int_{\mathbb{D}} |f^{'}(z)|^2 \omega(z) dm(z)$$

Proof. If $\varphi(0) = 0$ then we can consider $I_{(g,\varphi)}$ acting on $\mathcal{H}_\omega$, the closed subspaces of $\mathcal{H}_\omega$ consisting all function with $f(0) = 0$. Not that $I_{(g,\varphi)}$ has closed range if and only if be an bounded bellow. Using change of variable formula, we get

$$\begin{aligned} \|I_{(g,\varphi)} f\|^2_{\mathcal{H}_\omega} &= \int_{\mathbb{D}} |f^{'}(\varphi(z))|^2 |g(z)|^2 \omega(z) dm(z) \\ &= \int_{\mathbb{D}} |f^{'}(z)|^2 d\mu_{\omega,g,\varphi}(z) \end{aligned}$$

Thus in this case the proof is complete. If $\varphi(0) = a \neq 0$, define the function $\psi = \sigma_a \circ \varphi$. Then

$$\begin{aligned} \int_{\mathbb{D}} |f^{'}(\psi(z))|^2 |g(z)|^2 \omega(z) dm(z) &= \int_{\mathbb{D}} |f^{'} \circ \sigma_a(\varphi(z))|^2 |g(z)|^2 \omega(z) dm(z) \\ &= \int_{\mathbb{D}} |f^{'} \circ \sigma_a(z)|^2 d\mu_{\omega,g,\varphi}(z) \\ &\geq C \int_{\mathbb{D}} |f^{'}(\sigma_a(z))|^2 \omega(z) dm(z) \end{aligned}$$

By Lemma **??**and change of variable formula we get

$$\int_{\mathbb{D}} |f^{'}(\sigma_a(z))|^2 \omega(z) dm(z) \geq (\frac{1 - |a|^2}{4})^2 \int_{\mathbb{D}} |f^{'}(z)|^2 \omega(z) dm(z)$$

Since $\varphi = \sigma_a \circ \psi$ the proof of converse part is similar. ☐

Example 3.4. *Suppose that $I_{(g,\varphi)} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be abounded operator and φ is an authomorphism of $\mathbb{D}$. Then put $g(z) = \frac{\varphi^{'}(z)}{1-|\varphi(z)|^2}$. If $\varphi(0) = 0$ schwarz Lemma implies that $|\varphi^{-1}(z)| \leq |z|$. Since ω is non-increasing, $\omega(\varphi^{-1}(z)) = \omega(|\varphi^{-1}(z)|) \geq \omega(|z|) = \omega(z)$. now,*

$$\begin{aligned} \int_{\mathbb{D}} |f^{'}(z)|^2 d\mu_{\omega,g,\varphi}(z) &= \int_{\mathbb{D}} |f^{'}(\varphi(z))|^2 |g(z)|^2 \omega(z) dm(z) \\ &= \int_{\mathbb{D}} |f^{'}(\varphi(z))|^2 |\frac{\varphi^{'}(z)}{1 - |\varphi(z)|^2}|^2 \omega(z) dm(z) \\ &\geq \frac{1}{4} C \int_{\mathbb{D}} |f^{'}(z)|^2 \omega(z) dm(z) \end{aligned}$$

If $\varphi(0) \neq 0$ then the same argument be applied.

In this section we charactterize the boundedness, compactness and investigated closed range of the generalized composition operators on the space $\mathcal{H}_\omega$ using the generalized counting Nevanlinna function. Also we give a Hiklbert - Schmidtcharacterization for this operators

Theorem 3.5. *Let ω be an admissible weight and $\varphi \in \mathcal{H}_\omega$. If*

$$\sup_{|z|<1} \frac{N_{\omega,g,\varphi}(z)}{\omega(z)} \quad < \quad \infty \tag{1}$$

Then $I_{(g,\varphi)}$ is bounded operator on $\mathcal{H}_\omega$.

Proof. Suppose that (2.1) is satisfied. The bounded ness of $I_{g,\varphi}$ follows from the change of variable formula:

$$\begin{aligned} ||I_{g,\varphi}f||^2_{\mathcal{H}_\omega} &= |I_{g,\varphi}f(0)| \\ &+ \int_{\mathbb{D}} |f'(\varphi(z))|^2 |g(z)|^2 \omega(z) dm(z) \\ &= \int_{\varphi(\mathbb{D})} |f'(z)|^2 N_{\varphi,\omega,g}(z) dm(z) \\ &\leq c \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dm(z) \\ &\leq c||f||^2_{\mathcal{H}_\omega} \end{aligned}$$

□

Theorem 3.6. *Let ω be an admissible weight and $\varphi \in \mathcal{H}_\omega$. If*

$$\lim_{|z|\to 1^-} \frac{N_{\omega,g,\varphi}(z)}{\omega(z)} \quad = \quad 0 \tag{2}$$

Then $I_{g,\varphi}$ is compact operator on $\mathcal{H}_\omega$.

Proof. Assume that (2.2) is satisfied. Let $(f_n)_n$ be a sequence in the unit ball of $\mathcal{H}_\omega$ converging to 0 weakly. It suffices to show that $||I_{g,\varphi}f_n||_{\mathcal{H}_\omega} \to 0$ as $n \to \infty$. The weak convergence of f_n implies that $f_n(z) \to 0$ and $f'_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $\epsilon > 0$ there exists $\rho_\epsilon \in (\frac{1}{2}, 1)$ such that

$$N_{\varphi,\omega,g} \leq \epsilon\omega(z) \qquad for \quad \rho_\epsilon < |z| < 1$$

By the change of variable formula

$$\begin{aligned} ||I_{g,\varphi}f_n||^2_{\mathcal{H}_\omega} &= |I_{g,\varphi}f_n(0)| \\ &+ \int_{\mathbb{D}} |f'_n(\varphi(z))|^2 |g(z)|^2 \omega(z) dm(z) \\ &= \int_{\varphi(\mathbb{D})} |f'_n(z)|^2 N_{\varphi,\omega,g}(z) dm(z) \\ &\leq \int_{\rho_\epsilon(\mathbb{D})} |f'_n(z)|^2 N_{\varphi,\omega,g}(z) dm(z) \\ &+ \epsilon \int_{\varphi(\mathbb{D})\setminus\rho_\epsilon(\mathbb{D})} |f'_n(z)|^2 \omega(z) dm(z) \\ &\leq \epsilon \end{aligned}$$

The conclusion easy follows since f'_n uniformly converges to 0 on the closed disk $\rho_\epsilon(\overline{\mathbb{D}})$. □

Theorem 3.7. *Let ω be an admissible weight and $I_{g,\varphi}$ be abounded operator on $\mathcal{H}_\omega$. Then the operator $I_{g,\varphi}$ has closed range if and only if there exists a constant $C > 0$ such that for all $f \in \mathcal{H}_\omega$*

$$\int_{\mathbb{D}} |f^{'}(z)|^2 \tau_{\varphi,\omega,g}(z)\omega(z)dm(z) \geq$$
$$C \int_{\mathbb{D}} |f^{'}(z)|^2 \omega(z)dm(z)$$

Proof. If $\varphi(0) = 0$ then we can consider $I_{g,\varphi}$ acting on $\mathcal{H}_\omega$, the closed subspaces of $\mathcal{H}_\omega$ consisting all function with $f(0) = 0$. Not that $I_{g,\varphi}$ has closed range if and only if be an bounded bellow. Using change of variable formula, we get

$$\begin{aligned} \|I_{g,\varphi} f\|^2_{\mathcal{H}_\omega} &= \int_{\mathbb{D}} |f^{'}(\varphi(z))|^2 |g(z)|^2 \omega(z)dm(z) \\ &= \int_{\mathbb{D}} |f^{'}(z)|^2 N_{\varphi,\omega,g}(z)dm(z) \\ &= \int_{\mathbb{D}} |f^{'}(z)|^2 \tau_{\varphi,\omega,g}(z)\omega(z)dm(z) \end{aligned}$$

Thus in this case the proof is complete. If $\varphi(0) = a \neq 0$, define the function $\psi = \sigma_a \circ \varphi$. Then

$$\begin{aligned} \int_{\mathbb{D}} |f^{'}(\psi(z))|^2 |g(z)|^2 \omega(z)dm(z) &= \int_{\mathbb{D}} |f^{'} \circ \sigma_a(\varphi(z))|^2 |g(z)|^2 \omega(z)dm(z) \\ &= \int_{\mathbb{D}} |f^{'} \circ \sigma_a(z)|^2 \tau_{\varphi,\omega,g}(z)\omega(z)dm(z) \\ &\geq C \int_{\mathbb{D}} |f^{'}(\sigma_a(z))|^2 \omega(z)dm(z) \end{aligned}$$

By change of variable formula we get

$$\int_{\mathbb{D}} |f^{'}(\sigma_a(z))|^2 \omega(z)dm(z) \geq (\frac{1-|a|^2}{4})^2 \int_{\mathbb{D}} |f^{'}(z)|^2 \omega(z)dm(z)$$

Since $\varphi = \sigma_a \circ \psi$ the proof of converse part is similar. $\square$

Example 3.8. *Suppose that $I_{g,\varphi} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be abounded operator and φ is an authomorphism of $\mathbb{D}$. Then put $g(z) = \varphi^{'}(z)$. If $\varphi(0) = 0$ schwarz Lemma implies that $|\varphi^{-1}(z)| \leq |z|$. Since ω is non-increasing, $\omega(\varphi^{-1}(z)) = \omega(|\varphi^{-1}(z)|) \geq \omega(|z|) = \omega(z)$. now,*

$$\begin{aligned} \int_{\mathbb{D}} |f^{'}(z)|^2 \tau_{\varphi,\omega,g}(z)\omega(z)dm(z) &= \int_{\mathbb{D}} |f^{'}(\varphi(z))|^2 |g(z)|^2 \omega(z)dm(z) \\ &= \int_{\mathbb{D}} |f^{'}(\varphi(z))|^2 |\varphi^{'}(z)|^2 \omega(z)dm(z) \\ &\geq \int_{\mathbb{D}} |f^{'}(z)|^2 \omega(z)dm(z) \end{aligned}$$

If $\varphi(0) \neq 0$ then the same argument be applied.

Theorem 3.9. *Let ω be a wight. Then $I_{g,\varphi} : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is Hilbert - Schmidt if and only if*

$$\int_{\mathbb{D}} \|R_z\| N_{\varphi,\omega,g}(z)\omega(z)dm(z) < \infty$$

, where R_z is the reproducing kernel of the wieghted Bergman space $\mathcal{A}^2_\omega$.

Proof. Note that $\{\frac{z^n}{\|z^n\|_{\mathcal{H}_\omega}}\}$ is an orthonormal basis for $\mathcal{H}_\omega$. This implies that

$$\begin{aligned}
\sum_{n=1}^{\infty} \|I_{g,\varphi}(\frac{z^n}{\|z^n\|_{\mathcal{H}_\omega}})\| &= \sum_{n=1}^{\infty} \int_{\mathbb{D}} \frac{n^2|\varphi(z)|^{2(n-1)}}{\|z^n\|^2_\omega} |g(z)|^2 \omega(z)dm(z) \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{D}} \frac{n^2|z|^{2(n-1)}}{\|z^n\|^2_\omega} N_{\varphi,\omega,g}(z)\omega(z)dm(z) \\
&= \sum_{n=1}^{\infty} \int_{\mathbb{D}} \frac{|z|^{2n}}{p_n} N_{\varphi,\omega,g}(z)\omega(z)dm(z) \\
&= \int_{\mathbb{D}} \|R_z\| N_{\varphi,\omega,g}(z)\omega(z)dm(z)
\end{aligned}$$

This completes the proof. □

Theorem 3.10. *Let ω_1 and ω_2 are the admissible weights and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. If*

$$\sup_{|z|<1} \frac{N_{\varphi,\omega_2,g}(z)}{\omega_1(z)} < \infty \tag{3}$$

Then $I_{g,\varphi} : \mathcal{H}_{\omega 1} \to \mathcal{H}_{\omega 2}$ is bounded operator.

Proof. Suppose that (2.3) is satisfied. The bounded ness of $I_{g,\varphi}$ follows from the change of variable formula:

$$\begin{aligned}
||I_{g,\varphi} f||^2_{\mathcal{H}_{\omega_2}} &= |I_{g,\varphi} f(0)| \\
&+ int_{\mathbb{D}} |f'(\varphi(z))|^2 |g(z)|^2 \omega_2(z)dm(z) \\
&= \int_{\varphi(\mathbb{D})} |f'(z)|^2 N_{\varphi,\omega_2,g}(z)dm(z) \\
&\leq c \int_{\mathbb{D}} |f'(z)|^2 \omega_1(z)dm(z) \\
&\leq c||f||^2_{\mathcal{H}_{\omega_1}}
\end{aligned}$$

□

Theorem 3.11. *Let ω_1 and ω_2 are the admissible weights and $\varphi \in \mathcal{H}_\omega$. If*

$$\lim_{|z|\to 1^-} \frac{N_{\varphi,\omega_2,g}(z)}{\omega_1(z)} = 0 \tag{4}$$

Then $I_{g,\varphi} : \mathcal{H}_{\omega 1} \to \mathcal{H}_{\omega 2}$ is compact operator.

Proof. Assume that (2.4) is satisfied. Let $(f_n)_n$ be a sequence in the unit ball of $\mathcal{H}_{\omega_1}$ converging to 0 weakly. It suffices to show that $||I_{g,\varphi}f_n||_{\mathcal{H}_{\omega_2}} \to 0$ as $n \to \infty$. The weak convergence of f_n implies that $f_n(z) \to 0$ and $f_n'(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$. Let $\epsilon > 0$ there exists $\rho_\epsilon \in (\frac{1}{2}, 1)$ such that

$$N_{\varphi,\omega_2,g} \leq \epsilon\omega_1(z) \qquad for \quad \rho_\epsilon < |z| < 1$$

By the change of variable formula

$$\begin{aligned}
||I_{g,\varphi}f_n||^2_{\mathcal{H}_{\omega_2}} &= |I_{g,\varphi}f_n(0)| \\
&+ \int_{\mathbb{D}} |f_n'(\varphi(z))|^2|g(z)|^2\omega_2(z)dm(z) \\
&= \int_{\varphi(\mathbb{D})} |f_n'(z)|^2 N_{\varphi,\omega_2,g}(z)dm(z) \\
&\leq \int_{\rho_\epsilon(\mathbb{D})} |f_n'(z)|^2 N_{\varphi,\omega_2,g}(z)dm(z) \\
&+ \epsilon\int_{\varphi(\mathbb{D})\setminus\rho_\epsilon(\mathbb{D})} |f_n'(z)|^2\omega_1(z)dm(z) \\
&\leq \epsilon
\end{aligned}$$

The conclusion easy follows sine f_n' uniformly converges to 0 on the closed disk $\rho_\epsilon(\overline{\mathbb{D}})$. □

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G-Fusion Frames in Hilbert Spaces

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Abstract

In this paper, we investigate the notion of g-fusion frames in Hilbert spaces.We study perturbations of g-fusion frames.

1. Introduction

Frames for Hilbert space were first introduced by Duffin and Schaeffer [8] in 1952. Daubechies, Grossmann and Meyer [7] reintroduced frames, in 1986 [7] and considered from then. Frame theory has applications in signal processing, image processing, data compression and sampling theory.
Orthonormal bases are special case of frames in Hilbert space. Any element in Hilbert can be present as an infinite linear combination, not necessary unique, of the frame element. For more information, readers can refer to [3] and [9]. Some new type and generalization of frame were introduced by researcher such as fusion frames, g-frames, woven frames, etc. Frame of subspaces or fusion frames are a generalization of frames which were introduced by Cassaza and Kutyniok [4] in 2003 and were investigated in [1, 2, 5, 6, 10–12]. Generalized frames or in abbreviation g-frames were introduced by Sun [14] in 2006. Most recently, g-fusion frames in Hilbert space were introduced by Sadri et.al. [13].
In this paper, motivated and inspired by the above-mentioned works we introduce the concept of g-fusion frame. This frame includes g-frames and fusion frames. We study perturbations of g-fusion frames.

2. Basic definitions and Preliminaries

As a preliminary of frames, at the first, we mention fusion frames. Also we review g-frames, g-fusion frames and woven frames . Through of this paper, $\mathcal{I}$ is the indexing set where it can be finite or infinity countable set. Also, $\mathcal{H}$ and $\mathcal{H}_i$ are separable Hilbert spaces and $B(\mathcal{H}, \mathcal{H}_i)$ is the collection of all the bounded linear operators of $\mathcal{H}$ into $\mathcal{H}_i$. If $\mathcal{H} = \mathcal{H}_i$, then $B(\mathcal{H}, \mathcal{H})$ will be denoted by $B(\mathcal{H})$ and P is the orthogonal projection.

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Definition 2.1. Let $\{v_i\}_{i\in\mathcal{I}}$ be a family of real weights such that $v_i > 0$ for all $i \in \mathcal{I}$. A family of closed subspaces $\{W_i\}_{i\in\mathcal{I}}$ of a Hilbert space $\mathcal{H}$ is called a fusion frame (or frame of subspaces) for $\mathcal{H}$ with respect to weights $\{v_i\}_{i\in\mathcal{I}}$, if there exist constants $C, D > 0$ such that

$$C||f||^2 \leq \sum_{i\in\mathcal{I}} v_i^2 ||P_{W_i}(f)||^2 \leq D||f||^2, \qquad \forall f \in \mathcal{H}, \tag{1}$$

where P_{W_i} is the orthogonal projection of $\mathcal{H}$ to W_i. The constants C and D are called the lower and upper fusion frame bounds, respectively. If the right inequality in (1) holds, the family of subspace $\{W_i\}_{i\in\mathcal{I}}$ is called a Bessel sequence of subspaces with respect to $\{v_i\}_{i\in\mathcal{I}}$ with Bessel bound D. Also is called tight fusion frame with respect to $\{v_i\}_{i\in\mathcal{I}}$, if $C = D$ and is called Parseval fusion frame, if $C = D = 1$. We say $\{W_i\}_{i\in\mathcal{I}}$ an orthogonal fusion basis for $\mathcal{H}$, if $\mathcal{H} = \bigoplus_{i\in\mathcal{I}} W_i$.

Definition 2.2. The fusion frame $\{W_i\}_{i\in\mathcal{I}}$ with respect to some family of weights is called a Riesz decomposition of $\mathcal{H}$, if for every $f \in \mathcal{H}$, there is a unique choice of $f_i \in W_i$ so that $f = \sum_{i\in\mathcal{I}} f_i$.

For each family of subspaces $\{W_i\}_{i\in\mathcal{I}}$ of $\mathcal{H}$, the representation space:

$$\left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2} = \left\{ \{f_i\}_{i\in\mathcal{I}} | f_i \in W_i \ \ and \ \ \sum_{i\in\mathcal{I}} ||f_i||^2 < \infty \right\},$$

with inner product

$$\langle \{f_i\}_{i\in\mathcal{I}}, \{g_i\}_{i\in\mathcal{I}} \rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i \rangle,$$

is a Hilbert space. This space is needed in the studying of fusion systems.

Definition 2.3. Let $\{W_i\}_{i\in\mathcal{I}}$ be a fusion frame family for $\mathcal{H}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$. Then the analysis operator for $\{W_i\}_{i\in\mathcal{I}}$ with weights $\{v_i\}_{i\in\mathcal{I}}$ is defined by:

$$U_{W,v} : \mathcal{H} \to \left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2}, \qquad U_{W,v}(f) = \{v_i P_{W_i}(f)\}_{i\in\mathcal{I}}.$$

The adjoint of $U_{W,v}$ is called the synthesis operator, we denote $T_{W,v} = U^*_{W,v}$.
By elementary calculation, we have

$$T_{W,v} : \left(\sum_{i\in\mathcal{I}} \oplus W_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_{W,v}(\{f_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} v_i P_{W_i} f_i.$$

Like discrete frames, the fusion frame operator for $\{W_i\}_{i\in\mathcal{I}}$ with respect to $\{v_i\}_{i\in\mathcal{I}}$ is the composition of analysis and synthesis operators,

$$S_{W,v} : \mathcal{H} \to \mathcal{H}, \qquad S_{W,v}(f) = T_{W,v} U_{W,v}(f) = \sum_{i\in I} v_i^2 P_{W_i}(f), \quad \forall f \in \mathcal{H}.$$

Definition 2.4. Let $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$ be a family of Hilbert spaces. We call $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i), \ i \in \mathcal{I}\}$ a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in\mathcal{I}}$, or simply, a g-frame for H, if there exist two positive constants C, D such that

$$C||f||^2 \leq \sum_{i\in\mathcal{I}} ||\Lambda_i f||^2 \leq D||f||^2, \qquad \forall f \in \mathcal{H}. \tag{2}$$

The positive numbers C and D are called the lower and upper g-frame bounds, respectively. We call Λ a tight g-frame, if $C = D$ and we call it a Parseval g-frame, if $C = D = 1$. If only the second inequality holds, we call it g-Bessel sequence. If Λ is a g-frame, then the g-frame operator S_Λ is defined by

$$S_\Lambda f = \sum_{i\in\mathcal{I}} \Lambda_i^* \Lambda_i f, \qquad f \in \mathcal{H},$$

which is a bounded, positive and invertible operator such that

$$CI \leq S_\Lambda \leq DI,$$

and for each $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} f = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in\mathcal{I}} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = \sum_{i\in\mathcal{I}} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f.$$

The canonical dual g-frame for Λ is defined by $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathcal{I}}$ with bounds $\frac{1}{D}, \frac{1}{C}$. In other words, $\{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathcal{I}}$ and $\{\Lambda_i\}_{i\in\mathcal{I}}$ are dual g-frames with respect to each other.

It is easy to show that by letting $\mathcal{H}_i = W_i$, $\Lambda_i = P_{W_i}$ and $v_i = 1$, a fusion frame is a g-frame.
Let

$$\left(\sum_{i\in\mathcal{I}} \oplus\mathcal{H}_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in\mathcal{I}} | f_i \in \mathcal{H}_i \ \ and \ \sum_{i\in\mathcal{I}} \|f_i\|^2 < \infty\right\},$$

with the inner product defined by

$$\langle \{f_i\}_{i\in\mathcal{I}}, \{g_i\}_{i\in\mathcal{I}}\rangle = \sum_{i\in\mathcal{I}} \langle f_i, g_i\rangle,$$

is a Hilbert space.

Definition 2.5. Let $W = \{W_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, $\{v_i\}_{i\in\mathcal{I}}$ be a family of weights, i.e. $v_i > 0$ and $\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. We say $\Lambda := (\Lambda_i, W_i, v_i)$ is a generalized fusion frame (or g-fusion frame) for $\mathcal{H}$, if there exists $0 < A \leq B < \infty$ such that for each $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i\in\mathcal{I}} v_i^2 \|\Lambda_i P_{W_i} f\|^2 \leq B\|f\|^2. \tag{3}$$

We call Λ a Parseval g-fusion frame, if $A = B = 1$. When the right hand of (3) holds, Λ is called a g-fusion Bessel sequence for $\mathcal{H}$ with bound B. If $\mathcal{H}_i = \mathcal{H}$ for all $i \in \mathcal{I}$ and $\Lambda_i = I_\mathcal{H}$, then we get the fusion frame (W_i, v_i) for $\mathcal{H}$. Throughout this paper, Λ will be a triple (Λ_i, W_i, v_i) with $i \in \mathcal{I}$ unless otherwise stated.

Definition 2.6. Let Λ be a g-fusion frame for $\mathcal{H}$. Then, the analysis operator for Λ is defined by

$$U_\Lambda : \mathcal{H} \to \left(\sum_{i\in\mathcal{I}} \oplus\mathcal{H}_i\right)_{\ell^2}, \qquad U_\Lambda(f) = \{v_i \Lambda_i P_{W_i}(f)\}_{i\in\mathcal{I}}.$$

The adjoint of U_Λ is called the synthesis operator, we denote $T_\Lambda = U_\Lambda^*$.
By the elementary calculation, we have

$$T_\Lambda : \left(\sum_{i\in\mathcal{I}} \oplus\mathcal{H}_i\right)_{\ell^2} \to \mathcal{H}, \qquad T_\Lambda(\{f_i\}_{i\in\mathcal{I}}) = \sum_{i\in\mathcal{I}} v_i P_{W_i}\Lambda_i^* f_i.$$

The g-fusion frame operator Λ is the composition of analysis and synthesis operators,

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = T_\Lambda U_\Lambda(f) = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} f.$$

We have

$$\langle S_\Lambda f, f\rangle = \sum_{i\in\mathcal{I}} v_i^2 \|\Lambda_i P_{W_i} f\|^2.$$

Therefore

$$AI \leq S_\Lambda \leq BI.$$

This means that S_Λ is bounded, positive and invertible operator (with adjoint inverse). So, we have the reconstruction formula for any $f \in \mathcal{H}$

$$f = \sum_{i\in\mathcal{I}} v_i^2 P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} S_\Lambda^{-1} f = \sum_{i\in\mathcal{I}} v_i^2 S_\Lambda^{-1} P_{W_i}\Lambda_i^*\Lambda_i P_{W_i} f.$$

3. Main results

In this section we obtain important results about g-fusion frames.

Definition 3.1. Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame. Let $V = \{V_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, and $\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$, also let $0 \le \lambda_1, \lambda_2 < 1$. Let $\{c_i\}_{i\in\mathcal{I}}$ be an arbitrary sequence of positive numbers such that $\sum_{i\in\mathcal{I}}(v_ic_i)^2 < \infty$. We say that the family $\Gamma = (\Gamma_i, V_i, v_i)$ is a $(\lambda_1, \lambda_2, \{c_i\}_{i\in\mathcal{I}})-$perturbation of $\Lambda = (\Lambda_i, W_i, v_i)$ if we have

$$\|\Lambda_i P_{W_i}(f) - \Gamma_i P_{V_i}(f)\| \le \lambda_1\|\Lambda_i P_{W_i}(f)\| + \lambda_2\|\Gamma_i P_{V_i}(f)\| + c_i\|f\|, \qquad (f \in \mathcal{H}).$$

Now we obtain the following theorem under this definition.

Theorem 3.2. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame with g-fusion frame bounds A, B and $V = \{V_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, and $\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. let $\Gamma = (\Gamma_i, V_i, v_i)$ be a $(\lambda_1, \lambda_2, \{c_i\}_{i\in\mathcal{I}})-$perturbation of $\Lambda = (\Lambda_i, W_i, v_i)$. Suppose that*
$(1-\lambda_1)\sqrt{A} > (\sum_{i\in\mathcal{I}}(v_ic_i)^2)^{\frac{1}{2}}$. Then (Γ_i, V_i, v_i) is a g-fusion frame with g-fusion frame bounds

$$(\frac{(1-\lambda_1)\sqrt{A} - (\sum_{i\in\mathcal{I}}(v_ic_i)^2)^{\frac{1}{2}}}{1+\lambda_2})^2 \qquad and \qquad (\frac{(1+\lambda_1)\sqrt{B} + (\sum_{i\in\mathcal{I}}(v_ic_i)^2)^{\frac{1}{2}}}{1-\lambda_2})^2$$

.

Theorem 3.3. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame with g-fusion frame bounds A, B and $V = \{V_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, and $\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. If there exists an $0 < R < A$ such that*

$$\sum_{i\in\mathcal{I}} v_i^2\|A_i\Lambda_i P_{W_i}(f) - B_i\Gamma_i P_{V_i}(f)\|^2 \le R\|f\|^2$$

for all $f \in \mathcal{H}$.Suppose that

$$0 < A' = \inf_{i\in\mathcal{I}} A_i \le \sup_{i\in\mathcal{I}} A_i = A'' < \infty,$$

and

$$0 < B' = \inf_{i\in\mathcal{I}} B_i \le \sup_{i\in\mathcal{I}} B_i = B'' < \infty,$$

then (Γ_i, V_i, v_i) is a g-fusion frame with g-fusion frame bounds
$A(\frac{A'-\sqrt{\frac{R}{A}}}{B''})^2$ and $B(\frac{A''+\sqrt{\frac{R}{B}}}{B'})^2$.

Proposition 3.4. *Let (Λ_i, W_i, v_i) and (Γ_i, V_i, u_i) be g-fusion Bessel sequences with g-fusion Bessel bounds A and C, respectively, and $T_\Gamma^* T_\Lambda = id_{\mathcal{H}}$. Then (Λ_i, W_i, v_i) and (Γ_i, V_i, u_i) are g-fusion frame.*

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}\|f\|^4 &= (\langle T_\Lambda(f), T_\Gamma(f)\rangle)^2 \le \|T_\Lambda\|^2\|T_\Gamma\|^2 \\ &= (\sum_{i\in\mathcal{I}} v_i^2\|\Lambda_i P_{W_i}(f)\|^2)(\sum_{i\in\mathcal{I}} u_i^2\|\Gamma_i P_{V_i}(f)\|^2) \le C\|f\|^2(\sum_{i\in\mathcal{I}} v_i^2\|\Lambda_i P_{W_i}(f)\|^2).\end{aligned}$$

Therefore $\frac{1}{C}\|f\|^2 \le \sum_{i\in\mathcal{I}} v_i^2\|\Lambda_i P_{W_i}(f)\|^2$. Similarly we obtain a lower bound for (Γ_i, V_i, u_i). □

Theorem 3.5. *Let $\Lambda = (\Lambda_i, W_i, v_i)$ be a g-fusion frame with g-fusion frame bounds A, B and $V = \{V_i\}_{i\in\mathcal{I}}$ be a family of closed subspaces of $\mathcal{H}$, and $\Gamma_i \in B(\mathcal{H}, \mathcal{H}_i)$ for all $i \in \mathcal{I}$. Then the following are equivalent:*

(i) *(Γ_i, V_i, v_i) is a g-fusion frame.*

(ii) *There exists a constant $M > 0$, such that for all $f \in \mathcal{H}$, we have*

$$\begin{aligned}&\|\sum_{i\in\mathcal{I}} v_i\langle(\Lambda_i P_{W_i} - \Gamma_i P_{V_i})(f), (\Lambda_i P_{W_i} - \Gamma_i P_{V_i})(f)\rangle\| \\ &\le M \min\big(\|\sum_{i\in\mathcal{I}} v_i^2\langle\Lambda_i P_{W_i}(f), \Lambda_i P_{W_i}(f)\rangle\|, \|\sum_{i\in\mathcal{I}} v_i^2\langle\Gamma_i P_{V_i}(f), \Gamma_i P_{V_i}(f)\rangle\|\big). \qquad (4)\end{aligned}$$

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On a $p(x)$-biharmonic equation via variational method

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Abstract

This paper is concerned with the existence of two non-trivial weak solutions for a $p(x)$-biharmonic problem of the following form

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) + |u|^{p(x)-2}u = \lambda(x)|u|^{q(x)-2}u + \mu(x)|u|^{\gamma(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekeland's variational principle.

1. Introduction

In this paper, we study the following problem

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) + |u|^{p(x)-2}u = \lambda(x)|u|^{q(x)-2}u + \mu(x)|u|^{\gamma(x)-2}u & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$, $p(x)$, $q(x)$,$\gamma(x) \in C(\overline{\Omega})$, $\inf_{\overline{\Omega}} p(x) > 1$, $\inf_{\overline{\Omega}} q(x) > 1$ and $\inf_{\overline{\Omega}} \gamma(x) > 1$.

In recent years, the study of differential equations and variational problems with $p(x)$-growth conditions has been an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. In that context we refer the reader to Ruzicka [15], Zhikov [19] and the reference therein and also see [5, 6, 9, 10].

Fourth order equations appear in many contexts. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, and phase field models of multiphase systems (see [11]) and the references therein. In addition, this type of equation can describe the static from change of the beam, sport of rigid body. The study of fourth order equations with variable exponents is a new and interesting topic. We refer the readers to some recent works [2, 3, 12]. In [2], A. Ayoujil et al. firstly studied the spectrum of a fourth order elliptic equation with variable exponent.

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In [3], El Amrouss et al. studied a class of $p(x)$-biharmonic of the form

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = \lambda |u|^{p(x)-2}u + f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded domains in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $N \geq 1$, $\lambda \leq 0$ and some assumptions on the Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$. They obtained the existence and multiplicity of solutions.
L. Li et al. [12] have considered the above problem and using variational methods, under suitable assumptions on the Carathéodory function f, they established the existence of at least one solution and infinitely many solutions of the problem.
Inspired by the above references and the work of M. Mihăilescu et al. [14], the aim of the present paper is to study the existence of two non-trivial weak solutions for problem (1).

2. Notations and Preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces.
Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{h(x);\ h(x) \in C(\overline{\Omega}),\ h(x) > 1,\ \forall x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$, we define $h^+ = \max\{h(x) :\ x \in \overline{\Omega}\}$, $h^- = \min\{h(x) :\ x \in \overline{\Omega}\}$. For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \Big\{u :\ u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\},$$

endowed with the so-called *Luxemburg norm*

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space.

Proposition 2.1 (See [10]). *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leq k\right\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_N^{\alpha_N}}u$, with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)},$$

also becomes a separable and reflexive Banach space. For more details, we refer the reader to [7, 10, 13, 17]. Denote

$$p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if} \quad kp(x) < N, \\ +\infty & \text{if} \quad kp(x) \geq N \end{cases}$$

for any $x \in \overline{\Omega}$, $k \geq 1$.

Proposition 2.2 (See [10])**.** *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

If we replace $\leq$ with $<$, the embedding is compact.

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Note that the weak solutions of problem (1) are considered in the generalized Sobolev space

$$X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$$

equipped with the norm

$$\|u\| = \inf\left\{\mu > 0 : \int_\Omega \left(\left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + \lambda\left|\frac{u(x)}{\mu}\right|^{p(x)}\right) dx \leq 1\right\}.$$

Remark 2.3. *According to [18], the norm $\|\cdot\|_{2,p(x)}$ is equivalent to the norm $|\Delta.|_{p(x)}$ in the space X. Consequently, the norms $\|\cdot\|_{2,p(x)}$, $\|\cdot\|$ and $|\Delta\cdot|_{p(x)}$ are equivalent.*

Proposition 2.4 (See El Amrouss et al. [3])**.** *If we denote $\rho(u) = \int_\Omega (|\Delta u|^{p(x)} + |u|^{p(x)})\, dx$, then for $u, u_n \in X$, we have*

$$(a)\|u\| < 1(\text{respectively} = 1; > 1) \iff \rho(\text{u}) < 1(\text{respectively} = 1; > 1);$$
$$(b)\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-};$$
$$(c)\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq \rho(u) \leq \|u\|^{p^+};$$
$$(d)\|u_n\| \to 0\ (respectively \to \infty) \iff \rho(u_n) \to 0\ (respectively \to \infty).$$

The Euler-Lagrange functional associated to (1) is given by

$$I(u) = \int_\Omega \left(\frac{1}{p(x)}|\Delta u|^{p(x)} + |u|^{p(x)}\right) dx - \int_\Omega \frac{\lambda(x)}{q(x)}|u|^{q(x)}\, dx - \int_\Omega \frac{\mu(x)}{\gamma(x)}|u|^{\gamma(x)}\, dx.$$

It is easy to verify that $I \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous with the derivative given by

$$\langle I'(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v + |u|^{p(x)-2}uv)\, dx - \int_\Omega \lambda(x)|u|^{q(x)-1}uv\, dx - \int_\Omega \mu(x)|u|^{\gamma(x)-1}uv\, dx,$$

for all $u,\ v \in X$. Thus, we notice that we can seek weak solutions of (1) as critical point of the energetic functional I. Let us define the functional

$$J(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |u|^{p(x)})\, dx.$$

It is well known that J is well defined, even and C^1 in X. Moreover, the operator $L = J' : X \to X^*$ defined as

$$\langle L(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v + |u|^{p(x)-2}uv)\, dx$$

for all $u, v \in X$ satisfies the following assertions.

Proposition 2.5 (See El Amrouss et al. [3]).

(a') *L is continuous, bounded and strictly monotone.*
(b')*L is a mapping of (S_+) type, namely*

$$u_n \rightharpoonup u \ \textit{ and } \ \limsup_{n\to+\infty} L(u_n)(u_n-u)\le 0, \ \textit{ implies } \ u_n \to u.$$

(c') *L is a homeomorphism.*

3. Main results

In this section, we discuss the existence of two non-trivial weak solutions of (1) by using the mountain pass theorem of Ambrosetti and Rabinowitz and Ekland's variational principle. For simplicity, we use $C,\ C',\ c_i,\ \alpha_i,\ i=1,2,\ldots$ to denote the general positive constant (the exact value may change from line to line). Hereafter, $\lambda(x),\ \mu(x),\ q(x)$ and $\gamma(x)$ are always supposed to verify

$(\square\mathbf{M})_1$ $\lambda,\mu\in L^\infty(\Omega)$,

$(\square\mathbf{M})_2$ there exists an $x_0\in\Omega$ and two positive constants r and R with $0<r<R$ such that $\overline{B_R(x_0)}\subset\Omega$ and $\lambda(x)=0=\mu(x)$ for $x\in\overline{B_R(x_0)\backslash B_r(x_0)}$ while $\lambda(x)>0$ and $\mu(x)>0$ for $x\in\Omega\backslash\overline{B_R(x_0)\setminus B_r(x_0)}$,

$(\mathbf{Q}\square)_1$ $q,\gamma\in C_+(\overline{\Omega})$ and $1\le q(x),\gamma(x)<p_2^*(x)$ for any $x\in\overline{\Omega}$,

$(\mathbf{Q}\square)_2$ either $\max_{\overline{B_r(x_0)}}\{q,\gamma\}<p^-\le p^+<\min_{\overline{\Omega\backslash B_R(x_0)}}\{q,\gamma\}$, or $\max_{\overline{\Omega\backslash B_R(x_0)}}\{q,,\gamma\}<p^-\le p^+<\min_{\overline{B_r(x_0)}}\{q,\gamma\}$.

Theorem 3.1. *Assume that the conditions $(\square\mathbf{M})_1-(\square\mathbf{M})_2$ and $(\mathbf{Q}\square)_1-(\mathbf{Q}\square)_2$ are fulfilled. Then there exists $\eta^*>0$ such that problem (1) has at least two positive non-trivial weak solutions, provided that* $\max\{|\lambda|_{L^\infty(\Omega)},|\mu|_{L^\infty(\Omega)}\}<\eta^*$.

We confine ourselves to the case where the former condition of $(\mathbf{Q}\square)_2$ holds true. A similar proof can be made if the later condition holds true.

Lemma 3.2. *Let $q(x),\ \gamma(x),\ \lambda(x)$, and $\mu(x)$ be as in Theorem 3.1, then there exist $\rho>0$ and $\delta>0$ such that $I(u)\ge\delta>0$ for any $u\in X$ with $\|u\|=\rho$.*

Proof. Let us define $q_1,\gamma_1:\overline{B_r(x_0)}\to[1,\infty)$, $q_1(x)=q(x)$ and $\gamma_1(x)=\gamma(x)$ for any $x\in\overline{B_r(x_0)}$ and $q_2,\gamma_2:\overline{\Omega\backslash B_R(x_0)}\to[1,\infty)$, $q_2(x)=q(x)$ and $\gamma_2(x)=\gamma(x)$ for any $x\in\overline{\Omega\backslash B_R(x_0)}$. We also introduce the notations

$$\begin{aligned}
&q_1^-=\min_{x\in\overline{B_r(x_0)}}q_1(x), \qquad q_1^+=\max_{x\in\overline{B_r(x_0)}}q_1(x), \gamma_1^-=\min_{x\in\overline{B_r(x_0)}}\gamma_1(x)\\
&\gamma_1^+=\max_{x\in\overline{B_r(x_0)}}\gamma_1(x), \qquad q_2^-=\min_{x\in\overline{\Omega\backslash B_R(x_0)}}q_2(x), \qquad q_2^+=\max_{x\in\overline{\Omega\backslash B_R(x_0)}}q_2(x)\\
&\gamma_2^-=\min_{x\in\overline{\Omega\backslash B_R(x_0)}}\gamma_2(x), \qquad \gamma_2^+=\max_{x\in\overline{\Omega\backslash B_R(x_0)}}\gamma_2(x).
\end{aligned}$$

Then by relations $(\mathbf{Q}\square)_1$ and $(\mathbf{Q}\square)_2$ we have these following four cases

(i) $1\le q_1^-\le q_1^+<\gamma_1^-\le\gamma_1^+<p^-\le p^+<q_2^-\le q_2^+<\gamma_2^-\le\gamma_2^+<p_2^*(x)$,

(ii) $1\le q_1^-\le q_1^+<\gamma_1^-\le\gamma_1^+<p^-\le p^+<\gamma_2^-\le\gamma_2^+<q_2^-\le q_2^+<p_2^*(x)$,

(iii) $1\le\gamma_1^-\le\gamma_1^+<q_1^-\le q_1^+<p^-<p^+<q_2^-\le q_2^+<\gamma_2^-\le\gamma_2^+<p_2^*(x)$,

(iv) $1\le\gamma_1^-\le\gamma_1^+<q_1^-\le q_1^+<p^-\le p^+<\gamma_2^-\le\gamma_2^+<q_2^-\le q_2^+<p_2^*(x)$,

for any $x \in X$. Thus, we have

$$X \hookrightarrow L^{q_i^{\pm}}(\Omega), \quad i \in \{1,2\},$$
$$X \hookrightarrow L^{\gamma_i^{\pm}}(\Omega), \quad i \in \{1,2\}.$$

So, there exist positive constants C, C' such that

$$\int_\Omega |u|^{q_i^{\pm}} dx \leq C\|u\|^{q_i^{\pm}}, \quad \int_\Omega |u|^{\gamma_i^{\pm}} dx \leq C'\|u\|^{q_i^{\pm}}, \quad \forall u \in X,\ i \in \{1,2\}.$$

It follows that there exist two positive constants c_1 and c_2 such that

$$\begin{aligned}\int_{B_r(x_0)} |u|^{q_1(x)} dx &\leq \int_{B_r(x_0)} |u|^{q_1^-} dx + \int_{B_r(x_0)} |u|^{q_1^+} dx \\ &\leq \int_\Omega |u|^{q_1^-} dx + \int_\Omega |u|^{q_1^+} dx \\ &\leq c_1\Big(\|u\|^{q_1^-} + ||u||^{q_1^+}\Big), \end{aligned} \tag{3}$$

and

$$\begin{aligned}\int_{\Omega\setminus B_R(x_0)} |u|^{q_2(x)}\, dx &\leq \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^-}\, dx + \int_{\Omega\setminus B_R(x_0)} |u|^{q_2^+} dx \\ &\leq \int_\Omega |u|^{q_2^-}\, dx + \int_\Omega |u|^{q_2^+}\, dx \\ &\leq c_2\Big(\|u\|^{q_2^-} + \|u\|^{q_2^+}\Big). \end{aligned} \tag{4}$$

Similarly, we have

$$\int_{B_r(x_0)} |u|^{\gamma_1(x)}\, dx \leq c_3\Big(\|u\|^{\gamma_1^-} + \|u\|^{\gamma_1^+}\Big), \tag{5}$$

and

$$\int_{\Omega\setminus B_R(x_0)} |u|^{\gamma_2(x)} dx \leq c_4\Big(\|u\|^{\gamma_2^-} + \|u\|^{\gamma_2^+}\Big). \tag{6}$$

Let $c_5 = \max\{c_1, c_2\}$ and $c_6 = \max\{c_3, c_4\}$. By relations (3)-(6), for $\|u\|$ sufficiently small, noting Proposition 2.4, we have

$$\begin{aligned} I(u) \geq\ & \frac{1}{p^+}\int_\Omega (|\Delta u|^{p(x)} + |u|^{p(x)})dx - \int_{B_r(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)} dx - \int_{\Omega\setminus B_R(x_0)} \frac{\lambda(x)}{q(x)}|u|^{q(x)} \\ & - \int_{B_r(x_0)} \frac{\mu(x)}{\gamma(x)}|u|^{\gamma(x)} dx - \int_{\Omega\setminus B_R(x_0)} \frac{\mu(x)}{\gamma(x)}|u|^{\gamma(x)} dx \\ \geq\ & \frac{1}{p^+}\|u\|^{p^+} - \frac{|\lambda|_{L^\infty(\Omega)} c_5}{q^-}\Big(\|u\|^{q_1^-} + \|u\|^{q_1^+} + \|u\|^{q_2^-} + \|u\|^{q_2^+}\Big) \\ & - \frac{|\mu|_{L^\infty(\Omega)} c_6}{\gamma^-}\Big(\|u\|^{\gamma_1^-} + \|u\|^{\gamma_1^+} + \|u\|^{\gamma_2^-} + \|u\|^{\gamma_2^+}\Big) \\ \geq\ & \Big[\alpha_1\|u\|^{p^+} - \alpha_2|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_1^-} + \|u\|^{q_1^+})\Big] + \Big[\alpha_1\|u\|^{p^+} - \alpha_2|\lambda|_{L^\infty(\Omega)}(\|u\|^{q_2^-} \\ & + \|u\|^{q_2^+})\Big] + \Big[\alpha_1\|u\|^{p^+} - \alpha_3|\mu|_{L^\infty(\Omega)}(\|u\|^{\gamma_1^-} + ||u||^{\gamma_1^+})\Big] \\ & + \Big[\alpha_1\|u\|^{p^+} - \alpha_3|\mu|_{L^\infty(\Omega)}(\|u\|^{\gamma_2^-} + \|u\|^{\gamma_2^+})\Big], \end{aligned} \tag{7}$$

where $\alpha_1,\ \alpha_2$ and α_3 are positive constants. Since the functions $g_1 : [0,1] \to \mathbb{R}$ and $g_2 : [0,1] \to \mathbb{R}$ defined by

$$g_1(t) = \alpha_1 - \alpha_2 t^{q_2^- - p^+} - \alpha_2 t^{q_2^+ - p^+},$$
$$g_2(t) = \alpha_1 - \alpha_3 t^{\gamma_2^- - p^+} - \alpha_3 t^{\gamma_2^+ - p^+},$$

are positive in a neighborhood of the origin, it follows that there exist $0 < \rho_1 < 1$ and $0 < \rho_2 < 1$ such that $g_1(\rho_1) > 0$ and $g_2(\rho_2) > 0$.
Set $\rho = \min\{\rho_1, \rho_2\}$ and

$$\lambda^* = \min\Big\{1, \frac{\alpha_1}{2\alpha_2}\min\{\rho^{p^+ - q_1^-}, \rho^{p^+ - q_1^+}\}\Big\},$$
$$\mu^* = \min\Big\{1, \frac{\alpha_1}{2\alpha_3}\min\{\rho^{p^+ - \gamma_1^-}, \rho^{p^+ - \gamma_1^+}\}\Big\}.$$

Defining $\eta^* = \min\{\lambda^*, \mu^*\}$, we deduce for the cases $(\mathbf{i}) - (\mathbf{iv})$, there exists $\delta > 0$ such that for any $u \in X$ with $\|u\| = \rho$ we have $I(u) \geq \delta > 0$ provided thatmax$\{|\lambda|_{L^\infty(\Omega)}, |\mu|_{L^\infty(\Omega)}\} < \eta^*$. □

Lemma 3.3. *Let $q(x),\ \gamma(x),\ \lambda(x)$, and $\mu(x)$ be as in Theorem 3.1, then there exists $\psi \in X,\ \psi \neq 0$ such that* $\lim_{t\to+\infty} I(t\psi) \to -\infty$.

Proof. Let $\psi \in C_0^\infty(\Omega),\ \psi \geq 0$ and there exist $x_1 \in \Omega\backslash B_R(x_0)$ and $\epsilon > 0$ such that for any $x \in B_\epsilon(x_1) \subset (\Omega\backslash B_R(x_0))$ we have $\psi(x) > 0$. Thus, for $t > 1$ we have

$$\begin{aligned} I(t\psi) &= \int_\Omega \frac{1}{p(x)}(|\Delta t\psi|^{p(x)} + |t\psi|^{p(x)})\,dx - \int_\Omega \frac{\lambda(x)}{q(x)}|t\psi|^{q(x)}\,dx - \int_\Omega \frac{\mu(x)}{\gamma(x)}|t\psi|^{\gamma(x)}\,dx \\ &\leq \frac{t^{p^+}}{p^-}\int_\Omega(|\Delta\psi|^{p(x)} + |\psi|^{p(x)})\,dx - \int_{\Omega\backslash B_R(x_0)} \frac{\lambda(x)}{q(x)}|t\psi|^{q(x)}\,dx \\ &\quad - \int_{\Omega\backslash B_R(x_0)} \frac{\mu(x)}{\gamma(x)}|t\psi|^{\gamma(x)}\,dx \\ &\leq \frac{t^{p^+}}{p^-}\int_\Omega(|\Delta\psi|^{p(x)} + |\psi|^{p(x)})\,dx - t^{q_2^-}\int_{\Omega\backslash B_R(x_0)} \frac{\lambda(x)}{q(x)}|\psi|^{q(x)}\,dx \\ &\quad - t^{\gamma_2^-}\int_{\Omega\backslash B_R(x_0)} \frac{\mu(x)}{\gamma(x)}|\psi|^{\gamma(x)}\,dx \\ &\to -\infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

due to $p^+ < q_2^-, \gamma_2^-$ in the cases $(\mathbf{i}) - (\mathbf{iv})$. □

Lemma 3.4. *Let all conditions in Theorem 3.1 hold. Then there exists $\varphi \in X,\ \varphi \neq 0$ such that $I(t\varphi) < 0$ for $t > 0$ small enough.*

Proof. Let $\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0$ and there exist $x_2 \in B_r(x_0)$ and $\varepsilon > 0$ such that for any $x \in B_\varepsilon(x_2) \subset B_r(x_0)$ we have $\varphi(x) > 0$. For any $0 < t < 1$, we have

$$\begin{aligned} I(t\varphi) &= \int_\Omega \frac{1}{p(x)}(|\Delta t\varphi|^{p(x)} + |t\varphi|^{p(x)})\,dx - \int_\Omega \frac{\lambda(x)}{q(x)}|t\varphi|^{q(x)}\,dx - \int_\Omega \frac{\mu(x)}{\gamma(x)}|t\varphi|^{\gamma(x)}\,dx \\ &\leq \frac{1}{p^-}\int_\Omega t^{p(x)}(|\Delta\varphi|^{p(x)} + |\varphi|^{p(x)})\,dx - \int_{B_r(x_0)} \frac{\lambda(x)}{q_1(x)}|t\varphi|^{q_1(x)}\,dx \\ &\quad - \int_{B_r(x_0)} \frac{\mu(x)}{\gamma_1(x)}|t\varphi|^{\gamma_1(x)}\,dx \\ &\leq \frac{t^{p^-}}{p^-}\int_\Omega(|\Delta\varphi|^{p(x)} + |\varphi|^{p(x)})\,dx - t^{q_1^+}\int_{B_r(x_0)} \frac{\lambda(x)}{q_1(x)}|\varphi|^{q_1(x)}\,dx \\ &\quad - t^{\gamma_1^+}\int_{B_r(x_0)} \frac{\mu(x)}{\gamma_1(x)}|\varphi|^{\gamma_1(x)}\,dx. \end{aligned}$$

So in the cases (**i**) $-$ (**iv**), $I(t\varphi) < 0$, since $q_1^+, \gamma_1^+ < p^-$ □

Proof of Theorem 3.1. By Lemmas 3.2 and 3.3 and the mountain pass theorem of Ambrosetti and Rabinowitz [1], we deduce the existence of a sequence (u_n) such that

$$I(u_n) \to c_7 > 0 \quad \text{and } I'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{8}$$

We prove that (u_n) is bounded in X. Assume for the sake of contradiction, if necessary to a subsequence, still denote by (u_n), $\|u_n\| \to \infty$ and $\|u_n\| > 1$ for all n.
In the cases (**i**) and (**iii**), by Proposition 2.4, we may infer that for n large enough

$$\begin{aligned}
1 + c_8 + \|u_n\| &\geq I(u_n) - \frac{1}{q_2^-}\langle I'(u_n), u_n\rangle \\
&= \Big[\int_\Omega \frac{1}{p(x)}(|\Delta u_n|^{p(x)} + |u_n|^{p(x)})\,dx - \int_\Omega \frac{\lambda(x)}{q(x)}|u_n|^{q(x)}\,dx \\
&\quad - \int_\Omega \frac{\mu(x)}{\gamma(x)}|u_n|^{\gamma(x)}\,dx\Big] \\
&\quad - \frac{1}{q_2^-}\Big[\int_\Omega (|\Delta u_n|^{p(x)} + |u_n|^{p(x)})\,dx - \int_\Omega \lambda(x)|u_n|^{q(x)}\,dx \\
&\quad - \int_\Omega \mu(x)|u_n|^{\gamma(x)}\,dx\Big] \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{q_2^-}\Big)\int_\Omega (|\Delta u_n|^{p(x)} + |u_n|^{p(x)})\,dx \\
&\quad + \int_{B_r(x_0)} \lambda(x)(\frac{1}{q_2^-} - \frac{1}{q_1(x)})|u_n|^{q_1(x)}\,dx \\
&\quad + \int_{B_r(x_0)} \mu(x)(\frac{1}{q_2^-} - \frac{1}{\gamma_1(x)})|u_n|^{\gamma_1(x)}\,dx \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{q_2^-}\Big)\|u_n\|^{p^-} - \eta^*\Big(\frac{1}{q_1^-} - \frac{1}{q_2^-}\Big)\int_{B_r(x_0)} |u_n|^{q_1(x)}\,dx \\
&\quad - \eta^*\Big(\frac{1}{\gamma_1^-} - \frac{1}{q_2^-}\Big)\int_{B_r(x_0)} |u_n|^{\gamma_1(x)}\,dx \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{q_2^-}\Big)\|u_n\|^{p^-} - c_9\eta^*\Big(\frac{1}{q_1^-} - \frac{1}{q_2^-}\Big)\Big(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\Big) \\
&\quad - c_{10}\eta^*\Big(\frac{1}{\gamma_1^-} - \frac{1}{q_2^-}\Big)\Big(\|u_n\|^{\gamma_1^-} + \|u_n\|^{\gamma_1^+}\Big) \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{q_2^-}\Big)\|u_n\|^{p^-} - c_{11}\Big(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\Big) \\
&\quad - c_{12}\Big(\|u_n\|^{\gamma_1^-} + \|u_n\|^{\gamma_1^+}\Big).
\end{aligned} \tag{9}$$

Similarly if the cases (**ii**) and (**iv**), we can write

$$
\begin{aligned}
1 + c_{13} + \|u_n\| &\geq I(u_n) - \frac{1}{\gamma_2^-}\langle I'(u_n), u_n\rangle \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\gamma_2^-}\Big)\int_\Omega (|\Delta u_n|^{p(x)} + |u_n|^{p(x)})\,dx \\
&\quad + \int_{B_r(x_0)} \lambda(x)(\frac{1}{\gamma_2^-} - \frac{1}{q_1(x)})|u_n|^{q_1(x)}\,dx \\
&\quad + \int_{B_r(x_0)} \mu(x)(\frac{1}{\gamma_2^-} - \frac{1}{\gamma_1(x)})|u_n|^{\gamma_1(x)}\,dx \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\gamma_2^-}\Big)\|u_n\|^{p^-} - \eta^*\Big(\frac{1}{q_1^-} - \frac{1}{\gamma_2^-}\Big)\int_{B_r(x_0)} |u_n|^{q_1(x)}\,dx \\
&\quad - \eta^*\Big(\frac{1}{\gamma_1^-} - \frac{1}{\gamma_2^-}\Big)\int_{B_r(x_0)} |u_n|^{\gamma_1(x)}\,dx \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\gamma_2^-}\Big)\|u_n\|^{p^-} - c_{14}\eta^*\Big(\frac{1}{q_1^-} - \frac{1}{\gamma_2^-}\Big)\Big(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\Big) \\
&\quad - c_{15}\eta^*\Big(\frac{1}{\gamma_1^-} - \frac{1}{\gamma_2^-}\Big)\Big(\|u_n\|^{\gamma_1^-} + \|u_n\|^{\gamma_1^+}\Big) \\
&\geq \Big(\frac{1}{p^+} - \frac{1}{\gamma_2^-}\Big)\|u_n\|^{p^-} - c_{16}\Big(\|u_n\|^{q_1^-} + \|u_n\|^{q_1^+}\Big) - c_{17}\Big(\|u_n\|^{\gamma_1^-} \\
&\quad + \|u_n\|^{\gamma_1^+}\Big).
\end{aligned} \tag{10}
$$

But, (9) and (10) can not hold true since $p^- > 1$. Hence (u_n) is bounded in X. This information combined with the fact X is reflexive implies that there exists a subsequence, still denote by (u_n), and $u_1 \in X$ such that $u_n \rightharpoonup u_1$ in X. Since X is compactly embedded in $L^{q(x)}(\Omega)$ and $L^{\gamma(x)}(\Omega)$, it follows that $u_n \to u_1$ in $L^{q(x)}(\Omega)$ and $L^{\gamma(x)}(\Omega)$. Using Proposition 2.1 we deduce

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \lambda(x)|u_n|^{q(x)-2}u_n(u_n - u_1)dx = 0, \\
&\lim_{n\to\infty}\int_\Omega \mu(x)|u_n|^{\gamma(x)-2}u_n(u_n - u_1)dx = 0.
\end{aligned}
$$

This fact and relation (8) yield

$$
\lim_{n\to\infty}\int_\Omega \Big(|\Delta u_n|^{p(x)-2}\Delta u_n(\Delta u_n - \Delta u_1) + |u_n|^{p(x)-2}u_n(u_n - u_1)\Big), dx = 0.
$$

Using Proposition 2.5, we find that $u_n \to u_1$ in X. Then by relation (7) we have

$$
I(u_1) = c_7 > 0 \quad \text{and} \quad I'(u_1) = 0,
$$

that is, u_1 is a non-trivial weak solution of (1).

We hope to apply Ekeland's variational principle [4] in order to get another nontrivial weak solution of problem (1). Indeed, let $\eta^* > 0$ be defined as in Lemma 3.2 and assume that $\max\{|\lambda|_{L^\infty(\Omega)}, |\mu|_{L^\infty(\Omega)}\} < \eta^*$. By Lemma 3.2 it follows that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0) = \{\omega \in X : \|\omega\| < \rho\}$, we have

$$
\inf_{\partial B_\rho(0)} I > 0.
$$

By Lemma 3.4, there exists $\varphi \in X$ such that

$$
I(t\varphi) < 0 \quad \text{for} \quad t > 0 \quad \text{small enough.}
$$

Moreover, by (7) for $u \in B_\rho(0)$ we have

$$-\infty < c_{18} = \inf_{B_\rho(0)} I < 0.$$

We let now $0 < \varepsilon < \inf_{\partial B_\rho(0)} I - \inf_{B_\rho(0)} I$. Applying Ekeland's variational principle [4] to the functional $I : \overline{B_\rho(0)} \to \mathbb{R}$, we find $u_\varepsilon \in \overline{B_\rho(0)}$ such that

$$I(u_\varepsilon) < \inf_{\overline{B_\rho(0)}} I + \varepsilon$$
$$I(u_\varepsilon) < I(u) + \varepsilon\|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.$$

Since

$$I(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} I + \varepsilon \leq \inf_{B_\rho(0)} I + \varepsilon < \inf_{\partial B_\rho(0)} I,$$

we deduce that $u_\varepsilon \in B_\rho(0)$. Now, we define $K : \overline{B_\rho(0)} \to \mathbb{R}$ by $K(u) = I(u) + \varepsilon\|u - u_\varepsilon\|$. It is clear that u_ε is a minimum point of K and thus

$$\frac{K(u_\varepsilon + tv) - K(u_\varepsilon)}{t} \geq 0,$$

for small $t > 0$ and $v \in B_1(0)$. The above relation yields

$$\frac{I(u_\varepsilon + tv) - I(u_\varepsilon)}{t} + \varepsilon\|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle I'(u_\varepsilon), v\rangle + \varepsilon\|v\| > 0$ and we infer that $\|I'(u_\varepsilon)\| \leq \varepsilon$. We deduce that there exists a sequence $(v_n) \subset B_\rho(0)$ such that

$$I(v_n) \to c_{18} \quad \text{and} \quad I'(v_n) \to 0. \tag{11}$$

It is clear that (v_n) is bounded in X. Thus, there exists $u_2 \in X$ such that, up to a subsequence, (v_n) converges weakly to u_2 in X. Actually, with similar arguments as those used in the proof that the sequence $u_n \to u_1$ in X we can show that $v_n \to u_2$ in X. Thus, by relation (11),

$$I(u_2) = c_{18} < 0 \quad \text{and} \quad I'(u_2) = 0,$$

i.e., u_2 is a non-trivial weak solution for problem (1).
Finally, since

$$I(u_1) = c_7 > 0 > c_{18} = I(u_2),$$

we see that $u_1 \neq u_2$. Thus, problem (1) has two non-trivial weak solutions. □

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Some Results on Jensen's Inequality With Respect to M-Convex Functions

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Abstract

In this paper, we extended the Jensen's inequality for m-convex functions and we present some other results with respect to these functions.

1. Introduction and Preliminaries

As one of the important applications, the inequalities theory uses as the theoretical foundation of approximate methods (see[7]). This theory has been developed by C. F. Gacss, A. L. Cathy and P. L. Cebysey. We can see some of the properties of the m-convex functions, uniformly functions and etc, in the references ([1],[2],[8],[9],[10]). Some authors applied the extended Jensen inequality to obtain the lower bounds for the various entropy measures of discrete random variables [6]. We can see two converses of Jensen integral inequality for convex function in [5]. Here, we extend the Jensen's inequality for m-convex functions to obtain the new lower and upper bounds for Jensen's discrete inequality.

Definition 1.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

holds for any $x, y \in I$ and $\lambda \in [0, 1]$.
In the geometric sense, this means that for three distinct points P, Q and R on the graph of f with Q between P and R, then Q is on or below the chord PR.

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Definition 1.2. ([3], [4])Let $[0, c] \subset \mathbb{R}$ be a bounded closed interval with $c > 0$, and let $m \in [0, 1]$. A function $f : [0, c] \longrightarrow \mathbb{R}$ is said to be m-convex if the inequality

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y),$$

holds for any $x, y \in [0, c]$ and $t \in [0, 1]$.

Definition 1.3. Let $x_1, ..., x_n \in I$ and $p_1, ..., p_n \in [0, 1]$ be such that $\sum_{i=1}^{n} p_i = 1$. The sum

$$\sum_{i=1}^{n} p_i x_i$$

is called the convex combination of points x_i.

Theorem 1.4. *(**Jensen's Inequality**) Let $f : I \longrightarrow \mathbb{R}$ be a convex function. Then the inequality*

$$f(\sum_{i=1}^{n} p_i x_i) \leq \sum_{i=1}^{n} p_i f(x_i)$$

holds for every convex combination $\sum_{i=1}^{n} p_i x_i$ of points $x_i \in I$.

Theorem 1.5. *[11] Assume that f is a convex function on I, then*

$$\begin{aligned} 0 &\leq \max_{0 \leq \mu \leq \nu \leq n} \{p_\mu f(x_\mu) + p_\nu f(x_\nu) - (p_\mu + p_\nu) f(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu})\} \\ &\leq \sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i). \end{aligned}$$

Lemma 1.6. *[3] Let $f : [0, c] \longrightarrow \mathbb{R}$ be a m-convex function and $f(0) \leq 0$. Then $f(tx) \leq tf(x)$, for all $x \in [0, c]$ and $t \in [0, 1]$.*

2. main results

In this section, we use numbers $c > 0$, $m \in (0, 1]$ and $n \in \mathbb{N}$.
In the sequel, we shall prove some properties of m-convex functions.

Lemma 2.1. *Assume that $f : [0, c] \longrightarrow \mathbb{R}$ is a m-convex function. If $r, s, n \in \mathbb{N}$ and $r < s < n$ and $y \in [0, m^n c]$ then*

$$m^r f(\frac{y}{m^r}) \leq m^s f(\frac{y}{m^s}) \leq m^n f(\frac{y}{m^n}).$$

Proof. By the use of Lemma 1.6 it is obvious. □

Lemma 2.2. *Let f be a m-convex function and $x \in [0, c]$ $0 \leq p, q$ and $p + q = 1$, then for every $x \in [0, c]$*

1. *$f(px) \leq pf(x) + qmf(0)$ and $f(qx) \leq qf(x) + pmf(0)$,*
2. *$f(px) + f(qx) \leq f(x) + mf(0)$.*

Proof.
1. $f(px) = f(px + q \times 0) \leq pf(x) + mqf(\frac{0}{m}) = pf(x) + mqf(0)$, and similarly $f(qx) \leq qf(x) + pmf(0)$.
2. Suppose that $p + q = 1$. By using (1), we have

$$f(px) + f(qx) \leq pf(x) + qmf(0) + qf(x) + pmf(0) = f(x) + mf(0).$$

□

Lemma 2.3. *If f is a m-convex function such that $f(0) \leq 0$ and n is a natural number then f is a m^n-convex function.*

Proof. It is obvious. □

In the following we give the Jensen inequality for m-convex function.

Theorem 2.4. *Let $f : [0, c] \longrightarrow \mathbb{R}$ be a m-convex function. Then the inequality*

$$f(\sum_{i=0}^{n} p_i x_i) \leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) \tag{1}$$

holds for any convex combination $\sum_{i=0}^{n} p_i x_i$ of points $x_i \in [0, m^n c]$.

Proof. The proof can be done by applying mathematical induction. □

Theorem 2.5. *Let f be a m-convex function and r, s, n be natural numbers such that $0 \leq r < s \leq n$, then*

$$\begin{aligned} 0 &\leq \max_{0 \leq r < s \leq n} \{m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)})\} \\ &\leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i). \end{aligned}$$

Proof. Suppose that,

$$\begin{cases} q_i = p_i, \quad i \neq r, s \\ q_r = p_r + p_s \\ q_s = 0 \end{cases}$$

and

$$\begin{cases} y_i = x_i, \quad i \neq r \\ y_r = \frac{p_r x_r + p_s x_s}{p_r + p_s} \end{cases}$$

then $\sum_{i=0}^{n} q_i = 1$ and Theorem 2.4 follows that

$$\begin{aligned} f(\sum_{i=0}^{n} p_i x_i) &= f(\sum_{i=0}^{n} q_i y_i) \\ &\leq \sum_{i=0}^{n} m^i q_i f(\frac{y_i}{m^i}) \\ &= \sum_{i \neq r,s}^{n} m^i q_i f(\frac{y_i}{m^i}) + m^r q_r f(\frac{y_r}{m^r}) + m^s q_s f(\frac{y_s}{m^s}) \\ &= \sum_{i \neq r,s}^{n} m^i p_i f(\frac{x_i}{m^i}) + m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + x_r)}). \end{aligned}$$

Hence,

$$\begin{aligned} m^r p_r f(\frac{x_r}{m^r}) + m^s p_s f(\frac{x_s}{m^s}) - m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)}) \\ \leq \sum_{i=0}^{n} m^i p_i f(\frac{x_i}{m^i}) - f(\sum_{i=0}^{n} p_i x_i). \end{aligned}$$

Thus the theorem is proved. □

Corollary 2.6. *Let f be a m-convex function and $0 \leq r < s \leq n$ are arbitrary, then*

$$f(\sum_{i=0}^{n} p_i x_i) \leq \sum_{i \neq r,s} m^i p_i f(\frac{x_i}{m^i}) + m^r (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^r (p_r + p_s)})$$
$$\leq \sum_{i \neq r,s} m^i p_i f(\frac{x_i}{m^i}) + m^s (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^s (p_r + p_s)})$$
$$\leq \sum_{i \neq r,s} m^i p_i f(\frac{x_i}{m^i}) + m^n (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n (p_r + p_s)})$$
$$\leq m^n \{ \sum_{i \neq r,s} p_i f(\frac{x_i}{m^n}) + (p_r + p_s) f(\frac{p_r x_r + p_s x_s}{m^n (p_r + p_s)}) \}.$$

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Existence and multiplicity of solutions for Neumann boundary value problems involving nonlocal $p(x)$-Laplacian equations

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Abstract

In this article, we study the nonlocal $p(x)$-Laplacian problem of the following form

$$\begin{cases} M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})dx\Big)\Big(-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u+|u|^{p(x)-2}u\Big)=\lambda f(x,u) & \text{in } \Omega,\\ M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})dx\Big)|\nabla u|^{p(x)-2}\nabla\frac{\partial u}{\partial\nu}=\mu g(x,u) & \text{on } \partial\Omega, \end{cases}$$

By means of a direct variational approach and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence and multiplicity of solutions for the problem.

1. Introduction

In this paper, we are concerned with the following problem

$$\begin{cases} M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})dx\Big)\Big(-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u+|u|^{p(x)-2}u\Big)=\lambda f(x,u) & \text{in } \Omega,\\ M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)}+|u|^{p(x)})dx\Big)|\nabla u|^{p(x)-2}\nabla\frac{\partial u}{\partial\nu}=\mu g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\Omega}$ is the outer unit normal derivative, $p(x)\in C(\overline{\Omega})$, $p(x)>1$, $\forall x\in\overline{\Omega}$ and λ, $\mu\in\mathbb{R}$. Throughout the paper, we assume that $\lambda^2+\mu^2\neq 0$. The operator $-\Delta_{p(x)}u=\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x)\equiv p$ (a constant). An essential difference between them is that the p-Laplacian operator is $(p-1)$-homogeneous, that is, $\triangle_p(\lambda u)=\lambda^{p-1}\triangle_p u$ for every $\lambda>0$, but the $p(x)$-Laplacian operator, when $p(x)$ is not a constant, is not homogeneous. The study of problems involving variable exponent growth conditions has a strong motivation due to the fact that they can model various phenomena which arise in the study of elastic mechanics [25], electrorheological fluids [26] or image restoration [27].

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Problem (1) is called nonlocal because of the presence of the term M, which implies that the equation in (1) is no longer pointwise identities. This provokes some mathematical difficulties which make the study of such a problem particulary interesting. For the physical and biological meaning of the nonlocal coefficients we refer the reader to [1, 2, 5, 7] and the references therein.

2. Notations and preliminaries

For the reader's convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to [4, 6, 9, 10].
Let Ω be a bounded domain of $\mathbb{R}^N$, denote

$$C_+(\overline{\Omega}) = \{p(x);\ p(x) \in C(\overline{\Omega}),\ p(x) > 1,\ \forall x \in \overline{\Omega}\};$$
$$p^+ = \max\{p(x);\ x \in \overline{\Omega}\}, \quad p^- = \min\{p(x);\ x \in \overline{\Omega}\};$$
$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\},$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\Big\{\lambda > 0;\ \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\Big\},$$

with

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the natural norm

$$||u||_{W^{1,p(x)}(\Omega)} = |u(x)|_{L^{p(x)}(\Omega)} + |\nabla u(x)|_{L^{p(x)}(\Omega)}.$$

We remember that $(W^{1,p(x)}(\Omega), ||.||)$ is a reflexive and separable Banach space. In this paper we will use the following equivalent norm on $W^{1,p(x)}(\Omega)$:

$$||u|| = \inf\Big\{\lambda > 0;\ \int_\Omega \frac{|\nabla u(x)|^{p(x)} + |u|^{p(x)}}{\lambda^{p(x)}}dx \leq 1\Big\} \quad .$$

Proposition 2.1 (See [4, 10]). (**i**)*The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv|dx \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}$$

(**ii**)*If $p_1(x),\ p_2(x) \in C+\overline{\Omega},\ p_1(x) \leq p_2(x),\ \forall x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2 (See [10, 11]). *If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies*

$$|f(x,s) \leq a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \qquad \forall x \in \overline{\Omega},\ s \in \mathbb{R},$$

where $p_1(x),\ p_2(x) \in C_+(\overline{\Omega}),\ a(x) \in L^{p_2(x)}(\Omega),\ a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytsky operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $(N_f(u))(x) = f(x, u(x))$ is a continuous and bounded operator.

Proposition 2.3 (See [12]). *Set $\rho(u) = \int_\Omega |\nabla u(x)|^{p(x)} + |u|^{p(x)}dx$, then for $u,\ u_k \in W^{1,p(x)}(\Omega)$; we have*

$$(1)||u|| < 1(respectively = 1; > 1) \Longleftrightarrow \rho(u) < 1(respectively = 1; > 1);$$
$$(2) for\ u \neq o,\ ||u|| = \lambda \Longleftrightarrow \rho(\frac{u}{\lambda}) = 1;$$
$$(3) if\ ||u|| > 1, then ||u||^{p^-} \leq \rho(u) \leq ||u||^{p^+};$$
$$(4) if\ ||u|| < 1, then ||u||^{p^+} \leq \rho(u) \leq ||u||^{p^-};$$
$$(5)||u|| \to 0(respectively \to \infty) \Longleftrightarrow \rho(u) \to 0(respectively \to \infty).$$

Let us define, for every $x \in \Omega$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ \infty & \text{if} \quad p(x) \geq N. \end{cases}$$

Proposition 2.4 (See [10]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Proposition 2.5 (See [8]). *If we denote*

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \textit{if} \quad p(x) < N, \\ \infty & \textit{if} \quad p(x) \geq N, \end{cases}$$

then the embedding from $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact and continuous, where $q(x) \in C_+(\partial\Omega)$ and $q(x) < p_*(x)$ for $x \in \partial\Omega$.

3. Existence of solutions

In this paper, we denote by $X = W^{1,p(x)}(\Omega)$; $X^* = (W^{1,p(x)}(\Omega))^*$, the dual space and $\langle .,. \rangle$, the dual pair.

Lemma 3.1 (See [13]). *Denote*

$$I(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx, \quad \forall u \in X,$$

then $I(u) \in C^1(X, R)$ and the derivative operator I' of I is

$$\langle I'(u), v \rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)dx, \quad \forall u, \ v \in X,$$

and we have

(1) *I is a convex functional .*
(2) *$I' : X \to X^*$ is a bounded homeomorphism and strictly monotone operator,*
(3)*I' is a mapping of type (S_+), namely*

$$u_n \rightharpoonup u \textit{ and } \limsup_{n\to+\infty} I'(u_n)(u_n - u) \leq 0, \textit{ implies } u_n \to u.$$

The Euler-Lagrange functional associated to (1) is given by

$$J(u) = \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx \Big) - \lambda \int_\Omega F(x,u)dx - \mu \int_{\partial\Omega} G(x,u)d\sigma,$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$. Under proper assumptions on f and g, then

$$\langle J'(u), v \rangle = M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx \Big) \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)dx$$
$$-\lambda \int_\Omega f(x,u)vdx - \mu \int_{\partial\Omega} g(x,u)vd\sigma,$$

for all $u, \ v \in X$, then we know that the weak solution of (1) corresponds to the critical point of the functional J, where F and G are denoted by

$$F(x,t) = \int_0^t f(x,s)ds, \qquad G(x,t) = \int_0^t g(x,s)ds.$$

Hereafter, $f(x,t), \ g(x,t)$ and $M(t)$ are always supposed to verify the following assumption:

($\mathbf{f_0}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and there exist two constants $C_1 \geq 0$, $C_2 \geq 0$ such that

$$|f(x,t)| \leq C_1 + C_2|t|^{\alpha(x)-1}, \qquad \forall (x,t) \in \Omega \times \mathbb{R}$$

where $\alpha(x) \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$,

($\mathbf{f_1}$) There exist $M_1 > 0,\ \theta_1 > \frac{p^+}{1-\mu}$ such that for all $x \in \Omega$ and all $t \in \mathbb{R}$ with $|t| \geq M_1$,

$$0 < \theta_1 F(x,t) \leq t f(x,t),$$

where μ comes from ($\mathbf{m_1}$) below.

($\mathbf{f_2}$) $f(x,t) = o(|t|^{p^+-1})$ as $t \to 0$ uniformly with respect to $x \in \Omega$.

($\mathbf{f_3}$) $f(x,-t) = -f(x,t)$, for all $x \in \Omega$ and $t \in \mathbb{R}$.

($\mathbf{g_0}$) $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition and there exist two constants $C_1' \geq 0$, $C_2' \geq 0$ such that

$$|g(x,t)| \leq C_1' + C_2'|t|^{\beta(x)-1}, \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}$$

where $\beta(x) \in C_+(\overline{\Omega})$ and $\beta(x) < p_*(x)$ for all $x \in \partial\Omega$,

($\mathbf{g_1}$) There exist $M_2 > 0,\ \theta_2 > \frac{p^+}{1-\mu}$ such that for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$ with $|t| \geq M_2$,

$$0 < \theta_2 G(x,t) \leq t g(x,t),$$

where μ comes from ($\mathbf{m_1}$) below.

($\mathbf{g_2}$) $g(x,t) = o(|t|^{p^+-1})$ as $t \to 0$ uniformly with respect to $x \in \partial\Omega$.

($\mathbf{g_3}$) $g(x,-t) = -g(x,t)$, for all $x \in \partial\Omega$ and $t \in \mathbb{R}$.

($\mathbf{m_0}$) There exists $m_0 > 0$, such that $M(t) \geq m_0$.

($\mathbf{m_1}$) There exists $0 < \mu < 1$ such that $\widehat{M}(t) \geq (1-\mu)M(t)t$.

Remark 3.2. *Under the conditions f_0 and g_0, the functional J is of class $C^1(X,\mathbb{R})$.*

Remark 3.3. *For simplicity, we use $C,\ M,\ K,\ K_i$, to denote the general nonnegative or positive constant (the exact value may change from line to line).*

Theorem 3.4. *If M satisfies ($\mathbf{m_0}$) and ($\mathbf{f_0}$), ($\mathbf{g_0}$) hold and $\alpha^+, \beta^+ < p^-$, then (1) has a weak solution.*

Proof. From ($\mathbf{m_0}$) we have $\widehat{M}(t) \geq m_0 t$. For $(u_n) \in X$ such that $||u_n|| \to +\infty$, we have

$$\begin{aligned}
J(u_n) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big) - \lambda \int_\Omega F(x,u)dx - \mu \int_{\partial\Omega} G(x,u)d\sigma \\
&\geq m_0 \int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx - |\lambda| \int_\Omega C(1+|u_n|^{\alpha(x)})dx \\
&\quad - |\mu| \int_{\partial\Omega} C(1+|u_n|^{\beta(x)})d\sigma \\
&\geq \frac{m_0}{p^+}||u_n||^{p^-} - |\lambda|C||u_n||^{\alpha^+} - |\mu|C||u_n||^{\beta^+} - M \to \infty \quad \text{as} \quad ||u_n|| \to \infty,
\end{aligned}$$

so J is coercive since $\alpha^+, \beta^+ < p^-$.

By Propositions 2.4 and 2.5, it is easy to verify that J is weakly lower semicontinuous. So J has a minimum point u in X and u is a weak solution of (1). □

Corollary 3.5. *Under the assumptions in Theorem 3.4 and* $(\mathbf{m_1})$*, if one of the following conditions hold, (1) has a nontrivial weak solution.*

(1) *If* $\lambda,\ \mu \neq 0,$ *there exist two positive constants* $d_1,\ d_2 < \dfrac{p^-}{1-\mu}$ *such that*

$$\liminf_{t\to 0} \frac{sgn(\lambda)F(x,t)}{|t|^{d_1}} > 0, \quad \textit{for } x \in \Omega \textit{ uniformly},$$
$$\liminf_{t\to 0} \frac{sgn(\mu)G(x,t)}{|t|^{d_2}} > 0, \quad \textit{for } x \in \Omega \textit{ uniformly},$$

(2) *If* $\lambda = 0,\ \mu \neq 0,$ *there exists a positive constant* $d_2 < \dfrac{p^-}{1-\mu}$ *such that*

$$\liminf_{t\to 0} \frac{sgn(\mu)G(x,t)}{|t|^{d_2}} > 0, \quad \textit{for } x \in \Omega \textit{ uniformly},$$

(3) *If* $\lambda \neq 0,\ \mu = 0,$ *there exists a positive constant* $d_1 < \dfrac{p^-}{1-\mu}$ *such that*

$$\liminf_{t\to 0} \frac{sgn(\lambda)F(x,t)}{|t|^{d_1}} > 0, \quad \textit{for } x \in \Omega \textit{ uniformly},$$

Proof. From Theorem 3.4, we know J has a global minimum point u. We just need to show u is nontrivial. We only consider the case $\lambda, \mu \neq 0$ here. From (1), we know that for $0 < t < 1$ small enough, there exists a positive constant C such that

$$sgn(\lambda)F(x,t) \geq C|t|^{d_1}, \quad sgn(\mu)G(x,t) \geq C|t|^{d_2}.$$

When $t > t_0$ from $(\mathbf{m_1})$ we can easily obtain that

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{(1-\mu)}}} := Ct^{\frac{1}{(1-\mu)}},$$

where t_0 is an arbitrary positive constant. Choose $u_0 > 0$. For $0 < t < 1$ small enough, we have

$$\begin{aligned}
J(tu_0) &\leq C\Big(\int_\Omega \frac{1}{p(x)}(|t\nabla u_0|^{p(x)} + |tu_0|^{p(x)})dx\Big)^{\frac{1}{(1-\mu)}} - |\lambda| \int_\Omega sgn(\lambda)F(x,tu_0)dx \\
&\quad - |\mu| \int_{\partial\Omega} sgn(\mu)G(x,tu_0)d\sigma \\
&\leq C\Big(\frac{t^{p^-}}{p^-}\int_\Omega |u_0|^{p(x)}dx\Big)^{\frac{1}{(1-\mu)}} - |\lambda| \int_\Omega C|tu_0|^{d_1}dx \\
&\quad - |\mu| \int_{\partial\Omega} C|tu_0|^{d_2}d\sigma \\
&= K_1 t^{\frac{p^-}{1-\mu}} - |\lambda| K_2 t^{d_1} - |\mu| K_3 t^{d_2}.
\end{aligned}$$

Since $d_1, d_2 < \frac{p^-}{1-\mu}$, there exists $0 < t_0 < 1$ small enough such that $J(t_0u_0) < 0$. So the global minimum point u of J is nontrivial. $\square$

Definition 3.6. We say that J satisfies (PS) condition in X, if any sequence (u_n) such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence, where (PS) means Palais-Smale.

Remark 3.7. *We know that if we denote*

$$\phi(u) = -\lambda \int_\Omega F(x,u)dx, \qquad \psi(u) = -\mu \int_{\partial\Omega} G(x,u)d\sigma,$$

then by Propositions 2.2, 2.4 and 2.5, they are both weakly continuous and their derivative operators are compact. By Lemma 3.1, we deduce that $J' = I' + \phi' + \psi'$ is also of type (S_+). By [13], to verify that J satisfies the (PS) condition on X, it is enough to verify that any (PS) sequence is bounded.

Lemma 3.8. *If* $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{g_0})$, $(\mathbf{g_1})$, $(\mathbf{m_0})$, $(\mathbf{m_1})$ *hold and* $\lambda, \mu \geq 0$, *then* J *satisfies the (PS) condition.*

Proof. Suppose that $(u_n) \subset X$, $|J(u_n)| \leq C$ and $J'(u_n) \to o$. Then

$$\begin{aligned}
C + 1 &\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle + \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx\Big) - \lambda\int_\Omega F(x,u_n)dx - \mu\int_{\partial\Omega} G(x,u_n)d\sigma \\
&\quad - \frac{1}{\theta}\Big[M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx\Big)\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\
&\quad - \lambda\int_\Omega f(x,u_n)u_n dx - \mu\int_{\partial\Omega} g(x,u_n)u_n d\sigma\Big] + \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\
&\geq (1-\mu)M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx\Big)\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\
&\quad - \lambda\int_\Omega F(x,u_n)dx - \mu\int_{\partial\Omega} G(x,u_n)d\sigma - \frac{1}{\theta}\Big[M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx\Big) \\
&\quad - \lambda\int_\Omega f(x,u_n)u_n dx - \mu\int_{\partial\Omega} g(x,u_n)u_n d\sigma\Big] + \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\
&\geq m_0\Big(\frac{1-\mu}{p^+} - \frac{1}{\theta}\Big)||u_n||^{p^-} - \frac{1}{\theta}||J'(u_n)||_{X^*}||u_n|| - C \\
&\geq m_0\Big(\frac{1-\mu}{p^+} - \frac{1}{\theta}\Big)||u_n||^{p^-} - \frac{1}{\theta}||u_n|| - C,
\end{aligned}$$

where $\theta = \min\{\theta_1, \theta_2\}$ and we have supposed that $||u_n|| > 1$ for convenience. Since $\theta > \frac{p^+}{1-\mu}$, we know that (u_n) is bounded in X. □

Theorem 3.9. *If* M *satisfies* $(\mathbf{m_0})$, $(\mathbf{m_1})$ *and* $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_2})$, $(\mathbf{g_0})$, $(\mathbf{g_1})$, $(\mathbf{g_2})$ *hold and* $\alpha^-, \beta^- > p^+$; $\lambda, \mu \geq 0$, *then (1) has a nontrivial weak solution.*

Proof. Let us show that J satisfies the conditions of Mountain Pass Theorem (see Theorem 2.10 of [15]). By Lemma 3.8, J satisfies (PS) condition in X. Since

$$p^+ < \alpha^- \leq \alpha(x) < p^*(x), \quad \forall x \in \overline{\Omega}; \qquad p^+ < \beta^- \leq \beta(x) < p_*(x), \quad \forall x \in \partial\Omega,$$

we have $X \hookrightarrow L^{p^+}(\Omega)$, $X \hookrightarrow L^{p^+}(\partial\Omega)$. Then there exists a constant $C > 0$ such that

$$|u|_{L^{p^+}(\Omega)} \leq C||u||, \qquad |u|_{L^{p^+}(\partial\Omega)} \leq C||u||, \qquad \forall u \in X.$$

From $(\mathbf{f_0})$, $(\mathbf{f_2})$ and $(\mathbf{g_0})$, $(\mathbf{g_2})$, we have there exist an arbitrary constant $0 < t < 1$ and two positive constants (both denoted by $C(\epsilon)$) such that

$$\begin{aligned}
|F(x,t)| &\leq \epsilon|t|^{p^+} + C(\epsilon)|t|^{\alpha(x)}, \qquad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}, \\
|G(x,t)| &\leq \epsilon|t|^{p^+} + C(\epsilon)|t|^{\beta(x)}, \qquad \text{for all} \quad (x,t) \in \partial\Omega \times \mathbb{R}.
\end{aligned}$$

In view of $(\mathbf{m_0})$ and above inequalities, for $||u||$ sufficiently small, noting Proposition 2.3, we have

$$\begin{aligned}
J(u) &\geq \frac{m_0}{p^+}||u||^{p^+} - \lambda\int_\Omega F(x,u)dx - \mu\int_{\partial\Omega} G(x,u)d\sigma \\
&\geq \frac{m_0}{p^+}||u||^{p^+} - \lambda\int_\Omega (\epsilon|u|^{p^+} + C(\epsilon)|u|^{\alpha(x)})dx - \mu\int_{\partial\Omega} (\epsilon|u|^{p^+} + C(\epsilon)|u|^{\beta(x)})d\sigma \\
&\geq \frac{m_0}{p^+}||u||^{p^+} - (\lambda\epsilon C + \mu\epsilon C)||u||^{p^+} - \lambda C(\epsilon)||u||^{\alpha^-} - \mu C(\epsilon)||u||^{\beta^-}.
\end{aligned}$$

Choose $\epsilon > 0$ so small that $0 < \lambda\epsilon C + \mu\epsilon C < \frac{m_0}{2p^+}$, we obtain

$$J(u) \geq \frac{m_0}{2p^+}||u||^{p^+} - C(\lambda, \mu, \epsilon)C(||u||^{\alpha^-} + ||u||^{\beta^-}).$$

Since $\alpha^-, \beta^- > p^+$, there exist $r > 0$ small enough and $\delta > 0$ such that $J(u) \geq \delta > 0$ as $||u|| = r$.
On the other hand, we have known that the assumption $(\mathbf{f_1})$, $(\mathbf{g_1})$ implies the following assertion:

$$\begin{aligned} &F(x,t) \geq C|t|^{\theta_1} - M, \qquad \forall (x,t) \in \Omega \times \mathbb{R},\\ &G(x,t) \geq C|t|^{\theta_2} - M, \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}. \end{aligned}$$

For $t > 1$ large enough, we have

$$\begin{aligned} J(t\widetilde{u}) &= \widehat{M}\Big(\int_\Omega \frac{t^{p(x)}}{p(x)}(|\nabla\widetilde{u}|^{p(x)} + |\widetilde{u}|^{p(x)})dx\Big) - \lambda\int_\Omega F(x,\widetilde{u})dx - \mu\int_{\partial\Omega} G(x,\widetilde{u})d\sigma,\\ &\leq \Big(\frac{t^{p^+}}{p^-}\Big)^{\frac{1}{1-\mu}}\Big(\int_\Omega (|\nabla\widetilde{u}|^{p(x)} + |\widetilde{u}|^{p(x)})dx\Big)^{\frac{1}{1-\mu}} - \lambda C t^\theta \int_\Omega |\widetilde{u}|^\theta dx - \mu C t^\theta \int_{\partial\Omega} |\widehat{u}|^\theta d\sigma + C\\ &\to -\infty \quad \text{as} \quad t \to +\infty, \end{aligned}$$

due to $\theta = \min\{\theta_1, \theta_2\} > \frac{p^+}{1-\mu}$. □

Since X is a reflexive and separable Banach space, then X^* is too. There exist (see [28]) $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}\,\{e_j \ : j = 1, 2, ...\}}, \qquad X^* = \overline{\text{span}\,\{e_j^* \ : j = 1, 2, ...\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j,\\ 0 & \text{if} \quad i \neq j, \end{cases}$$

where $\langle ., . \rangle$ denote the duality product between X and X^*. We define

$$X_j = \text{span}\,\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Lemma 3.10. *(Fountain Theorem, see [15]). Let $J \in C^1(X, \mathbb{R})$ be an even functional, where $(X, ||.||)$ is a separable and reflexive Banach space. Suppose that for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A1) $\inf\{J(u) \ : \ u \in Z_k,\ ||u|| = r_k\} \to +\infty$ *as* $k \to +\infty$.
(A2) $\max\{J(u) \ : \ u \in Y_k,\ ||u|| = \rho_k\} \leq 0$.
(A3) *J satisfies the (PS) condition for every $c > 0$.*

Then J has an unbounded sequence of critical points.

Lemma 3.11 (See [14])**.** *If $\alpha(x) \in C_+(\overline{\Omega}),\ \alpha(x) < p^*(x),\ \forall x \in \overline{\Omega}$ and $\beta(x) \in C_+(\partial\Omega),\ \beta(x) < p_*(x),\ \forall x \in \partial\Omega$, denote*

$$\begin{aligned} \alpha_k &= \sup\{|u|_{L^{\alpha(x)}(\Omega)};\ ||u|| = 1,\ u \in Z_k\}\\ \beta_k &= \sup\{|u|_{L^{\beta(x)}(\partial\Omega)};\ ||u|| = 1,\ u \in Z_k\}, \end{aligned}$$

then $\lim_{k\to\infty} \alpha_k = 0,\ \lim_{k\to\infty} \beta_k = 0$.

Theorem 3.12. *If $(\mathbf{m_0})$, $(\mathbf{m_1})$, $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_3})$, $(\mathbf{g_0})$, $(\mathbf{g_1})$, $(\mathbf{g_3})$ hold and $\alpha^-, \beta^- > p^+$, $\lambda, \mu > 0$, then (1) has a sequence of solutions $(\pm u_k, \pm v_k)$ such that $J(\pm u_k, \pm v_k) \to +\infty$ as $k \to +\infty$.*

Proof. According to the assumptions on f and g, Remark 3.7, Lemma 3.8, J is an even functional and satisfies Palais-Smale condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that $(A1)$ and $(A2)$ holding. Thus, the conclusion can be obtained from Fountain theorem.
(**A1**) For any $(u) \in Z_k,\ ||u|| > 1$, we have

$$\begin{aligned} J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big) - \lambda\int_\Omega F(x,u)dx - \mu\int_{\partial\Omega} G(x,u)d\sigma \\ &\geq \frac{m_0}{p^+}||u||^{p^-} - \lambda\int_\Omega C(1+|u|^{\alpha(x)})dx - \mu\int_{\partial\Omega} C(1+|u|^{\beta(x)})d\sigma \\ &\geq \frac{m_0}{p^+}||u||^{p^-} - \lambda C\max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}\} - \mu C\max\{|u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\Omega)}\} - C \\ &\geq \frac{m_0}{p^+}||u||^{p^-} - C(\lambda,\mu)\max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta}(\Omega)}\} - C. \end{aligned}$$

If $\max\{|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta}(\Omega)}\} = |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}$, we have

$$J(u) \geq \frac{m_0}{p^+}||u||^{p^-} - C(\lambda,\mu)\alpha_k^{\alpha^+}||u||^{\alpha^+} - C.$$

At this stage, we fix r_k as follows:

$$r_k = \Big(\frac{\alpha^+ C(\lambda,\mu)\alpha_k^{\alpha^+}}{m_0}\Big)^{\frac{1}{p^- - \alpha^+}} \to +\infty \quad \text{as} \quad k \to +\infty.$$

Consequently, if $||u|| = r_k$ then

$$J(u) \geq m_0\Big(\frac{1}{p^+} - \frac{1}{\alpha^+}\Big)r_k^{p^-} - C \to +\infty \quad \text{as} \quad k \to +\infty,$$

due to $\alpha^+ > \alpha^- > p^+$.
(**A2**) From ($\mathbf{m_1}$), ($\mathbf{f_1}$) and ($\mathbf{g_1}$), we have

$$\begin{aligned} &\widehat{M}(t) \leq Ct^{\frac{1}{1-\mu}}, \\ &F(x,t) \geq C|t|^{\theta_1} - M, \quad \forall (x,t) \in \Omega\times\mathbb{R}, \\ &G(x,t) \geq C|t|^{\theta_2} - M, \quad \forall (x,t) \in \partial\Omega\times\mathbb{R}. \end{aligned}$$

Therefore, for any $u \in Y_k$ we have

$$\begin{aligned} J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big) - \lambda\int_\Omega F(x,u)dx - \mu\int_{\partial\Omega} G(x,u)d\sigma \\ &\leq \Big(\frac{1}{p^-}\Big)^{\frac{1}{1-\mu}}||u||^{\frac{p^+}{1-\mu}} - \lambda\int_\Omega (C|u|^{\theta_1} - M) - \mu\int_{\partial\Omega}(C|u|^{\theta_2} - M)d\sigma \\ &\leq \Big(\frac{1}{p^-}\Big)^{\frac{1}{1-\mu}}||u||^{\frac{p^+}{1-\mu}} - \lambda C\int_\Omega |u|^{\theta_1}dx - \mu C\int_{\partial\Omega}|u|^{\theta_2}d\sigma + K \to -\infty \quad \text{as} \quad ||u|| \to \infty, \end{aligned}$$

since $\theta_1, \theta_2 > \frac{p^+}{1-\mu}$ and $\dim Y_k < \infty$. So $(A2)$ holds. From the proofs of $(A1)$ and $(A2)$, we can choose $\rho_k > r_k > 0$. The proof is completed. □

4. The case of concave-convex nonlinearity

In this section, we will obtain much better results with f and g in a special form. We have the following theorem:

Theorem 4.1. *Assume the conditions* $(\mathbf{m_0})$ *and* $(\mathbf{m_1})$ *hold. And let* $\alpha(x) \in C_+(\overline{\Omega})$, $\beta(x) \in C_+(\partial\Omega)$, $\alpha(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, $\beta(x) < p_*(x)$ *for any* $x \in \partial\Omega$ *with* $\alpha^- > \frac{p^+}{1-\mu}$, $\beta^+ < p^-$ *and* $f(x,t) = |t|^{\alpha(x)-2}t$, $g(x,t) = |t|^{\beta(x)-2}t$, *then we have*

(**i**) *For every* $\lambda > 0$, $\mu \in \mathbb{R}$, *(1) has a sequence of weak solutions* $(\pm u_k)$ *such that* $J(\pm u_k) \to +\infty$ *as* $k \to +\infty$.
(**ii**) *For every* $\mu > 0$, $\lambda \in \mathbb{R}$, *(1) has a sequence of weak solutions* $(\pm v_k)$ *such that* $J(\pm v_k) \to 0$ *as* $k \to +\infty$.

We will use Lemma 3.10 to prove Theorem 4.1 (**i**) and the following "Dual fountain theorem" to prove Theorem 4.1 (**ii**), respectively.

Lemma 4.2. *(Dual Fountain Theorem, see [15]). Assume* $(\mathbf{A1})$ *is satisfied and there is* $k_0 > 0$ *so that, for each* $k \geq k_0$, *there exist* $\rho_k > r_k > 0$ *such that*

(**B1**) $a_k = \inf\{J(u) \,:\, u \in Z_k,\ ||u|| = \rho_k\} \geq 0$.
(**B2**) $b_k = \max\{J(u) \,:\, u \in Y_k,\ ||u|| = r_k\} < 0$.
(**B3**) $d_k = \inf\{J(u) \,:\, u \in Z_k,\ ||u|| \leq \rho_k\} \to 0$ *as* $k \to +\infty$.
(**B4**) J *satisfies the* $(PS)^*_c$ *condition for every* $c \in [d_{k_0}, 0)$.

Then J *has a sequence of negative critical values converging to* 0.

Definition 4.3. We say that J satisfies the $(PS)^*_c$ condition (with respect to (Y_n)), if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$, $J(u_{n_j}) \to c$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$, contain a subsequence converging to a critical point of J.

Lemma 4.4. *Assume that the conditions in Theorem 4.1 hold, then* J *satisfies the* $(PS)^*_c$ *condition.*

Proof. Suppose $(u_{n_j}) \subset X$ such that $n_j \to +\infty$, $u_{n_j} \in Y_{n_j}$ and $(J|_{Y_{n_j}})'(u_{n_j}) \to 0$. Assume $||u_{n_j}|| > 1$ for convenience. If $\lambda \geq 0$, for n large enough, we have

$$\begin{aligned}
C + 1 &\geq J(u_{n_j}) - \frac{1}{\alpha^-}\langle J'(u_{n_j}), (u_{n_j})\rangle + \frac{1}{\alpha^-}\langle J'(u_{n_j}), (u_{n_j})\rangle \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})dx\Big) - \lambda \int_\Omega F(x, u_{n_j})dx - \mu \int_{\partial\Omega} G(x, u_{n_j})d\sigma \\
&\quad - \frac{1}{\alpha^-}\Big[M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})dx\Big) - \lambda \int_\Omega f(x, u_{n_j})u_{n_j}dx \\
&\quad - \mu \int_{\partial\Omega} g(x,_{n_j})u_{n_j}d\sigma\Big] + \frac{1}{\alpha^-}\langle J'(u_{n_j}), u_{n_j}\rangle \\
&\geq \Big(\frac{1-\mu}{p^+} - \frac{1}{\alpha^-}\Big)M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})dx\Big)\int_\Omega (|\nabla u_{n_j}|^{p(x)} + |u_{n_j}|^{p(x)})dx \\
&\quad + \mu \int_{\partial\Omega} \Big(\frac{1}{\alpha^-} - \frac{1}{\beta(x)}\Big)|u_{n_j}|^{\beta(x)}d\sigma \\
&\geq m_0\Big(\frac{1-\mu}{p^+} - \frac{1}{\alpha^-}\Big)||u_{n_j}||^{p^-} - K||u_{n_j}||^{\beta^+}.
\end{aligned}$$

Since $p^- > \beta^+$ and $\alpha^- > \frac{p^+}{1-\mu}$, we deduce that (u_{n_j}) is bounded in X.
If $\lambda < 0$, for n large enough, we can consider the inequality below to get the boundedness of (u_{n_j}).

$$C + 1 \geq J(u_{n_j}) - \frac{1}{\alpha^+}\langle J'(u_{n_j}), u_{n_j}\rangle + \frac{1}{\alpha^+}\langle J'(u_{n_j}), u_{n_j}\rangle.$$

Going if necessary to a subsequence, we can assume $u_{n_j} \rightharpoonup u$ in X. As $X = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$. Hence

$$\begin{aligned}
\lim_{n_j \to +\infty}\langle J'(u_{n_j}), u_{n_j} - u\rangle &= \lim_{n_j \to +\infty}\langle J'(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \lim_{n_j \to +\infty}\langle J'(u_{n_j}), v_{n_j} - u\rangle \\
&= \lim_{n_j \to +\infty}\Big\langle (J|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\Big\rangle \\
&= 0.
\end{aligned}$$

As J' is of type (S_+), we can conclude $u_{n_j} \to u$, furthermore we have $J'(u_{n_j}) \to J'(u)$.
Let us prove $J'(u) = 0$ below. Taking $\omega_k \in Y_k$, notice that when $n_j \geq k$ we have

$$\begin{aligned}\langle J'(u), \omega_k\rangle &= \langle J'(u) - J'(u_{n_j}), \omega_k\rangle + \langle J'(u_{n_j}), \omega_k\rangle \\ &= \langle J'(u) - J'(u_{n_j}), \omega_k\rangle + \Big\langle (J|_{Y_{n_j}})'(u_{n_j}), \omega_k\Big\rangle.\end{aligned}$$

Going to the limit on the right side of the above equation reaches

$$\langle J'(u), \omega_k\rangle = 0, \qquad \forall \omega_k \in Y_k,$$

so $J'(u) = 0$, this show that J satisfies the $(PS)^*_c$ condition for every $c \in \mathbb{R}$. □

Proof of Theorem 4.1

(**i**) The proof is similar to that of Theorem 3.12 if we use the Fountain theorem, and the proof of the boundedness of (PS) sequence is same as in Lemma 4.4, we know that J satisfies (**A1**) and (**B4**), the assertion of conclusion can be obtained from Dual fountain theorem. Now, it remains to prove that there exist $\rho_k > r_k > 0$ such that if k is large enough (**B1**), (**B2**) and (**B3**) are satisfied.

(**B1**) Let $u \in Z_k$, then

$$\begin{aligned}J(u) &\geq \frac{m_0}{p^+}||u||^{p^+} - \frac{|\lambda|}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx - \frac{\mu}{\beta^-}\int_{\partial\Omega}|u|^{\beta(x)}d\sigma \\ &\geq \frac{m_0}{p^+}||u||^{p^+} - \frac{C|\lambda|}{\alpha^-}||u||^{\alpha^-} - \frac{\mu}{\beta^-}\max\Big\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\Big\}.\end{aligned}$$

There exist $0 < \rho_1 < 1$ small enough such that $\frac{C|\lambda|}{\alpha^-}||u||^{\alpha^-} \leq \frac{m_0}{p^+}||u||^{p^+}$ as $0 < \rho = ||u|| \leq \rho_1$. Then we have

$$J(u) \geq \frac{m_0}{p^+}||u||^{p^+} - \frac{\mu}{\beta^-}\max\Big\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\Big\}.$$

If $\max\Big\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\Big\} = |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}$, then

$$J(u) \geq \frac{m_0}{p^+}||u||^{p^+} - \frac{\mu}{\beta^-}\beta_k^{\beta^+}||u||^{\beta^+}.$$

Choose $\rho_k = \Big(\frac{2p^+\mu\beta_k^{\beta^+}}{m_0\beta^-}\Big)^{\frac{1}{p^+-\beta^+}}$, then

$$J(u) \geq \frac{m_0}{2p^+}(\rho_k)^{p^+} - \frac{m_0}{2p^+}(\rho_k)^{p^+} = 0.$$

Since $p^- > \beta^+$, $\beta_k \to 0$, we know $\rho_k \to 0$ as $k \to +\infty$.
If $\max\Big\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\Big\} = |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}$, we can do the same work as the case above. So (**B1**) is satisfied.

(**B2**) For $u \in Y_k$ with $||u|| \leq 1$, we have

$$\begin{aligned}J(u) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big) - \lambda\int_\Omega F(x,u)dx - \mu\int_{\partial\Omega} G(x,u)d\sigma \\ &\leq M\Big(\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx\Big)\int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx - \lambda\int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}dx \\ &\quad - \mu\int_{\partial\Omega}\frac{1}{\beta(x)}|u|^{\beta(x)}d\sigma \\ &\leq C||u||^{p^-} + \frac{|\lambda|}{\alpha^-}\int_\Omega |u|^{\alpha(x)}dx - \frac{\mu}{\beta^+}\int_{\partial\Omega}|u|^{\beta(x)}d\sigma.\end{aligned}$$

Since dim$Y_k = k$, conditions $\beta^+ < p^-$ and $p^+ < \frac{p^+}{1-\mu} < \alpha^-$ imply that there exists a $r_k \in (0, \rho_k)$ such that $J(u) < 0$ when $||u|| = r_k$. Hence $b_k = \max\{J(u) \,:\, u \in Y_k,\ ||u|| = r_k\} < 0$, so (**B2**) is satisfied.
(**B3**) Because $Y_k \cap Z_k \neq \emptyset$ and $r_k < \rho_k$, we have

$$d_k = \inf\{J(u) \,:\, u \in Z_k,\ ||u|| \leq \rho_k\} \ \leq b_k = \max\{J(u) \,:\, u \in Y_k,\ ||u|| = r_k\} < 0.$$

In view of the proof of (**B1**), we have

$$J(u) \geq -\frac{\mu}{\beta^-}\beta_k^{\beta^+}||u||^{\beta^+} \qquad \text{or} \qquad -\frac{\mu}{\beta^-}\beta_k^{\beta^-}||u||^{\beta^-}$$

Since $\beta_k \to 0$ and $\rho_k \to 0$ as $k \to +\infty$, (**B3**) is satisfied.
The conclusion of Theorem 4.1 (**ii**) is reached by the Dual fountain theorem.

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Compatible mapping and Coincidence point in a metric space with a partial order

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Abstract

Due to its possible applications, Fixed Point Theory has become one of the most useful branches of Nonlinear Analysis. In a very recent paper, Khojasteh et al. introduced the notion of simulation function to express different contractivity conditions in a unified way, and they obtained some fixed point results. In this paper, we consider a pair of compatible, continuous and nonlinear operators satisfying in generalized **Z**-contractions in a metric space endowed with a partial order. For this pair of operators, we establish coincidence and unique common fixed point results. Some applications of our obtained results are given. In addition, we provide an example to support our main result.

1. introduction

Fixed point theory is a very useful tool for several areas of mathematical analysis and its applications. Loosely speaking, there are three principal categories in this theory: the metric, the topological and the order-theoretic approach, where fundamental examples of these are: Banach's, Brouwer's and Tarski's theorems respectively.
In recent years, many results appeared related to metric fixed point theory in partially ordered sets. The first work in this direction was the 2004 paper of Ran and Reurings [7], where they established a fixed point result, which can be considered as a combination of two fixed point theoerms: Banach contraction principle and Knaster-Tarski fixed point theorem. Further, several results appeared in this direction, we mention [2, 3, 5, 6] and the references therein.
On the other hand, very recently Khojasteh et al. [4] introduced the concept of **Z**-contraction, by using a notion of simulation function. Consequently, fixed point results involving a **Z**-contraction are established in [4]. This approach has been of great importance to discuss various fixed point problems from an unifying point of view; see for instance [1, 8] and the references therein.
In this paper, we consider a pair of compatible, continuous and nonlinear operators satisfying in a generalized **Z**-contraction in a metric space endowed with partial order. For this kind of contraction , we establish coincidence and

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unique common fixed point results. Some applications of our obtained results are given. In addition, we provide an example to support our main result.
The class of simulation functions was introduced by Khojasteh et al. in [4] as follows.

Definition 1.1. Let (X, d) be a metric space. A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the folloing conditions

ζ_1) $\zeta(0, 0) = 0$

ζ_1) $\zeta(p, q) < q - p, \forall p, q > 0$;

ζ_2) if $\{p_n\}$ and $\{q_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = l > 0$, then

$$\limsup_{n\to\infty} \zeta(p_n, q_n) < 0.$$

The main result in [4] is the following.

Theorem 1.2. *Let (X, d) be a complete metric space and $f : X \to X$ be a* **Z***-contraction with respect to ζ, that is,*

$$\zeta(d(fx, fy), d(x, y)) \geq 0, \qquad \forall x, y \in X.$$

Then f has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges to this fixed point.

Successively, Argoubi et all.[1] point out the fact that condition (ζ_1) is not mentioned in the proof of theorem 1.2. Then, they provided the following definition that we use in this article.

Definition 1.3. A simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the conditions (ζ_2) and (ζ_3).

Before presenting our main fixed point results using simulation functions, we show a wide range of examples to highlight their potential applicability to the field of Fixed Point Theory. These examples can also be found in [4], but we include here to obtain some consequences of our main results.

Example 1.4. *Let $\zeta_i : [0, \infty) \times [0, \infty) \to \mathbb{R}, i = 1, 2$ be defined by*

1) $\zeta_1(p, q) = \psi(q) - \phi(p)$ for all $p, q \in [0, \infty)$, where $\psi, \phi : [0, \infty) \to [0, \infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$.

2) $\zeta_2(p, q) = \alpha q - p$ for all $p, q \in [0, \infty)$ is a particular case of ζ_1 with $\phi(t) = t$ and $\psi(t) = \alpha t$ for all $t \geq 0$ and $\alpha \in [0, 1)$.

2. Main result

Definition 2.1. Let (X, d) be a metric space and $S, T : X \to X$. If $v = Su = Tu$, for some u in X, then u is called a coincidence point of S and T.

Definition 2.2. ([11]). Let (X, d) be a metric space. The mappings $S, T : X \to X$ are compatible if and only if for any sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n$ then $\lim_{n\to\infty} d(STu_n, TSu_n) = 0$

Definition 2.3. ([10]). Let (X, d) be a metric space. The mappings $S, T : X \to X$ are weakly compatible if and only if for $Su = Tu$ for some that $u \in X$ implies that $STu = TSu$ or S and T commute at their coincidence points.

If S and T are compatible, then S and T are weakly compatible.

Definition 2.4. ([2]). Let $(X, \preceq)$ is a partially ordered set and $S, T : X \to X$. S is said to be T-non-decreasing if for $u, v \in X$,

$$Tu \preceq Tv \Longrightarrow Su \preceq Sv.$$

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose that there exists a simulation function ζ and $S, T : X \to X$ such that*

$$\zeta(d(Sx, Sy), d(Tx, Ty)) \geq 0 \quad \textit{for all } x, y \in X \textit{ such that } Tx \preceq Ty. \tag{1}$$

We suppose the following hypotheses:

i) $SX \subseteq TX$,

ii) S *is* T*-nondecreasing,*

iii) S *and* T *are continuous,*

iv) the pair $\{S, T\}$ *compatible.*

If there exists $x_0 \in X$ such that $Tx_0 \preceq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if $Tu \preceq TTu$ and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

Proof. Using the theorem condition, we have $x_0 \in X$ such that $Tx_0 \preceq Sx_0$. Since $SX \subseteq TX$, then there exists $x_1 \in X$ such that $Tx_1 = Sx_0$ and $Tx_0 \preceq Sx_0 = Tx_1$. Since S is T-non-decreasing, we have $Sx_0 \preceq Sx_1$. Continuing this process, we construct the sequence $\{x_n\}$ with the following conditions

$$Sx_n = Tx_{n+1} \quad \forall\, n \geq 0, \tag{2}$$

and

$$Tx_0 \preceq Sx_0 = Tx_1 \preceq Sx_1 = Tx_2 \preceq Sx_2 ... \preceq Sx_{n-1} = Tx_n \preceq Sx_n = Tx_{n+1} \preceq ... \tag{3}$$

If two consecutive members of the sequences $\{Sx_n\}$ or $\{Tx_n\}$ are equal, then the conclusion of the theorem follows. So we have

$$d(Sx_n, Sx_{n+1}) \neq 0, \quad d(Tx_n, Tx_{n+1}) \neq 0, \qquad \forall n \geq 0. \tag{4}$$

If for some $n \in \mathbb{N}$, we assume that $d(Tx_{n-1}, Tx_n) < d(Tx_n, Tx_{n+1})$, then by property (ζ_1) of simulation function, (2), (3) and (4) we have

$$\begin{aligned} 0 &\leq \zeta(d(Sx_{n-1}, Sx_n), d(Tx_{n-1}, Tx_n)) \\ &= \zeta(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)) \\ &< d(Tx_{n-1}, Tx_n) - d(Tx_n, Tx_{n+1}) < 0. \end{aligned} \tag{5}$$

This contradiction shows that

$$d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_n).$$

This implies that the sequence $\{d(Tx_{n-1}, Tx_n)\}$ is a monotone decreasing sequence of non-negetive real numbers and consequently there exists $r \geq 0$ such that $\{d(Tx_{n-1}, Tx_n)\} \to r$.
Suppose $r > 0$. By (3), we know that the elements Tx_n and Tx_{n+1} are comparable, so using the property (ζ_2) of a simulation function, with $p_n = d(Sx_n, Sx_{n+1})$ and $q_n = d(Sx_{n-1}, Sx_n)$ we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d(Sx_{n-1}, Sx_n), d(Tx_{n-1}, Tx_n)) \\ &= \limsup_{n\to\infty} \zeta(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)) < 0, \end{aligned}$$

which is a contradiction and hence

$$\lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = 0 \tag{6}$$

The next step is to show that the sequence $\{Tx_n\}$ is cauchy. By contradiction and by lemma 2.1 of [3], then there exists an $\epsilon > 0$ and $\{Tx_{m(k)}\}, \{Tx_{n(k)}\} \subset \{Tx_n\}$ with $n(k) > m(k) \geq k,\ \forall k \in \mathbb{N}$, such that

$$\lim_{n\to\infty} d(Tx_{m(k)}, Tx_{n(k)}) = \lim_{n\to\infty} d(Tx_{m(k)+1}, Tx_{n(k)+1}) = \epsilon, d(Tx_{m(k)}, Tx_{n(k)}) \geq \epsilon. \tag{7}$$

Then we can assume that

$$d(Tx_{m(k)+1}, Tx_{n(k)+1}) > 0 \qquad \forall k \in \mathbb{N} \tag{8}$$

Again by (3), we know that the elements $Tx_{m(k)}$ and $Tx_{n(k)}$ are comparable, so using (7), (8) and the property (ζ_2) of a simulation function, with $p_n = d(Tx_{m(k)+1}, Tx_{n(k)+1})$ and $q_n = d(Tx_{m(k)}, Tx_{n(k)})$, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(d(Sx_{m(k)}, Sx_{n(k)}), d(Tx_{m(k)}, Tx_{n(k)})) \\ &= \limsup_{n\to\infty} d(Tx_{m(k)+1}, Tx_{n(k)+1}), d(Tx_{m(k)}, Tx_{n(k)})) < 0, \end{aligned}$$

which is a contradiction. We conclude that the sequence $\{Tx_n\}$ is Cauchy sequence and hence $\{Tx_n\}$ is convergent in the complete metric space (X, d). Then, there exists $u \in X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = u. \tag{9}$$

Since S and T are compatible, this implies that

$$\lim_{n\to\infty} (S(Tx_n), T(Sx_n)) = 0. \tag{10}$$

From 9 and the continuity of S and T, we have

$$\lim_{n\to\infty} T(Tx_n) = Tu, \qquad \lim_{n\to\infty} S(Tx_n) = Su. \tag{11}$$

By the triangular inequality, we have:

$$d(Su, Tu) \leq (Su, S(Tx_n)) + d(S(Tx_n), T(Sx_n)) + d(T(Tx_{n+1}), Tu). \tag{12}$$

By 10 and 11, and letting $n \to \infty$, we obtain:

$$d(Su, Tu) \leq 0,$$

therefore

$$v = Su = Tu, \tag{13}$$

that is, u is coincidence point of S and T.
Since S and T compatible and therefore are weakly compatible, we have, $STu = TSu$. Then

$$Tv = TTu = TSu = STu = SSu = Sv. \tag{14}$$

If $Tv = v$ or $Sv = v$, then v is a common fixed point. Otherwise, i.e $Tv \neq v$ and $Sv \neq v$, by property (ζ_1) of a simulation function with $Tu \preceq TTu$

$$\begin{aligned} 0 &\leq \zeta(d(v, Sv), d(v, Tv)) \\ &= \zeta(d(Su, SSu), d(Tu, TTu)) < d(Tu, TTu) - d(Su, SSu). \end{aligned}$$

Using (13) and (14) in above inequality we have

$$d(Su, SSu) < d(Tu, TTu) = d(Su, SSu),$$

which is a contradiction.Therefore $Tv = v$ or $Sv = v$ and we conclude that $v = Sv = Tv$
Now, suppose that the set of fixed points of T is totally ordered. Assume on the contrary that $v = Sv = Tv$ and $v' = Sv' = Tv'$ but $v \neq v'$. Since v and v' contain a set of fixed points of T, without loss of generality we assume

that $Tv \preceq Tv'$. If $Sv = Sv'$ or $Tv = Tv'$, then $v = v'$, wich is a contraction. Otherwise, i.e $Sv \neq Sv'$ and $Tv \neq Tv'$, by property (ζ_1) of a simulation function, we have

$$\begin{aligned} 0 &\leq \zeta(d(Sv, Sv'), d(Tv, Tv')) \\ &= \zeta(d(v, v'), d(v, v')) < d(v, v') - d(v, v') = 0, \end{aligned}$$

wich is a contraction. Therefore S and T have a unique common fixed point.

Example 2.6. *We suppose that $\zeta(p, q) : [0, \infty) \times [0, \infty) \to \mathbb{R}$ with $\zeta(p, q) = q - \frac{p+2}{p+1}p$. Clearly ζ is a simulation function. Let $X = [0, \infty)$ be endowed with the metric $d : X \times X \to \mathbb{R}$ given by*

$$d(x, y) = \begin{cases} 0 & if x = y, \\ max\{x, y\} & if x \neq y. \end{cases} \tag{15}$$

Now, consider the usual order of real numbers and define the mappings $S, T : X \to X$ by $Sx = x$ and $Tx = 2x$, for all $x \in X$. Then, we have

$$\zeta(d(Sx, Sy), d(Tx, Ty)) = 2y - \frac{y+2}{y+1}y = \frac{2y(y+1) - y(y+2)}{y+1} = \frac{y^2}{y+1} \geq 0,$$

for all $x, y \in X$, with $x \leq y$. Therefore, the inequality (1) is satisfied.Thus, S and T satisfy all the hypotheses of Theorem 2.5. Here, $v = 0$ is a coincidence point as well as unique common fixed point of S and T.

If $T : X \to X$ is the identity mapping, we can deduce easily the following fixed point result.
The following result is an immediate consequence of Theorem 2.5.

Theorem 2.7. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose that there exists a simulation function ζ and $S : X \to X$ such that*

$$\zeta(d(Sx, Sy), d(x, y)) \geq 0 \quad \textit{for all } x, y \in X \textit{ such that } x \preceq y.$$

We suppose the following hypotheses:

i) S is a non-decreasing function,

ii) S is continuous,

If there exists $x_0 \in X$ such that $x_0 \preceq Sx_0$, then S has a unique fixed point.

Corollary 2.8. *Let $S, T : X \to X$ be mappings such that there exists two continuous functions $\phi, \psi : [0, \infty) \to [0, \infty)$ verifying $\psi(t) = \phi(t) = 0$ if and only if $t = 0$, $\psi < t \leq \phi(t)$ for all $t > 0$ and*

$$\phi(d(Sx, Sy)) \leq \psi(d(Tx, Ty)) \quad \textit{for all } x, y \in X, \textit{ such that } Tx \preceq Ty.$$

We suppose the following hypotheses:

i) $SX \subseteq TX$,

ii) S is T-nondecreasing,

iii) S and T are continuous,

iv) the pair $\{S, T\}$ compatible.

If there exists $x_0 \in X$ such that $Tx_0 \preceq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if $Tu \preceq TTu$ and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

The result follows from Theorems 2.5 by taking as simulation function

$$\zeta_1(p,q) = \psi(q) - \phi(p)$$

for all $p, q \geq 0$, which was introdued in Example 1.4.

Corollary 2.9. *Let $S, T : X \to X$ be mappings such that there exists $\alpha \in (0,1)$ verifying*

$$d(Sx, Sy) \leq \alpha d(Tx, Ty) \quad \textit{for all } x, y \in X, \ \textit{such that } Tx \preceq Ty.$$

We suppose the following hypotheses:

i) $SX \subseteq TX$,

ii) S is T-nondecreasing,

iii) S and T are continuous,

iv) the pair $\{S, T\}$ compatible.

If there exists $x_0 \in X$ such that $Tx_0 \preceq Sx_0$, then S and T have a coincidence point, that is, there exists $u \in X$ such that $Su = Tu$. Further, if $Tu \preceq TTu$ and the set of fixed points of T is totally ordered, then S and T have a unique common fixed point.

The result follows from Theorems 2.5 by taking as simulation function

$$\zeta_2(p,q) = \alpha q - p$$

for all $p, q \geq 0$, which was introdued in Example 1.4.

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Some Notes on Local Subspace Transitivity

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Abstract

An operator T on Banach space X is called transitive, if for every nonempty open subsets U,V of X, there is a positive integer n, such that $T^n(U)\cap V \neq \phi$. In the present paper, local subspace transitivite operators are introduced. Moreover the local subspace transitivity criterion is stated. Also, we show an operator may satisfies in the local subspace transitivity criterion without being topological transitive.

1. Introduction

An operator T for subspace M of a Banach space X over the field $\mathbb{C}$ of complex numbers is called subspace-hypercyclic or M-hypercyclic if there is a vector $x \in X$, so that the intersection $Orb(T,\ x) \cap M = \{T^n x : n \in \mathbb{N} \cup \{0\}\} \cap M$ is dense in M. If $X = M$, then the operator T is called hypercyclic and the vector x is a hypercyclic vector for T. Observe that in this case, the underlying Banach space X should be separable. Then it is well known and easy to show that an operator T is hypercyclic if and only if T is topologically transitive, to be precise, for every pair of nonempty open subsets U, V of X, there exists an integer $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$. Again let M be a pure subspace of X, then the operator T is called M-transitivity if for any pair U, V of nonempty relatively open subsets of M, there exists some integers $n \geq 0$, such that $T^n(U) \cap V \neq \emptyset$ and $T^n(M) \subseteq M$. it was confirmed that M-hypercyclicity is derived from M-transitivity, [5]. Le [4] showed that M-transitivity is not equivalent to M-hypercyclicity when M is a pure subspace of X.

It is worthwhile to mention that, with following remark we can ensure that there is an infinite subset P of $\mathbb{N}$, that T satisfies in M-transitive condition for every member of it.

Remark 1.1. *If $T \in B(X)$ is an M-transitive operator and U, V are nonempty relatively open subsets of M and if $n_0 \geq 0$, so that $T^{-n_0}(U) \cap V \neq \emptyset$ and $T^{n_0}(M) \subseteq M$, then two distinct vectors $x, y \in T^{-n_0}(U) \cap V$ and relatively open subsets O_x and O_y of M are assumed, so that $O_x, O_y \subset T^{-n_0}(U) \cap V$ and $O_x \cap O_y = \emptyset$. According to the assumptions, there is an integer $k \geq 1$, such that we have $T^k(M) \subseteq M$ and $T^{-k}(O_x) \cap O_y \neq \emptyset$. Thus, the set $V \cap T^{-(k+n_0)}(U)$ is nonempty and, clearly, $T^{(n_0+k)}(M) \subseteq M$. By repeating the above method, it can be ensured that there is an infinite subset P of $\mathbb{N}$, such that for every $n \in P$, $V \cap T^{-n}(U) \neq \emptyset$ and $T^n(M) \subseteq M$.*

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The concept of hypercyclicity was localized by introducing certain sets ,which is called J-sets, [2]. To be precise for a given vector $x \in X$ and an operator T;

$$J(x) = \{z \in X; \textit{there exist a sequence } \{z_n\} \subset X \textit{ and a strictly increasing sequence of positive integers } \{m_n\}, \textit{ such that } z_n \longrightarrow x \textit{ and } T^{m_n} z_n \longrightarrow z\}.$$

An operator $T \in B(X)$ is called a J-class operator if there exists a non-zero $x \in X$ such that $J(x) = X$. In this case, the vector x is called a J-class vector for T.
Similarly, we provide the following definition;

Definition 1.2. The M-extended limit set of vector x under an operator $T \in B(X)$ is the set of all $y \in M$ such that there is a sequence $\{z_n\}$ in subspace M and a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ such that $z_n \longrightarrow x$ and $T^{k_n} z_n \longrightarrow y$, and for all n, $T^{k_n}(M) \subseteq M$.
The M-extended limit set of vector x under an operator T is denoted by $\mathbb{J}_M(x)$.

See the paper [1] and the good book [3] for some details on hypercyclicity and J-class and related properties.

In this paper, we give an equivalent definition of $\mathbb{J}_M(x)$ and moreover, a sufficient condition is expressed as subspace $\mathbb{J}_M$-class criterion for an operator $T \in B(X)$ to be the subspace $\mathbb{J}_M$-class operator.
In all the following discussions, M is a nontrivial subspace of X.

2. Main Result

At the beginning of this section, we state an equivalent definition of $\mathbb{J}_M(x)$.

Theorem 2.1. *Assume that $T \in B(X)$ and $x \in M$. Then we have $\mathbb{J}_M^{tra}(x) = \mathbb{J}_M(x)$ where $\mathbb{J}_M^{tra}(x)$ is the set of all $y \in M$ such that for every relatively open neighborhoods U_x, V_y of vectors x, y in M respectively, and every positive integer n_0, there exists an integer $n > n_0$ such that $T^n(U_x) \cap V_y \neq \emptyset$ and $T^n(M) \subseteq M$.*

Proof. Assume that $y \in \mathbb{J}_M^{tra}(x)$, $k_0 = 1$, $N = k_{n-1}$ and set for every $n \in \mathbb{N}$:

$$U_{(x,n)} = B(x, \frac{1}{n}) \cap M, \qquad V_{(y,n)} = B(y, \frac{1}{n}) \cap M.$$

By the assumption, integer $k_n > N$ and vector $x_n \in U_{(x,n)}$ exist, so that $T^{k_n}(M) \subseteq M$ and $T^{k_n} x_n \in V_{(y,n)}$. Thus the sequences $\{k_n\}$ and $\{x_n\}$ are proposed by induction, so that we have $x_n \longrightarrow x$ and $T^{k_n} x_n \longrightarrow y$. This completes the proof of nontrivial side. □

For an operator $T \in B(X)$, if equality $\mathbb{J}_M(x) = M$ is established for a vector $x \in M$, then x is a subspace $\mathbb{J}$-class vector for subspace M under operator T. We then call T a $\mathbb{J}_M$-class operator.
According to the earlier theorem which in fact defines a localized M-transitive operator, it is concluded that $T \in B(X)$ is an M-transitive operator if and only if for all $x \in M$, the set $\mathbb{J}_M(x)$ is equal to the subspace M.
As hypercyclicity criterion is a useful tool for identifying hypercyclic operators and provides a sufficient condition to ensure that an operator is hypercyclic, we also intend to introduce the local subspace transitivity criterion. Also, we will provide an example that satisfies this criterion. For useful reference of example and properties of hypercyclic, subspace hypercyclic and transitive criterions one can see [6].

Theorem 2.2 (Local Subspace Transitivity Criterion). *Assume that T is an operator on Banach space X and Y is a dense subset of M. If for vector $x \in M$, there exists a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$, which satisfies the following conditions;*

1. *There is a sequence $\{z_n\} \subset M$, such that $z_n \longrightarrow x$ and $T^{k_n} z_n \longrightarrow 0$,*
2. *For every $y \in Y$, there is a sequence $\{w_n\} \subset M$, such that $w_n \longrightarrow 0$ and $T^{k_n} w_n \longrightarrow y$,*
3. *M is an invariant subspace under T^{k_n}, for every $n \in \mathbb{N}$,*

then $\mathbb{J}_M(x) = M$.

Proof. Assume that $y \in M$ and integer $N \geq 1$; also, for sufficiently small $\varepsilon > 0$, $W_1' := B(0, \frac{\varepsilon}{2})$ is the relatively open ball in M. Now, we set

$$U_x := W_1' + x, \qquad U_y := W_1' + y.$$

By the assumptions, there is an integer $k_1 \geq N$, such that

$$T^{k_1}(U_x) \cap W_1' \neq \emptyset \quad and \quad T^{k_1}(W_1') \cap U_y \neq \emptyset.$$

There are vectors $w_1, w_2, w_3, w_4 \in W_1'$, such that

$$T^{k_1}(w_1 + x) = w_2, \qquad T^{k_1}(w_3) = w_4 + y.$$

Consequently,

$$T^{k_1}(x + w_1 + w_3) = w_2 + w_4 + y$$

and

$$x_1 := x + w_1 + w_3 \in B(x, \varepsilon), \qquad y_1 := y + w_2 + w_4 \in B(y, \varepsilon).$$

Similarly, we can construct the sequences $\{x_n\},\ \{y_n\} \subset M$ and $\{k_n\} \subset \mathbb{N}$, $k_n - 1 > k_{n-1}$, such that for each $n \geq 2$;

$$x_n \in B(x, \frac{\varepsilon}{n}), \qquad y_n \in B(y, \frac{\varepsilon}{n})$$

and $T^{k_n} x_n \longrightarrow y$ as $n \longrightarrow \infty$. The above contents along with (3) show $y \in \mathbb{J}_M(x)$ and, since the choice of vector $y \in M$ is arbitrary, equality $\mathbb{J}_M(x) = M$ is obtained, so x is a subspace $\mathbb{J}$-class vector for subspace M under T. $\square$

In the following, an operator will be raised such that it satisfies the local subspace transitivity criterion however it is not a topological transitive operator. In fact, the unilateral weighted backward shift T on $\ell^2(\mathbb{N})$ with the bounded weight sequence $\{w_n \geq 0\}_{n\in\mathbb{N}}$ is topological transitive if and only if $\limsup_n (\prod_{i=1}^{n} w_i) = +\infty$, [3].

Example 2.3. *Consider weighted backward shift operator T on $\ell^2(\mathbb{N})$ given by:*

$$T(x^1, x^2, \cdots) = (2x^2, \frac{3}{2}x^3, \frac{4}{3}x^4, \cdots).$$

Also let Y' be the set of finite sequences with entries $z \in \mathbb{C}$ that $Re(z) \in \mathbb{Q},\ Im(z) \in \mathbb{Q}$ and subspace M of $\ell^2(\mathbb{N})$ is considered as:

$$M = \{\{x_n\} \in \ell^2(\mathbb{N})\ ;\ x_{2k} = 0,\ for\ all\ k \in \mathbb{N}\}.$$

Since $Y := Y' \cap M$ is dense in M, so there are strictly increasing sequence $\{2k\}_k$, sequence $\{x_k\} \subset Y$ that

$$x_k = (x^1, 0, x^3, 0, \cdots, x^{2k-1}, 0, 0, \cdots),$$

$x_k \longrightarrow 0$ as $k \longrightarrow \infty$ and $T^{2k} x_k = 0$.
Now, for the random member $y = (y^1, 0, y^3, 0, \cdots, y^{2m+1}, 0, 0, \cdots) \in Y$ and $k \geq 1$, we set;

$$w_{2k}(y) = (\underbrace{0, \cdots, 0}_{2k-times},\ \frac{y^1}{2k+1}, 0\ , \frac{3y^3}{2k+3},\ 0, \cdots,\ \frac{(2m+1)y^{2m+1}}{2(k+m)+1},\ 0,\ 0, \cdots).$$

Clearly, for every $k \in \mathbb{N} \cup \{0\}$, $w_{2k}(y)$ belongs to Y and the sequence $\{w_{2k}(y)\}$ is a sequence in $\ell^2(\mathbb{N})$. Since

$$||w_{2k}(y)||^2 = \sum_{j=1}^{2m} |\frac{j}{2k+j} y^j|^2 \leq \frac{4m^2}{(2k+1)^2} ||y||^2,$$

so $w_2k(y) \longrightarrow 0$, as $k \longrightarrow \infty$. Note that for $n \geq 1$:

$$T^n(x^1, x^2, x^3, \cdots) = \left((n+1)x^{n+1},\ \frac{1}{2}(n+2)x^{n+2},\ \frac{1}{3}(n+3)x^{n+3},\ \cdots\right),$$

thus

$$T^{2k}w_{2k}(y) = \Bigg((2k+1)\frac{1}{2k+1}y^1, 0, (\frac{2k+3}{3})(\frac{3}{2k+3})y^3, 0, \cdots, \\ (\frac{2(k+m)+1}{(2m+1)})(\frac{2m+1}{2(k+m)+1})y^{2m+1}, 0, 0, \cdots\Bigg) = y.$$

Hence the condition (2) holds. For each $k \geq 1$, condition (3) clearly holds; therefore, the operator T satisfies the local subspace transitivity criterion with respect to subspace M and $\mathbb{J}_M(0) = M$.

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Hyers-Ulam stabilities for operator weighted backward shift on Reproducing Kernel Hilbert Spaces

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Abstract

In this paper, we introduce the concept of some operators $T_{\lambda,\varphi}$ and $T_{\lambda,\omega}$ on weighted Hardy spaces H^2_β. And in the following, we investigate the boundedness of those operators on H^2_β. Then we prove that the stability for some operators $T_{\lambda,\varphi}$ and $T_{\lambda,\omega}$ on H^2_β.

1. Introduction and Preliminaries

The stability problem of functional equations had been first raised by Ulam. In 1941, Hyers gave a first affirmative answer to the question of Ulam for Banach spaces. Several stability problems for various functional equations on C^*-algebras have been investigated in (see [2–4]). In recent years, the stability of many differential, integral, operatorial, functional equations have been extensively investigated (see[1, 6]).
M. Obloza seems to be the first author to investigate Ulam stability of differential equations. Later C. Alsina and R. Ger proved that for every differentiable mapping $f : I \to R$ satisfying

$$|f'(x) - f(x)| \leq \varepsilon \qquad x \in I.$$

where $\varepsilon > 0$, is a given number and I is an open interval of $\mathbb{R}$, there exists a differentiable function $g : I \to R$ with the property

$$g'(x) = g(x) \qquad |f(x) - g(x)| \leq 3\varepsilon, \qquad x \in I.$$

Let A, B be normed spaces and consider a mapping $T : A \to B$. The operator T is Ulam stable if there exists $K \geq 0$ such that for every $\varepsilon > 0$, $f \in A$ and $g \in B$ such that

$$\|Tf - g\| \leq \varepsilon$$

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there exists $f_0 \in A$ with the properties $Tf_0 = g$ and

$$\|f - f_0\| \leq K \cdot \varepsilon.$$

We call such K an Ulam constant for T.

Definition 1.1. [5] Given a set X, we will say that H is a reproducing kernel Hilbert space(RKHS) on X over $\mathbb{F}$, provided that:

1. H is a vector subspace of F(X, F),
2. H is endowed with an inner product, $< \quad , \quad >$ making it into a Hilbert space,
3. for every $y \in X$, the linear evaluation functional, E_y : H $\to \mathbb{F}$, defined by $E_y(f) = f(y)$, is bounded.

Definition 1.2. [5] The function k_y is called the reproducing kernel for the point y. The $2-$variable function defined by

$$K(x, y) = k_y(x)$$

is called the reproducing kernel for H.

2. The boundedness of the operators $T_{\lambda,\varphi}$ and $T_{\lambda,\omega}$ on weighted Hardy spaces

Throughout this section, let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic maps, M_φ be the multiplication operator on weighted Hardy spaces.

Definition 2.1. Γ is called a weighted backward shift, such that

$$\Gamma e_n = \begin{cases} \omega_n e_{n-1}, & \text{for } n > 0; \\ \\ 0, & \text{for n=0.} \end{cases}$$

with

$$\omega_n = \frac{n\beta_{n-1}}{\beta_n}.$$

Define the operators $T_{\lambda,\varphi}$ and $T_{\lambda,\omega}$ on H^2_β by

$$T_{\lambda,\varphi} f(z) = (M_\varphi f)(z) + \lambda f(z) \qquad (f \in H^2_\beta,\ z \in \mathbb{D} \quad and \quad \lambda \in \mathbb{C});$$

$$T_{\lambda,\omega}(f)(z) = \Gamma(f)(z) + \lambda(M_\varphi f)(z) \qquad (f \in H^2_\beta,\ z \in \mathbb{D} \quad and \quad \lambda \in \mathbb{C}).$$

Proposition 2.2. *Let $T_{\lambda,\varphi}$ be the operator on weighted hardy space H^2_β. Then $T_{\lambda,\varphi}$ is a bounded on H^2_β if and only if $\varphi \in L^\infty$*

Proof. Necessity. Suppose that $T_{\lambda,\varphi}$ is bounded on H^2_β. For each $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}
&\|T_{\lambda,\varphi} e_n\|_\beta = \|M_\varphi e_n + \lambda e_n\|_\beta \\
&= \|\varphi(z)\frac{z^n}{\beta_n} + \lambda\frac{z^n}{\beta_n}\|_\beta \\
&= \|(\varphi(z) + \lambda)\frac{z^n}{\beta_n}\|_\beta \\
&= |\varphi(z) + \lambda|\|\frac{z^n}{\beta_n}\| \\
&> |\varphi(z)| - |\lambda|
\end{aligned}$$

Hence $|\varphi(z)| - |\lambda| \leq \|T_{\lambda,\varphi} e_n\|_\beta < \infty$, then we have $|\varphi(z)| \leq M + |\lambda| \quad \forall z$.
Hence $\varphi \in L^\infty$
Sufficiency. We suppose $\varphi \in L^\infty$. We have

$$\begin{aligned}
&\|T_{\lambda,\varphi} e_n\|_\beta = \|M_\varphi e_n + \lambda e_n\|_\beta \\
&= \|\varphi(z)\frac{z^n}{\beta_n} + \lambda\frac{z^n}{\beta_n}\|_\beta \\
&= \|(\varphi(z) + \lambda)\frac{z^n}{\beta_n}\|_\beta \\
&= |\varphi(z) + \lambda|\| \\
&\leq \|\varphi\|_\infty + |\lambda|
\end{aligned}$$

So, $T_{\lambda,\varphi}$ is bounded on H^2_β. The proof is complete. □

Proposition 2.3. *Let $T_{\lambda,\omega}$ be the operator on weighted hardy space H^2_β. Then following statements are equivalent:*

1. *$T_{\lambda,\omega}$ is a bounded on H^2_β*
2. *The sequence $\left\{\frac{n\beta_{n-1}}{\beta_n}\right\}_{n=1}^{\infty}$ is bounded, where the weight $\beta = \{\beta_n\}$ and $\varphi \in L^\infty$*

Proof. $1 \Rightarrow 2$ Suppose that $T_{\lambda,\omega}$ is bounded on H^2_β. For $n \in \mathbb{Z}^+$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
&\|T_{\lambda,\omega} e_n\|_\beta = \|\Gamma e_n + \lambda M_\varphi e_n\|_\beta \\
&= \|\omega e_{n-1} + \lambda M_\varphi e_n\|_\beta \\
&= \|\frac{n\beta_{n-1}}{\beta} e_{n-1} + \lambda\varphi(z) e_n\|_\beta \\
&= \sqrt{\left(\frac{n\beta_{n-1}}{\beta}\right)^2 + |\lambda|^2(\varphi(z))^2}.
\end{aligned}$$

Hence $T_{\lambda,\omega}$ is bounded. So, we have

$$\left\{\frac{n\beta_{n-1}}{\beta_n}\right\}_{n=1}^{\infty} \leq T_{\lambda,\omega} < \infty,$$

and

$$\varphi(z) \leq T_{\lambda,\omega} < \infty \qquad \forall z \in \mathbb{D} \Rightarrow \varphi \in L^\infty.$$

$2 \Rightarrow 1$ We suppose that the sequence $\left\{\frac{n\beta_{n-1}}{\beta_n}\right\}_{n=1}^{\infty}$ is bounded. Let $S = sup\left\{\frac{n\beta_{n-1}}{\beta_n} : n \geq 1\right\}$, then $S < +\infty$. For every holomorphic polynomial $f = \Sigma_{n=0}^{l} < f, e_n > e_n \in H^2_\beta$, we get

$$\begin{aligned}
\Gamma f &= \sum_{n=1}^{l} \omega_{n+1} < f, e_{n+1} > e_{n+1} \\
&= \sum_{n=1}^{l} \frac{n\beta_n}{\beta_{n+1}} < f, e_n > \frac{z^{n+1}}{\beta_{n+1}} \\
&= \sum_{n=1}^{l} \frac{n\beta_n}{\beta_{n+1}} < f, e_n > e_{n+1} \\
&= \sum_{n=0}^{l-1} \frac{n\beta_{n-1}}{\beta_n} < f, e_n > e_n
\end{aligned}$$

It follows above equation, that

$$\|\Gamma f\|\beta = \| \sum_{n=0}^{l-1} \left(\frac{n\beta_{n-1}}{\beta_n}\right)^2 | < f, e_n > |^2 \|\beta \leq S^2 \|f\|_\beta^2$$

Given the above equation

$$\begin{aligned}\|T_{\lambda,\omega} f\|_\beta &= \|\Gamma f + \lambda M_\varphi f\|_\beta \leq \|\Gamma_n f\|_\beta + \|\lambda M_\varphi f\|_\beta \\ &\leq S\|f\|_\beta + |\varphi(z)| \|f\|_\beta \leq (S + \|\varphi\|_\infty)\|f\|_\beta\end{aligned}$$

Since the polynomials are dense in H^2_β, it follows that $T_{\lambda,\omega}$ is bounded on H^2_β. The proof is complete. □

3. The Hyers- Ulam Stability of weighted backward shift operator

Theorem 3.1. *Let Γ_{e_n} be a weighted backward shift operator. Then following statements are equivalent:*

1. *Γ_{e_n} is Hyers-Ulam stable*
2. *The sequence ω_n is bounded.*

Now, we investigate the Ulam-Hyers stability for operator weighted m-backward shift.

Corollary 3.2. *Consider the operator Γ_{e_n} for m (briefly m-backward operator). In this case, m-backward operator is Hyers-Ulam stable iff $\omega_{n-m} = \left\{\frac{(n-m)!\beta_n}{n!\beta_{n-m}}\right\}_{n=1}^{\infty}$ is bounded.*

Example 3.3. *Let $\beta_0 = \beta_1 = 1$ in H^2_β. If β_{i-1} has been defined and i satisfies $2^{2k} \leq i \leq 2^{2k+1}$ for some integer K and let $\beta_i = \frac{\beta_{i-1}}{2}$ and since $\beta_1 \leq 1$ for all $i \geq 0$. Therefore for 2^{2k}, m-backward shift operator the best Hyers-Ulam stability constant 1.*

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A partial answer to a known conjecture

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Abstract

In this work, we introduce a new generalized class of contractive multi valued mappings which is a generalization of many known results such as Banach (1922). Nadlcr (1969). C'iric(1974). Reich (1983). Mizoguclii and Takahashi (1989). Daffer-Kaneko (1995), Rhoades (21)01), Rouhani and Moradi (2010). Amini-Harandi (2010) and Moradi and Khojasteh (2011). Finally, a partial answer to the con-jecture which is introduced by Rouhani and Moradi is given.

1. Introduction

In 1922, Banach established the most famous fundamental fixed point theorem (so- called the Banach contraction principle) which has played an important role in various fields of applied mathematical analysis. It is known that the Ba-nach contraction principle has been extended and generalized in many various different directions by several authors(see [3]-[5]). An interesting direction of research is the extension of the Banach contraction principle to multi valued maps, known as Nadlcr's fixed point theorem [8], Mizoguchi-Takahashi's fixed point theorem [6]. M. Berinde and V. Berinde [1], ciric, Reich [2], Daffer and Kaneko [3]. Rhoades [9], Rouhani and Moradi [10]. Amini-Harandi. [5], Moradi and Khojasteh [7], Du [4] and references therein.

Theorem 1.1. *([6], Mizoguchi and Takahashi) Let {X. d) be a complete metric space,* $\phi\colon [0,\infty) \to [0,1)$*be a MT-function and* $T : X \to CB(X)$ *be a multi valued map.*

$$H(Tx,Ty) \leq \phi(d(x,y)).d(x,y),$$

For all $x, y \in X$*.Then* $F(T) \neq \varphi$*.*

A mapping $T\colon X \to X$is said to be a weak contraction if there exists $0 \leq \alpha < 1$such that

$$d(Tx,Ty) \leq \alpha M(x,y),$$

for all $x, y \in X$,where

$$M\left(x,y\right) : = \max\left\{d\left(x,y\right).d\left(x,Tx\right),d\left(y,Ty\right),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

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Two multi valued mapping $T\colon X \to CB(X)$are called generalized weak contraction if there exists $0 \le \alpha < 1$ such that

$$M\left(x,y\right):\ =\max\left\{d\left(x,y\right).d\left(x,Tx\right),d\left(y,Ty\right),\frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

Two multi valued mapping $T\colon X \to CB(X)$are called generalized weak contraction if there exists $0 \le \alpha < 1$ such that

Also two mapping $T,S\colon X \to CB(X)$are called generalized weak contractive if there exists a map $\phi\colon [0,+\infty) \to [0,+\infty)$ with $\phi(0)=0$ and $\phi(t)>0$ for all t > 0 such that

$$H(Tx,Sy) \le M_{T,S}(x,y) - \phi(M_{T,S}(x,y)), \forall x,y \in X$$

2. Main results

Definition 2.1. A function $\vartheta\colon IR \times IR \to IR$is called GMT function if the following conditions hold:

$$1) \circ < \vartheta(t,s) < 1 \forall S,t > \circ$$

2) For any bounded sequence $\{t_n\} \subset (\circ,+\infty)$and any no increasing sequence $\{S_n\} \subset (\circ,+\infty)$, it holds $\lim_{n\to\infty} \sup\, \vartheta\left(t_n,S_n\right) < 1$.

We denote the set of all GMT function by GMT (R).

Example 2.2. *Let $\vartheta\colon [\circ,1)$ be an MT-function then $\vartheta(t,s)=\vartheta(s)$ is a GMT- function.*

Theorem 2.3. *Let (X,d) be a complete metric space and. let $T,S\colon X \to CB(X)$ and there exists $\vartheta \in GMT(R)$such that*

$$H(Tx,Sy) \le \vartheta(H(Tx,Sy),M_{T,S}(x,y)M_{T,S}(x,y) \tag{2.1}$$

for each $x,y \in X$.Then T, S has a common fixed point.

Example 2.4. *Suppose that $X=[0,1]\cup\{4\}$and let $T,S\colon X \to CB(X)$ be defined as follows:*
$Tx=\left[0,\frac{x}{4}\right]$ *and* $Sy=\left\{\frac{y}{4}\right\}$
Assume

$$\phi\left(t\right)=\begin{cases} t/2 t\in[0,1], \\ 3t/4 t>1,\end{cases}$$

and $\vartheta\left(t,s\right)=1-\frac{\phi(s)}{s}$ for all $t,s>0$. For any bounded sequence $\{t_n\}\subset(0,+\infty)$ and any non-increasing sequence $\{s_n\}\subset(0,+\infty)$, it holds

$$\limsup_{n\to\infty}\vartheta\left(t_n,s_n\right)=\limsup_{n\to\infty}\left(1-\frac{\phi(s_n)}{s_n}\right)<1$$

First suppose $0\le x\le 1$ and $0\le y\le 1$then

$$H\left(Tx,Sy\right)=\max\left\{|\frac{y}{4}-\frac{x}{4}|,\frac{y}{4}\right\}$$

$$\le\frac{1}{2}\max\left\{|y-x|,|y-\frac{y}{4}|\right\}$$

$$\le\frac{1}{2}M_{T,S}\left(x,y\right)$$

$$=\vartheta\left(H\left(Tx,Sy\right),M_{T,S}\left(x,y\right)\right)M_{T,S}\left(x,y\right).$$

In the other case, suppose $x=4$ and $0\le y\le 1$, then

$$H\left(Tx, Sy\right) = \max\left\{|\frac{y}{4} - 1|, \frac{y}{4}\right\}$$
$$\leq \frac{1}{4}\max\left\{|4 - y, y\right\}$$
$$\leq \frac{1}{2}M_{T,S}\left(x, y\right)$$
$$= \vartheta\left(H\left(Tx, Sy\right), M_{T,S}\left(x, y\right)\right)M_{T,S}\left(x, y\right).$$

Other cases are easily verified as the above arguments. Henceforth, T is WGMT- contraction and enjoys all conditions of Theorem 2.1. Also, T.S has a common fixed point {0}.

Problem (A):
Let (X. d) be a complete metric space and let $T, S\colon X \to CB(X)$be two mappings such that for all $x, y \in X$,

$$H(Tx, Sy) \leq M_{T,S}(x, y) - (M_{T,S}(x, y)), \forall x, y \in X$$

(i.e. generalize ϕ-weak contraction) where $\phi\colon [0, +\infty) \to [0, +\infty)$ is l.s.c. with $\phi(0) = 0, \phi(t) < t$and $\phi(t) > 0$, for all $t > 0$. Then does T and S have a common fixed point?
In the following theorem a partial solution to Problem (A) is given as an application of 'Theorem 2.1.

Corollary 2.5. *Let (X, d) be a complete metric space and let $T, S\colon X \to CB(X)$ be two mappings such that for all $x; y \in X$,*

$$H(Tx, Sy) \leq M_{T,S}(x, y) - \phi(M_{T,S}(x, y)), \forall x, y \in X$$

(i.e. generalize ϕ-weak contraction) where $\phi\colon [0, +\infty) \to [0, +\infty)$, with $\phi(0) = 0$, $\phi(t) < t$ and $\phi \in W_{lsc}(R)$. Then there exists a unique point $x \in X$ such that Fix $(T, S) \neq \emptyset$.

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General Jensen s-Functional on Banach Algebras

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Abstract

In this paper, we introduce the concept of Jensen s-functional equation on Banach algebras. Then we prove Hyers-Ulam stability and hyperstability of Jensen s-functional by using fixed point theorem on Banach algebras.

1. Introduction and Preliminaries

A classical question in the sense of functional equation says that "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation? " Ulam raised the stability of functional equations and Hyers [3] in 1941 was the first one which gave an affirmative answer to the question of Ulam for additive mapping between Banach spaces. In 1978, Th. M. Rassias [8] by replacing control function Hyers theorem from ε to $\theta(\|a\|^r + \|b\|^r)$, introduced new concept of stability. A generalization of the theorem of Rassias was obtained by Găvruta [2] by replacing a general control function $\varphi : X \times X \longrightarrow [0, \infty)$. In 1996, Isac and Rassias were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (for example see [1, 4, 5]).
Park in 2015, [6, 7] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. And in the following we defined general Jensen s-functional equation on Banach algebras.

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Throughout this paper , Let $\mathfrak{A}$ be a Banach algebra.
Consider the generalized Jensen s-functional equation

$$3f\Big(\frac{a+b+c}{3}\Big)-f(a)-f(b)-f(c)=s\Big(f(a+b+c)+f(a)-f(a+b)-f(a+c)\Big) \tag{1}$$

where $s\neq 0,\pm 1$ is a real number.
In this paper, we solve above equation and show that a function which satisfies its is additive. We also prove its Hyers-Ulam stability by using B. Margolis and J. B. Diaz theorem on Banach algebras.

Theorem 1.1. *Let (A,d) be a complete generalized metric space and let $F:A\to A$ be a strictly contractive mapping with Lipschitz constant $\beta<1$. Then for each given element $a\in A$, either*

$$d(F^i a, F^{i+1}a)=\infty$$

for all nonnegative integers i or there exists a positive integer i_0 such that
(1) $d(F^i a, F^{i+1}a)<\infty, \qquad \forall i\geq i_0$;
(2) *the sequence $\{F^i a\}$ converges to a fixed point b^* of F;*
(3) *b^* is the unique fixed point of F in the set $B=\{b\in A \mid d(F^{i_0}a,b)<\infty\}$;*
(4) $d(b,b^*)\leq \frac{1}{1-\beta}d(b,Fb)$ *for all $b\in B$.*

2. General Jensen s-Functional

Throughout the section, let s is real number and $s\neq 0,\pm 1$.
To prove the main theorems, we need the following proposition. Firstly, in the next proposition, we prove that f is a additive mapping.

Proposition 2.1. *If a mapping $f:\mathfrak{A}\to\mathfrak{A}$ satisfies*

$$3f\Big(\frac{a+b+c}{3}\Big)-f(a)-f(b)-f(c)=s\Big(f(a+b+c)+f(a)-f(a+b)-f(a+c)\Big) \tag{2}$$

for all $a,b,c\in\mathfrak{A}$. Then the mapping $f:\mathfrak{A}\to\mathfrak{A}$ is additive.

Proof. Assume that $f:\mathfrak{A}\to\mathfrak{A}$ satisfies in (2).
Let $a=b=c=0$ in (2), we have $f(0)=0$.
Now, by putting $b=c=0$ in (2), we have

$$3f(a)=f(3a). \tag{3}$$

By using (3) and putting $a=b=c$ in (2), we have

$$2f(a)=f(2a). \tag{4}$$

Again putting $b=-a$ and $c=0$, we get

$$f(-a)=-f(a).$$

Finally, replace $c=0$ in (2) and using (3), we have

$$f(a+b)=f(a)+f(b).$$

Thus $f:\mathfrak{A}\to\mathfrak{A}$ is additive. □

Suppose that δ be mapping from $\mathfrak{A}^3$ into $[0,\infty)$, for all $a,b,c\in\mathfrak{A}$ such that

$$\delta(\frac{a}{3},\frac{b}{3},\frac{c}{3})\leq\frac{\beta}{3}\delta(a,b,c) \tag{5}$$

for some constant $0 < \beta < 1$, and if we take $a = b = c = 0$, then $\delta(0,0,0) = 0$.
It follows (5) that

$$\lim_{i\to\infty} 3^i\delta(\frac{a}{3^i}, \frac{b}{3^i}, \frac{c}{3^i}) = 0 \tag{6}$$

for all $a, b, c \in \mathfrak{A}$.
In the following, we prove Hyers-Ulam stability of generalized Jensen s-functional equation on Banach algebras.

Theorem 2.2. *Let $f : \mathfrak{A} \to \mathfrak{A}$ be a mapping for which there exist functions $\delta : \mathfrak{A}^3 \to [0,\infty)$ satisfying (5) and*

$$\Big\|3f\Big(\frac{a+b+c}{3}\Big) - f(a) - f(b) - f(c) - s\Big(f(a+b+c) + f(a) - f(a+b) - f(a+c)\Big)\Big\| \leq \delta(a,b,c), \tag{7}$$

for all $a, b, c \in \mathfrak{A}$. Then there exist a unique additive mapping $J : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|f(a) - J(a)\| \leq \frac{\beta}{(1-\beta)}\delta(a,0,0), \tag{8}$$

for all $a \in \mathfrak{A}$.

Theorem 2.2 generalized the result of Rassias, whenever we define

$$\delta(a,b,c) := \theta\Big(\|a\|^r + \|b\|^r + \|c\|^r\Big),$$

for all $\theta \in \mathbb{R}^+$ and $r \neq 1$.

Corollary 2.3. *Let $r \neq 1$ and θ be nonegative real numbers and $f : \mathfrak{A} \to \mathfrak{A}$ be a mapping satisfying $f(0) = 0$ and*

$$\Big\|3f\Big(\frac{a+b+c}{3}\Big) - f(a) - f(b) - f(c) - s\Big(f(a+b+c) + f(a) - f(a+b) - f(b+c)\Big)\Big\| \leq \theta(\|a\|^r + \|b\|^r + \|c\|^r)$$

for all $a, b, c \in \mathfrak{A}$. Then there exists a unique additive mapping $J : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|f(a) - J(a)\| \leq \frac{2\theta}{2^r - 2}\|a\|^r, \qquad \text{for} \quad r > 1$$

$$\|f(a) - J(a)\| \leq \frac{2\theta}{2 - 2^r}\|a\|^r, \qquad \text{for} \quad r < 1$$

for all $a \in \mathfrak{A}$.

In next Theorem, we investigate hyperstability of functional equation (1), by Găvruta's control function.

Theorem 2.4. *Suppose there exist function $\delta : \mathfrak{A}^3 \to [0,\infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{3^i}\delta(0, 3^i b, 3^i c) = 0, \tag{9}$$

for all $b, c \in \mathfrak{A}$. Moreover, suppose that $f : \mathfrak{A} \to \mathfrak{A}$ is mapping such that

$$\Big\|3f\Big(\frac{a+b+c}{3}\Big) - f(a) - f(b) - f(c) - s\Big(f(a+b+c) + f(a) - f(a+b) - f(a+c)\Big)\Big\| \leq \delta(0,b,c), \tag{10}$$

for all $a, b, c \in \mathfrak{A}$, then f is an additive equation.

Now, in the following corollary we prove hyperstability of (1), by Rassias's control function.

Corollary 2.5. *Let $\theta, r \in \mathbb{R}^+$ with $r > 1$ and $f : \mathfrak{A} \to \mathfrak{A}$ is mapping such that*

$$\Big\|3f\Big(\frac{a+b+c}{3}\Big) - f(a) - f(b) - f(c) - s\Big(f(a+b+c) + f(a) - f(a+b) - f(a+c)\Big)\Big\| \leq \theta(\|b\|^r + \|c\|^r)$$

for all $b, c \in \mathfrak{A}$, then f is an additive equation.

Conclusions

In this paper, the authors introduced generalized Jensen s-functional equation. And by Using a fixed point theorem with Găvruta's control function it is proved that generalized Jensen s-functional equation on Banach algebras can be stable and hyperstable.

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Fixed point results for extended b-metric spaces

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Abstract

In this paper, we obtain fixed point results under some rational contractive conditions for extended b-metric spaces and some examples are provided here to illustrate the usability of the obtained results.

1. Introduction

Recall (see [3]) that a b-metric b on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$b(x,z) \leq s[b(x,y) + b(y,z)], \quad x,y,z \in X, \tag{1}$$

for some fixed $s \geq 1$. Motivated by this, the following further generlization has been recently presented by Parvaneh and Ghoncheh.

Definition 1.1. [6] Let X be a (nonempty) set. A function $d : X \times X \to [0,\infty)$ is an extended b-metric (p-metric, for short) if there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for all $t \geq 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$ such that for all $x,y,z \in X$, the following conditions hold:

(p_1) $d(x,y) = 0$ iff $x = y$,

(p_2) $d(x,y) = d(y,x)$,

(p_3) $d(x,z) \leq \Omega(d(x,y) + d(y,z))$.

In this case, the pair (X,d) is called a p-metric space, or an extended b-metric space.

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It should be noted that each b-metric is a p-metric, with $\Omega(t) = st$ for some $s \geq 1$, while each metric is a p-metric, with $\Omega(t) = t$. More general examples of p-metrics can be constructed using the following easy proposition.

Proposition 1.2. *Let (X, b) be a b-metric space with coefficient $s \geq 1$ and let $d(x, y) = \xi(b(x, y))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then, d is a p-metric with $\Omega(t) = \xi(st)$.*

Taking various functions ξ in the previous proposition, we can obtain a lot of examples of p-metrics. We state just a few of them which will be used later in the text.

Example 1.3.
1. *If $\xi(t) = e^t - 1$, we get $d(x, y) = e^{b(x,y)} - 1$ and $\Omega(t) = e^{st} - 1$. Note that $\Omega^{-1}(t) = \frac{1}{s}\ln(1+t)$.*
2. *If $\xi(t) = \sinh t$, we get $d(x, y) = \sinh b(x, y)$ and $\Omega(t) = \sinh(st)$. Note that $\Omega^{-1}(t) = \frac{1}{s}\sinh^{-1} t$.*
3. *If $\xi(t) = te^t$, then $d(x, y) = b(x, y)e^{b(x,y)}$ and $\Omega(t) = ste^{st}$. Note that in this case $\Omega^{-1}(t) = \frac{1}{s}W(t)$, for $t \geq 0$, where W is the Lambert W-function (see, e.g., [1]).*
4. *If $\xi(t) = t\cosh t$, then $d(x, y) = b(x, y)\cosh b(x, y)$ and $\Omega(t) = st\cosh(st)$, for $t \geq 0$.*

Note that such functions ξ and Ω generate p-metric spaces which are usually not b-metric spaces. For instance, in the case (2) of the previous example (if $b(x, y) = (x - y)^2$), it was shown in [1] that there is no s such that d is a b-metric with parameter s.

Definition 1.4. [6] Let (X, d) be a p-metric space. Then a sequence $\{x_n\}$ in X is called:
(a) p-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$;
(b) p-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(c) The p-metric space (X, d) is p-complete if every p-Cauchy sequence in X p-converges.

We will need the following simple lemma about the p-convergent sequences.

Lemma 1.5. *[6] Let (X, d) be a p-metric space with the function Ω, and suppose that $\{x_n\}$ and $\{y_n\}$ p-converge to x, y, respectively. Then, we have*

$$(\Omega^2)^{-1}(d(x, y)) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq \Omega^2(d(x, y)).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$$\Omega^{-1}(d(x, z)) \leq \liminf_{n\longrightarrow\infty} d(x_n, z) \leq \limsup_{n\longrightarrow\infty} d(x_n, z) \leq \Omega(d(x, z)).$$

2. Main results

In the rest of the paper, (X, d) will always be a p-metric space with function Ω.

2.1. Fixed point results via generalized Geraghty functions

In the rest of the paper, (X, d) will always be a p-metric space with function Ω.

2.2. Fixed point results via generalized Geraghty functions

Let $\mathcal{B}_\Omega$ denote the class of all functions $\beta : [0, \infty) \to [0, \Omega^{-1}(1))$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \Omega^{-1}(1) \text{ implies that } t_n \to 0, \text{ as } n \to \infty.$$

Definition 2.1. Let $(X, d, \preceq)$ be a partially ordered p-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type I if there exists $\beta \in \mathcal{B}_\Omega$ such that

$$\Omega(d(fx, fy)) \leq \beta(M(x, y))M(x, y) \tag{2}$$

holds for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)d(y, fy)}{1 + d(fx, fy)}\right\}.$$

A partially ordered p-metric space $(X, d, \preceq)$ is said to have the sequential limit comparison property (s.l.c. property) if for every nondecreasing sequence $\{x_n\}$ in X, the convergence of $\{x_n\}$ to some $x \in X$ yields that $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2.2. *Let $(X, d, \preceq)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f is a rational Geraghty contraction of type I. If*
(I) f is continuous, or
(II) $(X, d, \preceq)$ has the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Definition 2.3. Let $(X, d, \preceq)$ be a partially ordered p-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type II if there exists $\beta \in \mathcal{B}_\Omega$ such that

$$\Omega(d(fx, fy)) \leq \beta(M(x, y))M(x, y)$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned}&M(x,y)\\&= \max\left\{d(x,y), \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+\Omega[d(x,fx)+d(y,fy)]}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\right\}.\end{aligned}$$

Theorem 2.4. *Let $(X, d, \preceq)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f is a rational Geraghty contractive mapping of type II. If*
(I) f is continuous, or
(II) $(X, d, \preceq)$ has the s.l.c. property.
Then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Definition 2.5. Let $(X, d, \preceq)$ be a partially ordered p-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type III if there exists $\beta \in \mathcal{B}_\Omega$ such that

$$\Omega(d(fx, fy)) \leq \beta(M(x, y))M(x, y) \tag{3}$$

for all $x, y \in X$ with $x \preceq y$, where

$$\begin{aligned}M(x,y) = \max\Bigg\{&d(x,y), \frac{d(x,fx)d(y,fy)}{1+\Omega[d(x,y)+d(x,fy)+d(y,fx)]},\\&\frac{d(x,fy)d(x,y)}{1+\Omega(d(x,fx))+\Omega^3[d(y,fx)+d(y,fy)]}\Bigg\}.\end{aligned}$$

Theorem 2.6. *Let $(X, d, \preceq)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f is a rational Geraghty contractive mapping of type III. If*
(I) f is continuous, or
(II) $(X, d, \preceq)$ has the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Corollary 2.7. *Let $(X, b, \preceq)$ be a partially ordered b-complete b-metric space with parameter $s \geq 1$, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$sb(fx, fy)e^{b(fx,fy)}e^{sb(fx,fy)e^{b(fx,fy)}} \leq rM(x, y)$$

for all $x, y \in X$ with $x \preceq y$, where $0 \leq r < \Omega^{-1}(1) = \frac{1}{s}W(1)$, $\Omega(t) = ste^{st}$, and

$$M(x,y) = \max\Big\{b(x,y)e^{b(x,y)}, \frac{b(x,fx)e^{b(x,fx)}b(y,fy)e^{b(y,fy)}}{1+b(x,y)e^{b(x,y)}}, \frac{b(x,fx)e^{b(x,fx)}b(y,fy)e^{b(y,fy)}}{1+b(fx,fy)e^{b(fx,fy)}}\Big\},$$

or

$$M(x,y) = \max\Big\{b(x,y)e^{b(x,y)}, \frac{b(x,fx)e^{b(x,fx)}b(x,fy)e^{b(x,fy)} + b(y,fy)e^{b(y,fy)}b(y,fx)e^{b(y,fx)}}{1+s[b(x,fx)e^{b(x,fx)}+b(y,fy)e^{b(y,fy)}]e^{s[b(x,fx)e^{b(x,fx)}+b(y,fy)e^{b(y,fy)}]}}, \frac{b(x,fx)e^{b(x,fx)}b(x,fy)e^{b(x,fy)} + b(y,fy)e^{b(y,fy)}b(y,fx)e^{b(y,fx)}}{1+b(x,fy)e^{b(x,fy)}+b(y,fx)e^{b(y,fx)}}\Big\}.$$

If f is continuous, or $(X, b, \preceq)$ has the s.l.c. property, then f has a fixed point.

Example 2.8. *Let $X = [0, 1.2)$ be equipped with the p-metric $d(x, y) = |x-y|^2$ for all $x, y \in X$, where $\Omega(t) = e^t-1$, with $\Omega^{-1}(t) = \ln(1 + t)$ (Example 1.3.(1)).*
Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leq x$, a mapping $f : X \to X$ by

$$f(x) = (\frac{1}{4})\ln(x^2 + 1)$$

and a function $\beta \in \mathcal{B}_\Omega$ by $\beta(t) = \frac{1}{2} < 0.69 \approx \Omega^{-1}(1)$. For all $x, y \in X$ with $x \preceq y$, we have:

$$\begin{aligned} d(fx, fy) &= |\frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(y^2+1)|^2 \\ &\leq \frac{1}{4}|x-y|^2 \\ &= \frac{1}{4}d(x,y), \end{aligned}$$

therfor

$$\begin{aligned} \Omega d(fx, fy) &= (e^{|\frac{1}{4}\ln(x^2+1)-\frac{1}{4}\ln(y^2+1)|^2} - 1) \\ &\leq \frac{1}{4}(e^{|x-y|^2} - 1) \\ &= \frac{1}{4}(e^{d(x,y)} - 1) \leq \frac{1}{2}d(x,y) = \beta(d(x,y))d(x,y), \end{aligned}$$

So, from Theorem 2.2, f has a fixed point.

Example 2.9. *Let $X = \{0, 1, 2, 3\}$ be equipped with the p-metric generated from the standard metric $m(x, y) = |x-y|$ as in Corollary 2.7, i.e. take $\Omega(t) = e^t - 1$. Define a partial order on X by*

$$\preceq = \{(0,0), (1,1), (2,2), (3,3), (1,0), (3,1), (3,0), (2,0)\},$$

and consider the mapping $f : X \to X$ given by

$$f : \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then, X is a partially ordered p-complete p-metric space and f is a nondecreasing mapping. Take the function $\beta \in \mathcal{B}_\Omega$ given by

$$\beta(t) = \begin{cases} \ln 2 \cdot e^{-0.04\,t}, & t > 0, \\ \delta, & t = 0, \end{cases}$$

where $0 < \delta < \ln 2$ *(i.e., take* $\gamma = 0.04$ *in Corollary 2.7). We will check the condition* (**??**) *of this corollary. Considering elements* $x, y \in X$ *with* $x \preceq y$*, the following cases are nontrivial.*
(1) $x = 3$, $y = 1$. *Then we have* $d(fx, fy) = e - 1$, $M(x, y) \geq d(x, y) = e^2 - 1 = 6.35$ *and*

$$\begin{aligned}\Omega(d(fx, fy)) &= e^{d(fx,fy)} - 1 = e^{e-1} - 1 = 3.35 < 3.41 = \ln 2 \cdot 6.35\, e^{-0.04 \cdot 6.35} \\ &= \beta(d(x, y)) \cdot d(x, y) \leq \beta(M(x, y)) \cdot M(x, y)\end{aligned}$$

(note that $t\beta(t)$ *is an increasing function for* $0 < t < 25 = \gamma^{-1}$*).*
(2) $x = 3$, $y = 0$. *Then we have* $d(fx, fy) = e - 1$, $M(x, y) \geq d(x, y) = e^3 - 1 \approx 20$ *and*

$$\begin{aligned}\Omega(d(fx, fy)) &= e^{e-1} - 1 = 3.35 < 6.2 = \ln 2 \cdot 20\, e^{-0.04 \cdot 20} \\ &= \beta(d(x, y)) \cdot d(x, y) \leq \beta(M(x, y)) \cdot M(x, y).\end{aligned}$$

(3) $x = 2$, $y = 0$. *Then we have* $d(fx, fy) = e - 1$, $M(x, y) \geq d(x, y) = e^2 - 1 = 6.35$ *and*

$$\begin{aligned}\Omega(d(fx, fy)) &= e^{e-1} - 1 = 3.35 < 3.41 = \ln 2 \cdot 6.35\, e^{-0.04 \cdot 6.35} \\ &= \beta(d(x, y)) \cdot d(x, y) \leq \beta(M(x, y)) \cdot M(x, y).\end{aligned}$$

Hence, all the conditions of Corollary 2.7 are fulfilled and the mapping f *has a (unique) fixed point (which is* $u = 0$*). Note that the same conclusion could not be obtained if the space without partial order were used. Indeed, in this case we should have also the following case to consider.*
(4) $x = 2$, $y = 1$. *Then* $d(fx, fy) = d(x, y) = e - 1$ *and*

$$M(x, y) = \max \left\{ e - 1, \frac{(e-1)^2}{e}, \frac{(e-1)^2}{e} \right\} = e - 1,$$

and it is trivial to check that the condition (**??**) *does not hold. In fact, none of the known fixed point results can be used to obtain the conclusion in this case.*

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Fixed point theorems for generalized b-metric spaces and application

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Abstract

In this paper, we obtain fixed point theorems under some comparison function and application.

1. Introduction

Recall (see [1]) that a b-metric b on a set X is a generalization of standard metric, where the triangular inequality is replaced by

$$b(x,z) \leq s[b(x,y)+b(y,z)], \quad x,y,z \in X, \tag{1}$$

for some fixed $s \geq 1$. Motivated by this, the following further generlization has been recently presented by Parvaneh and Ghoncheh.

Definition 1.1. [7] Let X be a (nonempty) set. A function $d : X \times X \to [0,\infty)$ is an extended b-metric (p-metric, for short) if there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $\Omega^{-1}(t) \leq t \leq \Omega(t)$ for all $t \geq 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$ such that for all $x,y,z \in X$, the following conditions hold:

(p_1) $d(x,y) = 0$ iff $x = y$,

(p_2) $d(x,y) = d(y,x)$,

(p_3) $d(x,z) \leq \Omega(d(x,y)+d(y,z))$.

In this case, the pair (X,d) is called a p-metric space, or an extended b-metric space.

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It should be noted that each b-metric is a p-metric, with $\Omega(t) = st$ for some $s \geq 1$, while each metric is a p-metric, with $\Omega(t) = t$. More general examples of p-metrics can be constructed using the following easy proposition.

Proposition 1.2. *Let (X, b) be a b-metric space with coefficient $s \geq 1$ and let $d(x, y) = \xi(b(x, y))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing continuous function with $t \leq \xi(t)$ for $t \geq 0$ and $\xi(0) = 0$. Then, d is a p-metric with $\Omega(t) = \xi(st)$.*

Taking various functions ξ in the previous proposition, we can obtain a lot of examples of p-metrics. We state just a few of them which will be used later in the text

Example 1.3.
1. *If $\xi(t) = e^t - 1$, we get $d(x, y) = e^{b(x,y)} - 1$ and $\Omega(t) = e^{st} - 1$. Note that $\Omega^{-1}(t) = \frac{1}{s}\ln(1+t)$.*
2. *If $\xi(t) = \sinh t$, we get $d(x, y) = \sinh b(x, y)$ and $\Omega(t) = \sinh(st)$. Note that $\Omega^{-1}(t) = \frac{1}{s}\sinh^{-1} t$.*
3. *If $\xi(t) = te^t$, then $d(x, y) = b(x, y)e^{b(x,y)}$ and $\Omega(t) = ste^{st}$. Note that in this case $\Omega^{-1}(t) = \frac{1}{s}W(t)$, for $t \geq 0$, where W is the Lambert W-function (see, e.g., [**?**]).*
4. *If $\xi(t) = t\cosh t$, then $d(x, y) = b(x, y)\cosh b(x, y)$ and $\Omega(t) = st\cosh(st)$, for $t \geq 0$.*

Note that such functions ξ and Ω generate p-metric spaces which are usually not b-metric spaces. For instance, in the case (2) of the previous example (if $b(x, y) = (x - y)^2$), it was shown in [**?**] that there is no s such that d is a b-metric with parameter s.

Definition 1.4. [7] Let (X, d) be a p-metric space. Then a sequence $\{x_n\}$ in X is called:
(a) p-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$;
(b) p-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(c) The p-metric space (X, d) is p-complete if every p-Cauchy sequence in X p-converges.

We will need the following simple lemma about the p-convergent sequences.

Lemma 1.5. *[7] Let (X, d) be a p-metric space with the function Ω, and suppose that $\{x_n\}$ and $\{y_n\}$ p-converge to x, y, respectively. Then, we have*

$$(\Omega^2)^{-1}(d(x, y)) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq \Omega^2(d(x, y)).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$ we have

$$\Omega^{-1}(d(x, z)) \leq \liminf_{n\longrightarrow\infty} d(x_n, z) \leq \limsup_{n\longrightarrow\infty} d(x_n, z) \leq \Omega(d(x, z)).$$

2. Main results

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered p-complete p-metric space. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f is a rational Geraghty contraction of type I. If*
(I) f is continuous, or
(II) $(X, d, \preceq)$ has the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Lemma 2.2. *If $\psi \in \Psi$, then the following are satisfied:*
(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Theorem 2.3. *Let $(X, d, \preceq)$ be a partially ordered p-complete p-metric space, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$\Omega[d(fx, fy)] \leq \psi(M(x, y)) \tag{2}$$

for some $\psi \in \Psi$ and for all $x, y \in X$ with $x \preceq y$, where

$$M(x,y) = \max\Big\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\Big\}.$$

If f is continuous, or $(X, d, \preceq)$ has the s.l.c. property, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Corollary 2.4. *Let $(X, b, \preceq)$ be a partially ordered b-complete b-metric space with parameter s, and let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$s \cdot b(fx,fy)\cosh(b(fx,fy))\cosh(s \cdot b(fx,fy)\cosh(b(fx,fy))) \leq \psi(M(x,y)),$$

where

$$M(x,y) = \max\Big\{b(x,y)\cosh(b(x,y)), \frac{b(x,fx)\cosh(b(x,fx))b(y,fy)\cosh(b(y,fy))}{1+b(x,y)\cosh(b(x,y))},$$
$$\frac{b(x,fx)\cosh(b(x,fx))b(y,fy)\cosh(b(y,fy))}{1+b(fx,fy)\cosh(b(fx,fy))}\Big\}$$

for some $\psi \in \Psi$ and all $x, y \in X$ with $x \preceq y$. If f is continuous, or $(X, b, \preceq)$ has the s.l.c. property, then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Remark 2.5. *As any b-metric is a p-metric with $\Omega(t) = st$ for all $t \in [0, \infty)$, so our results modify the obtained results in [**?**] and several other articles.*

3. Examples

Example 3.1. *Let $X = [0, 2.5]$ be also equipped with the p-metric $d(x,y) = e^{|x-y|} - 1$ for all $x, y \in X$, where $\Omega(t) = e^t - 1$. Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leq x$, a mapping $f : X \to X$ by*

$$fx = \frac{x}{2}e^{-x}$$

and a function $\psi \in \Psi$ by $\psi(t) = e^{\frac{1}{2}t} - 1$. It is easy to see that $f(X) = [0, 0.358] \subseteq X$ and $\psi(t) < t$ for all $t \in X$. For all $x, y \in X$ with $x \preceq y$, by Mean Value Theorem, we have

$$\Omega[d(fx,fy)] = e^{e^{\frac{1}{2}|xe^{-x}-ye^{-y}|}-1} - 1 \leq e^{e^{\frac{1}{2}(|x-y|)}-1} - 1$$
$$\leq e^{\frac{1}{2}(e^{|x-y|}-1)} - 1 = e^{\frac{1}{2}d(x,y)} - 1 = \psi(d(x,y)),$$

So, from Theorem 2.3, f has a fixed point.

4. Application to existence of local solutions for first-order periodic problems

In this section we present an application to existence of a solution for a periodic problem which is a consequence of Theorem 2.3. This kind of application first appeared in [**?**].
Let $X = C(I)$ be the set of all real continuous functions on $I = [0, T]$ where $T < 2.5$. Obviously, this space with the p-metric given by

$$d(x,y) = e^{\max_{t\in I}|x(t)-y(t)|} - 1$$

for all $x, y \in X$ is a p-complete p-metric space with $\Omega(t) = e^t - 1$. Secondly, X can also be equipped with a partial order given by

$$x \preceq y \text{ iff } x(t) \leq y(t) \text{ for all } t \in I.$$

Moreover, as in [**?**], it can be proved that $(X, \preceq)$ enjoys the s.l.c. property.

Consider the following first-order periodic boundary value problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(0) = x(T), \end{cases} \tag{3}$$

where $t \in I$ and $f : I \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. A lower solution for (3) is a function $\alpha \in C^1(I)$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)) \\ \alpha(0) \leq \alpha(T), \end{cases}$$

where $t \in I$.
Assume that there exists $\lambda > 0$ such that, for all $x, y \in X$ and $t \in I$, we have

$$|f(t, x(t)) + \lambda x(t) - f(t, y(t)) - \lambda y(t)| \leq \frac{\lambda}{2}(|x(t) - y(t)|).$$

Problem (3) can be rewritten as

$$\begin{cases} x'(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t) \equiv F(t, x(t)) \\ x(0) = x(T), \end{cases}$$

where $t \in I$. It is well known that this problem is equivalent to the integral equation

$$x(t) = \int_0^T G(t, s) F(s, x(s))\, ds,$$

where G is the Green's function given as

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s \leq t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t \leq s \leq T. \end{cases}$$

Now define an operator $S : X \to X$ by

$$Sx(t) = \int_0^T G(t, s) F(s, x(s))\, ds.$$

The mapping S is nondecreasing [1]. Note that if $u \in C^1(I)$ is a fixed point of S then u is a solution of (3).
Let $x, y \in X$. Then we have

$$\begin{aligned}
\Omega\big(d(Sx, Sy)\big) &= e^{e^{\max_{t \in I} |S(x(t)) - S(y(t))|} - 1} - 1 \\
&= e^{e^{\max_{t \in I} |\int_0^T G(t,s) F(s, x(s))\, ds - \int_0^T G(t,s) F(s, y(s))\, ds|} - 1} - 1 \\
&\leq e^{e^{\max_{t \in I} \int_0^T |G(t,s)|\, |F(s, x(s)) - F(s, y(s))|\, ds} - 1} - 1 \\
&\leq e^{e^{\max_{t \in I} \int_0^T |G(t,s)| \frac{\lambda}{2} |x(t) - y(t)|\, ds} - 1} - 1 \\
&\leq e^{e^{\frac{\lambda}{2} \max_{t \in I} |x(t) - y(t)| \left(\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}\, ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}\, ds \right)} - 1} - 1 \\
&= e^{e^{\frac{\lambda}{2} \max_{t \in I} |x(t) - y(t)| \frac{1}{\lambda(e^{\lambda T}-1)} \left(e^{\lambda T} - e^{\lambda(T-t)} + e^{\lambda(T-t)} - 1 \right)} - 1} - 1 \\
&= e^{\frac{1}{2} d(x, y)} - 1 = \psi\big(d(x, y)\big),
\end{aligned}$$

wherefrom $\Omega(d(Sx, Sy)) \leq \psi(M(x, y))$, where

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, Sx) d(y, Sy)}{1 + d(x, y)}, \frac{d(x, Sx) d(y, Sy)}{1 + d(Sx, Sy)} \right\}.$$

Finally, let α be a lower solution for (3). In [**?**], it was shown that $\alpha \preceq S(\alpha)$.
Hence, the hypotheses of Theorem 2.3 are satisfied with $\psi(t) = e^{\frac{1}{2}t} - 1$. Therefore, there exists a fixed point $\hat{x} \in C(I)$ such that $S\hat{x} = \hat{x}$. This $\hat{x}$ is then a solution of problem (3).

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Norm Estimates for Extended Subclasses of Starlike Functions

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Abstract

We investigate on some subclasses of analytic fuctions defined by subordination. Also, we give estimates of $\sup_{|z|<1}(1-|z|^2)|\frac{f''(z)}{f'(z)}|$, for functions belonging to extended class of starlike functions. For a locally univalent analytic function f defined on $\Delta = \{z \in \mathbb{C} : |Z| < 1\}$, we consider the pre-Schwarzian norm by $\|T_f\| = \sup_{|z|<1}(1-|z|^2)|\frac{f''(z)}{f'(z)}|$. In this paper, we find the sharp norm estimate for the functions f in the extended classes of starlike functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\Delta = \{z : z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ that are univalent in Δ.
Also, let S^* denote the class of starlike functions that is defined as

$$S^* = \left\{ f \in \mathcal{S};\ Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ \ z \in \Delta \right\}.$$

Let f and g be analytic in Δ. The function f is said to be *subordinate* to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w in Δ satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$[4].
If g is univalent, then $f \prec g$ if and only if $f(0) = 0$ and $f(\Delta) \subset g(\Delta)$.

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For the first time, Ma and Minda [5] extended the class of starlike functions by using of subordination method. In fact, they introduced $S^*(\phi)$ as follow:

$$S^*(\phi) = \Big\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec \phi(z) \Big\},$$

where, ϕ is analytic and $Re\,\{\phi\} > 0,\ \phi(0) = 1,\ \phi'(0) > 0.$
For a locally univalent analytic function f, the pre-Schwarzian derivative of f is defined by

$$T_f = \frac{f''(z)}{f'(z)}$$

and its norm is defined by

$$\|T_f\| = \sup_{|z|<1} \big(1 - |z|^2\big) \Big| \frac{f''(z)}{f'(z)} \Big|$$

It is essential to note that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent in Δ. It is also to be noted that if $f \in \mathcal{S}$, then $\|T_f\| \le 6$. Conversely, it follows from Becker's theorem, is if $f \in \mathcal{A}$ and if $\|T_f\| \le 1$ then $f \in \mathcal{S}$. The results are sharp [1, 2]. For functions belonging to the class of convex functions, $\|T_f\| \le 4$. According to Yamashita [9], if $f \in S^*(\alpha)$, then $\|T_f\| \le 6 - 4\alpha$. Bhowmik et al. [3] have obtained the estimate of the norm as $4 < \|T_f\| < 2\alpha + 2$ for functions in the class of concave univalent functions of order α.
The pre-Schwarzian derivative T_f and its norm, $\|T_f\|$, have important meanings in the theory of the Teichmuller space
.
Recently, many researchers motivated by the work of Ma and Minda[5], introduced and studied some interesting extended class of starlike functions by choosing suitable ϕ.

For our main purpose in this paper, we need definitions of some extended classes of starlike functions. S_l^*, S_e^*, and S_c^* denote the classes of extended starlike functions, respectively, Following [6–8] these are defined as:

$$S_l^* = \Big\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2}\ \ z \in \Delta \Big\}, \tag{2}$$

$$S_e^* = \Big\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec e^z\ \ z \in \Delta \Big\}, \tag{3}$$

$$S_c^* = \Big\{ f \in \mathcal{S};\ \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2\ \ z \in \Delta \Big\}, \tag{4}$$

.

2. Main Results

In this section, we obtain upper bound of $\|T_f\|$ for functions belonging to some extended class of starlike functions.

Theorem 2.1. *Let the function $f(z)$ given by (1) be in the class S_l^*. Then $\|T_f\| \le 2$.*

Proof. By (2), we have,

$$\frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2}, \quad z \in \Delta,$$

now, by the definition of subordination, yields that

$$\frac{zf'(z)}{f(z)} = w(z) + \sqrt{1+w(z)^2} \tag{5}$$

where, $w(z)$ is Schwartz function. Applying the Schwarz-Pick lemma, we get

$$\left|w^{'}(z)\right| \leq \frac{1-|w(z)|^2}{1-|z|^2}, \quad z \in \Delta.$$

Logarithmic differentiation of (5) gives

$$\frac{1}{z} + \frac{f^{''}(z)}{f^{'}(z)} - \frac{f^{'}(z)}{f(z)} = \frac{w^{'}(z)}{\sqrt{1+w^2(z)}}. \tag{6}$$

From (5) and (6), we obtain the pre-Schwarzian derivative of f as follows:

$$T_f(z) = \frac{f^{''}(z)}{f^{'}(z)} = \frac{1}{z}\left(w(z) + \sqrt{1+w^2(z)}\right) - \frac{1}{z} + \frac{w^{'}(z)}{\sqrt{1+w^2(z)}}.$$

By using the Schwarz-pick lemma and triangle inequality we conclude that,

$$\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1}{|z|}\left(|w(z)| + \sqrt{1+w^2(z)} + 1\right) + \frac{1-|w(z)|^2}{(1-|z|^2)\sqrt{1+w^2(z)}}, \tag{7}$$

Multiplying the inequality (7) by $(1-|z|^2)$, we get

$$(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(|w(z)| + \sqrt{1+w^2(z)} + 1\right) + \frac{1-|w(z)|^2}{\sqrt{1+w^2(z)}}. \tag{8}$$

Therefore, by using the inequality $|w(z)| \leq |z|$ for all $z \in \Delta$, we see that

$$(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(|z| + \sqrt{1+|z|^2} + 1\right) + 1 + |z|. \tag{9}$$

Taking the supremum value from both sides in the unit disc, the inequality (9) becomes

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \sup_{z\in\Delta}(1+|z|).$$

This completes the proof. □

Theorem 2.2. *Let the function $f(z)$ given by (1) be in the class S_e^*. Then $\|T_f\| \leq 2$.*

Proof. Since $f \in S_e^*$, we have

$$\frac{zf^{'}(z)}{f(z)} \prec e^z.$$

Using the definition of subordination, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zf^{'}(z)}{f(z)} = e^{w(z)}. \tag{10}$$

By logarithmic differentiation of (10), we ge

$$\frac{1}{z} + \frac{f^{''}(z)}{f^{'}(z)} - \frac{f^{'}(z)}{f(z)} = w^{'}(z). \tag{11}$$

Also, equation (10) gives us,

$$\frac{f^{'}(z)}{f(z)} = \frac{1}{z} e^{w(z)}. \tag{12}$$

Equations (11) and (12) give us the pre-Schwarzian derivative of f as follows

$$T_f(z) = \frac{f^{''}(z)}{f^{'}(z)} = \frac{1}{z}\left(e^{w(z)} - 1\right) + w^{'}(z).$$

By applying triangle inequality and the Schwarz-Pick lemma, we obtain

$$\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1}{|z|}\left(e^{|w(z)|} - 1\right) + \frac{1-|w(z)|^2}{1-|z|^2}, \quad z \in \Delta,$$

For all $z \in \Delta$, we have

$$(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(e^{|w(z)|} - 1\right) + 1 - |w(z)|^2.$$

Since, $w(z)$ is Schwarz function and $|w(z)| \leq |z|$, hence,

$$(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(e^{|z|} - 1\right) + 1 + |z|.$$

We take the lower limit as $z \to 1^{-1}$ to obtain the following inequality

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{f^{''}(z)}{f^{'}(z)}\right| \leq \sup_{z\in\Delta}\left\{\frac{1-|z|^2}{|z|}\left(e^{|z|} - 1\right) + 1 + |z|\right\}.$$

This completes the proof. □

Theorem 2.3. *Let the function $f(z)$ given by (1) be in the class S_c^*. Then $\|T_f\| \leq \frac{8}{3}$.*

Proof. In order to prove this theorem, we use a similar procedure. Since $f \in S_c^*$, we have

$$\frac{zf^{'}(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2.$$

By using subordination method we see that,

$$\frac{zf^{'}(z)}{f(z)} = 1 + \frac{4}{3}w(z) + \frac{2}{3}w(z)^2, \tag{13}$$

where $w(z)$ is Schwarz function. By logarithmic differentiation of (13), we get

$$\frac{1}{z} + \frac{f^{''}(z)}{f^{'}(z)} - \frac{f^{'}(z)}{f(z)} = \frac{\frac{4}{3}w^{'}(z)\left(1 + w(z)\right)}{1 + \frac{4}{3}w(z) + \frac{2}{3}w(z)^2}. \tag{14}$$

From (13), we have

$$\frac{f^{'}(z)}{f(z)} = \frac{1}{z}\left(1 + \frac{4}{3}w(z) + \frac{2}{3}w(z)^2\right). \tag{15}$$

By the help of the Schwarz-Pick lemma and triangle inequality, from (14) and (15), we obtain

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1}{|z|}\left(\frac{4}{3}|w(z)| + \frac{2}{3}|w(z)|^2\right) + \frac{4}{3}\left(\frac{1-|w(z)|^2}{1-|z|^2}\right)\left|\frac{1+w(z)}{1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2}\right|. \tag{16}$$

Consequently, for all $z \in \Delta$, we conclude that,

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(\frac{4}{3}|w(z)| + \frac{2}{3}|w(z)|^2\right) + \frac{4}{3}\left(1-|w(z)|^2\right)\left|\frac{1+w(z)}{1+\frac{4}{3}w(z)+\frac{2}{3}w(z)^2}\right|. \tag{17}$$

By the inequality $|w(z)| \leq |z|$ for all $z \in \Delta$, we obtain

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z| + \frac{2}{3}|z|^2\right) + \frac{4}{3}\left(1+|z|\right). \tag{18}$$

Hence, the limit of $(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$ and $\frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z| + \frac{2}{3}|z|^2\right) + \frac{4}{3}\left(1+|z|\right)$ exist when z tends to 1 from the left and it equals the supremum of $(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|$ and $\frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z| + \frac{2}{3}|z|^2\right) + \frac{4}{3}\left(1+|z|\right)$. Therefore

$$\sup_{z\in\Delta}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq \sup_{z\in\Delta}\left\{\frac{1-|z|^2}{|z|}\left(\frac{4}{3}|z| + \frac{2}{3}|z|^2\right) + \frac{4}{3}\left(1+|z|\right)\right\}.$$

This completes the proof. □

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Some Results of Contractive Mapping Theorem in Partially Ordered Metric Spaces

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Abstract

In this paper, we prove that an operator K satisfying certain rational contration condition has a fixed point in a partially ordered metric space. Also, we establish some coincidence, common fixed point fixed point theorem for monotone f-non decreasing self mapping satisfying certain rational type contraction in the context of metric spaces endowed with partial order.

1. Introduction

The Banach contraction principle is one of the most versatile result in fixed point theory and approximation theory. It plays an important role in solving many existing problems in pure and applied mathematics. There is a vast literature dealing with technical extensions and generalizations of Banach contraction principle, some instance of these works are in [1, 2]. Besides, this famous classical theorem gives an iteration process through which we can obtain better approximation to the fixed point. It renders a key role in solving system of linear algebra equation involving iteration process. Iteration procedures are using in nearly every banach of applied mathematics covergence proof and also in estimating the process of errors, very often by an application of banach's fixed point theorem.
In recent time, fixed points of mapping in ordered metric spaces are of great use in many branches of mathematical analysis for solving nonlinear equations. The first result in this direction was initiated by work [9]. Ran and Reurings [8] studied the existence of fixed points for certain mapping in partially ordered metric spaces and applied their results to matrix equations. The results of Ran and Reurigs [8] were extended by Nieto et al.[6, 7] for non decreasing mapping and obtained the solutions of certain partial differential equations with periodic boundary conditions. While Agarwal et al. [3] have discussed some new results for generalized contractions in partially ordered metric spaces. There have been a lot of generalizations and improvements of the results to obtain fixed point, common fixed point results for single valued and multivalued operators in various ordered spaces with topological properties, some of which are [4, 5].

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The purpose of this paper is to establish some fixed point results of a mapping satisfying cerrtain rational condition in partially ordered metric space.
The following definitions are frequently used in results given in upcoming sections.

Definition 1.1. The triple $(X, d, \preceq)$ is called partialy ordered metric space, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a metric space.

Definition 1.2. If (X, d) is a complete metric space, then triple $(X, d, \preceq)$ is called complete partialy ordered metric space.

Definition 1.3. Let $(X, \preceq)$ be a partially ordered set. A self-mapping $f : X \to X$ is said to be stricttly increasing if $f(x) \prec f(y)$, for all $x, y \in X$ with $x \prec y$ and is also said to be strictly decreasing if $f(x) \succ f(y)$, for all $x, y \in X$ with $x \prec y$.

Definition 1.4. A point $x \in A$, where A is a non-empty subset of a metric space (X, d) is called common fixed (coincidence) point of two self-mapping f and K if $fx = Kx = x(fx = Kx)$.

Definition 1.5. The two self-mapping f and K defined over a subset A of a metric space (X, d) are called commuting if $fKx = Kfx$ for all $x \in A$.

Definition 1.6. Two self-mappings f and K defined over a subset $A \subset X$ are compatible, if for any sequence $\{x_n\}$ with $lim_{n\to+\infty} fx_n = lim_{n\to+\infty} Kx_n = \mu$, for some $\mu \in A$ then $lim_{n\to+\infty}(Kfx_n, fKx_n) = 0$

Definition 1.7. Two self-mappings f and K defined over a subset $A \subset X$ are said to be weakly compatible, if they commute at their coincidence points. i. e., if $fx = Kx$ then $fKx = Kfx$.

Definition 1.8. Let f and K be two self-mapping defined over a partially ordered set $(X, \preceq)$. A mapping K is called a monotone f non-decreasing if

$$fx \preceq fy \ \ impliesKx \preceq Ky, \ for \ \ all \ \ x, \ y \in X.$$

Definition 1.9. Let A be a non-empty subset of a partially ordered set $(X, \preceq)$. If very two elements of A are comparable then it is called well ordered set.

Definition 1.10. A partially ordered metric space $(X, d, \preceq)$ is called on ordered complete, if for each convergent sequence $\{x_n\}_{n=0}^{+\infty} \subset X$, one of the following condition holds

- if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ implies $x_n \preceq x$, for all $n \in \mathbb{N}$ that is, $x = sup\{x_n\}$ or
- if $\{x_n\}$ is a nonincreasing sequence in X such that $x_n \to x$ implies $x \preceq x_n$, for all $n \in \mathbb{N}$ that is, $x = inf\{x_n\}$.

2. Main results

We start this section with the following theorem.

Theorem 2.1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space and let K be a non-decreasing, continuous self-mapping defined on X. Suppose that a self-mapping K satisfies the following condition:*

$$d(Kx, Ky) = \begin{cases} \lambda d(x, y) + \mu \dfrac{d(x, Kx)d(x, Ky) + d(y, Kx)d(y, Ky)}{d(y, Kx) + d(x, Ky)} & if \ \ A \neq 0, \\ 0 & if \ \ A = 0, \end{cases}$$

for all $x, y \in X$ with $y \preceq x$, where $A = d(y, Kx) + d(x, Ky)$ and λ, μ are non-negative reals such that $\lambda + \mu < 1$. If there exists $x_0 \in X$ with $x_0 \preceq Kx_0$, then K has a fixed point.

Example 2.2. *Let $X = [0,1]$ with the usual metric and usual order $\leq$. We define an operator* $Kx = \dfrac{2x+3}{4(x^2+x+\frac{5}{4})}$.

It is clear that K is continuous on $[0,1]$. Now, $\lambda = \frac{16}{25}$ and any $\mu \in [0,1)$ such that $\lambda+\mu<1$. Without loss of generality, we assume that $x \leq y$. So, we have

$$\begin{aligned} d(Kx,Ky) &= \frac{1}{4}\Big|\frac{2x+3}{x^2+x+\frac{5}{4}} - \frac{2y+3}{y^2+y+\frac{5}{4}}\Big| \\ &= \Big|\frac{2xy(y-x)+3(y-x)(x+y)+3(y-x)-\frac{5}{2}(y-x)}{4(x^2+x+\frac{5}{4})(y^2+y+\frac{5}{4})}\Big| \\ &= \Big|\frac{2xy+3(x+y)+\frac{1}{2}}{4(x^2+x+\frac{5}{4})(y^2+y+\frac{5}{4})}\Big||x-y| \\ &\leq \frac{16}{25}|y-x| = \frac{16}{25}d(x,y) \end{aligned}$$

for all $x,y \in X$. Also, there exists $x_0 = 0 \in X$ such that $x_0 = 0 \leq Kx_0$ is satisfied. This shows that conditions of Theorem 2.1 hold and K has a fixed point $\frac{1}{2} \in [0,1]$.

We may remove the continuity criteria on K in Theorem 2.1 as follows.

Theorem 2.3. *Let $(X,d,\preceq)$ be a complete partially ordered metric space and let K be a non-decreasing self-mapping defined on X. Suppose that a self-mapping K satisfies the following condition:*

$$d(Kx,Ky) = \begin{cases} \lambda d(x,y) + \mu\dfrac{d(x,Kx)d(x,Ky)+d(y,Kx)d(y,Ky)}{d(y,Kx)+d(x,Ky)} & if \;\; A \neq 0, \\ 0 & if \;\; A = 0, \end{cases}$$

for all $x,y \in X$ with $y \preceq x$, where $A = d(x,Kx)+d(y,Ky)$ and λ,μ are non-negative reals with $\lambda+\mu<1$. And also suppose that X has the (OC) property. If there exists $x_0 \in X$ with $x_0 \preceq Kx_0$, then K has a fixed point.

Now we prove the sufficient condition for the uniqueness of the fixed point in Theorem 2.1 and Theorem 2.3, that is, U : for any $y,z \in X$, there exists $x \in X$ which is comparable to y and z.

Theorem 2.4. *Adding the above mentioned condition to the hypothesis of Theorem 2.1 (or Theorem 2.3), one obtains the uniqueness of the fixed point of K.*

We get the following fixed point theorem in partially ordered metric spaces if we take $\lambda = 0$ in the before theorems.

Theorem 2.5. *Let $(X,d,\preceq)$ be a complete partially ordered metric space and let K be a non-decreasing self-mapping defined on X. Suppose that a self-mapping K satisfies the following condition:*

$$d(Kx,Ky) = \begin{cases} \mu\dfrac{d(x,Kx)d(x,Ky)+d(y,Kx)d(y,Ky)}{d(y,Kx)+d(x,Ky)} & if \;\; A \neq 0, \\ 0 & if \;\; A = 0, \end{cases}$$

for all $x,y \in X$ with $y \preceq x$, where $A = d(y,Kx)+d(x,Ky)$ and μ are non-negative real with $0 \leq \mu < 1$. Suppose also that either K is continuous or X satisfies the condition (OC). If there exists $x_0 \in X$ with $x_0 \preceq Kx_0$, then K has a fixed point.

If $(X,\preceq)$ satisfies the conditionused in Theorem 2.4, then the uniqueness of a fixed point can be proved.
In the following, we prove some coincidence, common fixed point theorems in the context of ordered metric space.

Theorem 2.6. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self mapping f and K on X are continuous, K is a monotone f-nondecreasing, $K(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Kx, Ky) \leq \alpha \frac{d(fx, Kx)d(fy, Ky)}{d(Kx, fy)} + \beta[d(fx, Kx) + d(fy, Ky)] + \gamma d(fx, fy)$$

for all $x, y \in X$ with $f(x) \neq f(y)$ are compareable, where $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq K(x_0)$ and the mapping K and f are compatible, then K and f have a coincidence point in X.

Corollary 2.7. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self mapping f and K on X are continuous, K is a monotone f-nondecreasing, $K(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Kx, Ky) \leq \alpha \frac{d(fx, Kx)d(fy, Ky)}{d(fx, fy)} + \beta[d(fx, Kx) + d(fy, Ky)]$$

for all $x, y \in X$ with $f(x) \neq f(y)$ are compareable, where $\alpha, \beta \in [0, 1)$ with $0 \leq \alpha + 2\beta < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq K(x_0)$ and the mapping K and f are compatible, then K and f have a coincidence point in X.

Corollary 2.8. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self mapping f and K on X are continuous, K is a monotone f-nondecreasing, $K(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Kx, Ky) \leq \beta[d(fx, Kx) + d(fy, Ky)] + \gamma d(fx, fy)$$

for all $x, y \in X$ with $f(x) \neq f(y)$ are compareable, where $\beta, \gamma \in [0, 1)$ with $0 \leq 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq K(x_0)$ and the mapping K and f are compatible, then K and f have a coincidence point in X.

We may remove the continuity criteria of K in Theorem 2.6 is still valid by assuming the following hypothesis in X: If $\{x_n\}$ is nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2.9. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the self mapping f and K on X are continuous, K is a monotone f-nondecreasing, $K(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Kx, Ky) \leq \alpha \frac{d(fx, Kx)d(fy, Ky)}{d(fx, fy)} + \beta[d(fx, Kx) + d(fy, Ky)] + \gamma d(fx, fy)$$

for all $x, y \in X$ with $f(x) \neq f(y)$ are compareable for some $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq K(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

If $f(x)$ is a complete subset of X then K and f have a coincidence point in X. Further, if K and f are weakly compatible, then K and f have a common fixed point in X. Moreover, the set of common fixed points of K and f is well ordered if and only if K and f have one and only one common fixed point in X.

Corollary 2.10. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that f and K are self mapping on X, K is a monotone f-nondecreasing, $K(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Kx, Ky) \leq \alpha \frac{d(fx, Kx)d(fy, Ky)}{d(fx, fy)} + \beta[d(fx, Kx) + d(fy, Ky)]$$

for all $x, y \in X$ with $f(x) \neq f(y)$ are compareable, where $\alpha, \beta, \in [0, 1)$ with $0 \leq \alpha + 2\beta < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq K(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.
If $f(X)$ is a complete subset of X, then K and f have a coincidence point in X. Further, if K and f are weakly compatible, then K and f have a coincidence point in X. Moreover, the set of common fixed points of K and f is well ordered if and only if K and f have one and only one common fixed point in X.

Corollary 2.11. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that f and K are self mapping on X, K is a monotone f-nondecreasing, $K(X) \subseteq f(X)$ and satisfying the following condition*

$$d(Kx, Ky) \leq \beta[d(fx, Kx) + d(fy, Ky)] + \gamma d(fx, fy)$$

for all $x, y \in X$ with $f(x) \neq f(y)$ are compareable, where $\beta, \gamma \in [0, 1)$ with $0 \leq 2\beta + \gamma < 1$. If there exists a point $x_0 \in X$ such that $f(x_0) \preceq K(x_0)$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.
If $f(X)$ is a complete subset of X, then K and f have a coincidence point in X. Further, if K and f are weakly compatible, then K and f have a coincidence point in X. Moreover, the set of common fixed points of K and f is well ordered if and only if K and f have one and only one common fixed point in X.

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Unique Fixed point Theorem for Rational forward Contraction in Partially Ordered Asymmtric Metic Spaces

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Abstract

In this paper, we prove that an operator K satisfying certain rational forward contration condition has a fixed point in a partially ordered asymmetric metric spaces.

1. Introduction

The Banach contraction principle is one of the most versatile result in fixed point theory and approximation theory. It plays an important role in solving many existing problems in pure and applied mathematics. There is a vast literature dealing with technical extensions and generalizations of Banach contraction principle, some instance of these works are in [1, 2]. Besides, this famous classical theorem gives an iteration process through which we can obtain better approximation to the fixed point. It renders a key role in solving system of linear algebra equation involving iteration process. Iteration procedures are using in nearly every banach of applied mathematics covergence proof and also in estimating the process of errors, very often by an application of banach's fixed point theorem.
In recent time, fixed points of mapping in ordered metric spaces are of great use in many branches of mathematical analysis for solving nonlinear equations. The first result in this direction was initiated by work [9]. Ran and Reurings [8] studied the existence of fixed points for certain mapping in partially ordered metric spaces and applied their results to matrix equations. The results of Ran and Reurigs [8] were extended by Nieto et al.[6, 7] for non decreasing mapping and obtained the solutions of certain partial differential equations with periodic boundary conditions. While Agarwal et al. [3] have discussed some new results for generalized contractions in partially ordered metric spaces. There have been a lot of generalizations and improvements of the results to obtain fixed point, common fixed point results for

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single valued and multivalued operators in various ordered spaces with topological properties, some of which are [4, 5].
The purpose of this paper is to establish fixed point result of a mapping satisfying cerrtain rational forward contraction condition in partially ordered asymmetric metric spaces.
The following definitions are frequently used in results given in upcoming sections.

Definition 1.1. A function $d : X \times X \to \mathbb{R}$ is an asymmetric metric and (X, d) is an asymmetric metric space if

(i) For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$,

(ii) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$. Henceforth, (X, d) shall be an asymmetric metric space.

Definition 1.2. A mapping $K : X \to X$ is said forward (backward) contraction when there exists $0 < \alpha < 1$ such that

$$d(Kx, Ky) \leq \alpha d(x, y) \ \ (d(Kx, Ky) \leq \alpha d(y, x))$$

for each $x, y \in X$.

Definition 1.3. A sequence $\{x_k\}_{k\in\mathbb{N}}$ forward converges to $x_0 \in X$, respectively backward converges to $x_0 \in X$ if and only if

$$lim_{k\to\infty} d(x_0, x_k) = 0, \ \ respectively, \ \ lim_{k\to\infty} d(x_k, x_0) = 0.$$

Then we write $x_k \xrightarrow{f} x_0$, $x_k \xrightarrow{b} x_0$ respectively.

Definition 1.4. A set $S \subset X$ is forward complete if every forward Couchy sequence is forward convergent.

Definition 1.5. The triple $(X, d, \preceq)$ is called partialy ordered asymmetric metric space, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a asymmetric metric space.

Definition 1.6. If (X, d) is a complete asymmetric metric space, then triple $(X, d, \preceq)$ is called complete partialy ordered asymmetric metric space.

Definition 1.7. Let $(X, \preceq)$ be a partially ordered set. A self-mapping $f : X \to X$ is said to be strictly increasing if $f(x) \prec f(y)$, for all $x, y \in X$ with $x \prec y$ and is also said to be strictly decreasing if $f(x) \succ f(y)$, for all $x, y \in X$ with $x \prec y$.

Definition 1.8. A point $x \in A$, where A is a non-empty subset of a asymmetric metric space (X, d) is called common fixed (coincidence) point of two self-mapping f and K if $fx = Kx = x(fx = Kx)$.

Definition 1.9. The two self-mapping f and K defined over a subset A of a asymmetric metric space (X, d) are called commuting if $fKx = Kfx$ for all $x \in A$.

Definition 1.10. Two self-mappings f and K defined over a subset $A \subset X$ are compatible, if for any sequence $\{x_n\}$ with $lim_{n\to+\infty} fx_n = lim_{n\to+\infty} Kx_n = \mu$, for some $\mu \in A$ then $lim_{n\to+\infty}(Kfx_n, fKx_n) = 0$

Definition 1.11. Two self-mappings f and K defined over a subset $A \subset X$ are said to be weakly compatible, if they commute at their coincidence points. i. e., if $fx = Kx$ then $fKx = Kfx$.

Definition 1.12. Let f and K be two self-mapping defined over a partially ordered set $(X, \preceq)$. A mapping K is called a monotone f non-decreasing if

$$fx \preceq fy \qquad implie \ \ Kx \preceq Ky, \ for \ \ all \ \ x, \ y \in X.$$

Definition 1.13. Let A be a non-empty subset of a partially ordered set $(X, \preceq)$. If very two elements of A are comparable then it is called well ordered set.

Definition 1.14. A partially ordered asymmetric metric space $(X, d, \preceq)$ is called on ordered complete, if for each convergent sequence $\{x_n\}_{n=0}^{+\infty} \subset X$, one of the following condition holds

- if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ implies $x_n \preceq x$, for all $n \in \mathbb{N}$ that is, $x = sup\{x_n\}$ or
- if $\{x_n\}$ is a nonincreasing sequence in X such that $x_n \to x$ implies $x \preceq x_n$, for all $n \in \mathbb{N}$ that is, $x = inf\{x_n\}$.

2. Main results

Theorem 2.1. *Let $(X, d, \preceq)$ be a complete partially ordered asymmetric metric space and let K be a non-decreasing continuous self-mapping defined on X. Suppose that a self-mapping K satisfies*

$$d(Kx, Ky) = \begin{cases} \lambda d(x,y) + \mu \dfrac{d(x,Kx)d(x,Ky) + d(y,Kx)d(ky,y)}{d(y,Kx) + d(x,Ky)} & if \;\; A \neq 0, \\ 0 & if \;\; A = 0, \end{cases} \tag{1}$$

for all $x, y \in X$ with $y \preceq x$, where $A = d(y, Kx) + d(x, Ky)$ and λ, μ are non-negative reals such that $\lambda + \mu < 1$. If there exists $x_0 \in X$ with $x_0 \preceq Kx_0$, then K has a fixed point.

Proof. By assumption, there exists $x_0 \in X$ with $x_0 \preceq Kx_0$. If $x_0 = Kx_0$, then the proof is finshed. So, we suppose that $x_0 \prec Kx_0$. Since k is a non-decreasing mapping, we get

$$x_0 \prec Kx_0 \preceq K^2x_0 \preceq ... \preceq K^m x_0 \preceq K^{m+1}x_0 \preceq ...$$

by iteration. Put $x_{n+1} = kx_n$. If there exists $n_0 \in N$ such that $x_{n_0} = x_{n_0+1}$, then from $x_{n_0} = x_{n_0+1} = Kx_{n_0}$, we get x_{n_0} is fixed point, and the proof is finished. Suppose that $x_n \neq x_{n+1}$ for $n \in N$. Since the points x_n and x_{n-1} are compareable for all $n \in N$, we have the following two cases.

Case 1 . If $A = d(y, Kx) + d(x, Ky) \neq 0$, then using the contractive condition (2.1), we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Kx_n, Kx_{n-1}) \\ &\leq \lambda d(x_n, x_{n-1}) + \mu \frac{d(x_n, Kx_n)d(x_n, Kx_{n-1}) + d(x_{n-1}, Kx_n)d(Kx_{n-1}, x_{n-1})}{d(x_{n-1}, Kx_n,) + d(x_n, Kx_{n-1})} \\ &\leq \lambda d(x_n, x_{n-1}) + \mu \frac{d(x_n, x_{n+1})d(x_n, x_n) + d(x_{n-1}, x_{n+1})d(x_n, x_{n-1})}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} \\ &\leq \lambda d(x_n, x_{n-1}) + \mu \frac{d(x_{n-1}, x_{n+1})d(x_n, x_{n-1})}{d(x_{n-1}, x_{n+1})} \\ &\leq (\lambda + \mu) d(x_n, x_{n-1}). \end{aligned}$$

Hence, we drive that $d(x_{n+1}, x_n) \leq h^n d(x_0, x_1)$, where $h = (\lambda + \mu) < 1$. Moreover, by the triangular inequality, we have, for $m \geq n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq (h^{m-1} + h^{m-2} + ... + h^n) d(x_1, x_0) \leq \frac{h^n}{1-h} d(x_1, x_0) \end{aligned}$$

and this proves that $d(x_m, x_n) \to 0$ as $m, n \to \infty$. So, $\{x_n\}$ is a forward Cauchy sequence and, since X is a forward complete metric space, there exists $z \in X$ such that $x_n \xrightarrow{f} z$ as $n \to \infty$. Further, the continuity of K implies $Kz = K(lim_{n\to\infty} x_n) = lim_{n\to\infty} Kx_n = lim_{n\to\infty} x_{n+1} = z$. Thus z is a fixed point.

Case2 . If $A = d(y, Kx) + d(x, Ky) = 0$, then $d(x_{n+1}, x_n) = 0$. This implies that $x_n = x_{n+1}$, a forward contraction. Thus there exists a fixed point z of K.

□

Example 2.2. *Let $X = \mathbb{R} \geq 0$ be a nonempty set. Consider the map $d : X \times X \to \mathbb{R} \geq 0$ defined by*

$$d(x, y) = \begin{cases} y - x & y \geq x, \\ \dfrac{1}{4}(x - y) & x > y \end{cases}$$

It is easy to show that (X,d) is an asymmetric metric space. Consider $K: X \to X$ by $Kx = \frac{1}{2}x$. Clearly, K is a forward cotraction, whereas it is not backward. For this, let α be an arbitary with $0 < \alpha < 1$. Set $x = 2^{-2\alpha}$ and $y = 2^{-\alpha}$. Then we have

$$\begin{aligned}
d(Kx, Ky) &= \frac{1}{2}|(2^{-\alpha} - 2^{-2\alpha})| \\
&\leq \frac{1}{4}|(2^{-2\alpha} - 2^{-\alpha})| + |\mu \frac{d(x,Kx)d(x,Ky) + d(y,Kx)d(y,Ky)}{d(y,Kx) + d(x,Ky)}| \\
&\leq \frac{1}{4}|(2^{-2\alpha} - 2^{-\alpha})| \\
&\leq \frac{1}{4}d(x,y)
\end{aligned}$$

for all $x, y \in X$. This shows that conditions of Theorem (2.1) hold and K has a fixed point.

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Split common fixed point problem for a finite family of generalized demimetric mappings

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Abstract

In this paper, we study the split common fixed point problem for a finite family of generalized demimetric mappings in Hilbert spaces. We introduce an iterative scheme which does not require any prior knowledge of operator norm and prove the strong convergence of our iterative scheme for approximating a solution of this problem. As application we use our algorithm for solving the multiple-set split feasibility problem.

1. Introduction and Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $\mathcal{A} : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and let $\{C_i\}_{i=1}^m$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^m$ be a family of nonempty closed convex subsets in $\mathcal{K}$. The multiple-set split feasibility problem (MSSFP) was introduced by Censor et al. (2005) [1] and is formulated as finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{m} C_i \quad and \quad \mathcal{A}x^\star \in \bigcap_{i=1}^{m} Q_i.$$

The multiple-set split feasibility problem with $m = 1$ is known as the split feasibility problem (introduced by Censor and Elfving (1994) ([2])).
Let C be a nonempty, closed, and convex subset of $\mathcal{H}$. For each point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \qquad \forall y \in C.$$

The mapping $P_C : \mathcal{H} \to C$ is called the metric projection of $\mathcal{H}$ onto C and is characterized by the following two properties: $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \leq 0, \qquad \forall x \in \mathcal{H}, y \in C.$$

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Let E be a normed space and let $U : E \to E$ be a nonlinear mapping. A point $x \in E$ such that $Ux = x$ is called a fixed point of U. The set of fixed points of nonlinear mapping U shall be denoted by $Fix(U)$. That is, $Fix(U) := \{x \in E : Ux = x\}$. Since every closed convex subset of a Hilbert space is the fixed point set of its associating projection, the MSSFP becomes a special case of the split common fixed point problem (SCFPP) [3] of finding a point $x^\star$ with the property:

$$x^\star \in \bigcap_{i=1}^{m} Fix(S_i) \quad and \quad \mathcal{A}x^\star \in \bigcap_{i=1}^{m} Fix(T_i).$$

where $S_i : \mathcal{H} \to \mathcal{H}$, and $T_i : \mathcal{K} \to \mathcal{K}(i = 1, ..., m)$ are nonlinear operators.
The split feasibility problem and the split common fixed point problem have received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory and control theory. For examples, one can refer to [4, 5] and related literature. Various algorithms have been invented to solve it (see, for example, [6] and references therein).
We recall the following definitions concerning mapping $T : \mathcal{H} \to \mathcal{H}$. The mapping $T : \mathcal{H} \to \mathcal{H}$ is called:
• Contraction, if there exists a constant $0 \le k < 1$ such that

$$\|T(x) - T(y)\| \le k\|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

• Nonexpansive, if

$$\|Tx - Ty\| \le \|x - y\|, \qquad \forall x, y \in \mathcal{H}.$$

Definition 1.1. Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping with $Fix(T) \neq \emptyset$. Then $I - T$ is said to be demiclosed at zero if $\{x_n\}$ is a sequence in C converges weakly to x and $(I - T)x_n$ converges strongly to zero, then $(I - T)x = 0$.

In 2018, Kawasaki and Takahashi [7], introduced a new general class of mappings, called generalized demimetric mappings as follows:

Definition 1.2. Let ζ be a real number with $\zeta \neq 0$. A mapping $T : \mathcal{H} \to \mathcal{H}$ with $Fix(T) \neq \emptyset$ is called generalized demimetric, if

$$\zeta\langle x - x^\star, x - Tx\rangle \ge \|x - Tx\|^2,$$

for all $x \in \mathcal{H}$ and $x^\star \in Fix(T)$. This mapping T is called ζ-generalized demimetric.

Such a class of mappings is fundamental because it includes many types of nonlinear mappings arising in applied mathematics and optimization, see [7, 8] for details.
In this paper, we introduce a new algorithm for solving the split common fixed point problem for a finite family of generalized demimetric mappings in Hilbert spaces. We derive a strong convergence theorem of the proposed iterative algorithm under appropriate situations. Our results improved and extend the corresponding results announced by many others.

2. Main result

In this section, we present our algorithm for solving the split common fixed points problem.

Theorem 2.1. *Let $\mathcal{H}$ and $\mathcal{K}$, be real Hilbert spaces. Let for $i = 1, 2, ..., m$, $\mu^{(i)} \neq \emptyset$ and $T^{(i)} : \mathcal{H} \to \mathcal{H}$ be a finite family of $\mu^{(i)}$−generalized demimetric mappings such that $T^{(i)} - I$ is demiclosed at 0. Let for $i = 1, 2, ..., m$, $\zeta^{(i)} \neq \emptyset$ and $S^{(i)} : \mathcal{K} \to \mathcal{K}$ be a finite family of $\zeta^{(i)}$− generalized demimetric mappings such that $S^{(i)} - I$ is demiclosed at 0. Let $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator such that $A \neq 0$. Suppose that $\Omega = \{x^* \in \bigcap_{i=1}^m Fix(T^{(i)})) : Ax^* \in \bigcap_{i=1}^m Fix(S^{(i)})\} \neq \emptyset$. Assume that f be a contraction of $\mathcal{H}$ into itself with constant $k \in (0, 1)$. For $x_1 \in \mathcal{H}$, let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} z_n = x_n - \sum_{i=1}^m l^{(i)}\alpha_n^{(i)}\,\theta_n^{(i)}\,A^*(Ax_n - S^{(i)}Ax_n) \\ y_n = z_n - \sum_{i=1}^m r^{(i)}\beta_n^{(i)}\,\gamma_n^{(i)}(I - T^{(i)})z_n, \\ x_{n+1} = (1 - \xi_n)y_n + \xi_n\,f(y_n), \qquad \forall n \ge 1, \end{cases} \tag{1}$$

where $l^{(i)} = \frac{\zeta^{(i)}}{|\zeta^{(i)}|}$, $r^{(j)} = \frac{\mu^{(j)}}{|\mu^{(j)}|}$, $\{\tau_n^{(i)}\} \subset [a^{(i)}, b^{(i)}] \subset (0, \frac{2l^{(i)}}{\zeta^{(i)}})$,

$$\theta_n^{(i)} = \frac{\tau_n^{(i)} \|Ax_n - S^{(i)}Ax_n\|^2}{\|A^*(Ax_n - S^{(i)}Ax_n)\|^2 + \rho_n^{(i)}}$$

and $\{\rho_n^{(i)}\}$ *is a sequence of positive real numbers for each* $i = 1, 2, ..., m$. *Let* $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}, \{\gamma_n^{(i)}\}$, *and* $\{\xi_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n^{(i)}\}_{i=1}^m \subset (0, 1)$, $\sum_{i=1}^m \alpha_n^{(i)} = 1$ *and* $\liminf_n \alpha_n^{(i)} > 0$;

(ii) $\{\beta_n^{(i)}\}_{i=1}^m \subset (0, 1)$, $\sum_{i=1}^m \beta_n^{(i)} = 1$ *and* $\liminf_n \beta_n^{(i)} > 0$;

(iii) $0 < d^{(i)} \leq \gamma_n^{(i)} < 2\frac{r^{(i)}}{\mu^{(i)}}$ *for* $i \in \{1, 2, ..., m\}$;

(iv) $\lim_{n\to\infty} \xi_n = 0$ *and* $\sum_{n=0}^\infty \xi_n = \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to a point* $x^\star \in \Omega$ *which solves the variational inequality;*

$$\langle x^\star - f(x^\star), x - x^\star \rangle \geq 0, \qquad \forall x \in \Omega. \tag{2}$$

3. Application

As application we utilize our main result for solving the multiple-set split feasibility problem in Hilbert spaces.

Theorem 3.1. *Let* $\mathcal{H}$ *and* $\mathcal{K}$*, be real Hilbert spaces and let* $A : \mathcal{H} \to \mathcal{K}$ *be bounded linear operators such that* $A \neq 0$. *Let* $\{C_i\}_{i=1}^m$ *be a finite family of nonempty closed convex subsets of* $\mathcal{H}$ *and let* $\{Q_i\}_{i=1}^m$ *be a finite family of nonempty closed convex subsets of* $\mathcal{K}$. *Suppose that* $\Omega = \bigcap_{i=1}^m C_i \cap A^{(-1)}Q_i \neq \emptyset$. *Assume that* f *be a contraction of* $\mathcal{H}$ *into itself with constant* $k \in (0, 1)$. *For* $x_1 \in \mathcal{H}$, *let* $\{x_n\}$ *be a sequence defined by:*

$$\begin{cases} z_n = x_n - \sum_{i=1}^m \alpha_n^{(i)}\, \theta_n^{(i)}\, A^*(Ax_n - P_{Q_i}Ax_n) \\ y_n = z_n - \sum_{i=1}^m \beta_n^{(i)}\, (I - P_{C_i})z_n, \\ x_{n+1} = (1 - \xi_n)y_n + \xi_n\, f(y_n), \qquad \forall n \geq 1, \end{cases} \tag{3}$$

where $\{\tau_n^{(i)}\} \subset [a^{(i)}, b^{(i)}] \subset (0, 2)$,

$$\theta_n^{(i)} = \frac{\tau_n^{(i)} \|Ax_n - P_{Q_i}Ax_n\|^2}{\|A^*(Ax_n - P_{Q_i}Ax_n)\|^2 + \rho_n^{(i)}}$$

and $\{\rho_n^{(i)}\}$ *is a sequence of positive real numbers for each* $i = 1, 2, ..., m$. *Let* $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}$, *and* $\{\xi_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n^{(i)}\}_{i=1}^m \subset (0, 1)$, $\sum_{i=1}^m \alpha_n^{(i)} = 1$ *and* $\liminf_n \alpha_n^{(i)} > 0$;

(ii) $\{\beta_n^{(i)}\}_{i=1}^m \subset (0, 1)$, $\sum_{i=1}^m \beta_n^{(i)} = 1$ *and* $\liminf_n \beta_n^{(i)} > 0$;

(iii) $\lim_{n\to\infty} \xi_n = 0$ *and* $\sum_{n=0}^\infty \xi_n = \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to a point* $x^\star \in \Omega$.

Proof. We know that P_{C_i} and P_{Q_i} are $1-$generalized demimetric mappings. Also we know that $P_{C_i} - I$ and $P_{Q_i} - I$ are demiclosed at 0. Note that $Fix(P_{C_i}) = C_i$ and $Fix(P_{Q_i}) = Q_i$. Now putting $\gamma_n^{(i)} = 1$ in Theorem 2.1 we obtained the desired result. □

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Operators commuting with certain module actions

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Abstract

In this note, we study bounded linear operators associated with unitary representations which commute with certain module actions.

1. Introduction and preliminaries

Throughout G is a locally compact group with the unit e, a fixed left Haar-measure. The left Haar-integral of a complex-valued Haar-measurable function f on G will be denoted by $\int_G f(x)\,dx$. The convolution product of two complex-valued functions f and g on G is defined as follows.

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dx,$$

when the integral makes sense. As usual, $L^1(G)$ denote the group algebra of G as defined in [3]. The notation l_x is the left translation operator by $x \in G$; i.e., $l_x f(y) = f(xy)$ for all complex-valued function f on G. Note that $L^1(G)$ is a left G-module with the action $x \cdot \phi = l_{x^{-1}}\phi$ for all $x \in G$ and $\phi \in L^1(G)$. Let $L^\infty(G)$ is usual Lebesgue space as defined in [3] equipped with the essential supremum $\|\cdot\|_\infty$. Then $L^\infty(G)$ can be identified by the first dual space of $L^1(G)$ under the pairing

$$\langle f, \phi\rangle = \int_G f(x)\phi(x)\,dx \qquad (f \in L^\infty(G),\, \phi \in L^1(G)).$$

Moreover, the dualization of the left G-module action on $L^1(G)$ makes $L^\infty(G)$ as a right G-module as follows

$$\langle f \cdot x, \phi\rangle = \langle f, x \cdot \phi\rangle \quad (f \in L^\infty(G),\, x \in G).$$

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We can also consider $L^\infty(G)$ as a right Banach $L^1(G)$-module by the following action.

$$f \cdot \phi = \int_G f \cdot x \, \phi(x) \, dx \qquad (f \in L^\infty(G), \, \phi \in L^1(G)).$$

Let also, $LUC(G)$ denote the C^*-algebra of left uniformly continuous functions; i.e., $f \in LUC(G)$ when the map $x \mapsto l_x f$ from G into $L^\infty(G)$ is norm continuous.
In recent years, many authors have extensively studied the behavior and relations of G-module and $L^1(G)$-module maps, in the sense of the map commute with the translations, convolutions and conjugations; see for example [5–8]. Special attention has focused on such operators on $L^\infty(G)$. As known, any bounded linear operator on $L^\infty(G)$ that commutes with convolution from the left also commutes with left translations; see [7]. Here, we study such notions with an emphasis on unitary representations.
All over this paper, (π, H_π) is a unitary representation of a locally compact group G. As mentioned in [1], $Tr(H_\pi)$, denotes all of the trace-class operators on H_π with norm $\|T\|_1 = tr|T|$, take the role played by $L^1(G)$ in the theory of amenable groups and the left action of G on $L^1(G)$ being replaced by the following left action of G on $Tr(H_\pi)$.

$$x \cdot_\pi S = \pi(x) S \pi(x)^{-1} \qquad \big(x \in G, \, S \in Tr(H_\pi)\big).$$

Moreover, $Tr(H_\pi)$ is an isometric Banach G-module by Lemma 2.1 of [1]. Also, $B(H_\pi)$ is known as the dual space of $Tr(H_\pi)$ by the duality $T(S) = tr(ST)$ for all $T \in B(H_\pi)$ and $S \in Tr(H_\pi)$. Clearly, $T \cdot_\pi x = \pi(x)^{-1} T \pi(x)$ for each $T \in B(H_\pi)$ and $x \in G$. These facts imply that $B(H_\pi)$ is a right Banach $L^1(G)$-module as follows.

$$T \cdot_\pi \phi = \int_G T \cdot_\pi x \, \phi(x) \, dx \qquad (T \in B(H_\pi), \, \phi \in L^1(G)).$$

Since the map $x \mapsto T \cdot_\pi x$ from G into $B(H_\pi)$ is not necessarily norm-continuous, $B(H_\pi)$ is not Banach as a G-module, in general. So, one has considered the set of all $T \in B(H_\pi)$ for which $G \longrightarrow B(H_\pi)$, $x \longmapsto T \cdot_\pi x$ is norm-continuous, $UCB(\pi)$. Elements in $UCB(\pi)$ are called G-continuous operators. Moreover, the Cohen's factorization theorem implies that

$$B(H_\pi) \cdot_\pi L^1(G) = UCB(\pi) \cdot_\pi L^1(G) = UCB(\pi).$$

See [1] for more details and the survey article. For any $M \in B(H_\pi)^*$ and $T \in B(H_\pi)$, we can define a complex-valued function MT on G by

$$MT(x) = \langle M, \, T \cdot_\pi x \rangle \quad (x \in G).$$

Obviously, MT is bounded by $\|M\|\|T\|$. Besides,

$$l_x MT = (M)(T \cdot_\pi x) \quad (x \in G).$$

Suppose that $M \in B(H_\pi)^*$. Then the linear operator $\rho_M : UCB(\pi) \longrightarrow LUC(G)$ given by $T \longmapsto MT$ is well-defined due to [2, Lemma 2.2]. Furthermore, let $T \in UCB(\pi)$ and $\phi \in L^1(G)$. Then $\langle MT, \, \phi \rangle = \langle M, \, T \cdot_\pi \phi \rangle$ by directly calculaion. Therefore, $\rho_M(T \cdot_\pi \phi) = \rho_M(T) \cdot \phi$. Also, $\rho_M(T \cdot_\pi x) = \rho_M(T) \cdot x$ for all $x \in G$. These simple properties of ρ_M are a motivating force for this research. We extend them by the following definition that is the starting point of our path to express the main results in this note.

Definition 1.1. Let (π, H_π) be a unitary representation of a locally compact group G, and let $\gamma : B(H_\pi) \longrightarrow L^\infty(G)$ be a bounded linear operator.

(a) γ is said *to commute with the action as $L^1(G)$-module* if

$$\gamma(T \cdot_\pi \phi) = \gamma(T) \cdot \phi \qquad (T \in B(H_\pi), \, \phi \in L^1(G)). \tag{1}$$

(b) γ is said *to commute with the action as G-module* if

$$\gamma(T \cdot_\pi x) = \gamma(T) \cdot x \qquad (T \in B(H_\pi), \, x \in G), \tag{2}$$

Remark 1.2. *Suppose that $M \in B(H_\pi)^*$. We do not yet whether $MT \in L^\infty(G)$ for all $T \in B(H_\pi)$ or not. Therefore, we can not define safely the operator ρ_M from $B(H_\pi)$ into $L^\infty(G)$ by $\rho_M(T) = MT$. But as will be seen, there exist such operators. For instance, the map γ_M defined by $\langle \gamma_M(T), \, \phi \rangle = \langle M, \, T \cdot_\pi \phi \rangle$ for all $T \in B(H_\pi)$ and $\phi \in L^1(G)$ satisfies in the both of 1 and 2.*

2. The results

We commence the note by the following result that shows 1 and 2 coincide when the operator γ restricts to $UCB(\pi)$. Before starting, note that for all $M \in UCB(\pi)^*$ and $T \in UCB(\pi)$, we can also define the complex-valued function MT by $\overline{M}T$ on G, where $\overline{M}$ is any Hahn-Banach extension of M. Since the Hahn-Banach extension is not unique, in general, we use again the notation ρ_M instead of $\rho_{\overline{M}}$ for unification.

Theorem 2.1. *Let $(\pi,\, H_\pi)$ be a unitary representation of a locally compact group G, and let $\gamma : UCB(\pi) \longrightarrow L^\infty(G)$ be a bounded linear operator. Then each of the following statements implies that the range of γ lies in $LUC(G)$. Also, they are equivalent.*

(a) *γ commutes with the action as $L^1(G)$-module,*
(b) *$\gamma = \rho_M$ for some $M \in UCB(\pi)^*$,*
(c) *γ commutes with action as G-module.*

Proof. Let $T \in UCB(\pi)$. If (a) holds, then $\gamma(T) = \gamma(S \cdot_\pi \phi) = \gamma(S) \cdot \phi$ for some $S \in UCB(\pi)$ and $\phi \in L^1(G)$ that yields $\gamma(T) \in LUC(G)$. If (b) holds, then $\gamma(T) = \rho_M(T) = MT \in LUC(G)$. Finally, if (c) holds and $x_\alpha \longrightarrow x$ in G, then

$$\begin{aligned}
\|l_{x_\alpha}\gamma(T) - l_x\gamma(T)\|_\infty &= \|\gamma(T)\cdot x_\alpha - \gamma(T)\cdot x\|_\infty \\
&= \|\gamma(T \cdot_\pi x_\alpha) - \gamma(T \cdot_\pi x)\|_\infty \\
&\leq \|\gamma\|\,\|T \cdot_\pi x_\alpha - T \cdot_\pi x\| \\
&\longrightarrow 0.
\end{aligned}$$

It follows that $\gamma(T) \in LUC(G)$.
Now, for equivalency of them, we can confirm (a) and (c) if (b) holds, as noted earlier. Suppose that (a) holds and (ϕ_i) is a bounded approximate identity of $L^1(G)$. Then $(\gamma^*(\phi_i))$ is bounded in $UCB(\pi)^*$, where γ^* is the usual adjoint of γ. Let now $M \in UCB(\pi)^*$ be a weak*-cluster point of $(\gamma^*(\phi_i))$. So, we may assume that $\gamma^*(\phi_i) \longrightarrow M$ in the weak*-topology of $UCB(\pi)^*$. Let $T \in UCB(\pi)$. Then for each $\phi \in L^1(G)$, we have

$$\begin{aligned}
\langle \rho_M(T),\, \phi\rangle &= \langle MT,\, \phi\rangle = \langle M,\, T \cdot_\pi \phi\rangle \\
&= \lim_i \langle \gamma^*(\phi_i),\, T \cdot_\pi \phi\rangle = \lim_i \langle \phi_i,\, \gamma(T \cdot_\pi \phi)\rangle \\
&= \lim_i \langle \phi_i,\, \gamma(T)\cdot\phi\rangle = \lim_i \langle \gamma(T)\cdot\phi,\, \phi_i\rangle \\
&= \lim_i \langle \gamma(T),\, \phi * \phi_i\rangle = \langle \gamma(T),\, \phi\rangle.
\end{aligned}$$

Therefore, part (b) holds. Assume now, γ is commuting with the action as G-module. Take $M = \gamma^*(\delta_e) \in UCB(\pi)^*$, where $\delta_e(f) = f(e)$ for all $f \in LUC(G)$. Then for each $T \in UCB(\pi)$ and $x \in G$, we have

$$\begin{aligned}
\gamma(T)(x) &= \big(\gamma(T)\cdot x\big)(e) = \langle \delta_e,\, \gamma(T)\cdot x\rangle \\
&= \langle \delta_e,\, \gamma(T \cdot_\pi x)\rangle = \langle \gamma^*(\delta_e),\, T \cdot_\pi x\rangle \\
&= \langle M,\, T \cdot_\pi x\rangle = MT(x).
\end{aligned}$$

It follows that $\gamma(T) = MT = \rho_M(T)$ for all $T \in UCB(\pi)$ and so, $\gamma = \rho_M$. One shows the implication (c) into (b). $\square$

As mentioned earlier, every bounded linear operator on $L^\infty(G)$ commuting with the action as $L^1(G)$-module commute also, with the action as G-module. Here, we have the following result.

Proposition 2.2. *Let $(\pi,\, H_\pi)$ be a unitary representation of a locally compact group G, and let γ be a bounded linear operator from $B(H_\pi)$ into $L^\infty(G)$ that is commuting with the action as $L^1(G)$-module. Then γ commutes with the action as G-module.*

Proof. Suppose that $T \in B(H_\pi)$, $x \in G$ and $\phi \in L^1(G)$. One can easily check that $(T \cdot_\pi x) \cdot_\pi \phi = T \cdot_\pi (x \cdot \phi)$. Let furthermore, (ϕ_i) is an approximate identity for $L^1(G)$. Then

$$\begin{aligned}
\langle \gamma(T \cdot_\pi x),\, \phi \rangle &= \lim_i \langle \gamma(T \cdot_\pi x),\, \phi_i * \phi \rangle \\
&= \lim_i \langle \gamma(T \cdot_\pi x) \cdot \phi_i,\, \phi \rangle \\
&= \lim_i \langle \gamma\big((T \cdot_\pi x) \cdot_\pi \phi_i\big),\, \phi \rangle \\
&= \lim_i \langle \gamma\big(T \cdot_\pi (x \cdot \phi_i)\big),\, \phi \rangle \\
&= \lim_i \langle \gamma(T) \cdot (x \cdot \phi_i),\, \phi \rangle \\
&= \lim_i \langle \gamma(T),\, (x \cdot \phi_i) * \phi \rangle \\
&= \lim_i \langle \gamma(T),\, x \cdot (\phi_i * \phi) \rangle \\
&= \lim_i \langle \gamma(T) \cdot x,\, \phi_i * \phi \rangle \\
&= \langle \gamma(T) \cdot x,\, \phi \rangle.
\end{aligned}$$

Therfore, γ commutes with the action as G-module. □

It is tempting to know whether the converse of Proposition 2.2 is valid or not. It's known that in the same style of operators on $L^\infty(G)$, the converse fails. So, it turns out that the converse fails here, too. Due to Theorem 2.1, for take an example, one can consider a non-discrete group G and the left unitary representation $(\lambda,\, L^2(G))$.
We end the work with the following example that shows the converse of Proposition 2.2 has been unable to confirm in general.

Example 2.3. *Let G be either $(\mathbb{R},\, +)$ or any infinite compact abelian group. We show that there exists a bounded linear operator γ from $B\big(L^2(G)\big)$ into $L^\infty(G)$ such that γ commutes the action as G-module; whereas, $\gamma(T \cdot_\lambda \phi) \neq \gamma(T) \cdot \phi$ for some $T \in B\big(L^2(G)\big)$ and $\phi \in L^1(G)$. Toward this end, first, recall that for each $f \in L^\infty(G)$, the map $\tau : f \mapsto T_f$ is an isometric embedding of $L^\infty(G)$ into $B\big(L^2(G)\big)$, where T_f is the multiplication operator on $L^2(G)$ by f. It is rutin checking that $T_f \cdot_\lambda \phi = T_{f \cdot \phi}$ for each $f \in L^\infty(G)$ and $\phi \in L^1(G)$. On the other hand, G satisfies in conditions of Theorem 4.1 of [8]. So, the following statements hold for some bounded linear operators Ψ on $L^\infty(G)$ such that*

(a) *Ψ commutes the action as G-module.*
(b) *each $\Psi(f)$ is a constant function for all $f \in L^\infty(G)$.*
(c) *$\Psi(f \cdot \phi) \neq \Psi(f) \cdot \phi$ for some $f \in L^\infty(G)$ and some continuous function ϕ with compact support.*

Take now, $\gamma = \Psi \circ \tau_l^{-1}$, where τ_l^{-1} is the left inverse of τ. Note that G is non-discrete and so, $B(L^2(G)) \neq UCB(\lambda)$ by using of [4, , Remark 3.11 (i)]. However, it follows that

$$\gamma(T_f \cdot_\lambda x) = \Psi(f \cdot x) = \Psi(f) \cdot x = \gamma(T_f) \cdot x$$

for each $f \in L^\infty(G)$ and $\phi \in L^1(G)$. Besides,

$$\gamma(T_f \cdot_\lambda \phi) = \Psi(f \cdot \phi) \neq \Psi(f) \cdot \phi = \gamma(T_f) \cdot \phi$$

for each f and ϕ that satisfy part (c) in the above.

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Hermite-Hadamard Type Inequalities for Lipschitzian Bifunctions In Polar Coordinates

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Abstract

Some Hermite-Hadamard type inequalities for Lipschitzian bifunctions are obtained by the use of polar coordinates. Also bifunctions whose the partial derivative is Lipschitzian are considered.

1. Introduction and Preliminaries

Suppose that $\mathcal{D}(\mathcal{C},\mathcal{R})$ is a closed disk in the plane centered at the point $\mathcal{C}=(a,b)$ having the radius $\mathcal{R}>0$. In [2], the Hermite-Hadamard inequality for a convex function defined on $\mathcal{D}(\mathcal{C},\mathcal{R})$ was presented as the following:

Theorem 1.1. *If the mapping $\mathcal{W}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ is convex on $\mathcal{D}(\mathcal{C},\mathcal{R})$, then one has the inequality*

$$\mathcal{W}(\mathcal{C})\leq\frac{1}{\pi\mathcal{R}^2}\int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{W}(x,y)dxdy\leq\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{W}(\psi)dl(\psi),\tag{1}$$

where $\partial(\mathcal{C},\mathcal{R})$ is the circle centered at the point $\mathcal{C}=(a,b)$ with radius $\mathcal{R}$. The above inequalities are sharp.

Note that the classic form of Hermite-Hadamard inequality (see [4, 5, 7]) for a real valued convex function f defined on $[a,b]$ is as follows:

$$f\Big(\frac{a+b}{2}\Big)(b-a)\leq\int_a^b f(x)dx\leq(b-a)\frac{f(a)+f(b)}{2}.\tag{2}$$

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Generally, in the literature associated to any Hermite-Hadamard type inequality there exist two inequalities which we call them trapezoid and mid-point type inequalities. The names "trapezoid" and "mid-point" comes from two classic inequalities (Due to their geometric interpretation) related to the Hermite-Hadamard inequality obtained in [3] and [6], respectively:

$$\left|\int_a^b f(x)dx-(b-a)\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{8}(b-a)^2\big(|f'(a)|+|f'(b)|\big), \tag{3}$$

and

$$\left|\int_a^b f(x)dx-(b-a)f\Big(\frac{a+b}{2}\Big)\right|\leq\frac{1}{8}(b-a)^2\big(|f'(a)|+|f'(b)|\big), \tag{4}$$

where $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ is a differentiable mapping on $I^\circ, a,b\in I^\circ$ with $a<b$ and $|f'|$ is convex on $[a,b]$.
Recently in [1] the authors obtained the trapezoid and mid-point type inequality related to (1) as the following, respectively.

Theorem 1.2. *Consider a set $I\subset\mathbb{R}^2$ with $\mathcal{D}(\mathcal{C},\mathcal{R})\subset I^\circ$. Suppose that the mapping $\mathcal{W}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ has continuous partial derivatives in the disk $\mathcal{D}(\mathcal{C},\mathcal{R})$ with respect to the variables ρ and φ in polar coordinates. If for any constant $\varphi\in[0,2\pi]$, the function $\left|\frac{\partial\mathcal{W}}{\partial\rho}\right|$ is convex with respect to the variable ρ on $[0,\mathcal{R}]$ then:*

$$\left|\frac{1}{2\pi\mathcal{R}}\int_{\partial(\mathcal{C},\mathcal{R})}\mathcal{W}(\gamma)dl(\gamma)-\frac{1}{\pi\mathcal{R}^2}\int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{W}(x,y)dxdy\right|\leq\frac{1}{6\pi}\int_{\partial(\mathcal{C},\mathcal{R})}\left|\frac{\partial\mathcal{W}}{\partial r}\right|(\gamma)dl(\gamma), \tag{5}$$

and

$$\left|\frac{1}{\pi\mathcal{R}^2}\int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})}\mathcal{W}(x,y)dxdy-\mathcal{W}(C)\right|\leq\frac{2}{3\pi}\int_{\partial(\mathcal{C},\mathcal{R})}\left|\frac{\partial\mathcal{W}}{\partial r}\right|(\gamma)dl(\gamma). \tag{6}$$

Note that inequality (5) is sharp.

In this paper, we obtain some trapezoid and mid-point type inequalities related to (1) for Lipschitzian mappings defined on the disk $\mathcal{D}(\mathcal{C},\mathcal{R})$ in a plane. Also we investigate trapezoid and mid-point type inequalities in the case that in polar coordinates (ρ,φ), the derivative of considered function with respect to the variable ρ is Lipschitzian.

2. Main Results

We start with the definition of $\mathcal{L}$-Lipschitzian bifunctions:

Definition 2.1. [8] A function $\mathcal{W}:I\subset\mathbb{R}^2\to\mathbb{R}$ is said to satisfy a Lipschitz condition (briefly $\mathcal{L}$-Lipschitzian) on I with respect to a norm $||.||$, if there exists a constant $\mathcal{L}>0$ such that

$$|\mathcal{W}(X_1)-\mathcal{W}(X_2)|\leq\mathcal{L}||X_1-X_2||,$$

for any $X_1,X_2\in I$.

If $\mathcal{W}:\mathcal{D}(\mathcal{C},\mathcal{R})\to\mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{L}>0$ and the Euclidean norm $||.||$, then for any $X_1=(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)$ and $X_2=(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)$ we have

$$\begin{aligned}|\mathcal{W}(X_1)-\mathcal{W}(X_2)|&=\Big|\mathcal{W}(a+\rho_1\cos\varphi_1,b+\rho_1\sin\varphi_1)-\mathcal{W}(a+\rho_2\cos\varphi_2,b+\rho_2\sin\varphi_2)\Big|\\&\leq\mathcal{L}||(\rho_1\cos\varphi_1-\rho_2\cos\varphi_2,\rho_1\sin\varphi_1-\rho_2\sin\varphi_2)||=\mathcal{L}\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\varphi_1-\varphi_2)},\end{aligned}$$

for any $\rho_1,\rho_2\in[0,\mathcal{R}]$ and $\varphi_1,\varphi_2\in[0,2\pi]$. Also it is obvious that if $\mathcal{W}:I\subseteq\mathbb{R}^2\to\mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{L}>0$ on I, then it is continuous and so integrable on I.

2.1. $\mathcal{W}$ is Lipschitzian

The first result of this section is the trapezoid type inequality related to (1) for the case that our considered function is Lipschitzian. We start with a lemma.

Lemma 2.2. *Define a function $\mathcal{W} : \mathcal{D}(\mathcal{C},\mathcal{R}) \to \mathbb{R}$ as*

$$\mathcal{W}(X) = \mathcal{W}\big(a + \rho cos\varphi, b + \rho sin\varphi\big) = \mathcal{L}(\mathcal{R} - \rho),$$

for fixed $\mathcal{L} > 0$ and all $0 \leq \rho \leq \mathcal{R}$, $0 \leq \varphi \leq 2\pi$. Then the function $\mathcal{W}$ is $\mathcal{L}$-Lipschitzian

Theorem 2.3. *Suppose that the mapping $\mathcal{W} : \mathcal{D}(\mathcal{C},\mathcal{R}) \to \mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{L} > 0$ and the Euclidean norm $||.||$. Then:*

$$\left| \frac{1}{2\pi\mathcal{R}} \int_{\partial(\mathcal{C},\mathcal{R})} \mathcal{W}(\psi) dl(\psi) - \frac{1}{\pi\mathcal{R}^2} \int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{W}\big(x,y\big) dxdy \right| \leq \frac{\mathcal{L}\mathcal{R}}{3}, \tag{7}$$

where $\partial(\mathcal{C},\mathcal{R})$ is the boundary of $\mathcal{D}(\mathcal{C},\mathcal{R})$ and $\psi : [0, 2\pi] \to \mathbb{R}^2$ is its corresponding curve. Also inequality (7) is sharp.

The following result is the mid-point type inequality related to (1) for Lipschitzian functions defined on a closed disk.

Theorem 2.4. *Suppose that the mapping $\mathcal{W} : \mathcal{D}(\mathcal{C},\mathcal{R}) \to \mathbb{R}$ is Lipschitzian with respect to a constant $\mathcal{L} > 0$ and the Euclidean norm $||.||$. Then:*

$$\left| \frac{1}{\pi\mathcal{R}^2} \int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{W}\big(x,y\big) dxdy - \mathcal{W}(\mathcal{C}) \right| \leq \frac{2\mathcal{L}\mathcal{R}}{3}. \tag{8}$$

Furthermore inequality (8) is sharp.

In the following example for a given function it is illustrated that how we can obtain a Lipschitz constant $\mathcal{L}$ for a real valued bifunction defined on a disk.

Example 2.5. *Consider $\mathcal{W}(x,y) = (x-a)^n + (y-b)^n$, $(x,y) \in \mathcal{D}(\mathcal{C},\mathcal{R})$, $n \in \mathbb{N}$. We find a Lipschitz constant for $\mathcal{W}$ as follows:*
For $X_1, X_2 \in \mathcal{D}(\mathcal{C},\mathcal{R})$ consider the path $\eta : [0,1] \to \mathcal{D}(\mathcal{C},\mathcal{R})$ from X_2 to X_1 in $\mathcal{D}(\mathcal{C},\mathcal{R})$ as

$$\eta(s) = sX_1 + (1-s)X_2,$$

for $s \in [0,1]$. The fundamental theorem of calculus implies that:

$$\big|\mathcal{W}(X_1) - \mathcal{W}(X_2)\big| = \big|\mathcal{W}(\eta(1)) - \mathcal{W}(\eta(0))\big| = \bigg| \int_0^1 \frac{d\mathcal{W}(\eta(s))}{ds} ds \bigg|.$$

Also the chain rule for differentiation implies that:

$$\frac{d\mathcal{W}(\eta(s))}{ds} = \nabla\mathcal{W}(\eta(s)).\eta'(s) = \nabla\mathcal{W}(\eta(s))(X_1 - X_2),$$

where ∇f is the gradient vector of $\mathcal{W}$. So

$$\begin{aligned} &\bigg| \int_0^1 \frac{d\mathcal{W}(\eta(s))}{ds} ds \bigg| = \bigg| \int_0^1 \nabla\mathcal{W}(\eta(s))(X_1 - X_2) ds \bigg| \leq ||X_1 - X_2|| \int_0^1 ||\nabla\mathcal{W}(\eta(s))|| ds \\ &\leq ||X_1 - X_2|| \sup_{w \in \mathcal{D}(\mathcal{C},\mathcal{R})} ||\nabla\mathcal{W}(w)||, \end{aligned}$$

which implies that

$$|\mathcal{W}(X_1) - \mathcal{W}(X_2)| \leq ||X_1 - X_2|| \sup_{w \in \mathcal{D}(\mathcal{C},\mathcal{R})} ||\nabla \mathcal{W}(w)||.$$

Now we conclude that $\mathcal{L} = \sup_{w \in \mathcal{D}(\mathcal{C},\mathcal{R})} ||\nabla \mathcal{W}(w)||$ *(if exists) is a Lipschitz constant for* $\mathcal{W}$. *Therefore for any* $(x, y) \in \mathcal{D}(\mathcal{C}, \mathcal{R})$ *we have*

$$\nabla \mathcal{W}(x, y) = \big(n(x-a)^{n-1}, n(y-b)^{n-1}\big),$$

and then by the use of polar transformation we get

$$\begin{aligned}||\nabla \mathcal{W}(w)|| &= \sqrt{(n(x-a)^{n-1})^2 + (n(y-b)^{n-1})^2} = n\sqrt{(\rho^2 \cos^2 \varphi)^{n-1} + (\rho^2 \sin^2 \varphi)^{n-1}} \\ &\leq n\sqrt{(\rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi)^{n-1}} = n\rho^{n-1} \leq n\mathcal{R}^{n-1}.\end{aligned}$$

So we can choose $\mathcal{L} = \sup_{w \in \mathcal{D}(\mathcal{C},\mathcal{R})} ||\nabla \mathcal{W}(w)|| = n\mathcal{R}^{n-1}$ *as a Lipschitz constant for* $\mathcal{W}$ *on* $\mathcal{D}(\mathcal{C}, \mathcal{R})$.

2.2. $\dfrac{\partial \mathcal{W}}{\partial \rho}$ *is Lipschitzian*

In this part, we investigate the trapezoid and mid-point type inequalities in the case that in polar coordinates (ρ, φ), the partial derivative of considered function with respect to the variable ρ is Lipschitzian in the Euclidean norm $||.||$.

Theorem 2.6. *Consider a set* $I \subset \mathbb{R}^2$ *with* $\mathcal{D}(\mathcal{C}, \mathcal{R}) \subset I^\circ$ *and a mapping* $\mathcal{W} : \mathcal{D}(\mathcal{C}, \mathcal{R}) \to \mathbb{R}$ *such that* $\dfrac{\partial \mathcal{W}}{\partial \rho}$ *(partial derivative of* $\mathcal{W}$ *with respect to the variable* ρ *in polar coordinates) is Lipschitzian with respect to a constant* $\mathcal{L} > 0$ *and the Euclidean norm* $||.||$. *Then:*

$$\left|\frac{1}{2\pi\mathcal{R}} \int_{\partial(\mathcal{C},\mathcal{R})} \mathcal{W}(\psi)dl(\psi) - \frac{1}{\pi\mathcal{R}^2} \int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{W}(x, y)dxdy\right| \leq \frac{\mathcal{L}\mathcal{R}^2}{4}. \tag{9}$$

The following is a trapezoid type inequality for the case that the partial derivative of considered function with respect to the variable "ρ" is Lipschitzian with respect to the Euclidean norm $||.||$.

Theorem 2.7. *Consider a set* $I \subset \mathbb{R}^2$ *with* $\mathcal{D}(\mathcal{C}, \mathcal{R}) \subset I^\circ$ *and a mapping* $\mathcal{W} : \mathcal{D}(\mathcal{C}, \mathcal{R}) \to \mathbb{R}$ *such that* $\dfrac{\partial \mathcal{W}}{\partial \rho}$ *(partial derivative of* f *with respect to the variable* ρ *in polar coordinates) is Lipschitzian with respect to a constant* $\mathcal{L} > 0$ *and the Euclidean norm* $||.||$. *Then:*

$$\left|\frac{1}{\pi\mathcal{R}^2} \int\int_{\mathcal{D}(\mathcal{C},\mathcal{R})} \mathcal{W}(x, y)dxdy - \mathcal{W}(\mathcal{C})\right| \leq \frac{3\mathcal{L}\mathcal{R}^2}{4}. \tag{10}$$

Example 2.8. *Consider* $a, b > 0$, $0 < \mathcal{R} \leq \min\{a, b\}$ *and* $0 \leq \rho \leq \mathcal{R}$. *For* $n \in \mathbb{N}$ *and polar function* $\mathcal{W}(\rho, \varphi) = (a-\rho)^n + (b-\rho)^n$ *which is defined on* $\mathcal{D}\big((a, b), \mathcal{R}\big)$, *by some calculations we can conclude that*

$$\nabla\Big(\frac{\partial \mathcal{W}}{\partial \rho}\Big)(\rho, \varphi) = n(n-1)\big((a-\rho)^{n-2} + (b-\rho)^{n-2}, 0\big),$$

and then

$$\mathcal{L} = \sup_{0 \leq \rho \leq \mathcal{R}, 0 \leq \varphi \leq 2\pi} \Big|\Big|\nabla\Big(\frac{\partial \mathcal{W}}{\partial \rho}\Big)(\rho, \varphi)\Big|\Big| = n(n-1)(a^{n-2} + b^{n-2}),$$

is a Lipschitz constant for $\mathcal{W}$. *Then*

$$\left|A\Big((a-\mathcal{R})^n, (b-\mathcal{R})^n\Big) - A(a^n, b^n)\right| \leq \frac{n(n-1)A(a^{n-2}, b^{n-2})\mathcal{R}^2}{2},$$

where $A(a, b) = \frac{a+b}{2}$ *is arithmetic mean of* a *and* b.

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A new version of Kakutani fixed point theorem with applications

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Abstract

The aim of this paper is to extend the Kakutani's fixed point theorem from locally convex topological vector spaces to linear topological spaces. Finally, as an application, an existence result of a solution for quasi-equilibrium problem is given.

1. Introduction

Fixed-point theory is an important branch of nonlinear analysis. It is used to investigate the conditions under which single-valued or multivalued mappings have solutions. Numerous problems occuring in different branches of mathematics, such as differential equations, optimization theory and variational analysis (see, for instance, [1, 2]).
In mathematical analysis, the Kakutani's fixed-point theorem [8] is a fixed point theorem for set-valued mapping. It provides sufficient conditions for a set valued mapping defined on a convex, compact subset of a Euclidean space to have a fixed point, i.e. a point which is mapped to a set containing it. The Kakutani fixed point theorem is a generalization of Brouwer fixed point theorem. The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact convex subsets of Euclidean spaces. Kakutani's theorem extends this to set-valued functions.
The theorem was developed by Kakutani in 1941 [8] and was used by Nash in his description of Nash equilibria [10]. The existence of Nash equilibria point is closely related to fixed point theory, intersection theorems for a family of sets and existence of maximal elements for set valued mappings.
The quasi equilibrium problem (QEP) was introduced and studied by Noor and Oettli [11]. The quasi-equilibrium problem is a mixed problem of fixed point and equilibrium problem (EQ) (for more details of EQ the reader can

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consulate to [4, 6, 7, 11]) is to find $x \in K$ such that $x \in A(x)$ and $f(x, y) \geq 0, \forall y \in A(x)$, where K is a subset of a topological vector space and $A : K \to 2^K$, the set of all subsets of K, is a set-valued mapping.
In the book of Aubin and Cellina [3](see, Pages 85,213,232) is proved the existence of equilibrium point, i.e. Brouwer fixed point Theorem, which is equivalent to the Kakutani's Theorem. From an applied point of view in economic dynamics, the method described in [9] is based on calulating locally conjugate mappings of compositions of set-valued mappings, where the duality problem is formulated for the problem under consideration.
The first aim of this paper is to extend the Kakutani's theorem from locally convex topological vector space to topological vector space. Finally, as an application, an existence result of a solution for quasi-equilibrium problem is given.
The rest of this section we recall some definitions and results which are needed in the next section.

Definition 1([12]) Let K be a non-empty subset of topological vector space X. A set-valued mapping $T : K \to 2^X$ is called a KKM-mapping if for every finite subset $\{x_1, x_2, ..., x_n\}$ *of* K, $conv\{x_1, x_2, ..., x_n\}$(=$\{\sum_{i=1}^n t_i x_i : t_i \geq 0, \sum_{i=1}^n t_i = 1\}$) is contained in $\bigcup_{i=1}^n T(x_i)$, where $conv$ denotes the convex hull.

The KKM-mappings were first considered by Knaster, Kuratowski and Mazurkiewicz (KKM) [12] in 1920, in order to guarantee the finite intersection property for values of the mapping.

Lemma 1 ([5]) Let K be a nonempty subset of a topological vector space X and $F : K \longrightarrow 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\cap_{x\in B} F(x)$ is compact. Then $\cap_{x\in K} F(x) \neq \emptyset$.

Remark that if $F : K \longrightarrow 2^X$ is a KKM-mapping with closed values in K, then the family $\{Fx : x \in X\}$ of sets has the finite intersection property.

2. Main results

In this section, we apply the KKM-mapping to obtain the existence of solutions for QEP.

Theorem 1 Let K be a compact convex subset of a Hausdorff topological vector space and $A : K \to 2^K$ be a nonempty set-valued mapping such that for every $y \in K$, $A^{-1}(y) = \{x \in K : y \in A(x)\}$ is open in K. Then there exists $x^* \in K$such that $x^* \in (convA(x^*) = \{\sum_{i=1}^n t_i x_i : t_i \geq 0, \sum_{i=1}^n t_i = 1, x_i \in A(x^*)\})$.
One can consider Theorem 1 as a topological vector space version of Kakutani's fixed point theorem which is a generalization of Brouwer's fixed point theorem.

Theorem 2 Let K be a compact convex subset of a Hausdorff topological vector space and $A : K \to 2^K$ be a nonempty set-valued mapping with convex values such that for every $y \in K$, $A^{-1}(y)$ is open in K. Then A has a fixed point, that is, there exists $x^* \in K$such that $x^* \in A(x^*)$.
In the next result, by using Theorem 1, a new proof of Ky Fan's Lemma 1 is provided.

Lemma 2 Let K be a compact convex subset of a topological vector space and $T : K \to 2^K$ *be a* KKM-mapping with closed values in K. Then $\bigcap_{x\in K} T(x) \neq \emptyset$.
The next result is a new version of Lemma 2 by relaxing the compactness of the convex set K.

Theorem 3 *Let K be a non-empty convex subset of a topological vector space and $A : K \to 2^K$ be a KKM-mapping with closed values in K. Further, assume that there exist a compact convex subset B_0 of K and a compact subset N_0 of K such that for each $x \in K \backslash N_0$, there exists $u \in B_0$ that $x \notin A(u)$. Then $\bigcap_{u\in K} A(u) \neq \emptyset$.*
We are now ready to present the first application of this section.

Theorem 4 Let $f : C \times C \to$ R be a function and $K : C \to 2^C$ be a set-valued mapping, where C is a non-empty compact and convex subset of a Hausdorff topological vector space of X. Assume that the following conditions are satisfied:

i) for all $x \in C$, $K(x)$ *is a nonempty closed and convex set and* $f(x,x) \geq 0$*;*
ii) for each $x \in C$ and $y \in K(x)$ the set $\{w \in K(x) : f(w,y) < 0\}$ is *open and convex;*
iii) for all $x \in C$, $y \in K(x)$, the set $\{w \in K(x) : f(w,y) \geq 0\}$ is convex.
iv) $K^{-1}(y) \bigcap \{x \mid y \in EP(f,\ K(x))\}$ is *open.*
Then the solution set of quasi equilibrium problem $(QEP(f,K))$ is *non-empty and compact.*
Theorem 5 *Let* $A : K \to 2^K$ *be a set-valued mapping, where* K *is a non-empty convex subset of a topological vector space of* X*. Assume that*

i) For each $x \in K$, $A(x)$*is a convex set;*
ii) *For every* $x \in K$, $x \notin A(x)$*;*
iii) For each $y \in K$, $A^{-1}(y) = \{x \in K : y \in A(x)\}$ is *open in* K*;*
iv) There exist *non-empty compact and convex subset* $B \subseteq K$*and non-empty compact subset* $N \subseteq K$ *such that*

$$A(x) \bigcap B \neq \phi, \qquad \forall x \in K \backslash N.$$

Then there exists $x^* \in K$ such that $A(x^*) = \phi$.
The following example illustrates the previous theorem.

Example 1 Let $X = \mathbb{R}$ the real line, $K = [0,1]$ and $A : K \to 2^K$ be defined by $A(x) = (0,x)$. It is obvious that for each x of K the $A(x)$ is convex and $x \notin A(x)$. It is clear that $A^{-1}(y) = \emptyset$ when $y = 0,1$ and $A^{-1}(y) = (y,1]$ for $y \in (0,1)$ which are open in K. Hence A satisfies all the assumptions of Theorem 2 and $A(x^* = 0) = \emptyset$.

Remark 1 By reviewing the proof of Theorem 5, one can replace conditions (i) and (ii) of it by the following condition:

$$x \notin convA(x), \qquad \forall x \in K.$$

Because in line seven of the proof we obtained

$$x_i \in A(z), \qquad \forall i = 1,2,...,n.$$

where $z = \sum_{i=1}^{n} \lambda_i x_i$, $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i, so

$$x_i \in convA(z).$$

Hence $z = \sum_{i=1}^{n} \lambda_i x_i \in convA(z)$ and the rest of the proof can be continued. Moreover, it follows from condition (ii) that the set-valued mapping A is never a *KKM* mapping. Hence we cannot use Ky Fan's Lemma 1 for the mapping A.

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Existence Theorems of Solutions for Generalized Operator Equilibrium Problems

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Abstract

In this paper, a system of generalized operator equilibrium problems(for short, SGOEP) in the setting of topological vector spaces is introduced. Moreover, using Ky Fan's lemma an existence result for the generalized operator equilibrium problem(for short, GOEP) is established. The results of the paper can be viewed as a generalization and improvement of the corresponding results in this area.

1. Introduction

Throughout the paper, unless otherwise specified, we use the following notations.
Let I be an index set, for each $i \in I$, let X_i and Y_i stand for topological vector spaces(for short, t.v.s.) and $L(X_i, Y_i)$, the space of all continuous linear operators from X_i into Y_i. Consider a family of nonempty convex subset $\{K_i\}_{i\in I}$ with K_i in $L(X_i, Y_i)$. The symbol $\Pi_{j\in I}K_j$ denotes the cartesian product of K_j. So for each $f \in \Pi_{j\in I}K_j$, we have $f = (f_j)_{j\in I}$, where $f_j \in K_j$.
For each $i \in I$, let $C_i : \Pi_{j\in I}K_j \to 2^{Y_i}$ be a set-valued mapping such that, for each $f \in \Pi_{j\in I}K_j$, $C_i(f)$ is closed, pointed convex cone such that $e_i \in intC_i(f)$ (we recall that a subset $C_i(f)$ of Y_i is convex cone and pointed whenever $\lambda C_i(f) + (1-\lambda)C_i(f) \subseteq C_i(f)$, for all $0 < \lambda < 1$, $2C_i(f) \subseteq C_i(f)$ and $C_i(f) \cap -C_i(f) = \{0_{Y_i}\}$, resp.), for more details see [7].
Also for each $i \in I$, let $F_i : \Pi_{j\in I}K_j \times K_i \to 2^{Y_i}$ be a set-valued mapping. We consider the following problem which we call system of generalized operator equilibrium problem(for short, SGOEP):
Find $f^* = (f_j^*)_{j\in I} \in \Pi_{j\in I}K_j$ such that for each $i \in I$,

$$F_i(f^*, g_i) \nsubseteq -C_i(f^*),\ \forall g_i \in K_i. \tag{1}$$

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We remark that, for suitable choices of I, F_i, K_i, X_i, Y_i and C_i, SGOEP (1) reduces to the preoblems presented in [1, 8] and the references therein.
When I is singelton, that is $F_i = F$, $X_i = X$, $Y_i = Y$, $K_i = K \subseteq L(X,Y)$, $C_i = C : K \to 2^Y$, then (1) reduces to the following problem which is called a generalized operator equilibrium problem(for short, GOEP) and studied in [8]:
Find $f^* \in K$ such that

$$F(f^*, g) \nsubseteq -C(f^*), \quad \forall g \in K. \tag{2}$$

Now, we recall some concepts and results which are used in the sequel.

Definition 1.1. [2] Let X and Y be two topological spaces. A set valued mapping $G : X \longrightarrow 2^Y$ is called

(i) upper semicontinuous(u.s.c.) at $x \in X$ if for each open set V containing $G(x)$, there is an open set U containing x such that for each $t \in U, G(t) \subseteq V$; G is said to be u.s.c. on X if it is u.s.c. at all $x \in X$;

(ii) lower semicontinuous(l.s.c.) at $x \in X$ if for each open set V with $G(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $t \in U, G(t) \cap V \neq \emptyset$; G is said to be l.s.c. on X if it is l.s.c. at all $x \in X$;

(iii) closed if the graph of G, that is, the set $\{(x, y) : x \in X, y \in G(x)\}$, is a closed set in $X \times Y$;

(iv) compact if the closure of range G, that is, $clG(X)$, is compact, where $G(X) = \bigcup_{x \in X} G(x)$.

Remark 1.2. *One can see that if $G(x)$ is compact and G is u.s.c., then for any net $\{x_\alpha\} \subseteq X$ such that $x_\alpha \longrightarrow x$ and for every $y_\alpha \in G(x_\alpha)$ there exist $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \longrightarrow y$.*

The nonlinear scalarization mapping that has a crucial role in the paper, was first introduced in [6] in order to apply to study the vector optimization theory and vector equilibrium problems.

Definition 1.3. [6, 10] Let X be a topological vector space with the convex and pointed cone C. The formula

$$\xi_e(x) := inf\{r \in \mathbb{R} : re - x \in C\},$$

where $x \in X$ and $e \in intC$, defines a mapping from X into $\mathbb{R}$(The real line) and is called the nonlinear scalarization mapping on X(with respect to C and e).

The following lemma characterizes some of the important properties of the nonlinear scalarization mapping which are used in the sequel.

Lemma 1.4. *[3, 5, 9] Let X be a t.v.s. and C be a closed, pointed convex cone of X with $e \in intC$. Then for each $r \in \mathbb{R}$ and $x \in X$ the following statements are satisfied:*

(i) $\xi_e(x) = min\{r \in \mathbb{R} : re - x \in C\}$.

(ii) $\xi_e(x) \leq r \Longleftrightarrow re - x \in C$.

(iii) $\xi_e(x) < r \Longleftrightarrow re - x \in intC$.

(iv) $\xi_e(x) = r \Longleftrightarrow x \in re - \partial C$, *where ∂C is the topological boundary of C.*

(v) $y_2 - y_1 \in C \Longrightarrow \xi_e(y_1) \leq \xi_e(y_2)$.

(vi) The mapping ξ_e is continuous, positively homogeneous and subadditive(that is sublinear) on X.

For proving an existence result of an eqeuilibrium problem, Ky Fan's lemma plays a key role. We are going now to state it. Before stating it we need the following definition.

Definition 1.5. [4] Let K be a nonempty subset of topological vector space X. A set-valued mapping $T : K \to 2^X$ is called a KKM-mapping if, for every finite subset $\{x_1, x_2, ..., x_n\}$ of K, $conv\{x_1, x_2, ..., x_n\}$ is contained in $\bigcup_{i=1}^{n} T(x_i)$, where $conv$ denotes the convex hull.

Ky Fan in 1984 obtained the following result, which is known as Ky Fan's lemma.

Lemma 1.6. *(Ky Fan-1984) [4] Let K be a nonempty subset of topological vector space X and $T : K \to 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B} T(x)$ is compact. Then*

$$\bigcap_{x\in K} T(x) \neq \emptyset.$$

2. Main results

The following maximal element theorem which proved by Ky Fan's lemma will be used in establishing some existence results in this paper.

Theorem 2.1. *For each $i \in I$, let K_i be a nonempty convex subset of $L(X_i, Y_i)$ and let $\Gamma_i : \Pi_{j\in I}K_j \to 2^{K_i}$ be a set-valued mapping satisfying the following conditions:*

(i) $\forall i \in I$ and $\forall f = (f_j)_{j\in I} \in \Pi_{j\in I}K_j$; $f_i \notin conv\Gamma_i(f)$, where f_i is the ith projection of f;

(ii) $\forall i \in I$ and $\forall g_i \in K_i$; $\Gamma_i^{-1}(g_i)$ is open in $\Pi_{j\in I}K_j$;

(iii) There exist a nonempty compact subset D of $\Pi_{j\in I}K_j$ and a nonempty compact convex subset $E_j \subseteq K_j$, $\forall j \in I$ such that $\forall f \in \Pi_{j\in I}K_j \setminus D$ there exists $j \in I$ such that $\Gamma_j(f) \cap E_j \neq \emptyset$.

Then there exists $f^* \in \Pi_{j\in I}K_j$ such that $\Gamma_j(f^*) = \emptyset$, for each $j \in I$.

Following the same arguments as in the proof of Theorem 2.1, we can get the following result.

Theorem 2.2. *Let all assumptions of Theorem 2.1 and the following conditions hold:*

(i) $\forall i \in I$ and $\forall f \in \Pi_{j\in I}K_j$; $\Gamma_i(f)$ is convex;

(ii) $\forall i \in I$ and $\forall g_i \in K_i$, $\Gamma_i^{-1}(g_i)$ is open in $\Pi_{j\in I}K_j$;

(iii) There exist a nonempty compact subset D of $\Pi_{j\in I}K_j$ and a nonempty compact convex subset $E_i \subseteq K_i$, $\forall i \in I$, such that $\forall f \in \Pi_{j\in I}K_j \setminus D$ there exists $i \in I$ with $\Gamma_i(f) \cap E_i \neq \emptyset$.

Then

(a) if $\exists i \in I$ such that $\Gamma_i(f) \neq \emptyset, \forall f \in \Pi_{j\in I}K_j$, then there exist $i \in I$ and $f \in \Pi_{j\in I}K_j$ such that $f_i \in \Gamma_i(f)$.

(b) if $\forall i \in I$ and $\forall f \in \Pi_{j\in I}K_j, f_i \notin \Gamma_i(f)$, then there exists $f^* \in \Pi_{j\in I}K_j$ such that $\Gamma_i(f^*) = \emptyset$, for each $i \in I$.

Now applying the properties of nonlinear scalarization mapping and Lemma 2.1, we prove the following existence theorem for SGOEP.

Theorem 2.3. *For each $i \in I$, let K_i be nonempty and convex subset of $L(X_i, Y_i)$ and $F_i : \Pi_{j\in I}K_j \times K_i \longrightarrow 2^{Y_i}$ be a mapping satisfying the following conditions:*

(i) $\forall i \in I$ and $\forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j$, $F_i((f_j)_{j\in I}, f_i) \nsubseteq -C((f_j)_{j\in I})$, where f_i is the ith component of $(f_j)_{j\in I}$;

(ii) $\forall i \in I$ and $\forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j$, the mapping $g_i \longrightarrow \xi_{e_i} oF_i((f_j)_{j\in I}, g_i)$ is $\mathbb{R}_+$-natural quasi convex;

(iii) $\forall i \in I$ *and* $\forall g_i \in K_i$*, the set*

$$\{(f_j)_{j\in I} \in \Pi_{j\in I}K_j : F_i((f_j)_{j\in I}, g_i) \nsubseteq -C((f_j)_{j\in I})\},$$

is closed in $\Pi_{j\in I}K_j$*;*

(iv) There exist a nonempty compact subset D *of* $\Pi_{j\in I}K_j$ *and a nonempty compact convex subset* $E_i \subseteq K_i$*,* $\forall i \in I$ *such that* $\forall (f_j)_{j\in I} \in \Pi_{j\in I}K_j \setminus D$*; there exist* $i \in I$ *and* $g_i^* \in E_i$ *with*

$$F_i((f_j)_{j\in I}, g_i^*) \subset -C_i((f_j)_{j\in I}).$$

Then the solution set of SGOEP is nonempty and relatively compact.

The next result is a special case of Theorem 2.3 when I is singelton.

Theorem 2.4. *Let* X *and* Y *be two t.v.s. and* K *be a nonempty convex subset of* $L(X,Y)$*,* C *be a closed, pointed convex cone in* Y *with* $e \in intC$ *and also* $F : K \times K \longrightarrow 2^Y$ *be a set-valued mapping with nonempty values. Assume that the following conditions hold:*

(i) for all $f \in K$*,* $F(f,f) \nsubseteq -C(f)$*;*

(ii) for all fixed $f \in K$*, the mapping* $g \longrightarrow \xi_e oF(f,g)$ *is* $\mathbb{R}_+-$*natural quasi convex;*

(iii) for all $g \in K$*, the set*

$$\{f \in K : F(f,g) \subseteq -C(f)\},$$

is open in K*;*

(iv) there exist a nonempty compact convex subset D *of* K *and a nonempty compact subset* E *of* K *such that for each* $f \in K \setminus D$*, there exists* $g \in E$ *satisfying* $F(f,g) \subseteq -C(f)$*.*

Then the solution set of $GOEP$ *is nonempty and relatively compact.*

Remark 2.5. *If* $F(f,.)$ *is* $C(.)-$*natural quasi convex, then the mapping* $g \longrightarrow \xi_e oF(f,g)$ *is* $\mathbb{R}_+-$*natural quasi convex. Therefore Theorem 2.4 is valid when one replaces* $\xi_e oF(f,.)$ *by* $F(f,.)$*.*

The next example shows that although Theorem 2.4 is true when $F(f,.)$ is $C(.)-$natural quasi convex but condition (ii) is sharper than it.

Example 2.6. *Assume that*

$$f(x) = \begin{cases} |x| & x \in Q \cap [-1,1] \\ 2|x|+1 & x \in Q^c \cap [-1,1]. \end{cases}$$

Define the mapping $F : [-1,1] \longrightarrow 2^{\mathbb{R}^2}$ *by*

$$F(x) = [f(x), f(x)+1] \times [3,4],$$

$\xi_e oF(x) = [3,4]$*, where* $C = \{(x,y) : x,y \geq 0\}$ *and* $e = (1,1) \in intC$*.*

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Fixed point theorems in ordered non-Archimedean fuzzy metric spaces

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Abstract

In this paper, we extend very recent fixed point theorems in the setting of ordered non-Archimedean fuzzy metric spaces. We present some fixed point theorems for self-mappings satisfying generalized (ϕ, ψ)-contraction condition in partially ordered complete non-Archimedean fuzzy metric spaces. On the other hand, we consider a more general class of auxiliary functions in the contractivity condition and we extend recent fixed point theorems for complete ordered non-Archimedean fuzzy metric spaces. Also, we present a few examples to illustrate the validity of the results obtained in the paper.

1. Introduction

Fixed points of mappings satisfying contractive conditions in generalized metric spaces are highly useful in large number of mathematical problems of pure and applied mathematics. There are two well-known extensions of the notion of metric space to frameworks in which imprecise models are considered: fuzzy metric spaces (see [12]) and probabilistic metric spaces [14, 15]. These two concepts are very similar, but they are different in nature. The concept of a fuzzy metric space was introduced in different ways by some authors (see [3, 4]). Gregori and Sapena [4] introduced the notion of fuzzy contractive mappings and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. Mihet [10] developed the class of fuzzy contractive mappings of Gregori and Sapena, considered these mappings in non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michalek, and obtained a fixed point theorem for fuzzy contractive mappings. Lots of different types of fixed point theorems has been presented by many authors by expanding the Banach's result, simultaneously (see [5, 16]). Recently, Sun and Yang introduced the concept of fuzzy metric spaces and proved two common fixed point theorems for four mappings (see [5]).

Recently, many fixed point theorems have been presented for probabilistic metric space $(X, F, *)$, where F is a distance distribution function. Many of them were inspired by the corresponding results on metric spaces. One of the

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most attractive and effective ways to introduce contractivity conditions in the probabilistic framework is based on considering some terms like in the following expression:

$$\frac{1}{F(x,y,t)} - 1, \quad where \quad x, y \in X \; and \; t > 0$$

(see [4, 10, 17]).

In this paper, we consider the more general contractivity conditions, replacing the function $t \to \frac{1}{t} - 1$ by an appropriate function h to establish the existence of fixed points for a self-mapping and common fixed points and coincidence points for two self-mappings in ordered complete fuzzy metric space. Our results generalize Theorem 2.1 and 2.2 of [2] and the corollaries of [6, 11].

2. Preliminaries

Before giving our main results, we recall some basic concepts and results in metric space and fuzzy metric spaces.

Definition 2.1. [1] A point $\nu \in X$, is called coincidence (common fixed) point for two self-mappings T and S, if $T\nu = S\nu$ ($\nu = T\nu = S\nu$).

Definition 2.2. [9] A metric space X with a partially ordered relation $\preceq$ is called a partially ordered metric space and is denoted by $(X, \preceq)$.

Definition 2.3. [9] Let $(X, \preceq)$ be a partially ordered metric space.
(i) If any two elements of X are comparable, then it is called a well-ordered set.
(ii) A self-mapping T on X is said to be monotone nondecreasing, if $T(\nu) \preceq T(\mu)$ for all $\nu, \mu \in X$ with $\nu \preceq \mu$.
(iii) Let T and S be two self-mappings on X. Then T is called monotone S-nondecreasing, if $Tx \preceq Ty$ for all $x, y \in X$ with $Sx \preceq Sy$.

Definition 2.4. [8] Let (X, d) be a metric space.
(i) Two self-mappings T and S on X are called compatible, if for all sequence $\{x_n\}$ with $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$, then $\lim_{n\to\infty} d(TSx_n, STx_n) = 0$.
(ii) A pair of self-mappings (T, S) on X is called weakly compatible, if they commute at their coincidence points, i.e. $T\nu = S\nu$ implies $TS\nu = ST\nu$.

Definition 2.5. [15] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous $t-$norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative,
(TN-2) $*$ is continuous,
(TN-3) $*(a, 1) = a$ for all $a \in [0,1]$,
(TN-4) $*(a, b) \leq *(c, d)$ whenever $a, b, c, d \in [0,1]$ with $a \leq c, b \leq d$.

Definition 2.6. [5] A fuzzy metric space is a triple $(X, F, *)$ where X is a nonempty set, $*$ is a continuous $t-$norm and F is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(FM-1) $F(x, y, t) > 0$ for all $t > 0$,
(FM-2) $F(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $F(x, y, t) = F(y, x, t)$ for all $t > 0$,
(FM-4) $F(x, y, t + s) \geq F(x, z, s) * F(z, y, t)$ for all $s, t > 0$,
(FM-5) $F(x, y, .) : (0, \infty) \to [0, 1]$ is continuous.

If the triangular inequality (FM-4) is replaced by

$$F(x,y,max\{s,t\}) \geq F(x,z,s) * F(z,y,t)$$

for all $x, y, z \in X$ and all $s, t > 0$ or equivalently,

$$F(x,y,t) \geq F(x,z,t) * F(z,y,t), \tag{1}$$

then the triple $(X, F, *)$ is called a non-Archimedean fuzzy metric space [7].

Example 2.7. *Let (X, d) be a metric space. Then the triple $(X, F, *)$ is a fuzzy metric space on X where $*(a, b) = ab$ for all $a, b \in [0, 1]$ and $F(x, y, t) = t/(t + d(x, y))$ for all $x, y \in X$ and all $t > 0$. We call this F as the standard fuzzy metric induced by the metric d. Even if we define $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$, the triple $(X, F, *)$ will be a fuzzy metric space.*

Definition 2.8. [5] Let $\{x_n\}$ be a sequence in a fuzzy (or a non-Archimedean fuzzy) metric space $(X, F, *)$. We say that:

- $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} F(x_n, x, t) = 1$; i.e., for all $t > 0$ and all $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n, x, t) > 1 - \lambda$ for all $n \geq n_0$ (in such a case, we write $\{x_n\} \to x$);
- $\{x_n\}$ is a Cauchy sequence if and only if for all $t > 0$ and all $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n, x_m, t) > 1 - \lambda$ for all $n, m \geq n_0$. $\{x_n\}$ is a G-Cauchy sequence if and only if for all $t > 0$ and all $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $F(x_n, x_{n+p}, t) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$ (i.e., $\lim_{n\to\infty} F(x_n, x_{n+p}, t) = 1$).
- The fuzzy (or the non-Archimedean fuzzy) metric space $(X, F, *)$ is called complete (G-complete) if every Cauchy (G-Cauchy) sequence is convergent.

Lemma 2.9. *[5] Let $(X, F, *)$ be a fuzzy metric space. Then $F(x, y, t)$ is nondecreasing with respect to t for all $x, y \in X$.*

Lemma 2.10. *[5] Let $(X, F, *)$ be a fuzzy metric space. Then F is a continuous function on $X^2 \times (0, \infty)$.*

It is easy to prove that a $F(x, y, t)$ in a non-Archimedean fuzzy metric space $(X, F, *)$ is also nondecreasing with respect to t and continuous for all $x, y \in X$.

Definition 2.11. [5] A (complete) fuzzy metric space $(X, F, *)$ with a partially ordered relation $\preceq$ is called a (complete) partially ordered fuzzy metric space and denoted by $(X, F, *, \preceq)$.

The following families of auxiliary functions were considered in [13].

Definition 2.12. Let Φ be the family of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:
(1) $\phi(t) = 0$ if and only if $t = 0$;
(2) $\lim_{t\to\infty} \phi(t) = \infty$;
(3) ϕ is continuous at $t = 0$.

Definition 2.13. Let Ψ be the class of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:
(1) ψ is nondecreasing;
(2) $\psi(0) = 0$
(3) for a sequence $\{a_n\}$ in $[0, \infty)$ whit $\{a_n\} \to 0$, $\{\psi^n(a_n)\} \to 0$ (ψ^n denotes the nth-iterate of ψ)

It worths mentioning that $\psi \in \Psi$ is continuous at $t = 0$. (Proposition 7 of [13])

The following family of auxiliary functions were considered in [13].

Definition 2.14. Let $\mathcal{H}$ be the family of all functions $h : (0,1] \to [0,\infty)$ satisfying the following conditions,
($\mathcal{H}_1$) for all sequence $\{a_n\}$ in $(0,1]$, $\{a_n\} \to 1$ if and only if $\{h(a_n)\} \to 0$;
($\mathcal{H}_2$) for all sequence $\{a_n\}$ in $(0,1]$, $\{a_n\} \to 0$ if and only if $\{h(a_n)\} \to \infty$.

The previous conditions are guaranteed whenever $h : (0,1] \to [0,\infty)$ is a strictly decreasing bijection such that h and h^{-1} are continuous (in a broad sense, it is sufficient to assume the continuities of h and h^{-1} on the extremes of the respective domains). For instance, this is the case of the function $h(t) = 1/t - 1$ for all $t \in (0,1]$. However, the functions in $\mathcal{H}$ need not to be continuous nor monotone.

Proposition 2.15. *[13] If $h \in \mathcal{H}$, then $h(1) = 0$. Furthermore, $h(t) = 0$ if and only if $t = 1$.*

3. Main results

In this section, we present an extension of fixed point theorems in several ways: the metric space is more general, the contractivity condition is better and the involved auxiliary functions form a wider class.

Theorem 3.1. *Let $(X, F, *, \preceq)$ be a partially ordered G-complete non-Archimedean fuzzy metric space and let $T : X \to X$ be a continuous and nondecreasing mapping with regards to $\preceq$. Suppose that there exist $c \in (0,1), \phi \in \Phi, \psi \in \Psi$ and $h \in \mathcal{H}$ such that*

$$h(F(Tx, Ty, \phi(ct))) \leq \psi(h(M(x,y))) \tag{2}$$

for all $x, y \in X$ with $x \preceq y$ and all $t > 0$ and

$$M(x,y) = max\left\{F(x,y,\phi(t)), \frac{F(x,Tx,\phi(t)) * F(y,Ty,\phi(t))}{1 + F(Tx,Ty,\phi(t))}\right\}. \tag{3}$$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ and also $\lim_{t\to\infty} F(x_0, Tx_0, t) = 1$, then T has at least one fixed point in X.

Proof. If there exists $x_0 \in X$ such that $Tx_0 = x_0$, then the proof is finished. Suppose $x_0 \in X$, such that $x_0 \prec Tx_0$ and $\lim_{t\to\infty} F(x_0, Tx_0, t) = 1$, then construct the sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \geq 0$. Since T is nondecreasing, by using mathematical induction, we get the following

$$\begin{aligned} x_0 \prec Tx_0 = x_1 \preceq Tx_1 = x_2 \preceq .... \preceq Tx_{n-1} = x_n \\ \preceq Tx_n = x_{n+1} \preceq .... \end{aligned} \tag{4}$$

If for some $n_0 \in \mathbb{N}$, $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ then x_{n_0} is a fixed point of T and we have nothing to prove.
Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since $x_n \succ x_{n-1}$ for all $n \geq 1$, by (2) we have

$$\begin{aligned} h(F(x_n, x_{n+1}, \phi(ct))) &= h(F(Tx_{n-1}, Tx_n, \phi(ct))) \\ &\leq \psi(h(M(x_{n-1}, x_n))), \end{aligned} \tag{5}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= max\left\{F(x_{n-1}, x_n, \phi(t)), \frac{F(x_{n-1}, Tx_{n-1}, \phi(t)) * F(x_n, Tx_n, \phi(t))}{1 + F(Tx_{n-1}, Tx_n, \phi(t))}\right\} \\ &= max\left\{F(x_{n-1}, x_n, \phi(t)), \frac{F(x_{n-1}, x_n, \phi(t)) * F(x_n, x_{n+1}, \phi(t))}{1 + F(x_n, x_{n+1}, \phi(t))}\right\}. \end{aligned} \tag{6}$$

Since

$$\frac{F(x_{n-1}, x_n, \phi(t)) * F(x_n, x_{n+1}, \phi(t))}{1 + F(x_n, x_{n+1}, \phi(t))} \leq F(x_{n-1}, x_n, \phi(t)),$$

by (6) we have

$$M(x_{n-1}, x_n) = F(x_{n-1}, x_n, \phi(t)),$$

and hence from (5) again we have

$$h(F(x_n, x_{n+1}, \phi(ct))) \leq \psi(h(F(x_{n-1}, x_n, \phi(t)))) \tag{7}$$

for all $t > 0$ and all $n \geq 1$.
We claim that $\lim_{n\to\infty} F(x_n, x_{n+1}, s) = 1$ for all $s > 0$. In order to prove it, let $s > 0$ be arbitrary. As $\lim_{r\to\infty} c^r s = 0$ and ϕ is continuous at $t = 0$, then $\lim_{r\to\infty} \phi(c^r s) = \phi(0) = 0$. Since $s > 0$, there exists $r \in \mathbb{N}$ such that

$$\phi(c^r s) \leq s.$$

Let $n \in \mathbb{N}$ be such that $n > r$. Applying the contractivity (7), it follows that

$$h(F(x_n, x_{n+1}, \phi(c^r s))) \leq \psi(h(F(x_{n-1}, x_n, \phi(c^{r-1} s)))). \tag{8}$$

Repeating this argument, we find that

$$h(F(x_{n-1}, x_n, \phi(c^{r-1} s))) \leq \psi(h(F(x_{n-2}, x_{n-1}, \phi(c^{r-2} s)))).$$

As ψ is nondecreasing, then

$$\psi(h(F(x_{n-1}, x_n, \phi(c^{r-1} s)))) \leq \psi^2(h(F(x_{n-2}, x_{n-1}, \phi(c^{r-2} s)))). \tag{9}$$

Combining inequalities (8) and (9), we deduce that

$$\begin{aligned} h(F(x_n, x_{n+1}, \phi(c^r s))) &\leq \psi(h(F(x_{n-1}, x_n, \phi(c^{r-1} s)))) \\ &\leq \psi^2(h(F(x_{n-2}, x_{n-1}, \phi(c^{r-2} s)))). \end{aligned}$$

By repeating this argument n times, we have

$$\begin{aligned} h(F(x_n, x_{n+1}, \phi(c^r s))) &\leq \psi^n(h(F(x_0, x_1, \phi(c^{r-n} s)))) \\ &\leq \psi^n(h(F(x_0, x_1, \phi(\frac{s}{c^{n-r}})))), \end{aligned} \tag{10}$$

for all $n > r$. As a consequence,

$$\begin{aligned} \lim_{n\to\infty} \frac{s}{c^{n-r}} = \infty \Rightarrow &\lim_{n\to\infty} \phi(\frac{s}{c^{n-r}}) = \infty \\ \Rightarrow &\lim_{n\to\infty} F(x_0, x_1, \phi(\frac{s}{c^{n-r}})) = 1 \\ \Rightarrow &\lim_{n\to\infty} h(F(x_0, x_1, \phi(\frac{s}{c^{n-r}}))) = 0. \end{aligned}$$

As the sequence $\{a_n = h(F(x_0, x_1, \phi(\frac{s}{c^{n-r}})))\} \to 0$ we have $\{\psi^n(a_n)\} \to 0$. Since $h \in \mathcal{H}$, by (10) we deduce that

$$\lim_{n\to\infty} h(F(x_n, x_{n+1}, \phi(c^r s))) = 0.$$

In particular, as $h \in \mathcal{H}$, condition $(\mathcal{H}_1)$ implies that

$$\lim_{n\to\infty} F(x_n, x_{n+1}, \phi(c^r s)) = 1.$$

Taking into account $\phi(c^r s) < s$, we observe that

$$F(x_n, x_{n+1}, \phi(c^r s)) \leq F(x_n, x_{n+1}, s) \leq 1.$$

Therefore,

$$\lim_{n\to\infty} F(x_n, x_{n+1}, s) = 1$$

which means that $\{x_n\}$ is a G-Cauchy sequence in X, [13, Lemma 15]. Since X is G-complete, there exists $x \in X$ such that $\{x_n\} \to x$.
Also, the continuity of T implies that

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Therefore, x is a fixed point of T in X. □

Example 3.2. *Let $X = [0, 1)$, $*(a, b) = min\{a, b\}$, and*

$$F(x, y, t) = \begin{cases} 1, & \textit{if } x = y \\ \frac{1}{1+max\{x,y\}}, & \textit{otherwise}, \end{cases}$$

where $x, y \in X$ and $t > 0$. It is easy to prove that $(X, F, *, \preceq)$ is a complete partially ordered non-Archimedean fuzzy metric space with usual ordering. Define $T : X \to X$ by $T(x) = x/2$ for all $x \in X$. Assume that $\psi(t) = \phi(t) = t$ for all $t \in [0, \infty)$ and let $h : (0, 1] \to [0, \infty)$ be a strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous (for instance, $h(t) = 1/t - 1$, $t \in (0, 1]$, but any other function verifying these properties yields the same result). In this context, the contractivity conditions (2) and (3) are equivalent to

$$\begin{aligned} & h(F(Tx, Ty, \phi(ct))) \le \psi(h(M(x, y))) \\ \Leftrightarrow\ & h(F(Tx, Ty, ct)) \le h(M(x, y)) \\ \Leftrightarrow\ & F(Tx, Ty, ct) \ge M(x, y) \ge F(x, y, t), \end{aligned}$$

for all $c \in (0, 1)$ and $x, y \in X$, whit $x \neq y$ and for all $t > 0$,

$$\begin{aligned} F(Tx, Ty, ct) &= F(\frac{x}{2}, \frac{y}{2}, ct) \\ &= \frac{1}{1 + max\{\frac{x}{2}, \frac{y}{2}\}} \\ &\ge \frac{1}{1 + max\{x, y\}} \\ &= F(x, y, t). \end{aligned}$$

In the case $x = y$ it is trivial. As a result, the contractivity condition is verified. Also, all the assumptions made in Theorem 3.1 are satisfied and hence, it guarantees that T has a unique fixed point (which is $x = 0$).

By weakening the continuity property of a map T in Theorem 3.1, we have the following result.

Theorem 3.3. *In Theorem 3.1 let X has the property that, for every nondecreasing sequence $\{x_n\}$ with $\{x_n\} \to x$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$, i.e., $x = sup\, x_n$. Then a non-continuous map T has a fixed point in X.*

The uniqueness of an existing fixed point in Theorems 3.1 and 3.3, can be obtained, if the set of fixed pints of T, $Fix(T)$, is well-ordered.

Theorem 3.4. *If in Theorems 3.1 and 3.3, $Fix(T)$, is well-ordered and $\lim_{t\to\infty} F(x, y, t) = 1$ for all $x, y \in Fix(T)$ and also $h \in \mathcal{H}$ is decreasing, then T has a unique fixed point in X.*

We have the following results, which are the generalizations of Theorems 3.1 and 3.3 in the partially ordered non-Archimedean fuzzy metric spaces.

Theorem 3.5. *Let $(X, F, *, \preceq)$ be a partially ordered non-Archimedean fuzzy metric space and suppose $T, S : X \to X$ are continuous mappings such that*
(i) for some $c \in (0, 1)$, $\phi \in \Phi$, $\psi \in \Psi$, and $h \in \mathcal{H}$ with

$$h(F(Tx, Ty, \phi(ct))) \le \psi(h(M_S(x, y))) \tag{11}$$

for all $x, y \in X$ *with* $Sx \preceq Sy$ *and all* $t > 0$ *and*

$$M_S(x,y) = max\left\{F(Sx,Sy,\phi(t)), \frac{F(Sx,Tx,\phi(t)) * F(Sy,Ty,\phi(t))}{1+F(Tx,Ty,\phi(t))}\right\} \tag{12}$$

(ii) $TX \subseteq SX$ *and* SX *is a G-complete subspace of* X*,*
(iii) T *is a monotone* S*-nondecreasing mapping,*
(iv) T *and* S *are compatible.*

If there exists $x_0 \in X$ *such that* $Sx_0 \preceq Tx_0$ *and* $\lim\limits_{t\to\infty} F(Sx_0, Tx_0, t) = 1$*, then* T *and* S *have a coincidence point in* X*.*

Replacing the condition of being weakly compatible instead of compatibility in Theorem 3.5, we obtain the following result.

Corollary 3.6. *Assume in Theorem 3.5,* $\lim\limits_{t\to\infty} F(Tx,Ty,t) = 1$ *for all coincidence points of* T *and* S *and* $h \in \mathcal{H}$ *is decreasing. If* X *has the property that for every nondecreasing sequence* $\{Sx_n\}$ *in* X *such that* $\lim\limits_{n\to\infty} Sx_n = Sx$ *implies that* $Sx_n \preceq Sx$ *for all* $n \in \mathbb{N}$*, that is* $Sx = sup\ Sx_n$*. If* T *and* S *are weakly compatible for every coincidence point* ν *of* T *and* S *with* $S\nu \preceq S(S\nu)$*, then* T *and* S *have common fixed point in* X*. Furthermore, the set of common fixed point of* T *and* S *is well-ordered if and only if* T *and* S *have one common fixed point in* X*.*

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Some theorems in Menger probabilistic metric spaces

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Abstract

In this paper, we extend very recent fixed point theorems in the setting of Menger probabilistic metric spaces. We present some fixed point theorems for self mappings satisfying generalized (ϕ, ψ)-contraction condition in Menger probabilistic metric spaces. On the other hand, we consider a more general class of auxiliary functions in the contractivity condition. The aim of this paper is to prove the existence of coincidence points, coupled coincidence points, common fixed point and common coupled fixed points of auxiliary functions in the contractivity condition which include both reciprocal and weakly reciprocal continuous mappings on Menger probabilistic metric spaces.

1. Introduction

The concept of a Menger probabilistic metric space was initiated by Menger [8]. The idea of Menger was ti use a distribution function instead of a nonnegative number for the value of a metric. In recent times, many fixed point theorems have been presented in the setting of probabilistic metric space $(X, F, \triangle)$ that F is a distance distribution function. Many of them were inspired by their corresponding results on metric spaces. One of the most attractive, effective ways to introduce contractivity conditions in the probabilistic framework is based on considering some terms like in the following expression:

$$\frac{1}{F_{x,y}(t)} - 1, \quad where \quad x, y \in X \; and \; t > 0$$

(see [9, 15]).

In this paper, we consider the more general contractivity conditions replacing the function $t \to \frac{1}{t} - 1$ by an appropriate function h to establish an existence of a fixed point and its uniqueness of a self mapping and common fixed point, coincidence point for two self mappings in Menger probabilistic metric space. Also, we will establish an existence of a coupled coincidence point and common coupled fixed point for two mappings in Menger probabilistic metric space. Our results generalize Theorem 2.1 and 2.2 of [2] and the Corollaries of [3, 4, 10].

Before giving our main results, we recall some basic definitions and facts which will be used further on.

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Definition 1.1. [6] A function $f : (-\infty, \infty) \to [0, 1]$ is called a distribution function, if it is nondecreasing and left continuous with $\inf_{x\in\mathbb{R}} f(x) = 0$. If in addition $f(0) = 0$, then f is called a distance distribution function. Furthermore, a distance distribution function f satisfying $\lim_{t\to\infty} f(t) = 1$ is called a Menger distance distribution function.

The set of all Menger distance distribution functions is denoted by Λ^+.

Definition 1.2. [6] A triangular norm (abbreviated, T-norm) is a binary operation $\triangle$ on $[0, 1]$, which satisfies the following conditions:
(a) $\triangle$ is associative and commutative,
(b) $\triangle$ is continuous,
(c) $\triangle(a, 1) = a$ for all $a \in [0, 1]$,
(d) $\triangle(a, b) \leq \triangle(c, d)$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Among the important examples of a T-norm we mention the following two T-norms: $\triangle_p(a, b) = ab$ and $\triangle_m(a, b) = \min\{a, b\}$. The T-norm $\triangle_m$ is the strongest T-norm, that is, $\triangle \leq \triangle_m$ for every T-norm $\triangle$.

Definition 1.3. [5] A triangular norm $\triangle$ is said to be of H-type ($Had\check{z}i\acute{c}$ type) if a family of functions $\{\triangle^n(t)\}_{n=1}^{+\infty}$ is equicontinuous at $t = 1$, that is,

$$\forall\varepsilon \in (0, 1), \exists\delta \in (0, 1) : t > 1 - \delta \Rightarrow \triangle^n(t) > 1 - \varepsilon \quad (n \geq 1),$$

where $\triangle^n : [0, 1] \to [0, 1]$ is defined as follows:

$$\triangle^1(t) = \triangle(t, t), \quad \triangle^n(t) = \triangle(t, \triangle^{n-1}(t)), \quad n = 2, 3, \ldots .$$

Obviously, $\triangle^n(t) \leq t$ for any $n \in \mathbf{N}$ and $t \in [0, 1]$.

Definition 1.4. [14] A Menger probabilistic metric space (abbreviated, Menger PM space) is a triple $(X, F, \triangle)$ where X is a nonempty set, $\triangle$ is a continuous T-norm and F is a mapping from $X \times X$ into Λ^+ such that, if $F_{p,q}$ denotes the value of F at the pair (p, q),the following conditions hold:
(PM_1) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$ $(p, q \in X)$,
(PM_2) $F_{p,q}(t) = F_{q,p}(t)$ for all $t > 0$ and $p, q \in X$,
(PM_3) $F_{p,r}(s + t) \geq \triangle(F_{p,q}(s), F_{q,r}(t))$ for all $p, q, r \in X$ and every $s > 0, t > 0$.

Definition 1.5. [14] A sequence $\{x_n\}$ in Menger PM space X is said to converge to a point x in X (written as $x_n \to x$), if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there is an integer $N(\epsilon, \lambda) > 0$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$, for all $n \geq N(\epsilon, \lambda)$. The sequence is said to be Cauchy sequence if for each $\epsilon > 0$ and $\lambda \in (0, 1)$, there is an integer $N(\epsilon, \lambda) > 0$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$, for all $n, m \geq N(\epsilon, \lambda)$. A Menger PM space $(X, F, \triangle)$ is said to be complete if every Cauchy sequence in X converges to a point of X. Also, the sequence is said to be G-Cauchy sequence if for each $\epsilon > 0$ and $\lambda \in (0, 1)$, there is an integer $N(\epsilon, \lambda) > 0$ such that $F_{x_{n+p},x_n}(\epsilon) > 1 - \lambda$, for all $n \geq N(\epsilon, \lambda)$ and $p \in \mathbb{N}$. A Menger PM space $(X, F, \triangle)$ is said to be G-complete if every G-Cauchy sequence in X converges to a point of X.

It is easy to see that, for $\tilde{a} = (x, y), \tilde{b} = (u, v) \in X^2 = X \times X$, the function $\tilde{F}$ from X^2 into Λ^+, is a distribution function:

$$\tilde{F}_{\tilde{a},\tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\} \quad \textit{for all } t > 0.$$

Lemma 1.6. *[16] If $(X, F, \triangle)$ is a complete Menger PM space, then $(X^2, \tilde{F}, \triangle)$ is also a complete Menger PM space.*

Definition 1.7. [1] **(i)** Let f and g be two maps from X into Y. We say f and g have a coincidence point, if there exists a point x in X such that $fx = gx$.
(ii) Let f and g be two self maps on X. We say $x \in X$ is a common fixed point of f and g, if $fx = gx = x$.
(iii) An element $(x, y) \in X \times X$ is called a coupled point of the mapping $T : X \times X \to X$, if $T(x, y) = x$ and $T(y, x) = y$.
(iv) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = gx$ and $T(y, x) = gy$.
(v) An element $(x, y) \in X \times X$ is called a common coupled fixed point of the mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = gx = x$ and $T(y, x) = gy = y$.

Definition 1.8. [4] Let f and g be two self maps of a Menger PM space $(X, F, \triangle)$. Then f and g are said to be Menger compatible if $\lim_{n\to\infty} F_{fgx_n,gfx_n}(t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x \in X$.

Two self mappings f and g of a metric space (X, d) are called R-weakly commuting of type-(A_g) [12], if there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all $x \in X$. Similarly, two self mappings f and g of a metric space (X, d) are called R-weakly commuting of type-(A_f) [12], if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all $x \in X$.

In 2007, Kohali and Vashistha [7] introduced the notions of R-weakly commuting mappings in probabilistic metric spaces as follow:

Definition 1.9. Two self mappings f and g of a Menger PM space $(X, F, \triangle)$ are called R-weakly commuting of type-(MA_g), if there exists some real number $R \geq 0$ such that $F_{ffx,gfx}(t) \geq F_{fx,gx}(\frac{t}{R})$ for all $t > 0$ and $x \in X$.

In 1998, Pant [11] introduced the concept of reciprocal continuity for the pair of single valued maps. In the following, we have the same definition but in a Menger PM space X.

Definition 1.10. Two self mappings f and g of a Menger PM space X are called reciprocally continuous , if $\lim_{n\to\infty} gfx_n = gx$ and $\lim_{n\to\infty} fgx_n = fx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

Note that a pair of mappings which is reciprocally continuous need not be continuous even on their common fixed point (see for example [11]).

Pant et al. [12] generalized reciprocal continuity by introducing the notion of weakly reciprocal continuity for a pair of single valued maps as follows but in metric space (X, d).

Definition 1.11. [4] Two self mappings f and g of a Menger PM space X are called weakly reciprocally continuous , if $\lim_{n\to\infty} gfx_n = gx$ or $\lim_{n\to\infty} fgx_n = fx$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

It seems important to note that reciprocal continuity implies weak reciprocal continuity, but the converse is not true (see Example 7 [3]).

Definition 1.12. Let Φ be the family of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying:

(1) $\phi(t) = 0$ if, and only if, $t = 0$;
(2) $\lim\limits_{t\to\infty} \phi(t) = \infty$;
(3) ϕ is continuous at $t = 0$.

Definition 1.13. Let Ψ be the class of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(1) ψ is nondecreasing;
(2) $\psi(0) = 0$;
(3) if $\{a_n\} \subset [0, \infty)$ is a sequence such that $\{a_n\} \to 0$, then $\{\psi^n(a_n)\} \to 0$(where ψ^n denotes the nth-iterate of ψ).

We shall remind that ψ is continuous at $t = 0$ for functions in Ψ. (Proposition 7 of [13])

The following family of auxiliary functions are introduced in [13].

Definition 1.14. Let $\mathcal{H}$ be the family of all functions $h : (0, 1] \to [0, \infty)$ satisfying:

$(\mathcal{H}_1)$ if $\{a_n\} \subset (0, 1]$, then $\{a_n\} \to 1$ if, and only if, $\{h(a_n)\} \to 0$;
$(\mathcal{H}_2)$ if $\{a_n\} \subset (0, 1]$, then $\{a_n\} \to 0$ if, and only if, $\{h(a_n)\} \to \infty$.

The previous conditions are guaranteed when $h : (0, 1] \to [0, \infty)$ is a strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous (in a broad sense, it is sufficient to assume the continuities of h and h^{-1} on the extremes of the respective domains). For instance, this is the case of the function $h(t) = 1/t - 1$ for all $t \in (0, 1]$. However, the functions in $\mathcal{H}$ need not be continuous nor monotone.

Proposition 1.15. *[13] If $h \in \mathcal{H}$, then $h(1) = 0$. Furthermore, $h(t) = 0$ if, and only if, $t = 1$.*

2. Main Results

In this section we present an extension of fixed point theorems in several ways: the metric space is more general, the contractivity condition is better and the involved auxiliary functions form a wider class.

Theorem 2.1. *Let $(X, F, \triangle)$ be a Menger PM space with a T-norm $\triangle$ of H-type, T, S are two self maps of X such that for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$ satisfying*

$$h(F_{Tx,Ty}(\phi(ct))) \leq \psi(h(M_S(x, y))), \tag{1}$$

for any $x, y \in X$ and all $t > 0$ and

$$M_S(x, y) = max\{F_{Sx,Sy}(\phi(t)), \frac{\triangle(F_{Sx,Tx}(\phi(t)), F_{Sy,Ty}(\phi(t)))}{1 + F_{Tx,Ty}(\phi(t))}\}, \tag{2}$$

with $T(X) \subseteq S(X)$, then T and S have a coincidence point in X if either
(a) X is G-complete and S is surjective ; or
$(b$ X is G-complete and S is continuous and T and S are Menger compatible; or
(c) $S(X)$ is G-complete; or
(d) $T(X)$ is G-complete.
Furthermore, if $h \in \mathcal{H}$ is decreasing, the coincidence point is unique, i.e. $Sp = Sq$ whenever $Sp = Tp$ and $Sq = Tq$ $(p, q \in X)$.

Corollary 2.2. *Let $(X, F, \triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type and T a self mapping of X satisfying (1) for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$ with $S = I$, the identity map on X. Then T has a fixed point and at this fixed point T is continuous.*

Example 2.3. *Let $X = [0, \infty)$. Define $F : X \times X \to \Lambda^+$ by*

$$F_{x,y}(t) = \begin{cases} \epsilon_{max\{x,y\}}(t), & if \quad x \neq y \\ 1, & if \quad x = y. \end{cases}$$

for all $x, y \in X$ and for all $t > 0$, such that

$$\epsilon_a(t) = \begin{cases} 0, & if \quad 0 \leq t \leq a \\ 1, & if \quad a < t \leq \infty. \end{cases}$$

It is easy to see that $(X, F, \triangle_p)$ is a G-complete Menger PM space (see Example 12,[13]). Let $T : X \to X$ be the self-mapping defined by $Tx = \frac{x}{2}$, for all $x \in X$.
Now, consider self-mappings ϕ and ψ on $[0, \infty)$ defined by $\psi(t) = \phi(t) = t$, for all $t \in [0, \infty)$, and let $h : (0, 1] \to [0, \infty)$ be whatever strictly decreasing bijection between $(0, 1]$ and $[0, \infty)$ such that h and h^{-1} are continuous (for instance, $h(t) = 1/t - 1$ for all $(0, 1]$, but any other function verifying these properties yields the same result). In this context, the contractivity conditions (1) and (2) are equivalent to

$$\begin{aligned} &h(F_{Tx,Ty}(\phi(ct))) \leq \psi(h(M(x, y))) \\ &\Leftrightarrow h(F_{Tx,Ty}(ct)) \leq h(M(x, y)) \\ &\Leftrightarrow F_{Tx,Ty}(ct) \geq M(x, y) \geq F_{x,y}(t). \end{aligned}$$

For all $x, y \in X$, $t > 0$ and for some $c \in (0,1)$. Let $x \neq y$ and by setting $c = \frac{1}{2}$, we get

$$\begin{aligned} F_{Tx,Ty}(ct) &= F_{\frac{x}{2},\frac{y}{2}}(\frac{t}{2}) \\ &= \epsilon_{max\{\frac{x}{2},\frac{y}{2}\}}(\frac{t}{2}) \\ &= \begin{cases} 0, & if \quad 0 \leq \frac{t}{2} \leq max\{\frac{x}{2},\frac{y}{2}\} \\ 1, & if \quad max\{\frac{x}{2},\frac{y}{2}\} < \frac{t}{2} \end{cases} \\ &= \begin{cases} 0, & if \quad 0 \leq t \leq max\{x, y\} \\ 1, & if \quad max\{x, y\} < t \end{cases} \\ &= F_{x,y}(t). \end{aligned}$$

It is clear, if $x = y$. As a result, the contractivity condition is verified. Also, all the assumptions made in Theorem 2.1 or Corollary 2.2 are satisfied and hence, it guarantees that T has a unique fixed point (which is $x = 0$) and it is continuous at fixed point.

Definition 2.4. [4] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and g are Menger compatible if

$$\lim_{n\to\infty} F_{gT(x_n,y_n),T(gx_n,gy_n)}(t) = 1,$$

for all $t > 0$ and

$$\lim_{n\to\infty} F_{gT(y_n,x_n),T(gy_n,gx_n)}(t) = 1,$$

for all $t > 0$, whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} gx_n = x,$$

and

$$\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} gy_n = y,$$

for all $x, y \in X$.

Corollary 2.5. *Let $(X, F, \triangle)$ be a Menger PM space with a T-norm $\triangle$ of H-type, $G : X \times X \to X$ and $f : X \to X$ are two mappings such that for some $c \in (0,1)$, $\phi \in \Phi$, $\psi \in \Psi$, and $h \in \mathcal{H}$*

$$h(F_{G(x,y),G(u,v)}(\phi(ct))) \leq \psi(h(M^*_f((x,y),(u,v)))), \tag{3}$$

for all $(x, y), (u, v) \in X \times X$ and all $t > 0$ and

$$M^*_f((x,y),(u,v)) = max\{min\{F_{fx,fu}(\phi(t)), F_{fy,fv}(\phi(t))\}, \frac{\triangle(min\{F_{fx,G(x,y)}(\phi(t)), F_{fy,G(y,x)}(\phi(t))\}, min\{F_{fu,G(u,v)}(\phi(t)), F_{fv,G(v,u)}(\phi(t))\})}{1 + min\{F_{G(x,y),G(u,v)}(\phi(t)), F_{G(y,x),G(v,u)}(\phi(t))\}}\}, \tag{4}$$

with $G(X \times X) \subseteq f(X)$. Then G and f have a coupled coincidence point if either one of the conditions (a) or (b) or (c) in Theorem (2.1) holds, or $G(X \times X)$ is G-complete. Furthermore, if $h \in \mathcal{H}$ is decreasing, the coupled coincidence value is unique.

Following similar arguments as in proof of Corollary (2.2) and (2.5), we can deduce the next result. we omit the details of the proof.

Corollary 2.6. *Let $(X, F, \triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, $G : X \times X \to X$ is a mapping such that for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$,*

$$h(F_{G(x,y),G(u,v)}(\phi(ct)) \leq \psi(h(M^*((x, y), (u, v)))),$$

for all $(x, y), (u, v) \in X \times X$ and all $t > 0$ and

$$M^*((x, y), (u, v)) = max\{min\{F_{x,u}(\phi(t)), F_{y,v}(\phi(t))\},$$
$$\frac{\triangle(min\{F_{x,G(x,y)}(\phi(t)), F_{y,G(y,x)}(\phi(t))\}, min\{F_{u,G(u,v)}(\phi(t)), F_{v,G(v,u)}(\phi(t))\})}{1 + min\{F_{G(x,y),G(u,v)}(\phi(t)), F_{G(y,x),G(v,u)}(\phi(t))\}}.$$

Then G has a coupled point and at this coupled point G is continuous.

Theorem 2.7. *Let $(X, F, \triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, T, S are two weakly reciprocally continuous self maps of X satisfying (1) and (2) for some $c \in (0, 1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$, with $T(X) \subseteq S(X)$, then T and S have a coincidence point in X (if h is decreasing, T and S have a common fixed point in X,) if either*
(a) T and S are Menger compatible; or
(b) T and S are R-weakly commuting of type-(MA_S); or
(c) T and S are R-weakly commuting of type-(MA_T).

Now, we first recall the concept of weakly commuting of two mappings $T : X \times X \to X$ and $g : X \to X$ on a Menger PM space X.

Definition 2.8. [4] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and g are called R-weakly commuting of type-(MA_g), if there exists some real number $R \geq 0$ such that

$$F_{T(T(x,y),T(y,x)),gT(x,y)}(t) \geq F_{T(x,y),gx}(\frac{t}{R}),$$

and

$$F_{T(T(y,x),T(x,y)),gT(y,x)}(t) \geq F_{T(y,x),gy}(\frac{t}{R}),$$

for all $t > 0$ and $(x, y) \in X \times X$.

Definition 2.9. [4] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and g are called R-weakly commuting of type-(MA_T), if there exists some real number $R \geq 0$ such that

$$F_{T(gx,gy),ggx}(t) \geq F_{T(x,y),gx}(\frac{t}{R}),$$

and

$$F_{T(gy,gx),ggy}(t) \geq F_{T(y,x),gy}(\frac{t}{R}),$$

for all $t > 0$ and $(x, y) \in X \times X$.

Reciprocal continuity and weakly reciprocal continuity are generalized for a pair of single valued maps in Menger PM space as follows.

Definition 2.10. [4] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and f are called reciprocally continuous , if $\lim_{n\to\infty} fT(x_n, y_n) = fx$, $\lim_{n\to\infty} fT(y_n, x_n) = fy$ and $\lim_{n\to\infty} T(fx_n, fy_n) = T(x, y)$, whenever $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} fy_n = y$ for some $(x, y) \in X \times X$.

Definition 2.11. [4] Let $(X, F, \triangle)$ be a Menger PM space and $T : X \times X \to X$ and $g : X \to X$. Then T and f are called weakly reciprocally continuous , if $\lim_{n\to\infty} fT(x_n, y_n) = fx$ and $\lim_{n\to\infty} fT(y_n, x_n) = fy$ or $\lim_{n\to\infty} T(fx_n, fy_n) = T(x, y)$, whenever $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $\lim_{n\to\infty} T(x_n, y_n) = \lim_{n\to\infty} fx_n = x$ and $\lim_{n\to\infty} T(y_n, x_n) = \lim_{n\to\infty} fy_n = y$ for some $(x, y) \in X \times X$.

Theorem 2.12. *Let $(X, F, \triangle)$ be a G-complete Menger PM space with a T-norm $\triangle$ of H-type, $G : X \times X \to X$ and $f : X \to X$ are two weakly reciprocally continuous mappings satisfying (3) and (4) for some $c \in (0,1), \phi \in \Phi, \psi \in \Psi$, and $h \in \mathcal{H}$, with $G(X \times X) \subseteq f(X)$, then G and f have a coupled coincidence point in X (if h is decreasing, G and f have a common coupled fixed point in X,) if either*
(a) G and f are Menger compatible; or
(b) G and f are R-weakly commuting of type-(MA_f); or
(c) G and f are R-weakly commuting of type-(MA_G).

Example 2.13. *Let $X = \{2^n : n \in \mathbb{N}\} \bigcup \{0\}$ and define the mapping $F : X \times X \to \Lambda^+$ by $F_{x,y}(0) = 0$ for all $x, y \in X$, $F_{x,x}(t) = 1$ for all $x \in X$ and $t > 0$,*

$$F_{x,y}(t) = \begin{cases} \frac{3}{5}, & \text{if} \quad 0 < t \le |x-y|, \\ 1, & \text{if} \quad t > |x-y|. \end{cases}$$

for all $x, y \in X$ with $x \ne y$. It is easy to see that $(X, F, \triangle_m)$ is a complete Menger PM space.
Let $G : X \times X \to X$ and $f : X \to X$ be two mappings defined by

$$G(x,y) = 0,$$

for all $x, y \in X$ with $xy = 0$,

$$G(2,y) = 0,$$

for all $y \in X$,

$$G(x,y) = x,$$

for all $x, y \in X$ with $x \ne y$ and $x \ne 2$ and

$$f(0) = 0, \quad f(2^n) = 2^{n+1}$$

for each $n \in \mathbb{N}$. It is easy to see that $G(X \times X) = f(X) = \{2^{n+1} : n \in \mathbb{N}\} \bigcup \{0\}$ and so $f(X)$ is complete. We also see that G and f are weakly reciprocally continuous and compatible.
Now, consider self-mappings ϕ and ψ on $[0,\infty)$ defined by $\psi(t) = \phi(t) = t$, for all $t \in [0,\infty)$, and let $h : (0,1] \to [0,\infty)$ be whatever strictly decreasing bijection between $(0,1]$ and $[0,\infty)$ such that h and h^{-1} are continuous. In this context, the contractivity conditions (3) and (4) are equivalent to

$$\begin{aligned} & h(F_{G(x,y),G(u,v)}(\phi(ct))) \le \psi(h(M_f^*((x,y),(u,v)))) \\ \Leftrightarrow\ & h(F_{G(x,y),G(u,v)}(ct)) \le h(M_f^*((x,y),(u,v))) \\ \Leftrightarrow\ & F_{G(x,y),G(u,v)}(ct) \ge M_f^*((x,y),(u,v)) \\ \Leftrightarrow\ & F_{G(x,y),G(u,v)}(ct) \ge min\{F_{fx,fu}(t), F_{fy,fv}(t)\}. \end{aligned} \tag{5}$$

If $c = \frac{1}{2}$, for all $x, y, u, v \in X$, if $xy = 0$ and $uv = 0$, then G and f satisfy (5). For all $x, y, u, v \in X$ with $xy \ne 0$ or $uv \ne 0$ and $t > 0$, if $\frac{1}{2} > |G(x,y) - G(u,v)|$, then we have

$$F_{G(x,y),G(u,v)}(\frac{t}{2}) = 1 \ge min\{F_{fx,fu}(t), F_{fy,fv}(t)\}.$$

Next, assume that $\frac{1}{2} \le |G(x,y) - G(u,v)|$. We show the condition (5) by the following cases:
***(I)** $xy = 0, u = 2^n, v = 2^m$. For all $t > 0$, $\frac{t}{2} < |G(x,y) - G(u,v)| = 2^n$ implies that $t < 2^{n+1} = f(u)$ and so*

$$F_{G(x,y),G(u,v)}(\frac{t}{2}) = \frac{3}{5} = min\{F_{fx,fu}(t), F_{fy,fv}(t)\}.$$

***(II)** $xy \ne 0$ and $uv \ne 0$. Let $x = 2^k, y = 2^l, u = 2^n$ and $v = 2^m$ for each $k, l, n, m \in \mathbb{N}$. For all $t > 0$, $\frac{t}{2} < |G(x,y) - G(u,v)| = |2^k - 2^n|$ implies that $t < |2^{k+1} - 2^{n+1}| = |f(x) - f(u)|$ and so*

$$F_{G(x,y),G(u,v)}(\frac{t}{2}) = \frac{3}{5} = min\{F_{fx,fu}(t), F_{fy,fv}(t)\}.$$

By the cases above, (3) holds for all $x, y, u, v \in X$ and all $t > 0$. Therefore, by Theorem (2.12), G and f have a common coupled fixed point in X. i.e., there exist $x^*, y^* \in X$ such that $G(x^*, y^*) = f(x^*) = x^*$ and $G(y^*, x^*) = f(y^*) = y^*$. In fact, $x^* = y^* = 0$

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Existence of three solutions for a system via a theorem of Ricceri

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Abstract

In this article, using a theorem for differentiable functionals due to Ricceri, we establish the existence of at least two solutions for a class of systems of second order partial differential equations with Dirichlet boundary conditions.

1. Introduction

In [7], the author has proved the following theorem:
Let X be a topological space, I a real interval and $\Psi : X \times I \to Y = \mathbb{R}$ a real-valued function. If Ψ is lower semicontinuous and inf-compact in X, quasiconcave and continuous in I and satisfies $\sup_I \inf_X \Psi < \inf_X \sup_I \Psi$, then there exists $\lambda^* \in I$ such that $\Psi(\cdot, \lambda^*)$ has at least two global minima.
In papers [5] and [8], this theorem was extended to the case where Y is an arbitrary convex set in a vector space. Applications of the results of these articles can be found in [3, 4, 10].
Ricceri in [3] established following theorem:

Theorem 1.1. *Let X be a topological space, $(Y, \langle\cdot,\cdot\rangle)$ a real Hilbert space, $T \subseteq Y$ a convex set dense in Y and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ two functions such that, for each $y \in T$, the function $x \to I(x) + \langle\varphi(x), y\rangle$ is lower semicontinuous and* inf*-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$ such that*

(φ_1) *x_0 is a global minimum of both functions I and $\|\varphi(\cdot)\|$;*

(φ_2) $\inf_{x\in X}\langle\varphi(x), \varphi(x_0)\rangle < \|\varphi(x_0)\|^2$.

Then, for each convex set $S \subseteq T$ dense in Y, there exists $\tilde{y} \in S$ such that the functional $x \to I(x) + \langle\varphi(x), \tilde{y}\rangle$ has at least two global minima in X.

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An application of this theorem in solving a system of elliptic equations is also presented.
In this paper, employing a special case of this theorem, we show that a system involving Kirchhoff-type operator has at least three weak solutions.
In the past years, problems involving Kirchhoff-type operators have been studied in many papers, we refer interested readers to [1, 2, 6, 9, 11].

2. Main results

Let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary. We denote by $\mathcal{A}$ the class of all functions $\Phi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which are measurable in Ω, C^1 in $\mathbb{R}^2$ and satisfy

$$\sup_{(x,u,v)\in\Omega\times\mathbb{R}^2} \frac{|\Phi_u(x,u,v)| + |\Phi_v(x,u,v)|}{1 + |u|^m + |v|^m} < +\infty$$

where Φ_u (resp. Φ_v) denoting the derivative of Φ with respect to u (resp. v) and $m > 0$ with $m < \frac{n+2}{n-2}$ when $n > 2$.
Here, we are interested in the problem

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 dx)\Delta u = \Phi_u(x,u,v) & \text{in } \Omega, \\ -(a + b\int_\Omega |\nabla v|^2 dx)\Delta v = \Phi_v(x,u,v) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Phi \in \mathcal{A}$, $a, b \in \mathbb{R}$ with $a, b > 0$.
A weak solution of (1) is any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$(a + b\int_\Omega |\nabla u|^2 dx)\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega \Phi_u(x,u(x),v(x))\varphi(x)dx,$$
$$(a + b\int_\Omega |\nabla v|^2 dx)\int_\Omega \nabla v(x)\nabla\psi(x)dx = \int_\Omega \Phi_v(x,u(x),v(x))\psi(x)dx$$

for all $\varphi, \psi \in H_0^1(\Omega)$.
Let $I_\Phi : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$I_\Phi(u,v) = \frac{1}{2}\Big(a\big(\int_\Omega |\nabla u|^2 dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla u|^2 dx\big)^2 + a\big(\int_\Omega |\nabla v|^2 dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla v|^2 dx\big)^2\Big) - \int_\Omega \Phi(x,u(x),v(x))dx$$

for all $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Hence, the critical points of the functional I_Φ are the weak solutions of the problem (1).
Our main result is as follow:

Theorem 2.1. *Let $F, G, H, K \in \mathcal{A}$, with $K(x,0,0) = 0$ for all $x \in \Omega$, satisfy the following conditions:*

(k_1) *there is $\eta \in (0, \frac{a\lambda_1}{2})$ such that $K(x,s,t) \leq \eta(s^2+t^2)$ for all $x \in \Omega$, $s,t \in \mathbb{R}$, where $\lambda_1 = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}$;*

(f_1) $\lim_{s^2+t^2\to+\infty} \frac{\sup_{x\in\Omega}(|F(x,s,t)|+|G(x,s,t)|+|H(x,s,t)|)}{s^2+t^2} = 0$;

(f_2) *one has* $\operatorname{meas}(\{x \in \Omega : |F(x,0,0)|^2 + |G(x,0,0)|^2 + |H(x,0,0)|^2 > 0\}) > 0$ *and*

$$|F(x,0,0)|^2 + |G(x,0,0)|^2 + |H(x,0,0)|^2 \leq |F(x,s,t)|^2 + |G(x,s,t)|^2 + |H(x,s,t)|^2$$

for all $x \in \Omega$, $s,t \in \mathbb{R}$;

(f_3) *one has*

$$\text{meas}(\{x \in \Omega : \inf_{(s,t)\in\mathbb{R}^2} (|F(x,0,0)|F(x,s,t) + |G(x,0,0)|^2 G(x,s,t) + |H(x,0,0)|^2 H(x,s,t)) < |F(x,0,0)|^2 + |G(x,0,0)|^2 + |H(x,0,0)|^2\}) > 0$$

Then, for every convex set $S \subseteq L^\infty(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega)$ *dense in* $L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$, *there exists* $(\alpha,\beta,\gamma) \in S$ *such that the problem*

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = \alpha(x)F_u(x,u,v) + \beta(x)G_u(x,u,v) \\ \qquad\qquad +\gamma(x)H_u(x,u,v) + K_u(x,u,v) & \text{in } \Omega, \\ -(a+b\int_\Omega |\nabla v|^2dx)\Delta v = \alpha(x)F_v(x,u,v) + \beta(x)G_v(x,u,v) \\ \qquad\qquad +\gamma(x)H_v(x,u,v) + K_v(x,u,v) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)\times H_0^1(\Omega)$ *of the functional*

$$(u,v) \to \frac{1}{2}\Big(a\big(\int_\Omega |\nabla u|^2dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla u|^2dx\big)^2 + a\big(\int_\Omega |\nabla v|^2dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla v|^2dx\big)^2\Big) - \int_\Omega (\alpha(x)F(x,u,v) + \beta(x)G(x,u,v) + \gamma(x)H(x,u,v) + K(x,u(x),v(x)))dx.$$

Next, we wish to point out two remarkable particular cases of Theorem 2.1.

Theorem 2.2. *Let* $K \in \mathcal{A}$, *with* $K(x,0,0) = 0$ *for all* $x \in \Omega$, *satisfies* (k_1). *Then, for every convex set* $S \subseteq L^\infty(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega)$ *dense in* $L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$, *there exists* $(\alpha,\beta,\gamma) \in S$ *such that the problem*

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = (\alpha(x)\cos(uv) - \beta(x)\sin(uv) + \gamma(x))v + K_u(x,u,v) & \text{in } \Omega, \\ -(a+b\int_\Omega |\nabla v|^2dx)\Delta v = (\alpha(x)\cos(uv) - \beta(x)\sin(uv) + \gamma(x))u + K_v(x,u,v) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)\times H_0^1(\Omega)$ *of the functional*

$$(u,v) \to \frac{1}{2}\Big(a\big(\int_\Omega |\nabla u|^2dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla u|^2dx\big)^2 + a\big(\int_\Omega |\nabla v|^2dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla v|^2dx\big)^2\Big) - \int_\Omega (\alpha(x)sin(u(x)v(x)) + \beta(x)cos(u(x)v(x)) + \gamma(x)u(x)v(x) + K(x,u(x),v(x)))dx.$$

Proof. Apply Theorem 2.1 to the functions $F, G, H : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(s,t) = \sin(st)$, $G(s,t) = \cos(st)$ and $H(s,t) = st$ for all $(s,t) \in \mathbb{R}^2$. $\square$

Corollary 2.3. *Let* $F, G, H : \mathbb{R} \to \mathbb{R}$ *belong to* $\mathcal{A}$ *and assume that* F, G, H *are twice differentiable at* 0 *and such that*

$$0 < |F(0)|^2 + |G(0)|^2 + |H(0)|^2 = \inf_{s\in\mathbb{R}}(|F(s)|^2 + |G(s)|^2 + |H(s)|^2)$$

$$F''(0)F(0) + G''(0)G(0) + H''(0)H(0) < 0. \tag{2}$$

Then, for every convex set $S \subseteq L^\infty(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega)$ *dense in* $L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$, *there exists* $(\alpha,\beta,\gamma) \in S$ *such that the problem*

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2dx)\Delta u = \alpha(x)F'(u) + \beta(x)G'(u) + \gamma(x)H'(u) & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ *of the functional*

$$(u,v) \to \frac{1}{2}\Big(a\big(\int_\Omega |\nabla u|^2dx\big) + \frac{b}{2}\big(\int_\Omega |\nabla u|^2dx\big)^2\Big) - \int_\Omega (\alpha(x)F(u(x)) + \beta(x)G(u(x)) + \gamma(x)H(u(x)))dx.$$

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Infinitely many solutions for a nonlinear equation with Hardy potential

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Abstract

In this article, by using critical point theory, we prove the existence of infinitely many weak solutions for a nonlinear problem with Hardy potential. Indeed, intervals of parameters are determined for which the problem admits an unbounded sequence of weak solutions.

1. Introduction

In this work, we discuss the existence of infinitely many weak solutions for the following problem

$$\begin{cases} -\Delta_p^3 u + \frac{|u|^{p-2}u}{|x|^{3p}} = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$ and p is a constant with $1 < p < \frac{N}{3}$. The operator $\Delta_p^3 u := div\left(\Delta(|\nabla\Delta u|^{p-2}\nabla\Delta u)\right)$ is the p-triharmonic operator.

In recent years, the study of boundary value problems involving the triharmonic operator has been considered see for instance [9, 11].

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Rahal [9] studied the existence of weak solutions to the following nonlinear Navier boundary value problem involving the $p(x)$-Kirchhoff type triharmonic operator

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)\Delta^3_{p(x)}u = \lambda\zeta(x)|u|^{\alpha(x)-2}u - \lambda\xi(x)|u|^{\beta(x)-2}u, & x\in\Omega,\\ u=\Delta u=\Delta^2 u=0, \qquad x\in\partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N>3)$ is a bounded domain with smooth boundary, λ is a positive parameter, $p\in C^0(\overline{\Omega})$ with $1<p(x)<\frac{N}{3}$ for any $x\in\overline{\Omega}$ and $\zeta,\xi,\alpha,\beta\in C^0(\overline{\Omega})$.

In [4], presenting a version of the infinitely many critical points theorem of Ricceri (see [10, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Strum-Liouville problem, having discontinuous nonlinearities, has been established. In such an approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used in several works in order to obtain existence results for different kinds of problems (see, for instance, [1–3, 5–7, 12] and references therein).

The rest of this paper is organized as follows. In Section 2, some known definitions and results on Lebesgue and Sobolev spaces, which will be used in sequel, are collected. Moreover, the abstract critical points theorem (Lemma 2.1) is recalled. In Section 3, we state the main result and its proof.

2. Preliminaries

Here and in the sequel Ω is an open bounded subset of $\mathbb{R}^N$ and

$$X := W^{3,p}(\Omega)\cap W^{1,p}_0(\Omega)$$

endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla\Delta u|^p\,dx\right)^{\frac{1}{p}} \tag{2.2}$$

for $u\in X$.

Corresponding to f and g, we introduce the functions $F,G:\Omega\times\mathbb{R}\to\mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \qquad G(x,t) := \int_0^t g(x,\xi)\,d\xi$$

for all $x\in\Omega$ and $t\in\mathbb{R}$.

For every $u\in X$, let us define $\Phi,\Psi : X\to\mathbb{R}$ by putting

$$\Phi(u) := \frac{\|u\|^p}{p} + \frac{1}{p}\int_\Omega \frac{|u(x)|^p}{|x|^{3p}}dx, \qquad \Psi(u) = \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx.$$

By standard arguments, we have that Φ and Ψ are Gâteaux differentiable and whose derivative are

$$\Phi'(u)(v) = \int_\Omega |\nabla\Delta u|^{p-2}\nabla\Delta u\cdot\nabla\Delta v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^{3p}}uv dx$$

$$\Psi'(u)(v) = \int_\Omega [f(x,u(x)) + \frac{\mu}{\lambda}g(x,u(x))]v(x)\,dx$$

for any $u, v \in X$. Fixing $q \in [1, p^* := \frac{pN}{N-3p})$, from the Sobolev embedding there exists a positive constant c_q such that $\|u\|_{L^{p^*}(\Omega)} \leq c_q\|u\|$ for $u \in X$. Thus the embedding $X \hookrightarrow L^q(\Omega)$ is compact.

We recall Hardy inequality in X, which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^{3p}} dx \leq \frac{1}{H} \int_\Omega |\nabla \Delta u(x)|^p dx,$$

where $H = (\frac{[N(p-1)+p](N-p)(N-3p)}{p^3})^p$.

Finally, a weak solution of problem (1.1) is a function $u \in X$ such that

$$\begin{aligned}&\int_\Omega |\nabla \Delta u|^{p-2} \nabla \Delta u \cdot \nabla \Delta v dx + \int_\Omega \frac{|u(x)|^{p-2}}{|x|^{3p}} uv dx \\ &- \lambda \int_\Omega f(x, u(x)) v(x)\, dx - \mu \int_\Omega g(x, u(x)) v(x)\, dx = 0\end{aligned}$$

for all $v \in X$, it is obvious that our goal is to find critical points of the functional I_λ. For achieving this aim, our main tool is the following critical point theorem of Ricceri [10, Theorem 2.5] (see also [4] for a refined version).

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\left(\sup_{v \in \Phi^{-1}(-\infty, r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I := \Phi - \lambda \Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(b_1) *I possesses a global minimum, or*

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

3. Main results

In this section, we present our main results.

Theorem 3.1. *Assume that*

(A_1) $F(x,t) \geq 0$ *for every* $(x,t) \in \Omega \times [0,+\infty[$;

(A_2) *there exists* $s > 0$ *such that, if we put*

$$\alpha := \liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t} \int_\Omega F(x,\xi)\,dx}{t^p},$$

$$\beta := \limsup_{t\to+\infty} \frac{\int_{B(0,\frac{s}{2})} F\left(x,\frac{t}{h}\right)\,dx}{t^p},$$

where $\alpha < R\beta$, $h > 1$ *is a constant,* $R = \frac{h^p}{\sigma c_q^p}$ *and*

$$\sigma = \frac{H+1}{H}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{128}{3s^4}(N+2)r - \frac{64}{s^3}(N+1) + \frac{32}{3sr^2}(N-1)\right|^p r^{N-1}dr.$$

Then, for each $\lambda \in \left(\frac{1}{pc_q^p R\beta}, \frac{1}{pc_q^p\alpha}\right)$ *and for every* $g \in C^0(\overline{\Omega}\times\mathbb{R})$ *whose potential* $G(x,t) := \int_0^t g(x,\xi)\,d\xi$ *for all* $(x,t) \in \overline{\Omega}\times[0,+\infty[$, *is a non-negative function satisfying the condition*

$$g_\infty := \limsup_{\xi\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t}\int_\Omega G(x,\xi)\,dx}{t^p} < +\infty, \tag{3.3}$$

if we put

$$\mu_{g,\lambda} := \frac{1}{pc_q^p g_\infty}\left(1 - \lambda p c_q^p \alpha\right),$$

where $\mu_{g,\lambda} = +\infty$ *when* $g_\infty = 0$, *problem* (1.1) *has an unbounded sequence of weak solutions for every* $\mu \in [0,\mu_{g,\lambda})$ *in* X.

Proof. We want to apply Lemma 2.1(b) with $X = W^{3,p}(\Omega)\cap W_0^{1,p}(\Omega)$ endowed with the norm introduced in (2.2). For fix $\lambda \in (\lambda_1,\lambda_2)$ and $\mu \in (0,\mu_{g,\lambda})$, we take Φ, Ψ as in the previous section. Similar arguments as those used in [1] and assumption (A_2), imply that

$$\gamma \leq \liminf_{n\to+\infty}\varphi(r_n) \leq pc_q^p\Big(\alpha + \frac{\mu}{\lambda}g_\infty\Big) < +\infty, \tag{3.4}$$

and consequently $\lambda < \frac{1}{\gamma}$.

Let λ be fixed. We claim that the functional I is unbounded from below. Since

$$\frac{1}{\lambda} < \frac{ph^p}{\sigma}\beta,$$

there exist a sequence $\{\tau_n\}$ of positive numbers and $\eta > 0$ such that $\lim_{n\to+\infty}\tau_n = +\infty$ and

$$\frac{1}{\lambda} < \eta < \frac{ph^p}{\sigma}\frac{\int_{B(0,\frac{s}{2})} F(x,\frac{\tau_n}{h})\,dx}{\tau_n^p} \tag{3.5}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define $w_n \in X$ by

$$w_n(x) := \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(0,s), \\ \frac{\tau_n}{h}\left(\frac{16}{3s^4}\rho^4 - \frac{64}{3s^3}\rho^3 + \frac{24}{s^2}\rho^2 - \frac{32}{3s}\rho + \frac{8}{3}\right) & \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \\ \frac{\tau_n}{h} & \text{if} \quad x \in B(0,\frac{s}{2}), \end{cases} \tag{3.6}$$

where $\rho = dist(x,0) = \sqrt{\sum_{i=1}^{N}(x_i - x_i^0)^2}$. Then, we have

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{64}{3s^4}\rho^2 - \frac{64}{s^3}\rho + \frac{48}{s^2} - \frac{32}{3s\rho}\right)x_i & \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

$$\frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if} \;\; x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{64}{3s^4}(\rho^2 + 2x_i^2) - \frac{64}{s^3}\left(\frac{\rho^2 + x_i^2}{\rho}\right) + \frac{48}{s^2} + \frac{32}{3s}\left(\frac{x_i^2 - \rho^2}{\rho^3}\right)\right) & \\ \quad \text{if} \;\; x \in B(0,s) \setminus B(0,\frac{s}{2}), & \end{cases}$$

$$\sum_{i=1}^{N} \frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 \quad \text{if} \;\; x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{64\rho^2}{3s^4}(N+2) - \frac{64\rho}{s^3}(N+1) + \frac{48}{s^2}N - \frac{32}{3s\rho}(N-1)\right) \\ \quad \text{if} \;\; x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

$$\frac{\partial \Delta w_n(x)}{\partial x_i} = \begin{cases} 0 \quad \text{if} \;\; x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{128}{3s^4}(N+2)x_i - \frac{64}{s^3\rho}(N+1)x_i + \frac{32}{3s\rho^3}(N-1)x_i\right) \\ \quad \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

and

$$|\nabla \Delta w_n(x)| = \frac{\tau_n}{h}\left|\frac{128}{3s^4}(N+2)\rho - \frac{64}{s^3}(N+1) + \frac{32}{3s\rho^2}(N-1)\right|.$$

For any fixed $n \in \mathbb{N}$, one has

$$\begin{aligned} \Phi(w_n) &\le \frac{H+1}{pH}\int_{B(0,s)\setminus B(0,\frac{s}{2})} |\nabla \Delta w_n(x)|^p dx \\ &= \frac{H+1}{pH}\left(\frac{\tau_n}{h}\right)^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{128}{3s^4}(N+2)r - \frac{64}{s^3}(N+1) + \frac{32}{3sr^2}(N-1)\right|^p r^{N-1}dr = \frac{\sigma\tau_n^p}{ph^p}. \end{aligned} \tag{3.7}$$

On the other hand, bearing (A_1) in mind and since G is non-negative, from the definition of Ψ, we infer

$$\Psi(w_n) = \int_\Omega \left[F(x, w_n(x)) + \frac{\mu}{\lambda}G(x, w_n(x))\right] dx \ge \int_{B(0,\frac{s}{2})} F(x, \frac{\tau_n}{h})\, dx. \tag{3.8}$$

By (3.5), (3.7) and (3.8), we observe that

$$I(w_n) \le \frac{\sigma\tau_n^p}{ph^p} - \lambda\int_{B(0,\frac{s}{2})} F(x, \frac{\tau_n}{h})\, dx < \frac{\sigma}{ph^p}(1 - \lambda\eta)\tau_n^p \tag{3.9}$$

for every $n \in \mathbb{N}$ large enough. Since $\lambda\eta > 1$ and $\lim_{n\to+\infty}\tau_n = +\infty$, we have

$$\lim_{n\to+\infty} I(w_n) = -\infty.$$

Then, the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of I_λ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty,$$

and the conclusion is achieved. $\square$

Now, we present the following consequence of Theorem 3.1 with $\mu = 0$.

Theorem 3.2. *Let all the assumptions in the Theorem 3.1 hold. Then, for each*

$$\lambda \in \left(\frac{1}{pc_q^p R\beta}, \frac{1}{pc_q^p \alpha}\right)$$

the problem

$$\begin{cases} -\Delta_p^3 u + \frac{|u|^{p-2}u}{|x|^{3p}} = \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega \end{cases} \tag{3.10}$$

has an unbounded sequence of weak solutions in X.

Here, we point out the following consequence of Theorem 3.1.

Corollary 3.3. *Let the assumption* (A_1) *in the Theorem 3.1 holds. Suppose that*

$$\alpha < \frac{1}{pc_q^p}, \qquad \beta > \frac{1}{pc_q^p R}.$$

Then, the problem

$$\begin{cases} -\Delta_{p(x)}^3 u + \frac{|u|^{p-2}u}{|x|^{3p}} = f(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega \end{cases}$$

has an unbounded sequence of weak solutions in X.

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Unifying form of the stability functions of secend derivative methods for ODEs

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Abstract

Extensions of the linear multistep methods for the numerical solution of ordinary differential equations by aiming increasing the order of convergence and wider region of stability have been carried out in some directions. The use of the second derivative of the solution and equipping the methods with superfuture point technique have led to efficient methods. In this paper, we analyze the stability of a family of such methods and give a general formula for their stability functions. This formula while facilitates the stability analysis, assists that how maneuvering on the structure of the method can be useful in improving its stability properties.

1. Introduction

Several attempts have been made to derive efficient numerical methods for solving stiff initial value problems (IVPs) in ordinary differential equations (ODEs)

$$\begin{aligned} &y'(x) = f(x, y(x)), \qquad x \in [x_0,\, X], \\ &y(x_0) = y_0, \end{aligned} \tag{1}$$

where $f : [x_0,\, X] \times \mathbb{R}^m \to \mathbb{R}^m$ and m is the dimensionality of the system. Most of the constructed methods in the class of linear multistep methods (LMMs) are improvements on backward differentiation formulae (BDF) by using some techniques such as higher derivatives of the solutions, off-step points and super future points. Among the main directions in searching for higher order A-stable methods, the use of the second derivative of the solution has been one of the effective techniques for the construction of methods of higher order with extensive region of stability. Second derivative multistep methods (SDMMs) [1] and second derivative BDF methods (SDBDF) [3] were the first second derivative LMMs, meanwhile they have been also base for other modifications of second derivative methods in this class. The SDMMs equipped with super future point technique have led to successful methods. A class of SDMMs equipped with the super future point technique based on SDBDF methods, so called ESDMMs, and

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their modification, MESDMMs, were constructed in [5]. Furthermore, some perturbations of these methods which improve their stability properties while preserve their order, were studied in [2]. In an another attempt, to implement the methods in parallel computers, a scheme was investigated in [4] based on SDBDF possessing super future point technique, so-called PMESDMM, which let them be faster on the vast majority of the problem. For each of the above-mentioned methods analyzing the stability properties goes through the increasingly complicated calculations. In this paper, we are going to derive a general formula that generates the stability functions of the implicit advanced step-point SDMMs (IASS) encompassing SDBDF, ESDMM, MESDMM and PMESDMM. Such general formulae can provide us with a glimpse of the theoretical and computational difficulties encountered during the investigation of multi-stage methods. A similar general formula for a group of implicit advanced step-point methods incorporating only the first derivative of the solution has been given in [6].

2. Second derivative LMMs

In this section, we brifely recall the structure of some Second derivative LMMs.

2.1. SDBDF methods

SDBDF methods, inspired from BDF, have been designed such that in which the structure of the stability function allows to get better absolute stability properties than general form of SDMMs. A k-step SDBDF takes the form

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\beta_k f(y_{n+k}) + h^2\gamma_k g(y_{n+k}), \tag{2}$$

in which, $\alpha_k = 1$ and the other coefficients are chosen such that the method has order $p = k+1$.

2.2. ESDMM Methods

The ESDMM is an implicit scheme which uses two SDBDF predictors and one implicit SDMM corrector given by the formula

$$\sum_{j=0}^{k}\widehat{\alpha}_j y_{n+j} = h\widehat{\beta}_k f_{n+k} + h^2(\widehat{\gamma}_k g_{n+k} - \widehat{\gamma}_{k+1} g_{n+k+1}). \tag{3}$$

Here $\widehat{\alpha}_k = 1$ and the coefficients $\widehat{\alpha}_j$, $j = 0,1,\ldots,k-1$, $\widehat{\beta}_k$, $\widehat{\gamma}_k$, $\widehat{\gamma}_{k+1}$ are chosen so that (3) has the order $p = k+2$. The coefficients of the k-step methods of class (3) are given in [5]. Assuming that the solution values of $y_n, y_{n+1}, \ldots, y_{n+k-1}$ are available, the ESDMM approach goes as follows:

- **Stage 1.** Use the SDBDF (2) as predictor to compute $\overline{y}_{n+k}$ as

$$\overline{y}_{n+k} + \sum_{j=0}^{k-1}\alpha_j y_{n+j} = h\beta_k \overline{f}_{n+k} + h^2\gamma_k \overline{g}_{n+k}, \tag{4}$$

  where $\overline{f}_{n+k} = f(x_{n+k}, \overline{y}_{n+k})$ and $\overline{g}_{n+k} = g(x_{n+k}, \overline{y}_{n+k})$.

- **Stage 2.** Use the SDBDF (2) as predictor to compute $\overline{y}_{n+k+1}$ as

$$\overline{y}_{n+k+1} + \alpha_{k-1}\overline{y}_{n+k} + \sum_{j=0}^{k-2}\alpha_j y_{n+j+1} = h\beta_k \overline{f}_{n+k+1} + h^2\gamma_k \overline{g}_{n+k+1}, \tag{5}$$

  where $\overline{f}_{n+k+1} = f(x_{n+k+1}, \overline{y}_{n+k+1})$ and $\overline{g}_{n+k+1} = g(x_{n+k+1}, \overline{y}_{n+k+1})$.

- **Stage 3.** Compute y_{n+k} as the solution of the corrector

$$y_{n+k} + \sum_{j=0}^{k-1}\widehat{\alpha}_j y_{n+j} = h\widehat{\beta}_k f_{n+k} + h^2(\widehat{\gamma}_k g_{n+k} - \widehat{\gamma}_{k+1}\overline{g}_{n+k+1}).$$

The overall k-step ESDMM is of order $p = k+2$.

2.3. MESDMM Methods

In each stage of ESDMM scheme, the algebraic equations are solved using a modified form of Newton. The Jacobian matrix for stages 1 and 2 is the same but it differs for Stage 3. In order to unify the Jacobin matrix in all three stages with the aim of reducing the computational cost, a modification of ESDMM, so-called MESDMM, was introduced in which the third stage is replaced by the following formula

- **Stage** $3^{\star}$. Compute y_{n+k} as the solution of the corrector

$$\begin{aligned} y_{n+k} + \sum_{j=0}^{k-1} \widehat{\alpha}_j y_{n+j} = & h(\widehat{\beta}_k - \beta_k)\overline{f}_{n+k} + h\beta_k f_{n+k} + h^2(\widehat{\gamma}_k - \gamma_k)\overline{g}_{n+k} \\ & -h^2\widehat{\gamma}_{k+1}\overline{g}_{n+k+1} + h^2\gamma_k g_{n+k}. \end{aligned} \tag{6}$$

This modification does not affect on the order of methods while there is an improvement on the stability regions for all values of k in MESDMMs.

2.4. PMESDMM Methods

A class of methods possessing a parallel feature has been introduced in [4]. These three-stage methods which are based on MESDMMs, so-called PMESDMMs, may grant the possibility of their efficiently using on a parallel computer. Assuming that the solution values $y_n, y_{n+1}, \ldots, y_{n+k-1}$ are available, the PMESDMMs take the following form

- **Stage 1.** Use the SDBDF (2) as predictor to compute $\overline{y}_{n+k}$.
- **Stage 2.** Use the following predictor to compute $\overline{y}_{n+k+1}$

$$\overline{y}_{n+k+1} + \sum_{j=0}^{k-1} \overline{\alpha}_j y_{n+j} = h\overline{\beta}_{k+1} f(x_{n+k+1}, \overline{y}_{n+k+1}) + h^2\overline{\gamma}_{k+1} g(x_{n+k+1}, \overline{y}_{n+k+1}), \tag{7}$$

  where the coefficients $\overline{\alpha}_j$, $j = 0, 1, \ldots, k-1$, $\overline{\beta}_{k+1}$ and $\overline{\gamma}_{k+1}$, reported in [4], are chosen so that (7) has order $k+1$.

- **Stage 3.** Compute the corrected solution y_{n+k} using (6).

The parallel feature is because of independent of the second predictor of the first one. The overall k-step PMESDMMs is of order $p = k+2$. The numerical experiments reported in [4] indicate that the accuracy of the PMESDMMs is very satisfactory. Even a little improvement happens in the stability properties of PMESDMMs respect to MESDMMs.

3. A general formula for the stability functions of SDBDF and IASS methods

In this section, we introduce a general formula which generates the stability functions of SDBDF and IASS methods without needing to go through the increasingly complicated calculations for each case.

Theorem 3.1. *Suppose that*

i) α_j, β_k *and* γ_k *are the coefficients of SDBDF* (2)*;*

ii) $\widehat{\alpha}_j$, $\widehat{\beta}_k$, $\widehat{\gamma}_k$ *and* $\widehat{\gamma}_{k+1}$ *are the coefficients of ESDBDF* (3)*;*

iii) $\overline{\alpha}_j$, $\overline{\beta}_{k+1}$ *and* $\overline{\gamma}_{k+1}$ *are the coefficients of SDBDF* (7)*.*

Then, for any permitted order and step-point k, the stability functions of the distinct schemes ESDMMs, MESDMMs and PMESDMMs, collectively named IASS methods, together with the stability function of SDBDF method can be obtained from the general formula

$$\Phi(w,z)=\sum_{j=0}^{k}C_j(z)w^j, \tag{8}$$

where

$$\begin{aligned} C_k(z) &= 1-z(\beta_k+b_k)-z^2(\gamma_k+c_k),\\ C_j(z) &= \Big(1-\theta+\mu\frac{z(\widehat{\beta}_k-\beta_k)+z^2(\widehat{\gamma}_k-\gamma_k)}{A}\Big)\alpha_j+\theta\widehat{\alpha}_j+\theta z^2\overline{\gamma}_{k+1}\Big(\nu d_j+\frac{(\nu-1)\overline{\alpha}_j}{\overline{A}}\Big), \end{aligned} \tag{9}$$

in which the other coefficients for each method are given in Table 1. The largest values of step-point k for SDBDF and IASS methods take $k=10$ and $k=12$, respectively.

Table 1. The coefficients in (9).

	b_k	c_k	θ	μ	ν
SDBDF	0	0	0	0	free
ESDMMs	$\widehat{\beta}_k-\beta_k$	$\widehat{\gamma}_k-\gamma_k$	1	0	1
MESDMMs	0	0	1	1	1
PMESDMMs	0	0	1	1	0

Proof. For the proof, one can obtain the stability function for each of the mentioned methods separately by the standard linear stability analysis and then verify the general formula (8) with the coefficients (9). □

The general formula given in Theorem 3.1 provides the expected results for the stability functions which can be of substantial assistance in stability analysis of the methods. Also, by using this general formula and developing a MATLAB code, one can plot the stability regions and drive the angles of $A(\alpha)$-stability of the methods. Such a general formula is also important as they can provide ideas for designing new algorithms with better stability properties.

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Application of barycentric interpolation in the construction of numerical methods for ODEs

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Abstract

The linear barycentric interpolants and their modifications, unlike the customary polynomial interpolants, have some interesting properties which make them to be useful in the construction of numerical methods for some functional equations. We are going to introduce a class of methods based on linear barycentric rational interpolants for the numerical solution of ordinary differential equations. The proposed methods have better accuracy and stability properties in comparison with the similar methods constructed based on polynomial interpolants.

1. Introduction

The traditional numerical methods for solving initial value problems (IVPs) in ordinary differential equations (ODEs) in the form

$$\begin{aligned} &y'(t) = f(y(t)), \quad t \in [t_0, \overline{T}], \\ &y(t_0) = y_0, \end{aligned} \tag{1}$$

are based on the classical polynomial interpolation. These interpolations are ill-conditioned and lead to Runge's phenomenon if the interpolation nodes are equispaced. The linear barycentric rational interpolants (LBRIs), however, which have been first introduced by Berrut [5] and generalized by Floater and Hormann [8], have some interesting convergence and stability properties. For these futures of LBRIs, they have been recently used to construct efficient methods for the numerical solution of functional equations such as ODEs [2, 4, 7] and Volterra integral equations [1, 3, 6].

In this paper, we are going to design a class of methods based on LBRIs. To do this, extended second derivative backward differentiattion formulus (ESDBDF) methods [9] are considered as the base algorithm. The introduced algorithms have better stability properties that the base methods.

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2. The linear barycentric rational interpolants

For the given support points (t_j, y_j), $j = 0, 1, \dots, n$, LBRIs are based on a blend of the local polynomial interpolants taking the form

$$r_n(t) = \frac{\sum_{j=0}^{n-d} \lambda_j(t) p_j(t)}{\sum_{j=0}^{n-d} \lambda_j(t)},$$

with the blending functions

$$\lambda_j(t) = \frac{(-1)^j}{(t-t_j)\cdots(t-t_{j+d})}, \quad j = 0, 1, \dots, n-d.$$

Here $d \le n$ is a fixed nonnegative integer and p_j stands for the unique polynomial of degree at most d with $p_j(t_k) = y_k$, $k = j, j+1, \dots, j+d$. By representing the blended polynomials p_j in Lagrange form and rearranging the sums, LBRIs can be written in the form

$$r_n(t) = \sum_{j=0}^{n} \frac{\beta_j}{t-t_j} y_j \Bigg/ \sum_{j=0}^{n} \frac{\beta_j}{t-t_j},$$

in which the barycentric weights β_j for equispaced nodes are

$$\beta_j = \frac{(-1)^{j-d}}{2^d} \sum_{i \in J_j} \binom{d}{j-i}, \quad J_j := \big\{ i \in \{0, 1, \dots, n-d\} : \, j-d \le i \le j \big\}. \tag{2}$$

This form is more efficient in view of the numerical computations.

3. Methods based on LBRIs

Methods based on LBRIs, including RBDF [4], REBDF (and MREBDF) [7], and RSDBDF methods [2], have been recently introduced.

3.1. BDF-type methods based on LBRIs (RBDF methods)

Consider a uniform grid $t_0 < t_1 < \cdots < t_N = T$ for the given interval $[t_0, T]$ with the constant stepsize $h = t_{j+1} - t_j$, $j = 0, 1, \dots, N-1$. Suppose that m and n, $n \le m$ be positive integers. Then, the barycentric rational finite differences (RFD) formula for approximating the first derivative of y at the points t_{m-n}, t_{m-n+1}, …, t_m can be written as

$$y'(t_{m-n+i}) \approx \frac{1}{h} \sum_{j=0}^{n} d_{i,j} y(t_{m-n+j}), \quad i = 0, 1, \dots, n,$$

where

$$d_{i,j} = \begin{cases} \dfrac{\beta_j}{(i-j)\beta_i}, & j \ne i, \\[2ex] -\displaystyle\sum_{l=0,\, l \ne i}^{n} d_{i,l}, & j = i. \end{cases}$$

Applying the left one-sided RFD formula to the IVP (1) yields the RBDF method as follows

$$\sum_{j=0}^{n} d_{n,j} y_{m-n+j} = h f(y_m), \quad m = n, n+1, \dots, N.$$

The order of this family of the methods is d in case of even $n-d$, and $d+1$ in case of odd $n-d$. In order to implement RBDF methods, the set of starting values $y_1, y_2, \ldots, y_n$ is calculated using the RFD formula at the mesh points t_m, $m = 1, 2, \ldots, n$, as

$$\sum_{j=0}^{n} d_{m,j} y_j = hf(y_m), \quad m = 1, 2, \ldots, n.$$

3.2. SDBDF-type methods based on LBRIs (RSDBDF methods)

The class of the n-step SDBDF-type methods based on LBRIs for the numerical solution of IVPs (1) is introduced as

$$\sum_{j=0}^{n} \widehat{\alpha}_j y_{m-n+j} = hC_{n,d}^{(2)} f(y_m) - h^2 C_{n,d}^{(1)} g(y_m), \; m = n, n+1, \ldots, N, \tag{3}$$

where $\widehat{\alpha}_j = d_{nj}^{(1)} C_{n,d}^{(2)} - d_{nj}^{(2)} C_{n,d}^{(1)}$,

$$d_{ij}^{(1)} = \begin{cases} \dfrac{\beta_j}{\beta_i} \dfrac{1}{i-j}, & j \neq i, \\ -\displaystyle\sum_{\substack{l=0 \\ l \neq i}}^{n} d_{il}^{(1)}, & j = i. \end{cases}$$

$$d_{ij}^{(2)} = \begin{cases} 2\Big(d_{ij}^{(1)} d_{ii}^{(1)} - \dfrac{d_{ij}^{(1)}}{i-j}\Big), & j \neq i, \\ -\displaystyle\sum_{\substack{l=0 \\ l \neq i}}^{n} d_{il}^{(2)}, & j = i. \end{cases}$$

$$C_{n,d}^{(1)} = \begin{cases} -\dfrac{n-d+1}{2d+4}, & n-d \text{ is odd}, \\ -\dfrac{1}{d+1}, & n-d \text{ is even}, \end{cases}$$

$$C_{n,d}^{(2)} = C \cdot \begin{cases} -\dfrac{n-d+1}{d+2}, & n-d \text{ is odd}, \\ -\dfrac{2}{d+1}, & n-d \text{ is even}. \end{cases}$$

with

$$C = (-1)^{n-d+1} \sum_{m=1}^{d} \frac{1}{m} + d! \sum_{i=0}^{n-d-1} (-1)^i \prod_{k=i}^{i+d} \frac{1}{n-k}.$$

Here the function g stands for the second derivative of the solution (1) given by $g := f_y f$. For more details, one should study [2].

4. ESDBDF-type methods based on LBRIs (RESDBDF methods)

Using the future-step point technique, the class of ESDBDF methods has been introduced in [10]. By equipping RSDBDF with this technique, we will be able to introduce a new class of second derivative multistep methods which will be referred to as RESDBDF methods. Therefore, the class of the n-step RESDBDF methods for the numerical solution of IVPs (1) is introduced as

$$\sum_{j=0}^{n} \alpha_j y_{m-n+j} = h\beta_n f(y_m) + h^2 \gamma_n g(y_m) + h^2 \gamma_{n+1} g(y_{m+1}), \tag{4}$$

in which

$$\begin{aligned}
\alpha_j &= C^{(3)}_{n,d}\Big(C^{(2)}_{n,d}d^{(1)}_{n,j} - C^{(1)}_{n,d}d^{(2)}_{n,j}\Big) - C^{(12)}_{n,d}\left(\widetilde{C}^{(2)}_{n,d}d^{(2)}_{n,j} - C^{(2)}_{n,d}\widetilde{d}^{(2)}_{n,j}\right),\\
\beta_n &= C^{(3)}_{n,d}C^{(2)}_{n,d},\\
\gamma_n &= -C^{(3)}_{n,d}C^{(1)}_{n,d} - C^{(12)}_{n,d}\widetilde{C}^{(2)}_{n,d},\\
\gamma_{n+1} &= C^{(12)}_{n,d}C^{(2)}_{n,d}
\end{aligned}$$

and

$$\widetilde{d}^{(2)}_{n,j} = \frac{\sum_{i=0}^{n}\frac{\beta_i}{n+1-i}d^{(2)}_{ij}}{\sum_{i=0}^{n}\frac{\beta_i}{n+1-i}}.$$

In order to implement the RESDBDF, we design the three-stage scheme as

(i) Compute $\overline{y}_{m+k}$ as the solution of the n-step RSDBDF

$$\widehat{\alpha}_n\overline{y}_m + \sum_{j=0}^{n-1}\widehat{\alpha}_j y_{m-n+j} = hC^{(2)}_{n,d}f(\overline{y}_m) - h^2C^{(1)}_{n,d}g(\overline{y}_m). \tag{5}$$

(ii) Compute $\overline{y}_{m+k+1}$ as the solution of the the n-step RSDBDF

$$\widehat{\alpha}_n\overline{y}_{m+1} + \widehat{\alpha}_{n-1}\overline{y}_m + \sum_{j=0}^{n-2}\widehat{\alpha}_j y_{m+1-n+j} = hC^{(2)}_{n,d}f(\overline{y}_{m+1}) - h^2C^{(1)}_{n,d}g(\overline{y}_{m+1}). \tag{6}$$

(iii) Compute y_{m+k} as the solution of the n-step RESDBDF

$$\sum_{j=0}^{n}\alpha_j y_{m-n+j} = h\beta_n f(y_m) + h^2\gamma_n g(y_m) + h^2\gamma_{n+1}g(\overline{y}_{m+1}). \tag{7}$$

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An effective scheme based on three-dimensional orthonormal Legendre polynomial for solving Fredholm integral equations

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Abstract

In this paper, an efficient computational method is provided for the numerical solution of three-dimensional integral equations based on the Legendre scaling functions. Firstly, the definitions and features of the Legendre scaling functions are presented. In the following, with the aid of these features as well as numerical integration techniques, the three-dimensional Fredholm integral equations have been converted into an algebraic equation system. Finally, three test examples are mentioned to illustrate the superiority of the Legendre scaling function method over other numerical methods.

1. Introduction

Integral equations have amazingly far been reaching applications in the numerous areas, such as mathematics, physics, engineering and economics. The multi-dimensional integral equations are integration which is carried out about different variables. Our study is concerned with the Fredholm three-dimensional Integral Equation of second kind expressed as

$$u(x,y,z) = f(x,y,z) + \int_0^1\int_0^1\int_0^1 k(r,s,t,x,y,z)u(r,s,t)drdsdt, \tag{1}$$

$$(x,y,z) \in \Omega = [0,1]\times[0,1]\times[0,1],$$

where $f(x,y,z), k(r,s,t,x,y,z)$ are known continuous functions described on Ω and Ω^2, and $u(x,y,z)$ is unknown function which we will find it approximately.
Strategies for solving three-dimensional integral equations are very important as they are show up in the formulation of Mathematica. Since such equations are typically troublesome to solve analytically, the object of this research is to reach an appropriate scheme of analytical solutions for treating these equations.
In this paper, we chose Legendre polynomials as the basis functions for solving three-dimensional Fredholm integral equations and our computational results indicate that this kind of problem can be solved effectively using the process.

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The numerical methods for solving two-dimensional integral equations were employed by numerous studies. A system of integral equations in [1] has already been provided with Hybrid Legendre .Block-Pulse as a useful implement and bivariate Chebyshev collocation methods for mixed type Volterra-Fredholm integral equations has been applied in [2]. Many authors have used the Legendre technique to solve numerically one-dimensional and two-dimensional integral equations [4–7]. Numerous times the Legendre wavelet has also been used to solve ordinary differential equations as well as partial differential equations[8–11], moreover the Legendre wavelet is also used to solve fractional derivative problems [12–14]. Numerical solutions for high-dimensional Fredholm integral equation have been developed using Barycentric interpolation collocation methods in [15]. Another method for three-dimensional Fredholm integral equations was also tested with the aid of three-dimensional Jacobi polynomials by concentrating on low cost computational solution in [16].
This paper structure is divided into five sections: In Section 2 Legendre scaling function are introduced. In the section 3 and 4 we will discuss the methodology and how to approximate a function using scaling function. In the section 5 the efficiency of the approach is illustrated through numerical examples and comparison with other method's. Finally, conclusions of this study is given in the section 6.

2. Legendre scaling functions

In the past decades, Legendre Polynomials have received considerable attention for their applications. One of the most popular applications is Legendre scaling functions which is constructed by Legendre Polynomials. Orthogonality of these functions gives capability to turn integral equation into the system of linear and nonlinear equations.
Legendre polynomials of degree m in the interval $[-1,1]$ are defined as follows:

$$\begin{aligned}
&p_0(x) = 1\\
&p_1(x) = x\\
&.\\
&.\\
&.\\
&p_{m+1}(x) = (1+\tfrac{m}{m+1})p_m(x) - (1-\tfrac{m}{m+1})p_{m-1}(x), \qquad m = 1,2,3,...
\end{aligned}$$

Three-dimensional Legendre scaling functions can also be defined on the interval $[0,1]\times[0,1]\times[0,1]$ in this way:

$$\psi_{i,j,k}(x,y,z) = \begin{cases} \sqrt{2i+1}\sqrt{2j+1}\sqrt{2k+1}p_i(2x-1)p_j(2x-1)p_k(2x-1) & (x,y,z)\in\Omega \\ 0 & O.W. \end{cases}$$

where

$$\Omega = [0,1]\times[0,1]\times[0,1].$$

Note that three-dimensional Legendre scaling function are orthonormal set on Ω.

3. Function approximation

Assume that $u(x,y,z)$ is a function in $L^2(\Omega)$, then it can be approximated as follows:

$$u(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C_{i,j,k}\psi_{i,j,k}(x,y,z) \tag{2}$$

Where $C_{i,j,k}$ is called the scaling function coefficient and is determined in this way

$$C_{i,j,k} = \int_0^1\int_0^1\int_0^1 u(x,y,z)\psi_{i,j,k}(x,y,z)dxdydz.$$

Instead of (2), one can use the following approximation

$$u(x,y,z)=\sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{l}C_{i,j,k}\psi_{i,j,k}(x,y,z) \tag{3}$$

At this point, it is necessary to pay attention to the following matters:

1. The first feature is the orthogonality of Legendre functions, so

$$\int_0^1\int_0^1\int_0^1 \psi_{i_1,j_1,k_1}(x,y,z)\psi_{i_2,j_2,k_2}(x,y,z)dxdydz=\begin{cases}1 & i_1=i_2, j_1=j_2, k_1=k_2\\ 0 & O.W.\end{cases}$$

2. According to relation (3), we have that

$$u(x,y,z)\simeq C^T\Psi(x,y,z),$$

   where

$$\Psi(x,y,z)=\left[\psi_{0,\dots,0}(x,y,z),...,\psi_{0,\dots,l}(x,y,z),...\psi_{l,\dots,0}(x,y,z),...,\psi_{l,\dots,l}(x,y,z)\right]^T,$$

   and

$$C=\left[C_{0,\dots,0},...,C_{0,\dots,l},...,C_{l,\dots,0},...,C_{l,\dots,l}\right]^T.$$

3. Integral of multiplying the Legendre functions is the identity matrix

$$\int_0^1\int_0^1\int_0^1 \Psi(x,y,z)\Psi^T(x,y,z)dxdydz=I.$$

4. methodlogy

Considering the three-dimensional integral equation of the Fredholm type such as this

$$u(x,y,z)=f(x,y,z)+\int_0^1\int_0^1\int_0^1 k(r,s,t,x,y,z)u(r,s,t)drdsdt. \tag{4}$$

By using the following approximation for unknown functions $u(x,y,z)$ and with the aid of Legendre scaling function as well as numerical integration techniques for known functions f, k we have

$$u(x,y,z)=C^T\Psi(x,y,z)=$$
$$\sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m}C_{i,j,k}(x,y,z)\psi_{i,j,k}(x,y,z)=\Psi^T(x,y,z)C, \tag{5}$$
$$f(x,y,z)\simeq F^T\Psi(x,y,z), \tag{6}$$

$$k(r,s,t,x,y,z)=\Psi(x,y,z)K\Psi^T(x,y,z), \tag{7}$$

where $K_{(m+1)^3\times(m+1)^3}$ and $F_{(m+1)^3\times 1}$ are known matrices.
By substituting (5), (6) and (7) in (4) we have that

$$\psi^T(x,y,z)C=\psi^TF+\int_0^1\int_0^1\int_0^1 \psi^T(x,y,z)K\psi(x,y,z)\psi^TCdrdsdt,$$

$$\Rightarrow C=F+KC\Rightarrow C=(I-K)^{-1}F.$$

5. Computational illustrations

Throughout this section, we will present numerical results on several examples from the proposed method. Three examples are discussed below. We compared our method with approaches from [3]. Analysis of the errors demonstrates that the method described in this article is much more efficient and reliable. Note that the numerical results associated with the examples were achieved using **MATLAB R2013a**.

Example 5.1. *Consider the following three-dimensional integral equation of Fredholm type as:*

$$u(x,y,z) - \frac{1}{2}\int_{\Omega} x\cos(y+z)u(r,s,t)dtdsdr = f(x,y,z),$$

where $\int_{\Omega} = \int\int\int$ *and,*

$$f(x,y,z) = x^2z + y^2z + 0.008181230850\cos(y+z).$$

The exact solution is with relation $u(x,y,z) = x^2z + y^2z$*. The absolute errors in this solution provided in **Table 1** and **Table 2** while the error of the Legendre scaling function system in our paper overcomes the radial base methods such as GA, MQ and IMQ.*

Table 1. Maximum absolute error(MAE) for Exp. 5.1.

	GA method in [3]	MQ method in [3]	IMQ method in [3]	Present method
$n = 44$	$5000 - pointMC$	$5000 - pointMC$	$5000 - pointMC$	$m = 2$
MAE	$2.69e-03$	$1.94e-03$	$3.23e-3$	$6.93e-5$

Example 5.2. *Consider the following three-dimensional integral equation of Fredholm type as:*

$$u(x,y,z) - \frac{1}{2}\int_{\Omega} \frac{u(r,s,t)}{1+x+y+z}dtdsdr = f(x,y,z),$$

where

$$f(x,y,z) = z\cos(x+y) - \frac{0.081008167157}{1+x+y+z}.$$

The exact solution is given by the relation $u(x,y,z) = z\cos(x+y)$*. The absolute errors in this solution are given in **Table 3** and **Table 4**. We compared the obtained Lgendre scaling function errors with the GA and MQ method errors in [3], which indicates our provided method is accurate and efficient.*

Table 2. Absolute errors for Example 5.1.

m	1	2
(x,y,z)	Present method	Present method
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	8.92×10^{-6}	7.77×10^{-7}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	3.62×10^{-5}	7.22×10^{-6}
$(\frac{1}{8},\frac{1}{8},\frac{1}{8})$	3.75×10^{-4}	4.37×10^{-6}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	1.44×10^{-3}	6.32×10^{-5}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	1.27×10^{-3}	6.93×10^{-5}

Table 3. Maximum absolute error(MAE) for Ex. 5.2.

	GA method in [3]	MQ method in [3]	MQ method in [3]	Present method
$n=19$	$1000-pointMC$	$1000-pointMC$	$4000-pointMC$	$m=2$
MAE	$3.85e-02$	$1.46e-02$	$1.44e-2$	$4.51e-4$

Table 4. Absolute errors for Example 5.2

m	1	2
(x,y,z)	Present method	Present method
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	2.74×10^{-4}	8.27×10^{-6}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	3.25×10^{-3}	6.75×10^{-5}
$(\frac{1}{8},\frac{1}{8},\frac{1}{8})$	1.20×10^{-2}	2.42×10^{-4}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	1.44×10^{-2}	4.51×10^{-4}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	1.83×10^{-2}	1.33×10^{-4}

Example 5.3. *Consider the following three-dimensional integral equation of Fredholm type as:*

$$u(x,y,z)-\frac{1}{2}\int_{\Omega}\cos(x+y+z)u(r,s,t)dtdsdr=f(x,y,z),$$

where

$$f(x,y,z)=\sin(x+y+z)-0.129776032028\cos(x+y+z),$$

the unique solution is given by $u(x,y,z)=\sin(x+y+z)$. The absolute errors in our solution were given in ***Table 5*** *and* ***Table 6***.

Table 5. Maximum absolute error(MAE) for Ex. 5.3.

	GA method in [3]	MQ method in [3]	IMQ method in [3]	Present method
$n=29$	$3000-pointMC$	$3000-pointMC$	$3000-pointMC$	$m=2$
MAE	$7.47e-03$	$8.75e-03$	$1.02e-3$	$6.36e-4$

6. Conclusion

In this paper, the three-dimensional Fredholm integral equation is converted by the Legendre scaling function into an algebraic equation system. This approach is attractive and simple because of the Legendre scaling function orthogonality. The efficiency of the Legendre scaling function method is verified by a comparison of this method with the radial basis functions method. With the aid of Legendre scaling function, we can concentrate on solving nonlinear multi-dimensional integral equations as a subject for future study.

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Table 6. Absolute errors for Example 5.3

m	1	2
(x,y,z)	Present method	Present method
$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	1.73×10^{-6}	8.71×10^{-5}
$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	8.37×10^{-5}	3.33×10^{-4}
$(\frac{1}{8},\frac{1}{8},\frac{1}{8})$	1.07×10^{-3}	7.19×10^{-4}
$(\frac{1}{16},\frac{1}{16},\frac{1}{16})$	1.43×10^{-3}	1.34×10^{-4}
$(\frac{1}{32},\frac{1}{32},\frac{1}{32})$	1.60×10^{-3}	6.36×10^{-4}

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A numerical approach on nonlinear Shrodinger equation by the homotopy analysis method

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Abstract

In this article, by making use of the homotopy analysis method(HAM), solutions of nonlinear Schrodinger equation are excatly obtained in the form of convergent Taylor series. Also, through an example, the results of the desired method are examined.

1. Intruduction

The Schrodinger equation is an important and widely used partial differential equation in many different sciences such as plasma physics and chemistry.
One of the remarkable forms of the nonlinear Schrodinger equations is power law nonlinearity which was studied by Wazwaz in [4] by applying the tanh-coth method.
In this paper, Schrodinger equation with a power law nonlinearity of the form

$$i\frac{\partial w}{\partial t}+a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^4 w}{\partial x^4}+c|w|^{2n}w=0, n\geq 2, w(x,0)=f(x), i^2=-1, \tag{1}$$

where a, b, c are real constant is considered and $w=w(x,t)$ is a complex unknown function [3,4].

2. Main Idea

In this section, we consider Eq. (1) , as follows:

$$i\frac{\partial w}{\partial t}+a\frac{\partial^2 w}{\partial x^2}+b\frac{\partial^4 w}{\partial x^4}+cw^{n+1}\bar{w}^n=0, w(x,0)=f(x), i^2=-1. \tag{2}$$

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we consider:

$$L[\Phi(x,t,q)] = i\frac{\partial\Phi(x,t,q)}{\partial t}, \quad L(c) = 0, \tag{3}$$

where c is a real constant and

$$N[\Phi(x,t,q)] = i\frac{\partial\Phi(x,t,q)}{\partial t} + a\frac{\partial^2\Phi(x,t,q)}{\partial x^2} + b\frac{\partial^4\Phi(x,t,q)}{\partial x^4} + c\Phi^{n+1}(x,t,q)\bar{\Phi}^n(x,t,q). \tag{4}$$

and$H(t) = 1$ and zeroth-order deformation equation is:

$$(1-q)L[\Phi(x,t,q) - w_0] = qhN[\Phi(x,t,q)]. \tag{5}$$

The mth-order deformation equation:

$$L[w_m - \chi_m w_{m-1}] = hR_m(w_{m-1}), \tag{6}$$

which

$$R_m(w_{m-1}) = i\frac{\partial w_{m-1}}{\partial t} + a\frac{\partial^2 w_{m-1}}{\partial x^2} + b\frac{\partial^4 w_{m-1}}{\partial x^4} +$$
$$c\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_1-k_2}\cdots\sum_{k_{n+1}}^{k_1-k_2-\ldots-k_{2n}}\sum_{\alpha_1=0}^{m-1-k_1}\sum_{\alpha_2=0}^{m-1-k_1-\alpha_1}\cdots$$
$$\sum_{\alpha_{n-1}=0}^{m-1-k_1-\alpha_1-\ldots-\alpha_{n-2}} w_{k_2}w_{k_3}\ldots w_{k_{n+1}}w_{(k_1-k_2-\ldots-k_{n+1})}\bar{w}_{\alpha_1}\bar{w}_{\alpha_2}\ldots\bar{w}_{\alpha_{n-1}}\bar{w}_{(m-1-k_1-\alpha_1-\ldots-\alpha_{n-1})}. \tag{7}$$

Here, it is mentioned that the number of summations will be $2n$. Then the m-th order deformation equation is:

$$w_m = \chi_m w_{m-1} + \frac{h}{i}\int_0^t R_m(w_{m-1})dt + c, \quad m \geq 1, \tag{8}$$

$$w_0(x,t) = f(x),$$
$$R_1(w_0) = c \mid f(x) \mid^n f(x) + af^{(2)}(x) + bf^{(4)}(x),$$

and

$$w_1(x,t) = -hi(c \mid f(x) \mid^n f(x) + af^{(2)}(x) + bf^{(4)}(x))t, ...$$

For more details about the HAM, we refer readers to [1,2].
Also, in order to prove the convergence of the method, the following theorem is proposed.

Theorem 2.1. *(Convergence of the HAM) If the series solution*

$$w(x,t) = w_0(x,t) + w_1(x,t) + \ldots,$$

generated from the HAM is convergent, it converges to the exact solution of the Eq.(2).

Proof. suppose the serises

$$\sum_{m=0}^{\infty} w_m(x,t)$$

be convergent. So, we consider

$$w(x,t) = \sum_{m=0}^{\infty} w_m(x,t).$$

In this case, we will get,

$$\lim_{m\to\infty} w_m(x,t) = 0. \tag{9}$$

So

$$\sum_{m=1}^{n} [w_m(x,t) - \chi_m w_{m-1}(x,t)] = w_n(x,t) \tag{10}$$

By using Eq.(10), we will have:

$$\sum_{m=1}^{\infty} [w_m(x,t) - \chi_m w_{m-1}(x,t)] = \lim_{n\to\infty} w_n(x,t) = 0, \tag{11}$$

then we can write :

$$\sum_{m=1}^{\infty} L[w_m(x,t) - \chi_m w_{m-1}(x,t)] = L(\sum_{m=1}^{\infty} (w_m(x,t) - \chi_m w_{m-1}(x,t)) = 0. \tag{12}$$

By applying

$$L[w_m(x,t) - \chi_m w_{m-1}] = hH(t)R_m(w_{m-1}) \tag{13}$$

we get:

$$\sum_{m=1}^{\infty} L[w_m(x,t) - \chi_m w_{m-1}] = hH(t) \sum_{m=1}^{\infty} R_m(w_{m-1}). \tag{14}$$

Moreover, we know $h, H(t) \neq 0$ then

$$\sum_{m=1}^{\infty} [R_m(w_{m-1})] = 0 \tag{15}$$

According to the Eq.(7) and after summarizing ,it can be written:

$$\begin{gathered}\sum_{m=1}^{\infty} [R_m(w_{m-1})] = \\ i\frac{\partial}{\partial t} \sum_{m=0}^{\infty} w_m + a\frac{\partial^2}{\partial t^2} \sum_{m=0}^{\infty} w_m + b\frac{\partial^4}{\partial t^4} \sum_{m=0}^{\infty} w_m + \\ c\sum_{k_2=0}^{\infty} w_{k_2} \sum_{k_3=0}^{\infty} w_{k_3} \sum_{k_4=0}^{\infty} w_{k_4} \dots \sum_{k_{n+1}=0}^{\infty} w_{k_{n+1}} \sum_{k_1=0}^{\infty} w_{k_1} \\ \sum_{\alpha_1=0}^{\infty} \bar{w}_{\alpha_1} \sum_{\alpha_2=0}^{\infty} \bar{w}_{\alpha_2} \dots \sum_{\alpha_{n-1}=0}^{\infty} \bar{w}_{\alpha_{n-1}} \sum_{m=0}^{\infty} \bar{w}_m\end{gathered} \tag{16}$$

From Eqs.(15)(16), and the formula

$$\sum_{m=0}^{\infty} \bar{w_m} = \overline{\sum_{m=0}^{\infty} w_m}$$

We can consider that

$$w(x,t) = \sum_{m=0}^{\infty} w_m(x,t)$$

is the exact solution of the Eq.(2). □

3. Example

In this part ,we solve a Schrodinger equation with power law nonlinearity with the HAM by applying Eqs.(7),(8) and display some numerical results ,also an h-curve is drawn.

Example 3.1. *Consider the following PDE:*

$$iw_t + 1/2w_{xx} - 1/2w_{xxxx} + 2|w|^4w = 0, \quad w(x,0) = e^{ix}. \tag{17}$$

after applying the HAM, we will obtain:

$$w_0(x,t) = e^{ix},$$
$$w_1(x,t) = -htie^{ix},$$
$$w_2(x,t) = -(hte^{xi}(2h+2-hti)i)/2,$$
$$w_3(x,t) = -(hte^{xi}(-h^2t^2i + 6h^2t + 6h^2i + 6ht + 12hi + 6i))/6, ...$$

if $h = -1$ we get:

$$w_0(x,t) = e^{ix},$$
$$w_1(x,t) = te^{xi}i,$$
$$w_2(x,t) = -(t^2e^{xi})/2 = (it)^2e^{xi}/2!,$$
$$w_3(x,t) = -(t^3e^{xi}i)/6 = (it)^3e^{xi}/3!, ...$$

so, we have:

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) + \ldots =$$
$$e^{ix} + te^{xi}i + (it)^2e^{xi}/2! + (it)^3e^{xi}/3! + \ldots = e^{i(x+t)}.$$

which is the exact solution of the equation. See Fig.1 that shows the convergence area of the equation at the point $(1,1)$ and $-1.5 < h < 0$.

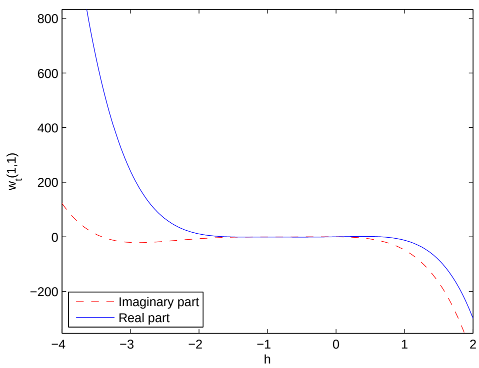

Fig.1.The h-curve of 5-approximation for the example when x=1 and t=1.

See Table 1 that shows the error of the HAM at $x = 1$ and different value of t, when $n = 15$ and $h = -1$.

Table 1. The error of the HAM at $x = 1$.

t	0.2	0.4	0.6	0.8	1
$error$	1.1102e−016	1.1444e−016	1.2912e−016	1.2413e−015	4.7647e−014

See Table 2 which shows the error of the HAM at point (2,0.7) when $h = -1$.
Fig.2 compares the imaginary part of approximation solution and imaginary part of exact solution and Fig.3 presents the real part of approximation solution and exact solution when $n = 7$, $x \in [-2, 2]$, $t \in [0, 1]$, $h = -1$.

Table 2. The error of the HAM

n	error
2	5.664393231101619e-002
4	$1.393798067426435e{-}003$
6	$1.629160728583589e{-}005$
8	$1.109812227816635e{-}007$
10	$4.946498263544871e{-}010$
12	$1.554355314496644e{-}012$
14	$3.652363777994156e{-}015$

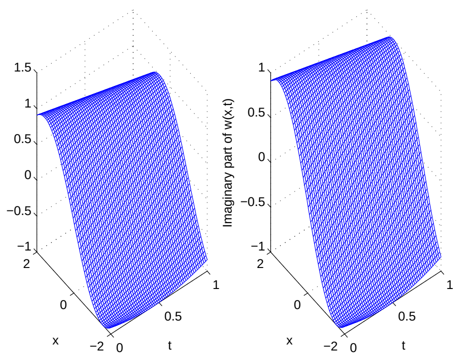

Fig.2.Imaginary part of 7-approximation(left) and imaginary part of exact solution

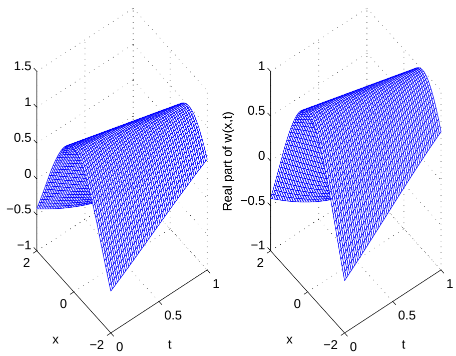

Fig.3.Real part of 7-approximation(left) and real part of exact solution

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A new numerical technique for solving a class of fractional variational problems

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Polynomial basis functions

Abstract

The manuscript deals with a numerical method of the a class of fractional variational problems(FVPs) based on the Caputo fractional derivative by the ritz approximation. To use this method, we transform the FVPs into an optimization problem and obtain the system of nonlinear algebraic equations. By polynomial basis functions, we approximate solutions. Then we have the coefficients of polynomial expansions by solving the system of nonlinear equations. Some numerical examples are included to demonstrate the theoretical results and the performance of the numerical approximation.

1. Introduction

The fractional calculus is an important research field in several different areas such as chemistry, physics, biology, economics and control theory [3, 6, 8]. It has its origin more than 300 years ago when L'Hopital asked Leibniz what should be the meaning of a derivative of non-integer order. After that episode, several more famous mathematicians contributed to the development of fractional calculus: Abel, Fourier, Liouville, Riemann, Riesz [11, 20]. In the last decades, considerable research has been done in fractional calculus. Generally, the analytical solutions of most fractional differential equations are not easy, even impossible. Therefore, seeking numerical solutions of these equations becomes more and more important [15, 16]. This is particularly true in the area of the calculus of variations which has a long history, and it has been used almost in every field where energy principles are applicable [4, 5]. In a lot of problems arising in physics, mechanics, geometry, quantization, control theory and others, it is necessary to determine the maximal or minimal of a certain functional. Because of the important role of this problem in science and engineering, considerable attentions has been received on these kinds of problems. Such problems are called variational problems. Riewe [18, 19] was the first to propose Euler-Lagrange equations for the variational problems with fractional derivatives. In[5], the fractional Euler-Lagrange equation has been used to formulate fractional variational problems. In [14] an extension of Riewe's fractional Hamiltonian formulation is obtained for fractional constrained systems. Agrawal [4] proposed a general finite element formulation for a class of FVPs. Malinowska and Torres[13] presented the necessary and sufficient optimality conditions for problems of the fractional calculus of variations with a Lagrangian depending on the free-end-points. In [7] a discrete-time fractional calculus of variational on the time

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scale is introduced for solving FVPs. Yousefi et al. [21], obtained necessary conditions which must be satisfied to make the fractional variational problems with completely free boundary conditions having extremum. Wang and Xiao [22], presented the fractional discrete Euler-Lagrange equation and the fractional variational integrators for a class of fractional variational problems. Khader and Hendy [9] introduced a general fractional Chebyshev finite difference formulation for solving FVPs. Their method is based on the combination of a useful properties of Chebyshev polynomials approximation and finite difference method. Khader [10]introduced an approximate formula for the Caputo fractional derivative using Rayleigh–Ritz method and chain rule for solving a wide class of FVPs.

2. Solution of fractional variational problems

In this section, we give a numerical technique for obtaining the extremal values of functionals of the general form

$$J[u] = \int_0^1 F(t, u(t), D^{\beta}u(t), D^{\alpha}u(t))dt, n-1 < \alpha \leq n, 0 \leq \beta \leq \alpha,$$

with the boundary conditions

$$u^{(j)}(0) = \kappa_j, u^{(j)}(1) = \eta_j, j = 0, 1, ..., n-1.$$

Here F is a linear or nonlinear function. Fractional derivatives are taken in the Caputo sense that defind[1]

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^{(n)}(\tau)}{(t-\tau)^{-n+\alpha+1}}d\tau$$

The method consists of conversion fractional optimal control problem to optimization problem and expanding the solution by polynomial basis functions with unknown coefficients.
We approximate $u(t)$ as

$$u(t) \cong u_m(t) = \sum_{i=0}^{m} c_i t^{j+1}(t-1)^{j+1}\phi_i(t) + w(t), \qquad j = 0, 1, ..., n, \tag{1}$$

where $\phi_i(t)$ are polynomial basis functions and c_i are unknown coefficients. In following, we determine $w(t)$ as $w^{(j)}(0) = u^{(j)}(0)$ and $w^{(j)}(1) = u^{(j)}(1)$.
Now we have the following optimal problem

$$J[c_0, c_1, ..., c_m] = \int_0^1 F(t, u_m(t), D^{\beta}u_m(t), D^{\alpha}u_m(t))dt, \tag{2}$$

If $c_k$are decided by the optimizing function J, then by (1), we obtain functions which approximate the optimum value of J in (2). To find unknowns c_k, $k = 0, 1, ..., m$ in $u_m(t)$, according to the necessary conditions of optimization for (2), we have

$$\frac{\partial J}{\partial c_k} = 0, \quad k = 0, ..., m. \tag{3}$$

Then by solving the above system of m algebraic equations (3), we obtain c_k, $k = 0, 1, ..., m$. The approached demonstrated here relies on the Ritz method. Then with solving this problem by mathematica software, we obtain c_k. The method presented here is based on the Ritz method. We refer the interested reader to [2] for more information.

3. Illustrative examples

To demonstrate the effectiveness of the method, here we consider a fractional variational problems. The following example demonstrate that the desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative example show that this method in comparison to other methods has high accuracy and is easily implemented.

Example 3.1. *Consider the following FVP: find the extremum of the functional [12]*

$$J[u] = \int_0^1 (D^{\frac{1}{4}}u(t) + D^{\frac{5}{2}}u(t) - f(t))^2 dt, \tag{4}$$

where

$$f(t) = \frac{5\sqrt{\pi}\Gamma(\frac{7}{4})t^{\frac{9}{4}}}{2\Gamma(\frac{3}{4})\Gamma(\frac{13}{4})} + \frac{15\sqrt{\pi}t^{\frac{5}{4}}}{8\Gamma(\frac{9}{4})},$$

under the following boundary conditions

$$u(0) = 0, u'(0) = 0,$$
$$u(1) = 1, u'(1) = \tfrac{5}{2}.$$

The exact solution is given by $u(t) = t^{\frac{5}{2}}$.
By applying our method with different values of m, we obtain the numerical results. Fig. 1 shows the absolute error of this problem obtained by the present method with $m = 8$. From Fig. 1, we can see that the present method provides accurate results.

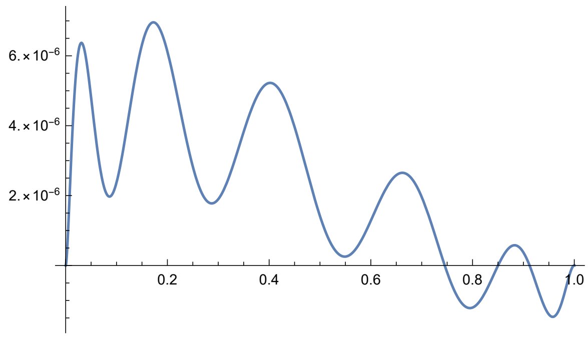

Fig.1. The absolute error between exact and numerical solution for $m = 8$.

The following table shows the values of minimum J for different values of approximations.

	$m = 3$	$m = 5$	$m = 8$
J	1.38821×10^{-6}	2.14559×10^{-7}	2.64512×10^{-8}

Example 3.2. *Consider the following FVP: find the extremum of the functional [17]*

$$J(u) = \frac{1}{2}\int_0^1 (D^{\alpha}u(t))^2 dt, \qquad 0 \le \alpha \le 1, \tag{5}$$

under the following boundary conditions

$$u(0) = 0, u(1) = 1.$$

A closed-form solution for this problem is given in [9]

$$u(t) = \frac{1}{2\alpha - 1}\int_0^t \frac{dx}{[(1-x)(t-x)]^{1-\alpha}}.$$

For $\alpha = 1$, the exact solution of this problem is $u(t) = t$. We apply the Ritz Approximation explained in Section 2 for solving numerically (5) and obtain $u(t) = t$.
Fig. 3 represents the approximate solutions of u(t) for $\alpha = 0.5, 0.7, 0.9, 1$ with $m = 5$ in comparison with the exact solution $u(t)$. Numerical results are presented to demonstrate the effectiveness of the proposed method.

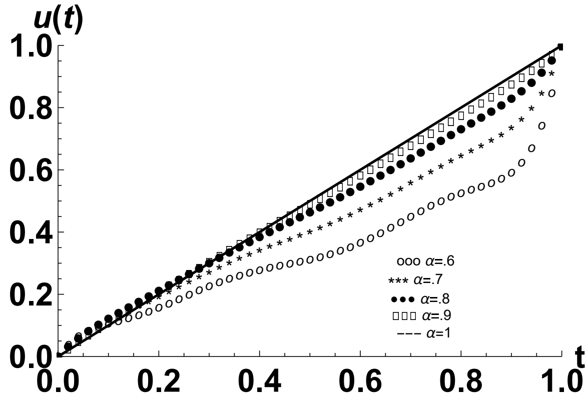

Fig. 2. Comparison of $u_m(t)$ for $m = 5$ with $\alpha = 0.6, 0.7, 0.8, 0.9, 1$ and the exact solution.

Example 3.3. *Consider the following FVP: find the extremum of the functional [4]*

$$J[u] = \int_0^1 (\frac{1}{2}(D^\alpha u(t))^2 - (t))dt, \qquad 0 \le \alpha \le 1, \tag{6}$$

under the following boundary conditions

$$u(0) = 0, u(1) = 0.$$

For $\alpha = 1$, the exact solution of this problem is $u(t) = \frac{t}{2}(1-t)$. We apply the Ritz Approximation explained in Section 2 for solving numerically (6) and for $\alpha = 1$ obtain $u_5(t) = \frac{t}{2}(1-t)$.
Fig. 3 represents the approximate solutions of u(t) for $\alpha = 0.6, 0.7, 0.8, 0.9, 1$ with $m = 5$ in comparison with the exact solution $u(t)$. Numerical results are presented to demonstrate the effectiveness of the proposed method.

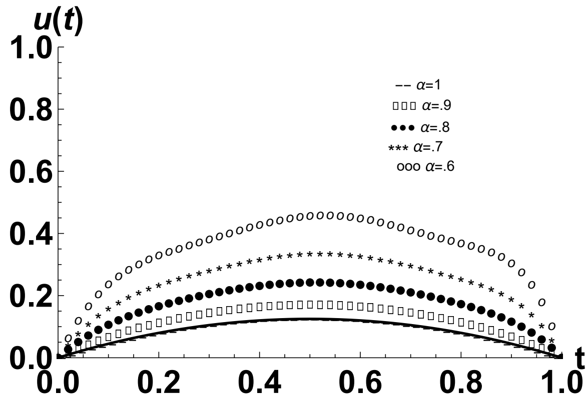

Fig.3. Comparison of $u_m(t)$ for $m = 5$ with $\alpha = 0.6, 0.7, 0.8, 0.9, 1$ and the exact solution.

4. Conclusion

This paper presents a simple and effective approach to solve a wide class of fractional variational problems. The desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative example show that this method has high accuracy and is easily implemented. The method will be expected to deal with other fractional problems such as fractional inverse problems, fractional optimal problems and other problems, which will be discussed in a future papers.

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Solving two-dimensional nonlinear Volterra integral equations using Rationalized Haar functions

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Abstract

In this paper, we have introduced a computational method for a class of two-dimensional nonlinear Volterra integral equations, based on the expansion of the solution as a series of Haar functions. To achieve this aim it is necessary to define the integral operator. The Banach fixed point theorem guarantees that under certain assumptions this operator has a unique fixed point, we have introduced an orthogonal projection and by interpolation property, we have achieved an operational matrix of integration. Also, by using the Banach fixed point theorem, we get an upper bound for the error of our method. Since our examples in this article are selected from different references, so should be the numerical results obtained here can be compared with other numerical methods.

1. Introduction and preliminaries

The first work for the solution of two-dimensional linear Volterra integral equations(VIE) has been done by Brunner and Kauthen [2], who introduced collocation and iterated collocation methods. Kauthen has extended this study to the case of linear Volterra-Fredholm integral equations (VFIE) [8] and Brunner has considered in [11], the case of nonlinear VIE. For example, radiation transfer problems, different kinds of Cauchy problems for certain partial differential equations (e.g. the telegraph equation), or the Darboux problem can also be reduced to an equation of this form. Several methods are used to approximate this equation, for example, Nystrom trapezoidal, iterated collocation, and Galerkin combined with extrapolation, basis of block-pulse functions, besides projection methods and Euler method, other kinds of computational algorithms have been used to solve this equation. The aim of this work is to present a numerical method for approximating the solution of nonlinear second-kind Volterra integral equation as follows:

$$u(t,s)=f(t,s)+\int_0^s\int_0^t W_1(t,s,x,y,u(x,y))dxdy+\alpha\int_0^s W_2(t,s,y,u(t,y))dy+\beta\int_0^t W_3(t,s,x,u(x,s))dx, \tag{1}$$

where $x,y,t,s\in[0,1], u\in X=C([0,1]^2)$, and $\alpha,\beta\in R$ and

$$f:[0,1]^2\to\mathbb{R}^2,$$

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$$W_1 : [0,1]^4 \times \mathbb{R}^2 \to \mathbb{R}^2,$$
$$W_2, W_3 : [0,1]^3 \times \mathbb{R}^2 \to \mathbb{R}^2,$$

are assumed to be known continuous functions satisfying the Lipschitz condition, that is, there exist $M_1, M_2, M_3 \geqslant 0$ such that:

$$|W_1(t,s,x,y,v_1(x,y)) - W_1(t,s,x,y,v_2(x,y))| \leqslant M_1|v_1 - v_2|,$$
$$|W_2(t,s,y,v_1(t,y)) - W_2(t,s,y,v_2(x,y))| \leqslant M_2|v_1 - v_2|,$$
$$|W_3(t,s,x,v_1(x,s)) - W_3(t,s,x,v_2(x,y))| \leqslant M_3|v_1 - v_2|,$$

where $v_1, v_2 \in \mathbb{R}^2$, and the unknown function to be determined is $u : [0,1]^2 \to \mathbb{R}^2$.
The numerical solution of equation (1) is computed by using Rationalized Haar functions. The orthogonal set of Haar wavelets is a group of square waves with magnitude $-2^{j/2}$, $2^{j/2}$ and 0 for $j = 0, 1, ...,$ [9]. Lynch and Reis's [5] have Rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. This modification results in what is called the Rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [14]. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitude $+1$, -1, and 0.The numerical solution of nonlinear one-dimensional Fredholm equations using a basis of Haar functions has been considered by Razzaghi and Ordokhani in [17]. Certain types of partial differential equations can be reformulated as two-dimensional Volterra integral equation, for example in [7], the Darboux problem has been considered for $a, b, a_0, b_0 \in [0, \infty)$ as

$$D_{xy}z(x,y) = f(x,y,z(x,y)), \quad (x,y) \in \ E = (0,a] \times (0,b],$$
$$z(x,y) = \phi(x,y) \ \ for \ \ (x,y) \in ([-a_0,a] \times [-b_0,b]) \setminus E,$$

where $D_{xy}z = \frac{\partial^2 z}{\partial x \partial y}$, $B = [-a_0, 0] \times [-b_0, b]$, and

$$f : \overline{E} \times C(B,R) \to R, \quad z : E^0 \cup E \to R,$$

and in [12] it has been shown that if $E = (0,1] \times (0,1]$ this problem is equivalent to

$$u(t,s) = u(t,0) + u(0,s) + \int_0^s \int_0^t W(x,y,u(x,y))dxdy - u(0,0). \tag{2}$$

Another example can be cited as a telegraph equation

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx},$$

where $c^2 = \frac{1}{LC}$, $\alpha = \frac{G}{C}$, $\beta = \frac{R}{L}$, which consists of a resistor of resistance Rdx, a coil of inductance, Ldx, a resistor of conductance Gdx or a capacitor of capacitance Cdx.
In [15] it has been shown that the telegraph equation can be reduced to an equation of the form (1), if $\alpha, \beta \neq 0$, and

$$W_1(t,s,x,y,u(x,y)) = g_1(x,y,u(x,y)),$$
$$W_2(t,s,y,u(t,y)) = g_2(t,y,u(t,y)),$$
$$W_3(t,s,x,u(x,s)) = g_3(x,s,u(x,s)).$$

The numerical results presented in that paper show a fast convergence of another method when applied to integral equations. To achieve this aim it is necessary to define the integral operator, $T : (X, \|.\|_\infty) \to (X, \|.\|_\infty)$. By applying this operator in Eq (1), we have

$$T(u(t,s)) = f(t,s) + \int_0^s \int_0^t W_1(t,s,x,y,u(x,y))dxdy + \alpha \int_0^s W_2(t,s,y,u(t,y))dy + \beta \int_0^t W_3(t,s,x,u(x,s))dx. \tag{3}$$

The Banach fixed point theorem guarantees that under certain assumptions [1], T has an unique fixed point; that is, the two-dimensional Volterra integral equation has exactly one solution. Indeed, suppose further that W_1, W_2, W_3 are Lipschitz functions with respect to their fifth and fourth variables with Lipschitz constants $M_1, M_2, M_3 > 0$ and $M_1+|\alpha|M_2+|\beta|M_3 < 1$, then the operator T is contractive with contraction number $q = M_1+|\alpha|M_2+|\beta|M_3$, and thus T has an unique fixed point u. Moreover, $u = \lim_{n\to\infty} T^n(u_0)$, where u_0 is any continuous function on $[0, 1]$. Since, in general it is not possible to calculate u explicitly from the sequence of functions $\{T^n(u)\}_{n\in\mathbb{N}}$, so we define a new sequence of functions, denoted by $\{u_i\}_{i\in\mathbb{N}}$, obtained recursively by using interpolation and RH basis.

2. Properties of the Rationalized Haar functions

Definition 2.1. The RH wavelet is the function defined on the real line $\mathbb{R}$ as follows:

$$H(t) = \begin{cases} 1, & 0 < t \leq \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1, \\ 0, & otherwise. \end{cases} \tag{4}$$

Definition 2.2. The RH functions $h_n(t)$ for any $n = 1, 2, \ldots$, where $n = 2^i+j$, for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, 2^i-1$, are defined by

$$h_n(t) = H(2^i t - j) = \begin{cases} 1 & j.2^{-i} \leq t < (j+\frac{1}{2})2^{-i} \\ -1 & (j+\frac{1}{2})2^{-i} \leq t < (j+1)2^{-i} \\ 0 & \text{otherwise.} \end{cases}$$

Also, we define $h_0(t) = 1$ for all $t \in [0, 1)$. Here the integer $2^i, i = 0, 1, \ldots$, indicates the level of the wavelet and $j = 0, 1, \ldots, 2^i - 1$ is the translation parameter. Note that the basic multiplication properties of RH functions are as follows:

$$h_0(t)h_q(t) = h_q(t),$$

for $q \in \mathbb{Z}^+ \cup \{0\}$, and for $0 < l < q$, we have

$$h_l(t)h_q(t) = \begin{cases} h_q(t), & if\ h_q\ occurs\ during\ the\ positive\ half-wave\ of\ h_l, \\ -h_q(t), & if\ h_q\ occurs\ during\ the\ negative\ half-wave\ of\ h_l, \\ 0, & otherwise. \end{cases} \tag{5}$$

Also, the square of any RH function is a block-pulse with magnitude 1 during both positive and negative half-waves of RH function. It can be shown that the sequence $\{h_n\}_{n=0}^{\infty}$ is a complete orthogonal system in $L^2[0, 1]$ and also for $n = 2^j + k$, for $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^j - 1$, we can expand $f(t) \in C[0, 1]$ with RH function as

$$f(t) = \sum_{n=0}^{\infty} a_n h_n(t), \tag{6}$$

where

$$a_n = 2^j \int_0^1 f(t)h_n(t)dx.$$

Thus the series $\sum_n 2^j \langle f, h_n\rangle h_n$, will converge to f, see e.g. [16], if (6) truncated up to its first m terms that $m = 2^{\lambda+1}$, $\lambda = 0, 1, ...$ where $j = 0, 1, ..., \lambda$ then:

$$f(t) = \sum_{n=0}^{m-1} a_n h_n(t) = \mathbf{A}^T\mathbf{h}(t), \tag{7}$$

where $\mathbf{A} = [a_0, a_1, ..., a_{m-1}]^T$, and $\mathbf{h}(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T$. It is known that

$$< h_l(t), h_q(t) >= \int_0^1 h_l(t)h_q(t)dt = \begin{cases} 2^{-i}, & if \ \ l = q = 2^i + j \ with \ i \in \mathbf{Z}^+ \cup \{0\} \ and \ j \in J_i \\ 1, & if \ \ l = q = 0, \\ 0, & if \ \ l \neq q, \end{cases} \tag{8}$$

where $J_i = \{0, 1, \ldots, 2^i - 1\}$ for any $i \in \mathbf{Z}^+ \cup \{0\}$. Consequently, all the RH functions $h_l(t), \ \ l \in \mathbf{Z}^+ \cup \{0\}$, are orthogonal to each other. The integration of the $h_l(t)$ is given by

$$\int_0^1 h_l(t)dt = \begin{cases} 1, & if \ \ l = 0, \\ 0, & if \ \ l \neq 0. \end{cases} \tag{9}$$

Also, any function $f(t, s)$ of two variables in X can be similarly approximated in terms of RH functions as

$$f(t, s) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} f_{ij}h_{ij}(t, s) = \mathbf{F}^T \mathbf{h}(t, s), \tag{10}$$

that two-dimensional RH functions are defining as $h_{ij}(t, s) = h_i(t)h_j(s)$, and

$$\mathbf{F} = [f_{00}, f_{01}, ..., f_{m-1,m-1}]^T_{(m-1)\times(m-1)},$$
$$\mathbf{h}(t, s) = [h_{00}, h_{01}, ..., h_{m-1,m-1}]^T_{(m-1)\times(m-1)}(t, s).$$

The RH function coefficients f_{ij} are given by

$$f_{ij} = \frac{\langle f(t, s), h_{ij}(t, s)\rangle}{\|h_{ij}(t, s)\|_2^2}.$$

3. Numerical approximation of the solution

We now describe the idea of our proposed numerical method. The first point lies in the operator formulation of the two-dimensional nonlinear Volterra integral equationss. Using an initial function $u_0 \in C[0, 1]$, and since in general we cannot calculate $T(u_0)$, explicitly we approximate this function as follows:
for each $t, s \in [0, 1]$ and $m \in \mathbb{N}$, that $m = 2^{i+1}$ for $i = 0, 1, ...$,we define recursively

$$u_i(t, s) = f(t, s)+\int_0^s \int_0^t W_1(t, s, x, y, u_{i-1}(x, y))dxdy+\alpha \int_0^s W_2(t, s, y, u_{i-1}(t, y))dy+\beta \int_0^t W_3(t, s, x, u_{i-1}(x, s))dx. \tag{11}$$

In this section we assume:

$$\psi_1^{(i-1)}(t, s, x, y) := W_1(t, s, x, y, u_{i-1}(x, y)), \tag{12}$$
$$\psi_2^{(i-1)}(t, s, y) := W_2(t, s, y, u_{i-1}(t, y)), \tag{13}$$
$$\psi_3^{(i-1)}(t, s, x) := W_3(t, s, x, u_{i-1}(x, s)), \tag{14}$$

we can expand $\psi_i^{(i-1)}$ for $i = 1, 2, 3$ in terms of RH functions as

$$\psi_1^{(i-1)}(t, s, x, y) = h^T(t, s)K_1h(x, y),$$
$$\psi_2^{(i-1)}(t, s, y) = h^T(t, s)K_2h(t, y),$$
$$\psi_1^{(i-1)}(t, s, x) = h^T(t, s)K_3h(x, s),$$

if Q_m is an orthogonal projection with following interpolation property we have

$$Q_m(\psi_1^{(i-1)}(t, s, x, y)) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \sum_{r=0}^{m-1} \sum_{q=0}^{m-1} k^1_{ijrq}h_{ij}(t, s)h_{rq}(x, y), \tag{15}$$

$$Q_m(\psi_2^{(i-1)}(t,s,y)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k_{ijrq}^2 h_{ij}(t,s)h_{rq}(t,y), \tag{16}$$

$$Q_m(\psi_3^{(i-1)}(t,s,x)) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{r=0}^{m-1}\sum_{q=0}^{m-1} k_{ijrq}^3 h_{ij}(t,s)h_{rq}(x,s). \tag{17}$$

Thus K_1, K_2, K_3 are block matrices of the form

$$K_s = [K_s^{(i,j)}]_{i,j=0}^{m-1}, \quad s = 1,2,3, \tag{18}$$

where

$$K_s^{(i,j)} = [K_{ijrq}^s]_{i,j,r,q=0}^{m-1}, \quad s = 1,2,3, \tag{19}$$

and for coefficients $K_{ijrq}^s,\ s = 1,2,3$ we have:

$$k_{ijrq}^1 = \frac{\langle W_1(t,s,x,y,u(x,y)), h_{rq}(x,y)\rangle, h_{ij}(t,s)\rangle}{\langle h_{ij}(t,s), h_{ij}(t,s)\rangle\langle h_{rq}(x,y), h_{rq}(x,y)}, \tag{20}$$

$$k_{ijrq}^2 = \frac{\langle W_2(t,s,y,u(t,y)), h_{rq}(t,y)\rangle, h_{ij}(t,s)\rangle}{\langle h_{ij}(t,s), h_{ij}(t,s)\rangle\langle h_{rq}(t,y), h_{rq}(t,y)}, \tag{21}$$

$$k_{ijrq}^3 = \frac{\langle W_3(t,s,x,u(x,s)), h_{rq}(x,s)\rangle, h_{ij}(t,s)\rangle}{\langle h_{ij}(t,s), h_{ij}(t,s)\rangle\langle h_{rq}(x,s), h_{rq}(x,s)}. \tag{22}$$

The row vector below has been introduced by Chen and Hsiao [6]

$$\mathbf{h}(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T. \tag{23}$$

Now we have

$$\int_0^t \mathbf{h}(s)ds = \mathbf{Ph}(t), \tag{24}$$

where $\mathbf{P} = \mathbf{P}_{k\times k}$ is called the operational matrix of integration, that $k = m-1$. Chen and Hsiao have shown that the following recursive formula holds:

$$\mathbf{P} = \frac{1}{2k}\begin{pmatrix} 2k\,\mathbf{P}_{(\frac{k}{2})\times(\frac{k}{2})} & -\hat{H}_{(\frac{k}{2})\times(\frac{k}{2})} \\ \hat{H}^{-1}_{(\frac{k}{2})\times(\frac{k}{2})} & \mathbf{0} \end{pmatrix}, \tag{25}$$

The Haar coefficient matrix $\hat{H}_{m\times m}$ is defined as:

$$\hat{H}_{m\times m} = [\mathbf{h}(\frac{1}{2m}), \mathbf{h}(\frac{3}{2m}), \ldots, \mathbf{h}(\frac{2m-1}{2m})]. \tag{26}$$

For example, the first eight RH functions can be written in matrix form as

$$\hat{H}_{8\times 8} = \begin{pmatrix} \mathbf{h}_0(t) \\ \mathbf{h}_1(t) \\ \vdots \\ \mathbf{h}_7(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{27}$$

Which $\hat{H}_{1\times 1} = [1]$, $\mathbf{P}_{1\times 1} = [\frac{1}{2}]$, and

$$\hat{H}^{-1}_{k\times k} = (\frac{1}{k}).\hat{H}^{\mathbf{T}}_{k\times k}.\mathbf{diag}\Big(1,1,2,2,\underbrace{2^2,\ldots,2^2}_{2^2},\underbrace{2^3,\ldots,2^3}_{2^3},\ldots,\underbrace{\frac{k}{2},\ldots,\frac{k}{2}}_{\frac{k}{2}}\Big). \tag{28}$$

Thus for the nonlinear two-dimensional Volterra integral equations we have:

$$u_i(t,s) = f(t,s)+\int_0^s\int_0^t Q_m(\psi_1^{(i-1)}(t,s,x,y))dxdy+\alpha\int_0^s Q_m(\psi_2^{(i-1)}(t,s,y))dy+\beta\int_0^t Q_m(\psi_3^{(i-1)}(t,s,x))dx. \tag{29}$$

4. Error analysis

In this section, by using the Banach fixed point theorem, we get an upper bound for the error of our method. Suppose that, $f:[0,1]\to \mathbf{R}$ be an arbitrary continuous function, we define

$$||f||_\infty = sup\{|f(x)|; x\in[0,1]\}. \tag{30}$$

If we generalize this norm for two-dimensional function $f(x,t):[0,1]^2\to\mathbf{R}^2$, we have

$$||f||_\infty = sup\{|f(x,y)|; (x,y)\in[0,1]\times[0,1]\}. \tag{31}$$

Lemma 4.1. *Let $W_1\in C([0,1]^4\times\mathbb{R}^2)$, and $W_2, W_3\in C([0,1]^3\times\mathbb{R}^2)$ are Lipschitz functions with respect to their fifth and fourth variables, with Lipchitz constants M_1, M_2 and M_3, then T has an unique fixed point and for all $u_0\in C([0,1]^2)$*

$$\|u-T^i(u_0)\|_\infty \le \|T(u_0)-u_0\|_\infty \times \sum_{j=i}^{\infty} q^j, \tag{32}$$

where $q = M_1+|\alpha|M_2+M_3|\beta|<1$, and u is the fixed point of T.

Proof: For $u,v\in C([0,1]^2)$, we have:

$$\begin{aligned}
|T(u(t,s))-T(v(t,s))| = {} & |\int_0^s\int_0^t (W_1(t,s,x,y,u(x,y))-W_1(t,s,x,y,v(x,y)))dxdy\\
&+\alpha\int_0^s (W_2(t,s,y,u(t,y))-W_2(t,s,y,v(t,y)))dy\\
&+\beta\int_0^t (W_3(t,s,x,u(x,s))-W_3(t,s,x,v(x,s)))dx|\\
&\le\int_0^s\int_0^t |W_1(t,s,x,y,u(x,y))-W_1(t,s,x,y,v(x,y))|dxdy\\
&+|\alpha|\int_0^s |W_2(t,s,y,u(t,y))-W_2(t,s,y,v(t,y))|dy\\
&+|\beta|\int_0^t |W_3(t,s,x,u(x,s))-W_3(t,s,x,v(x,s))|dx\\
&\le M_1\int_0^s\int_0^t |u(x,y)-v(x,y)|dxdy + M_2|\alpha|\int_0^s |u(t,y)-v(t,y)|dy\\
&+M_3|\beta|\int_0^t |u(x,s)-v(x,s)|dx\\
&\le M_1||u-v||_\infty + M_2|\alpha|||u-v||_\infty + |\beta|M_3||u-v||_\infty\\
&\le (M_1+|\alpha|M_2+|\beta|M_3)||u-v||_\infty.
\end{aligned}$$

By induction, for all $n\in\mathbb{N}$ we have

$$\|T^n(u)-T^n(v)\|_\infty \le q^n\|u-v\|_\infty,$$

since $q < 1$ we have:

$$\sum_{n=1}^{\infty} \|T^n(u) - T^n(v)\|_\infty < \infty.$$

Thus T has a unique fixed point which means that (3) has a unique solution and (32) follows from the Banach fixed-point theorem. □

Theorem 4.2. *Assume that* $\psi_1^{(i-1)} \in C([0,1]^4)$*, and* $\psi_2^{(i-1)}, \psi_3^{(i-1)} \in C([0,1]^3)$ *and* $\{u_i\}_{i\geq 1}$ *is a subset of* $C([0,1]^2)$*, and* $W_1 \in C([0,1]^4 \times \mathbb{R}^2)$*,* $W_2, W_3 \in C([0,1]^3 \times \mathbb{R}^2)$ *are Lipschitz functions with respect to their fifth and fourth variables, then we have*

$$\|u - u_i\|_\infty \leq \|T(u_0) - u_0\|_\infty \sum_{j=i}^{\infty} q^j + c, \tag{33}$$

where c *is a constant*

Proof: If

$$L_{i-1} = \max\{\|\frac{\partial \psi_k^{i-1}}{\partial t}\|_\infty, \|\frac{\partial \psi_k^{i-1}}{\partial s}\|_\infty \|\frac{\partial \psi_k^{i-1}}{\partial x}\|_\infty, \|\frac{\partial \psi_k^{i-1}}{\partial y}\|_\infty\},$$

for $k = 1,2,3$ and $m = 2^{i+1}$ for $i = 1,\ldots,$ then

$$\begin{aligned}
\|T(u_{i-1}) - u_i\|_\infty \leq & \|\int_0^s \int_0^t \psi_1^{(i-1)}(t,s,x,y) - Q_m(\psi_1^{(i-1)}(t,s,x,y))dxdy\|_\infty \\
& + |\alpha| \, \|\int_0^s \psi_2^{(i-1)}(t,s,y) - Q_m(\psi_2^{(i-1)}(t,s,y))dy\|_\infty \\
& + |\beta| \, \|\int_0^t \psi_3^{(i-1)}(t,s,x) - Q_m(\psi_3^{(i-1)}(t,s,x))dx\|_\infty \\
\leq & \|\psi_1^{i-1} - Q_m(\psi_1^{i-1})\|_\infty + |\alpha| \, \|\psi_2^{i-1} - Q_m(\psi_2^{i-1})\|_\infty + |\beta| \, \|\psi_3^{i-1} - Q_m(\psi_3^{i-1})\|_\infty.
\end{aligned}$$

If we define

$$g(t,s,x,y) := \psi^{i-1} - Q_m(\psi^{i-1}),$$

by using interpolating property and the mean-value theorem for four variables with $t_0, s_0, x_0, y_0 = 0$ and

$$\begin{aligned}
t_i &= \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}}, \quad for \;\; i = 2^{n_1} + v_1, \\
s_j &= \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}}, \quad for \;\; j = 2^{n_2} + v_2, \\
x_k &= \frac{1}{2^{n_3+1}} + \frac{v_3}{2^{n_3}}, \quad for \;\; k = 2^{n_3} + v_3, \\
y_l &= \frac{1}{2^{n_4+1}} + \frac{v_4}{2^{n_4}}, \quad for \;\; l = 2^{n_4} + v_4,
\end{aligned}$$

where $n_1, n_2, n_3, n_4 \geq 1$, and $i, j, k, l \leq m-1$, we have

$$\begin{aligned}
\|\psi^{i-1} - Q_m(\psi^{i-1})\|_\infty =& \|g(t_i, s_j, x_k, y_l) + \frac{\partial g}{\partial t}(\xi, \gamma, \tau, \upsilon)(\xi - t_i) + \frac{\partial g}{\partial s}(\xi, \gamma, \tau, \upsilon)(\gamma - s_j) + \\
& \frac{\partial g}{\partial x}(\xi, \gamma, \tau, \upsilon)(\tau - x_k) + \frac{\partial g}{\partial y}(\xi, \gamma, \tau, \upsilon)(\upsilon - y_l)\|_\infty \\
=& \|(I - Q_m)\frac{\partial \psi^{i-1}}{\partial t}(\xi, \gamma, \tau, \upsilon) + (I - Q_m)\frac{\partial \psi^{i-1}}{\partial s}(\xi, \gamma, \tau, \upsilon) \\
& + (I - Q_m)\frac{\partial \psi^{i-1}}{\partial x}(\xi, \gamma, \tau, \upsilon) + (I - Q_m)\frac{\partial \psi^{i-1}}{\partial y}(\xi, \gamma, \tau, \upsilon)\|_\infty \\
& \max\{\|\xi - t_i\|_\infty, \|\gamma - s_j\|_\infty, \|\tau - x_k\|_\infty, \|\upsilon - y_l\|_\infty\} \\
\leq& \frac{2}{2^i}\|(I - Q_m)\|_\infty \|\|\frac{\partial \psi^{i-1}}{\partial t}(\xi, \gamma, \tau, \upsilon) + \frac{\partial \psi^{i-1}}{\partial s}(\xi, \gamma, \tau, \upsilon) \\
& + \frac{\partial \psi^{i-1}}{\partial x}(\xi, \gamma, \tau, \upsilon) + \frac{\partial \psi^{i-1}}{\partial y}(\xi, \gamma, \tau, \upsilon)\|_\infty \leq \frac{8L_{i-1}}{2^i}.
\end{aligned}$$

The same proof holds for ψ_k^{i-1} for $k = 2, 3$. We have

$$\|T(u_{i-1}) - u_i\|_\infty \leq (4 + 3|\alpha| + 3|\beta|)\frac{2L_{i-1}}{2^i}. \tag{34}$$

For certain constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i > 0$ that $i \geq 1$, if

$$(4 + 3|\alpha| + 3|\beta|)\frac{2L_{i-1}}{2^i} < \varepsilon_k \ \ for \ \ k = 1, 2, ..., i,$$

we have

$$\|T(u_{i-1}) - u_i\|_\infty < \varepsilon_i, \quad i \geq 1. \tag{35}$$

By applying the triangle inequality, we achieve

$$\|u - u_i\|_\infty \leq \|u - T^i(u_0)\|_\infty + \sum_{j=1}^{i} q^j \|T(u_{j-1}) - u_j\|_\infty. \tag{36}$$

From (32) and (35) we conclude

$$\|u - u_i\|_\infty \leq \|T(u_0) - u_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j}\varepsilon_j. \tag{37}$$

5. Numerical examples

We illustrate the behavior of our numerical method by three examples, in this section. To define the sequence of approximating functions $\{u_i\}_{i\in\mathbb{N}}$ we have taken an initial function $u^0 \in C([0,1]^2)$ and $m = 2^{j+1}$ for all $j = 0, 1, ...$ We exhibit the absolute errors committed in points $(x_i, t_i) \in [0,1]^2$, when we approximate the exact solution u iteratively. In this section points are proposed as $(x_i, t_i) = (\frac{1}{2^i}, \frac{1}{2^i})$ for $i = 1, 2, ...6$, numerical results obtained here can be compared with other numerical methods.

Example 5.1. *For the first example, consider the special case of Cauchy problem equation on* $X = [0,1] \times [0,1]$ *(see*

[15])

$$
\begin{aligned}
u(x,t) = \frac{1}{4}\int_0^t\int_0^x & sin(u(y,z))cos(\frac{y-z}{2})sin(\frac{y+z}{2}) + cos(u(y,z))sin(\frac{y-z}{2})cos(\frac{y+z}{2})dydz \\
& -\frac{1}{4}\int_0^x sin(u(y,t)) + cos(u(y,t)) - sin(u(y,0)) - cos(u(y,0))dy \\
& +\frac{1}{4}\int_0^t cos(u(x,z)) + cos(u(x,z)) - cos(u(0,z)) + sin(u(0,z))dz \\
& +sin^2(\frac{x}{2}) - sin^2(\frac{t}{2}),
\end{aligned}
$$

with exact solution $u(x,t) = sin(\frac{x+t}{2})sin(\frac{x-t}{2})$. Numerical results for Example 1 are displayed in Table 1.

Table 1. Numerical results for Example 1.

(x,t)	Legendre polynomials method([13]) $M=6$	Presented method $m=16$	Presented method $m=32$
(0.0,0.2)	1.5×10^{-6}	0	0
(0.2,0.4)	6.4×10^{-6}	2.19×10^{-6}	5.89×10^{-7}
(0.3,0.6)	5.6×10^{-6}	2.93×10^{-6}	7.36×10^{-7}
(0.4,0.8)	3.5×10^{-6}	3.53×10^{-6}	9.29×10^{-7}
(0.8,1.0)	3.3×10^{-6}	2.80×10^{-6}	7.26×10^{-7}

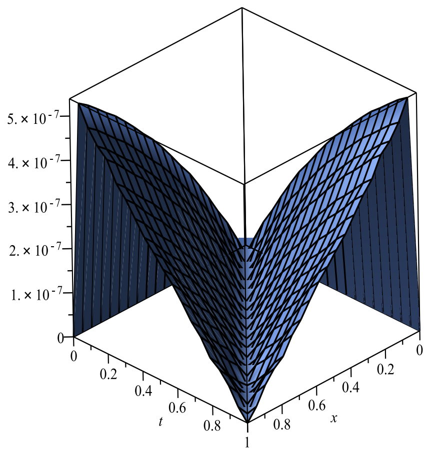

Fig. 1. Comparison of absolute errors for Example 1 with grid=[20,20]

Example 5.2. *Let us consider the nonlinear two dimensional Volterra integral equation of the second kind (see [13],[3]).*

$$u(x,t) = x + t - \frac{1}{12}xt(x^3 + 4x^2t + 4xt^2 + t^3) + \int_0^t\int_0^x (x+t-y-z)u^2(y,z)dydz, \tag{38}$$

with exact solution $u(x,t) = x + t$, and $0 \le x,t < 1$. Numerical results for Example 2 are displayed in Table 2.

Table 2. Numerical results for Example 2.

$(x,t)=(\frac{1}{2^i},\frac{1}{2^i})$	Legendre polynomials method ([13]) $M=2$	Haar wavelet method ([3]) $m=32$	Presented method $m=16$
i=1	3.5×10^{-3}	3.1×10^{-2}	2.15×10^{-4}
i=2	4.5×10^{-4}	3.1×10^{-2}	6.71×10^{-4}
i=3	6.1×10^{-4}	3.1×10^{-2}	2.08×10^{-5}
i=4	5.7×10^{-4}	3.1×10^{-2}	6.38×10^{-7}
i=5	3.6×10^{-4}	3.1×10^{-2}	1.73×10^{-8}
i=6	2.0×10^{-4}	2.2×10^{-9}	7.76×10^{-10}

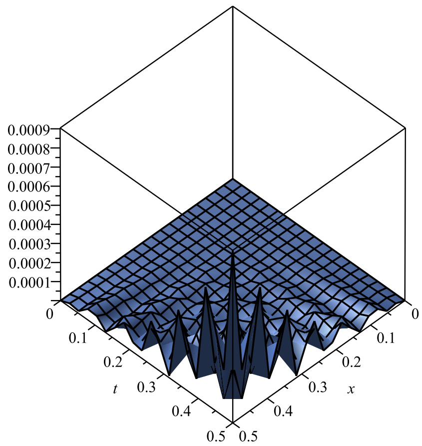

Fig. 2. Comparison of absolute errors for Example 2 with grid=[20,20]

Example 5.3. *Finally let us, consider the following nonlinear two-dimensional Volterra integral equation (see [10],[4],[3])*

$$u(x,t)=x^2+t^2-\frac{1}{45}xt(9x^4+10x^2t^2+9t^4)+\int_0^t\int_0^x u^2(y,z)dydz, \tag{39}$$

with exact solution $u(x,t)=x^2+t^2$, and $0\le x,t<1$. For Example 3, we have $\alpha=\beta=0$ and $W_1(t,s,x,y,u)=u^2$, where u is the unknown function. Since in this example $u(x,t)=x^2+t^2$, and $(x,t)\in[0,1)\times[0,1)$, we have $0\le u<2$ and the Lipschitz constant for W_1 is

$$q=M_1=max|2u|<4 \quad ,that \;\; 0\le u<2$$

which does not satisfy the sufficient condition prescribed in Lemma 4.1. But, since in this example $u(x,t)=x^2+t^2$, and $0\le x,t<\frac{1}{2}$, the sufficient condition prescribed in Lemma 4.1, is satisfied. Numerical results for Example 3 are displayed in Table 3.

Table 3. Numerical results for Example 3.

$(x,t)=(\frac{1}{2^i},\frac{1}{2^i})$	2D-BPFs method([10]) $m=32$	Haar wavelet method ([3]) $m=32$	Presented method $m=32$
i=2	1.0×10^{-1}	1.6×10^{-2}	5.90×10^{-5}
i=3	7.0×10^{-2}	8.5×10^{-3}	9.06×10^{-7}
i=4	5.8×10^{-2}	4.6×10^{-3}	1.29×10^{-8}
i=5	5.3×10^{-2}	2.6×10^{-3}	1.43×10^{-10}
i=6	2.1×10^{-4}	1.6×10^{-4}	2.00×10^{-12}

6. Conclusions

We have introduced a new method for solving two-dimensional nonlinear Volterra integral equations, based on the expansion of the solution as a series of Haar functions. As an advantage, this method does not require the solution of algebraic systems and in this case, we use a small number of basis functions to obtain high accuracy.

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Numerical treatment of two-dimensional nonlinear mixed Volterra-Fredholm integral equations using wavelets

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Abstract

In this study, we suggest a numerically practical algorithm to approximate the solution of two-dimensional nonlinear mixed Volterra-Fredholm integral equations. This method is based on two-dimensional Legendre wavelets to reduce these nonlinear integral equations to a system of nonlinear algebraic equations. The main characteristic of this approach is high accuracy and computational efficiency of performing which are the consequences of Legendre wavelets properties. The convergence analysis and error bound of the proposed Legendre wavelets method is investigated. Numerical examples confirm that the Legendre wavelets collocation method is accurate and reliable for solving nonlinear two-dimensional integral equations.

1. Introduction

In recent years, several authors have written a number of papers which establish numerical techniques for finding an approximation of the two-dimensional nonlinear integral equations. These methods can be categorized into radial basis functions methods (see, e.g. [1, 2]), triangular functions method (see, e.g. [6]), Legendre polynomials method (see, e.g. [8]), hybrid functions method (see, e.g. [7]) and the rationalized Haar functions method (see, e.g. [3, 4]). Consider the following two-dimensional nonlinear mixed Volterra-Fredholm integral equation of the second kind

$$f(x,y) = g(x,y) + \int_0^y \int_0^1 k\big(x,y,s,t,f(s,t)\big)dsdt \quad (x,y) \in [0,1] \times [0,1], \tag{1}$$

where the functions g and k are known continuous functions defined on $[0,1]\times[0,1]$ and $\Omega = [0,1]^4 \times \mathbb{R}$, respectively, and $f(x,y)$ is an unknown real-valued function to be determined. Volterra-Fredholm integral equations arise in a variety of applications in many fields including modeling of the spatio-temporal development of an epidemic, theory of parabolic initial boundary value problems, population dynamics, and Fourier problems [5, 9, 10].

The objective of this study is to propose an effective approach based on a new set of two-dimensional orthogonal functions which are extensions of one-dimensional Legendre wavelets. The present method consists of reducing the solution of Eq. (1) to a nonlinear system of algebraic equations by using the numerical integration and collocation method. The analysis of the accuracy estimation of the method is given and the performance of the proposed method is illustrated by means of a numerical example.

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2. Two-dimensional Legendre wavelets

2.1. Definition

Consider the well-known Legendre polynomials, which are orthogonal with respect to the weight function $w(x) = 1$ and derived from the following recursive formula

$$L_0(x) = 1,$$
$$L_1(x) = x,$$
$$L_m(x) = \tfrac{2m-1}{m} x L_{m-1}(x) - \tfrac{m-1}{m} L_{m-2}(x), \quad m = 2, 3, \ldots.$$

Legendre wavelets $\psi_{n,m}(x) = \psi(k, \hat{n}, m, x)$ have four arguments, $k = 2, 3, \ldots, \hat{n} = 2n-1, n = 1, 2, 3, \ldots, 2^{k-1}, m = 0, 1, 2, \ldots, M-1$ is the order of Legendre polynomials and M is a fixed positive integer. They are defined on the interval $[0, 1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{(m+\frac{1}{2})}\, 2^{\frac{k}{2}} L_m(2^k x - \hat{n}), & \frac{\hat{n}-1}{2^k} \leqslant x < \frac{\hat{n}+1}{2^k} \\ 0, & elsewhere. \end{cases} \tag{2}$$

Two-dimensional Legendre wavelet, $\psi_{n_1 m_1 n_2 m_2}(x, y)$, are defined on $[0, 1) \times [0, 1)$ as:

$$\begin{cases} \sqrt{(m_1+\frac{1}{2})(m_2+\frac{1}{2})}\, 2^{\frac{k_1+k_2}{2}} L_{m_1}(2^{k_1} x - \hat{n}_1) L_{m_2}(2^{k_2} y - \hat{n}_2), & \frac{\hat{n}_1-1}{2^{k_1}} \leqslant x < \frac{\hat{n}_1+1}{2^{k_1}} \\ & \frac{\hat{n}_2-1}{2^{k_2}} \leqslant y < \frac{\hat{n}_2+1}{2^{k_2}} \\ 0, & elsewhere. \end{cases} \tag{3}$$

where $\hat{n}_1 = 2n_1 - 1, \hat{n}_2 = 2n_2 - 1, n_1 = 1, 2, 3, \ldots, 2^{k_1-1}, n_2 = 1, 2, 3, \ldots, 2^{k_2-1}, m_1 = 0, 1, \ldots, M_1 - 1, m_2 = 0, 1, \ldots, M_2 - 1$, and k_1, k_2, M_1, M_2 are arbitrary positive integers. Here L_{m_1} and L_{m_2} are Legendre polynomials of order m_1 and m_2, respectively.

2.2. Function approximation

A function $f(x, y)$ defined over $[0, 1) \times [0, 1)$ may be expanded as

$$f(x, y) = \sum_{i_1=1}^{\infty} \sum_{j_1=0}^{\infty} \sum_{i_2=1}^{\infty} \sum_{j_2=0}^{\infty} c_{i_1 j_1 i_2 j_2} \psi_{i_1 j_1 i_2 j_2}(x, y). \tag{4}$$

If the infinite series in Eq. (4) is truncated, then Eq. (4) can be written as

$$f(x, y) \simeq \sum_{i_1=1}^{2^{k_1-1}} \sum_{j_1=0}^{M_1-1} \sum_{i_2=1}^{2^{k_2-1}} \sum_{j_2=0}^{M_2-1} c_{i_1 j_1 i_2 j_2} \psi_{i_1 j_1 i_2 j_2}(x, y) = \mathbf{C}^T \square(x, y) = \Psi^T(x) \mathbf{F} \Psi(y), \tag{5}$$

where $\Psi(x)$ and $\Psi(y)$ are $2^{k_1-1}M_1 \times 1$ and $2^{k_2-1}M_2 \times 1$ matrices respectively given by

$$\Psi(x) = \Big[\psi_{10}(x), \ldots, \psi_{1M_1}(x), \psi_{20}(x), \ldots, \psi_{2M_1}(x), \ldots, \psi_{2^{k_1-1}0}(x), \ldots, \psi_{2^{k_1-1}M_1}(x)\Big]^T, \tag{6}$$

$$\Psi(y) = \Big[\psi_{10}(y), \ldots, \psi_{1M_2}(y), \psi_{20}(y), \ldots, \psi_{2M_2}(y), \ldots, \psi_{2^{k_2-1}0}(y), \ldots, \psi_{2^{k_2-1}M_2}(y)\Big]^T. \tag{7}$$

F is a $2^{k_1-1}M_1 \times 2^{k_2-1}M_2$ matrix whose elements can be calculated from

$$\int_0^1 \int_0^1 \psi_{i_1 j_1}(x) \psi_{i_2 j_2}(y) f(x, y)\, dy dx,$$

with $i_1 = 1, \ldots, 2^{k_1-1}$, $j_1 = 0, \ldots, M_1 - 1$, $i_2 = 1, \ldots, 2^{k_2-1}$, $j_2 = 0, \ldots, M_2 - 1$.

3. Solution of two-dimensional nonlinear integral equation

In this section, we consider the two-dimensional nonlinear Volterra-Fredholm integral equation (1) and perform the expansion of the unknown function $f(x, y)$ using two-dimensional Legendre wavelets expansion given in Eq. (5) such that $k_1 = k_2 = k$, $M_1 = M_2 = M$, and $N = 2^{k-1}$. Then we have

$$f(x,y) \simeq \bar{f}_{N,M}(x,y) = \sum_{i_1=1}^{N}\sum_{j_1=0}^{M-1}\sum_{i_2=1}^{N}\sum_{j_2=0}^{M-1} f_{i_1j_1i_2j_2}\psi_{i_1j_1i_2j_2}(x,y) = \mathbf{F}^T\square(x,y), \tag{8}$$

Substituting Eq. (8) into Eq. (1) gives

$$\bar{f}_{N,M}(x,y) = g(x,y) + \int_0^y\int_0^1 k\big(x,y,s,t,\bar{f}_{N,M}(s,t)\big)dsdt \tag{9}$$

Now, we discretize Eq. (9) at the set of collocation nodes (x_i, y_j) for $i, j = 1, 2, \ldots, NM$ as follows:

$$g(x_i,y_j) = \bar{f}_{N,M}(x_i,y_j) - \int_0^{y_j}\int_0^1 k\big(x_i,y_j,s,t,\bar{f}_{N,M}(s,t)\big)dsdt \tag{10}$$

where

$$x_i = \frac{1}{2}\Big(\cos(\frac{(2i-1)\pi}{2NM}) + 1\Big), \quad i = 1,2,\ldots,NM \tag{11}$$

$$y_j = \frac{1}{2}\Big(\cos(\frac{(2j-1)\pi}{2NM}) + 1\Big), \quad j = 1,2,\ldots,NM \tag{12}$$

are zeros of the shifted Chebyshev polynomials $T_{NM}(2x-1)$ and $T_{NM}(2y-1)$, respectively. Now the Gauss-Legendre quadrature formula is employed to approximate the integral operator in Eq. (10). For this purpose, linear transformations must be applied with the following forms

$$\tau = 2s-1, \quad s \in [0,1], \quad \eta = \frac{2}{y_j}t - 1, \quad t \in [0, y_j]. \tag{13}$$

Then

$$g(x_i,y_j) = \bar{f}_{N,M}(x_i,y_j) - \frac{y_j}{4}\int_{-1}^1\int_{-1}^1 k\Big(x_i,y_j,(\frac{1}{2}(\tau+1),\frac{y_j}{2}(\eta+1),\bar{f}_{N,M}(\frac{1}{2}(\tau+1),\frac{y_j}{2}(\eta+1))\Big)d\tau d\eta, \tag{14}$$

By applying the Legendre–Gauss–Lobatto integration formula, we obtain

$$g(x_i,y_j) = \bar{f}_{N,M}(x_i,y_j) - \frac{y_j}{4}\sum_{n=1}^{r_1}\sum_{m=1}^{r_2} w_n w_m\Big(k(x_i,y_j,(\frac{1}{2}(\tau_m+1),\frac{y_j}{2}(\eta_n+1),\bar{f}_{N,M}(\frac{1}{2}(\tau_m+1),\frac{y_j}{2}(\eta_n+1))\Big),$$
$$i = 1,2,\ldots,NM, \quad j = 1,2,\ldots,NM. \tag{15}$$

where τ_m and η_n are the Legendre–Gauss points, zeros of Legendre polynomials of degrees r_1 and r_2 in $[-1,1]$, respectively, and w_ms and w_ns are the corresponding weights. Eq. (15) generates a nonlinear system of $(NM)^2$ algebraic equations that can be solved using Newton's iteration method.

4. Convergence analysis

In this section we obtain an error estimate for the best approximation of the function f based on Legendre wavelets and describe the convergence behavior of the proposed numerical method. For this purpose, we present the following results. The first theorem provides an error term for the best approximation of $f(x, y)$.

Table 1. Absolute errors for Example 5.1.

$(x,y)=(\frac{1}{2^l},\frac{1}{2^l})$	$k=2,M=4$	$k=2,M=6$	$k=2,M=8$
$l=1$	2.54×10^{-4}	2.34×10^{-6}	3.21×10^{-7}
$l=2$	3.13×10^{-5}	8.47×10^{-7}	4.49×10^{-9}
$l=3$	1.12×10^{-5}	9.17×10^{-8}	5.38×10^{-10}
$l=4$	2.49×10^{-6}	3.44×10^{-8}	5.92×10^{-11}
$l=5$	1.57×10^{-6}	5.71×10^{-9}	7.63×10^{-11}
$l=6$	3.11×10^{-7}	5.92×10^{-9}	2.81×10^{-11}
$\|e_{N,M}(x,y)\|_\infty$	3.54×10^{-4}	6.74×10^{-6}	5.71×10^{-7}

Theorem 4.1. *Suppose that $f\in C^M([0,1]\times[0,1])$ is a real-valued function. Then $f_{NM}(x,y)=\mathbf{C}^T\Box(x,y)$ approximates $f(x,y)$ with the following error bound*

$$\left\|f(x,y)-\mathbf{C}^T\Box(x,y)\right\|_2\leqslant\frac{\gamma2^M}{N^MM!},\qquad \gamma=\max_{\substack{x,y\in[0,1]\\0\leqslant k\leqslant M}}\left|\frac{\partial^Mf(x,y)}{\partial x^k\partial y^{M-k}}\right|. \tag{16}$$

Theorem 4.2. *Consider the Eq. (1) and suppose that $k\in C^1(\Omega)$, $\Omega=[0,1]^4\times\mathbb{R}$, with $C_0=\sup_\Omega|k_z(x,y,s,t,z)|<\infty$. Moreover, let $f\in C^M([0,1]\times[0,1])$ $(M>2)$ be the exact solution of the Eq. (1). Then, we have,*

$$\|f(x,y)-\bar{f}_{NM}(x,y)\|_2\leqslant\frac{2^M}{N^NM!}\Big(\gamma+C_1\|A^{-1}\|_2(NM)^2(\ln(M)+1)\Big), \tag{17}$$

where $\bar{f}_{NM}(x,y)$ is the approximation of f and

$$A=\Big[\psi_{i_1j_1i_2j_2}(x_i,y_j)-\int_0^{y_j}\int_0^1k_z(x_i,y_j,s,t,\zeta)\psi_{i_1j_1i_2j_2}(s,t)dsdt\Big],\quad 1\leqslant i_1,i_2\leqslant N,\ 0\leqslant j_1,j_2\leqslant M-1,$$

that assumed to be nonsingular.

5. Numerical example

In this section, we apply the presented method for a numerical example (selected from [3]) to demonstrate the accuracy and effectiveness of the method.

Example 5.1. *Consider the following two-dimensional nonlinear Volterra-Fredholm integral equation of second kind:*

$$f(x,y)=g(x,y)+\int_0^y\int_0^1\frac{x(1-s^2)}{(1+y)(1+t^2)}(1-\exp(-f(s,t)))dsdt,\qquad(x,y)\in[0,1)\times[0,1),$$

where

$$g(x,y)=-\log(1+\frac{xy}{1+y^2})+\frac{xy^2}{8(1+y)(1+y^2)},$$

with the exact solution $f(x,y)=-\log(1+\frac{xy}{1+y^2})$. For different values of k,M, the absolute values of errors are presented in Table 1. Also, a comparison of maximum absolute errors is given in Table 2, which confirms that the proposed method in this paper is effective than the method given in [3].

Table 2. Comparison of maximum absolute errors for Example 5.1.

Presented method		Haar wavelets method [3]	
$k=2, M=2$	3.27×10^{-2}	$2M=4$	1.6×10^{-3}
$k=2, M=4$	3.54×10^{-4}	$2M=8$	4.4×10^{-4}
$k=2, M=8$	5.71×10^{-7}	$2M=16$	1.2×10^{-4}

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The second kind shifted Chebyshev reproducing kernel method for solving two-point fractional boundary value problems

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Abstract

In this paper, we construct some reproducing kernels based on the shifted Chebyshev polynomials of the second kind and introduce an efficient numerical method for solving the nonlinear two-point fractional boundary value problems based on the constructed kernels. In fact, we find the best approximation for the solution of the problem in a finite-dimensional space. The fractional derivatives are described in Caputo sense. An illustrative example is provided to confirm the reliability and effectiveness of the proposed method.

1. Introduction

In this paper, we consider the following nonlinear fractional two–point boundary value problems:

$$\begin{cases} D_t^{\alpha} y(t) = f(t, y(t)), & t \in (0,1) \\ y(0) = 0, y(1) = 0, \end{cases} \tag{1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, D_t^{α} is the left-sided Caputo fractional derivative of order $\alpha \in (1,2)$, denoted by

$$D^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds, \qquad t > 0, \tag{2}$$

where $n - 1 < \alpha < n, n \in \mathbb{N}^+$. Fractional differential equations have many applications in physics, chemistry, engineering, finance, and other sciences that have been developed in the last few decades [2]. The investigation of the existence and uniqueness of nonlinear fractional boundary value problems can be found in[2, 5]. Since the fractional differential equations have no closed-form solution in general, we require to design efficient numerical methods solving them. Many numerical methods have been developed to solve fractional differential equations, for example see [5] and the references therein. Here, we propose an efficient method based on the reproducing kernels constructed by

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second kind shifted Chebyshev polynomials. The numerical methods based on the reproducing kernels method have successfully been applied to several problems, for example see [1] and the references therein. The convergence of the proposed iterated technique is proved and an example is presented to illustrate the validity and efficiency of the presented method. The proposed iterative method is fast and easy to implement.

1.1. Second kind shifted Chebyshev reproducing kernel

Definition 1.1. The Chebyshev polynomials $U_n(t)$ of the second kind are defined as

$$U_n(t) = \frac{sin(n+1)\theta}{sin\theta},$$

where $t = cos\theta$ and $0 \leq \theta \leq \pi$ [4]. They are orthogonal polynomials on $[-1, 1]$ with respect to the following inner product

$$(U_n(t), U_m(t)) = \int_{-1}^{1} \sqrt{1-t^2}U_n(t)U_m(t)dt = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{cases} \tag{3}$$

$U_n(t)$ can be generated using the following recurrence relation

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \qquad n = 2, 3, ... \tag{4}$$

with $U_0(t) = 1, U_1(t) = 2t$.

In order to use these polynomials on the interval $[0, T]$ the shifted chebyshev polynomials of the second kind are defined as $U_n^*(t) = U_n(2t - 1)$. The shifted version are orthogonal on $[0, 1]$ as follows

$$(U_n^*(t), U_m^*(t))_\omega = \int_{0}^{1} \sqrt{t-t^2}U_n^*(t)U_m^*(t)dt = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{8}, & n = m, \end{cases} \tag{5}$$

where $\omega(t) = \sqrt{t - t^2}$ is the weight function. $U_n^*(t)$ can be generated using the following recurrence relation

$$U *_n (t) = 2(2t - 1)U_{n-1}^*(t) - U_{n-2}^*(t), \qquad n = 2, 3, ... \tag{6}$$

with $U_0(t) = 1, U_1(t) = 4t - 2$.

Definition 1.2. Let $P_n[0, 1] := span\{1, t, t^2, ..., t^n\}$, equipped with the following inner product

$$(u, v)_\omega = \int_{0}^{1} \sqrt{t-t^2} \times u(t)v(t)dt, \qquad \forall u, v \in P_n[0, 1],$$

and the norm $\|u\|_{P_n} = \sqrt{(u, u)_\omega}$.

Theorem 1.3. *$P_n[0, 1]$ is a finite dimensional reproducing kernel space and its reproducing kernel is $K_n(t, s) = \frac{8}{\pi} \sum_{i=0}^{n} U_i^*(t)U_i^*(s)$, which has the following property*
$I.K_n(t, .) \in P_n[0, 1], \qquad \forall t \in [0, 1],$
$II.(f(.), K_n(t, .))_\omega = f(t), \qquad \forall f \in P_n[0, 1], \forall t \in [0, 1]$ *(The reproducing property).*

2. The numerical method

In this section, an iterative method based on the second kind shifted Chebyshev reproducing kernel is presented. We can convert (1) into an equivalent Volterra-Fredholm integral equations, stated in the following theorem [2, 5].

Theorem 2.1. *Assume that $1 < \alpha < 2$, y is a function with absolutely continuous first derivative, and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then we have that $y \in C^1[0, T]$ is a solution of the boundary value problem (1) if and only if it is a solution of the following integral equation,*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s, y(s))ds - \frac{t}{T\Gamma(\alpha)} \int_{0}^{T} (T-s)^{\alpha-1} f(s, y(s))ds. \tag{7}$$

To overcome the nonlinearity of the problem (7), we solve the following linearized problem in each iteration

$$y^{m+1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y^m(s))ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y^m(s))ds. \tag{8}$$

If f has the Lipschitz condition, we can prove the convergence of the linearized scheme [2, 5]. Now, we solve the equivalent equations (8) instead of the fractional boundary value problem (1). For $n \geq 2$, let $\{t_i\}_{i=0}^n$ be $n+1$ distinct nodes in $[0,1]$ and let $\psi_i(t) = K_n(t_i, t)$. It can be easily shown that $\{\psi_i\}_{i=0}^n$ is a basis for $P_n[0,1]$ [3]. The orthonormal system $\{\bar{\psi}_i\}_{i=0}^n$ of $P_n[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i\}_{i=0}^n$, as

$$\bar{\psi}_i(t) = \sum_{k=0}^{i} \beta_{ki} \psi_k(t), \qquad i = 0, 1, ..., n.$$

The numerical approximation of the solution of (8) in the $P_n[0,1]$ will be obtained by the following iterative scheme:

$$\begin{cases} y_0(t) = 0, \\ B_i^m = \sum_{k=0}^i \beta_{ki} F_k^m, \\ y_{n+1}^{m+1}(t) = \sum_{i=0}^n B_i \bar{\psi}_i(t), \end{cases} \tag{9}$$

where

$$F_k^m = \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} f(s, y^m(s))ds - \frac{t_k}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y^m(s))ds.$$

Theorem 2.2. *If $\{t_k\}_{k=0}^n$ be $n+1$ distinct nodes in $[0,1]$, then the approximate solution $y_{n+1}^{m+1}(t)$ derived from (9) is the best approximation of the exact solution of (8) in $P_n[0,1]$.*

Proof. $P_n[0,1]$ is a finite dimensional subspace of a reproducing kernel space $\mathcal{H}$. Let $\hat{y}$ be the orthogonal projection of y onto $P_n[0,1]$. From the best approximation theorem we know that $\hat{y}$ is the best approximation of y in $P_n[0,1]$ in the sense that

$$\|y - \hat{y}\|_\omega < \|y - v\|_\omega,$$

for all v in $P_n[0,1]$ distinct from $\hat{y}$. Now we can see easily that $y_{n+1}^{m+1}(t)$ derived from (9) is the orthogonal projection of y^{m+1} onto $P_n[0,1]$.

$$\begin{aligned} \hat{y}(t) &= \sum_{i=0}^n (y^{m+1}(s), \bar{\psi}_i(s))_\omega \bar{\psi}_i(t) \\ &= \sum_{i=0}^n (y^{m+1}(s), \sum_{k=0}^i \beta_{ki} \psi_k(s))_\omega \bar{\psi}_i(t) \\ &= \sum_{i=0}^n \sum_{k=0}^i \beta_{ki} (y^{m+1}(s), \psi_k(s))_\omega \bar{\psi}_i(t) \\ &= \sum_{i=0}^n \sum_{k=0}^i \beta_{ki} (y^{m+1}(s), K(t_k, s))_\omega \bar{\psi}_i(t) \\ &= \sum_{i=0}^n \sum_{k=0}^i \beta_{ki} y^{m+1}(t_k) \bar{\psi}_i(t), \end{aligned}$$

and from (8) we have

$$y^{m+1}(t_k) = \frac{1}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} f(s, y^m(s))ds - \frac{t_k}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, y^m(s))ds.$$

□

3. Numerical results

In this section, we present some numerical results to illustrate the efficiency of the proposed second kind shifted Chebyshev reproducing kernel method. The approximated solutions are compared with the exact solution.

Example 3.1. *Consider the following nonlinear fractional boundary value problem:*

$$\begin{cases} D_t^\alpha y(t) = sin(t) y^2(t) + g(t), \qquad t \in (0,1) \\ y(0) = y(1) = 0. \end{cases} \tag{10}$$

The exact solution of this problem is given by $y(t) = t - t^2$. Using the presented method, the numerical results for various α, number of data points n and number of iteration m are given in figures 1 and 2. The reported results show high accuracy even when we have used the proposed method with a relatively small number of data points and iterations.

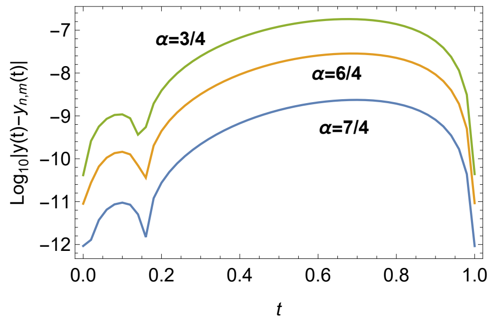

Fig. 1. The absolute error in logarithmic scale with the various α, $n = 10$ and $m = 10$ iterations.

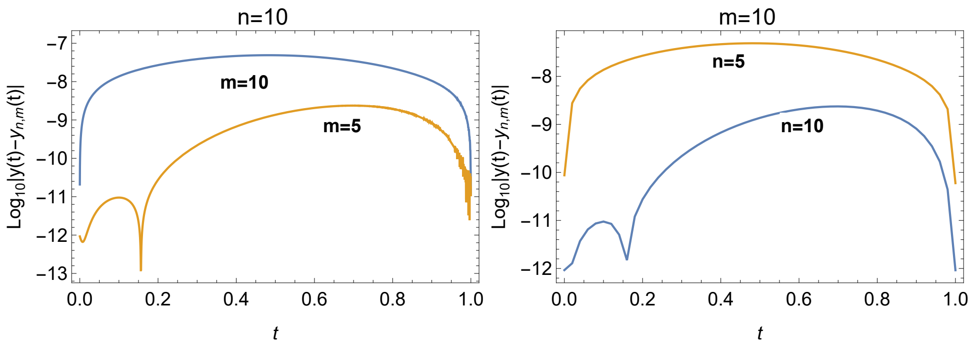

Fig. 2. The absolute error in logarithmic scale for $\alpha = \frac{7}{4}$ and various n and m.

4. Conclusions

In this paper, an efficient numerical method for solving a two-point fractional boundary value problem is presented. The numerical approach is based on the reproducing kernel space method. The second kind shifted Chebyshev polynomials are used to construct the reproducing kernel. In fact, we will find the best approximation for the solution of the problem in a finite-dimensional space. To show how good and accurate the presented method is, the results of numerical experiments are compared with the analytical solutions. The reported results and comparison confirm the accuracy and applicability of the proposed method.

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A quasi-wavelet spectral method for solving Fredholm-Hammerstein integral equations in complex plane

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Abstract

This paper presents a quasi-wavelets spectral collocation method to obtain the approximate solution of nonlinear Fredholm integral equations in complex plane. The periodic quasi-wavelets and their main properties are used to reduce the solution of a nonlinear integral equation to a complex system of algebraic equations. The rate of approximation solution converging to the exact solution is given and numerical examples confirm the accuracy and ease of implementation of the method.

1. Introduction

We consider the following nonlinear integral equation

$$y(t) = f(t) + \int_0^{2\pi} k(t,s)g(s,y(s))ds, \quad t \in [0, 2\pi] \tag{1}$$

where $y(t)$ is an unknown complex-valued function to be determined and f, k and g are given continuous complex-valued periodic functions. The real type of Eq. (1) was introduced for the first time by A. Hammerstein in [6]. Equations of this type appear in many applications. For example, They arise as a reformulation of the nonlinear two-point boundary value problems in magnetohydrodynamics [1].

Several authors have written a number of papers which establish numerical techniques for finding an approximation of the real type Hammerstein integral equations. Some of the schemes which have considered the solution of the linear and nonlinear cases of Eq. (1), numerically, are projection methods [7, 9, 10], continuous and discrete time spline collocation method [8], the Adomian decomposition method [12], wavelets method [11, 15, 16] and the Sinc collocation method [13]. Complex type of integral equations are much more difficult to solve than real integral equations. Therefore, only a few authors have tried to overcome these difficulties [2-5, 14].

In this paper, we will consider the use of periodic quasi-wavelets in a collocation method for numerical solution of Eq. (1). The analysis of the accuracy estimation of the method is given and the performance of the proposed method is illustrated by means of a numerical example.

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2. Review of periodic quasi-wavelets

In this section, we shall state an overview of the basic formulation of the periodic quasi-wavelets based on B-spline functions [3, 4, 14]. We first introduce the periodic spline functions.

Definition 2.1. The periodic B-spline function $B_p^{n,m}(x)$ with period T is defined by

$$B_p^{n,m}(x) = (K_m)^n \sum_{l\in\mathbb{Z}} \Big(\frac{\sin(l\pi/K_m)}{l\pi}\Big)^{n+1} \exp\Big(\frac{i2\pi lx}{T}\Big), \quad x \in \mathbb{R} \tag{2}$$

whose degree is n, and corresponding step length is $h_m = \frac{T}{K_m}, (m \in \mathbb{Z},\ m \geq 0)$, where $T = hq$, $K_m = 2^m q$ in which h is a positive real number and $n, q \in \mathbb{N}$ such that $q \geq n + 1$.

From the basic properties of spline functions the following holds:

(a) $V_0 \subset V_1 \subset ... \subset V_m \subset V_{m+1} \subset ...$;

(b) $\{V_m\}$ is dense in $L_p^2[0,T]$, that is $\overline{\bigcup_{m\in\mathbb{Z},m\geq 0} V_m} = L_p^2[0,T]$;

(c) $\Big\{B_p^{n,m}(x - jh_m), j = 0, ..., K_m - 1\Big\}$ forms a Riesz basis of V_m,

where

$$V_m = Span\Big\{B_p^{n,m}(x - jh_m), j = 0, ..., K_m - 1\Big\}. \tag{3}$$

$B_p^{n,m}(x - jh_m), j = 0, ..., K_m - 1$ is a basis for V_m, but it is not orthogonal. We construct an orthogonal basis for V_m. For this purpose, we define

$$A_v^{n,m}(x) = C_v^{n,m} \sum_{l=0}^{K_m-1} \exp\Big(\frac{i2\pi lv}{K_m}\Big) B_p^{n,m}(x - lh_m), \quad x \in \mathbb{R}, \tag{4}$$

where

$$C_v^{n,m} = \Big\{t_0 + 2\sum_{\lambda=1}^{q} t_\lambda \cos(\lambda v h_m)\Big\}^{-1/2}, \tag{5}$$

and

$$t_\lambda = B_p^{2q+1,m}(\lambda h_m). \tag{6}$$

The Fourier expansion of $A_v^{n,m}(x)$ is given by

$$A_v^{n,m}(x) = C_v^{n,m}(K_m)^{n+1} \sum_{\lambda\in\mathbb{Z}} \Big(\frac{\sin(v\pi/(K_m)}{(v + \lambda K_m)\pi}\Big)^{n+1} \exp\Big(\frac{i2\pi(v + \lambda K_m)x}{T}\Big) \tag{7}$$

From Eq. (7), we have the following lemma:

Lemma 2.2. *The set of functions $\{A_v^{n,m}(x)\}_{v=0}^{K_m-1}$ is an orthonormal basis for V_m .i.e.,*

$$\langle A_{v_1}^{n,m}, A_{v_2}^{n,m}\rangle = \delta_{v_1,v_2} \tag{8}$$

where $v_1, v_2 = 0, ..., K_m - 1$, and δ_{v_1,v_2} denotes the Kronecker delta.

Proposition 2.3. *Setting $P_m f(t) = \sum_{l=0}^{K_m-1}\langle f, A_l^{n,m}\rangle A_l^{n,m}(t)$, one can have the followings for all $f \in L_p^2[0,T]$*

(1) $\|P_m\| \leq 1$,

(2) $\|f - P_m f\|_2 \leq C\omega_{n+1}(f, h_m)$, *where C is a constant and*

$$\omega_{n+1}(f, h_m) = \sup_{0<h\leq h_m} \|\triangle_h^{n+1} f\|, \ \ \triangle_h^r f(t) = r!h^r[t, t + h, ...t + rh]f.$$

Here $L_p^2[0,T]$ denotes the space of all square integrable complex-valued functions on $[0,T]$ with period T by the following inner product:

$$\langle f, g\rangle = \frac{1}{T}\int_0^T f(x)\overline{g(x)}dx. \tag{9}$$

3. Methodolgy

In this section, we apply the periodic quasi-wavelets constructed on $[0, 2\pi]$ to approximate the integral kernel and then obtain the numerical solutions by means of the degenerate kernel scheme and the collocation method. For this purpose, first assume

$$z(t) := g(t, y(t)), \quad t \in [0, 2\pi] \tag{10}$$

By substituting (10) into (1) we get

$$y(t) = f(t) + \int_0^{2\pi} k(t,s)z(s)ds, \quad t \in [0, 2\pi] \tag{11}$$

which concludes that the new unknown $z(t)$ satisfies the nonlinear integral equation

$$z(t) = g\Big(t, f(t) + \int_0^{2\pi} k(t,s)z(s)ds\Big), \quad t \in [0, 2\pi]. \tag{12}$$

Suppose that $\{A_j^{n,m}\}$ are the periodic quasi-wavelets described in Section 2. Then the kernel $k(t, s)$ is approximated by a degenerate kernel

$$k_m(t,s) = \sum_{i,j=0}^{K_m-1} \alpha_{ij}^m A_i^{n,m}(t) A_j^{n,m}(s) \tag{13}$$

where the coefficients α_{ij}^m are given by

$$\alpha_{ij}^m = \langle A_i^{n,m}(t), \langle k(t,s), A_j^{n,m}(s)\rangle\rangle. \tag{14}$$

Also, $z(t)$ is approximated by a linear combination of periodic quasi-wavelets:

$$z_m(t) = \sum_{l=0}^{K_m-1} a_l^m A_l^{n,m}(t) \qquad t \in [0, 2\pi]. \tag{15}$$

Substituting (13) and (15) into (12) and using the orthonormality of $\{A_l^{n,m}; 0 \le l \le K_m - 1\}$ we find that

$$\sum_{l=0}^{K_m-1} a_l^m A_l^{n,m}(t) = g\Big(t, f(t) + \sum_{i,j=0}^{K_m-1} a_j^m \alpha_{ij}^m A_i^{n,m}(t)\Big), \tag{16}$$

where the coefficients $a_l^m; 0 \le l \le K_m - 1$ are determined by collocating Eq. (16) at the collocation points τ_i^m :

$$\sum_{l=0}^{K_m-1} a_l^m A_l^{n,m}(\tau_i^m) = g\Big(\tau_i^m, f(\tau_i^m) + \sum_{i,j=0}^{K_m-1} a_j^m \alpha_{ij}^m A_i^{n,m}(\tau_i^m)\Big), \quad 0 \le i \le K_m - 1 \tag{17}$$

Eq. (17) generates a closed set of K_m algebraic nonlinear equations for a_l^m which can be solved by a suitable iterative method. By substituting the approximation z_m into the right-hand side of (11), the required approximated solution $y(t)$ for Eq. (1), can be obtained as

$$y_m(t) := f(t) + \int_0^{2\pi} k(t,s)z_m(s)ds. \tag{18}$$

4. Convergence analysis

In this section, we show that the approximation z_m converges under suitable conditions to an exact solution of (11). We assume that the solution $y^*(t)$ to be determined is geometrically isolated, in other words, there is some ball $B(y^*, \delta) = \{y \in L_p^2[0, T] : \|y - y^*\| \le \delta\}$, with $\delta > 0$, that contains no solution of Eq. (1) other than y^*.

Table 1. Absolute values of error for Example 5.1.

t_j	$m = 1$	$m = 2$
1	7.92872×10^{-03}	2.70904×10^{-14}
2	8.44997×10^{-03}	3.88034×10^{-14}
3	1.03184×10^{-02}	3.41905×10^{-14}
4	8.29130×10^{-03}	2.49492×10^{-14}
5	8.06694×10^{-03}	3.73776×10^{-14}
6	1.02886×10^{-02}	3.72215×10^{-14}

Theorem 4.1. *Under the assumptions given in [15, 16], there exists a neighborhood of z^* in which for sufficiently large m equation (16) has a unique solution z_m and the following estimate holds:*

$$\|z^* - z_m\| \leq \alpha \|z^* - P_m z^*\| + \beta \|(\mathcal{K} - \mathcal{K}_m) z^*\|, \tag{19}$$

where α, β are independent of m.

Corollary 4.2. *If $z^* \in H^s[0, 2\pi]$, then with L_2-norm, we have*

$$\|z^* - z_m\| = O\big(2^{-ms}\big), \tag{20}$$

where $s < r$ and r is the Holder index of the functions $A^{n,m}$ defined in Section 2.

Proposition 4.3.

$$\|y^* - y_m\| \leq \|\mathcal{K}\|.\|z^* - z_m\|. \tag{21}$$

5. Numerical example

In this section, a numerical example was solved by our method with the collocation points chosen to be

$$\tau_i^m = \frac{2\pi}{K_m} i, \quad i = 0, \ldots, K_m - 1,$$

and the basis functions

$$\{A_i^{n,m}\}, \quad i = 0, \ldots, K_m - 1,$$

are taken as periodic quasi-wavelets, in which the natural number n is the degree of quasi-wavelets. Also, we applied the norm

$$\|y^* - y_m\| = \max_j |y^*(t_j) - y_m(t_j)|, \quad t_j = \frac{\pi(j-1)}{500}, \quad j = 1, 2, \ldots, 1000.$$

Example 5.1. *consider the following nonlinear complex integral equation*

$$y(t) = f(t) + \int_0^{2\pi} \sin(t - s) y^2(s) ds \tag{22}$$

where $f(t)$ is selected so that $y(t) = e^{\cos(t)} + i \sin(t)$ is the exact solution. Equation (22) was solved by our method with the periodic quasi-wavelet of order $n = 2$. We make a simulation and display of the numerical results in Table 1 and Figs. 1. Better approximation is expected by choosing $m = 2$, which we get

$$\|z^* - z_m\| = 1.65012 \times 10^{-7}, \quad \|y^* - y_m\| = 4.33201 \times 10^{-14}.$$

It is observed that y_m converges to y^* faster than z_m to z^*.

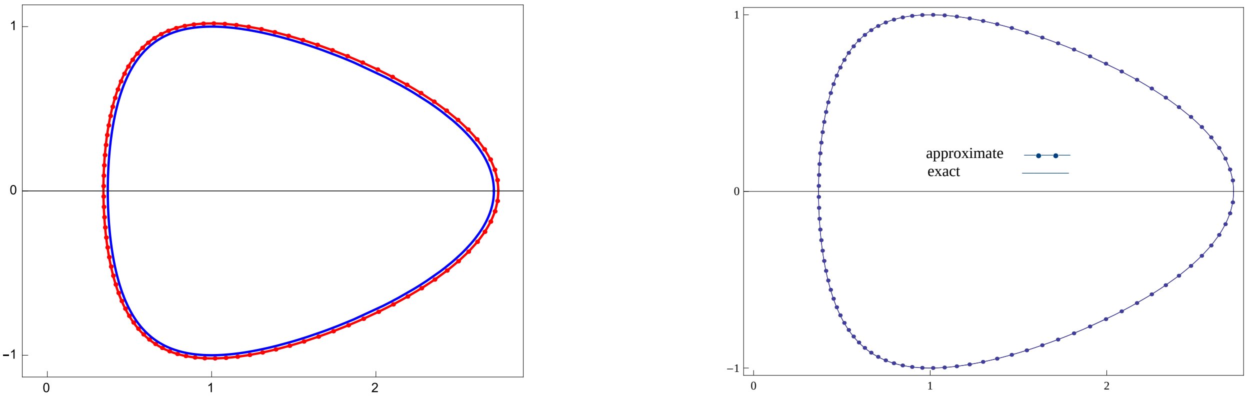

Fig. 1. Parametric plot of the approximate and exact solutions for $m = 1$ (left) and $m = 2$ (right) in Example 5.1.

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Optimal Control of spacecraft controlling path based on transformed Legendre polynomials and collocation method

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Abstract

In this article, an efficacious approach for finding the numerical solution of infinite horizon optimal control of the dynamic system which is modeled for control of a spacecraft's path with a quadratic performance index and various weight coefficients is presented. This paper is based on Legendre polynomials and their derivative operational matrix to convert the OCP to a mathematical programming problem that can be solved with various well-developed optimization algorithms. Numerical results demonstrate the efficiency of the propounded approach.

1. Introduction

Optimal control problems with infinite horizon come in some of the well-known areas of application, such as management science and economics [3], aerospace engineering [2], chemical reaction systems [4] and biology [5]. Such problems are defined on an infinite time domain, $[0,\infty)$. To study the methods that have been used so far to solve infinite horizon optimal control problems see [7, 8] and references therein.
Let $\beta(t)$ be the control of pitch angle. The differential equation that describes the motion of spacecraft is given by [9]

$$\mathcal{C}\frac{d^2}{dt^2}[\beta(t)] = \eta(t), \tag{1}$$

where $\mathcal{C}$ is the angular moments inertia and $\eta(t)$ is the torque caused by the gas jets. By considering $x_1(t) := \beta(t)$ and $x_2(t) := \beta'(t)$ as the state variables and $u(t) := \frac{\eta}{\mathcal{C}}$ as control function. The state equations are given by

$$x_1^{'}(t) = x_2(t), \tag{2}$$
$$x_2^{'}(t) = u(t).$$

Our goal is by using small acceleration to hold the angular position close to zero. The objective function is considered as follows:

$$J = \int_0^{\infty} [q_{11}x_1^2(t) + q_{22}x_2^2 + Ru^2(t)]dt, \tag{3}$$

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where q_{11}, q_{22} and R are positive weighting function.
This paper aims to present a method for an efficient solution of optimal control problems with infinite horizon modeled spacecraft path. For this purpose, at first, the infinite horizon optimal control problem is converted to a finite horizon problem by using a change of variable. After that, an appropriate finite or discrete representation of the solution of the problem which is based on Legendre functions is determined. Next, the infinite horizon optimal control problem is transcribed to a mathematical programming problem by using this discrete representation of the solution and collocating the differential equations on Chebyshev nodes. The obtained mathematical programming problem, then, can be solved by well-developed optimization algorithms.
This paper is organized as follows: In Section 2, we discuss some necessary preliminaries and basic results. Section 3, is devoted to the proposed method. In Section 4, the proposed method is applied to the given optimal control problem and numerical results are shown.

2. Preliminary Considerations

The Legendre polynomials $L_m(t)$ are defined on the interval $[-1, 1]$ as follows [6]:

$$L_0(t) = 1,\ \ L_1(t) = t,\ \ L_{m+1}(t) = \frac{2m+1}{m+1} t L_{m-1}(t) - \frac{m}{m+1} L_{m-1}(t),\ \ m = 1, 2, \cdots. \tag{4}$$

They are also complete on this interval. A function $f(t)$ defined on an interval $[-1, 1]$, may be approximated by Legendre polynomials and is denoted by $f_M(t)$ as

$$f(t) \approx f_M(t) = \sum_{i=0}^{M} f_i L_i(t) = F^T L(t), \tag{5}$$

where

$$F = [f_0, f_1, \cdots, f_M]^T \qquad L(t) = [L_0(t), L_1(t), \cdots, L_M(t)]^T. \tag{6}$$

The operational matrix of derivative in interval $[-1, 1]$ is defined as follows [6]:

$$\frac{dL(t)}{dt} = DL(t), \tag{7}$$

in which D is the $(M + 1)(M + 1)$ operational matrix of derivative given as

$$d_{ij} = \begin{cases} 2j+1, & j = i-k \begin{cases} k = 1, 3, \ldots, M & \text{M is odd,} \\ k = 1, 3, \ldots, M-1 & \text{M is even,} \end{cases} \\ 0, & otherwise. \end{cases} \tag{8}$$

3. Proposed Method

In the discretization of the controlled system, we utilize both the operational matrix of derivative and the Legendre approximation of control, state, and state rate functions. We use the Legendre-Gauss quadrature rule to approximate the performance index.
For discretization of dynamic system (3), we utilize logarithmic map $t = -Ln(\frac{1-\tau}{2}) = \phi(\tau)$ to transform semi-infinite horizon interval $[0, \infty]$ to $[-1, 1]$. The new function y_1, y_2 and u are defined on interval $[0, \infty]$ by

$$x_1(t) = x_1(\phi(\tau)) = y_1(\tau),\ \ x_2(t) = x_2(\phi(\tau)) = y_2(\tau),\ \ u(t) = u((\phi(\tau))) = v(\tau).\ \ t \in [0, \infty],\ \ \tau \in [-1, 1]. \tag{9}$$

From $x_2(t) = x_1^{'}(t)$ in (2), relations given in (9) and utilizing Chain rule in derivatives, we obtain $y_2(\tau) = (1 - \tau)y_1^{'}(\tau)$, so we have

$$y_2^{'}(\tau) = -y_1^{'}(\tau) + (1-\tau)y_1^{''}(\tau) \tag{10}$$

Table 1. CPU time

m	6	8	10	12
CPU TIME	0.21875	0.359375	0.60975	1.20375

From equation (2), it will be concluded that $u(t) = x_2^{'}(t) = x_1^{''}(t)$, so we obtain

$$v(\tau) = (1-\tau)y_2^{'}(\tau). \tag{11}$$

At last, from equations (11) and (10), the following equation will be resulted

$$v(\tau) = -(1-\tau)y_1^{'}(\tau) + (1-\tau)^2 y_1^{''}(\tau). \tag{12}$$

For discretization of (3), we suppose

$$y_1(\tau) = Y_1^T L(\tau), \quad y_1^{'}(\tau) = Y_1^T DL(\tau), \quad y_1^{''}(\tau) = Y_1^T D^2 L(\tau),$$

By substituting (12) and (3) in objective functional (3) and utilizing logical transformation $t = -Ln(\frac{1-\tau}{2})$, we gain

$$\int_0^\infty 4x_1^2(t) + 0.1u^2(t)dt = \int_{-1}^1 \left(4y_1^2(\tau) + 0.1\left(-(1-\tau)y_1^{'}(\tau) + (1-\tau)^2 y_1^{''}(\tau)\right)^2\right)\frac{1}{1-\tau}d\tau \tag{13}$$

By utilizing Gauss-Legendre (GL) quadrature formula and (3), we obtain

$$\sum_{j=0}^{M} \frac{\omega_j}{1-\tau_j}\left(4(Y_1^T L(\tau_j))^2 + 0.1\left(-(1-\tau_j)Y_1^T DL(\tau_j) + (1-\tau_j)^2 Y_1^T D^2 L(\tau_j)\right)^2\right) \tag{14}$$

where τ_js are GL nodes, zeros of Legendre polynomials L_{M+1} in the interval $[-1,1]$ and w_js are the corresponding weights. While explicit formulas for quadrature nodes are not known, the weights can be expressed in closed form by the following relation [1]

$$\omega_j = \frac{2}{((1-(\tau_j)^2)(L_{m+1}^{'}(\tau_j))^2}. \tag{15}$$

Also, the initial values $x_1(0) = 10$ and $x_2(0) = 0$ is converted to

$$Y_1^T L(-1) = 10, \;\; Y_1^T DL(-1) = 0. \tag{16}$$

Therefore, the optimal control problem given in equations (2) and (3) is approximated to a nonlinear optimization problem with equation (14) as the objective functional and equation (16) as constraints. Finally, we can utilize many well-developed optimization algorithms to solve the resulted optimization problem.

4. Numerical Discussion

In this part, we illustrate the performance of the propounded technique with three numeral examples.The following numerical implementations are performed by using Mathematica 10.4 software and AMD A6- 4400M APU 2.70 GHz package. In this paper, by considering $q_{11} = 4$, $q_{22} = 0$, $x_1(0) = 10$, $x_2(0) = 0$ and some values of R, we dissolve considered optimal control problem in (2) and (3). A manned spacecraft has been shown in figure (1). The figures (2) and (3) are demonstrated the state functions $(x_1(t), x_2(t))$ and control function $u(t)$ for $m = 12$ and $R = 0.1$ respectively. Figures (4) and (5) are also shown state and control functions for $m = 12$ and $R = 10$ respectively. The optimal value of objective functional for $m = 12$ is 224.937. The total CPU times utilized with Mathematica software for this example are given in Table. Increasing R places a heavier penalty on acceleration and control energy expenditure.

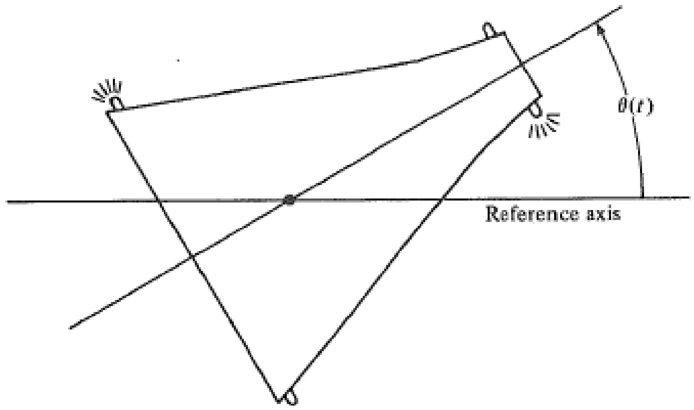

Fig. 1. Attitude Control of the Spacecraft

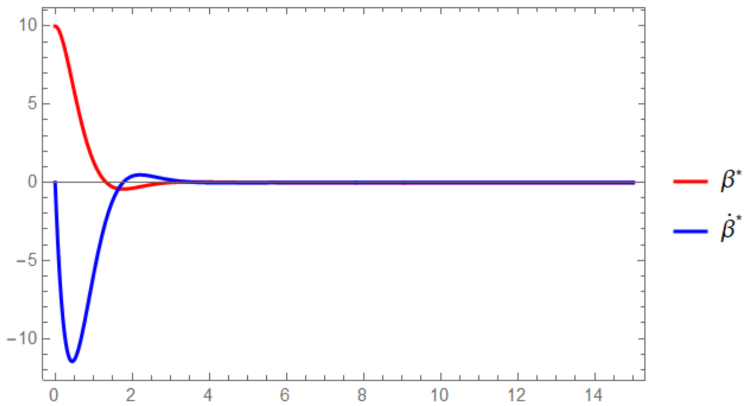

Fig. 2. Position and Velocity as functions of time for $R = 0.1$

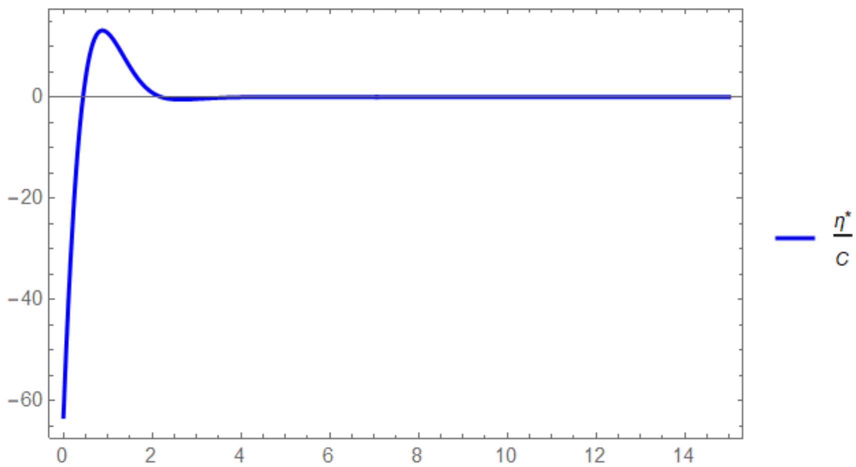

Fig. 3. Acceleration as a function of time for $R = 0.1$

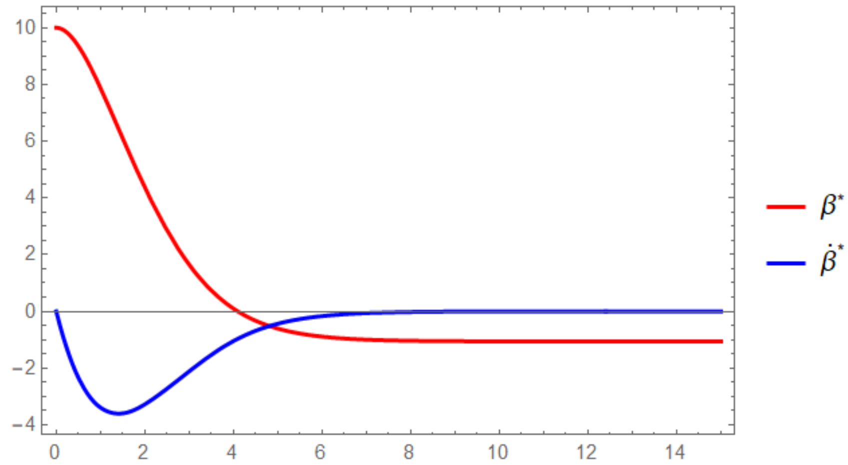

Fig. 4. Position and Velocity as functions of time for R = 10

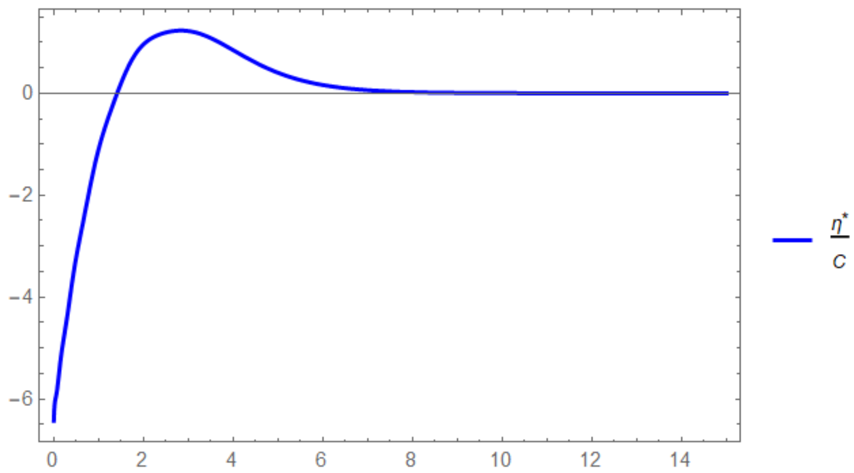

Fig. 5. Acceleration as a function of time for R = 10

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The Use of the Trapezoidal Method for Solving the Tacoma Narrows Bridge Model

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Abstract

In this paper, an efficient method is developed for approximate solution of a benchmark non-smooth dynamical system. In the proposed method, the trapezoidal method is utilized for solving the Tacoma Narrows Bridge equation. For this purpose, at first, the integral form of the dynamical equation is considered. Afterward, the obtained integral equation is discretized by the trapezoidal method. The accuracy and performance of the proposed method are examined by means of some numerical experiments.

1. Introduction

One of the most challenging problems in engineering which has been attracted the attention of many engineers, physicists, and mathematicians is the Tacoma Narrows Bridge problem. The Tacoma Narrows Bridge was a suspension bridge in the State of Washington which suffered collapse in a strong wind on the morning of November 7, 1940 (see Fig. 1). In many physics textbooks, the event is presented as an example of elementary forced mechanical resonance, but it was more complicated in reality. Accordingly, many researchers considered the reasons of the collapse which we can refer to the remarkable reference [1] in this regard. This paper will not discuss the reasons of this collapse, but instead a simple mathematical model of the problem is considered. It should be noted that the presented model is a very simplified one-dimensional model which can not consider all of the role-playing parameters of the problem. The interested readers are referred to [6] for more complicated models.
Now, consider the following Tacoma Narrows Bridge equation which is taken from [3, 5]. The problem is

$$\begin{aligned}&m\ddot{y}(t) = g(t) + F(y), \;\; 0 \le t \le 3\pi,\\ &y(0) = 0, \; \dot{y}(0) = 1,\end{aligned}$$

where

$$F(y) = \begin{cases} -\alpha y, & y \ge 0, \\ -\beta y, & y < 0, \end{cases}$$

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Fig. 1. Collapse of the Tacoma Narrows Bridge.

and the constant parameters are considered as $m = 1$, $\alpha = 4$, $\beta = 1$ and $g(t) = \sin(4t)$. For more details on the history and how to model this problem, the interested readers are referred to [3, 4, 6]. The problem has the following analytical solution

$$y(t) = \begin{cases} (\frac{2}{3} - \frac{1}{6}\cos(2t))\sin(2t), & 0 \le t \le \frac{\pi}{2}, \\ (\frac{7}{5} - \frac{4}{15}\sin(t)\cos(2t))\cos(t), & \frac{\pi}{2} \le t \le \frac{3\pi}{2}, \\ (-\frac{11}{15} - \frac{1}{6}\cos(2t))\sin(2t), & \frac{3\pi}{2} \le t \le 2\pi, \\ (-\frac{23}{15} - \frac{4}{15}\cos(t)\cos(2t))\sin(t), & 2\pi \le t \le 3\pi. \end{cases}$$

It is noted that, this problem is actually a non-smooth dynamical system which has a smooth solution. It is simple to show that, by using a substitution

$$y_1 = y,$$
$$y_2 = \dot{y},$$

the order of the problem is reduced to one and the following system of first-order initial value problems is derived

$$\left\{\begin{array}{l} \dot{y}_1(t)=y_2(t), \\ \dot{y}_2(t)=\frac{1}{m}(g(t)-\left\{\begin{array}{ll} \alpha y_1(t), & y_1(t)\geq 0, \\ \beta y_1(t), & y_1(t)<0, \end{array}\right. , \\ y_1(0)=0,\;\; y_2(0)=1. \end{array}\right. \tag{1}$$

The non-smooth dynamical systems whose right hand side of their dynamical systems or trajectories may not be differentiable everywhere are utilized to model a wide variety of phenomenon, especially in mechanical and control systems. It is necessary to mention that, due to non-smoothness in the right hand side of their dynamical systems or in their solution, the numerical approximation of non-smooth dynamical systems is a very difficult task. The aim of this paper is to present a method for an efficient numerical solution of the non-smooth dynamical equation of the Tacoma Narrows Bridge model. The next section is about introducing this method.

2. The proposed approach

Let's go back to the non-smooth initial value problem (1). In particular, the non-smooth initial value problem (1) can be considered as an initial value problem of the form

$$\left\{\begin{array}{l} \dot{\mathbf{y}}(t)=\mathbf{f}(t,\mathbf{y}(t)),\;\; t_0\leq t\leq t_f, \\ \mathbf{y}(t_0)=\mathbf{y}_0, \end{array}\right. \tag{2}$$

where, $\mathbf{y}=[y_1,\ldots,y_p]^T:[t_0,t_f]\rightarrow\mathbb{R}^p$ and $\mathbf{f}=[f_1,\ldots,f_p]^T:[t_0,t_f]\times\mathbb{R}^p\rightarrow\mathbb{R}^p$. It is worthwhile to note that, in the Eq. (2), the function $\mathbf{f}(t,\mathbf{y}(t))$ is a non-smooth function with respect to t or $\mathbf{y}$. Furthermore, it is assumed that, the Eq. (2) has a unique solution. Now, by integrating the dynamical equations in the Eq. (2) from t_0 to t, the equivalent system of Volterra integral equations is induced as

$$\mathbf{y}(t)=\mathbf{y}(t_0)+\int_{t_0}^{t}\mathbf{f}(\tau,\mathbf{y}(\tau))d\tau=\mathbf{y}_0+\int_{t_0}^{t}\mathbf{f}(\tau,\mathbf{y}(\tau))d\tau,\;\; t_0\leq t\leq t_f. \tag{3}$$

In the following, the trapezoidal integral formula for approximating the Volterra integral equations (3) is reviewed. For this purpose, at first, an equally spaced grid

$$t_i=ih,\quad i=0,1,\ldots,N,$$

is considered, where, $hN\leq t_f$ and $h(N+1)>t_f$. Now, for $n>0$, we can write

$$\mathbf{y}(t_n)=\mathbf{y}_n=\mathbf{y}_0+\int_{t_0}^{t_n}\mathbf{f}(\tau,\mathbf{y}(\tau))d\tau,\quad n=1,2,\ldots,N. \tag{4}$$

As a general approach, the integral term in the Eq. (4) can be approximated by the numerical integration such as

$$\int_{t_0}^{t_n}\mathbf{f}(\tau,\mathbf{y}(\tau))d\tau\simeq h\sum_{j=0}^{n}w_{n,j}\mathbf{f}(t_j,\mathbf{y}(t_j)),\quad n=1,2,\ldots,N, \tag{5}$$

where, the quadrature weights $hw_{n,j}$ are allowed to vary with the grid point t_n. So, the Eq. (4) is approximated by

$$\mathbf{y}_n\simeq\mathbf{y}_0+h\sum_{j=0}^{n}w_{n,j}\mathbf{f}(t_j,\mathbf{y}(t_j)),\quad n=1,2,\ldots,N. \tag{6}$$

Obviously, the Eq. (6), defines $\mathbf{y}_n$ implicitly. In other words, the Eq. (6) is a set of algebraic equations which can be solved by the root finding methods.

It is noted that, in this paper, the Eq. (6) is solved by the simple fixed point iteration method [2] where h is supposed to be sufficiently small. As we can see in the numerical illustrations section, using the fixed point iterations greatly increased the speed of the method. So, as a result, the Eq. (6) will find the following form

$$\mathbf{y}_n^{(k+1)} \simeq \mathbf{y}_0 + h\sum_{j=0}^{n-1} w_{n,j}\mathbf{f}(t_j, \mathbf{y}_j) + hw_{n,n}\mathbf{f}(t_n, \mathbf{y}_n^{(k)}), \quad k = 0, 1, \dots \tag{7}$$

for some given initial estimation of $\mathbf{y}_n^{(0)}$. There are many possible schemes for being in the Eq. (6). In this paper, the fantastic trapezoidal numerical method will be used. It is worthwhile to note that, the trapezoidal rule has the form

$$\int_{\alpha}^{\alpha+h} F(s)ds \simeq \frac{h}{2}[F(\alpha) + F(\alpha + h)].$$

So, in Eq. (5) we have

$$\int_{t_0}^{t_n} \mathbf{f}(\tau, \mathbf{y}(\tau))d\tau \simeq \frac{h}{2}\mathbf{f}(t_0, \mathbf{y}_0) + h\sum_{j=1}^{n-1}\mathbf{f}(t_j, \mathbf{y}_j) + \frac{h}{2}\mathbf{f}(t_n, \mathbf{y}_n),$$

and consequently looking at the Eq. (7), the trapezoidal method will lead to the following iterative equation

$$\mathbf{y}_n^{(k+1)} \simeq \mathbf{y}_0 + \frac{h}{2}\mathbf{f}(t_0, \mathbf{y}_0) + h\sum_{j=1}^{n-1}\mathbf{f}(t_j, \mathbf{y}_j) + \frac{h}{2}\mathbf{f}(t_n, \mathbf{y}_n^{(k)}), \quad k = 0, 1, \dots \tag{8}$$

It is emphasized again that, the Eq. (8) is solved by the simple fixed point iteration method where h is supposed to be sufficiently small. This leads the proposed method being very fast.

3. Numerical illustrations

This section is devoted to the numerical illustrations and the effectiveness of the presented method is shown. Noted that, all computations are performed on a 2.53 GHz Core i5 PC Laptop with 4 GB of RAM running in MATLAB R2016a. Now, consider the non-smooth initial value problem (1) again. This problem is solved by using the presented method. The approximated solution for $N = 6000$ discretization points is shown in Figure 2 alongside the exact solution, and the absolute error of the approximated solution on the interval $0 \leq t \leq 3\pi$. Also, for exploring the dependence of the error of the approximated solution on the parameter N, the presented method is applied on this problem for various values of N. In Figure 3, an overview of the rate of convergence by plotting the Euclidean norm of error, E_N, as a function of N can be seen. Obviously, if N increases, then the Euclidean norm of error will become smaller. Furthermore to better show the efficiency of the method, the CPU time of the presented method versus N is shown in Figure 4 and the log-linear graph for better vision, is plotted. As we can see, the presented method has desirable speed.

4. Conclusion

In this paper, the fantastic trapezoidal method is proposed for the numerical solution of the non-smooth Tacoma Narrows Bridge equation. The method can be easily applied to any type of non-smooth initial value problems. According to the numerical illustrations, the accuracy and the speed of the method is satisfactory. Furthermore, by using the simple iteration method for solving the root finding problem appeared in the method, the CPU time of the method is significantly reduced. Further research in the usage of the presented method to solve the non-smooth boundary value problems will be interesting.

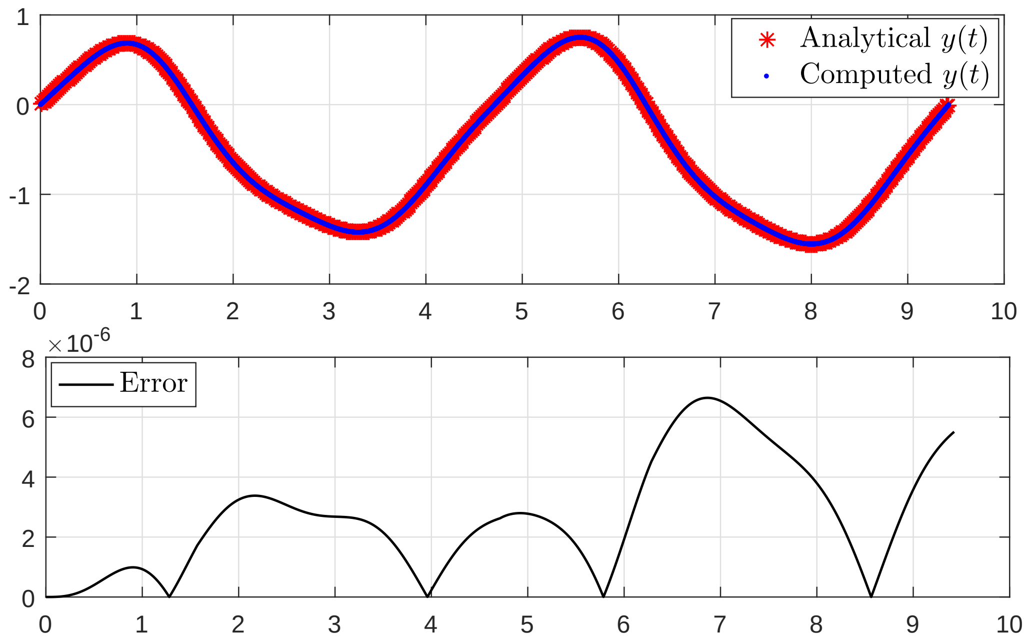

Fig. 2. Solution of the Tacoma Narrows Bridge equation.

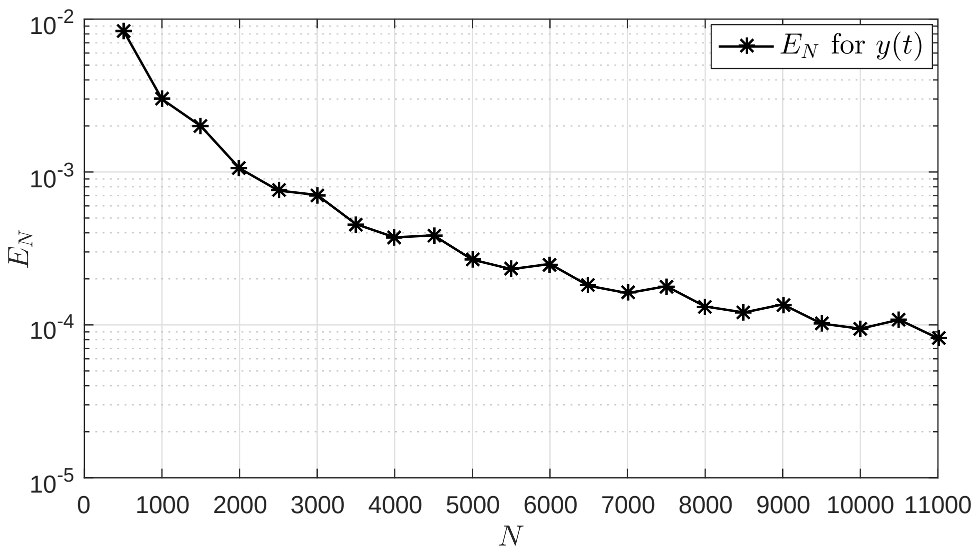

Fig. 3. Euclidean norm of error versus N.

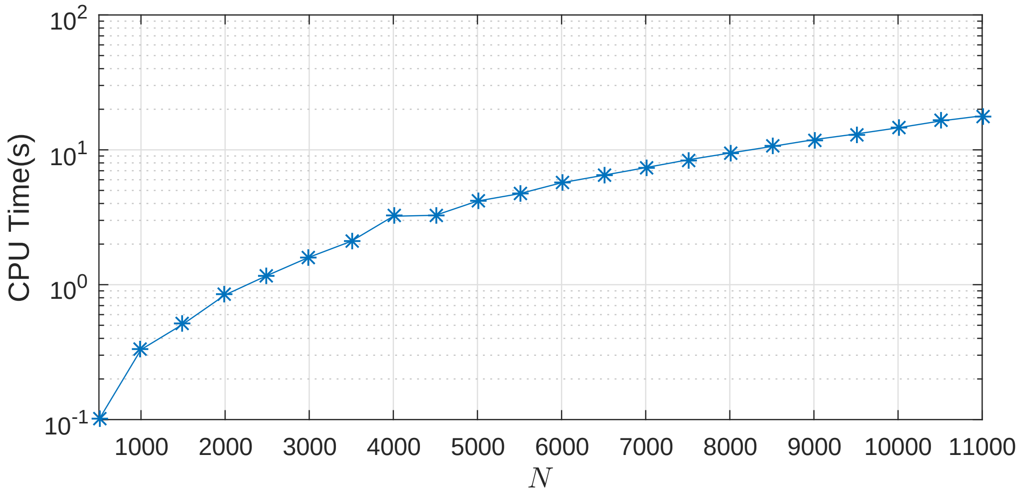

Fig. 4. CPU time of the presented method versus N.

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A New Combination of the Poisson and Exponential Probability Distributions

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Article Info

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Abstract

In this paper, an area-biased form of the single parameter Poisson-exponential distribution (PED) is obtained by area biasing the discrete Poisson-exponential distribution (PED) introduced by Fazal and Bashir. Poisson-exponential distribution is an important discrete distribution which has many applications in countable data sets. The first four moments (about origin) and the central moments (about mean) have been obtained and hence expression for coefficient of variation (CV), skewness, kurtosis and index of dispersion are derived. To estimate the parameters of area-biased Poisson-exponential distribution (ABPED), maximum likelihood method (MLE) and method of moments (MOM) are also developed. The goodness of fit for ABPED has been discussed using three real data sets the fit shows a better fit over size-biased Poisson-Lindley distribution (SBPLD).

1. Introduction

Fazal and Bashir [1] have obtained discrete Poisson-exponential distribution (PED) for modelling count data with probability mass function (p.m.f):

$$P(X=x)=\frac{\theta}{(1+\theta)^{x+1}},\theta>0,x=0,1,2,\ldots\ \ldots \tag{1}$$

The Poisson-exponential distribution (PED) in (1) is a mixture of Poisson and exponential distribution when the parameter of Poisson distribution (λ) follows exponential distribution. The first four moments (about origin) and the variance of (PED) obtained by Fazal and Bashir [1] are given as:

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$$\mu_1' = \frac{1}{\theta}$$

$$\mu_2' = \frac{2+\theta}{\theta^2}$$

$$\mu_3' = \frac{\theta^2+6\theta+6}{\theta^3}$$

$$\mu_4' = \frac{\theta^3+14\theta^2+36\theta+24}{\theta^4}$$

$$\mu_2 = \frac{1+\theta}{\theta^2} \tag{2}$$

The mathematical properties and estimation of parameter have been discussed by Fazal and Bashir and its application proves that it is a good replacement of Poisson distribution and Lindley distribution. The size-biased form of (PED) has been discussed by Fazal and Bashir and its goodness of fit gives quite satisfactory fit over size-biased Poisson distribution, size-biased Lindley distribution and size-biased geometric distribution. The mixture of Poisson and size-biased exponential distribution has been discussed by Fazal and Bashir with properties and applications. The size-biased and area-biased distributions were discussed earlier by Fisher c when sample observations have unequal probability of selection therefore we apply weights to the distribution to model bias. If the random variable 'x' had pdf $f(x, \theta); x = 0, 1, 2, \ldots..., ; \theta \geq 0$, then the weighted distribution is of the form:

$$P(x;\theta) = \frac{x^m f_o(x;\theta)}{\mu_m'} \tag{3}$$

For m=1 and m=2 we get the size-biased and area-biased distributions respectively. Area-biased distributions are applicable for sampling in forestry, medical sciences, psychology etc. Different discrete mixed distributions have been size-biased and discussed with their applications in real data sets. Shankar & Kumar [3] obtained size-biased Poisson-Garima distribution with mathematical properties to analyze genetics data sets. Shankar [4] introduced size-biased Poisson-Shankar distribution with applications. Shankar & Fasshaye [5] considered the size–biased form of Poisson-Sujata distribution which was first introduced by Shankar for modelling count data in various fields of knowledge. Shakila & Mujahid Rasul [6] derived the Poisson area-biased Lindley distribution with its applications in biological data to prove that it gives a better fit than Poisson distribution. Shankar & Fassahe [7] proposed the size-biased form of Poisson-Amarenda distribution and its applications proved that it is a good replacement of size-biased Poisson distribution (SBPD), size-biased Poisson-Lindley distribution (SBPLD) and Size-biased Poisson-Sujhata Distribution (SBPSD). Shakila and Mujahid [8] proposed the size-biased form of Poisson-Janardhan distribution and derived its mathematical properties, whereas Janardhan distribution is a two parameter distribution obtained by Rama & Mishra [9] as a mixture of exponential and gamma distribution. Rama & Mishra obtained the size-biased form of Qaussi Poisson-Lindley distribution of which size-biased poisson-Lindley distribution is a particular case (SBPLD). Ahmed & Munir [10] have discussed few size-biased discrete distributions and their generalizations with properties and application. The size-biased version of Poisson-Lindley distribution has been discussed by Ghitanni & Mutairi [11] and the new distribution introduced in this paper i.e area-biased Poisson-exponential distribution (ABPED) gives more satisfactory fit as compared to size-biased Poisson-Lindley distribution. The mathematical properties and estimation of parameters has been discussed and goodness of fit is also presented [12-16].

2. Area–biased poisson exponential distribution

Using (1), (2), (3) the pmf of the area-biased Poisson-exponential distribution can be obtained as:

$$P(x;\theta)=\frac{x^2 f_0(x;\theta)}{\mu_2'}=\frac{x^2\theta/(1+\theta)^{x+1}}{(2+\theta)/\theta^2} \tag{4}$$

where $\mu_2'=\dfrac{(2+\theta)}{\theta^2}$ is the second raw moment of discrete Poisson-exponential distribution. With simplifications we get the pmf of area-biased Poisson-exponential distribution with parameter θ as:

$$P(X=x)=\frac{x^2\theta^3}{(1+\theta)^{x+1}(2+\theta)} \qquad \theta>0\ ,\ x=1,2,3,4,\ldots\ldots\ldots \tag{5}$$

Graphs of area– biased Poisson-exponential distribution for different values of θ are shown in Figure 1.

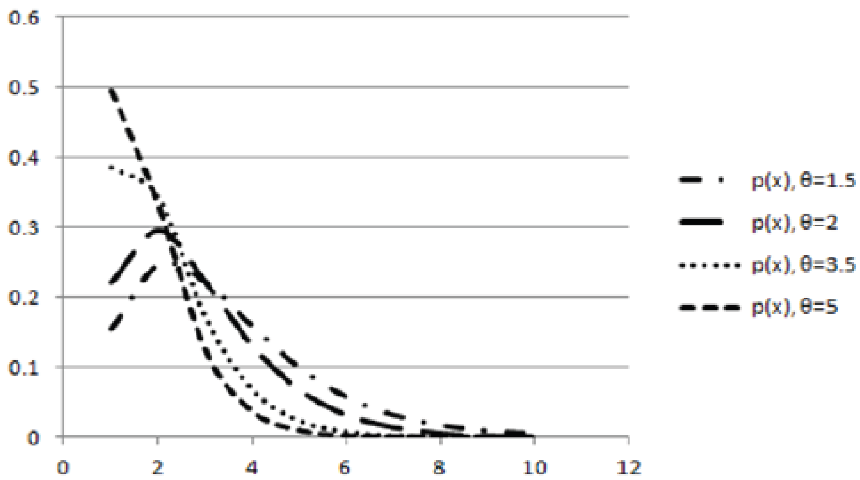

Fig. 1. Graphs of area–biased Poisson-exponential distribution for different values of θ

3. Moments and moment based measures of area-biased Poisson-exponential distribution

We start the mathematical derivations with moments and moment measures. The first four raw moments of area-biased Poisson- exponential distribution (ABPED) are:

$$\mu_1'=\frac{\theta^2+6\theta+\theta 6}{\theta(\theta+2)}$$

$$\mu_2'=\frac{\theta^2+12\theta+12}{\theta^2} \tag{6}$$

$$\mu_3'=\frac{\theta^4+30\theta^3+150\theta^2+240\theta+120}{\theta^3(\theta+2)}$$

$$\mu_4'=\frac{\theta^4+60\theta^3+420\theta^2+720\theta+360}{\theta^4}$$

The mean moments of ABPED are obtained by using the relationship between moments about mean and moments about origin:

$$\mu_2 = \frac{4\theta^3 + 16\theta^2 - 24\theta + 12}{\theta^2(\theta+2)^2}$$

$$\mu_3 = \frac{4\theta^5 + 28\theta^4 - 336\theta^3 + 168\theta^2 + 144\theta + 48}{\theta^3(\theta+2)^3}$$

$$\mu_4 = \frac{760\theta^7 + 938\theta^6 + 6200\theta^5 + 329920\theta^4 + 976320\theta^3 + 1326720\theta^2 + 650880\theta + 720}{\theta^4(\theta+2)^4} \tag{7}$$

The harmonic mean of area-biased Poisson-exponential distribution is:

$$H.M = \frac{\theta}{\theta+2} \tag{8}$$

The coefficient of variation (C.V), coefficient of skewness, coefficient of kurtosis and index of dispersion of area-biased Poisson-exponential distribution ABPED are obtained respectively as:

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{4\theta^3 + 16\theta^2 - 24\theta + 12}}{\theta^2 + 6\theta + 6}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\sqrt{4\theta^5 + 28\theta^4 - 336\theta^3 + 168\theta^2 + 144\theta + 48}}{\left(4\theta^3 + 16\theta^2 - 24\theta + 12\right)^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{760\theta^7 + 938\theta^6 + 6200\theta^5 + 329920\theta^4 + 976320\theta^3 + 1326720\theta^2 + 650880\theta + 720}{\left(4\theta^3 + 16\theta^2 - 24\theta + 12\right)^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{4\theta^3 + 16\theta^2 - 24\theta + 12}{\theta(\theta+2)(\theta^2 + 6\theta + 6)} \tag{9}$$

For the area-biased Poisson-exponential distribution, (ABPED), from (9) it can be seen that the model is positively skewed and leptokurtic. To study the characteristics and comparative behavior of ABPED and SBPLD, a table of $\mu_1', \mu_2, C.V, \sqrt{\beta_1}, \beta_2, and\ \gamma$ for varying values of the parameter θ, has been prepared and presented in the Tables 1 & 2.

Table 1. Values of θ for ABPED

	1	2	3	4	5	6
μ_1'	4.3333	2.75	2.2	1.9167	1.7429	1.625
μ_2	0.8889	0.5625	0.4267	0.3264	0.2547	0.2031
C.V	0.2176	0.3521	0.4199	0.4497	0.4614	0.4637
$\sqrt{\beta_1}$	2.4749	2.3754	1.4434	0.7824	0.3155	0.03754
β_2	5254.97	621.85	245.63	155.66	121.86	106.53
γ	0.2051	0.3409	0.3879	0.3877	0.37096	0.3494

The comparative graphs of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of ABPED and SBPLD are shown in Figure 2.

Table 2. Values of θ for SBPLD

	1	2	3	4	5	6
μ_1'	3.66667	2.25	1.8	1.583333	1.457143	1.375
μ_2	5.55556	1.9375	1.093333	0.743056	0.556735	0.442708
C.V	0.642824	0.61864	0.580903	0.544425	0.512061	0.483901
$\sqrt{\beta_1}$	1.318047	1.49478	1.649924	1.790721	1.921224	2.043701
β_2	5.4744	6.057232	6.599941	7.118613	7.625214	8.125813
γ	1.515152	0.861111	0.607407	0.469298	0.382073	0.32197

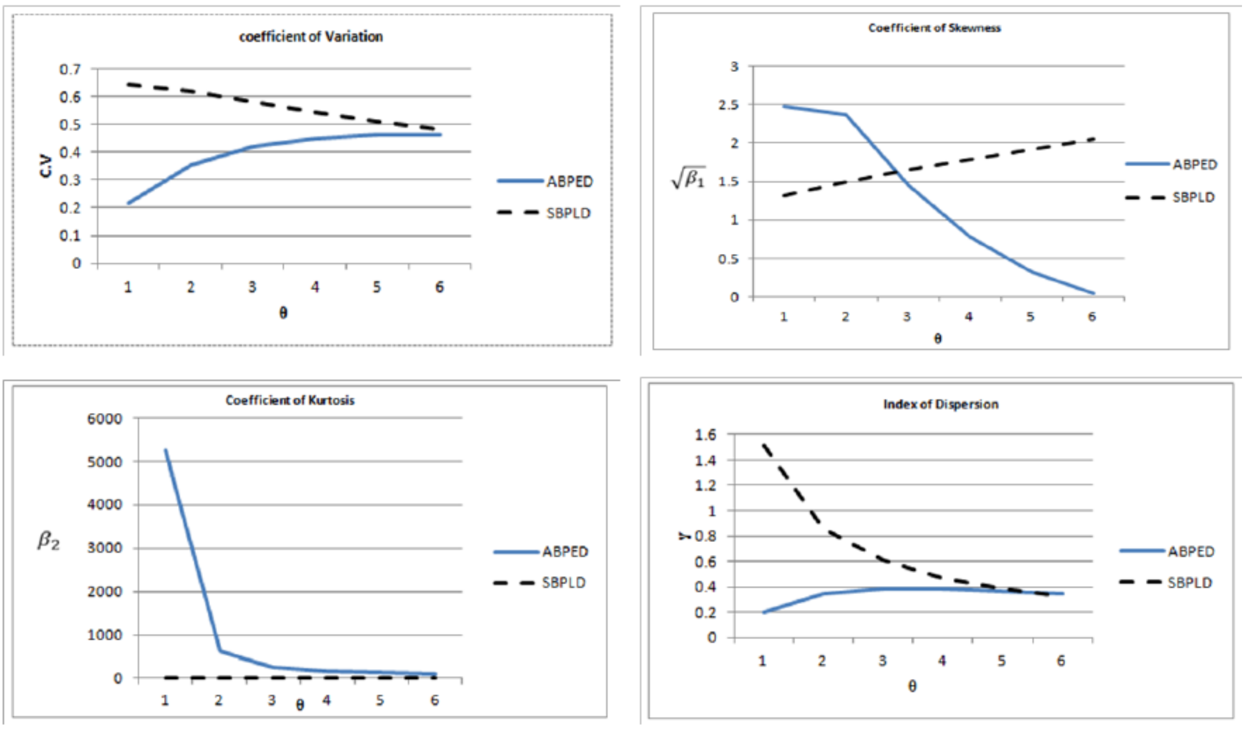

Fig. 2. Graphs of $C.V, \sqrt{\beta_1}, \beta_2$ and γ for ABPED and SBPLD

4. Reliability measures

Using pmf of ABPED from (4), we have:

$$\frac{P(x+1,\theta)}{P(x.\theta)} = \frac{\left(1+\frac{1}{x}\right)^2}{(1+\theta)} \quad (10)$$

Which is a decreasing function of x, therefore ABPED is uni-modal and has an increasing failure rate.

5. Generating functions of ABPED

Probability generating function: the probability generating function of ABPED can be obtained as:

$$P_x(t) = E(t^x) = \sum_{x=1}^{\infty} \frac{x^2\theta^3 t^x}{(1+\theta)^{x+1}(2+\theta)}$$

$$= \frac{\theta^3 t^2 + t(\theta^4+\theta^3)}{\theta^4+5\theta^3+9\theta^2+7\theta-t^3(\theta+2)+3t^2(\theta^2+3\theta+2)-3t(\theta^3+4\theta^2+5\theta+2)+2} \quad (11)$$

Fisher information matrix: If:

$x \sim f(x/\theta)$ where $f(x/\theta) = \dfrac{x^2\theta^3}{(1+\theta)^{x+1}(2+\theta)}$

is the pmf of area–biased Poisson-exponential distribution with $\theta > 0$, then:

$$I_x(\theta) = \frac{20\theta^4 - 8\theta^3 - 192\theta^2 - 309\theta - 144}{\theta^4(1+\theta)^2(2+\theta)^3} \quad (12)$$

is the Fisher Information Matrix of ABPED.

Moment generating function: the moment generating function of ABPED is obtained as:

$$M_x(t) = E(e^{tx}) = \frac{\theta^3 e^{2t} + (\theta^4+\theta^3)e^t}{\theta^4+5\theta^3+9\theta^2+7\theta-(\theta+2)e^{3t}+(3\theta^2+9\theta+6)e^{2t}-(3\theta^3+12\theta^2+15\theta+6)e^t} \quad (13)$$

6. Estimation of parameters

Method of moment (MOM) estimate: let $x_1, x_2, x_3 \ldots\ldots x_n$ be a sample of size n from ABPED then equating the population parameter to the sample mean we obtain the MOM estimate $\tilde{\theta}$ of θ of ABPED as:

$$\tilde{\theta} = \frac{-(\bar{x}-3)+\sqrt{(\bar{x}-3)^2+6(\bar{x}-1)}}{(\bar{x}-1)} \quad (14)$$

where $\bar{x}$ is the sample mean.

Maximum likelihood estimate (MLE): let $x_1, x_2, x_3 \ldots\ldots x_n$ be a sample of size n from ABPED), the MLE estimate $\tilde{\theta}$ of θ is obtained as:

$$\tilde{\theta} = \frac{-(\bar{x}-3)+\sqrt{(\bar{x}-3)^2+6(\bar{x}-1)}}{(\bar{x}-1)} \quad (15)$$

7. Goodness of fit for ABPED

The area-biased Poisson-exponential distribution has been fitted to a number of countable data sets, and compared with size–biased Poisson-Lindley distribution. The following examples are used to illustrate a few situations generating the area–biased distribution and their applications. Microsoft Excel has been used to facilitate the use of the size-biased models to real life data. The MOM and MLE estimates are used to fit the distributions and is presented in the tables below. The data sets include number of observations of size distributions i.e. small groups in various public situations reported by James [12], Coleman and James[13], and Simnoff [14], thunderstorm data sets reported by Shankar et al., for all these datasets the ABPED distribution gives much closer fit than SBPLD (Tables 3-6).

Table 3. Counts of group of people in public Places on a spring afternoon in Portland

Size of groups	Observed frequency	Expected frequency	
		SBPLD	ABPED
1	1486	1532.5	1480.381
2	694	630.6	704.9073
3	195	191.9	188.8048
4	37	51.3	39.95664
5	10	12.8	7.43203
6	1	3.9	1.273997
TOTAL	2423	2423	2422.756
Estimation of parameters		$\tilde{\theta} = 4.5224$	$\tilde{\theta} = 7.4302$
$\div^2$		13.766	1.2166
d.f		3	2
p-value		<0.01	0.74903
AICc		3.2732	2.99996
BIC		2.0649	1.7918

Table 4. Counts of shopping groups-Eugene, spring, department store and public market

Size of groups	Observed frequency	Expected frequency	
		SBPLD	ABPED
1	316	323	312.1584
2	141	132.5	148.1144
3	44	40.2	39.53136
4	5	10.7	8.336454
5	4	3.6	1.545125
Total	510	510	
Estimation of parameters		$\tilde{\theta} = 4.5224$	$\tilde{\theta} = 7.0435$
χ^2		3.021	0.9728
d.f		2	2
p-value		0.4	0.6148
AICc		3.51453	3.33331
BIC		1.79064	1.60942

Table 5. Counts of play Groups-Eugene, spring, public playground D

Size of groups	Observed frequency	Expected frequency	
		SBPLD	ABPED
1	305	314.4	302.7967
2	144	134.4	150.5768
3	50	42.5	42.1199
4	5	11.8	9.309185
5	2	3.1	1.808334
6	1	0.8	0.323734
Total	507	507	506.9
Estimation of parameters		$\tilde{\theta} = 4.3179$	$\tilde{\theta} = 7.0435$
χ^2		6.415	2.8126
d.f		2	2
p-value		0.043	0.2451
AICc		3.2809	3.000006
BIC		2.0727	1.79177

Table 6. Frequency of thunderstorm events containing X thunderstorms at Cape kennedy for May

X	Fo	Expected frequency	
		SBPD	ABPED
1	87	83.2	83.95544
2	25	30.5	29.27444
3	5	5.6	5.741838
4	3	0.7	0.889832
Total	120	120	119.8615
Estimation of parameters		$\tilde{\theta} = 0.36667$	$\tilde{\theta} = 10.4715$
χ^2		1.624	1.0167
d.f		1	1
p-value		0.2025	0.3133

8. Conclusions

Area-biased Poisson-exponential distribution (ABPED) has been derived by area biasing the Discrete Poisson- exponential distribution (PED) introduced by Fazal and Bashir. The discussion on estimation and applications of the area–biased distribution demonstrates that ABPED has a practical use to real life data. Form AIC and BIC measures the proposed area-biased model appears to offer substantial improvements in fit over the size-biased Poisson-Lindley model. Also the fitting in these tables reveal that the area-biased distribution provides us with better fits in the situations where zero-class is missing gives a better fit than size-biased Poisson-Lindley distribution (SBPLD) and size-biased Poisson distribution (SBPD) therefore we conclude that area-biased Poisson-exponential distribution is a good replacement of (SBPLD).

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Evaluating induced abortion in women between 15-49 years old using Warner random response method

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Article Info

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ratio
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Abstract

The main objective of current study is to obtain induced abortion rate among married women between 15 to 49 years old. Although this issue is the priority of national medical science research related to women's health and fertility, it is difficult to obtain real data to this respect. This cross-sectional research with sample size of 550 people was conducted using simple sampling from married women between 15 to 49 years old (as patients or in company with them) who referred to Aliyeh Garden and Abbaspour town medical centers. Since real data collection by questionnaires was difficult to perform, random response method was applied. The induced abortion ratio and variance were obtained. This rate is significantly different from values resulting from previous studies using Z-test ($p<0.001$).
Regarding the results of current study, it seems that this statistical method gives a higher and more realistic evaluation of induced abortion level in women between 15 to 49 years old. In order to achieve more accurate data, other studies using this method and a larger sample size are proposed, particularly among high-risk women.

1. Introduction

Investigation of frequency and complications of induced abortion is the first priority of medical science research on women's fertility and health. However, it is very difficult to obtain real information to this respect. If we raise the respective questions in questionnaires, most of people avoid giving correct answers due to fear, distrust and feeling of guilt etc. direct interviews with people may bring about similar problems and decrease exact and precise answers. In health evaluations or planning, similar conditions frequently occur where the researcher needs real information. Common data collection methods can not give sufficient and exact information. Sometimes, it may seem immoral to directly ask questions from the interviewer's or respondents' point of view.
In order to meet such requirements, random response method was developed. This method applies in statistical investigations to decrease or remove response errors. Warner (1965) first introduced this method. In this method data is collected indirectly. The respondent randomly selects two or more questions, at least one of which is the main question. She responds the selected question (without showing the questioner which question she has selected). As the response type and scope are the same for each question, the respondent's privacy will be observed. However, the obtained data

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is sufficient to evaluate the critical parameters under question; 2) this method is very effective in evaluation of such parameters and is more precise than direct questions.
This method applies in medical research to evaluate contraband alcoholic drinks [3], drug use [4,6,8]. Alcohol use [4,7,9,10], smoking and sexual activities in adolescents [4] and induced abortion [11]. This is a relatively temporary method for attracting the respondents' confidence [7,8,10,11]. In 1971, a study was conducted on privileged classes in Iran to investigate frequency of induced and spontaneous abortions in which induced abortion rate was 16.7%.
Another investigation in the same article using devices such as catheter, knitting mill, feather and strong caustic substances diagnosed 4.8% of all abortions referring to the hospital as incomplete induced abortion. 25% of patients belonged to low social classes suffering from acute uterine hemorrhage subsequent to abortions with dangerous and ineffective kits. In women from privileged social classes, the degree of both application of contraception remedies and induced abortion was higher. Women from lower social classes less frequently used contraception remedies or induced abortion because they were very expensive and unavailable.
In 1976, abortion and infertility laws were changed in Iran. Although, since 1972, abortions were performed in health centers and clinics, the official permission was issued regarding 3 following cases in 1976:
1- Social and medical reasons
2- If the fetus is younger than 12 weeks and parents' written consent is obtained.
3- In cases where survival of fetus endangers mother's life.
These rules were revised and changed after Islamic revolution. In a study, it became clear that in spite of changes in population policies during 1987-88 about illegality of abortion, privileged people could afford and provide scientific abortion methods but poor people resorted to infectious abortions [14]. No values were presented to prove this claim. Najaf Zare et.al, (2001) conducted a similar study in Shiraz in which induced abortion ratio was 2.86% with variance 718%. Since updated information is not available to this respect and traditional methods are not precise, we decided to apply this method to estimate induced abortion ratio in women between 15 to 49 years old referring to health centers in Minab.

2. Materials and Methods

This cross-sectional study was conducted in December, 2012. Subjects of study were women, 15-49 years old, admitted at Clinic No. 102 in Minab as patients or her fellow. Using random response method, some questions were asked about patients' history of induced abortion. Considering confidence level 95% and assuming induced abortion frequency about 2%, based on previous studies [15] and the expected assumed rate 5%, the minimum sample size was estimated by inverse binominal method 330 to find at least 10 cases of induced abortion.
Eq.)
X: number of people with induced abortion in an n-people sample
Since the number of people who would answer the main question was 6% of sample size, total number of sample was estimated 550 people. Simple sampling was conducted on population of study. Our method was as follows: 55 uniform cards were prepared on 35 of which this two-option question was written:
Have you ever had induced abortion?
a) Yes b)No
It was the main question of study with only one correct answer. An unrelated question was written on 20 remaining cards: "Is your ID card number even or odd?"
a) Yes b)No
This question was chosen so that its two options had specific and uniform distribution. In other words, probability of each unrelated answer was evident.
Some papers were distributed among subjects, on which the following statement was written:
Please check your response a) yes b) no
Subjects of study were justified and convinced in 5- to 10-people groups and their confidence was obtained about lack of questioner's knowledge about their response and the content of question. Induced abortion was defined as follows: " a miscarriage with curettage, use of intrauterine kits, manipulation of uterus by a doctor or a midwife or a person without medical qualification, or taking abortion-inducing medicine". We deranged 55-card bunches and asked the first person under study to randomly select a card and check her response on the answer sheet and throw it into a box

prepared to this end. Then we put the selected card among other cards, deranged and offered them to the next person. Using this method, it was not clear for the questioner that which questions the respondent had responded to.

3. Findings

The number of responses to each question was as follows:
a) Yes b) No
Analyses were conducted by substituting the resulting numbers from manual operations into the respective equation. Our study variable had two conditions. In other words, total probability of each response was 1.
π : the respective parameter which is the ration or percentage of population with the respective characteristic.
θ : the ration of cards on which the critical question was written.
ρ : probability of "yes" response to the unrelated question, 0.5
n: sample size=550
x: the number of people with "yes" response (whether to the main or unrelated question)
The objective is to estimate π, that is, probability or ratio of induced abortion.
In current study, probability of attempting to induced abortion by a woman during her conjugal life was estimated ……. This estimation is not diagonal ad its variance is calculated as follows.
If we assume this ratio in society equal to zero, z-test will be significant for the obtained ratio ($p < 0.1$)

4. Conclusion

Assumed induced abortion rate was 5%. As it can be seen, this rate was….% in current research. However, this study is not based on sampling from total population of society and this rate is different among different populations. In order to estimate abortions across the country, sampling should include all existing population groups across the area of study. Estimation of this rate may be of a greater importance among high risk populations in terms of health planning and death preventions. We hope that this study and introduction of random response methods can be useful in future work to answer similar health- and treatment-related questions.

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Parameter estimation of competing risks model under type-II generalized hybrid censoring scheme for exponential distribution

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Abstract

Time-to- failure under different causes of failure is known as a competing risks model. Practice, competing risks data can be appeared in different applications such as engineering fields or biological and medical lifetime studies as well as other related areas. Also, the causes of failure, which are competing may be partially observed. In this paper, we adopted the competing risks model with partially observed causes of failure when the latent failure times follow exponential distribution under type-II generalized hybrid censoring scheme. The maximum likelihood estimators of the model parameters with the associated confidence intervals are discussed. The results are discussed using a simulated data set for illustration purposes. Finally, the simulation experiments are performed to assess the proposed method.

1. Introduction

The problem of deriving the efficient statistical procedures under different life testing experiments for the unknown interesting quantities has considerable on the basis of different censoring methodologies. In the literature, there are a various censoring schemes which can be used in reliability analysis Hybrid censoring scheme (HSC) is more conventional for different situations than other censoring schemes such as type-I and type-II censoring schemes. In HSC, the experiment is terminated at some specified conditions of time and a number of failures. HSC can be appeared in different types such as type-I HCS and type-II HCS. Let, X_m denoted to the mth failure time and τ denoted to the prescribed test termination time. In type-I HCS, the test terminated at a random time $\eta = \min\{X_m, \tau\}$. The larger lifetimes of testing units than the time s have shown that a few number of failures can be observed through the experiment. Then, the statistical inference is done with a low precision result. Therefore, type-II HCS is suggested by [2] and applied to obtain the exact distribution of the MLE and exact lower confidence bound to mean of the exponential model, whereas the random termination time of the test is changed to satisfy that, at least m failures is observed.
The type-I and type-II HCSs are still having the small number of failure and test in a large period of time, respectively. These problems have been handled with generalized hybrid censoring scheme (GHCS), see [3]. The experiments under GHCS has guarantees not only controls within a proper testing period, but present at least fixed number of failures in

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testing procedure. Also, the statistical inference is more efficiency under GHCS due to more observed failures. The simple types of GHCS are known by type-I and type-II GHCSs described as follows.
For the machansim of type-II GHCS, consider a life-testing experiment in which n identical units are put to test, with priors integers $m \in 1, 2, \dots, n$; and times $0 < \tau_1 < \tau_2 < \infty$. The time to failure T_i is recorded until the time τ_1 is reached. If the failure time $T_m < \tau_1$; the experiment is terminated at τ_1. But if $\tau_1 < T_m < \tau_2$ then, the experiment is terminated at T_m. Also, if $\tau_2 < T_m$ then, the experiment is terminated at τ_2. So, the terminated test time η and the number D of the observed failure times, $\mathbf{t} = (t_{1:n}, t_{1:n}, \dots, t_{D:n})$ is defined as (η, D) and given by

$$(\eta, D) = \begin{cases} (\tau_2, D) & 1 \leq D \leq m \quad \text{if} \quad \tau_1 < \tau_2 < T_m, \\ (T_m, m) & \text{if} \quad \tau_1 < T_m < \tau_2, \\ (\tau_1, D) & m \leq D \leq n \quad \text{if} \quad T_m < \tau_1. \end{cases} \tag{1}$$

In type-II HCS the experiment has guaranteed that it will be completed by time τ_2. Hence, the time τ_2 is the allow time that the researcher is willing to complete the experiment.
If the experimenter need to remove a survival unit from the test other than the final point, then generalized progressive censoring schemes (GPCSs) are applied. The different conventional types of censoring schemes are included in GPCSs as a special cases, such as type-I censoring, type-II censoring, progressive type-I censoring, progressive type-II censoring and some other HCSs. For extensive reviews of progressive censoring schemes, one could refer to the works of [4] and [5], as well as the reference therein.
The competing risks model is appeared when units, are failing with different causes of failure, one of them has caused the failure. Then, this model is aimed to assess one cause of failure with respect to other causes of failure. The outcome in competing risks models is described, the time-to-failure T and the corresponding cause of failure ρ. So, the time T present a continuous random variable, but failure cause ρ is a fixed number labeled by $1, \dots, p$, where p is the number of causes. Hence, the bivariate distribution is applied as the basic probabilistic framework with continuous and discrete random variables given by T and ρ, respectively. Practice, the framework of the competing risks model has appeared in a several fields. In a medical field, ρ present the cause of death and T is the individual age. The time T in economics field, may be present a spent time on the unemployment register, but ρ is the reason for de-registering. Also, in the manufacturing field, the time T is the usage of the machine and ρ may be the cause of the breakdown of a machine. And, in the reliability field, T is a time running from start-up to breakdown and ρ may be the faulty component in a system.
Exponential distribution is widely used in electrical products, and its probability density function can be described as

$$f(t) = \theta e^{-\theta t}, \qquad t, \theta > 0, \tag{2}$$

where θ is the scale parameter.

2. Model description

Suppose that, $T_1, T_2, \dots, T_n$ be n identical independent distributed (i.i.d.) lifetime of n testing units under life testing experiment. Then, the latent failure time under two independent causes of failure of any unit is defined by

$$T_i = \min\{T_{i1}, T_{i2}\}, \qquad i = 1, 2, \dots, n. \tag{3}$$

Under type-II GHSC, suppose the prefixed number m and two ideal test times (τ_1, τ_2), are detrimental. Firstly, when the experiment is running, the failure times and the corresponding cause of failure $(T_{i:m:n}, \rho_i)$, $i = 1, 2, \dots, D$ are recorded, where D denote to number of failure units until terminated time of the experiment ρ; where (η, D) is defined by (1). When the failure time $T_m < \tau_1$, the experiment is terminated at $\eta = \tau_1$. But if $\tau_1 < T_m < \tau_2$ then, the experiment is terminated at $\eta = T_m$. Also, if $\tau_1 < \tau_2 < T_m$ then, the experiment is terminated at $\eta = \tau_2$. So, the observed failure times are: $\mathbf{t} = \{(t_{1:n}, \rho_1), (t_{2:n}, \rho_2), \dots, (t_{D:n}, \rho_D)\}$ and the corresponding cause of failure is defined with indicator value ρ_i

$$\rho_i = j, \qquad j = 1, 2, 3, \quad i = 1, 2, \dots, D, \tag{4}$$

where the value 1 of ρ_i mean that failure under the first cause, 2 mean failure under the second cause and, 3 mean failure cannot be determined its cause. Under some restricted the complex operating environment of units some failure causes cannot be detected clearly, then, we applied the partially observed causes of failure competing risks model.

From the type-II GHS sample of size D,**t**, the joint likelihood function from [1] is given by

$$L(\mathbf{t} \mid \omega) \quad = \quad \frac{n!}{(n-D)!} S(\eta)^{n-D} \prod_{i=1}^{D} [f_1(t_i) S_2(t_i)]^{I(\rho_i=1)} [f_2(t_i) S_1(t_i)]^{I(\rho_i=2)} [f(t_i)]^{I(\rho_i=3)}, \tag{5}$$

where ρ_i is defined by (4) and $f(.)$ and $S(.)$ denoted to density and survival function of $T_i = \min\{T_{i1}, T_{i2}\}$ and

$$I(\rho_i = j) = \begin{cases} 1 & \rho_i = j, \\ 0 & else \end{cases} \tag{6}$$

Hence, under the consideration that the observed sample is of type-II GHSC competing risk sample drawn from exponential and two independent failure causes, say $T_{ik} \sim Exp(\theta_k)$, $k = 1, 2$. Also the random variable $T_i = \min\{T_{i1}, T_{i2}\}$, $i = 1, 2, \ldots, n$ are independent and distributed as exponential with scale parameter $\theta_1 + \theta_2$. Therefore, the likelihood function of θ_1 and θ_2 can be expressed by

$$L(\theta_1, \theta_2 \mid \mathbf{t}) \propto \theta_1^{m_1} \theta_2^{m_2} (\theta_1 + \theta_2)^{m_3} e^{-\eta(n-D)(\theta_1+\theta_2)} \prod_{i=1}^{D} e^{-(\theta_1+\theta_2)t_i}, \tag{7}$$

where $m_j = \sum_{i=1}^{D} I(\rho_i = j)$, $j = 1, 2, 3$ and η is given by (1).

Remark 2.1. *The proposed model has suggested that failure time is recorded for some units with unknown cause of failure is distributed with exponential with $\theta_1 + \theta_2$ as scale parameter.*

Remark 2.2. *The discrete random variable m_3 is distributed with a Bernoulli distribution with masking probability p, where $0 \leq p \leq 1$. Hence, the values 1 and 0 denote, respectively, to the failures with unknown and known cause of failure.*

Remark 2.3. *The discrete random variables m_1 and m_2 which describe the number of units fails under first and second causes of failure are distributed as binomial distributions with sample size $(D - m_3)$, and with probability of success $\frac{\theta_1}{\theta_1+\theta_2}$ and $\frac{\theta_2}{\theta_1+\theta_2}$, respectively.*

3. ML estimation

In this section, we present point estimation and asymptotic confidence intervals of the model parameters for given two independent causes of failure and observed sample of size D, **t**. The indicator causes of failure ρ_i, $i = 1, 2, \ldots, D$ are defined by (4). From the log-likelihood function, after taking the first derivative with respected to θ_1 and θ_2; and equating to zero, we have

$$\frac{\partial \ell(\theta_1, \theta_2 \mid \mathbf{t})}{\partial \theta_j} \quad = \quad \frac{m_j}{\theta_j} - (n-D)\eta + \frac{m_3}{\theta_1 + \theta_2} \sum_{i=1}^{D} t_i = 0, \quad j = 1, 2. \tag{8}$$

The ML estimates of θ_1 and θ_2 can not be obtained in closed form, then some numerical techniques need to be employed to solve the non-linear equation (8).

3.1. Interval estimation

In this subsection, the approximate confidence intervals (CIs) of the parameters are constructed using the asymptotic normality of MLE. The minus expectation of second derivatives of the log-likelihood function has defined the Fisher information matrix (FIM) with respect to model parameters. Generally, in more situations the expectation of the second derivative is more serious. Then, the observed FIM present a suitable approximation which can be used to construct interval estimation as follows.
The oserved information matrix F at the ML estimate of the vector parameter $\Theta = \{\theta_1, \theta_2\}$ define by $\hat{F}$. Now, under standard regularity conditions the asymptotic distribution theory of MLE tell us that $\hat{\Theta} = \{\hat{\theta}_1, \hat{\theta}_2\}$ can be distributed as

Table 1. Point and 95% interval ML estimate.

Parameters	MLE	95% CI	Length
$\theta_1 = 0.4$	0.3985	(0.2816,0.5154)	0.2338
$\theta_2 = 0.2$	0.2032	(0.1187,0.2877)	0.1690

a bivariate normal distribution with mean $\Theta = \{\theta_1, \theta_2\}$ and variance covariance matrix $F^{-1}(\hat{\Theta})$. Then, $100(1-\alpha)\%$ approximate interval estimate of $\Theta = \{\theta_1, \theta_2\}$ is given by

$$\hat{\theta}_1 \pm \gamma_{\alpha/2}\sqrt{\hat{F}_{11}}\ , \qquad \hat{\theta}_2 \pm \gamma_{\alpha/2}\sqrt{\hat{F}_{22}} \tag{9}$$

where $\gamma_{\alpha/2}$ present the standard normal values with probability tailed $\alpha/2$ and the values $\hat{F}_{11}$ and $\hat{F}_{22}$ are the diagonal of the matrix $F^{-1}(\hat{\Theta})$.
However, sometimes the lower bound of the confidence intervals (9) may be less than 0, which contradicts with the prerequisite $\theta_1, \theta_2 > 0$. Log-transformation and delta method are used in order to avoid that case.
Under normal property of the pivotal $\zeta = \frac{\log\hat{\Theta} - \log\Theta}{var(\log\hat{\Theta})}$ with mean 0 and variance 1, the $100(1-\alpha)\%$ approximate confidence interval of $\Theta = \{\theta_1, \theta_2\}$ is given by

$$\left(\frac{\hat{\theta}_j}{\exp\{\gamma_{\alpha/2}\sqrt{var(\log\hat{\theta}_j)}\}}, \hat{\theta}_j \exp\{\gamma_{\alpha/2}\sqrt{var(\log\hat{\theta}_j)}\}\right) \tag{10}$$

where $var(\log\hat{\theta}_j) = \frac{var(\hat{\theta}_j)}{\hat{\theta}_j}$ and $j = 1, 2$. For more details, see [6].

4. Simulation studies

This section devotes to carry out a simulation study to evaluate the performance of the proposed ML estimation method. We generate a random sample from exponential distribution with parameter $\theta_1 + \theta_2$ of size 50, where $\theta_1 = 0.4$ and $\theta_1 = 0.2$. Also, under the generalized type-II hybrid censoring competing risks with consideration $\tau_1 = 3$ and $\tau_2 = 7$ and $m = 15$. Under consideration that, $n = 50$, $m = 15$ and $D = 21$ with masking probability $p = 0.1$ generate a random sample of size 21 from Bernoulli distribution. The number of masking failure $m_3 = 7$. Then, generate m_1 and m_2 from a binomial distribution with probability of success $\frac{\theta_1}{\theta_1+\theta_2}$ and $\frac{\theta_2}{\theta_1+\theta_2}$, respectively from a sample of size $(D - m_3 = 14)$ to be $m_1 = 10$ and $m_2 = 4$.
The results of ML estimate are reported in Table 1 for point and 95% confidence interval estimates.

In addition, we measure the change of sample size n and affected sample size m as well as parameter vector. Also, we test the effect of the change of ideal test times (τ_1, τ_2). The values of the parameters are arbitrarily assigned. Different combinations of $(n, m, \tau_1, \tau_2, p)$ are considered. The simulation study is done with respect to 1000 simulated data sets. The tools that used to test the point estimate is the mean estimate (ME) with the associated mean squared error (MSE). But, the interval estimate test under the mean interval length (MIL) and probability coverage (PC). For The results of simulation study are reported in Table 2.

5. Conclusions

The problem of ”time-to-failure” under different causes of failure is commonly in reliability studying. On natural, causes of failure may be dependent, but in our modeling, we proposed the causes of failure are independent. In the model under consideration some of failure time is observed and its causes not observed, which is known as partially

Table 2. MEs, (MSEs) and MIEs, (CPs) for $\Theta = (1.0, 1.5)$

(τ_1, τ_2, p)	(n,m)	MLE		MSE		MIE		CP	
		θ_1	θ_2	θ_1	θ_2	θ_1	θ_2	θ_1	θ_2
(2.5,5.5,0.3)	(30,15)	1.325	1.725	(0.284)	(0.512)	2.547	3.412	(0.87)	(0.89)
	(30,25)	1.285	1.624	(0.235)	(0.4561)	2.518	3.385	(0.90)	(0.89)
	(50,25)	1.276	1.615	(0.226)	(0.4553)	2.509	3.377	(0.91)	(0.91)
	(50,40)	1.245	1.590	(0.180)	(0.350)	2.450	3.324	(0.92)	(0.90)
	(70,50)	1.201	1.549	(0.145)	(0.309)	2.438	3.307	(0.93)	(0.92)
(2.5,5.5,0.1)	(30,15)	1.308	1.704	(0.259)	(0.500)	2.541	3.404	(0.89)	(0.89)
	(30,25)	1.275	1.625	(0.226)	(0.4559)	2.504	3.371	(0.91)	(0.90)
	(50,25)	1.271	1.607	(0.218)	(0.4541)	2.501	3.364	(0.90)	(0.92)
	(50,40)	1.238	1.584	(0.168)	(0.337)	2.438	3.311	(0.91)	(0.93)
	(70,50)	1.197	1.544	(0.129)	(0.292)	2.424	3.291	(0.92)	(0.91)
(3.5,5.5,0.1)	(30,15)	1.312	1.698	(0.238)	(0.483)	2.525	3.382	(0.90)	(0.89)
	(30,25)	1.271	1.618	(0.211)	(0.442)	2.481	3.348	(0.92)	(0.89)
	(50,25)	1.269	1.603	(0.201)	(0.442)	2.484	3.351	(0.92)	(0.91)
	(50,40)	1.229	1.571	(0.149)	(0.328)	2.421	3.300	(0.92)	(0.93)
	(70,50)	1.191	1.537	(0.114)	(0.276)	2.403	3.275	(0.93)	(0.92)

observed causes of failure. This model is built under exponential lifetime distribution with type-II generalized hybrid censoring scheme. Classical approach is discussed. From the simulation results, we have reported some points described as follows:

- The proposed model under type-II generalized hybrid censoring scheme for competing risks model serve well for all choice of censoring schemes.
- The increasing of the affect sample size m is reducing the MSE and MIL.
- The small value of masking probability serves well than the large values.
- The result is better for increasing the minimum ideal test time τ_1.

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A note on the new aging notions

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Abstract

The problem of deriving bounds on distribution functions has been studied for a long time in the literature. This paper continue this line of research by implementing the new aging notion. In fact, we define the weighted new better (worse) than used in expectation and provide some results on it.

1. Introduction

The problem of deriving inequalities on distribution functions has been studied for a long time in the literature. These inequalities are useful in life testing and reliability modeling. Moreover, they may be also used to produce distribution-free bounds on reliability when certain information about the underlying life distribution is available. Another useful application of them is to characterize the behavior of certain classes of life distributions in terms of their hazard rates and lower order moments in the reliability theory. Since Markov's fundamental inequality, a number of improvements have been obtained under additional assumptions on the underlying distribution function. It states that for a nonnegative random variable with mean $\mu = \mathbb{E}(X)$, and survival probability $\overline{F}(t) = P(X > t)$ it holds that $\overline{F}(t) \leq \mu/t$ for $t > \mu$ and $\overline{F}(t) \leq 1$ for $0 < t \leq \mu$. Since then this inequality is improved in the literature. For instance, Haines and Singpurwalla [1] and Marshall and Proschan [3] have obtained improvements of comparable simplicity by restricting the distributions to new better than used in expectation (NBUE) new worse than used in expectation (NWUE), respectively. The aim of this paper is to introduce a generalization of these definitions and obtain some new results.

2. Main results

In this section, we introduce with some general theorems and aging notions that are later applied to obtain more specific results. Let X be the lifetime of a system or a component with the CDF $F(x)$, probability density function (PDF) $f(x)$ and survival function $\overline{F}(x) = 1 - F(x)$. Then, the residual or excess of X, given that it exceeds a threshold t, is denoted by $X_t = [X - t \mid X > t]$, where as usual $[X|B]$ denotes a random variable having the same distribution of X conditioned on B. The survival function of X_t is given as $\overline{F}_t(x) = \overline{F}(x+t)/\overline{F}(t),\ x, t > 0$, and so the PDF is

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$f_t(x) = f(x+t)/\overline{F}(t)$ for $x, t > 0$. Moreover, the *mean residual life* (MRL) function of X with finite mean $\mu = \mathbb{E}[X]$ is defined as

$$m(t) = \mathbb{E}[X - t | X > t], \quad t > 0. \tag{1}$$

We say that X is increasing (decreasing) MRL, denoted as IMRL(DMRL), if $m(t)$ is increasing (decreasing) in $t > 0$. Another ageing class of life distributions is defined in the follows definition. Aging notions and stochastic orders, as discussed by Shaked and Shanthikumar [5], have found important uses in many disciplines. We now introduce briefly some definitions of ageing concept and stochastic orders, which are most pertinent for the developments here. Throughout this paper, the terms 'increasing' and 'decreasing' are used in a non-strict sense.

Definition 2.1. Let X be a nonnegative random lifetime with survival function $\overline{F}(x)$, mean $\mu = \mathbb{E}[X] < \infty$, and MRL function $m(t)$. Then X is new better (worse) than used in expectation NBUE(NWUE) if $m(t) \leq (\geq)\mu$ for $t > 0$.

Now, we provide a generalization of the above definition. To this aim, let us assume an increasing nonnegative and differentiable function $\psi(x)$ such that $\psi'(x) = \phi(x) \geq 0$. Additionally, if the weight function $\phi(x)$ is increasing (decreasing) in $x > 0$, then $\psi(x)$ is convex (concave). We recall the weighted mean residual life (WMRL) function defined by

$$m_\psi(t) \quad = \quad \mathbb{E}[\psi(X) - \psi(t) | X > t] = \frac{1}{\overline{F}(t)} \int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x, \tag{2}$$

for all $t \geq 0$ such that $\overline{F}(t) > 0$; see Toomaj and Di Crescenzo [6]. It is clear that $m_\psi(0) = \mathbb{E}[\psi(X)]$. It is worth noting that this function appears in the hazard rate function of the weighted distribution see Eq. (2.1) of Nanda and Jain [4]. As pointed out by Toomaj and Di Crescenzo [6], we recall that X has increasing (decreasing) WMRL, denoted as WIMRL(WDMRL), if $m_\psi(t)$ is increasing (decreasing) in $t > 0$. Now, we define the following definition.

Definition 2.2. Let X be a nonnegative random variable with WMRL $m_\psi(t)$ such that $m_\psi(0) = \mathbb{E}[\psi(X)] < \infty$. We say that X is weighted new better (worse) than used in expectation WNBUE(WNWUE) if $m_\psi(t) \leq (\geq)m_\psi(0)$ for $t > 0$.

In particular, when $\psi(t) = t$, and hence $\phi(t) = 1$, then WNBUE(WNWUE) reduces to NBUE(NWUE) property. We immediately have the following theorem.

Theorem 2.3. *Under the conditions of Definition 2.2, if X is DWMRL(IWMRL), then X is WNBUE(WNWUE).*

Proof. Since X is DWMRL(IWMRL) by the assumption, then $m_\psi(t)$ is decreasing(increasing) in $t > 0$ and hence this completes the theorem.

□

Recently, Toomaj and Di Crescenzo [6] showed that the variance of a transformed random variable X can be represented in terms of WMRL function as follows:

$$\sigma^2_\psi(X) = Var(\psi(X)) = \mathbb{E}[m^2_\psi(X)],$$

provided that the expectation exists. In this case, if X is WNBUE(WNWUE), then we have

$$\sigma^2_\psi(X) \leq (\geq) m^2_\psi(0) = \mathbb{E}^2[\psi(X)],$$

or equivalently

$$\gamma_\psi(X) = \frac{\sigma_\psi(X)}{\mathbb{E}[\psi(X)]} \leq (\geq) 1.$$

The new definition will allows us to find several characterization and bounds for the survival function. Consider the following example.

Example 2.4. Let us assume a series system with lifetime $X_{1:n} = \min\{X_1, \ldots, X_n\}$ consisting of n independent and identically distributed absolutely continuous non-negative random variables $X_1, \ldots, X_n$ having the common CDF $F(x)$ and PDF $f(x)$. Denote by $\overline{F}_{1:n}(x) = [\overline{F}(x)]^n$, $x \geq 0$, the survival function of $X_{1:n}$. Hence, by setting $\psi(t) = F(t)$, and thus $\phi(t) = f(t)$, from (2) we obtain, for $t > 0$,

$$m_{F(X_{1:n})}(t) = \frac{1}{\overline{F}_{1:n}(t)} \int_t^\infty f(x)\overline{F}_{1:n}(x)\,\mathrm{d}x = \frac{1}{[\overline{F}(t)]^n} \int_t^\infty f(x)[\overline{F}(x)]^n\,\mathrm{d}x = \frac{\overline{F}(t)}{n+1}.$$

It is well-known that $F(X_{1:n}) \sim Beta(1, n)$ and so $\mu_\psi = \mathbb{E}[F(X_{1:n})] = 1/(n+1)$. It is clear that $m_{F(X_{1:n})}(t) \leq m_{F(X_{1:n})}(0)$ for all $t > 0$, which means that $X_{1:n}$ is WNBUE. Since

$$Var(F(X_{1:n})) = \frac{n}{(n+1)^2(n+2)},$$

so we have

$$\gamma(F(X_{1:n})) = \frac{\sqrt{Var(F(X_{1:n}))}}{\mathbb{E}[F(X_{1:n})]} = \sqrt{\frac{n}{n+2}} \leq 1.$$

It is not an easy task to prove the aging properties of WMRL directly in some case. In the following theorem, we provide sufficient conditions for the monotonicity of $m_\psi(t)$.

Theorem 2.5. *Let X be a nonnegative random variable with the hazard rate function $\lambda(x)$. If any of the following conditions hold:*

(i) *If $\phi(x)/\lambda(x)$ is decreasing (increasing) in $x \geq 0$;*

(ii) *If $\phi(x)$ is decreasing (increasing) in x, and if X is DMRL(IMRL);*

(iii) *If X is NBU(NWU) and $\phi(x)$ is decreasing (increasing) in $x > 0$;*

then X is WNBUE (WNWUE).

Proof. Under the conditions (i) and (ii), Theorems 1 and 2 of Toomaj and Di Cresenzo [6] implies that X is DWMRL(IWMRL) and hence we have the result due to Theorem 2.3. For the case (iii), from (2), we have

$$\begin{aligned} m_\psi(t) &= \int_0^\infty \phi(x+t)\frac{\overline{F}(x+t)}{\overline{F}(t)}\,\mathrm{d}x, \\ &\leq (\geq) \int_0^\infty \phi(x+t)\overline{F}(x)\,\mathrm{d}x, \\ &\leq (\geq) \int_0^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x = m_\psi(0),\ t > 0. \end{aligned}$$

The first inequality is obtained by noting that X is NBU(NWU) while the last inequality is obtained by the fact that $\phi(x)$ is decreasing (increasing) in $x > 0$. □

Another useful result is given in the next theorem. First, we recall the concept of total positivity which is applied to demonstrate monotonicity results in the reminder of this paper. For two subsets of the real line A and B, a non-negative function $K(x, y)$ defined on $A \times B$ is said to be *totally positive of order 2 (regular of order 2)*, denoted by TP_2 (RR_2), if $K(x_1, y_1)K(x_2, y_2) \geq (\leq)\, K(x_1, y_2)K(x_2, y_1)$, for all $x_1 \leq x_2$ in A and $y_1 \leq y_2$ in B. For further details we refer to Karlin [2].

Theorem 2.6. *Let X be a nonnegative random variable with the hazard rate function $\lambda(x)$. If $\psi(x)\lambda(x)/\phi(x)$ is decreasing in x, then X is WNWUE.*

Proof. We first prove that X is IWMRL. Since $\psi(t) \geq 0$ is increasing in t, from

$$m_\psi(t) = \psi(t)\frac{m_\psi(t)}{\psi(t)},$$

it is sufficient to prove that the following function is increasing in $t > 0$:

$$\frac{m_\psi(t)}{\psi(t)} = \frac{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x}{\psi(t)\overline{F}(x)} = \frac{\int_t^\infty \phi(x)\overline{F}(x)\,\mathrm{d}x}{\int_t^\infty [\psi(x)f(x) + \phi(x)\overline{F}(x)]\,\mathrm{d}x}. \tag{3}$$

Define now

$$\Psi(i,t) := \int_0^\infty \nu(i,x)\eta(x,t)\,\mathrm{d}x, \qquad i = 1, 2,$$

where

$$\nu(i,x) = \begin{cases} \psi(x)f(x) + \phi(x)\overline{F}(x), & i = 1 \\ \phi(x)\overline{F}(x), & i = 2, \end{cases} \qquad \text{and} \qquad \eta(x,t) = \mathbf{1}[x > t],$$

with $\mathbf{1}[\cdot]$ the indicator function, i.e. $\mathbf{1}[\pi] = 1$ when π is true, and $\mathbf{1}[\pi] = 0$ otherwise. Due to the assumption, $\nu(i,x)$ is TP_2 in $(i,x) \in \{1,2\} \times (0,\infty)$. On the other hand, it is easy to observe that $\eta(x,t)$ is TP_2 in $(x,t) \in (0,\infty)^2$. From the general composition theorem of Karlin [2], it follows that $\Psi(i,t)$ is TP_2 in $(i,t) \in \{1,2\} \times (0,\infty)$. This implies that $m_\psi(t)/\psi(t)$ is increasing in t and hence X is IWMRL. So, Theorem 2.3 gives the desired result. □

3. Concluding remarks

In this note, we provide a generalization of NBUE(NWUE) of definition and obtained some results on it. Moreover, we provided an application of the new definition.

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Composite Inverse Weibull-Generated Family and Pareto II Distribution

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Abstract

In this paper, a lifetime model is introduced and studied. We have named this model inverse Weibull-Pareto II distribution. First, we define the inverse Weibull-generated family of distributions, and then introduce inverse Weibull-Pareto II distribution using inverse Weibull-generated family and The Pareto II distribution. We find the reliability, hazard function, moment, quantile function, Rényi entropy, and order statistics for the inverse Weibull-Pareto II distribution. The maximum likelihood, least squares, and weighted least squares estimators are obtained. The bias and mean square error of the unknown parameter estimators are examined for the simulated data.

1. Introduction

The inverse Weibull (IW) distribution is proper for statistical analysis of lifetime in reliability engineering research. This distribution has been employed to an extended range of sites, including utilization in medication, reliability, and ecology. Erto and Ropone [4] represented that IW distribution grants goodness of fit to several data sets. Khan et al. [8] displayed the flexibility of the IW distribution and absorbing properties. According to Murty et al. [10], degradation phenomena of mechanical components such as dynamic components of the diesel engine can be effectively based on IW distribution. Calabria and pulcini [3] supplied exposition of IW distribution in the context of load strength relationship for a component. Shafiei et al. [12] displayed that IW distribution is a suitable model for hazard function unimodal and indicated it's as one of the typical distributions in supplementary risk matters.
The Pareto II (the Lomax) distribution was introduced to model business failure data by Lomax [9]. This distribution has gained wide usage in a diversity of fields such as income and wealth inequality, actuarial science, medical and biological sciences, engineering, lifetime, and reliability modeling.
The probability distribution function (PDF) and the cumulative distribution function (CDF) of the Pareto II distribution with two parameters, β (shape parameter) and θ (scale parameter) is as follows, respectively.

$$\begin{aligned} &f(x;\beta,\theta)=\beta\theta\left(1+\theta x\right)^{-(\beta+1)};x,\beta,\theta>0,\\ &F(x;\beta,\theta)=1-(1+\theta x)^{-\beta} \end{aligned} \tag{1}$$

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The r[th] raw moment and the quantile function derivatives are, respectively.

$$\mu_r^{'} = \frac{\gamma(\beta - r)\gamma(1+r)}{\theta^r \gamma(\beta)}; \beta > r, r = 1, 2, \ldots, Q\left(p; \beta, \theta\right) = \frac{1}{\theta}\left[(1-p)^{-\frac{1}{\beta}} - 1\right] \tag{2}$$

Among the statisticians who have presented extended new families of distributions by utilizing several techniques are Eugene et al. [5], Jones [7], Alzaatreh et al. [1], and Bourguignon et al. [2]. Bourguignon et al. [2] introduced the Weibull- generated family of distributions. Therefore, the CDF of the inverse Weibull- generated family of distributions can be defined as follows.

$$F\left(x; \lambda, \alpha, \psi\right) = \int_0^{\frac{G(x;\psi)}{\overline{G}(x;\psi)}} \alpha\lambda t^{-(\alpha+1)} e^{-\lambda t^{-\alpha}} dt = EXP\left[-\lambda \left(\frac{G\left(x;\psi\right)}{\overline{G}\left(x;\psi\right)}\right)^{-\alpha}\right]; x, \lambda, \alpha > 0, \tag{3}$$

where, λ and α are scale and shape parameters, respectively, and G (x; ψ) is the baseline CDF with parameter vector ψ and $\overline{G}\left(x;\psi\right) = 1 - \mathrm{G}\left(\mathrm{x};\psi\right)$.

2. The Inverse Weibull - Pareto II Distribution

Considering the CDF of the Pareto II in (1) as the baseline CDF in (3), the CDF of the inverse Weibull - Pareto II distribution (IWPD) is determined by

$$F\left(x;\phi\right) = EXP[-\lambda[(1+\theta x)^{\beta} - 1]^{-\alpha}]; x, \lambda, \alpha, \beta, \theta > 0 \tag{4}$$

where (λ, Θ) and (α, β) are scale and shape parameters, respectively and $\phi = (\lambda, \theta, \alpha, \beta)$ is set of parameters. The PDF of IWPD is obtained as follows.

$$f\left(x;\phi\right) = \alpha\beta\lambda\theta\left(1+\theta x\right)^{(\beta-1)}\left[(1+\theta x)^{\beta} - 1\right]^{-(\alpha+1)} EXP\left[-\lambda[(1+\theta x)^{\beta} - 1]^{-\alpha}\right]; x, \lambda, \alpha, \beta, \theta > 0. \tag{5}$$

By applying the Taylor's series expansion of the exponential function and the binomial expansion as follows,

$$e^u = \sum\nolimits_{i=0}^{\infty} \frac{u^i}{i!}, (1-w)^{-k} = \sum\nolimits_{i=0}^{\infty} \frac{\gamma(k+i)}{\gamma(k)i!} w^i, \tag{6}$$

then, the PDF in (5) can be expanded as follows,

$$f\left(x;\phi\right) = \alpha\beta\theta \sum\nolimits_{i,j=0}^{\infty} \frac{(-1)^i \lambda^{i+1} \gamma(\alpha + \alpha i + j + 1)}{i! j! \gamma(\alpha + \alpha i + 1)} \left(1+\theta x\right)^{-(\beta(\alpha+\alpha i+j)+1)}. \tag{7}$$

It can be summarized as follows,

$$f\left(x;\phi\right) = \sum\nolimits_{i,j=0}^{\infty} h_{i,j} l\left(x; \beta\left(\alpha + \alpha i + j\right), \theta\right), \tag{8}$$

where $h_{i,j} = \frac{\alpha(-1)^i \lambda^{i+1} \gamma(\alpha+\alpha i+j)}{i! j! \gamma(\alpha+\alpha i+1)}$ and $l\left(x; \beta\left(\alpha + \alpha i + j\right), \theta\right)$ indicates the PDF of the Pareto II distribution with two parameters $\beta\left(\alpha + \alpha i + j\right)$ and θ. The plots of the IWPD PDF for several parameter values are shown in Figure 1. in Appendix.

3. PROPERTIES OF THE IWPD

3.1. Reliability

The reliability (or survival) function and the hazard function related to IWPD are exposed through,

$$R\left(x;\phi\right) = 1 - EXP[-\lambda[(1+\theta x)^{\beta} - 1]^{-\alpha}], h\left(x;\phi\right) = \frac{\alpha\beta\lambda\theta\left(1+\theta x\right)^{(\beta-1)}\left[(1+\theta x)^{\beta} - 1\right]^{-(\alpha+1)}}{EXP\left[\lambda\left[(1+\theta x)^{\beta} - 1\right]^{-\alpha}\right] - 1}$$

The plots of the hazard function for different parameter values are shown in Figure 2. in Appendix.

3.2. Moments

The r[th] raw moment for IWPD by using (2) and (8) becomes
$\mu_r^{'}=\sum_{i,j=0}^{\infty}h_{i,j}\frac{\gamma(\beta(\alpha+\alpha i+j)-r)\gamma(1+r)}{\theta^r\gamma(\beta(\alpha+\alpha i+j))};\beta\left(\alpha+\alpha i+j\right)>r,r=1,2,\ldots.$
Thus, $E\left(X\right)=\mu_1^{'}=\sum_{i,j=0}^{\infty}h_{i,j}\frac{1}{\theta(\beta(\alpha+\alpha i+j)-1)}.$
The r[th] central moment for IWPD becomes

$$\mu_r=E(X-E(X))^r=\sum\nolimits_{i=0}^{r}(-1)^{r-i}\mu_i^{'}\left(\mu_1^{'}\right)^{r-i}. \tag{9}$$

The moment measures of skewness and kurtosis can be calculated Using (9).

3.3. Quantile function

Using CDF of IWPD (4), the quantile function and hence median and interquartile range (IQR) for IWPD can be expanded as follows
$Q\left(p;\phi\right)=\frac{1}{\theta}\left[\left(1+\left(\frac{-\lambda}{\log(p)}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\beta}}-1\right], Median=\frac{1}{\theta}\left[\left(1+\left(1.4427\lambda\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\beta}}-1\right],$
$IQR=\frac{1}{\theta}\left[\left(1+\left(3.4761\lambda\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\beta}}-1\right]-\frac{1}{\theta}\left[\left(1+\left(0.72135\lambda\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\beta}}-1\right].$

3.4. Rényi entropy

The entropy is a tool for measuring the amount of uncertainty contained in a random observation (Rény, [11]). The concept of entropy is applied in various statuses in engineering, physics, statistical mechanics, and other applied sciences. The Rényi entropy of the IWPD is determined by

$$\begin{aligned}I_R\left(\gamma\right)&=\frac{1}{1-\gamma}\ln\int_0^{\infty}f^{\gamma}\left(x\right)dx\\&=\frac{1}{1-\gamma}\ln\int_0^{\infty}\left(\alpha\beta\lambda\theta\right)^{\gamma}\left(1+\theta x\right)^{\gamma(\beta-1)}\left[\left(1+\theta x\right)^{\beta}-1\right]^{-\gamma(\alpha+1)}EXP\left[-\gamma\lambda\left[\left(1+\theta x\right)^{\beta}-1\right]^{-\alpha}\right]dx.\end{aligned}$$

It can be summarized using (6) as follows

$$I_R\left(\gamma\right)=\frac{1}{1-\gamma}\ln\left[\left(\alpha\beta\lambda\right)^{\gamma}\theta^{\gamma-1}\sum\nolimits_{i,j=0}^{\infty}\frac{\left(-1\right)^{i}\left(\gamma\lambda\right)^{i}\gamma(\gamma\left(\alpha+1\right)+\alpha i+j)}{i!j!\gamma(\gamma\left(\alpha+1\right)+\alpha i)(\gamma\left(1+\alpha\beta\right)+\beta\left(\alpha\mathrm{i}+\mathrm{j}\right)-1)}\right].$$

3.5. Order statistics

The PDF of the r^{th} order statistics for a random sample $X_1,X_2,\ldots,X_n$ from the IWPD using (4), (5), and (6) is derived by

$$f_{(r)}\left(x\right)=r\left(\begin{array}{c}n\\r\end{array}\right)\alpha\beta\lambda\theta\left(1+\theta x\right)^{(\beta-1)}\sum\nolimits_{i=0}^{n-r}\sum\nolimits_{j=0}^{\infty}\left(\begin{array}{c}n-r\\i\end{array}\right)\frac{\left(-1\right)^{i+j}}{j!}(\lambda(r+i))^j[(1+\theta x)^{\beta}-1]^{-(\alpha+\alpha j+1)}.$$

After simplification, it is obtained as

$$f_{(r)}\left(x\right)=r\left(\begin{array}{c}n\\r\end{array}\right)\alpha\beta\lambda\theta\sum\nolimits_{i=0}^{n-r}\sum\nolimits_{j,k=0}^{\infty}\left(\begin{array}{c}n-r\\i\end{array}\right)\frac{\left(-1\right)^{i+j}\gamma(\alpha+\alpha j+k+1)}{j!k!\gamma(\alpha+\alpha j+1)}(\lambda(r+i))^j\left(1+\theta x\right)^{-\beta(\alpha+\alpha j)+k-1}.$$

4. Estimation of parameters

4.1. The maximum likelihood (ML) estimation

Suppose $\mathrm{X_1}, \mathrm{X_2}, \ldots, \mathrm{X_n}$ be a random sample of size n from IWPD (5). The log-likelihood function is obtained as follows

$$l(\phi) = n\ln(\alpha\beta\lambda\theta) + (\beta-1)\sum_{i=0}^{n}\ln(1+\theta x_i) - (\alpha+1)\sum_{i=0}^{n}\ln\left[(1+\theta x_i)^{\beta}-1\right] - \sum_{i=0}^{n}\lambda\left[(1+\theta x_i)^{\beta}-1\right]^{-\alpha}.$$

The ML estimators of $\alpha, \beta, \lambda, and\theta$ are obtained by dissolving the nonlinear equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n}\ln\left[(1+\theta x_i)^{\beta}-1\right] + \sum_{i=0}^{n}\lambda\left[(1+\theta x_i)^{\beta}-1\right]^{-\alpha}\ln\left[\lambda\left((1+\theta x_i)^{\beta}-1\right)\right] = 0,$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=0}^{n}\ln(1+\theta x_i) - (\alpha+1)\sum_{i=0}^{n}\frac{(1+\theta x_i)^{\beta}\ln(1+\theta x_i)}{(1+\theta x_i)^{\beta}-1} + \sum_{i=0}^{n}\lambda\alpha\left[(1+\theta x_i)^{\beta}-1\right]^{-(\alpha+1)}(1+\theta x_i)^{\beta}\ln(1+\theta x_i) = 0,$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=0}^{n}\left[(1+\theta x_i)^{\beta}-1\right]^{-\alpha} = 0,$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + (\beta-1)\sum_{i=0}^{n}\frac{x_i}{(1+\theta x_i)} - (\alpha+1)\sum_{i=0}^{n}\frac{\beta(1+\theta x_i)^{\beta-1}x_i}{(1+\theta x_i)^{\beta}-1} + \lambda\alpha\sum_{i=0}^{n}\left[(1+\theta x_i)^{\beta}-1\right]^{-(\alpha+1)}\beta(1+\theta x_i)^{\beta-1}x_i = 0.$$

These equations can be solved numerically like Newton-Raphson.

4.2. The Least Squares and Weighted Least Squares estimation

In this section, the Least Squares (LS) and Weighted Least Squares (WLS) estimators of the unknown parameters for IWPD based on the method proposed by Swain et al. [13] are obtained. let $\mathrm{X_1}, \mathrm{X_2}, \ldots, \mathrm{X_n}$ be a random sample of size n from a CDF and $\mathrm{X_{(1)}} \leq \mathrm{X_{(2)}} \leq \ldots \leq \mathrm{X_{(n)}}$ be its order statistics, can be written (Johnson et al. [6])
$E\left(F\left(\mathrm{X_{(i)}}\right)\right) = \frac{i}{n+1}, Var\left(F\left(\mathrm{X_{(i)}}\right)\right) = \frac{i(n+1-i)}{(n+1)^2(n+2)}$.
By minimizing $\sum_{i=1}^{n}[F\left(X_{(\mathrm{i})}\right) - E\left(F\left(X_{(\mathrm{i})}\right)\right)]^2$, the LS estimators of the unknown parameters is obtained.
Then, minimize $\sum_{i=1}^{n}\left[EXP\left(-\lambda\left[\left(1+\theta x_{(\mathrm{i})}\right)^{\beta}-1\right]^{-\alpha}\right) - \frac{i}{n+1}\right]^2$, respect to $\alpha, \beta, \lambda, \theta$.
By minimizing $\sum_{i=1}^{n}v_i[F\left(X_{(\mathrm{i})}\right) - E\left(F\left(X_{(\mathrm{i})}\right)\right)]^2$ where $v_i = [Var\left(F\left(X_{(\mathrm{i})}\right)\right)]^{-1}$, the WLS estimators of the unknown parameters is obtained.

5. Simulation Study

To compare the three estimation methods, we simulated random samples using software R. The conclusions have been truncated after four decimal places. The means squared error (MSE) and the average bias are calculated in the ML, LS, and WLS estimators for each set of sample size and parameters. We provide the results of simulation study for $(\alpha, \beta, \lambda, \theta) = \{(1.3, 3.2, 1.1, 2.1), (1.2, 3.8, 1.2, 1.7)\}$ with using 1000 iterations per sample size, n $= 50, 100, 150$. Table1 display the simulation results. It shows that MSEs decrease while sample size increases, and MSEs of the ML method are often lower than the other two methods.
Table1. The average bias and MSEs for ($\widehat{\alpha}$,$\hat{\beta}$, $\hat{\lambda}$, $\hat{\theta}$) using ML, LS, and WLS estimators.

6. Conclusion

In this paper, we introduce a four-parameter distribution called the inverse Weibull - Pareto II distribution. Several statistical properties of this distribution such as reliability, hazard function, moment, quantile function, Rényi entropy, and order statistics were discussed. Estimation of parameters of the distribution is derived through maximum likelihood, least squares, and weighted least squares methods.
The simulation results display that estimation performance is acceptable.

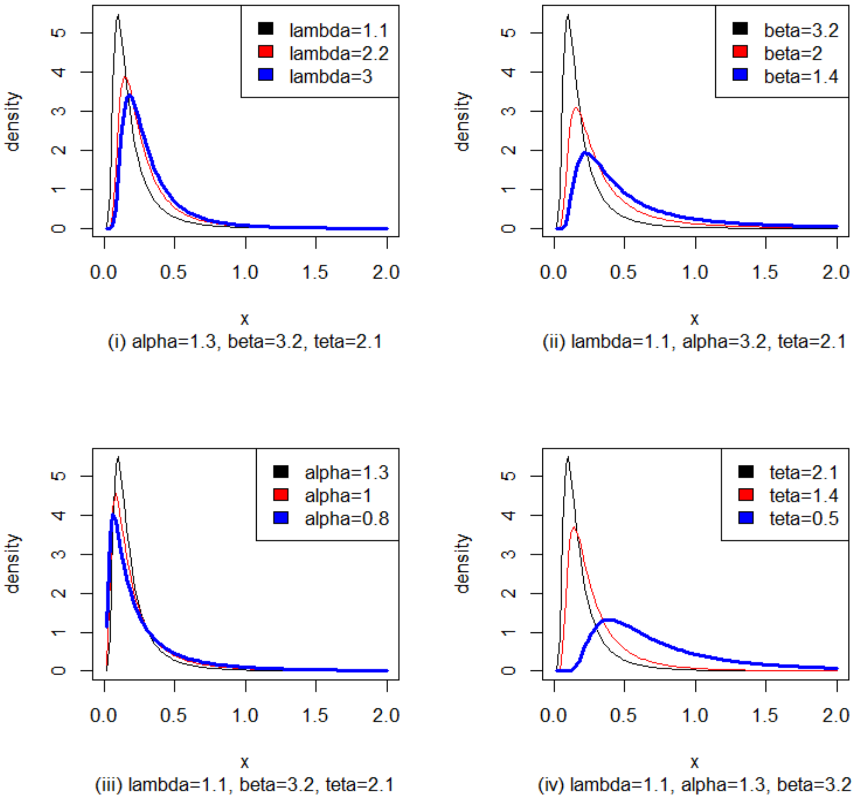

Fig. 1. The plots of the IWPD pdf for different parameter values.

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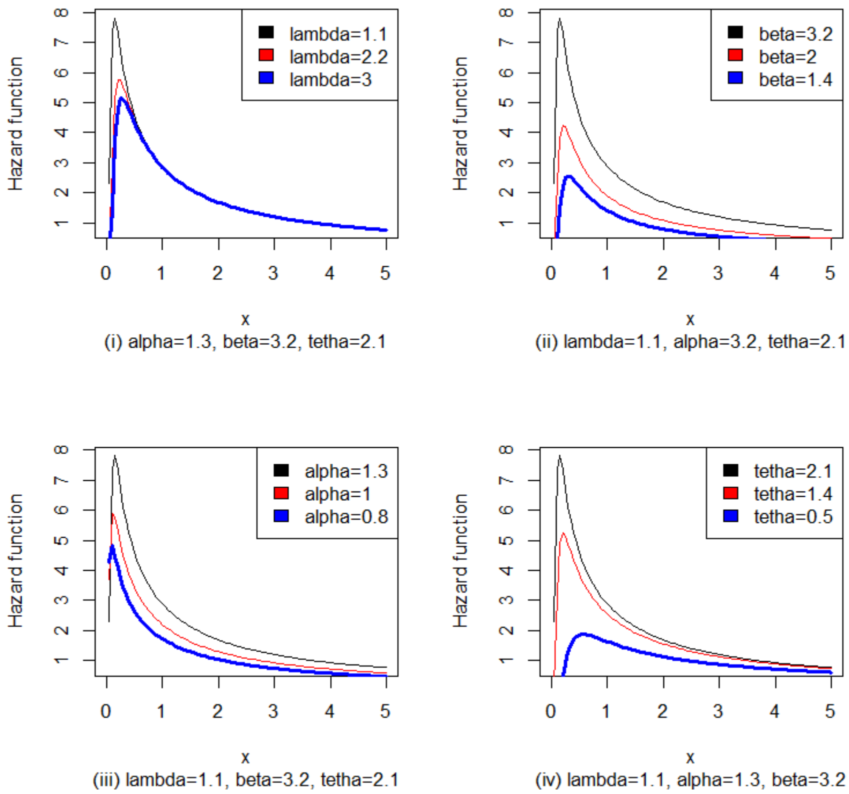

Fig. 2. The plots of the hazard function related to IWPD for different parameter values.

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Inference on stress-strength reliability for the generalized exponential distribution based on hybrid censoring scheme

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Abstract

Stress-strength reliability is a measure to compare the lifetimes of two systems. One of the most important subjects in the field of life testing is the stress-strength reliability, which always refers to the quantity $P(Y < X)$ in any statistical literature. It resamples a system with random strength (X) that is subjected to a random strength (Y) such that a system fails in case the stress exceeds the strength. In this study, we consider stress-strength reliability where the strength (X) and stress (Y) follow generalized exponential distribution. The maximum likelihood estimator of stress-strength reliability and its asymptotic distribution, along with a asymptotic confidence interval and two parametric bootstrap confidence intervals are obtained. A simulation study is performed to compare different estimation procedures and various hybrid censoring schemes. The simulation studies show that asymptotic confidence intervals provide more accurate average lengths of confidence intervals and higher probabilities to contain the true value of the parameter.

1. Introduction

In reliability theory, inferences of stress-strength reliability $R = P(Y < X)$, where, X and Y have independent distributions, is a general problem of interest. For example, in mechanical reliability of a system, if X is the strength of a component which is subject to stress Y, then R is a measure of system performance. The system fails, if at any time the applied stress exceeds than its strength. The model of stress-strength has found applications in many statistical problems, including quality control, engineering statistics, medical statistics and biostatistics, among others [6]. The problem of the estimation of stress-strength model has received considerable attention in the statistical literature. In connection of classical Mann-Whitney statistic, Birnbaum [3] introduced stress-strength model. Since then, a lot of work has been done on the estimation of stress-strength model for different distributions from the both frequentist and Bayesian approaches in complete sample case. An excellent monograph by Kotz et al. [8] provides a comprehensive treatment of different stress-strength models. Some recent works on stress-strength model can be found in Kundu and Gupta [11], Rezaei et al. [14], Babayi et al. [2], Sharma [15], etc.
Most of the inferences for stress-strength model have been carried out under complete sample case and very little work has been done based on censored data. Specially, stress-strength model is unexplored based on hybrid censored

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data. For example, Lio and Tsai [12] studied estimation of stress-strength parameter for Burr XII distribution based on progressively first failure censored samples, Kumar et al. [10] discussed estimation of the stress-strength parameter for Lindley distribution using progressively first failure censoring. Asgharzadeh et al. [1] studied estimation for Weibull distribution based on hybrid censored samples.
In life testing experiments experimenter often does not have complete control on the experiment in hand and items put on test are often lost or removed from the experiment before the completion of the experiment. In this case available data are censored. In literature the most common censoring schemes are Type-I and Type-II censoring schemes which are popularly used in life testing experiments. In Type-I censoring scheme the experiment is terminated after a pre-fixed time and in Type-II censoring scheme experiment is terminated after getting a pre-specified number of failures. A new censoring scheme was introduced by Epstein [5] which is the mixture of Type-I and Type-II censoring schemes and called it hybrid censoring scheme. In recent years, the hybrid censoring scheme has received considerable attention in the reliability theory and life testing experiments.
The hybrid censoring scheme can be described as follows: Let n identical units are put on life testing experiment and their lifetimes are assumed to be independently and identically distributed (i.i.d.) random variables with probability density function (pdf) $f(X)$ and cumulative distribution function (cdf) $F(x)$.
Let $X_{1:n} < X_{2:n} < \cdots < X_{r:n} < \cdots < X_{n:n}$ denote the ordered lifetimes of the experimental units. The test is terminated when a pre-specified number r out of n units have failed or a pre-specified time T has been reached. It is also assumed that the failed items are not replaced. In hybrid censoring scheme, the experiment is terminated at $\min\{X_{r:n}, T\}$. Thus, under hybrid censoring scheme available data may be in one of the following forms:

- **Case I:** $x_{1:n}, x_{2:n}, \ldots, x_{r:n}$, if $\;x_{r:n} \le T$;
- **Case II:** $x_{1:n}, x_{2:n}, \ldots, x_{m:n}$, if $\;0 \le m < r,\; x_{m:n} < T < x_{m+1:n}$,

where, m denotes the number of observed failures that occur before the time point T. Note that $x_{m+1:n}, x_{m+2:n}, \ldots, x_{r:n}$ are not observed in case II. On combining both of the cases, the likelihood function for hybrid censored sample, is given by

$$L(x_{1:n}, x_{2:n}, \ldots, x_{d:n}) = A \prod_{i=1}^{d} f(x_{i:n}) \big(1 - F(c)\big)^{n-d}, \tag{1}$$

where, $A = n(n-1)(n-2)\ldots(n-d+1)$, $c = \min\{x_{r:n}, T\}$ and $d = \sum_{i=1}^{r} I\{x_{i:n} \le c\}$, here I is an indicator function.
The cdf of the generalized exponential (GE) distribution, is given by

$$F(x) = (1 - e^{-\lambda/x})^{\alpha}, \tag{2}$$

where, $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. The GE distribution has increasing or decreasing hazard rate depending on the shape parameter. The two-parameter GE distribution is a particular member of the three-parameter exponentiated Weibull distribution, introduced by Mudholkar and Srivastava [13].
This article considers the problem of point and interval estimation of the stressstrength reliability $R = P(Y < X)$ under the assumption that X and Y both are independent GE random variables based on hybrid censored data. Let $X \sim GE(\alpha, \lambda)$ and $Y \sim GE(\beta, \lambda)$ be independent random variables, the stress-strength reliability is given by

$$P(Y < X) = \int_0^{\infty} F_Y(x) f_X(x) dx = \frac{\beta}{\alpha + \beta}. \tag{3}$$

Proposition 1.1. *(i) R is independent of λ, and (ii) when $\alpha = \beta$, $R = 0.5$, i.e.,in this case X and Y are i.i.d. and there is an equal chance that Y is smaller than X.*

The rest of the paper is organized as follows: In Section 2, the maximum likelihood estimator (MLE) of stress-strength reliability is derived. Section 3 deals with the asymptotic and two parametric bootstrap confidence intervals. In section 4, a simulation study is performed to compare different estimation procedures and various hybrid censoring schemes. Finally, conclusions and a brief discussion on the paper are given in Section 5.

2. Maximum Likelihood Estimation

Let $(x_1, x_2, \ldots, x_{d_1}) = (x_{1:n_1}, x_{2:n_1}, \ldots, x_{d_1:n_1})$ be a hybrid censored sample of size d_1 from $GE(\alpha, \lambda)$ with censoring scheme (r_1, T_1) and $(y_1, y_2, \ldots, y_{d_2}) = (y_{1:n_2}, y_{2:n_2}, \ldots, y_{d_2:n_2})$ be a independent hybrid censored sample of size d_2 from $GE(\beta, \lambda)$ with censoring scheme (r_2, T_2). Then the likelihood function without constant terms is given by

$$\begin{aligned} L(x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}; \alpha, \beta, \lambda) \quad = \quad & \alpha^{d_1} \beta^{d_2} \lambda^{d_1+d_2} \prod_{i=1}^{d_1} \frac{1}{x_i^2} \prod_{j=1}^{d_2} \frac{1}{y_j^2} e^{-\lambda(\sum_{i=1}^{d_1} \frac{1}{x_i} + \sum_{j=1}^{d_2} \frac{1}{y_j})} \\ & \prod_{i=1}^{d_1} (1 - e^{-\frac{\lambda}{x_i}})^{\alpha-1} \prod_{j=1}^{d_2} (1 - e^{-\frac{\lambda}{y_j}})^{\beta-1} \\ & \big(1 - (1 - e^{-\frac{\lambda}{c_1}})^{\alpha}\big)^{n_1-d_1} \big(1 - (1 - e^{-\frac{\lambda}{c_2}})^{\beta}\big)^{n_2-d_2}, \end{aligned} \tag{4}$$

where $c_k = \min(r_k, T_k)$, $d_k = \sum_{i=1}^{r_k} I x_{i:n} \le c_k$ and $k = 1, 2$. Since, MLEs do not exist for $d_1 = d_2 = 0$, therefore d_1 and d_2 both are assumed greater than zero. Therefore, the log likelihood function denote by ℓ. MLEs of the parameters α, β and λ, respectively, are the solutions of the following non-linear equations:

$$\frac{\partial \ell}{\partial \alpha} = \frac{d_1}{\alpha} + \sum_{i=1}^{d_1} \log(1 - e^{-\frac{\lambda}{x_i}}) - (n_1 - d_1) \log(1 - e^{-\frac{\lambda}{c_1}}) \frac{(1 - e^{-\frac{\lambda}{c_1}})^{\alpha}}{1 - (1 - e^{-\frac{\lambda}{c_1}})^{\alpha}}, \tag{5}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{d_2}{\beta} + \sum_{j=1}^{d_2} \log(1 - e^{-\frac{\lambda}{y_j}}) - (n_2 - d_2) \log(1 - e^{-\frac{\lambda}{c_2}}) \frac{(1 - e^{-\frac{\lambda}{c_2}})^{\beta}}{1 - (1 - e^{-\frac{\lambda}{c_2}})^{\beta}}, \tag{6}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} = \; & \frac{d_1 + d_2}{\lambda} - \sum_{i=1}^{d_1} \frac{1}{x_i} - \sum_{j=1}^{d_2} \frac{1}{y_j} + (\alpha - 1) \sum_{i=1}^{d_1} \frac{e^{-\frac{\lambda}{x_i}}}{x_i(1 - e^{-\frac{\lambda}{x_i}})} + (\beta - 1) \sum_{j=1}^{d_2} \frac{e^{-\frac{\lambda}{y_j}}}{y_j(1 - e^{-\frac{\lambda}{y_j}})} \\ & (n_1 - d_1)\alpha e^{-\frac{\lambda}{c_1}} \frac{(1 - e^{-\frac{\lambda}{x_i}})^{\alpha-1}}{c_1(1 - (1 - e^{-\frac{\lambda}{c_1}})^{\alpha})} - (n_2 - d_2)\beta e^{-\frac{\lambda}{c_2}} \frac{(1 - e^{-\frac{\lambda}{c_2}})^{\beta-1}}{c_2(1 - (1 - e^{-\frac{\lambda}{c_2}})^{\beta})}. \end{aligned} \tag{7}$$

The $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ can not be obtained in closed form, then some numerical techniques need to be employed to solve the non-linear equations (5)-(7).
Therefore, the MLE of R can be obtained using invariance property of MLEs as

$$\hat{R} = \frac{\hat{\beta}}{\hat{\alpha} + \hat{\beta}} \tag{8}$$

3. Different Confidence Intervals

In this Section, asymptotic confidence interval of R is constructed based on the asymptotic distribution of $\hat{R}$. Also,the use of two parametric bootstrap confidence intervals of R are proposed.

3.1. Asymptotic Confidence Interval

In this subsection, we derive the asymptotic confidence interval of R based on the approximate asymptotic variance-covariance matrix, which is given by

$$I^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = \Big[- \frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j} \Big]^{-1}, \quad i, j = 1, 2, 3,$$

where $\Theta = (\alpha, \beta, \lambda)$ and I is Fisher information matrix, which is not provided due to page restrictions.

Now, we find the approximate estimate of the variance of $\hat{R}$, using the delta method. Let define $G = [\frac{\partial R}{\partial \alpha}, \frac{\partial R}{\partial \beta}, \frac{\partial R}{\partial \lambda}]' = \frac{1}{(\alpha+\beta)^2}[-\beta, \alpha, 0]'$. Thus, an approximate estimate of $Var(\hat{R})$ is given by

$$\hat{V}ar(\hat{R}) = [GI^{-1}G']_{\hat{\alpha},\hat{\beta},\hat{\lambda}}.$$

Now, using the asymptotic normality property of MLEs, the $\hat{R}$ is asymptotically normal distributed with mean R and variance $\hat{V}ar(\hat{R})$. Therefore, the asymptotic $(1-\gamma)100\%$ confidence interval for R is given by

$$\left(\hat{R} - z_{\gamma/2}\sqrt{\hat{V}ar(\hat{R})},\ \hat{R} + z_{\gamma/2}\sqrt{\hat{V}ar(\hat{R})}\right), \tag{9}$$

where $-z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.
Since, $0 < R < 1$, a better confidence interval may be obtained using transformed confidence interval. Here, we use the logit transformation for the confidence interval estimation as suggested by Krishnamoorthy and Lin [9].
Let $\hat{\rho} = \ln \frac{\hat{R}}{1-\hat{R}}$ be the MLE of $\rho = \ln \frac{R}{1-R}$, using the asymptotic normality property of MLEs and the delta method, the asymptotic $(1-\gamma)100\%$ confidence interval for ρ is given by

$$\hat{\rho}_L < \rho < \hat{\rho}_U,$$
$$\hat{\rho}_L = \ln \frac{\hat{R}}{1-\hat{R}} - z_{\gamma/2}\frac{\sqrt{\hat{V}ar(\hat{R})}}{\hat{R}(1-\hat{R})},$$
$$\hat{\rho}_U = \ln \frac{\hat{R}}{1-\hat{R}} + z_{\gamma/2}\frac{\sqrt{\hat{V}ar(\hat{R})}}{\hat{R}(1-\hat{R})}.$$

Thus, the two sided equal tail asymptotic $(1-\gamma)100\%$ confidence interval for R is obtained as

$$\left(\frac{e^{\hat{\rho}_L}}{1+e^{\hat{\rho}_L}},\ \frac{e^{\hat{\rho}_U}}{1+e^{\hat{\rho}_U}}\right). \tag{10}$$

3.2. Bootstrap Confidence Intervals

Here, we propose the use of two parametric bootstrap confidence intervals. The two bootstrap methods that are widely used in practice are (i) the percentile bootstrap (boot-p) method proposed by Efron [4], and (ii) the bootstrap-t (boot-t) method proposed by Hall [7]. The boot-t confidence interval is developed based on a Studentized 'pivot' and requires an estimator of the variance of the MLE of R. We use the following algorithms for two parametric bootstrap confidence intervals for the stress-strength reliability R.

Boot-p method:

Step 1. Generate a hybrid censored sample $x = (x_1, x_2, \ldots, x_{d_1})$ with pre-fixed censoring scheme (r_1, T_1) of size d_1 from $GE(\alpha, \lambda)$ and generate another hybrid censored sample $y = (y_1, y_2, \ldots, y_{d_2})$ with censoring scheme (r_2, T_2) of size d_2 from $GE(\beta, \lambda)$. Compute the $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$.

Step 2. Generate a bootstrap sample $x^* = (x_1^*, x_2^*, \ldots, x_{d_1}^*)$ with pre-fixed censoring scheme (r_1, T_1) of size d_1 from $GE(\hat{\alpha}, \hat{\lambda})$ and generate a bootstrap sample $y^* = (y_1^*, y_2^*, \ldots, y_{d_2}^*)$ with censoring scheme (r_2, T_2) of size d_2 from $GE(\hat{\beta}, \hat{\lambda})$. Compute the $\hat{R}^*$ using equation (8).

Step 3. Repeat step 2, NBOOT times.

Step 4. Now, the approximate $(1-\gamma)100\%$ boot-p confidence interval of R is given by

$$\left(\hat{R}_{Bp}[\gamma/2],\ \hat{R}_{Bp}[1-\gamma/2]\right).$$

Boot-t method:

Table 1. Censoring schemes for simulation study.

(n,r,T)	(30,20,1.5)	(30,25,1.5)	(30,20,2.5)	(30,25,2.5)	(50,30,2)	(50,40,2)	(50,30,3)	(50,40,3)
CS	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8

Step 1. Same as in boot-p method..

Step 2. Same as in boot-p method.

Step 3. Compute the following statistic $T^* = \sqrt{d_1}(\hat{R}^* - \hat{R})/\sqrt{\hat{V}ar(\hat{R}^*)}$. Compute $\hat{V}ar(\hat{R}^* = \hat{B}^*/d_1$

Step 4. Repeat steps 2 and 3, NBOOT times.

Step 5. Now, the approximate $(1-\gamma)100\%$ boot-t confidence interval of R is given by

$$\big(\hat{R}_{Bt}[\gamma/2],\ \hat{R}_{Bt}[1-\gamma/2]\big).$$

4. Simulation Studies

This Section deals with the simulation study to compare the performance of different estimation procedures under various hybrid censoring schemes. The MLE of R along with average estimate (AE) and mean squared errors (MSE) are obtained. Also, the asymptotic and two types of bootstrap confidence in terms of average lengths and coverage probabilities are compared. Different parameter values, various censoring schemes and different sample sizes are considered. One set of true values of parameters $\alpha = 1.5$, $\beta = 2$, $\lambda = 1$ so that $R = 0.5714$ is taken.
Also, eight hybrid censoring schemes are considered and given in Table 1. The AEs and MSEs of ML estimator are obtained over 1000 pairs of hybrid censored samples generated from GE distribution. The average length of 95% asymptotic confidence interval based on logit scale, boot-p and boot-t confidence intervals of stress-strength parameter R are obtained. Here, $NBOOT = 1000$ for bootstrap methods is considered.
The results of the simulation study are presented in Tables 2 and 3, respectively.
Table 2 shows that as the effective sample size increases the AEs become close to their true values and MSEs decrease. Again, as the true value of R increases AEs depart from their true values and the MSEs decrease. In general, the combination of censoring schemes S_6 and S_8 give best results in comparison to other censoring schemes in terms of AEs, MSEs, average length of different confidence intervals.
Tables 3 presents the average confidence and credible lengths with corresponding coverage probabilities. The nominal level for the confidence and the credible intervals is 0.95 in each case. This Table shows that the average length of asymptotic and bootstrap confidence intervals narrow down as effective sample sizes increase. Also, the boot-t confidence intervals are wider than the asymptotic and boot-p confidence intervals. The asymptotic confidence intervals provide the smallest average credible lengths for different censoring schemes. Also, it is evident that the asymptotic confidence intervals provide the highest coverage probabilities in most cases considered.

5. Conclusions

In this article, the problem of classical estimation of stress-strength reliability $R = P(Y < X)$ for generalized exponential distribution using hybrid censored samples was considered. The hybrid censoring scheme is an operational censoring scheme and very useful in real life applications. Different estimation methods for estimating the stress-strength reliability in the case of different shapes and common scale unknown parameters of GE distribution were considered. The MLE of R and its asymptotic distribution was computed. Also, two parametric bootstrap confidence intervals were proposed and it was observed that the asymptotic confidence interval works the best even for small

Table 2. The AE and MSE of the MLE of R

CS	AE	MSE
(S_1, S_2)	0.5733	0.0068
(S_1, S_3)	0.5792	0.0070
(S_1, S_4)	0.5698	0.0062
(S_2, S_1)	0.5789	0.0064
(S_2, S_3)	0.5788	0.0065
(S_2, S_4)	0.5705	0.0055
(S_3, S_1)	0.5729	0.0068
(S_3, S_2)	0.5623	0.0058
(S_3, S_4)	0.5699	0.0058
(S_4, S_1)	0.5796	0.0066
(S_4, S_2)	0.5719	0.0055
(S_4, S_3)	0.5821	0.0062
(S_5, S_6)	0.5668	0.0038
(S_5, S_7)	0.5708	0.0043
(S_5, S_8)	0.5684	0.0038
(S_6, S_5)	0.5784	0.0039
(S_6, S_7)	0.5779	0.0040
(S_6, S_8)	0.5745	0.0033
(S_7, S_5)	0.5753	0.0046
(S_7, S_6)	0.5656	0.0037
(S_7, S_8)	0.5650	0.0039
(S_8, S_5)	0.5794	0.0039
(S_8, S_6)	0.5726	0.0034
(S_8, S_7)	0.5734	0.0037

Table 3. The AL and CP of 95% asymptotic, bootstrap confidence intervals of R.

	$\hat{R}_{MLE}$		$\hat{R}_{Bp}$		$\hat{R}_{Bt}$	
CS	AL	CP	AL	CP	AL	CP
(S_1, S_2)	0.2842	0.925	0.3086	0.923	0.3314	0.910
(S_1, S_3)	0.2931	0.934	0.3154	0.925	0.3458	0.908
(S_1, S_4)	0.2806	0.941	0.3028	0.914	0.3244	0.905
(S_2, S_1)	0.2916	0.938	0.3080	0.935	0.3385	0.923
(S_2, S_3)	0.2907	0.944	0.3041	0.928	0.3363	0.913
(S_2, S_4)	0.2772	0.955	0.2889	0.927	0.3135	0.915
(S_3, S_1)	0.2915	0.935	0.3158	0.926	0.3433	0.909
(S_3, S_2)	0.2836	0.946	0.3023	0.935	0.3250	0.922
(S_3, S_4)	0.2767	0.944	0.2965	0.928	0.3175	0.914
(S_4, S_1)	0.2830	0.930	0.3030	0.924	0.3295	0.902
(S_4, S_2)	0.2718	0.949	0.2886	0.926	0.3097	0.912
(S_4, S_3)	0.2823	0.937	0.2988	0.930	0.3267	0.915
(S_5, S_6)	0.2266	0.949	0.2378	0.937	0.2472	0.930
(S_5, S_7)	0.2418	0.942	0.2540	0.932	0.2680	0.922
(S_5, S_8)	0.2260	0.942	0.2369	0.951	0.2463	0.944
(S_6, S_5)	0.2333	0.951	0.2418	0.931	0.2558	0.922
(S_6, S_7)	0.2333	0.940	0.2420	0.940	0.2560	0.926
(S_6, S_8)	0.2135	0.941	0.2214	0.941	0.2308	0.931
(S_7, S_5)	0.2412	0.932	0.2538	0.939	0.2680	0.925
(S_7, S_6)	0.2267	0.947	0.2379	0.951	0.2473	0.942
(S_7, S_8)	0.2265	0.935	0.2367	0.945	0.2465	0.932
(S_8, S_5)	0.2316	0.947	0.2410	0.940	0.2541	0.925
(S_8, S_6)	0.2111	0.941	0.2206	0.942	0.2289	0.932
(S_8, S_7)	0.2320	0.950	0.2410	0.936	0.2539	0.928

effective sample sizes. The performance of the point and interval estimates of R is examined by extensive simulations. Simulation results suggested that the performance of asymptotic confidence interval work very well and these can be used for all practical purposes.

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A New Weighted Circular Distribution

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Abstract

In this paper, a new unimodal, multimodal and skew-symmetric circular distribution, called the weighted cardiod distribution, is introduced. The structural properties of the probability density function, trigonometric moment, mean resultant lenght and mean direction of this distribution are discussed.

1. Introduction

The classical models for circular data, such as the von Mises, wrapped Cauchy, cardioid and wrapped normal distributions (Mardia and Jupp, 1999; Jammalamadaka and SenGupta, 2001), are all symmetric. Weighted sampling arises when the sampling mechanism records the unit sample proportion to a non-negative function, called weight function. This is a generalization of random sampling under which the recorded data is weighted sample rather than the original sample. Therefore the classical statistical techniques lead to non-valid results and must be modified. The original study of this biased sampling traces back to Fisher (1934).
In this paper, we adapted the approach of Patil and Rao (1987) in order to obtain a new weighted cardiod for the symmetric, asymmetric, unimodal and multimodal cases.

2. Weighted cardiod distribution

If Θ is a circular random variable having the cardiod distribution, its probability density function takes the form

$$f^w(\theta) = \frac{1}{2\pi}(1 + 2\rho\cos\theta), \tag{1}$$

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where $\theta \in [-\pi, \pi)$, $\rho \in [0, \frac{1}{2}]$.
Patil and Rao (1978) suggested that the class of probability density functions can be obtained by using

$$f^w(x) = \frac{w(x)f(x)}{E[w(X)]},$$

where $w(x)$ is a non-negative weight function and

$$E[w(X)] = \int w(x)f(x)dx < \infty.$$

Weighted circular distribution was the result of applying the above concept, In theorem 1, to the circular cases. Remember that a circular pdf is a non-negative periodic function with a period 2π, which has an integration interval of length 2π. To make it more special, in the study ahead, we assume that the distribution is defined over the interval $[-\pi, \pi)$.

Theorem 2.1. *Suppose that θ is a circular random variable with pdf $f(\theta)$. The pdf of the weighted circular random variable Θ^w is given by*

$$f^w(\theta) = \frac{w(\theta)f(\theta)}{E[w(\Theta)]}, \tag{2}$$

where $w(\theta)$ is a non-negative and periodic i.e. $w(\theta) = w(\theta + 2k\pi)$ for all integers k.

Proof. $f^w(\theta)$ is a density on $[-\pi, \pi)$ because $f(\theta)$ is a density on $[-\pi, \pi)$ and $w(\theta) > 0$ for $\theta \in [-\pi, \pi)$. But $f^w(\theta)$ is also circular density because

$$f^w(\theta + 2k\pi) = \frac{w(\theta + 2k\pi)f(\theta + 2k\pi)}{E[w(\Theta + 2k\pi)]} = \frac{w(\theta)f(\theta)}{E[w(\Theta)]} = f^w(\theta),$$

for all integers k. □

Let weight function, $w(\theta) = 1 + \lambda_1 \sin\theta + \lambda_2 \cos\theta$ for $\lambda_1, \lambda_2 \in [-1, 1]$, with $|\lambda_1| + |\lambda_2| \leq 1$, $\theta \in [-\pi, \pi)$. Then we define the density of what we shall refer to as a weighted cardiod distribution by

$$f^w(\theta) = \frac{(1 + \lambda_1 \sin\theta + \lambda_2 \cos\theta)\{1 + 2\rho\cos\theta\}}{2\pi(1 + \lambda_2\rho)}, \tag{3}$$

where $\theta \in [-\pi, \pi)$, $\rho \in [0, \frac{1}{2}]$.
The flexibility of this model is illustrated in Figure 1.

$$F^w(\theta) = \frac{1}{2\pi}\Big\{(\theta + \pi) + \frac{1}{1 + \lambda_2\rho}(\lambda_1(1 - \cos\theta + \frac{\rho}{2}(1 + \cos 2\theta) + \lambda_2(\sin\theta + \frac{\rho}{2}\sin 2\theta) + 2\rho\sin\theta)\Big\}.$$

Since the pth cosine moments of a cardioid distribution are $\alpha_1 = \rho$ and $\alpha_p = 0$ for $p = 2, 3, ...$, for $j = 1$ the trigonometric moments of a $SCWC$ distribution are

$$\alpha_1^w = \frac{\lambda_2 + 2\rho}{2(1 + \lambda_2\rho)}, \qquad \alpha_2^w = \frac{\lambda_2\rho}{2(1 + \lambda_2\rho)}, \qquad \alpha_p^w = 0 \ (p \geq 3)$$

and

$$\beta_1^w = \frac{\lambda_1}{2(1 + \lambda_2\rho)}, \qquad \beta_2^w = \frac{\lambda_1\rho}{2(1 + \lambda_2\rho)}, \qquad \alpha_p^w = 0 \ (p \geq 3).$$

Thus, the mean resultant lenghts is given by

$$\rho^w = \frac{1}{1 + \lambda_2\rho}\sqrt{(\frac{\lambda_2}{2} + \rho)^2 + \frac{\lambda_1^2}{4}}, \qquad \rho_2^w = \frac{1}{1 + \lambda_2\rho}\sqrt{\frac{\rho^2}{4}(\lambda_1^2 + \lambda_2^2)}, \qquad \rho_p^w = 0 \ (p \geq 3),$$

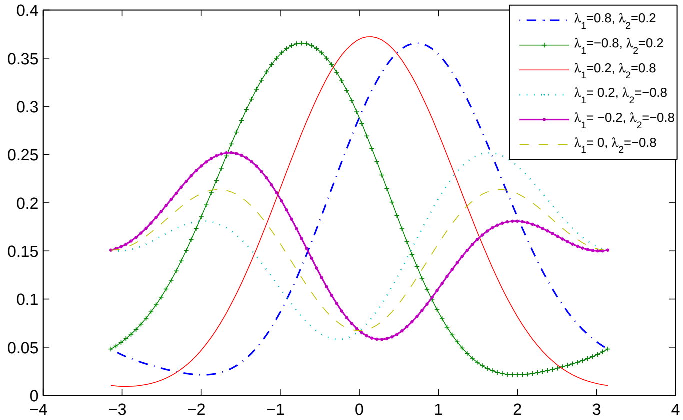

Fig. 1. Weighted cardioid densities with $\mu = 0$ and $\kappa = 1$.

and the mean direction is given by

$$\mu^w = \arg\{\lambda_2 + 2\rho + i\lambda_1\}, \qquad \mu_2^w = \arg\{\lambda_2\rho + i\lambda_1\rho\}, \qquad \mu_p^w = 0 \ (p \geq 3).$$

Now, we can represent the circular variance and circular standard deviation by

$$V^w = 1 - \rho^w = 1 - \frac{1}{1 + \lambda_2\rho}\sqrt{(\frac{\lambda_2}{2} + \rho)^2 + \frac{\lambda_1^2}{4}},$$

$$\begin{aligned} \sigma^w &= \{-2\log(1 - V^w)\}^{\frac{1}{2}} \\ &= \Big\{ -2\log\{\frac{1}{1 + \lambda_2\rho}\sqrt{(\frac{\lambda_2}{2} + \rho)^2 + \frac{\lambda_1^2}{4}}\}\Big\}^{\frac{1}{2}}. \end{aligned}$$

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Multivariate asymmetric distributions by elliptical copula

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Abstract

In this paper, we obtain the distribution of $\boldsymbol{Z} \overset{d}{=} (\boldsymbol{X}|Y > \mu_y)$ when $\boldsymbol{X}$ and Y are related through the elliptical copula. We explore our results in two special cases, the Gaussian as well as the T-copula. Using a simulation study, we compare the performance of our proposed distributions with the conventional skew-normal and skew-t distribution.

1. Introduction

1.1. Copula Function

Copula is a powerful way to model the dependence of a random vector. One key insight is due to the famous Sklar theorem: the distribution of any continuous random vector can be expressed using copula and the marginal distributions. It is easy to estimate the marginals of a random vector, so all we need is to estimate the copula function and this would lead to an estimator of the joint distribution.

Definition 1.1. [3].
Let $\boldsymbol{X} \in \mathbb{R}^d$ be a random vector and F be the distribution function of $\boldsymbol{X}$, i.e., $F(x) = P(X_1 \le x_1, ..., X_d \le x_d)$. Further, we denote $F_1, ..., F_d$ be the marginal distribution functions of $X_1, ..., X_n$. A copula is a function $C : [0,1]^d \rightarrow [0,1]$ with the following properties:
(A) Marginal. For any $i = 1, ..., d$, $C(u_i, 1) = u_i$.
(B) Isotonic. $C(u) \le C(v)$ if $u \le v$, where $u \le v$ means that $u_j \le v_j$ for all $j = 1, ..., d$.
(C) d-increasing. For any box $[a, b] \subset [0,1]^d$ with non-empty volume, $C([a, b]) > 0$.
Note that when there are d variables, C is often called the d-copula. A copula can be viewed as a distribution function of d-dimensional random vector U such that $U_j \sim Unif[0,1],\ j = 1, 2, ..., d$.

Theorem 1.2. ***Sklar's theorem***
For a random vector X with distribution function F and univariate marginal distribution functions $F_1, ..., F_d$. There exists a copula C such that $F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$. If X is continuous, then such a copula C is unique.

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1.2. Assymetric elliptical distributions

Definition 1.3. [2]. **Multivariate Gaussian copula**.
Let $\boldsymbol{R}$ be a symmetric and positive definite matrix with $diag(R) = (1, 1, ..., 1)^T$ and $\Phi_{\boldsymbol{R}}$ is the standardized multivariate normal distribution with correlation matrix $\boldsymbol{R}$, then the multivariate Gaussian copula is

$$C^{Ga}(u_1, u_2, ..., u_n) = \Phi_{\mathbf{R}}(\Phi^{-1}(u_1), \Phi^{-1}(u_2), ..., \Phi^{-1}(u_n)), \tag{1}$$

where Φ^{-1}, as usual, is the inverse of the standard univariate normal distribution function Φ.

Definition 1.4. [2]. **Multivariate Student's t copula**.
Let $\boldsymbol{R}$ be a symmetric and positive definite matrix with $diag(R) = (1, 1, ..., 1)^T$ and $t_{\boldsymbol{R},\nu}$ the standardized multivariate Student's t distribution with correlation matrix $\boldsymbol{R}$ and ν degrees of freedom, i.e.

$$C^T(u_1, u_2, ..., u_n) = T_{\mathbf{R},\nu}(T_\nu^{-1}(u_1), T_\nu^{-1}(u_2), ..., T_\nu^{-1}(u_n)), \tag{2}$$

where t_ν^{-1} is the inverse of the univariate c.d.f. of Student's t with ν degrees of freedom.

In the following, we will give a brief definition of skew normal, and skew t distribution.

Lemma 1.5. *[1]. A random variable Z is said to have a skew-symmetric (SS) distribution if its probability density function (PDF) can be written as*

$$f(z) = 2f_0(z)G(w(z)), \quad z \in \mathbb{R}$$

where f_0 the density function of a continuous random variable which is centrally symmetric around 0, and by G a distribution function such that $G(-x) = 1-G(x)$ for all real x. If $w(z)$ is a function from $\mathbb{R}^d$ to $\mathbb{R}$ such that $w(-z) = -w(z)$ for all $z \in \mathbb{R}^d$.

Skew-normal distribution [1]. Given a full-rank $d \times d$ covariance matrix $\Omega = (w_{rs})$, define $w = diag(w_1, ..., w_d) = diag(w_{11}, ..., w_{dd})^{1/2}$ and let $\bar{\Omega} = w^{-1}\Omega w$ be the associated correlation matrix; also let $\xi, \alpha \in \mathbb{R}^d$. A d-dimensional random variable Z is said to have a skew normal distribution if it is continuous with density function at $z \in \mathbb{R}^d$ of type

$$2\varphi_d(z - \xi; \Omega)\Phi(\alpha^T w^{-1}(z - \xi)),$$

We shall then write $Z \sim SN_d(\xi, \Omega, \alpha)$, referring to ξ, Ω, α as the location, dispersion and shape or skewness parameters, respectively.
Skew-t distribution [1]. The density function of the following form is called the skew-t density function

$$2t_d(z; \nu)T_1\big(\alpha^T w^{-1}(z - \xi)\big(\tfrac{\nu+d}{Q_z+\nu}\big)^{1/2}; \nu + d\big),$$

where $Q_z = (z - \xi)^T\Omega^{-1}(z - \xi)$ and $t_d(z; \nu)$ is the density function of a d-dimensional t variate with ν degrees of freedom, and $T_1(x; \nu + d)$ denotes the scalar t distribution function with $\nu + d$ degrees of freedom. We shall then write $Z \sim St_d(\xi, \Omega, \alpha, \nu)$.

2. Main results

In this paper, we talked about how to obtain the density function $\boldsymbol{Z} \stackrel{\text{d}}{=} (\boldsymbol{X}|Y > \mu_y)$ using the copula function and the marginal functions. Now, according to the hypotheses of the following two theorems, we try to calculate the density function $\boldsymbol{Z}$ in these cases.

Theorem 2.1. *Let $\boldsymbol{X}$ be a vector of random variables that have multivariate normal distribution with correlation matrix $\boldsymbol{\Omega}$ and Y has a distribution function $F_Y(y)$ so that are connected via the $Gussian$-copula with correlation matrix $\boldsymbol{R}$, then the distribution of $\boldsymbol{Z} \stackrel{\text{d}}{=} \boldsymbol{X}|Y > \mu_y$ is*

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = m_{\mu_y}\varphi_{\boldsymbol{X}}(\boldsymbol{z}, \boldsymbol{\Omega})\Phi(\frac{\alpha^{\boldsymbol{T}}\boldsymbol{z} - |\boldsymbol{\Omega}|\Phi^{-1}(F_Y(\mu_y))}{\sqrt{|\boldsymbol{R}||\boldsymbol{\Omega}|}} \quad (\boldsymbol{z} \in \mathbb{R}^d). \tag{3}$$

Corollary 2.2. *If Y has a normal distribution in the theorem 2.1, then the distribution of (3) is the same as standard skew-normal.*

Theorem 2.3. *Let $\boldsymbol{X}$ be a vector of random variables that have multivariate student's t-distribution with correlation matrix $\boldsymbol{\Omega}$ and ν degree freedom and Y has a distribution function $F_Y(y)$ so that are connected via the T-copula with correlation matrix $\boldsymbol{R}$, then the distribution of $\boldsymbol{Z} \overset{\mathrm{d}}{=} \boldsymbol{X}|Y > \mu_y$ is*

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = m_{\mu_y} t_{\nu,\boldsymbol{X}}(\boldsymbol{z},\boldsymbol{\Omega}) T(\frac{\alpha^{\boldsymbol{T}}\boldsymbol{z} - |\boldsymbol{\Omega}|T^{-1}(F_Y(\mu_y))}{\sqrt{|\boldsymbol{R}||\boldsymbol{\Omega}|}} \frac{\sqrt{\nu+d}}{\sqrt{\nu+\boldsymbol{z}^{\boldsymbol{T}}\boldsymbol{\Omega}^{-1}\boldsymbol{z}}}, \nu+d) \;\; (\boldsymbol{z} \in \mathbb{R}^d), \tag{4}$$

in the content provided $\boldsymbol{\Omega}$ is the correlation matrix of $\boldsymbol{X}$ and $\alpha^{\boldsymbol{T}} = (-|\boldsymbol{R}|a_{12} \ldots. -|\boldsymbol{R}|a_{1d})$ where $a_{1i},\ i = 2, 3, ..., d$, are the elements of the first row of matrix $\boldsymbol{R^{-1}}$.

Corollary 2.4. *If Y has a normal distribution in the theorem 2.3, then the distribution of (4) is the same as standard skew-t.*

3. Simulation study

In this section, the simulation is performed in two ways, first using the assumptions of Theorem 2.1 and then using the assumptions of Theorem 2.3 also we used a Monte Carlo simulation study. We generated 100 random pairs $(X_i, Y_i), i = 1, 2, ..., 1000$.
1) In the first scenario, we assume that X_1 and X_2 followed are normal distribution with correlation matrix $\rho_{X_1,X_2} = 0.6$ and Y came from χ^2_{10} distribution and they are connected via a Gaussian-copula with correlation $\rho_{X_1,Y} = 0.7$, $\rho_{X_2,Y} = 0.5$ and $\rho_{X_1,X_2} = 0.6$. We then repeated this procedure 5000 times. Table 1 shows the AIC and BIC values in four modes: skew-normal-copula, skew-normal, skew-t-copula and skew-t. According to the reported results, under the assumptions of Theorem 2.1, skew-t-copula performs better than the others.
2) In the second scenario, we assume that X_1 and X_2 followed student's t-distribution with 10 degree of freedom with correlation matrix $\rho_{X_1,X_2} = 0.5$ and Y came from a χ^2_{10} distribution and they are connected via a T-copula with correlation $\rho_{X_1,Y} = 0.8$, $\rho_{X_2,Y} = 0.6$ and $\rho_{X_1,X_2} = 0.5$. We then repeated this procedure 5000 times. A similar comparison is made in Table 1 and gives better results according to the assumptions of skew-normal-copula.

Table 1. AIC and BIC of skew-normal-copula , skew-normal , skew-t-copula and skew-t.

First comparison Estimation	skew-normal-copula	skew-normal	skew-t-copula	skew-t
AIC	293	292	285	286
BIC	320	320	312	313
Second comparison Estimation	skew-normal-copula	skew-normal	skew-t-copula	skew-t
AIC	216	217	218	220
BIC	244	244	245	248

Figure 1, depicts a visualization of this comparison.

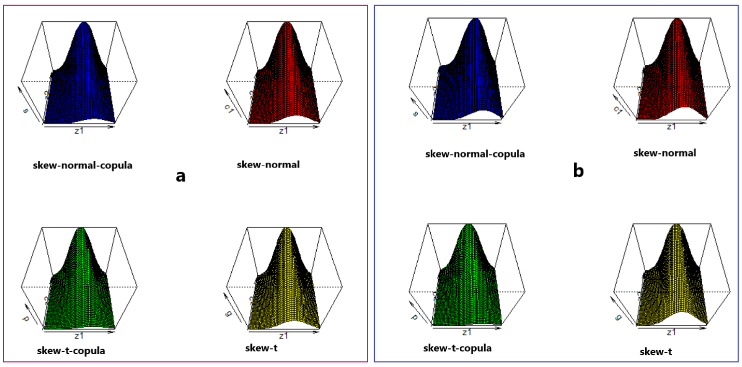

Fig. 1. Performance of skew-normal-copula , skew-normal , skew-t-copula and skew-t. **a**) first comparison, **b**) second comparison.

4. Conclusion

As mentioned, in many practical problems, the variables are interdependent and skewed. In this study, we sought to calculate $\boldsymbol{Z} \stackrel{\mathrm{d}}{=} (\boldsymbol{X}|Y > \mu_y)$, while the variables are related through elliptical copula functions, and as it can be seen in the simulation results, the presented cases perform better according to the two criterias AIC and BIC.

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On the Parameter Estimation of the Generalized Exponential Distribution Under Progressive Type-I Interval Censoring Scheme

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Abstract

Chen and Lio (Computational Statistics and Data Analysis 54: 1581-1591, 2010) proposed five methods for estimating parameters of the generalized exponential distribution under progressive type-I interval censoring scheme. Unfortunately, among them, the proposed EM algorithm is incorrect. Here, we propose the correct EM algorithm and compare its performance with the maximum likelihood estimators and that proposed by Chen and Lio (2010) in a simulation study.

1. Introduction

1.1. Generalized exponential (GE) distribution

The random variable X follows GE distribution if its probability density function (pdf) and distribution function are given by

$$f(x,\theta) = \alpha\big(1 - e^{-\lambda x}\big)^{\alpha-1} e^{-\lambda x}, \tag{1}$$

and

$$F(x,\theta) = \big(1 - e^{-\lambda x}\big)^{\alpha}, \tag{2}$$

where $\theta = (\alpha, \lambda)$ is parameter vector (α is the shape parameter and λ is the rate parameter). The family of GE distributions was introduced by Mudholkar and Srivastava (1993). For a comprehensive account of the theory and applications of GE distribution, we refer the readers to Gupta and Kundu (2007).

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1.2. Progressively type-I interval censoring scheme

Suppose n subjects are placed on a life testing simultaneously at time $t_0 = 0$ and under inspection at m pre-determined times $t_1 < t_2 < \cdots < t_m$ in which t_m is the time to terminate the life testing. At the i-th inspection time, t_i, the number, X_i, of failures within $(t_i, t_{i+1}]$ is recorded and R_i alive items are randomly removed from the life testing, for $i = 1, \ldots, m$. As pointed out by Chen and Lio (2010), since the number, Y_i, of surviving items is a random variable and the exact number of items withdrawn should not be greater than Y_i at time schedule t_i, then R_i could be determined by the pre-specified percentage of the remaining surviving units at t_i, or equivalently $R = \lfloor p_i Y_i \rfloor$; for $i = 1, \ldots, m$. Each progressively type-I interval censoring scheme is shown by $\{X_i, R_i, T_i\}_{i=1}^{m}$ where $n = \sum_{i=1}^{m} X_i + R_i$ is the sample size. If $Ri = 0$; for $i = 1, \ldots, m-1$, then the progressively type-I interval censoring scheme is equivalent to a type-I interval censoring scheme with sample $X_1, X2, \ldots, X_m, X_{m-1} = R_m$. Suppose a $\{X_i, R_i, T_i\}_{i=1}^{m}$ life testing scheme where n items each follows independently the cdf $F(., \theta)$ is under the test. The likelihood function is (see [1]) is

$$L(\theta) \propto \prod_{i=1}^{m} \big[F(t_i, \theta) - F(t_{i-1}, \theta)\big]^{X_i} \big[1 - F(t_i, \theta)\big]^{R_i}. \tag{3}$$

As the most common used tool, the maximum likelihood (ML) approach is employed to estimate the θ. But, equation (3) must be maximized through iterative algorithm such as Newton-Raphson to obtain the ML estimators and there is no guarantee that the Newton-Raphson method converges. Another technique is the expectation-maximization (EM) algorithm that always converges, see [5]. However, if practitioner is interested in the ML estimators, the first few steps of the EM algorithm can be used to get a good starting value for the Newton-Raphson algorithm, see [8].

1.3. EM algorithm

The EM algorithm, introduced by [3], is known as the popular method for computing the ML estimators when we encounter the incomplete data problem. In other word, the use of the EM algorithm involves cases that we are dealing with the latent variables, provided that the statistical model is formulated as a missing or latent variable problem. In what follows, we give a brief description of the EM algorithm. Let $\boldsymbol{\xi}$, $\boldsymbol{Z}$, and $\boldsymbol{\omega}$ denote the complete, unobservable variable, and observed data, respectively (complete data consists of observed values and unobservable variables, i.e., $\boldsymbol{\xi} = (\boldsymbol{Z}, \boldsymbol{\omega})$). The EM algorithm works by maximizing the conditional expectation $Q\left(\theta|\theta^{(t)}\right) = E\left(l_c(\theta; \boldsymbol{\xi})|\boldsymbol{\omega}, \theta^{(t)}\right)$ of complete data log-likelihood function given observed data and a current estimate $\theta^{(t)}$ of the parameter vector θ where $l_c(\theta; \boldsymbol{x})$ denotes the complete data log-likelihood function. Each iteration of the EM algorithm consists of two steps:

1. Expectation (E)-step: Computing $Q\left(\theta|\theta^{(t)}\right)$ at the t-th iteration.
2. Maximization (M)-step: Maximizing $Q\left(\theta|\theta^{(t)}\right)$ with respect to θ to get $\theta^{(t+1)}$.

The E-step and M-step are repeated until convergence occurs, see [3] and [6].

2. EM algorithm for GE family under progressive type-I interval censoring scheme

Suppose n failure times follow the GE distribution with pdf and cdf given by expressions (1) and (2), respectively. For convenience, let us to use the notations given by Chen and Lio (2010). So, let $t_{i,j}$; for $j = 1, \ldots, X_i$, denote the independent and identically distributed (iid) failure times in the subinterval $(t_{i-1}, t_i]$; for $i = 1, \ldots, m$ and $t^{*}_{I,j}$; for $j = 1, \ldots, R_i$, indicate on iid failure times of the randomly removed items alive at the end of the subinterval $(t_{i-1}, t_i]$; for $i = 1, \ldots, m$. Then, the complete data log-likelihood, $l_c(\theta)$, is (see [2])

$$l_c(\theta) \propto \sum_{i=1}^{m} \sum_{j=1}^{X_i} \log f(T_{i,j}, \theta) + \sum_{i=1}^{m} \sum_{j=1}^{R_i} \log f(T^{*}_{i,j}, \theta). \tag{4}$$

In expression (4), we show unobservable (or missing) variables by capital letters $T_{i,j}$ (for $i = 1, \ldots, m, j = 1, \ldots, x_i$) and $T^{*}_{i,j}$ (for $i = 1, \ldots, m; j = 1, \ldots, r_i$) in which $\sum_{i=1}^{m} x_i + r_i = n$. The progressive type-I censoring scheme is an incomplete data problem. The observed values are x_is and r_is; for $i = 1, \ldots, m$ and unobservable variables are $T_{i,j}$

(iid failure times during subinterval $(t_{i-1}, t_i])$ and $T^*_{i,j}$ (iid withdrawn survival times during subinterval $(t_{i-1}, t_i])$. Therefore, under EM algorithm framework mentioned in subsection 1.3, the vector of observed data, $\boldsymbol{\omega}$, is $\boldsymbol{\omega} = (x_1, \dots, x_m, r_1, \dots, r_m)$ and the vector of unobservable variables is, $\boldsymbol{Z} = (T_{i,1}, \dots, T_{i,x_i}, T^*_{i,1}, \dots, T^*_{i,r_i})$; for $i = 1, \dots, m$. Assuming that we are at t-th iteration, in order to implement the EM algorithm, we follow two steps given by the following.

- **E-step**: we need to compute the conditional expectation $Q\big(\theta\big|\theta^{(t)}\big) = E\big(l_c(\theta; \boldsymbol{\xi})\big|\boldsymbol{\omega}, \theta^{(t)}\big)$ of the complete data log-likelihood function. It follows, form (4), that

$$
\begin{aligned}
Q\big(\theta\big|\theta^{(t)}\big) =& \mathrm{C} + \sum_{i=1}^{m} E\Big(\sum_{j=1}^{X_i} \log f(T_{i,j}, \theta)\Big|\boldsymbol{\omega}, \theta = \theta^{(t)}\Big) + \sum_{i=1}^{m} E\Big(\sum_{j=1}^{R_i} \log f(T^*_{i,j}, \theta)\Big|\boldsymbol{\omega}, \theta = \theta^{(t)}\Big) \\
=& \mathrm{C} + \sum_{i=1}^{m} E\Big(\sum_{j=1}^{X_i} \log f(T_{i,j}, \theta)\Big|X_i = x_i, T_{i,j} \in (t_{i-1}, t_i], \theta = \theta^{(t)}\Big) \\
&+ \sum_{i=1}^{m} E\Big(\sum_{j=1}^{R_i} \log f(T^*_{i,j}, \theta)\Big|R_i = r_i, T^*_{i,j} \in [t_i, \infty), \theta = \theta^{(t)}\Big) \\
=& \mathrm{C} + \sum_{i=1}^{m}\sum_{j=1}^{x_i} E\big(\log f(T_{i,j}, \theta)\big|T_{i,j} \in (t_{i-1}, t_i], \theta = \theta^{(t)}\big) \\
&+ \sum_{i=1}^{m}\sum_{j=1}^{r_i} E\big(\log f(T^*_{i,j}, \theta)\big|T^*_{i,j} \in [t_i, \infty), \theta = \theta^{(t)}\big) \\
=& \big(\log(\alpha) + \log(\lambda)\big)\Big(\sum_{i=1}^{m}(x_i + r_i)\Big) - \lambda \sum_{i=1}^{m}\sum_{j=1}^{x_i} E\big(T_{i,j}\big|T_{i,j} \in (t_{i-1}, t_i], , \theta = \theta^{(t)}\big) \\
&- \lambda \sum_{i=1}^{m}\sum_{j=1}^{r_i} E\big(T^*_{i,j}\big|T^*_{i,j} \in [t_i, \infty), \theta = \theta^{(t)}\big) \\
&+ (\alpha - 1)\sum_{i=1}^{m}\sum_{j=1}^{x_i} E\big(\log\big(1 - e^{-\lambda T_{i,j}}\big)\big|T_{i,j} \in (t_{i-1}, t_i], \theta = \theta^{(t)}\big) \\
&+ (\alpha - 1)\sum_{i=1}^{m}\sum_{j=1}^{r_i} E\big(\log\big(1 - e^{-\lambda T^*_{i,j}}\big)\big|T^*_{i,j} \in [t_i, \infty), \theta = \theta^{(t)}\big),
\end{aligned} \tag{5}
$$

  where C is a constant independent of θ and $\theta^{(t)} = (\alpha^{(t)}, \lambda^{(t)})$. We note that the lifetimes of the r_i unobserved items during subinterval $(t_{i-1}, t_i]$ are conditionally independent, identically distributed, and follow the truncated GE distribution on interval $[t_i, \infty)$. Also, lifetimes of the x_i unobservable subjects during subinterval $(t_{i-1}, t_i]$ are conditionally independent, identically distributed, and follow the double-truncated GE distribution on subinterval $(t_{i-1}, t_i]$; for $i = 1, \dots, m$. Therefore, considering the right-hand side of (5), the required conditional expectations are:

$$E_{1i} = E\Big(T_{i,j}\Big|T_{i,j} \in (t_{i-1}, t_i], \theta^{(t)} = (\alpha^{(t)}, \lambda^{(t)})\Big) = \frac{\int_{t_{i-1}}^{t_i} u f\big(u, \theta^{(t)}\big) du}{F\big(t_i, \theta^{(t)}\big) - F\big(t_{i-1}, \theta^{(t)}\big)}, \tag{6}$$

$$\begin{aligned} E_{2i} &= E\Big(\log\big(1 - e^{-\lambda T_{i,j}}\big)\Big|T_{i,j} \in (t_{i-1}, t_i], \theta^{(t)} = (\alpha^{(t)}, \lambda^{(t)})\Big) \\ &= \frac{\int_{t_{i-1}}^{t_i} \log\big(1 - e^{-\lambda u}\big) f\big(u, \theta^{(t)}\big) du}{F\big(t_i, \theta^{(t)}\big) - F\big(t_{i-1}, \theta^{(t)}\big)}, \end{aligned} \tag{7}$$

$$E_{3i} = E\Big(T^*_{i,j}\Big|T^*_{i,j} \in [t_i, \infty), \theta^{(t)} = (\alpha^{(t)}, \lambda^{(t)})\Big) = \frac{\int_{t_i}^{\infty} u f\big(u, \theta^{(t)}\big) du}{1 - F\big(t_i, \theta^{(t)}\big)}, \tag{8}$$

$$\begin{aligned} E_{4i} &= E\Big(\log\big(1 - e^{-\lambda T^*_{i,j}}\big)\big|T^*_{i,j} \in [t_i, \infty), \theta^{(t)} = (\alpha^{(t)}, \lambda^{(t)})\Big) \\ &= \frac{\int_{t_i}^{\infty} \log\big(1 - e^{-\lambda u}\big) f\big(u, \theta^{(t)}\big) du}{1 - F\big(t_i, \theta^{(t)}\big)}, \end{aligned} \tag{9}$$

where $i = 1, \ldots, m$ and $t_0 = 0$.

- **M-step**: by substituting the computed conditional expectations E_{1i}, E_{2i}, E_{3i}, and E_{4i} given in (6)-(9) into the right-hand side of (5), we follow the EM algorithm by calculating the derivatives with respect to parameters as follows.

$$\frac{\partial Q\big(\theta|\theta^{(t)}\big)}{\partial \alpha} = \frac{\sum_{i=1}^{m}(x_i + r_i)}{\alpha} + \sum_{i=1}^{m} x_i E_{2i} + \sum_{i=1}^{m} r_i E_{4i}, \tag{10}$$

$$\frac{\partial Q\big(\theta|\theta^{(t)}\big)}{\partial \lambda} = \frac{\sum_{i=1}^{m}(x_i + r_i)}{\lambda} - \sum_{i=1}^{m} x_i E_{1i} - \sum_{i=1}^{m} r_i E_{3i}, \tag{11}$$

where $\sum_{i=1}^{m}(x_i + r_i) = n$. Equating the right-hand side of (10) and (11) to zero it turns out that

$$\alpha^{(t)} = -\frac{n}{\sum_{i=1}^{m} x_i E_{2i} + \sum_{i=1}^{m} r_i E_{4i}}, \tag{12}$$

and

$$\lambda^{(t)} = \frac{n}{\sum_{i=1}^{m} x_i E_{1i} + \sum_{i=1}^{m} r_i E_{3i}}. \tag{13}$$

The M-step is complete.

We mention that the EM algorithm proposed by Chen and Lio (2010) is incorrect since they took expectation form the complete data log-likelihood function after differentiating it with respect to parameters which in not usual in the EM framework. Using the starting values as $\theta^{(0)} = (\alpha^{(0)}, \lambda^{(0)})$ and repeating the E-step and M-step described as above the EM estimators are obtained. Compare the updated shape and rate parameters at t-th iteration given in (12) and (13) with those given by Chen and Lio (2010). It is known that the updated shape parameters are the same but there is a significant difference between updated rate parameter given here and that given in Chen and Lio (2010). Although, difference between rate parameters is theoretically significant, however we perform a simulation study in the next section to observe the differences visually.

3. Simulation study

Here, we perform a simulation study to compare the performance of three estimators including: EM algorithm, ML, and EM algorithm proposed by Chen and Lio (2010) for estimating the parameters of GE distribution when items lie under progressive type-I censoring scheme. For simulating a $\{X_i, R_i, T_i\}_{i=1}^{m}$ scheme we use the algorithm proposed by Chen and Lio (2010). We consider four scenarios as:

$$p_{(1)} = (0.25, 0.25, 0.25, 0.25, 0.50, 0.50, 0.50, 0.50, 1),$$
$$p_{(2)} = (0.50, 0.50, 0.50, 0.50, 0.25, 0.25, 0.25, 0.25, 1),$$
$$p_{(3)} = (0, 0, 0, 0, 0, 0, 0, 0, 1),$$

and $p_{(4)} = (0.25, 0, 0, 0, 0, 0, 0, 0, 1)$. Under each of above four scenarios, we simulate $n = 112$ observations from GE distribution with shape parameter $\alpha = 1.5$ and rate parameter $\lambda = 0.06$ and $m = 9$ pre-specified inspection times including: $t_1 = 5.5$, $t_2 = 10.5$, $t_3 = 15.5$, $t_4 = 20.5$, $t_5 = 25.5$, $t_6 = 30.5$, $t_7 = 40.5$, $t_8 = 50.5$, and termination time is $t_9 = 60.5$. These settings was used by Chen and Lio (2010). We run simulations for 1000 times when the ML method, proposed EM algorithm in this paper (called here EM), and proposed EM algorithm by Chen and Lio (2010) (called here EM-Chen) take part in the competition. We note that the starting values for implementing both of EM and EM-Chen algorithms are $\alpha^{(0)} = 1$ and $\lambda^{(0)} = 0.5$. The stopping criterion for both algorithms is $\max\{|\alpha^{(t+1)} - \alpha^{(t)}|, |\lambda^{(t+1)} - \lambda^{(t)}|\} \leq 0.000001$; for $t = 0, \ldots, 100$. The time series plots of the estimators are displayed in Figures (1)-(2). The summary statistics including bias and mean of squared errors (MSE) of estimators are given in Table 1. Recall that the EM and EM-Chen algorithms give the same estimators for the shape parameter and hence time series plot of $\hat{\alpha}_{EM-Chen}$ disappeared in left-hand side subfigures of Figures (1)-(2). As it is seen from Table 1, proposed EM algorithm outperforms EM-Chen algorithm under the first, second, and fourth scenarios in terms of bias, and it outperforms the EM-Chen algorithm in all four scenarios in the sense of MSE. Also, the EM algorithm shows better performance than the ML approach under the first scenario in the sense of both bias and MSE criteria.

Table 1. Bias and MSE of $\hat{\alpha}_{EM}$, $\hat{\lambda}_{EM}$, $\hat{\alpha}_{ML}$, $\hat{\lambda}_{ML}$, $\hat{\alpha}_{EM-Chen}$, and $\hat{\lambda}_{EM-Chen}$ under four settings $p_{(1)}$, $p_{(2)}$, $p_{(3)}$, and $p_{(4)}$.

scenario	Estimator	bias $\hat{\alpha}$	MSE $\hat{\alpha}$	bias $\hat{\lambda}$	MSE $\hat{\lambda}$
$p_{(1)}$	EM	-0.03470	0.03709	0.01747	0.00033
	ML	0.05680	0.10186	0.00119	0.00012
	EM-Chen	-0.03470	0.03709	0.04212	0.00203
$p_{(2)}$	EM	0.10301	0.05822	0.03990	0.00162
	ML	0.07546	0.16765	0.00222	0.00027
	EM-Chen	0.10301	0.05822	0.08084	0.00701
$p_{(3)}$	EM	-0.21885	0.05793	-0.00581	0.00005
	ML	0.05504	0.06842	0.00140	0.00006
	EM-Chen	-0.21885	0.05793	0.00518	0.00007
$p_{(4)}$	EM	-0.23154	0.06314	0.00364	0.00003
	ML	0.05017	0.06794	0.00101	0.00007
	EM-Chen	-0.23154	0.06314	0.01484	0.00027

4. Conclusion

We have discovered that the EM algorithm proposed by Chen and Lio (Computational Statistics and Data Analysis 54: 1581-1591, 2010) for estimating the parameters of generalized exponential distribution under progressive type-I censoring scheme is incorrect. Here, the corrected EM algorithm is proposed and then a comparison study have been made to discover differences. Theoretically there is no difference between shape estimators of our proposed EM algorithm and that proposed by Chen and Lio (2010). However, for the rate parameter the difference is quite significant. A simulation study have been performed to show visually the differences between performance of our proposed EM algorithm, maximum likelihood estimators, and EM algorithm proposed by Chen and Lio (2010). We note that both of our proposed EM algorithm and EM algorithm proposed by Chen and Lio (2010) converge under all four scenarios before 20 iterations.

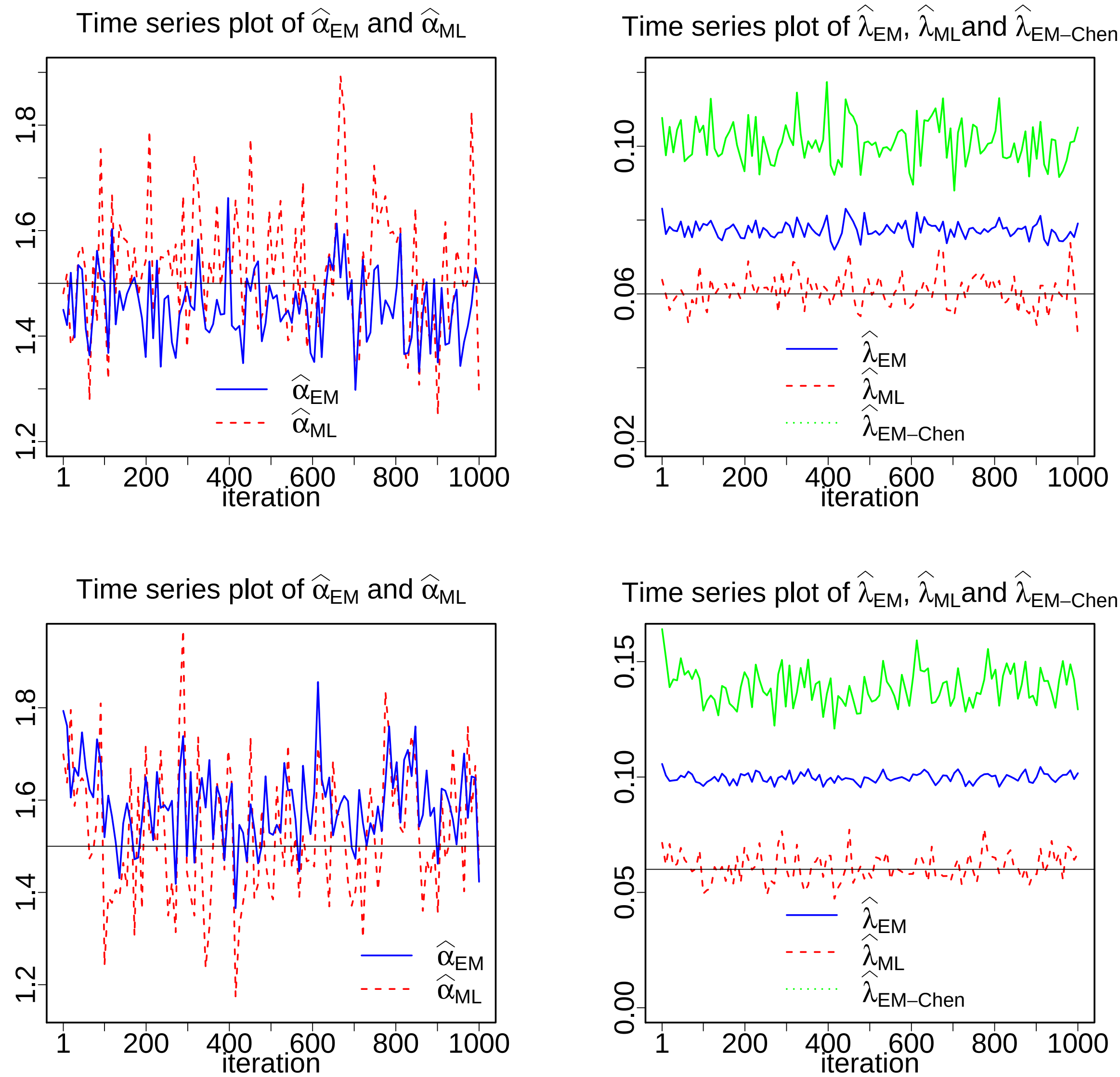

Fig. 1. Time series plot of $\hat{\alpha}_{EM}$, $\hat{\alpha}_{ML}$, and $\hat{\alpha}_{EM-Chen}$ under settings $p_{(1)}$ (top row) and $p_{(2)}$ (bottom row).

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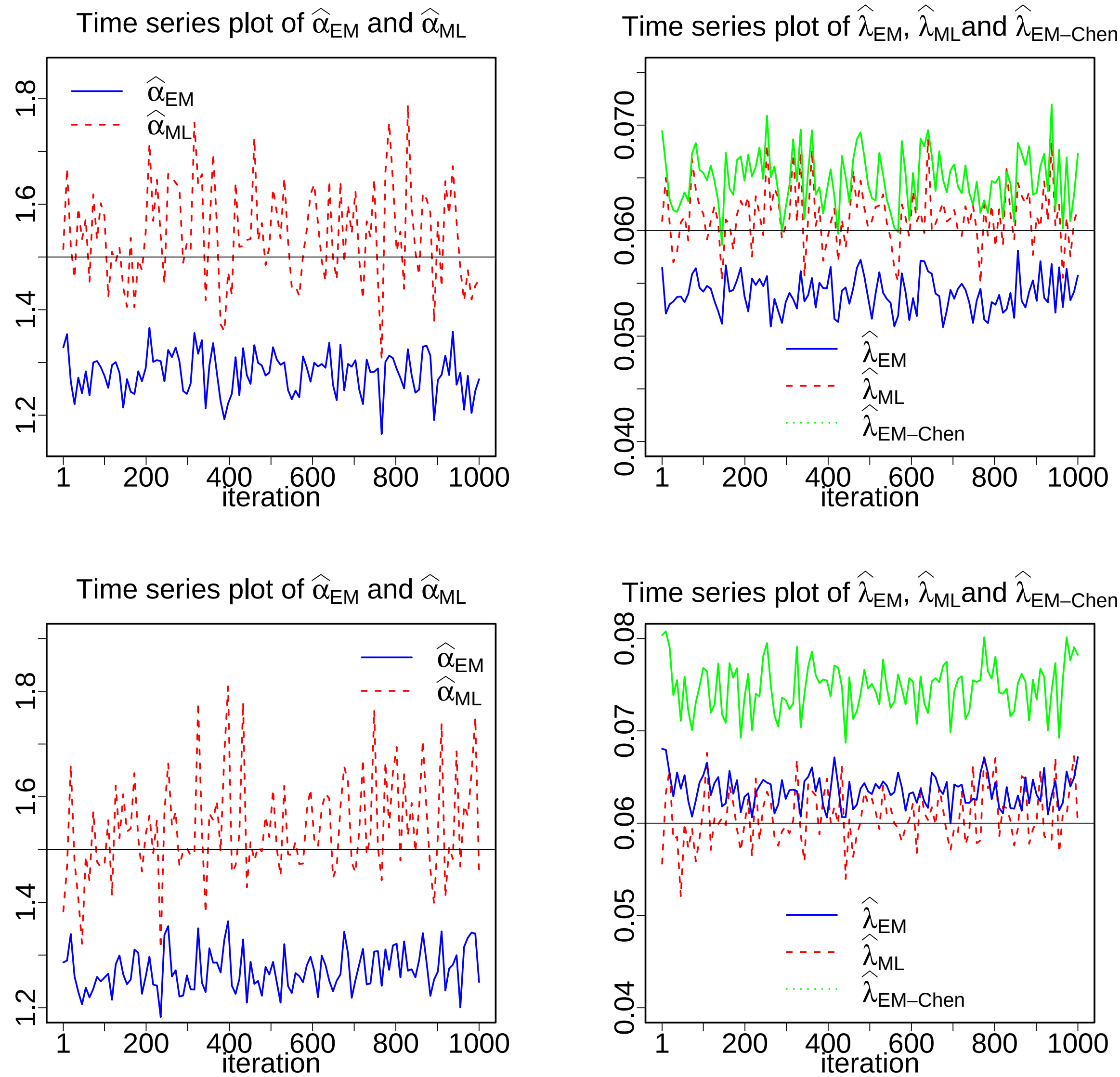

Fig. 2. Time series plot of $\hat{\alpha}_{EM}$, $\hat{\alpha}_{ML}$, and $\hat{\alpha}_{EM-Chen}$ under settings $p_{(3)}$ (top row) and $p_{(4)}$ (bottom row).

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Mixture of Cauchy distributions: Inference and Application

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Abstract

In this study, we use the skewed Cauchy distribution that is the central member of the class of stable distributions. The Cauchy distribution received much attention in several fields including economics, hydrology, image processing, physics, seismology, and signal processing. In this paper, we derive estimators for the parameters of the Cauchy and mixture of Cauchy distributions through the expectation-maximization (EM) algorithm. Performance of the presented EM algorithm is demonstrated through simulations and real data applications.

1. Introduction

The class of α-stable distributions includes a wide range of distributions that can control both of tails thickness and the skewness of the probability density function. So the α-stable distributions are becoming increasingly popular in many fields of studies. The Cauchy distribution, as the central member of the class of α-stable distributions, itself has many applications in a variety of fields such as econometric [5, 8, 33, 38], electronic [49], hydrology [24], image processing [17, 31], physics [28, 32, 44], seismology [22], signal processing [6, 14, 15], and tensor decomposition [50]. The probability density function of a Cauchy distribution has no closed-form expression and is represented through an integral form. The probability density function of Y can be represented as [37]:

$$f_Y(y|\boldsymbol{\theta}) = \frac{1}{\pi\sigma}\int_0^{+\infty} e^{-t}\cos\Big[\Big(\frac{y-\mu-\frac{2}{\pi}\beta\sigma\log\sigma}{\sigma}\Big)t + \beta\frac{2}{\pi}t\log t\Big]dt, \tag{1}$$

where $\boldsymbol{\theta} = (\beta, \sigma, \mu)^T$ is the parameter vector. Here, $\beta \in [-1, 1]$, $\sigma \in \mathbb{R}^+$, and $\mu \in \mathbb{R}$ account for the skewness, scale, and location parameters, respectively. We use the generic symbol $Ca(\beta, \sigma, \mu)$ to indicate the family of Cauchy distributions with probability density function given by (1). Evidently, $Ca(0, \sigma, \mu)$ is the ordinary symmetric Cauchy distribution with probability density function given by $f(y|\boldsymbol{\theta}) = \sigma/\big[\pi(\sigma^2 + (x-\mu)^2)\big]$. If $\beta = 1$ (or -1), we have the

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class of totally skewed to the right (or left) Cauchy distributions. The integral in right-hand side of (1) is computed numerically [36]. For standard case ($\mu = 0, \sigma = 1$), the probability density function of $Ca(\beta, \sigma, \mu)$ has been displayed in Figure 1. It should be noted that the probability density function of skew-Cauchy distribution induced by the Azzalini

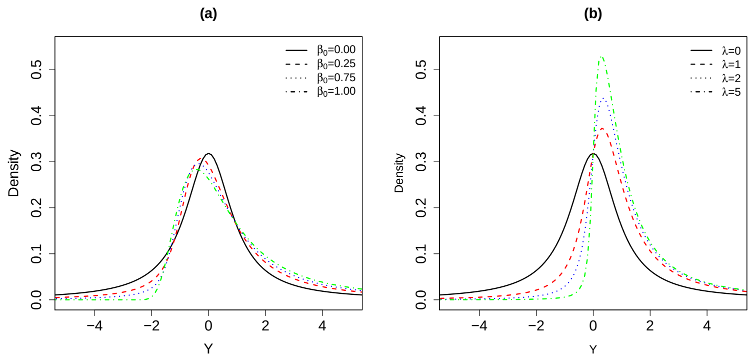

Fig. 1. The Probability density functions of the Cauchy induced by: (a) α-stable and (b) Azzalini (1985)'s skew-symmetric distributions.

(1985)'s methodology is given by [16]:

$$g(x) = \frac{1}{\pi\sigma\left(1+z^2\right)}\left\{1+\frac{2}{\pi}\,\text{arctan}(\lambda z)\right\},$$

where $z = (x - \mu)/\sigma$ and $-\infty < \lambda < +\infty$ is the skewness parameter. If $\lambda = 0$ and $\beta = 0$, then both of Cauchy distributions induced by α-stable and Azzalini (1985)'s method are the same. In what follows, some applications of Cauchy distribution in economics and hydrology are given.

- **Application in economics:** Studies show that a majority of financial variables follow a sharp peak and heavy-tailed distribution to which the Cauchy belongs [27]. Let P_t denotes the stock price at time t. The quantities P_t/P_{t-1} and $\log P_t/P_{t-1}$ are among the main variables for analyzing the stock market [18, 34]. If both of P_t and P_{t-1} independently come from a zero-mean normal distribution, then P_t/P_{t-1} follows a Cauchy distribution [3, 12, 30]. In practice, however, some deviation from symmetry is possible and so, it seems reasonable to suppose that a skewed Cauchy is an appropriate model for distribution of P_t/P_{t-1}. As an example, the monthly exchange rates between the US Dollar and the Tanzanian Shilling over period January 1975 to September 1997 has been studied in [38]. After fitting an α-stable to the log of monthly successive exchange rates, the maximum likelihood (ML) estimates with 95% confidence intervals are: $\alpha \in (1.088 \pm 0.185)$, $\beta \in (0.112 \pm 0.251)$, $\sigma \in (0.03 \pm 0.0055)$, and $\mu \in (0.005 \pm 0.00621)$. As it is seen, the confidence interval for α contains $\alpha = 1$, for which every α-stable distribution turns into a Cauchy distribution [37]. On the other hand, the confidence interval for β that is (-0.139, 0.363) has longer overlap with positive values leading us to believe that data are skewed to the right. So, based on this example, a Cauchy distribution with positive skewness parameter is an appropriate candidate for modelling the logarithm of monthly successive exchange rates in developing countries.

- **Application in hydrology:** Logarithm of the area-averaged rain rate over $L \times L$ square, denoted by $\log R_L$, is an important factor in hydrology for rain rate modelling. Now, distribution of R_L is restricted to be infinitely divisible and a suitable candidate for R_L is minus of a totally skewed to the left Cauchy distribution [24].

It is worth noting that the Cauchy distribution with probability density function given in (1) is infinitely divisible. The class of Cauchy distributions generated by skew-symmetric families are not generally infinite divisible [11]. The property of infinite divisibility received much attention in some study fields such as economics [13, 19] and hydrology [24].

The aim of this paper is twofold: (i) proposing the EM algorithm for computing the ML estimators of the Cauchy distributions and (ii) proposing the EM algorithm for computing the ML estimators of the mixture of Cauchy distributions. The remainder of this paper is organized as follows. Section 2 gives the EM algorithm for Cauchy distribution. The

EM algorithm for mixture of Cauchy distributions is presented in Section 3. Section 4 is devoted to the robust analysis and stopping criterion of the proposed EM algorithm for one-component and mixture models. Performance of the EM algorithm for estimating the parameters of Cauchy distribution in one- and two-component cases is demonstrated in Section 5 through simulation and real data analysis. We conclude the paper in Section 6.

2. EM algorithm for Cauchy distribution

In the following, Theorem 2.1 gives a new stochastic representation for class Ca(β,σ,μ). This representation plays main role for implementing the EM algorithm.

Theorem 2.1. *Suppose random variables $Y \sim Ca(\beta,\sigma,\mu)$, $Z, N \sim \mathcal{N}(0,1)$, and $P \sim Ca(1,1,0)$ are mutually independent. Then,*

$$Y \stackrel{d}{=} \eta \frac{N}{Z} + \lambda P + \delta, \tag{2}$$

where $\stackrel{d}{=}$ denotes the equality in distribution, $\eta = \sigma\left(1-|\beta|\right)$, and $\lambda=\sigma\beta$, and $\delta = \mu + \frac{2}{\pi}\lambda \log|\lambda|$.

Proof: Proof of Theorem 2.1 is given in Appendix A. The EM algorithm is the most popular approach for computing the maximum likelihood (ML) estimators of a statistical model parameters when we face with the missing or latent data problem [10, 35, 39]. This algorithm is an iterative approach and always converges [26]. The log-likelihood function of class $Ca(\beta,\sigma,\mu)$ has complicated form and hence computing the ML estimators is a so difficult task. Fortunately, as it is seen from (2), each Cauchy random variable can be represented as a complete data problem and admits a hierarchy given by the following.

$$\begin{aligned} Y_i|Z_i = z_i, P_i = p_i &\sim \mathcal{N}\Big(\delta + \lambda p_i, \frac{\eta^2}{z_i^2}\Big), \\ Z_i &\sim \mathcal{N}(0,1), \\ P_i &\sim Ca(1,1,0), \end{aligned} \tag{3}$$

where Z_i and P_i are independent, for $i = 1,\cdots,n$. Assume that $\boldsymbol{y} = (y_1,\cdots,y_n)^T$ constitutes a sequence of identically and independent realizations from $Ca(\beta,\sigma,\mu)$. Let $\boldsymbol{\xi} = \big(\boldsymbol{\xi}_1^T,\ldots,\boldsymbol{\xi}_n^T\big)^T = \big((y_1,p_1,z_1),\ldots,(y_n,p_n,z_n)\big)^T$ denote the vector of complete data wherein $\boldsymbol{z} = (z_1,\cdots,z_n)^T$ and $\boldsymbol{p} = (p_1,\cdots,p_n)^T$ are realizations of unobservable variables $\boldsymbol{Z}$ and $\boldsymbol{P}$, respectively. Based on hierarchy (3), the complete data log-likelihood function $l_c(\Theta)$ becomes

$$l_c(\Theta) = \mathrm{C} - \sum_{i=1}^{n} \log\frac{\eta}{z_i} - \frac{1}{2}\sum_{i=1}^{n}\Big(\frac{y_i-\delta-\lambda p_i}{\eta}\Big)^2 z_i^2,$$

where C is a constant independent of the parameters vector $\Theta = (\eta,\lambda,\delta)^T$. The conditional expectation $Q\big(\Theta\big|\Theta^{(t)}\big) = E\big(l_c(\Theta;\boldsymbol{p},\boldsymbol{z})\big|\Theta^{(t)},\boldsymbol{y}\big)$ of complete data log-likelihood function is given by

$$\begin{aligned} Q\big(\Theta\big|\Theta^{(t)}\big) =& \mathrm{C} - n\log\eta - \frac{1}{2\eta^2}\sum_{i=1}^{n}(y_i-\delta)^2 E\big(Z_i^2\big|\Theta^{(t)},y_i\big) \\ &+ \frac{\lambda}{\eta^2}\sum_{i=1}^{n}(y_i-\delta)E\big(Z_i^2P_i\big|\Theta^{(t)},y_i\big) - \frac{\lambda^2}{2\eta^2}\sum_{i=1}^{n}E\big(Z_i^2P_i^2\big|\Theta^{(t)},y_i\big), \end{aligned} \tag{4}$$

where $\Theta^{(t)} = \big(\eta^{(t)},\lambda^{(t)},\delta^{(t)}\big)^T$. In order to complete the E-step of the EM algorithm we need to compute the conditional expectations given in (4). We have

$$E_{ri}^{(t)} = E\Big(Z_i^2P_i^r\Big|\Theta^{(t)},y_i\Big) = \frac{2}{\pi^2\eta f\big(y|\boldsymbol{\theta}\big)}\int_{-\infty}^{+\infty}\int_{0}^{+\infty}\frac{p^r\exp\{-t\}\cos\Big(pt+\frac{2}{\pi}t\log t\Big)}{\Big[1+\big(\frac{y-\delta-\lambda p}{\eta}\big)^2\Big]^2}dpdt. \tag{5}$$

Details for approximating the required expectations for completing E-step are given in Appendix B. The steps of the EM algorithm are given by the following.

- **E-step**: Given a current guess of Θ, i.e., $\Theta^{(t)}$, compute $E_{ri}^{(t)}$ given in (5) for $r=0,1,2$ and $i=1,\cdots,n$.
- **M-step**: Update $\Theta^{(t)}$ as $\Theta^{(t+1)}$ by maximizing $Q\big(\Theta\big|\Theta^{(t)}\big)$ given in (4) with respect to δ, η, and λ. We have

$$\big(\eta^{(t+1)}\big)^2 =\frac{\sum_{i=1}^{n}\big(y_i-\delta^{(t)}\big)^2E_{0i}^{(t)}+\big(\lambda^{(t)}\big)^2\sum_{i=1}^{n}E_{2i}^{(t)}}{n}-2\frac{\lambda^{(t)}\sum_{i=1}^{n}\big(y_i-\delta^{(t)}\big)E_{1i}^{(t)}}{n}, \tag{6}$$

$$\lambda^{(t+1)} =\frac{\sum_{i=1}^{n}\big(y_i-\delta^{(t)}\big)E_{1i}^{(t)}}{\sum_{i=1}^{n}E_{2i}^{(t)}}, \tag{7}$$

$$\delta^{(t+1)} =\frac{\sum_{i=1}^{n}y_iE_{0i}^{(t)}-\lambda^{(t+1)}E_{1i}^{(t)}}{\sum_{i=1}^{n}E_{0i}^{(t)}}. \tag{8}$$

Both of E- and M-steps are repeated until convergence occurs.

Using the quantities given in (6)-(8), the parameter vector $\Theta^{(t)}$ is updated as $\Theta^{(t+1)}=\big(\eta^{(t+1)},\lambda^{(t+1)},\delta^{(t+1)}\big)^T$. Once we have obtained $\Theta^{(t+1)}$, the parameter vector $\boldsymbol{\theta}^{(t)}$ is updated as $\boldsymbol{\theta}^{(t+1)}=\big(\beta^{(t+1)},\sigma^{(t+1)},\mu^{(t+1)}\big)^T$. For this aim, the root of the equation $\beta\eta^{(t+1)}-\lambda^{(t+1)}(1-|\beta|)=0$ is considered as $\beta^{(t+1)}$. Also, we have $\sigma^{(t+1)}=\lambda^{(t+1)}/(1-|\beta^{(t+1)}|)$ and $\mu^{(t+1)}=\delta^{(t+1)}-2/\pi\beta^{(t+1)}\sigma^{(t+1)}\log|\beta^{(t+1)}\sigma^{(t+1)}|$, for $|\beta^{(t+1)}|\neq 1$.

Remark 2.2. *Computing the quantities given in right-hand side of (5), requires to evaluate three two-dimensional integrates in each iteration of the EM algorithm. For reduce the computational burden, we used the Monte Carlo approximation of these integrals.*

3. EM algorithm for mixture of Cauchy distributions

The mixture of α-stable distributions has been considered as a suitable model for statistical analysis of the phenomena with multimodal and heavy-tailed relative frequency [7, 41, 42, 46]. The works by [7] and [41] focused on estimating the parameters of the mixture of α-stable distributions that is computationally cumbersome. [42] proposed the Bayesian framework for estimating the parameters of the symmetric α-stable mixture model that possessing a lesser number of parameters an so, requires less computational effort compared with that of [7] and [41]. Additionally, [46] proposed the EM algorithm for estimating the parameters of the mixture of symmetric α-stable distributions that is faster than all these works. There is a considerable difference between the work of [46] and that we are dealing with in this work. In the work by [46], for each component, the parameter α varies in (0,2] and parameter β is zero. In contrast, in this work, for each component the parameter β varies in (-1,1) and parameter α is one. The probability density function $\mathcal{F}_Y(y|\boldsymbol{\Psi})$, of the Cauchy mixture model is represented by

$$\mathcal{F}_Y(y|\boldsymbol{\Psi})=\sum_{j=1}^{K}\omega_j f_Y(y|\boldsymbol{\theta}_j), \tag{9}$$

where the constant K denotes the number of components, $\boldsymbol{\theta}_j=\big(\beta_j,\sigma_j,\mu_j\big)^T$, for $j=1,\cdots,K$ is the parameter vector of the j-th component, $f_Y(y|\boldsymbol{\theta}_j)$ is the probability density function of the j-th component in class $Ca\big(\beta_j,\sigma_j,\mu_j\big)$, and $\boldsymbol{\Psi}=\big(\omega_1,\boldsymbol{\theta}_1^T,\cdots,\omega_K,\boldsymbol{\theta}_K^T\big)^T$ is the whole parameter vector. The vector of mixing parameters is shown by $\omega=\big(\omega_1,\cdots,\omega_K\big)^T$ in which w_js, for $j=1,\cdots,K$ are non-negative values summing up to one.

3.1. Identifiability of the mixture of Cauchy distributions

A finite mixture distribution is identifiable if distinct mixing distributions with finite support correspond to distinct mixtures. The concept of identifiability plays main role in parameter estimation and hypothesis testing. It is shown that the label switching can lead to difficulties when model-based clustering is carried out within the Bayesian paradigm [45], but when we are working within the EM algorithm framework, lack of identifiability can be resolved by defining some ordering on the mixing parameters [4].

Definition 3.1. [Holzmann et al. (2006)] Finite mixtures from the family $\mathcal{G} = \{f(x|\Lambda)|\ \Lambda = (\lambda_1, \cdots, \lambda_d)^T \in \mathbb{R}^d\}$ are identifiable if the relation of the form

$$\sum_{j=1}^{K} \omega_j f(x|\Lambda_j) = \sum_{j=1}^{K} \omega_j^{'} f(x|\Lambda_j^{'}),$$

where $\sum_{j=1}^{K} \omega_j = \sum_{j=1}^{K} \omega_j^{'} = 1$ and $\omega_j, \omega_j^{'} \geq 0$ for $j = 1, \cdots, K$, implies that there exists a permutation $q \in S_K$ such that $(\omega_j, \Lambda_j) = \left(\omega_{q(j)}^{'}, \Lambda_{q(j)}^{'}\right)$ for all j.

It is known that symmetric Cauchy mixture models are identifiable [20, Theorem 1]. For the Cauchy mixture model with probability density function given in (9) the following result holds.

Theorem 3.2. *Finite mixtures of the family* $\mathcal{G} = \left\{f_Y(y|\boldsymbol{\theta})|\ \boldsymbol{\theta} = (\beta, \sigma, \mu)^T : \beta \in [-1, 1], \sigma \in \mathbb{R}^+, \mu \in \mathbb{R}\right\}$ *are identifiable when* $\sigma\beta$ *is constant.*

Proof: Proof of Theorem 3.2 is given in Appendix C.

Corollary 3.3. *The class of Cauchy mixture models with probability density function given in (9) is identifiable if each components follows a symmetric ordinary Cauchy distribution, i.e., when* $\beta_j = 0$ *for* $j = 1, \cdots, K$.

As it is seen from Corollary 3.3, the call of Cauchy mixture models are not identifiable generally, but we have not observed any problem in simulation study due to the non-identifiability. It is worth noting that lack of identifiability can be resolved by defining some ordering on the mixing parameters [4].

3.2. Estimation of model parameters

The log-likelihood function of the Cauchy mixture model with probability density function given in (9) has complicated form and hence computing the ML estimators is a very difficult task. Fortunately each Cauchy random variable can be represented as a complete data problem that makes it easy to implement the EM algorithm. So, the EM algorithm can be applied for estimating the parameters of a Cauchy mixture model. We follow the method given by [39, 48] to represent the complete data log-likelihood function. The complete data related to the mixture model given in (9) is shown by $\boldsymbol{\xi} = \left(\boldsymbol{\xi}_1^T, \cdots, \boldsymbol{\xi}_n^T\right)^T = \left((y_1, p_1, z_1, \boldsymbol{b}_1^T), \cdots, (y_n, p_n, z_n, \boldsymbol{b}_n^T)\right)^T$ in which $\boldsymbol{y} = (y_1, \cdots, y_n)^T$ is vector of observed data, $\boldsymbol{p} = (p_1, \cdots, p_n)^T$ and $\boldsymbol{z} = (z_1, \cdots, z_n)^T$ are vectors of unobserved data, and $\boldsymbol{b} = (\boldsymbol{b}_1^T, \cdots, \boldsymbol{b}_n^T)^T$ where $\boldsymbol{b}_i = (b_{i1}, \cdots, b_{iK})^T$ as the realizations of the latent vector $\boldsymbol{B}_i = \left(B_{i1}, \cdots, B_{iK}\right)^T$. For each observed value such as y_i, for $i = 1, \cdots, n$, one of the components of $\boldsymbol{B}_i$ is one and others are zero. For instance, if y_i comes from the j-th component, then $B_{ij} = 1$ and $B_{ik} = 0$, for $k = 1, \cdots, K$ and $k \neq j$. If random variable Y has a probability density function of the form $\mathcal{F}_Y(y|\boldsymbol{\Psi})$, then Y admits a hierarchy given by the following.

$$\begin{aligned} Y_i\big|Z_i = z_i, P_i = p_i, B_{ij} = 1 &\sim \mathcal{N}\Big(\delta_j + \lambda_j p_i, \frac{\eta_j^2}{z_i^2}\Big), \\ P_i &\sim Ca(1, 1, 0), \\ Z_i &\sim \mathcal{N}(0, 1), \\ \boldsymbol{B}_i &\sim \mathcal{M}ultinomial(1, \omega_1, \ldots, \omega_K), \end{aligned} \tag{10}$$

for $j = 1, \ldots, K$ and $i = 1, \ldots, n$. Based on representation (10) the complete data log-likelihood function is

$$l_c(\Theta; \boldsymbol{p}, \boldsymbol{z}, \boldsymbol{b}) = \mathrm{C} + \sum_{i=1}^{n}\sum_{j=1}^{K} b_{ij} \log \omega_j - \sum_{i=1}^{n}\sum_{j=1}^{K} b_{ij} \left(\log \frac{\eta_j}{z_i} + \frac{\left(y_i - \delta_j - \lambda_j p_i\right)^2 z_i^2}{2\eta_j^2}\right), \tag{11}$$

where $\Theta = \left(\Theta_1^T, \ldots, \Theta_K^T\right)^T = \left((\omega_1, \eta_1, \lambda_1, \delta_1), \ldots, (\omega_K, \eta_K, \lambda_K, \delta_K)\right)^T$ and C is a constant independent of Θ. Taking the expectation after expanding and rearranging the right-hand side of (11), the conditional expectation $Q\left(\Theta\big|\Theta^{(t)}\right) =$

$E\big(l_c(\Theta;\boldsymbol{p},\boldsymbol{z},\boldsymbol{b})\big|\Theta^{(t)},\boldsymbol{y}\big)$ of the complete data log-likelihood function at the t-th iteration of the EM algorithm becomes

$$Q\big(\Theta|\Theta^{(t)}\big)=\sum_{i=1}^{n}\sum_{j=1}^{K}E_{1ij}^{(t)}\log\omega_j-\sum_{i=1}^{n}\sum_{j=1}^{K}E_{1ij}^{(t)}\log\eta_j-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{K}\Big(\frac{y_i-\delta_j}{\eta_j}\Big)^2E_{2ij}^{(t)}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{K}\frac{(y_i-\delta_j)\lambda_j}{\eta_j^2}E_{3ij}^{(t)}-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{K}\Big(\frac{\lambda_j}{\eta_j}\Big)^2E_{4ij}^{(t)},$$

where

$$E_{1ij}^{(t)}=E\left(B_{ij}\Big|\Theta^{(t)},y_i\right)=\frac{\omega_j^{(t)}f_Y\big(y_i|\boldsymbol{\theta}_j^{(t)}\big)}{\sum_{j=1}^{K}\omega_j^{(t)}f_Y\big(y_i|\boldsymbol{\theta}_j^{(t)}\big)},\tag{12}$$
$$E_{2ij}^{(t)}=E\Big(B_{ij}Z_i^2\Big|\Theta^{(t)},y_i\Big)=E_{1ij}^{(t)}E\Big(Z_i^2\Big|\Theta_j^{(t)},y_i\Big),\tag{13}$$
$$E_{3ij}^{(t)}=E\Big(B_{ij}Z_i^2P_i\Big|\Theta^{(t)},y_i\Big)=E_{1ij}^{(t)}E\Big(Z_i^2P_i\Big|\Theta_j^{(t)},y_i\Big),\tag{14}$$
$$E_{4ij}^{(t)}=E\Big(B_{ij}Z_i^2P_i^2\Big|\Theta^{(t)},y_i\Big)=E_{1ij}^{(t)}E\Big(Z_i^2P_i^2\Big|\Theta_j^{(t)},y_i\Big),\tag{15}$$

in which $\Theta_j^{(t)}=\big(\omega_j^{(t)},\eta_j^{(t)},\lambda_j^{(t)},\delta_j^{(t)}\big)^T$. The quantities $E\big(Z_i^2P^r\big|\Theta_j^{(t)},y_i\big)$, for $r=0,1,2$, given in the right-hand side of (13)-(15) are computed in the same fashion as described in (5). Both of E- and M-steps of the EM algorithm are given by the following.

- **E-step**: At the t-th iteration, given a guess of Θ such as $\Theta^{(t)}$, the quantities $E\big(Z_i^2P_i^r\big|\Theta_j^{(t)},y_i\big)$ given in (12)-(15) are computed for $r=0,1,2$, $j=1,\cdots,K$, and $i=1,\cdots,n$.
- **M-step**: At the t-th iteration, the M step maximizes $Q\big(\Theta\big|\Theta^{(t)}\big)$ with respect to Θ_j to obtain the elements of $\Theta_j^{(t+1)}$ as follows.

$$\omega_j^{(t+1)}=\frac{\omega_j^{(t)}f_Y\big(y_i|\boldsymbol{\theta}_j^{(t)}\big)}{\sum_{j=1}^{K}\omega_j^{(t)}f_Y\big(y_i|\boldsymbol{\theta}_j^{(t)}\big)},\tag{16}$$
$$\delta_j^{(t+1)}=\frac{\sum_{i=1}^{n}y_iE_{2ij}^{(t)}-\lambda_j^{(t)}\sum_{i=1}^{n}E_{3ij}^{(t)}}{\sum_{i=1}^{n}E_{2ij}^{(t)}},\tag{17}$$
$$\Big(\eta_j^{(t+1)}\Big)^2=\frac{\sum_{i=1}^{n}E_{2ij}^{(t)}\Big(y_i-\delta_j^{(t+1)}\Big)^2}{\sum_{i=1}^{n}E_{1ij}^{(t)}}+\frac{\Big(\lambda_j^{(t)}\Big)^2\sum_{i=1}^{n}E_{4ij}^{(t)}}{\sum_{i=1}^{n}E_{1ij}^{(t)}}-2\lambda_j^{(t)}\frac{\sum_{i=1}^{n}E_{3ij}^{(t)}\Big(y_i-\delta_j^{(t)}\Big)}{\sum_{i=1}^{n}E_{1ij}^{(t)}},\tag{18}$$
$$\lambda_j^{(t+1)}=\frac{\sum_{i=1}^{n}E_{3ij}^{(t)}\Big(y_i-\delta_j^{(t+1)}\Big)}{\sum_{i=1}^{n}E_{4ij}^{(t)}},$$

for $j=1,\ldots,K$. Using the quantities given in (16)-(18), the parameter vector $\Theta^{(t)}$ is updated as $\Theta_j^{(t+1)}=\big(\omega^{(t+1)},\eta^{(t+1)},\lambda^{(t+1)},\delta^{(t+1)}\big)^T$, for $j=1,\cdots,K$. Once we have obtained $\Theta^{(t+1)}$, the elements of the parameter vector $\boldsymbol{\Psi}^{(t)}$ are updated, for $j=1,\cdots,K$, as follows. The vector of mixing parameters is updated from (16). For updating $\boldsymbol{\theta}_j^{(t)}=\big(\beta_j^{(t)},\sigma_j^{(t)},\mu_j^{(t)}\big)^T$, note that the root of the equation $\beta\eta_j^{(t+1)}-\lambda_j^{(t+1)}(1-|\beta|)=0$ is considered as $\beta_j^{(t+1)}$. Also, we have $\sigma_j^{(t+1)}=\lambda_j^{(t+1)}/(1-|\beta_j^{(t+1)}|)$ and $\mu_j^{(t+1)}=\delta_j^{(t+1)}-2/\pi\beta_j^{(t+1)}\sigma_j^{(t+1)}\log|\beta_j^{(t+1)}\sigma_j^{(t+1)}|$, for $|\beta_j^{(t+1)}|\neq 1$.

Remark 3.4. *Analogous to the one-component case, computing the quantities given in right-hand sides of (13)-(15), requires to evaluate three two-dimensional integrates in each iteration of the EM algorithm for each component. For reduce the computational burden, we used the Monte Carlo approximation of these integrals.*

4. Initial values, robustness analysis, and stopping criterion

Here we perform some investigations about the initial values, robustness, and stopping criterion of the EM algorithm proposed in Section 2 (for one-component) and Section 3 (for mixture model).

4.1. Initial values and robustness analysis for one-component model

The initial (starting) values for implementing the EM algorithm play important role to reach convergence. It turns out that the estimators of the parameters based on empirical characteristic function (ECF) method suggested in [23] provide reasonable initial values for implementing the EM algorithm introduced in Section 2. In what follows, we compare the performance of the EM algorithm when the initial values are provided using two scenarios. In the first scenario, we use the ECF estimators as the initial values and in the second scenario the initial values are drawn from uniform distributions. In the second scenario, for implementing the EM algorithm in each run, the initial values of the parameters, i.e., β, σ, and μ follow $U(-1,1)$, $U(0.1,10)$, and $U(-10,10)$, respectively. During comparison study, we set $\beta=0.9$, $\sigma=1$, and $\mu=0$. The central processing unit (CPU) usage time is shown in Figure 2. As it is seen,

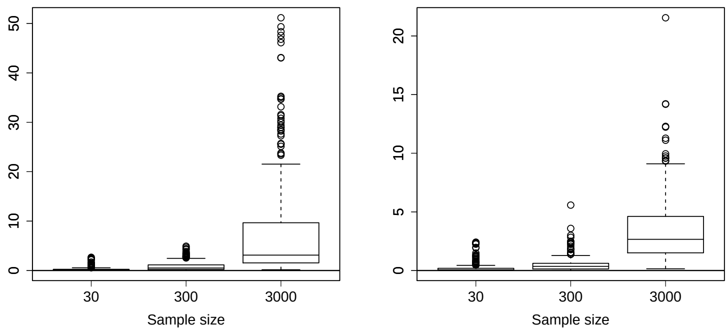

Fig. 2. Left-hand side: CPU usage time (in seconds) when the initial values are allowed to come from uniform distribution. Right-hand side: CPU usage time (in seconds) when the ECF estimators are used as the initial values.

the CPU usage time in left-hand side are clearly more scattered than those of the right-hand side. This means that the ECF estimators speed up the convergence of the EM algorithm. So, we use the ECF estimators as the initial values. Also, as a further investigation, the EM algorithm introduced in this work has a complexity of $O(n^2)$ where n is the sample size.

4.2. Stopping criterion for the one-component case

Suppose $l(\boldsymbol{\theta}^{(t)})$ denotes the log-likelihood value at t-th iteration of the EM algorithm and is defined as

$$l(\boldsymbol{\theta}^{(t)})=\sum_{i=1}^{n}\log f_Y\big(y_i\big|\boldsymbol{\theta}^{(t)}\big). \tag{19}$$

Based on Aitken's acceleration criterion, the convergence occurs id the predicted value, i.e.,

$$l_\infty=l(\boldsymbol{\theta}^{(t-1)})+\frac{1}{1-LR^{(t)}}\Big[l(\boldsymbol{\theta}^{(t)})-l(\boldsymbol{\theta}^{(t-1)})\Big],$$

is substantially larger than $l(\boldsymbol{\theta}^{(t)})$ or

$$LR^{(t)}=\frac{l(\boldsymbol{\theta}^{(t+1)})-l(\boldsymbol{\theta}^{(t)})}{l(\boldsymbol{\theta}^{(t)})-l(\boldsymbol{\theta}^{(t-1)})},$$

is close to zero [25]. Unfortunately, the Aitken's acceleration criterion dose not work reasonably well for the EM algorithm presented in this work, since in each iteration the expected values in E-step are approximated by the Monte Carlo simulation. So, we suggest the use of an algorithm given by the following for stopping the EM algorithm.

1. Suppose $\boldsymbol{\theta}^{(0)}$ denotes the initial value of the parameter vector. Set $\mathcal{R}^{(1)}_{\boldsymbol{\theta}^{(0)}} = 1$;
2. Set $g = 2$;
3. Suppose we are currently at $(100g + 1)$-th iteration of the EM algorithm. Evaluate

$$\mathcal{R}^{(g)}_{\boldsymbol{\theta}^{(100g+1)}} = \frac{l(\boldsymbol{\theta}^{(100g+1)}) - 2l(\boldsymbol{\theta}^{(100(g-1)+1)}) + l(\boldsymbol{\theta}^{(100(g-2)+1)})}{l(\boldsymbol{\theta}^{(100(g-1)+1)}) - l(\boldsymbol{\theta}^{(100(g-2)+1)})};$$

4. If

$$\min\left\{\mathcal{R}^{(g)}_{\boldsymbol{\theta}^{(g)}}, \frac{\mathcal{R}^{(g)}_{\boldsymbol{\theta}^{(g)}} - \mathcal{R}^{(g-1)}_{\boldsymbol{\theta}^{(g-1)}}}{\mathcal{R}^{(g-1)}_{\boldsymbol{\theta}^{(g-1)}}}\right\} < 0.05,$$

then go to the next step; otherwise set $g = g + 1$ and go to step (3) of the algorithm;
5. Accept $\boldsymbol{\theta}^{(100g+1)}$ as the EM estimators for the parameter vector and stop the EM algorithm.

4.3. Initial values and robustness analysis for mixture model

For estimating the parameters of the Cauchy mixture model using the EM algorithm proposed in Section 3, the initial values are determined by K-means clustering. For this purpose, before implementing the EM algorithm, observations are partitioned into K groups and assuming that observations within each group follow a Cauchy distribution, the ECF method is applied to each group for estimating the parameters of each group. The ratio of group sizes to the number of observations is considered as the initial values of the mixing parameters.

In order to investigate the robustness of the EM algorithm with respect to initial values, we simulate 600 samples of size 1000 under four scenarios as described in Table 1. For each scenario, the initial values come from uniform distribution over (a, b) as indicated in Table 1 by u_a^b. The results of simulations are shown in Figure 3. There are three curves in each sub-figure of Figure 3 including the estimated probability density function, the probability density function under true parameters of scenario, and probability density function under initial values of scenario. It should be noted that the averages of the estimated parameters after 600 runs are used to compute the estimated probability density function. We note that for drawing the probability density function of the Cauchy mixture models we have used software STABLE available at `http://www.robustanalysis.com`.

Table 1. Scenarios for checking the robustness of the EM algorithm when it is applied to two-component Cauchy mixture model. Note that $a \sim u_{0.2}^{0.8}$.

		Parameters							
Scenario	Parameter State	ω_1	ω_2	β_1	β_2	σ_1	σ_2	μ_1	μ_2
1	true value	0.7	0.3	0	0	0.5	0.5	-1	1
	initial value	a	1-a	$u_0^{0.9}$	$u_0^{0.9}$	u_3^5	u_3^5	u_{-15}^{-5}	u_5^{15}
2	true value	0.7	0.3	0	0	0.75	0.75	-1	1
	initial value	a	1-a	$u_0^{0.9}$	$u_0^{0.9}$	u_4^6	u_4^6	u_{-15}^{-5}	u_5^{15}
3	true value	0.7	0.3	0.9	0.9	0.5	0.5	-1	1
	initial value	a	1-a	$u_{-0.5}^{0.5}$	$u_{-0.5}^{0.5}$	u_3^5	u_3^5	u_{-15}^{-5}	u_5^{15}
4	true value	0.7	0.3	0.9	0.9	0.75	0.75	-1	1
	initial value	a	1-a	$u_{-0.5}^{0.5}$	$u_{-0.5}^{0.5}$	u_4^6	u_4^6	u_{-15}^{-5}	u_5^{15}

4.4. Stopping criterion for the mixture model

As a criterion for stopping the EM algorithm applied to the mixture of Cauchy distributions, we use the stopping criterion for the EM algorithm that has been applied to the Cauchy distribution introduced in Subsection 4.2.

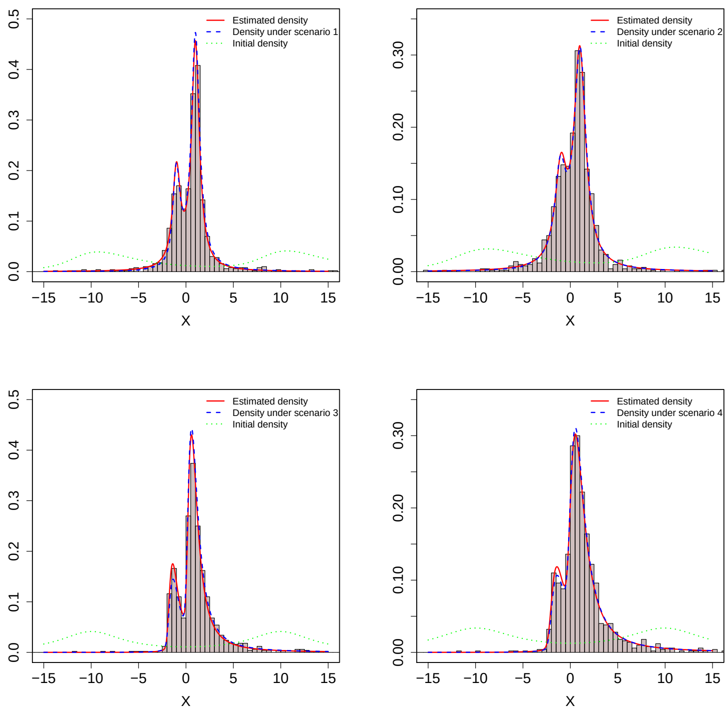

Fig. 3. Histogram of 1000 observations from two-component Cauchy mixture model under four scenarios. The first scenario (top left), second scenario (top right), third scenario (bottom left), and fourth scenario (bottom right). Superimposed are estimated probability density function (blue solid curve), computed probability density function under true parameters of scenario (red dashed curve), and probability density function under initial values of scenario (dotted black curve).

5. Simulation and real data analysis

Here, the performance of the EM algorithm proposed in Section 2 (for one-component) and Section 3 (for mixture model) will be demonstrated through simulations and real data applications. This section has four parts. In Subsection 5.1, we perform a simulation study to compare the performance of the EM and ML approaches for estimating the parameters of the class $Ca(\beta, \sigma, \mu)$. We use the software STABLE for computing the ML estimators. A simulation study is carried out in Subsection 5.2 in order to investigate performance of the EM algorithm for estimating parameters of the two-component Cauchy mixture model. Performance of the EM algorithm in one- and two-component cases will be demonstrated by applying it to the real data in Subsections 5.3 and 5.4, respectively. We note that the right-hand sides of (5) in both one- and multi-component cases is approximated by the Monte Carlo simulation.

5.1. One-component model validation via simulation

Here, we perform a simulation study to compare the performances of the EM and ML approaches for estimating the parameters of the class $Ca(\beta, \sigma, \mu)$. To do this, we set $\mu = 0, \sigma = 0.1, 2, 5,$ and $\beta = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.90$ and for each combination of these settings of parameters, a sample of 300 realizations are generated from class $Ca(\beta, \sigma, \mu)$. Comparisons between the EM and ML approaches are made based on the root of mean square error (RMSE). Since the required expectations in the E-step of the EM algorithm are approximated by the Monte Carlo simulation, so there is a small difference between RMSE of the EM and ML estimators as it is seen in Figure 4.

5.2. Mixture model validation via simulation

Here, we perform a simulation study to investigate the performance of the EM algorithm in estimating the parameters of two-component Cauchy mixture model. For this purpose, we generate a sample of size 1000 for 600 runs under each of four scenarios given by the following.

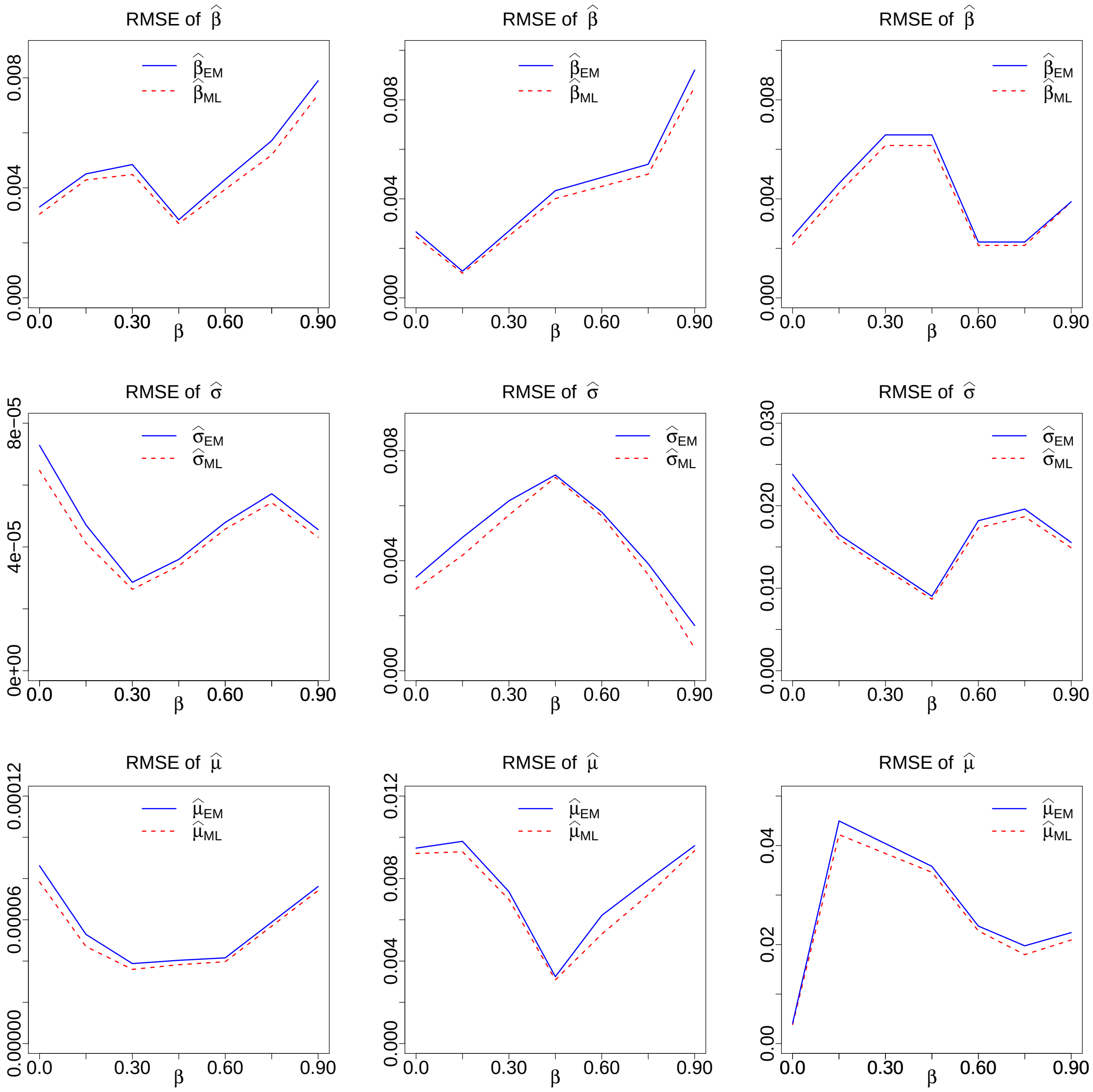

Fig. 4. The RMSE of estimators obtained through the EM and ML approaches. The estimators are obtained when realizations of size 300 are generated for from class $Ca(\beta, \sigma, \mu)$ for 600 runs. In each sub-figure, the subscripts ML and EM indicate that the estimators $\hat{\beta}$, $\hat{\sigma}$, and $\hat{\mu}$ are obtained using the EM algorithm (blue solid line) or the ML approach (red dashed line). Sub-figures in the first, second, and the third columns correspond to $\sigma = 0.10$, $\sigma = 2$, and $\sigma = 5$, respectively.

- $\boldsymbol{\beta} = (\beta_1, \beta_2)^T = (\beta, \beta)^T$ for $\beta = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.90$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^T = (0.25, 0.25)^T$, $\boldsymbol{\mu} = (\mu_1, \mu_2)^T = (-3, 3)^T$, and $\boldsymbol{\omega} = (\omega_1, \omega_2)^T = (0.5, 0.5)^T$.
- $\underline{\beta} = (\beta_1, \beta_2)^T = (\beta, \beta)^T$ for $\beta = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.90$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^T = (0.5, 0.5)^T$, $\boldsymbol{\mu} = (\mu_1, \mu_2)^T = (-3, 3)^T$, and $\boldsymbol{\omega} = (\omega_1, \omega_2)^T = (0.5, 0.5)^T$.
- $\underline{\beta} = (\beta_1, \beta_2)^T = (\beta, \beta)^T$ for $\beta = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.90$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^T = (0.5, 0.5)^T$, $\boldsymbol{\mu} = (\mu_1, \mu_2)^T = (-1, 1)^T$, and $\boldsymbol{\omega} = (\omega_1, \omega_2)^T = (0.7, 0.3)^T$.
- $\underline{\beta} = (\beta_1, \beta_2)^T = (\beta, \beta)^T$ for $\beta = 0.0, 0.15, 0.30, 0.45, 0.60, 0.75, 0.90$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)^T = (0.75, 0.75)^T$,

$\boldsymbol{\mu} = (\mu_1, \mu_2)^T = (-1, 1)^T$, and $\boldsymbol{\omega} = (\omega_1, \omega_2)^T = (0.7, 0.3)^T$.

The results of simulations are displayed in Figure 5 and Figure 6. We note that the ML approach that has been developed for the one-component stable distributions cannot be applied for the mixture of Cauchy distributions. Theorem 3.2 states that the class of the Cauchy mixture models are identifiable only under this condition that $\sigma\beta$ is constant, but we have not observed any source of non-identifiability during simulation study carried out in Subsection 4.3 (for robustness analysis) and herein. It should be noted that the problem of non-identifiability can be solved by defining some ordering on the mixing parameters [4].

5.3. One-component model validation using real data

Here, we consider the features related to large intensities (in Richter scale) of the earthquake recorded in seismometer locations in western North America between 1940 and 1980 [9, 21]. Among the features, we focus on 182 distances between seismological measuring station and epicenter of the earthquake (in km) as the variable of interest. This set of data can be found in package `nlme` developed for `R` environment. The histogram of distances superimposed by the estimated probability density function as well as the time series plots of the EM algorithm across iterations are shown in Figure 7. The EM-based estimators are $\hat{\mu} = 16.685$, $\hat{\sigma} = 11.423$, and $\hat{\beta} = 0.918$. The corresponding Kolmogorov-Smirnov (K-S) statistic is 0.0428. Furthermore, we consider other candidates for modelling this set of data including normal, skew normal, t, skew t, symmetric α-stable, and α-stable distributions. The K-S statistics correspond to these candidates are 0.2464, 0.2953, 0.1886, 0.0505, 0.178, and 0.034, respectively, leading us to believe that just α-stable works better than the Cauchy for modelling this set of data. This fact that α-stable provides better fit than the Cauchy distribution is not a strange outcome since α-stable is a wider class that involves the Cauchy distribution. We note that for modelling data using normal, skew normal, t, and skew t, we have used the package `sn` [1] and `mixsmsn` [40]. Also, for modelling data using α-stable distribution the package `STABLE`. All these packages have been developed for `R` environment.

5.4. Two-component mixture model validation using real data

Here, application of the two-component Cauchy mixture model will be illustrated using two sets of real data. The first set is related to the diagnostic tests on patients with cushing's syndrome [47] and the second set is the survival time in days of 72 guinea pigs infected with virulent tubercle bacilli [2]. For cushing's syndrome data that is available in `MASS` package developed for `R` environment, we focus on the urinary excretion rate (mg/24hr) of tetrahydrocortisone. We use the EM algorithm described in Section 3 for modelling this set of data using two-component Cauchy mixture model. To implement the EM algorithm, we use the initial values $\boldsymbol{\omega}_0 = (0.70, 0.30)^T$, $\boldsymbol{\sigma}_0 = (1, 3)^T$, $\boldsymbol{\beta}_0 = (0.95, 0.95)^T$, and $\boldsymbol{\mu}_0 = (3, 10)^T$. The updated parameters are plotted against iterations in Figure 8. The EM-based estimators of the parameters are $\hat{\boldsymbol{\omega}} = (0.481, 0.519)^T$, $\hat{\boldsymbol{\sigma}} = (0.967, 2.702)^T$, $\hat{\boldsymbol{\beta}} = (0.954, 0.920)^T$, and $\hat{\boldsymbol{\mu}} = (3.088, 9.849)^T$. The fitted distribution function shown in the left-hand side of Figure 9 captures general shape of the empirical distribution function well. The corresponding K-S statistic is 0.0852. For survival time of guinea pigs, we used initial values $\boldsymbol{\omega}_0 = (0.65, 0.35)^T$, $\boldsymbol{\sigma}_0 = (20, 55)^T$, $\boldsymbol{\beta}_0 = (0.20, 0.05)^T$, and $\boldsymbol{\mu}_0 = (110, 250)^T$. Applying the EM algorithm, the ML estimators are obtained as $\hat{\boldsymbol{\omega}} = (0.540, 0.460)^T$, $\hat{\boldsymbol{\sigma}} = (16.391, 39.119)^T$, $\hat{\boldsymbol{\beta}} = (0.123, 0.828)^T$, and $\hat{\boldsymbol{\mu}} = (107.166, 196.929)^T$. The corresponding K-S statistic is 0.0563. The fitted distribution function to the survival time of guinea pigs is shown in Figure 9 (right-hand side). As a further study, we demonstrate that two-component Cauchy mixture model (shown in Table 2 by MC) outperforms other candidates that can be considered for modelling above sets of data. These candidates consist of two-component mixtures of normal (MN), skew normal (MSN), Student's t (MT), skew Student's t (MST), and symmetric α-stable (MSαS). It turns out from Table 2 that MC provides better fit than other competitors for both survival time of guinea pigs and urinary excretion rate data. Furthermore, we followed

Table 2. Computed K-S statistic when two-component mixture models including normal (MN), skew normal (MSN), Student's t (MT), skew Student's t (MST), symmetric α-stable (MSαS), and Cauchy (MC) are fitted to survival time of guinea pigs and urinary excretion rate data.

	Candidate					
Dataset	MN	MSN	MT	MST	MSαS	MC
Survival time	0.0910	0.0672	0.1327	0.0776	0.071	0.0623
Urinary excretion rate	0.1337	0.1112	0.0886	0.1009	0.125	0.0798

the method of Louis (1982) for computing the observed Fisher information matrix (FIM) of the EM estimators, see Appendix D. By inverting observed FIM, we can find the standard error of the EM estimators. The standard errors of the estimators obtained using the EM algorithm applied to real data are given in Table 3.

Table 3. Computed standard errors of the estimators obtained using the EM algorithm applied to the real data.

	EM estimators							
	$\widehat{\omega_1}$	$\widehat{\beta_1}$	$\widehat{\sigma_1}$	$\widehat{\mu_1}$	$\widehat{\omega_1}$	$\widehat{\beta_2}$	$\widehat{\sigma_2}$	$\widehat{\mu_2}$
Survival time	0.0921	0.2877	3.9380	10.6309	0.1216	0.3992	10.7953	39.4761
Urinary excretion rate	0.1089	0.5298	0.2725	0.1354	0.1205	0.2078	1.0463	0.3773

6. Conclusion

In this paper, we have derived the expectation-maximization (EM) algorithm for estimating the parameters of Cauchy and mixture of Cauchy distributions. In both cases, performance of the EM algorithm has been shown by simulations and real data applications. It has been shown that the mixture of Cauchy distributions are robust with respect to the initial values. Also since Cauchy is a heavy-tail distribution, so the mixture of Cauchy distributions can be used appropriately for robust mixture modelling. Real data analysis demonstrated that a two-component Cauchy mixture model gives superior performance than other commonly used models such as mixture of normal, skew normal, t, and skew t distributions. Programs written in `R` environment for implementing the EM algorithm are available upon request.

Appendices

Appendix A. Proof of Theorem 2.1

Let T and P denote two independent random variables which follow $Ca(0,1,0)$ and $Ca(1,1,0)$, respectively. Define $Y = \eta T + \lambda P + \delta$ in which $\eta = \sigma(1-|\beta|)$, $\lambda = \sigma\beta$, and $\delta = \mu + 2/\pi\lambda \log|\lambda|$. We can write

$$\begin{aligned} E\exp(itY) =& E\exp\Big\{it\Big[\sigma(1-|\beta|)T + \lambda P + \delta\Big]\Big\} \\ =& E\exp\Big\{it\Big[\sigma(1-|\beta|)T + \delta\Big]\Big\} E\exp\Big\{it\lambda P\Big\} \\ =& \exp\Big\{-(1-|\beta|)|\sigma t| + it\delta\Big\} E\exp\Big\{it\lambda P\Big\} \\ =& \exp\Big\{-(1-|\beta|)|\sigma t| + it\mu + it\frac{2}{\pi}\sigma\beta\log|\sigma\beta|\Big\} E\exp\Big\{it\lambda P\Big\}. \end{aligned} \tag{A.1}$$

It follows from that

$$E\exp\Big\{it\lambda P\Big\} = \exp\Big\{-|\lambda t|\Big[1 + i\frac{2}{\pi}\text{sign}(\lambda)\text{sign}(t)\log|\lambda t|\Big]\Big\} = \exp\Big\{-|\sigma\beta t| - it\frac{2}{\pi}\sigma\beta\log|t| - it\frac{2}{\pi}\sigma\beta\log|\sigma\beta|\Big\}. \tag{A.2}$$

Substitute the right-hand side of (A.2) into the right-hand side of (A.1). It turns out that

$$\begin{aligned} E\exp(itY) =& \exp\Big\{-|\sigma t|\Big[1 + i\frac{2}{\pi}\beta\text{sign}(t)\log|t|\Big] + it\Big(\mu + \frac{2}{\pi}\sigma\beta\log|\sigma\beta| - \frac{2}{\pi}\sigma\beta\log|\sigma\beta|\Big)\Big\}, \\ =& \exp\Big\{-|\sigma t|\Big[1 + i\frac{2}{\pi}\beta\text{sign}(t)\log|t|\Big] + it\mu\Big\}, \end{aligned}$$

where the last expression is the chf of $Y \sim Ca(\beta,\sigma,\mu)$. Let N and Z denote two independent standard normal random variables. It is well-known that T can be represented as the ratio of N and Z. Therefore, $Y = \eta N/Z + \lambda P + \delta$ follows $Ca(\beta,\sigma,\mu)$. The proof is complete.

Appendix B. Computing $E\big(Z^2P^r\big|\Theta, y\big)$

Let $\mathcal{N}(a,b)$ denote a normal distribution with mean a and variance b, whose probability density function is shown by $g(.|a,b)$. The random variable Z follows $\mathcal{N}(0,1)$ and $f_Y(.)$ is the probability density function of $Y \sim Ca(\beta,\sigma,\mu)$. Also, $h(.)$ denotes the probability density function of $P \sim Ca(1,1,0)$. Define $I = E\big(Z^2P^r\big|\Theta, y\big)$, for $r = 0,1,2$. Based on hierarchy (3), the joint distribution of Z, P and Y can be represented as the product of $g\big(y\big|\delta+\lambda p, \eta^2/z^2\big)$, $g(z|0,1)$, and $h(p)$. So, we can write

$$I = \frac{\int_{\mathbb{R}}\int_{\mathbb{R}} z^2p^r g\big(y\big|\delta+\lambda p,\eta^2/z^2\big)g(z|0,1)h(p)dzdp}{f(y|\boldsymbol{\theta})} \tag{B.1}$$

Substituting the probability density functions $g\big(y\big|\delta+\lambda p,\eta^2/z^2\big)$ and $g(z|0,1)$ in the right-hand side of (B.1) and some algebraic simplifying, it follows that

$$I = \frac{1}{f\big(y|\boldsymbol{\theta}\big)}\int_{\mathbb{R}}\int_{\mathbb{R}} z^2p^r\frac{|z|e^{-\frac{z^2(1+q^2)}{2}}}{2\pi\eta}h(p)dzdp,$$

where $q = (y-\delta-\lambda p)/\eta$ and $\eta = \sigma\,(1-|\beta|)$. Use a change of variable of the form $z^2(1+q^2)/2 = w$ to see

$$I = \frac{2}{f\big(y|\boldsymbol{\theta}\big)}\int_{\mathbb{R}}\frac{p^rh(p)}{\pi\eta(1+q^2)^2}dp \tag{B.2}$$

where $r = 0,1,2$. It turns out from (1) that the probability density function of P that follows $Ca(1,1,0)$ can be represented as

$$h(p) = \frac{1}{\pi}\int_0^{+\infty}\exp\{-t\}\cos\Big(pt+\frac{2}{\pi}t\log t\Big)dt. \tag{B.3}$$

Substituting the right-hand side of (B.3) into the right-hand side of (B.2), we have

$$I = \frac{2}{\pi^2\eta f\big(y|\boldsymbol{\theta}\big)}\int_{-\infty}^{+\infty}\int_0^{+\infty}\frac{p^r\exp\{-t\}\cos\Big(pt+\frac{2}{\pi}t\log t\Big)}{\Big[1+\big(\frac{y-\delta-\lambda p}{\eta}\big)^2\Big]^2}dpdt. \tag{B.4}$$

The result follows.

Appendix C. Proof of Theorem 3.2

In order to investigate the identifiability of the Cauchy mixture model represented in (9) we follow the standard procedure given by Definition 3.1. Suppose $\mathcal{G} = \big\{f_Y(y|\boldsymbol{\theta}) \in Ca(\beta,\sigma,\mu)\big|\sigma\beta = \mathrm{C}, \beta\in[-1,1], \sigma\in\mathbb{R}^+, \mu\in\mathbb{R}\big\}$ where C denotes a constant. To start, assume that $\mu = 0$ and set $\boldsymbol{\gamma} = (\beta,\sigma)^T$. We order $\mathcal{G}$ by defining an ordering on $\boldsymbol{\gamma}$ as: $\boldsymbol{\gamma}_1 \prec \boldsymbol{\gamma}_2$ if $\sigma_2 > \sigma_1$ and $\beta_2 > \beta_1$. We can write

$$\begin{aligned}\lim_{t\to\infty}\frac{\phi_{\boldsymbol{\gamma}_2}(t)}{\phi_{\boldsymbol{\gamma}_1}(t)} &= \lim_{t\to\infty}\frac{\exp\Big\{-|\sigma_2 t|\,[1+i\frac{2}{\pi}\beta_2\,\mathrm{sgn}(t)\log|t|]\Big\}}{\exp\Big\{-|\sigma_1 t|\,[1+i\frac{2}{\pi}\beta_1\,\mathrm{sgn}(t)\log|t|]\Big\}}\\ &= \lim_{t\to\infty}\exp\Big\{-(\sigma_2-\sigma_1)|t| - i\frac{2}{\pi}(\sigma_2\beta_2-\sigma_1\beta_1)t\log|t|\Big\} = 0.\end{aligned} \tag{C.1}$$

Now, suppose that there exists a relation such that

$$\sum_{j=1}^{K}\pi_j f_Y(y|\boldsymbol{\theta}_j) = 0, \tag{C.2}$$

where $\pi_j \in \mathbb{R}$, $\boldsymbol{\theta}_j = (\beta_j, \sigma_j, \mu_j)^T$, and $f_Y(y|\boldsymbol{\theta}_j) \in Ca(\beta_j, \sigma_j, \mu_j)$ in which $\boldsymbol{\theta}_j$ is pairwise distinct. Applying the Fourier transform on the both sides of (C.2), we have

$$\sum_{j=1}^{K} \pi_j e^{it\mu_j} \phi_{\boldsymbol{\gamma}_j}(t) = 0. \tag{C.3}$$

For $m \geq 1$, define the ordering $\boldsymbol{\gamma}_1 \preceq \cdots \preceq \boldsymbol{\gamma}_K$ such that $\boldsymbol{\gamma}_1 = \cdots = \boldsymbol{\gamma}_m \preceq \boldsymbol{\gamma}_{m+1} \preceq \cdots \preceq \boldsymbol{\gamma}_K$. Divide both sides of (C.3) by $e^{it\mu_1}\phi_{\boldsymbol{\gamma}_1}(t)$ to see that

$$\pi_1 + \sum_{j=2}^{m} \pi_j e^{it(\mu_j-\mu_1)} + \sum_{j=m+1}^{K} \pi_j e^{it(\mu_j-\mu_1)} \frac{\phi_{\boldsymbol{\gamma}_j}(t)}{\phi_{\boldsymbol{\gamma}_1}(t)} = 0. \tag{C.4}$$

Recall that since $\lim_{t\to\infty} \phi_{\boldsymbol{\gamma}_2}(t)/\phi_{\boldsymbol{\gamma}_1}(t) = 0$, therefore the third term in (C.4) is zero when $t \to \infty$. Also, the second term goes to zero. This means that π_1=0 and an inductive argument completes the proof [20].

Appendix D. Observed Fisher information matrix (FMI)

Using the method proposed by [29], let us to rewrite the complete data log-likelihood function presented in (11) as follows.

$$L(\Theta|\boldsymbol{B}, \boldsymbol{y}) = \mathrm{C} + \sum_{i=1}^{n}\sum_{j=1}^{K} B_{ij} \log \omega_j + \sum_{i=1}^{n}\sum_{j=1}^{K} B_{ij} \log f_Y(y_i|\boldsymbol{\theta}), \tag{D.1}$$

where $f_Y(y_i|\boldsymbol{\theta})$ is the probability density function of $Y \sim Ca(\beta, \sigma, \mu)$ as defined in (1) and $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_K)^T$ is vector of mixing parameters. Further, $\boldsymbol{B} = \big(\boldsymbol{B}_1, \cdots, \boldsymbol{B}_n\big)^T$ denotes the vector of missing component labels as described in Subsection 3.2. As suggested by Louis (1985), the observed FIM $I_{\boldsymbol{y}}$, is represented as follows.

$$\begin{aligned}
I_{\boldsymbol{y}} =& -\sum_{i=1}^{n} E\Big(\frac{\partial^2 L(\Theta|\boldsymbol{B}, y_i)}{\partial\Theta\partial\Theta^T}\Big|y_i, \Theta^*\Big)\Big|_{\Theta=\Theta^*} - \sum_{i=1}^{n} E\Big(\frac{\partial L(\Theta|\boldsymbol{B}, y_i)}{\partial\Theta}\frac{\partial L(\Theta|\boldsymbol{B}, y_i)}{\partial\Theta^T}\Big|y_i, \Theta^*\Big)\Big|_{\Theta=\Theta^*} \\
&+ \sum_{i=1}^{n} E\Big(\frac{\partial L(\Theta|\boldsymbol{B}, y_i)}{\partial\Theta}\Big|y_i, \Theta^*\Big) E\Big(\frac{\partial L(\Theta|\boldsymbol{B}, y_i)}{\partial\Theta^T}\Big|y_i, \Theta^*\Big)\Big|_{\Theta=\Theta^*} \\
&= -I_1 - I_2 - I_3,
\end{aligned} \tag{D.2}$$

where $\Theta = \big(\Theta_1^T, \cdots, \Theta_K^T\big)^T = (\omega_1, \boldsymbol{\theta}_1^T, \cdots, \omega_K, \boldsymbol{\theta}_K^T\big)^T$ with $\boldsymbol{\theta}_j = \big(\beta_j, \sigma_j, \mu_j\big)^T$, for $j = 1, \cdots, K$. For computing $I_{\boldsymbol{y}}$ given in (D.2), we need to compute elements of the conditional expected complete data observed information I_1 (Hessian matrix of the complete data log-likelihood function (D.1)), matrix of gradient vector I_2, and matrix of I_3. To construct I_1, I_2, and I_3 we need the following quantities.

$$E\Big(\frac{\partial L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \omega_j}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}\frac{E^*_{ij}}{\omega^*_j}, \tag{D.3}$$

$$E\Big(\frac{\partial L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \beta_j}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}E^*_{ij}\frac{f_Y^{\beta_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}, \tag{D.4}$$

$$E\Big(\frac{\partial L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \sigma_j}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}E^*_{ij}\frac{f_Y^{\sigma_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}, \tag{D.5}$$

$$E\Big(\frac{\partial L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \mu_j}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}E^*_{ij}\frac{f_Y^{\mu_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}, \tag{D.6}$$

$$E\Big(\frac{\partial^2 L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \omega_j^2}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=-\sum_{i=1}^{n}\frac{E^*_{ij}}{\omega^{*2}_j}, \tag{D.7}$$

$$E\Big(\frac{\partial^2 L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \beta_j^2}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}E^*_{ij}\Big[\frac{f_Y^{\beta_j\beta_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}-\Big(\frac{f_Y^{\beta_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}\Big)^2\Big], \tag{D.8}$$

$$E\Big(\frac{\partial^2 L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \sigma_j^2}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}E^*_{ij}\Big[\frac{f_Y^{\sigma_j\sigma_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}-\Big(\frac{f_Y^{\sigma_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}\Big)^2\Big], \tag{D.9}$$

$$E\Big(\frac{\partial^2 L(\Theta|\boldsymbol{B},\boldsymbol{y})}{\partial \mu_j^2}\Big|\boldsymbol{y},\Theta^*\Big)\Big|_{\Theta=\Theta^*}=\sum_{i=1}^{n}E^*_{ij}\Big[\frac{f_Y^{\mu_j\mu_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}-\Big(\frac{f_Y^{\mu_j}(y_i|\boldsymbol{\theta}^*_j)}{f_Y(y_i|\boldsymbol{\theta}^*_j)}\Big)^2\Big], \tag{D.10}$$

where $\theta^*_j=(\beta^*_j,\sigma^*_j,\mu^*_j)^T$, for $j=1\cdots,K$, and the generic symbols $f_Y^{\beta_j}(y_i|\boldsymbol{\theta}_j)$ and $f_Y^{\beta_j\beta_j}(y_i|\boldsymbol{\theta}_j)$, respectively, refer to the first and second derivatives of Cauchy probability density function $f_Y(y_i|\boldsymbol{\theta}_j)$ given by (1) with respect to β_j. This pattern is valid for the parameters σ_j and μ_j, for $j=1,\cdots,K$. Moreover,

$$E^*_{ij}=E\left(B_{ij}\Big|\Theta^*,y_i\right)=\frac{\omega^*_j f_Y\left(y_i\big|\boldsymbol{\theta}^*_j\right)}{\sum_{j=1}^{K}\omega^*_j f_Y\left(y_i\big|\boldsymbol{\theta}^*_j\right)}.$$

For example, in what follows, we represent the matrices I_2 and I_3 when $K=2$. We note that the generic symbols f_i, $f_i^{\beta_j}$, and $f_i^{\beta_j\beta_j}$ have been used for $f_Y(y_i|\boldsymbol{\theta}^*_j)$, $f_Y^{\beta_j}(y_i|\boldsymbol{\theta}^*_j)$, and $f_Y^{\beta_j\beta_j}(y_i|\boldsymbol{\theta}^*_j)$, respectively. This pattern is valid for σ_j and μ_j (for $j=1,\cdots,K$).

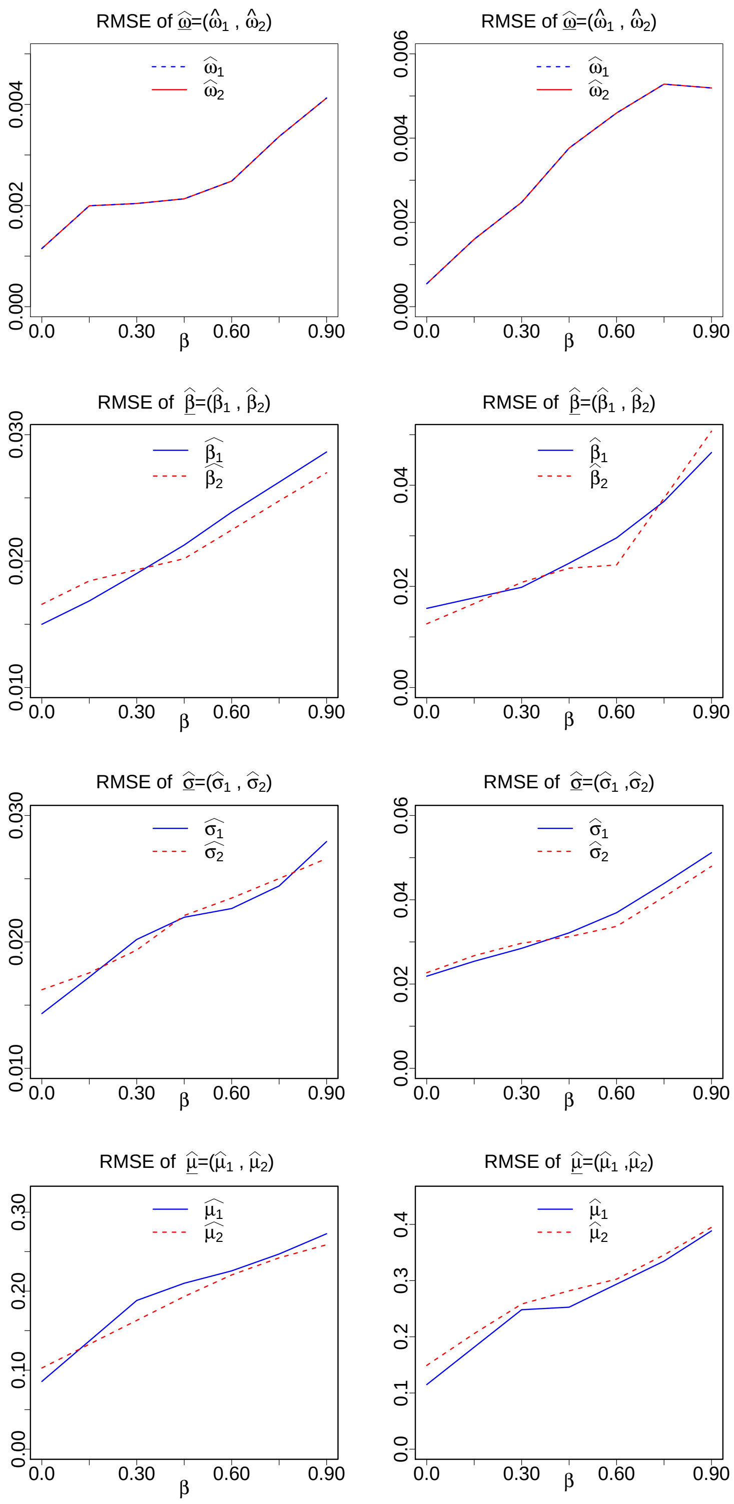

Fig. 5. The RMSE of $\hat{\boldsymbol{\omega}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\sigma}}$, and $\hat{\boldsymbol{\mu}}$ when EM algorithm is applied to the sample of size 1000 generated from two-component Cauchy mixture model for 600 runs. In each sub-figure, dashed red line and blue solid line refer to RMSE of the estimator for the first and the second components, respectively.

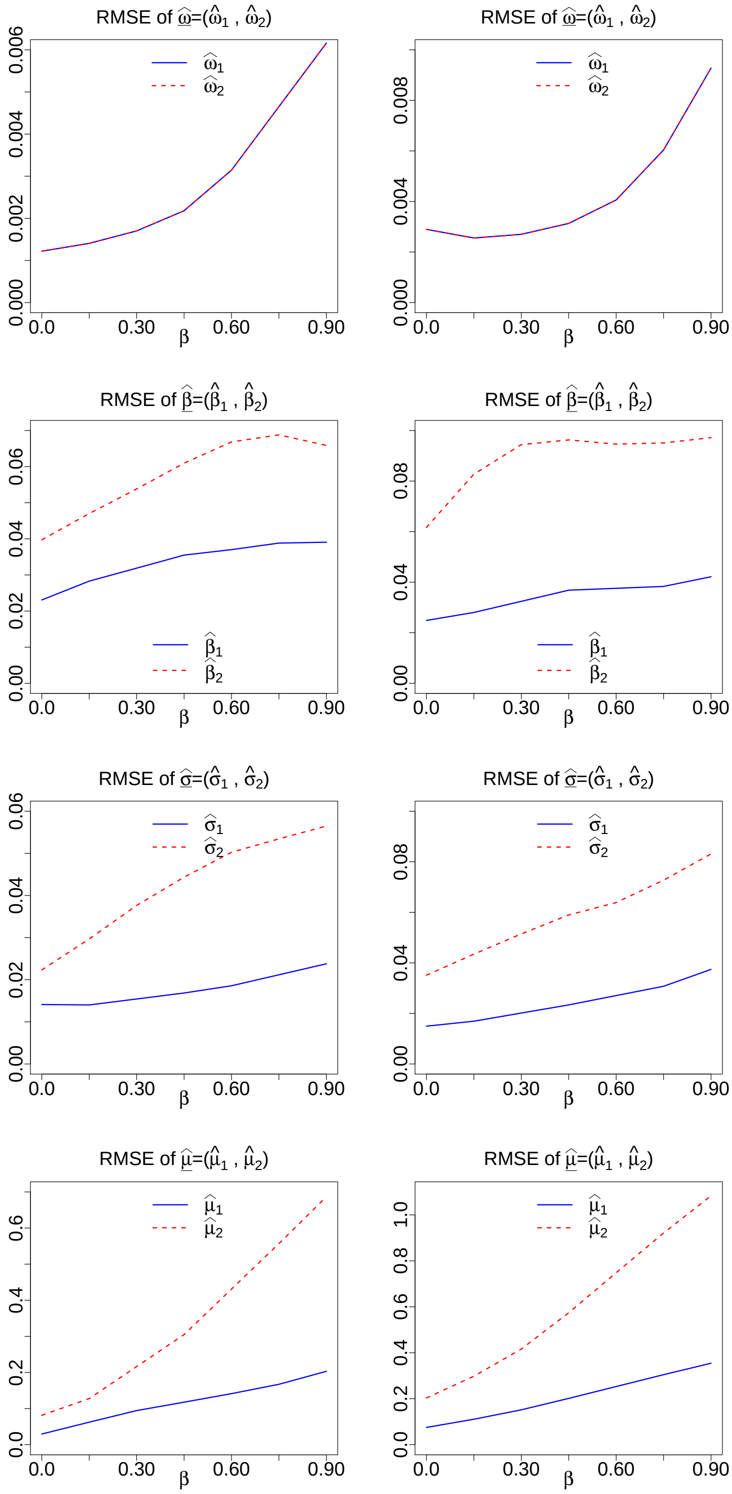

Fig. 6. The RMSE of $\hat{\boldsymbol{\omega}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\sigma}}$, and $\hat{\boldsymbol{\mu}}$ when EM algorithm is applied to the sample of size 1000 generated from two-component Cauchy mixture model for 600 runs. In each sub-figure, dashed red line and blue solid line refer to RMSE of the estimator for the first and the second components, respectively.

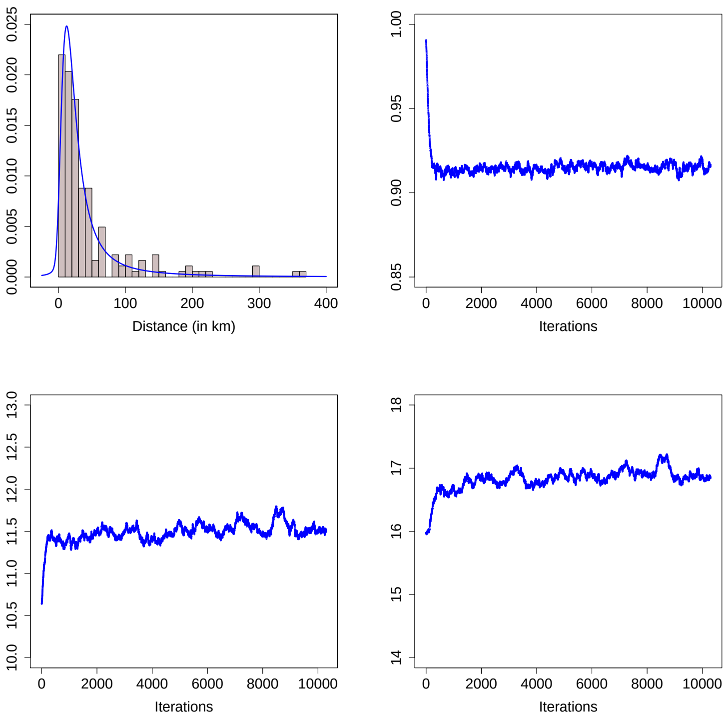

Fig. 7. Histogram of 182 distances from the seismological measuring station to the epicenter of the earthquake is shown in top left. Superimposed is the estimated probability density function with parameters $\hat{\mu} = 16.685$, $\hat{\sigma} = 11.423$, and $\hat{\beta} = 0.918$. Outputs of the EM algorithm for the skewness (top right), scale (bottom left), and location (bottom right) across iterations are shown. We note that oscillations in the movement of the output is due to the fact that expected values in the E-step are obtained by the Monte Carlo approximation. The time for implementing the EM algorithm for 10000 iterations is almost 490 seconds, but as it is seen the EM algorithm reaches convergence before 1000-th iteration.

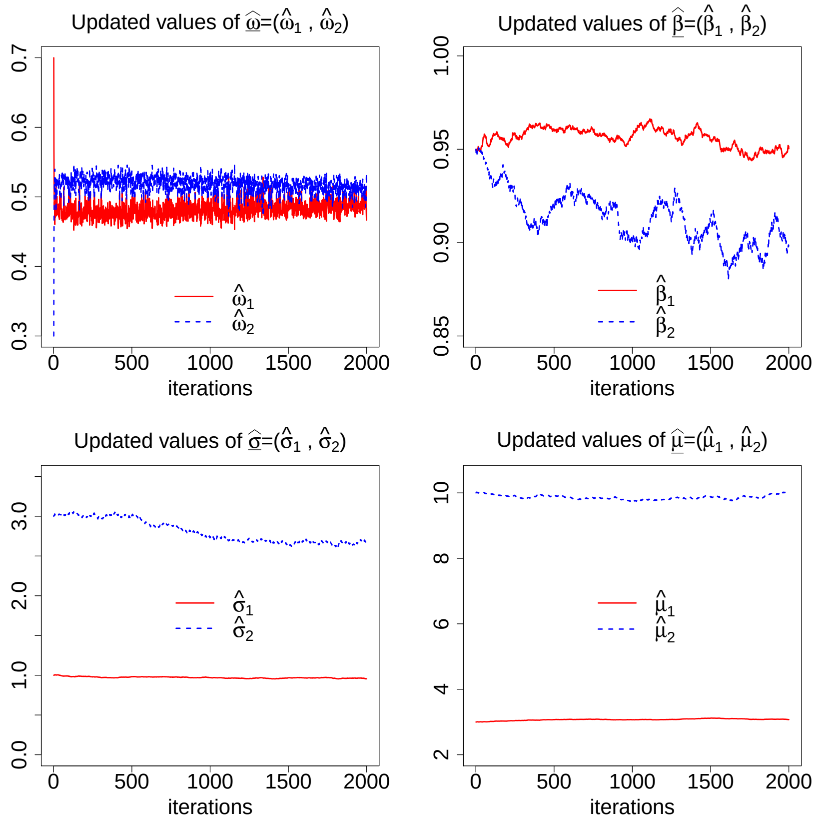

Fig. 8. Output of the EM algorithm across iterations when it is applied to the tetrahydrocortisone data. Outputs are for mixing parameters (top left), skewness parameters (top right), scale parameters (bottom left), and location parameters (bottom right). We note that oscillations in the movement of the output is due to the fact that expected values in the E-step are obtained by the Monte Carlo approximation.

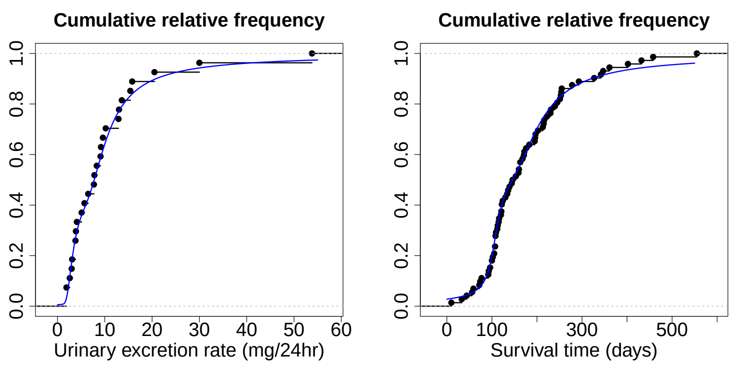

Fig. 9. Empirical cumulative relative frequency (distribution function) for urinary excretion rate of tetrahydrocortisone (left-hand side) and survival time (right-hand side) data. Estimated distribution function is shown by a blue solid curve.

$$I_2=\begin{bmatrix}
\sum_{i=1}^{n}\frac{E^*_{i1}}{\omega_1^{*2}} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\beta_1}}{\omega_1^* f_i} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\sigma_1}}{\omega_1^* f_i} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\mu_1}}{\omega_1^* f_i} & 0 & 0 & 0 & 0\\
\sum_{i=1}^{n}E^*_{i1}\frac{f^{\beta_1}}{\omega_1^* f_i} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\beta_1}f_i^{\beta_1}}{f_i^2} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\beta_1}f_i^{\sigma_1}}{f_i^2} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\beta_1}f_i^{\mu_1}}{f_i^2} & 0 & 0 & 0 & 0\\
\sum_{i=1}^{n}E^*_{i1}\frac{f^{\sigma_1}}{\omega_1^* f_i} & \sum_{i=1}^{n}\frac{E^*_{i1}f_i^{\beta_1}f_i^{\sigma_1}}{f_i^2} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\sigma_1}f_i^{\sigma_1}}{f_i^2} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\sigma_1}f_i^{\mu_1}}{f_i^2} & 0 & 0 & 0 & 0\\
\sum_{i=1}^{n}E^*_{i1}\frac{f^{\mu_1}}{\omega_1^* f_i} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\mu_1}f_i^{\beta_1}}{f_i^2} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\mu_1}f_i^{\sigma_1}}{f_i^2} & \sum_{i=1}^{n}E^*_{i1}\frac{f_i^{\mu_1}f_i^{\mu_1}}{f_i^2} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \sum_{i=1}^{n}\frac{E^*_{i2}}{\omega_2^{*2}} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\beta_2}}{\omega_2^* f_i} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\sigma_2}}{\omega_2^* f_i} & \sum_{i=1}^{n}E^*_{i2}f\frac{{}^{\mu_2}_{i}}{\omega_2^* f_i}\\
0 & 0 & 0 & 0 & \sum_{i=1}^{n}E^*_{i2}\frac{f^{\beta_2}}{\omega_2^* f_i} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\beta_2}f_i^{\beta_2}}{f_i^2} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\beta_2}f_i^{\sigma_2}}{f_i^2} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\beta_2}f_i^{\mu_2}}{f_i^2}\\
0 & 0 & 0 & 0 & \sum_{i=1}^{n}E^*_{i2}\frac{f^{\sigma_2}}{\omega_2^* f_i} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\beta_2}f_i^{\sigma_2}}{f_i^2} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\sigma_2}f_i^{\sigma_2}}{f_i^2} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\sigma_2}f_i^{\mu_2}}{f_i^2}\\
0 & 0 & 0 & 0 & \sum_{i=1}^{n}E^*_{i2}\frac{f^{\mu_2}}{\omega_2^* f_i} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\mu_2}f_i^{\beta_2}}{f_i^2} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\mu_2}f_i^{\sigma_2}}{f_i^2} & \sum_{i=1}^{n}E^*_{i2}\frac{f_i^{\mu_2}f_i^{\mu_2}}{f_i^2}
\end{bmatrix}$$

$$I_3=\begin{bmatrix}
\frac{E^*_{i1}}{\omega_1^{*2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & E^*_{i1}\Big[\frac{f_i^{\beta_1\beta_1}}{f_i}-\frac{(f_i^{\beta_1})^2}{f_i^2}\Big] & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & E^*_{i1}\Big[\frac{f_i^{\sigma_1\sigma_1}}{f_i}-\frac{(f_i^{\sigma_1})^2}{f_i^2}\Big] & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & E^*_{i1}\Big[\frac{f_i^{\mu_1\mu_1}}{f_i}-\frac{(f_i^{\mu_1})^2}{f_i^2}\Big] & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{E^*_{i2}}{\omega_2^{*2}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & E^*_{i2}\Big[\frac{f_i^{\beta_2\beta_2}}{f_i}-\frac{(f_i^{\beta_2})^2}{f_i^2}\Big] & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & E^*_{i2}\Big[\frac{f_i^{\sigma_2\sigma_2}}{f_i}-\frac{(f_i^{\sigma_2})^2}{f_i^2}\Big] & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & E^*_{i2}\Big[\frac{f_i^{\mu_2\mu_2}}{f_i}-\frac{(f_i^{\mu_2})^2}{f_i^2}\Big]
\end{bmatrix}$$

Data availability statement

The sets of data that support the findings of this study are openly available from the corresponding author upon the request.

Disclosure statement

The authors declare that they have no conflicts of interest.

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Analysis of joint Type-II censored competing risks data under a general proportional hazard rate model

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Abstract

In comparative reliability experiments, the joint censoring scheme is usually adopted for evaluating the performance of two identical products manufactured coming from different lines. In this paper, the analysis of joint Type-II censored competing risks data is considered. For this purpose, two random samples from two different lines are simultaneously put on a lifetime test and in order to save time and cost, a joint Type-II censoring scheme is conducted. Also, it is assumed that lifetime distribution of each competing cause of failure follows independently from a proportional hazard rate model with different proportionality parameter. Under this setup, the maximum likelihood estimator of the unknown parameters is discussed.

1. Introduction

Assessing the performance of products plays an essential role in manufacturing industries nowadays. But, there are situations in which the experimenters plan to compare several competing products. In these situations, comparative reliability life-tests are obviously most desirable. Also, in reliability life-tests, all lifetimes of all products on life-test cannot usually be observed due to the time limitation or the costs of experiments. Therefore, censored samples may occur in practice. When it is desirable to compare products manufactured by several lines in a combined manner, joint censoring schemes are suggested. There are several types of joint censoring schemes in reliability analysis. In this paper, we consider the joint Type-II censoring case. Consider products are being produced by two lines L_1 and L_2. Two independent samples of sizes n_1 and n_2 are randomly selected from these two lines and then placed simultaneously on a life test and the test terminates at the time of the r-th failure. When the r-th failure occurs, all of the surviving units $n_1 + n_2 - r$ are removed from the test. Notice that r is pre-fixed. Also, if $r = n_1 + n_2$, this censoring scheme is reduced to complete samples. The joint censored data have been discussed for the parametric inferences on different distributions. Balakrishnan and Rasouli [1] discussed the exact likelihood-based inferences based on the joint Type-II censored data coming from two independent exponential distributions. Also, see Rasouli and Balakrishnan [14], Parsi *et al.* [13], Doostparast *et al.* [6], Balakrishnan *et al.* [2] and Mondal and Kundu [11, 12].

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In reliability theory and survival analysis, when there exist more than one cause of failure (defined as risk factor), assessing the lifetime of products with an isolated risk factor is not usually possible. Hence, the experimenter needs to assess the effect of each risk factor in the presence of other risk factors. In these situations, the experimenter encounters the problem of competing risks. For example, the failure of a bearing assembly may be related to bearing failure or shaft failure. The data for such a competing risks model must come in a bivariate form composed of the lifetime of the unit and an indicator variable denoting which risk factor occurred for the unit. In practice, the risk factors may be statistically independent or dependent. In most situations, however, for analyzing a competing risks model, the risk factors are assumed to be independent. For more details, see, for example, David and Moeschberger [5] and Crowder [4]. In this paper, we consider a problem of competing risks involving k risk factors which are statistically independent. Many researchers have been interested the statistical inferences based on censored data in the presence of competing risks. For example, Mao *et al.* [8] and Mao *et al.* [9] studied the exact likelihood-based inferences based on the joint Type-II hybrid and the joint Type-I hybrid censored data arising from two independent exponential distributions, respectively.

Throughout this article, we suppose that the distibution functions (DFs) of lifetimes satisfy the proportional hazard rate model. Let X and Y be two random variables with hazard rate functions h_F and h_G, respectively. Then X and Y are said to satisfy the proportional hazard rate model, proposed by Cox [3], with proportionality constant $\theta > 0$, if $h_F(x) = \theta\, h_G(x)$ for all x. Or, equivalently,

$$\bar{F}(x) = \left(\bar{G}(x)\right)^{\theta}, \tag{1}$$

for all x, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$. This model includes several well-known lifetime distributions such as exponential, Rayleigh, Pareto, Weibull and so on and is a subclass of the one-parameter exponential family of distributions. Also, this model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic. For further details on proportional hazard rate models, one may refer to Marshall and Olkin [10].

In this paper, we present a brief overview of the considered model in Section 2. Then, we obtain the maximum likelihood estimators (MLEs) of the unknown parameters in Section 3. We show that these MLEs do not always exist. In Section 4, under some distributions belonging to proportional hazard rate model, the MLEs of parameters are derived.

2. Description of model

Assume that we have two production lines denoted by L_1 and L_2. Also, suppose that each unit from these two lines fails only by one of k specified fatal risk factors and the time-to-failure by competing risks follows independently from a proportional hazard rate model as given (1) with the same baseline DF G and the different proportionality constants θ. In other words, if T_{ij} is the lifetime of a test unit from line L_i ($i = 1, 2$) due to risk factor j ($j = 1, \ldots, k$), the DF and the probability density function (PDF) of T_{ij} under the model (1) are

$$F_{ij}(x) = 1 - \left(\bar{G}(x)\right)^{\theta_{ij}}; \;\; x > 0, \tag{2}$$

and

$$f_{ij}(x) = \theta_{ij}\, g(x) \left(\bar{G}(x)\right)^{\theta_{ij}-1}; \;\; x > 0, \tag{3}$$

respectively, where g is the baseline PDF. Since, we observe only the smaller of $\{T_{i1}, \ldots, T_{ik}\}$ for $i = 1, 2$, then the overall time-to-failure of a test unit is $T_i = \min(T_{i1}, \ldots, T_{ik})$. It is easy to show that the DF and the PDF of T_i for $i = 1, 2$ under the exponential distribution, from (2), are, respectively,

$$F_{T_i}(t) = 1 - \prod_{j=1}^{k} \left(\bar{F}_{ij}(t)\right) = 1 - \left(\bar{G}(t)\right)^{\lambda_i}; \;\; t > 0,$$

and

$$f_{T_i}(t) = \lambda_i\, g(t) \left(\bar{G}(t)\right)^{\lambda_i - 1}; \;\; t > 0,$$

where $\lambda_i = \sum_{j=1}^{k} \theta_{ij}$. Then, T_i follow a proportional hazard rate model with the baseline DF G and the proportionality constant λ_i.
After conducting a lifetime expriment with adopting the joint Type-II censoring scheme and k competing risks, the observed data are $(\mathbf{W}, \mathbf{Z})$ where $\mathbf{W} = (W_1, \dots, W_r)$ is the vector of observed lifetime data with $W_1 < \cdots < W_r$, $\mathbf{Z} = (\mathbf{Z}_1, \dots, \mathbf{Z}_r)$ where $\mathbf{Z}_h = \left(Z_h^{(11)}, \dots, Z_h^{(1k)}, Z_h^{(21)}, \dots, Z_h^{(2k)}\right)$ for $h = 1, \dots, r$ with $Z_h^{(ij)} = 1$ if h-th failure belongs to line L_i and due to risk factor j, otherwise $Z_h^{(ij)} = 0$ for $i = 1, 2$ and $j = 1, \dots, k$.
Based on the obtained data $(\mathbf{w}, \mathbf{z})$, the likelihood function of the vector of parameters θ is written as

$$L(\theta; \mathbf{w}, \mathbf{z}) = C \prod_{h=1}^{r} \left\{ \prod_{i=1}^{2} \prod_{j=1}^{k} \left(f_{ij}(w_h) \cdot \prod_{\substack{l=1 \\ l \neq j}}^{k} \bar{F}_{il}(w_h) \right)^{z_h^{(ij)}} \right\} \times \prod_{i=1}^{2} \left(\prod_{j=1}^{k} \bar{F}_{ij}(w_r) \right)^{n_i - m_i}, \tag{4}$$

where $m_i = \sum_{h=1}^{r} \sum_{j=1}^{k} z_h^{(ij)}$ for $i = 1, 2$ is the total number of observed failures from line L_i and C is the nomalizing constant given by

$$C = \frac{n_1!\, n_2!}{(n_1 - m_1)!\, (n_2 - m_2)!}.$$

3. Maximum likelihood estimates

Substituting Equations (2) and (3) into Equation (4), the likelihood function of the vector of parameters $\theta = (\theta_{11}, \dots, \theta_{1k}, \theta_{21}, \dots, \theta_{2k})$ is simplified to

$$L(\theta; \mathbf{w}, \mathbf{z}) = C \prod_{i=1}^{2} \prod_{j=1}^{k} \theta_{ij}^{m_{ij}} \prod_{h=1}^{r} \prod_{i=1}^{2} \prod_{j=1}^{k} \left(g(w_h) \big(\bar{G}(w_h)\big)^{\sum_{l=1}^{k} \theta_{il} - 1} \right)^{z_h^{(ij)}} \big(\bar{G}(w_r)\big)^{\sum_{i=1}^{2} \lambda_i (n_i - m_i)}, \tag{5}$$

where $m_{ij} = \sum_{h=1}^{r} z_h^{(ij)}$ is the total number of observed failures belonging to the line L_i which are failed due to risk factor j between r failures occured.
The log-likelihood function of θ is readily obtained from (5) as

$$\log L(\theta; \mathbf{w}, \mathbf{z}) = \log C + \sum_{i=1}^{2} \sum_{j=1}^{k} m_{ij} \log \theta_{ij} + \sum_{h=1}^{r} \sum_{i=1}^{2} \sum_{j=1}^{k} z_h^{(ij)} \Big[\log g(w_h) + \Big(\sum_{l=1}^{k} \theta_{il} - 1 \Big) \log \bar{G}(w_h) \Big] + \sum_{i=1}^{2} \lambda_i (n_i - m_i) \log \bar{G}(w_r).$$

Then, we readily obtain the likelihood equations as

$$\frac{\partial \log L(\theta; \mathbf{w}, \mathbf{z})}{\partial \theta_{ij}} = \frac{m_{ij}}{\theta_{ij}} + \sum_{h=1}^{r} \delta_h^{(i)} \log \bar{G}(w_h) + (n_i - m_i) \log \bar{G}(w_r) = 0\,; \quad i = 1, 2, \ j = 1, \dots, k, \tag{6}$$

where $\delta_h^{(i)} = \sum_{l=1}^{k} z_h^{(il)}$ for $i = 1, 2$ and $h = 1, \dots, r$ determines whether h-th failure belongs to the line L_i or not.
From Equation (6), the MLE of θ_{ij} is easily obtained as

$$\hat{\theta}_{ij} = m_{ij} \left[- \sum_{h=1}^{r} \delta_h^{(i)} \log \bar{G}(W_h) - (n_i - m_i) \log \bar{G}(W_r) \right]^{-1}, \tag{7}$$

conditional on $m_{ij} \geq 1$ for all $i = 1, 2$ and $j = 1, \dots, k$, that is, θ_{ij} is estimable if it is observed at least one failure belonging to the line L_i caused by risk factor j. Hence, the MLE of the vector of parameters $\theta = (\theta_{11}, \dots, \theta_{1k}, \theta_{21}, \dots, \theta_{2k})$ exists, provided $\mathbf{m} = \big(m_{11}, \dots, m_{1k}, m_{21}, \dots, m_{2k}\big)$ satisfies the following condition

$$\left\{ \mathbf{m} \,\middle|\, m_{ij} \geq 1 \text{ for all } i = 1, 2, \ j = 1, \dots, k \text{ and } \sum_{i=1}^{2} \sum_{j=1}^{k} m_{ij} = r \right\}.$$

4. Special cases

In reliability and life-testing applications, the exponential, Rayleigh and Pareto distributions are extensively used for modelling the lifetime data; See, for example, Lawless [7]. Therefore, in this paper, we will specifically consider the following three distributions. However, the conditional MLE obtained here is applicable to any proportional hazard rate model.

1. Exponential distribution with DF $F(x) = 1 - \exp\big\{-\theta x\big\}$, $x > 0$.
2. Rayleigh distribution with DF $F(x) = 1 - \exp\big\{-\theta x^2\big\}$, $x > 0$.
3. Pareto distribution with DF $F(x) = 1 - x^{-\theta}$, $x > 1$.

Then, from Equation (7), the conditional MLEs of parameters θ_{ij} ($i = 1, 2$ and $j = 1, \ldots, k$) under the exponential, Rayleigh and Pareto distributions are

$$\hat{\theta}_{ij} = m_{ij}\left[\sum_{h=1}^{r} \delta_h^{(i)} W_h + (n_i - m_i)W_r\right]^{-1},$$

$$\hat{\theta}_{ij} = m_{ij}\left[\sum_{h=1}^{r} \delta_h^{(i)} W_h^2 + (n_i - m_i)W_r^2\right]^{-1},$$

and

$$\hat{\theta}_{ij} = m_{ij}\left[\sum_{h=1}^{r} \delta_h^{(i)} \log W_h + (n_i - m_i)\log W_r\right]^{-1},$$

respectively.

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Arcihmedean copulas and goodness of fit test

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Abstract

The main endeavor of this paper is to compare the result of parameter estimation for Archimedean copulas by using Kendal coefficient and Goodness of fit test. It is seen that on modeling dependency structure of data by the GOF method, at the same time, we are able to estimate parameters and also test the compatibility of copulas to data.

1. Introduction

Copulas are used in modeling the dependence structure between variables, this modeling is irrespective of their marginal distributions. On the other hand copulas allow to choose different margins and merge the margins into a genuine multivariate distribution. Sklar (1959) fro the first time used the concept of copula and it has been introduced by in the following way,
A copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies:
(I) for every u, v in [0, 1], $C(u,0) = 0 = C(0,v)$, and $C(u,1) = u$ and $C(1,v) = v$;
(II) for every u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$.
Sklar's Theorem describes the importance of copulas as follows:
Let X and Y be random variables with joint distribution function H and marginal distribution functions F and G, respectively. Then there exists a copula C such that, $H(x,y) = C(F(x), G(y))$, for all x, y in $\mathbb{R}$. If F and G are continuous, then C is unique. Otherwise, the copula C is uniquely determined on $Ran(F) \times Ran(G)$. Conversely, if C is a copula and F and G are distribution functions, then the function H is a joint distribution function with margins F and G. This means that, one-dimensional margins of joint distribution functions are linked by copulas. For a more formal definition of copulas, the reader is referred to Nelsen (2006).
There are several parametric and non-parametric methods for estimating parameters of copulas. In this paper we compare the result of parameter estimation for Archimedean copulas by using Kendal confident and also Goodness of

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fit test. It can be an advantage for non-parametric GOF method, as it is able at the same time, to estimate parameters and also test the compatibility of copulas to data.
This paper is constructed as follows: Section 2 discusses related with estimation of copula parameters and copula selection methods. Section 3 explains the GOF method. An application study is given in Section 4 and finally Section 5 summarizes the conclusion of our work.

2. Copula selection methods

In multivariate statistical analysis of copulas one of the main topics is related with statistical inference on the dependence parameter. In the literature several methods proposed for estimation copula parameters. Genest and Rivest (1993) proposing a method which is based on concordance. Genest et al. (1995) proposed fully maximum likelihood (ML), pseudo maximum likelihood (PML). In 2005, Joe discussed on inference function for margins (IFM) and Tsukahara in 2005 proposed minimum distance (MD) method. There are some discussions about these methods by Kim et al. (2007) and also Najjari (2016).
As a result, PML estimator is better than ML and IFM in the most practical situations. simulation study by Kim et al. (2007) carried out that the PML method is conceptually almost the same as the IFM one. By using the PML method, any important statistical insights that would be gained by applying the IFM, would not be loosed. Therefore, the PML estimator is better than those of the ML and IFM in most practical situations. However, in high dimensional copulas ($n > 3$) ML, PML, IFM and MD methods, in time-consuming point of view, require so much computations. In these methods copula density function are used so that increases complexity of calculations, specially for $d > 2$[1] (Yan, 2007). Semi-parametric estimation of copula models based on the method of moments proposed by Brahimi and Necir in 2012. This method is quick and simple, nonetheless, Brahimi and Necir's method has its own complexity.
After estimating copula parameters, another step is in selecting the right copula that has the best fits to data. In the literature several methods proposed for selecting the best copula some of which are summarized as follows:
Most of methods for selecting right copula are based on a likelihood approach.For example, the Akaike Information Criteria (AIC), Pseudo-likelihood ratio test proposed by Chen and Fan (2005).
There are other methods of selecting the best copula which defines indicators of performance. Genest and Rivest (1993) proposed a method in Archimedean copulas as below,

$$K_\theta(t) = P(C(u, v\,|\theta) < t)$$

with its non-parametric estimation K_n, given by

$$K_n = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}(e_{jn} \leq t)$$

where $e_{jn} = (1/n)\sum_{k=1}^{n} \mathbf{1}(X_{1k} \leq X_{1j}, ..., X_{pk} \leq X_{pj})$. A copula that the function K_θ is closest to K_n, is the best one.
Choosing the best copula with minimizing the distance (L^2-norm, Kolmogorov, etc) from K_θ to the non-parametric estimation K_n suggested by Durrleman et al. (2000). Genest et al. (1995) proposed a GOF test statistic with a non-truncated version of Kendall process,

$$\mathbb{K}_n(t) = \sqrt{n}\{K_n(t) - K_{\theta_n}(t)\}$$

where θ_n denotes a robust estimation of θ. The expression for the statistic is simple and the test has nice properties.
In the p-dimensional copulas, Pollard (1979) presented χ^2 tests as bellow:

$$\chi^2 = \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{[o_{ij} - e_{ij}]^2}{e_{ij}}$$

[1]Dimensions

where n is the sample size, I, J are the numbers of classes. o_{ij} is the observed frequency of data in class ij and e_{ij} is the theoretical frequency of data for every class ij. Growing number of the classes increases power of the test. Kendall and Stuart(1983) discuss related with arbitrary choice of the subsets that divide the p-dimensional space $[0,1]^p$.
As examples, Dobric and Schmidt (2004) used this method in a financial application. Çelebioğlu (2003) in modeling student grades relies on this method. Najjari & Ünsal (2012) used this method in modeling meteorological data, Najjari et al. (2014) applied this method in modeling data of the Danube river, also see Şahin Tekin et al. (2014) and Kazemi Rad et al. (2021).

3. Goodness of fit test in selection of the right copula

All of the mentioned criteria the Section 2 rely on previous estimation of an optimal parameter set of copulas to select the right copula. In this section, a method is described in selecting the right copula which is independent of the chosen optimal parameter. On the other hand, at the same time, it is able to estimate the copula parameters and also it is able to select the right copula. For two dimensional data (X, Y) thid method is described as follows:
Time dependency of data are tested, then random samples $(X_1, Y_1), \cdots, (X_n, Y_n)$ are converted into normalized ranks in the usual fashion by setting $U_l = rank(X_l)/n$ and $V_l = rank(Y_l)/n$ for each $l \in \{1, 2, \cdots, n\}$. Then the data are grouped into $r \times r$ contingency table by using $r = round(\sqrt[4]{n})$.
Let observed frequencies is matrix $O_{I\times J}$, and matrix $E_{I\times J}$ consists of estimation of the expected frequencies ($I = J = r$) and in the matrix O, o_{ij} is an element of ith row and jth column, and e_{ij} is the expected value of the o_{ij}, which is calculated by multiplying the number of observations n with appropriate theoretical frequency estimated with copulas, where $i, j \in \{1, 2, \cdots, r\}$. Let copula parameters θ be in the form $\theta = (\theta_1, \theta_2, ..., \theta_s)$, where s is the number of copula parameters. C_θ for some $\theta \in \Theta$ (where Θ is parameters space), expected frequencies $e_{jk}(\theta)$ are computed for the contingency table. So $e_{ij}(\theta)$ is a function of the parameters θ and can be calculated as follows

$$e_{ij}(\theta) = n \times [C_\theta(u_i, v_j) - C_\theta(u_{i-1}, v_j) - C_\theta(u_i, v_{j-1}) + C_\theta(u_{i-1}, v_{j-1})]$$

where $i, j \in \{1, 2, \cdots, r\}$ and $e_{11}(\theta) = n \times C_\theta(u_1, v_1)$. So the standard GOF statistic value is as follows,

$$h(\theta) = \chi^2_\theta = \sum \frac{(o_{ij} - e_{ij}(\theta))^2}{e_{ij}(\theta)}. \tag{1}$$

In order to determine the right copula that fits the best to data, obviously expected frequencies estimated by copula must be close to observed frequencies. Clearly this fact occurs in point $\hat{\theta}$ (estimation of θ) that minimizes χ^2_θ in (1). Meanwhile, if $\chi^2_{\hat{\theta}}$ in (1) yields a low p-value by reference to the chi-squared distribution with $(r-1)\times(r-1)$ degrees of freedom, then $H_0 : C_{\hat{\theta}} \in C_\theta$ is rejected. On the other hand in the range of copula parameters minimum point of the function $h(\theta)$, is an estimating of the copula parameters and also a way in choosing the right copula that had best fits to data. Without loss of generality it can be assumed that the minimum value is accrued only on a single point of its parameters range and calculating minimum value of the function $h(\theta)$ of course is not complicated and easily applicable in multiparameter copulas. See also Najjari (2016).

4. Application

In our application we use minimum and maximum QFE data from 2010-2020 in Tehran-Iran which are 420 data and are available online at I.R OF IRAN METEOROLOGICAL ORGANIZATION (IRIMO). We remember that, QFE is the barometric altimeter setting that will cause an altimeter to read zero when at the reference datum of a particular airfield (in general, a runway threshold). In ISA temperature conditions the altimeter will read height above the datum in the vicinity of the airfield.
We consider six most widely used Archimedean families of copulas (Table 1, 2): Clayton, Gumbel and A12, A14, A15, A18 (this copula families numbered as 4.2.12, 4.2.14, 4.2.15, 4.2.18 respectively in the Nelsen's book [14]). In continue, related QFE data is arranged and has been divided to total data sample size plus one, as below (see [6], [2]):

$$u_i = \frac{R(x_i)}{n+1} \ , \ v_i = \frac{R(y_i)}{n+1} \ , \ i = 1, 2, 3, ..., n$$

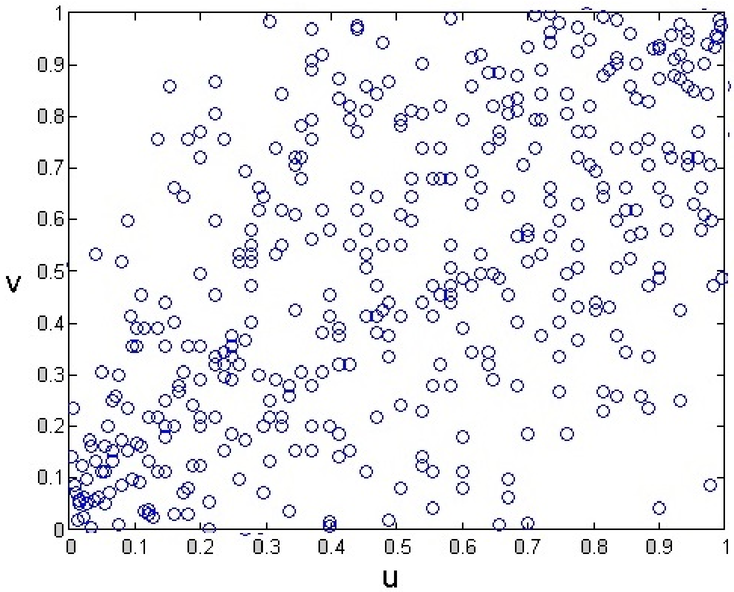

Fig. 1. Scatterplots for minimum and maximum QFE data.

where x_i is the minimum, and y_i is the maximum QFE data and $R(x_i)$ and $R(y_i)$ are (in ascending order) the rank of related data. Our final data (u and v) will be in interval $(0, 1)$. Figure 1 shows the scatterplots of the final data. Kendall's tau for this data is $\tau = 0.3894$. To apply GOF test, with using relation $\sqrt[4]{n}$, (n number of data), (see [2],

Table 1. Definition and parameter domain of the copulas used in this paper.

Family	$C(u, v; \theta)$	θ interval
Clayton	$\max([u^{-\theta} + v^{-\theta} - 1]^{-1/\theta}, 0)$	$[-1, \infty) - \{0\}$
Gumbel	$\exp(-[(-\ln u)^{\theta} + (-\ln v)^{\theta}]^{1/\theta})$	$[1, \infty)$
A12	$(1 + [(u^{-1} - 1)^{\theta} + (v^{-1} - 1)^{\theta}]^{1/\theta})^{-1}$	$[1, \infty)$
A14	$(1 + [(u^{-1/\theta} - 1)^{\theta} + (v^{-1/\theta} - 1)^{\theta}]^{1/\theta})^{-\theta}$	$[1, \infty)$
A15	$max\ ((1 - [(1 - u^{1/\theta})^{\theta} + (1 - v^{1/\theta})^{\theta}]^{1/\theta})^{\theta}, 0)$	$[1, \infty)$
A18	$max\ (1 + \theta / ln(e^{\theta/(u-1)} + e^{\theta/(v-1)})^{\square}, 0)$	$[2, \infty)$

[13]), we divided the range of two variables uniform transformation into 4 intervals each, therefore $df = 9$, and the critical point in GOF test is $\chi^2_{0.05,df} = 16.9190$. As Kendall's tau for this data is $\tau = 0.3894$, for each copulas family nonparametric estimation of family parameter applied, and then calculated the value of χ^2 test statistic. Result are in Table 3. In the GOF test method, GOF test statistic, we posed as a function of copula parameter, and then we calculated the minimum value of this function (see Figure 2). Hereby we will estimate copulas parameter and also we will find the right copula between copula families (see Table 3).
With respect to Table 3, obviously nonparametric estimation and the new method, selected Gumbel and A15 family

Table 2. Kandell's tau and it's domain of the copulas used in this paper.

Family	Kendall's tau	Kendall's tau interval
Clayton	$\frac{\theta}{\theta+2}$	$(0,1]$
Gumbel	$\frac{\theta-1}{\theta}$	$[0,1]$
A12	$1-\frac{2}{3\theta}$	$[1/3,1]$
A14	$1-\frac{4}{2+4\theta}$	$[1/3,1]$
A15	$1+\frac{4}{2-4\theta}$	$[-1,1]$
A18	$1-\frac{4}{3\theta}$	$[1/3,1]$

Table 3. Estimated copula parameter and GOF test statistic value.

Family	GOF		Nonparametric method	
	estimated parameter	χ^2	estimated parameter	χ^2
Clayton	**0.91**	**46.287**	**1.28**	**53.4373**
Gumbel	**1.59**	**8.4987**	**1.64**	**8.8204**
A12	**1.11**	**40.9745**	**1.09**	**41.0929**
A14	**1.23**	**34.1538**	**1.14**	**35.8295**
A15	**2.08**	**13.7028**	**2.14**	**14.3921**
A18	**2.21**	**1488.1**	**2.18**	**1507.6**

as right copulas, also Gumbel copula fits better to data, because it has lowest test statistic value. In Gumbel family, nonparametric estimation of copula parameter is $\theta = 1.64$ and χ^2 test statistic value is 8.8204, while the new method estimated copula parameter as $\theta = 1.59$ and χ^2 test statistic value as 8.4987. In A15 family, nonparametric estimation of copula parameter is $\theta = 2.14$ and χ^2 test statistic value is 14.3921, while the new method estimated copula parameter as $\theta = 2.08$ and χ^2 test statistic value as 13.7028.

5. Conclusion

In this study we used GOF test statistic, as a function on copula parameter. By this technique it is possible to have an estimation for copulas parameters. This means we don't need to rely on previous estimation of an optimal parameter set. Then with calculating minimum point of this function in the range of copula parameter, we will reach to both aims, estimating the copula parameter and choosing the right copula that fits the best to data.

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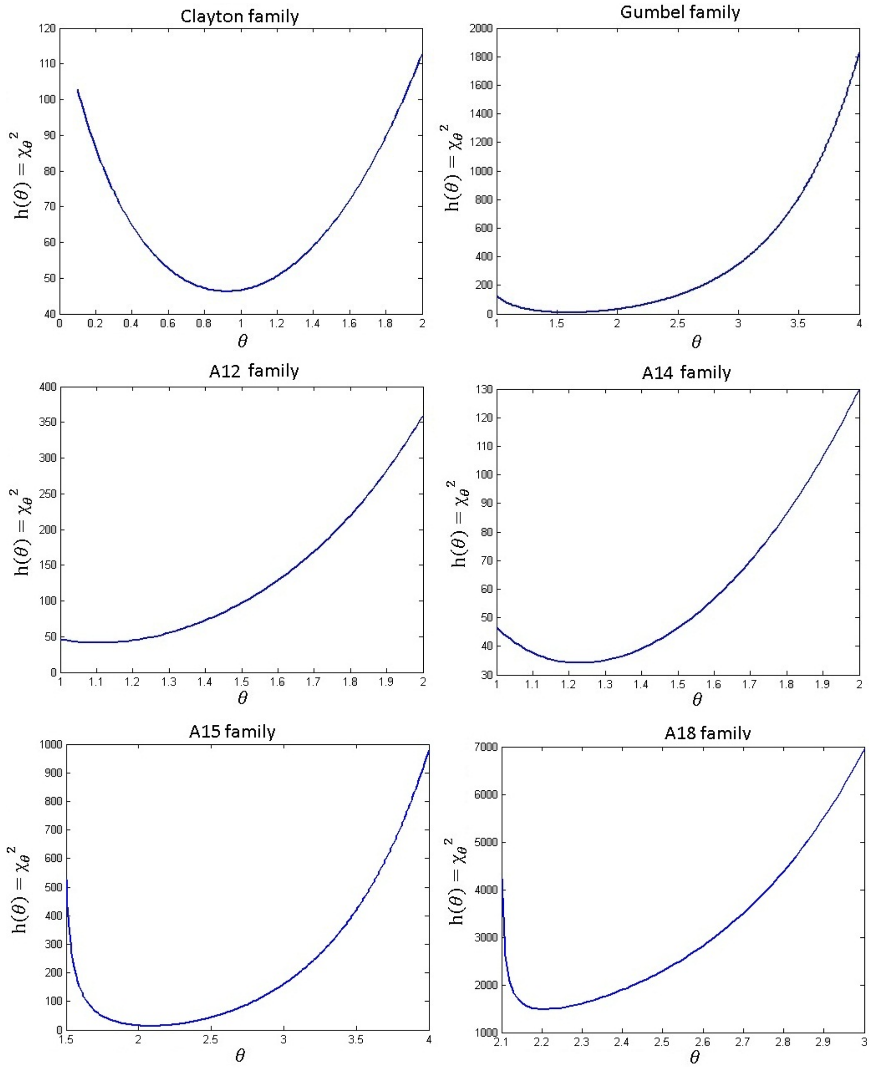

Fig. 2. GOF test statistic, as a function of copula parameter.

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Improved estimation in restricted measurement error models via Stein-rule procedure

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Article Info

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Abstract

In this paper, we propose a Stein-rule estimator for the measurement error model with normal distributions and when a set of equality constrains binding the regression coefficients is available. The asymptotic properties of the estimator are derived and the risk function under a specific quadratic loss function is studied. We also obtained a sufficient condition for the dominance of the proposed estimator over a consistent estimator. The results of a simulation study are presented to demonstrate the finite sample properties of the estimators.

1. Introduction

A basic assumption in classical linear regression is that the response and exploratory variables are measured correctly. However, in some practical situations, this assumption is violated and the variables are observed with measurement errors, see Fuller [8] and Carroll et al. [2] for more details.
In the presence of measurement errors, the usual regression tools do not lead to correct and valid model. In order to obtain consistent estimators of the parameters some prior information about is required. In some cases, some additional information about the regression coefficients is available in the form of exact restrictions which can be used to improve the efficiency of estimators, [6].
In the context of regression analysis, the family of Stein-rule estimators is widely used to overcome the inconsistency problem of the ordinary least square estimator, see for example [14], [9], [18], [16], [20], [1] and [13].
[3] proposed a Stein-rule estimator for the linear regression model with a set of linear restrictions on the regression coefficients.
In this study we consider a multivariate ultrastructural model which is a very general form of measurement error model. The ultrastructural model includes structural and functional variants as well as the classical regression model as its special cases. [18] proposed a Stein-rule estimator for the regression coefficients in the ultrastructural model which dominates a popular consistent estimator with respect to a specific loss function.
In this paper, we consider the ultrastructural model for the modeling of the measurement errors, and also assume that the additional information about unknown coefficients is the form of exact linear restrictions. We propose a restricted Stein-rule estimator that satisfies the given restriction and study asymptotic properties of the estimator.

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The paper is organized as follows. In Section 2, we first introduce the ultrastructural model as the general form of measurement error model, and describe the additional information on regression coefficients. We also present an estimator driving from Stein-rule procedure. Section 3 contains the main results of our work, where we obtain the asymptotic properties of the estimator and present a sufficient condition for proposed estimator to dominate a consistent estimator. In Section 4, we conduct a simulation study to investigate the proposed estimator.

2. The ultrastructural model and prior information

Let η be the $(n \times 1)$ vector of true values the study variable and $\Xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the $(n \times p)$ matrix of n true values on each of the p explanatory variables are related by

$$\eta = \Xi\beta, \tag{1}$$

where β is a $(p \times 1)$ vector of regression coefficients. The recorded values of the study and explanatory variables are observed with additive measurement errors, then we observe y and X as

$$y = \eta + \epsilon, \tag{2}$$

and

$$X = \Xi + \Delta, \tag{3}$$

where $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T$ is an $(n \times 1)$ vector of sum of equation errors and measurement errors that contaminates the study variable η and $\Delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$ is an $(n \times p)$ matrix of measurement errors that contaminates the explanatory variables.
Let M be the $(n \times p)$ matrix of mean of the variables in Ξ, then the true values of the explanatory variables (Ξ) can be written as

$$\Xi = M + \Phi, \tag{4}$$

where $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is the random matrix of order $(n \times p)$.
Equations (1)-(4) denote a multivariate ultrastructural model, see [7] and [5]. The ultrastructural model is a combination of structural and functional model. If the row vectors of Ξ have the same mean, i.e. when the all row vectors of M are identical, the ultrastructural model will reduce to the structural model. If the matrix of the explanatory variables (Ξ) is fixed but is contaminated with measurement error, the ultrastructural model will become the functional form of the measurement error model. The ultrastructural model will specify the classical linear regression model if the explanatory variables are fixed and observed without any measurement errors.
We also assume that we have some prior information on the regression coefficients which can be expressed in the form of linear restriction

$$r = R\beta, \tag{5}$$

where r is an $(m \times 1)$ known vector and R is an $(m \times p)$ known matrix of rank $m < (p)$.
We assume that ϵ_i, $(i = 1, \ldots, n)$ are independent and identically normally distributed with mean 0 and variance σ_ϵ^2. In addition, δ_i, $(i = 1, \ldots, n)$ are assumed to be independent and identically multivariate normally distributed with mean 0 and variance covariance matrix Σ_δ. Similarly, the vectors ϕ_i, $(i = 1, \ldots, n)$ are considered to be independent and identically multivariate normally distributed with mean 0 and variance covariance matrix Σ_ϕ. Furthermore, ϵ, Δ and Φ are assumed to be statistically independent of each other. We also assume that the limiting form of $n^{-1}M'M$ exists and is non-singular and limiting form of the matrix $n^{-1}M'e_n$ exists, where e_n is an $(n \times 1)$ vector of elements unity, as $n \to \infty$.
These assumption are required for the validity of the asymptotic approximation theory and they also avoid the possibility of any trend in the observations, see [17] and [19].

2.1. Estimators

Stein-rule estimator of the coefficients in the ultrastructural model has been proposed by [18] as the following form:

$$\hat{\beta}_S = \left[1 - \left(\frac{k}{n-p+2}\right)\frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{\hat{\beta}'\left(S - n\Sigma_\delta\right)\hat{\beta}}\right]\hat{\beta}, \tag{6}$$

where $S = X'X$, $k(\geq 0)$ is the shrinkage factor, and $\hat{\beta}$ is a consistent estimator for β under the ultrastructural linear measurement errors, which can be considered as $\hat{\beta} = \left(S - n\sigma_\delta^2 I_p\right)^{-1} X'Y$.

When some prior information about the regression coefficients in the ultrastructural model is available in the form of exact linear restriction, [19, 20] presented some consistent estimators of β. In this paper, we consider the following consistent estimator

$$\hat{\beta}_R = \hat{\beta} + \left(S - n\sigma_\delta^2 I_p\right)^{-1} R' R_{S\delta}^{-1}\left(r - R\hat{\beta}\right), \tag{7}$$

where $R_{S\delta} = R\left(S - n\sigma_\delta^2 I_p\right)^{-1} R'$.

Then similarly to the estimator (6), a Stein-rule restricted estimator of β under the model and the linear constraints (5), can be considered as follows:

$$b_{RS} = \left[1 - \left(\frac{k}{n-p+2}\right)\frac{(Y - X\hat{\beta}_R)'(Y - X\hat{\beta}_R)}{\hat{\beta}_R'\left(S - n\Sigma_\delta\right)\hat{\beta}_R}\right]\hat{\beta}_R. \tag{8}$$

Unfortunately, we have $Rb_{RS} \neq r$. To solve this problem we restrict the Stein-rule estimator of β, in the sense that we replace $\hat{\beta}$ with $\hat{\beta}_S$ in the restricted estimator (7). Therefore, the restricted Stein-rule estimator is obtained by

$$\hat{\beta}_{RS} = \hat{\beta}_S + \left(S - n\Sigma_\delta\right)^{-1} R' R_{S\delta}^{-1}\left(r - R\hat{\beta}_S\right). \tag{9}$$

Proposition 2.1. *The restricted Stein-rule estimator of regression coefficients in the ultra structural model with linear restriction can be written as follows:*

$$\hat{\beta}_{RS} = \left(I_p - \left(\frac{k}{n-p+2}\right)\frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{\hat{\beta}'\left(S - n\sigma_\delta^2 I_p\right)\hat{\beta}} G\right)\hat{\beta}_R, \tag{10}$$

where $G = \left(I_p - \left(S - n\Sigma_\delta\right)^{-1} R' R_{S\delta}^{-1} R\right)$.

When the shrinkage factor $k = 0$, the restricted Stein-rule estimator $\hat{\beta}_{RS}$ coincide with the restricted estimator $\hat{\beta}_R$.

3. Comparison of the estimators

In order to compare the estimator, we study the asymptotic properties of the estimators $\hat{\beta}_R$ and $\hat{\beta}_{RS}$ in this section. The consistent estimator of coefficients in the ultra structural model without prior information can be equivalently written as

$$\hat{\beta} = \beta + n^{-1/2}\left[I_p + n^{-1/2}\Sigma_\Xi^{-1} H\right]^{-1}\Sigma_\Xi^{-1} h,$$

where

$$H = n^{-1/2}\left[S - M'M - n\Sigma_\delta - n\Sigma_\phi\right], \tag{11}$$

$$h = n^{-1/2}\left[X'(\varepsilon - \Delta\beta) + n\Sigma_\delta\beta\right], \tag{12}$$

and

$$\Sigma_\Xi = n^{-1} M'M + \Sigma_\phi. \tag{13}$$

Using Neumann Series (see [15]), it can be proved that:

$$\hat{\beta} = \beta + n^{-1/2}\Sigma_{\Xi}^{-1}h - n^{-1}\Sigma_{\Xi}^{-1}H\Sigma_{\Xi}^{-1}h + O_p\left(n^{-3/2}\right). \tag{14}$$

Additionally using Neumann Series to expand $R_{S\delta}^{-1}$, and apply these expansion in the consistent restricted estimator, $\hat{\beta}_R$, we can show that

$$\begin{aligned}\hat{\beta}_R = {} & \beta + n^{-1/2}\Sigma_{\Xi}^{-1}h - n^{-1}\Sigma_{\Xi}^{-1}H\Sigma_{\Xi}^{-1}h \\ & - n^{-1/2}B\Sigma_{\Xi}^{-1}h + n^{-1}B\Sigma_{\Xi}^{-1}H\Sigma_{\Xi}^{-1}h \\ & - n^{-1}B\Sigma_{\Xi}^{-1}HB\Sigma_{\Xi}^{-1}h + n^{-1}\Sigma_{\Xi}^{-1}HB\Sigma_{\Xi}^{-1}h + O_p\left(n^{-3/2}\right),\end{aligned} \tag{15}$$

where $B = \Sigma_{\Xi}^{-1}R'(R\Sigma_{\Xi}^{-1}R')^{-1}R$.
Analogously to the estimator (10), we can obtain

$$\begin{aligned}\hat{\beta}_{RS} = {} & \hat{\beta}_R - n^{-1}k\theta(I-B)\beta + kn^{-3/2}\theta\left(\frac{\beta'H\beta + 2\beta'h}{\beta'\Sigma_{\Xi}\beta} - u\right)(I-B)\beta \\ & - n^{-3/2}k\theta(I-B)\Sigma_{\Xi}^{-1}h - n^{-3/2}k\theta(I-B)\Sigma_{\Xi}^{-1}HB\beta + O_p\left(n^{-2}\right).\end{aligned} \tag{16}$$

where

$$\theta = \frac{\sigma_{\varepsilon}^2 + \beta'\Sigma_{\delta}\beta}{\beta'\Sigma_{\Xi}\beta}, \tag{17}$$

and

$$u = \frac{(\varepsilon - \Delta\beta)'(\varepsilon - \Delta\beta)}{n^{1/2}(\sigma_{\varepsilon}^2 + \sigma_{\delta}^2\beta'\beta)} - n^{1/2}. \tag{18}$$

Proposition 3.1. *The restricted and restricted Stein-rule estimators asymptotically are unbiased, and then up to order $O(n^{-2})$, the bias vectors of the estimators $\hat{\beta}_R$ and $\hat{\beta}_{Rs}$ are obtained by*

$$\begin{aligned}B(\hat{\beta}_R) &= E[\hat{\beta}_R] - \beta \\ &= n^{-1}(I-B)\Sigma_{\Xi}^{-1}\left[(p+1-m)\Sigma_{\delta}\beta - B'\Sigma_{\delta}\beta\right. \\ &\qquad \left. + \Sigma_{\delta}(I-B)\Sigma_{\Xi}^{-1}\Sigma_{\delta}\beta + tr(\Sigma_{\delta}(I-B)\Sigma_{\Xi}^{-1})\Sigma_{\delta}\beta\right] \\ B(\hat{\beta}_{RS}) &= E[\hat{\beta}_{Rs}] - \beta \\ &= n^{-1}(I-B)\Sigma_{\Xi}^{-1}\left[(p+1-m)\Sigma_{\delta}\beta - B'\Sigma_{\delta}\beta + \Sigma_{\delta}(I-B)\Sigma_{\Xi}^{-1}\Sigma_{\delta}\beta\right. \\ &\qquad \left. + tr(\Sigma_{\delta}(I-B)\Sigma_{\Xi}^{-1})\Sigma_{\delta}\beta\right] - n^{-1}k\theta(I-B)\beta\end{aligned}$$

In the follow, we compare the mean squared error matrices of estimators $\hat{\beta}_R$ and $\hat{\beta}_{RS}$. For this purpose, the difference of the mean squared error matrices is consider as follows:

$$\Delta MSE = E\left[(\hat{\beta}_R - \beta)(\hat{\beta}_R - \beta)'\right] - E\left[(\hat{\beta}_{RS} - \beta)(\hat{\beta}_{RS} - \beta)'\right].$$

Theorem 3.2. *If the assumptions about the distribution of ϵ, Δ and Φ hold, and when $\Sigma_{\delta} = \sigma_{\delta}^2 I_p$ and $\Sigma_{\phi} = \sigma_{\phi}^2 I_p$,* *then we have*

$$\begin{aligned}
\Delta MSE = &-k^2n^{-2}\theta^2(I_p-B)\beta\beta'(I_p-B)' \\
&+ kn^{-2}\sigma_\delta^2\theta(I_p-B)[(p+1-m)+\sigma_\delta^2 tr((I-B)\Sigma_\Xi^{-1}) \\
&\qquad + \frac{\sigma_\phi^2\beta'\beta-2\sigma_\varepsilon^2-3\sigma_\delta^2\beta'\beta}{\beta'\Sigma_\Xi\beta}]\beta\beta'\Sigma_\Xi^{-1}(I_p-B)' \\
&- kn^{-2}\sigma_\delta^2\theta(I_p-B)\beta\beta'B\Sigma_\Xi^{-1}(I_p-B)' + kn^{-2}\sigma_\delta^4\theta(I_p-B)\beta\beta'\Sigma_\Xi^{-1}(I_p-B)'\Sigma_\Xi^{-1}(I_p-B)' \\
&- 2kn^{-2}\theta(I_p-B)\frac{\sigma_\varepsilon^2}{\beta'\Sigma_\Xi\beta}\beta\beta'(I_p-B)' - 2kn^{-2}\theta(I_p-B)\beta\beta'\Sigma_\Xi^{-1}(I_p-B)' \\
&+ kn^{-2}\theta(\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta)(I_p-B)\Sigma_\Xi^{-1}(I_p-B)' + kn^{-2}\sigma_\delta^2\theta(\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta)(I_p-B)\Sigma_\Xi^{-2}(I_p-B)' \\
&+ kn^{-2}\sigma_\delta^4\theta(I_p-B)\Sigma_\Xi^{-1}\beta\beta'\Sigma_\Xi^{-1}(I_p-B)' - kn^{-2}\sigma_\delta^2\theta(I_p-B)\Sigma_\Xi^{-1}\beta'B\beta(I_p-B)' \\
&- kn^{-2}\sigma_\delta^2\theta(I_p-B)\Sigma_\Xi^{-1}\beta\beta'B'(I_p-B)' - kn^{-2}\sigma_\delta^2(\sigma_\delta^2+\sigma_\phi^2)\theta(I_p-B)\Sigma_\Xi^{-1}\beta\beta'B'\Sigma_\Xi^{-1}(I_p-B)' \\
&- kn^{-2}\sigma_\delta^2(\sigma_\delta^2-\sigma_\phi^2)\theta(I_p-B)\Sigma_\Xi^{-1}\beta'B\beta\Sigma_\Xi^{-1}(I_p-B)' \\
&+ trancepose\ of\ the\ all\ of\ the\ above\ terms\ except\ of\ the\ first\ term + O_p\left(n^{-3}\right)
\end{aligned}$$

The identification range of k which consequence the difference of the mean squared error matrices is positive definite, is difficult. Then to compare the estimators, we consider the quadratic loss function

$$L(\tilde{\beta},\beta) = (\tilde{\beta}-\beta)'W(\tilde{\beta}-\beta),$$

where $\tilde{\beta}$ is an estimator of β and W is a positive definite weight matrix.

Theorem 3.3. *Let $W = (I_p-B')^{-1}\Sigma_\Xi(I_p-B)^{-1}$, and the assumptions of Theorem 3.2 hold, then the difference between the risks of the estimators under the quadratic loss is given by*

$$\begin{aligned}
R(\hat{\beta}_R) - R(\hat{\beta}_{RS}) = &-k^2n^{-2}\theta^2\beta'\Sigma_\Xi\beta \\
&+ 2kn^{-2}\sigma_\delta^2\theta[(p+1-m)+\sigma_\delta^2 tr((I-B)\Sigma_\Xi^{-1}) \\
&\qquad + \frac{\sigma_\phi^2\beta'\beta-2\sigma_\varepsilon^2-3\sigma_\delta^2\beta'\beta}{\beta'\Sigma_\Xi\beta}]\beta'\beta \\
&- 2kn^{-2}\sigma_\delta^2\theta(p+2)\beta'B\beta + 2kn^{-2}\sigma_\delta^4\theta\beta'\Sigma_\Xi^{-1}(I_p-B)'\beta \\
&- 4kn^{-2}\theta\sigma_\varepsilon^2 - 4kn^{-2}\theta\beta'\Sigma_\Xi\beta \\
&+ 2kn^{-2}\theta(\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta)[p+\sigma_\delta^2 tr(\Sigma_\Xi^{-1})] \\
&+ 2kn^{-2}\sigma_\delta^4\theta\beta'\Sigma_\Xi^{-1}\beta \\
&- 2kn^{-2}\sigma_\delta^2(\sigma_\delta^2+\sigma_\phi^2)\theta\beta'B'\Sigma_\Xi^{-1}\beta \\
&- 2kn^{-2}\sigma_\delta^2(\sigma_\delta^2-\sigma_\phi^2)\theta\beta'B\beta tr(\Sigma_\Xi^{-1}) + O\left(n^{-3}\right).
\end{aligned}$$

Thus we can conclude that if

$$\begin{aligned}
0 < k < 2[&p-2+(p+2+\sigma_\delta^2 tr(\Sigma_\Xi^{-1}))\sigma_\delta^2\frac{\beta'\beta-\beta'B\beta}{\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta} \\
&+ (1-m-\sigma_\delta^2 tr(B\Sigma_\Xi^{-1}))\frac{\sigma_\delta^2\beta'\beta}{\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta} \\
&+ (\sigma_\delta^4\beta'\Sigma_\Xi^{-1}(I_p-B)'\beta+\sigma_\delta^4\beta'(I_p-B)'\Sigma_\Xi^{-1}\beta)\frac{1}{\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta} \\
&+ \frac{\sigma_\delta^2\beta'\beta(\sigma_\phi^2\beta'\beta-\sigma_\varepsilon^2)}{\beta'\Sigma_\Xi\beta(\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta)} - \frac{2\beta'\Sigma_\Xi\beta}{(\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta)} + \sigma_\delta^2 tr(\Sigma_\Xi^{-1}) \\
&+ \sigma_\delta^2\sigma_\phi^2\frac{\beta'B\beta tr(\Sigma_\Xi^{-1})-\beta'B\Sigma_\Xi^{-1}\beta}{\sigma_\varepsilon^2+\sigma_\delta^2\beta'\beta}],
\end{aligned}$$

the estimator $\hat{\beta}_{RS}$ dominates the estimator $\hat{\beta}_R$ with respect to the quadratic loss function to order $O(n^{-2})$. Finally, if $\lambda_{max}(A)$ denotes the maximum eigenvalue of A, a sufficient condition for the dominance of $\hat{\beta}_{RS}$ over $\hat{\beta}_R$ can be obtained as follows:

$$0 < k < 2[p - 2 + (1-m) - \sigma_\delta^2 \lambda_{\max}(\Sigma_\Xi^{-1}) - 2\sigma_\delta^{-2}\lambda_{\max}(\Sigma_\Xi)]. \tag{19}$$

4. Simulation Study

In this section, we carry out a simulation study to investigate the behaviour of the estimators $\hat{\beta}_R$ and $\hat{\beta}_{RS}$. We suppose that the parameters β, R matrix and r vector in (5) have the following forms respectively,

$$\beta = \begin{pmatrix} 2.2 \\ 1.1 \\ 3 \\ 4.2 \\ 2.5 \end{pmatrix}, \quad \beta = \begin{pmatrix} -2.2 \\ -1.1 \\ -3 \\ -4.2 \\ -2.5 \end{pmatrix}, \quad \beta = \begin{pmatrix} 2.2 \\ -1.1 \\ 3 \\ -4.2 \\ 2.5 \end{pmatrix}, \quad R = \begin{pmatrix} 0.8 & 0.6 & 0.7 & 0.9 & 0.8 \\ 0.2 & 0.7 & 0.4 & 0.7 & 0.8 \\ 0.6 & 0.4 & 0.6 & 0.1 & 0.4 \end{pmatrix},$$

and $r = R\beta$. We choose a matrix M of mean values. For our chosen M,

$$\frac{1}{n}M'M = \begin{pmatrix} 11.158 & 6.240 & 2.133 & 7.146 & 7.039 \\ 6.240 & 9.111 & 1.195 & 6.886 & 7.159 \\ 2.133 & 1.195 & 11.970 & 4.180 & 7.138 \\ 7.146 & 6.886 & 4.180 & 19.890 & 6.147 \\ 7.039 & 7.159 & 7.138 & 6.147 & 14.739 \end{pmatrix}, \quad \text{when} \quad n = 25$$

and

$$\frac{1}{n}M'M = \begin{pmatrix} 7.516 & 2.463 & 2.387 & 3.171 & 3.474 \\ 2.463 & 10.498 & 4.356 & 4.690 & 2.319 \\ 2.387 & 4.356 & 13.634 & 4.810 & 7.074 \\ 3.171 & 4.690 & 4.810 & 15.208 & 0.802 \\ 3.474 & 2.319 & 7.074 & 0.802 & 12.946 \end{pmatrix}, \quad \text{when} \quad n = 45.$$

In the table 1, we numerically estimate the bias, k, risk and difference risks (D) of two estimators $\hat{\beta}_R$ and $\hat{\beta}_{RS}$ for above distributions of measurement errors by 10^5 samples with size of 25 and 45, respectively. The middle points of interval () is considered as the value of k. From the table 1, it is observed that the absolute bias of $\hat{\beta}_R$ and $\hat{\beta}_{RS}$ decreases as the sample size increases. The values of absolute bias and risk are smaller under the sample size $n = 45$ than under $n = 25$ which shows that the estimators under consideration are asymptotically unbiased even for this sample size. The estimator $\hat{\beta}_{RS}$ has smaller absolute bias and risk than $\hat{\beta}_R$ in both the small and large samples. The absolute bias and risk of two estimators increases when the measurement error variances are increased. The obtained results show that $\hat{\beta}_{RS}$ has much better performance compared to $\hat{\beta}_R$.

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Table 1. The finite sample performance of the estimators

	n=25		n=45	
$\sigma^2_\epsilon = 0.4, \sigma^2_\delta = 0.4, \sigma^2_\phi = 0.4$	$\hat{\beta}_R$	$\hat{\beta}_{RS}$	$\hat{\beta}_R$	$\hat{\beta}_{RS}$
$bias(\hat{\beta}_1)$	-0.01014	-0.00900	-0.00400	-0.00295
$bias(\hat{\beta}_2)$	-0.00600	-0.00604	-0.00377	-0.00382
$bias(\hat{\beta}_3)$	0.01577	0.01393	0.00608	0.00440
$bias(\hat{\beta}_4)$	0.00375	0.00308	0.00098	0.00036
$bias(\hat{\beta}_5)$	-0.00338	-0.00212	0.00040	0.00156
k	-	2.48388	-	2.405
$R(\hat{\beta}, \beta)$	0.07321	0.0246	0.07417	0.04567
$D(\hat{\beta}_R, \hat{\beta}_{RS})$		0.0485		0.0285
$\sigma^2_\epsilon = 0.8, \sigma^2_\delta = 0.8, \sigma^2_\phi = 0.8$				
$bias(\hat{\beta}_1)$	-0.01645	-0.01444	-0.00115	0.00018
$bias(\hat{\beta}_2)$	-0.00296	-0.00319	-0.00065	-0.00066
$bias(\hat{\beta}_3)$	0.02625	0.02298	0.00179	-0.00035
$bias(\hat{\beta}_4)$	0.00850	0.00725	0.00179	-0.00035
$bias(\hat{\beta}_5)$	-0.01386	-0.01143	-0.00041	0.00101
k	-	2.315	-	2.419
$R(\hat{\beta}, \beta)$	0.31160	0.2511	0.15039	0.0651
$D(\hat{\beta}_R, \hat{\beta}_{RS})$		0.0605		0.0852

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A New Test for Random Events of the Exponential Probability Distribution

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Abstract

A new statistical test procedure is described to evaluate whether a set of radioactive-decay data is compatible with the assumption that these data originate from the decay of a single radioactive species. Criteria to detect contributions from other radioactive species and from different event sources are given. The test is applicable to samples of exponential distributions with two or more events.

1. Introduction

In the observation of radioactive-decay events it is a question of basic interest whether the data are compatible with the assumption that the measured time values originate from the decay of a single radioactive species. In the present work we will present a test particularly adapted to low counting statistics, which is based on the second moment of the logarithmic decay-time distribution. The new test provides an additional tool to validate important discoveries based on low event numbers as e.g. in the search for new heavy elements. A radioactive nucleus is characterised by a certain decay probability per time $\mathrm{d}P/\mathrm{d}t=\lambda$. From a specific number n_0 of radioactive nuclei, each one decays independently of the others. Therefore, the number of remaining nuclei $n(t)$ decreases gradually as a function of time:

$$dn/dt = -\lambda n(t). \tag{1}$$

The solution of this differential equation is

$$n(t) = n_0 \exp(-\lambda t). \tag{2}$$

The number of decay events per time is given by

$$|dn/dt| = \lambda\, n_0 exp(-\lambda t). \tag{3}$$

This is the density distribution of radioactive decays of one species of nuclei.
In an experiment, the times $t_1, t_2, ..., t_i, ... , t_n$ of individual radioactive decays from a limited number of nuclei represent a sample of this density distribution. This sample is subject to statistical fluctuations. It is the task of a statistical analysis to deduce an estimate of the decay constant λ. This task may be complicated by the fact that the radioactive

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decays can only be observed in a limited time range, above a lower threshold $t_{\min}$ and below an upper threshold $t_{\max}$. In addition, events of other species which decay with different decay constants or background events which appear with a constant rate may be mixed in. An even more complex situation appears, if daughter nuclei produced in the primary decay are also radioactive. Elaborate methods have been developed to determine the decay constants λ of the contributing radioactive species and their statistical uncertainties (see e.g. ref. [1] and the references given therein). A short review on these methods with special emphasis on their applicability to low counting statistics allows us to develop a new procedure in order to test the compatibility of measured data with the assumption of a radioactive decay. This test might be helpful to decide whether an observed sample of events originates from the radioactive decay of a single radioactive species. It also gives a handle to discover events from other sources than radioactive decay.

2. 2. Analysis methods

2.1. First moment of the decay times

The first moment (average) of the density distribution (3) is

$$\bar{t} = \frac{\int_0^\infty t\lambda n_0 \exp(-\lambda t)\,\mathrm{d}t}{\int_0^\infty \lambda n_0 \exp(-\lambda t)\,\mathrm{d}t} = \frac{\left[\frac{-\lambda t-1}{\lambda^2}\exp(-\lambda t)\right]_0^\infty}{\left[\frac{1}{-\lambda}\exp(-\lambda t)\right]_0^\infty} = \frac{1}{\lambda} \tag{4}$$

That means that the first moment of the measured decay times

$$\bar{t}_{\text{exp}} = \frac{\sum_{i=1}^{n} t_i}{n} \tag{5}$$

is an estimate of the inverse of the decay constant $1/\lambda$. However, there are a few prerequisites for the application of this method:

1. The full time range must be covered by the measurement. ($t_{\min}$ must be very small and $t_{\max}$ must be very large compared to $1/\lambda$.)
2. Any contribution of other radioactive species and any background must be excluded.

The first moment of the measured decay times is easily evaluated and gives a good estimate for the decay constant, also in the case of low statistics, even for a single event. The second moment could be used for testing the compatibility of the data with a radioactive decay. However, it is inconvenient that its value strongly depends on the decay constant which can only roughly be determined from the data in the case of low statistics.

2.2. Exponential decay curve

In the conventional analysis procedure of radioactive-decay data, the individual decay times are sorted into a spectrum with time intervals of constant width δt. The channel m of the spectrum contains the number of events observed between the time t_m and the time $t_m+\delta t$. The expected shape of this spectrum is approximately equal to the density distribution (3):

$\delta n/\delta t \approx \mathrm{d}n/\mathrm{d}t$ (6)

This spectrum has the shape of an exponential function. When drawn in a logarithmic scale, the data points are expected to follow a straight line. The decay constant can be determined by a fit, e.g. by the least-squares method which minimises the sum of the quadratic deviations of measured and calculated numbers of events per time interval. Contributions of another radioactive species and of background events can be recognised and extracted by using a more complex fit function. The usage of this method is not so evident in the case of low statistics. Many time intervals may not contain any event, and the shape of the spectrum is dominated by statistical fluctuations. In this case, it is difficult to judge whether the spectrum contains decay events of a single nuclear species only. Another disadvantage of the method is that one needs a large number of channels to represent the mixture of different radioactive species with strongly differing life times.

2.3. Logarithmic time scale

An unconventional way to represent radioactive-decay data, first proposed in ref. [2], consists of sorting the individual decay times into a spectrum with time intervals δt which have a width that is proportional to the time t, that means $\delta t/t$ = constant. The representation in such logarithmic time bins allows storing the relevant information of decay times over a very large range of time with a moderate number of channels. The corresponding density distribution is given by

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}(\ln t)} = \frac{\mathrm{d}n}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}(\ln t)} = -n_0 \lambda t \exp\left(-\lambda t\right) = \\ -n_0 \exp\left(\ln\left(\lambda t\right)\right) \exp\left(-\exp\left(\ln\left(\lambda t\right)\right)\right) = \\ -n_0 \exp(\ln t + \ln\lambda) \exp(\exp\left(-\ln t + \ln\lambda\right))\end{aligned} \tag{7}$$

or if we introduce $\Theta = \ln t$:[1]
$\left|\frac{\mathrm{d}n}{\mathrm{d}\Theta}\right| = n_0 \exp\left(\Theta + \ln\lambda\right) \exp\left(-\exp\left(\Theta + \ln\lambda\right)\right)$. (8)
This is a bell-shaped, slightly asymmetric curve. It is obvious that the distribution does not depend in shape on the decay constant λ, it is just shifted by -lnλ=ln(1/λ). Only the height scales with the number of counts n_0. Its integral is equal to n_0. The maximum of this function is located at Θ_{max} which is given by:
$\frac{\mathrm{d}^2 n}{\mathrm{d}\Theta^2} = 0 \rightarrow \Theta_{\text{max}} = \ln\left(\frac{1}{\lambda}\right)$. (9)
The standard deviation (the square root of the second moment) of this curve is:

$$\sigma_\theta = \sqrt{\frac{\int_{-\infty}^{+\infty}\left(\Theta - \overline{\Theta}\right)^2 \left|\frac{\mathrm{d}n}{\mathrm{d}\Theta}\right| \mathrm{d}\Theta}{n}} \tag{10}$$

with

$$\overline{\Theta} = \frac{\int_{-\infty}^{+\infty}\Theta \left|\frac{\mathrm{d}n}{\mathrm{d}\Theta}\right| \mathrm{d}\Theta}{n} \tag{11}$$

The value of σ_θ is about 1.28. The fact that a radioactive decay curve has a universal shape in this representation gives us a handle to detect if a second radioactive species contributes to the spectrum. In this case, the standard deviation of the logarithm of the measured decay times

$$\sigma_{\theta_{\text{exp}}} = \sqrt{\frac{\sum_{i=1}^{n}\left(\Theta_i - \overline{\Theta}_{\text{exp}}\right)^2}{n}} \tag{12}$$

with

$$\overline{\Theta}_{\text{exp}} = \frac{\sum_{i=1}^{n}\Theta_i}{n} \tag{13}$$

is larger. If the standard deviation $\sigma_{\theta\text{exp}}$ is significantly smaller, this might be an indication that the decay-time spectrum is incomplete because the experiment was not sensitive to the whole range of decay times. If this possibility is excluded, this is a strong indication that the observed events or at least part of them do not originate from radioactive decays but from some other source.

3. Test procedure

As suggested in the preceding chapter, the universal shape of the logarithmic time distribution of radioactive decays offers the possibility for testing whether a set of measured data is compatible with the assumption that these data originate from the radioactive decay of a single nuclear species. On the basis of this idea, we will elaborate a test which compares the standard deviation of the measured logarithmic time distribution with theoretical expectations. Like the time values t_i of individual radioactive-decay events, also the standard deviation $\sigma_{\theta\text{exp}}$ of the logarithmic decay-time distribution of a specific experiment evaluated by equation (12) is subject to statistical fluctuations. The

[1] In these considerations we use t and λ as dimensionless numbers by implicitely introducing a time unit. The results of this work do not depend on the choice of this time unit.

expected magnitude of these fluctuations can be estimated. They give a measure for the expected deviation of the width of the measured logarithmic decay-time distribution from the expected value. The basic idea of the test we propose in the present work is best illustrated for the case of two observed decay events. Let us assume that their decay times differ by a factor of two or three. Considering the spread of an exponential decay, such behaviour looks quite "normal" to us. Now take another two decay events which differ by several orders of magnitude in time. This does not look like a "normal" behaviour. One is tempted to attribute the two events to two different radioactive species with different life times. A third sample may consist of two measured events with almost exactly the same time values. Also this sample does not look "normal", it is a fortuitous, not very probable result. It will be our task to find a quantitative description for the compatibility of such observations with the assumption of the decay of a single radioactive species. A closer view on the problem reveals that the expected distribution of the standard deviation $\sigma_{\theta\text{exp}}$ of logarithmic decay-time distributions (equation 12) differs systematically from the distribution of σ_θ values defined by equation (10). The reason for this difference is that $\overline{\Theta}$ in equation (10) is the "true" mean value of the distribution, while $\overline{\Theta}_{\text{exp}}$ is the estimate for the mean value deduced from the observed events (equation 13). Therefore, the standard deviation $\sigma_{\theta\text{exp}}$ is systematically smaller than the value of σ_θ, especially for small numbers of events. Some expected characteristic properties of the distribution of $\sigma_{\theta\text{exp}}$ values have been calculated with Monte-Carlo techniques. A number n of random decay times $t_1,\ldots, t_n$ from a given exponential distribution were chosen at random and analysed according to equations (12) and (13) to determine a statistical sample of $\sigma_{\theta_{\text{exp}}}$. This procedure was repeated many times. These samples of $\sigma_{\theta_{\text{exp}}}$ are denoted by $x_j(n), j = 1$ to k, in the following. From a large number of samples ($k \to \infty$) for different values of n, the expectation value E_n, the standard deviation σ_n, and the relative skewness γ_n of the distribution of the quantity $\sigma_{\theta_{\text{exp}}}$ as a function of n were calculated from the relations:

$$E_n = \lim_{k\to\infty}\left\{\frac{\sum_{j=1}^{k} x_j(n)}{k}\right\}, \tag{14}$$

$$\sigma_n = \sqrt{\lim_{k\to\infty}\left\{\frac{\sum_{j=1}^{k}\left(x_j(n)-E_n\right)^2}{k}\right\}}, \tag{15}$$

and

$$\gamma_n = \lim_{k\to\infty}\left\{\frac{\sum_{j=1}^{k}\left(x_j(n)-E_n\right)^3}{k}\right\}/\sigma_n^{\frac{3}{2}} \tag{16}$$

The resulting values are listed in table 1. (Of course, only a finite value of k could be realised in the numerical calculation. This explains the slightly irregular behaviour of the values.) By normalising the distributions obtained for $\sigma_{\theta_{\text{exp}}}$ from these samples and integrating up to levels of 5 % and 95 %, respectively, one obtains the limits which comprise the range of $\sigma_{\theta_{\text{exp}}}$ values which can be accepted with a 90 % significance level to belong to the radioactive decay of a single radioactive species. These limits are also listed in table 1. Experimental values of $\sigma_{\theta_{\text{exp}}}$ falling below the lower limit can be rejected with an error chance below 5 % to originate from radioactive decay. If the experiment was sensitive to the whole decay-time range, at least part of the events originates from another kind of source, may be from some periodic noise. Experimental values of $\sigma_{\theta_{\text{exp}}}$ falling above the upper limit can be rejected with an error chance below 5 % to belong to the decay of a single radioactive species. If any background can be excluded, there is probably another radioactive species with another life time which contributes to the observed sample. The test can easily be extended to other values of the significance level.

4. Examples

Finally, we would like to illustrate the test procedure with two examples. In the discovery of the nucleus $^{271}110$, five events with alpha energies close to 10.74 MeV have been observed []. The decay times measured were 0.6 ms, 1.8 ms, 4.4 ms, 0.5 ms, and 2.6 ms. The analysis of these events results in a value of 0.84 for $\sigma_{\theta_{\text{exp}}}$. This value falls well between the limits (0.41 and 1.9) defined by a significance level of 90 % given in table 1 for 5 events. Thus, these events are consistent with the assumption, that they originate from the decay of one radioactive species, namely the same state of $^{271}110$.

Table 1. Expected properties of the distribution of the standard deviation $\sigma_{\theta\exp}$ of logarithmic decay-time distributions, defined by equations (12) and (13) for given numbers *n* of observed events. The values have been calculated by Monte-Carlo techniques. This explains their slightly irregular behaviour. Experimental values falling below the lower limit can be rejected with an error chance below 5 % to originate from radioactive decay. Experimental values falling above the upper limit can be rejected with an error chance below 5 % to belong to the decay of a single radioactive species.

Number of events *n*	Expectation value E_n for $\sigma_{\theta_{\exp}}$	Expected standard deviationσ_n of $\sigma_{\theta_{\exp}}$	Expected relative skewness γ_n	Lower limit of $\sigma_{\theta_{\exp}}$	Upper limit of $\sigma_{\theta_{\exp}}$
1	0	0	0	-	-
2	0.69	0.58	1.42	0.04	1.83
3	0.89	0.55	1.24	0.19	1.91
4	0.98	0.50	1.13	0.31	1.92
5	1.04	0.47	1.12	0.41	1.90
6	1.08	0.44	1.10	0.48	1.89
7	1.11	0.42	0.99	0.52	1.87
8	1.13	0.40	0.96	0.58	1.85
9	1.15	0.38	0.95	0.62	1.84
10	1.16	0.37	0.90	0.65	1.82
11	1.17	0.35	0.84	0.67	1.81
12	1.18	0.34	0.84	0.70	1.79
13	1.19	0.33	0.82	0.72	1.77
14	1.19	0.32	0.78	0.73	1.77
15	1.20	0.31	0.78	0.75	1.76
16	1.20	0.30	0.76	0.77	1.75
17	1.21	0.30	0.74	0.78	1.74
18	1.22	0.29	0.72	0.79	1.73
19	1.22	0.28	0.69	0.80	1.72
20	1.22	0.28	0.68	0.81	1.71
30	1.24	0.23	0.57	0.89	1.64
40	1.25	0.20	0.55	0.94	1.60
50	1.25	0.20	0.55	0.98	1.57
60	1.26	0.17	0.44	1.00	1.54
70	1.26	0.15	0.45	1.02	1.53
80	1.27	0.15	0.45	1.04	1.51
90	1.27	0.14	0.40	1.05	1.50
100	1.27	0.13	0.37	1.06	1.49
$n \to \infty$	1.28	$1.3 / \sqrt{n}$	$\to 0$	$1.28 - 2.15/\sqrt{n}$	$1.28 + 2.15/\sqrt{n}$

Figure 1 demonstrates the time distribution of these events on a logarithmic time scale. The expected logarithmic decay-time distribution (Eq. 7) with $\ln\left(\frac{1}{\lambda}\right) = 0.84$ (using a time unit of 1 ms) is shown in addition. This graphical presentation gives an illustration on the scattering of the decay times of a radioactive nucleus.

As a second example we chose the three decay chains, attributed to the alpha-decay cascade of the nucleus $^{293}118$ as reported in ref. [4]. Since the decay times of the second, the third and the forth decay in these chains are rather close in time, we apply the test to the totality of these 9 observed decay times. The values are given in table 2. The data originate from the decay of at least 3 different states. Therefore, we expect some broadening of the distribution if compared to the decay of a single state. This means that only deviations of $\sigma_{\theta_{\exp}}$ below the lower limit of the confidence interval are significant. They would give some indication that the source of the events is different from a radioactive decay.

The resulting value of $\sigma_{\theta_{\exp}} = 0.467$, calculated from the values in table 2 with equations (12) and (13), is clearly lower than the lower limit of the 90 % confidence interval given in table 1 which was determined to be 0.62. According to our criterion, the assumption that these events originate from radioactive decays is statistically rejected with an error

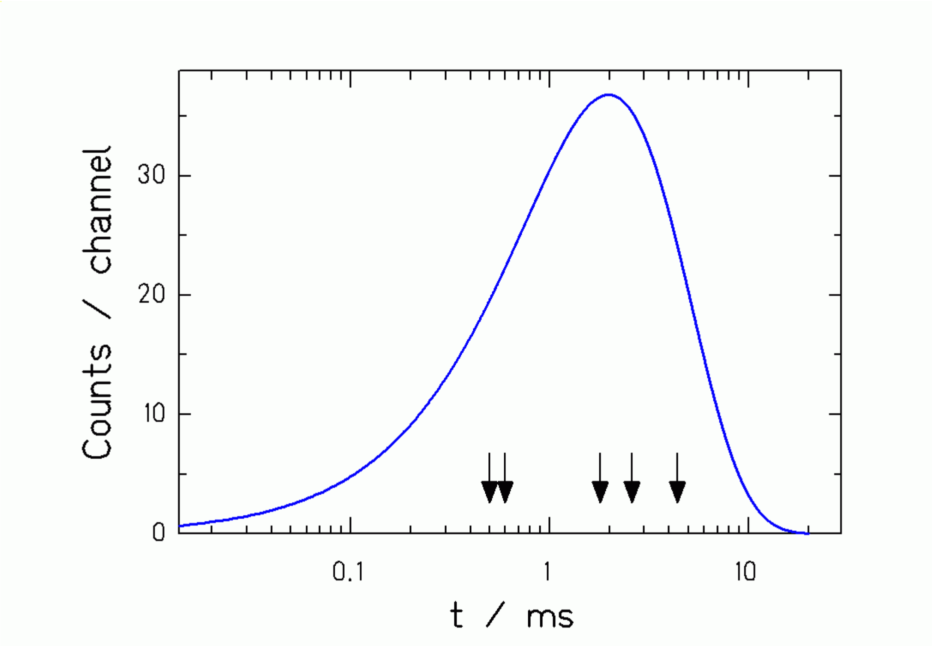

Fig. 1. Logarithmic decay-time distribution of 5 events observed in the α decay of $^{271}110$ with alpha energies close to 10.74 MeV. Data are taken from ref. [3]. The curve shows the logarithmic decay-time distribution (Eq. 7). The units on the ordinate are arbitrary.

probability of less then 5 %. This value gives the probability for an error of the first kind, i.e. the probability that the rejection is not justified. We would like to stress that the level of 5% does not provide strong evidence for the interpretation of these data. The test just yields an additional criterion for the judgement of the data, to be combined with the other experimental information available.

Table 2. Measured decay times of the second, third and forth alpha decay of the three decay chains, attributed to the decay of $^{293}118$ in ref. [4].

	First chain	Second chain	Third chain
α_2	1.243 ms	1.207 ms	0.310 ms
α_3	0.708 ms	0.741 ms	1.047 ms
α_4	1.201 ms	1.750 ms	0.939 ms

5. Conclusion

We have developed a procedure to test the hypothesis that a set of data originates from the decay of a single radioactive species. Larger fluctuations indicate that there is a continuous background or that one or more additional radioactive species with different half lives contribute to the data. Smaller fluctuations indicate that at least part of the data cannot be attributed to a radioactive decay but rather originates from a periodic noise. The test is particularly suited for small event numbers. It is applicable to any random variable governed by an exponential distribution.

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Left Censored Data Analysis from the Generalized Exponential Probability Distribution

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Abstract

The generalized exponential distribution proposed by Gupta and Kundu (1999) is an important lifetime distribution in survival analysis. In this paper, we consider the maximum likelihood estimation procedure of the parameters of the generalized exponential distribution when the data are left censored. We obtain the maximum likelihood estimators of the unknown parameters and also obtain the Fisher Information matrix. Simulation studies are carried out to observe the performance of the estimators in small sample.

1. Introduction

The generalized exponential (GE) distribution (Gupta and Kundu; 1999) has the cumulative distribution function (CDF)

$$F(x;\, \alpha,\, \lambda) = \left(1 - e^{-\lambda\, x}\right)^{\alpha};\, \alpha, \lambda,\, x > 0,$$

with the corresponding probability density function (PDF) given by

$$f(x;\, \alpha,\, \lambda) = \alpha\, \lambda\, \left(1 - e^{-\lambda\, x}\right)^{\alpha-1} e^{-\lambda\, x}, \text{for} x > 0.$$

Here α and λ are the shape and scale parameters respectively. GE distribution with the shape parameter α and the scale parameter λ will be denoted by $GE(\alpha,\, \lambda)$. It is known that the shape of the PDF of the two-parameter GE distribution is very similar to the corresponding shapes of gamma or Weibull distributions. It has been observed in Gupta and Kundu (1999) that the two-parameter $GE(\alpha,\, \lambda)$ can be used quite effectively in analyzing many lifetime data, particularly in place of two-parameter Gamma or two-parameter Weibull distributions. The two-parameter $GE(\alpha,\, \lambda)$ can have increasing and decreasing failure rate depending on the shape parameter. The readers are referred to Raqab (2002), Raqab and Ahsanullah (2001), Zheng (2003) and the references cited there for some recent developments on GE distribution. Although several papers have already appeared on the estimation of the parameters of GE distribution for complete sample case, see for example the review article of Gupta and Kundu (2006b), but not much attention has been paid in case of censored sample. The main aim of this is to consider the statistical analysis of the unknown

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parameters when the data are left censored from a GE distribution. We obtain the maximum likelihood estimators (MLEs) of the unknown parameters of the GE distribution for left censored data. It is observed that the MLEs can not be obtained in explicit form and the MLE of the scale parameter can be obtained by solving a non-linear equation. We propose a simple iterative scheme to solve the non-linear equation. Once the MLE of the scale parameter is obtained, the MLE of the shape parameter can be obtained in explicit form. We have also obtained the explicit expression of the Fisher information matrix and it has been used to construct the asymptotic confidence intervals of the unknown parameters. Extensive simulation study has been carried to observe the behavior of the proposed methods for different sample sizes and for different parameter values and it is observed that the performances of the proposed methods are quite satisfactory. There is a widespread application and use of left-censoring or left-censored data in survival analysis and reliability theory. For example, in medical studies patients are subject to regular examinations. Discovery of a condition only tells us that the onset of sickness fell in the period since the previous examination and nothing about the exact date of the attack. Thus the time elapsed since onset has been left censored. Similarly, we have to handle left-censored data when estimating functions of exact policy duration without knowing the exact date of policy entry; or when estimating functions of exact age without knowing the exact date of birth. A study on the "Patterns of Health Insurance Coverage among Rural and Urban Children" (Coburn, McBride and Ziller, 2001) faces this problem due to the incidence of a higher proportion of rural children whose spells were "left censored" in the sample (i.e., those children who entered the sample uninsured), and who remained uninsured throughout the sample. Yet another study (Danzon, Nicholson and Pereira, 2004) which used data on over 900 firms for the period 1988-2000 to estimate the effect on phase-specific (phases 1, 2 and 3) biotech and pharmaceutical R&D success rates of a firm's overall experience, its experience in the relevant therapeutic category, the diversification of its experience across categories, the industry's experience in the category, and alliances with large and small firms, saw that the data suffered from left censoring. This occurred, for example, when a phase 2 trial was initiated for a particular indication where there was no information on the phase 1 trial. Application can also be traced in econometric model, for example, for the joint determination of wages and turnover. Here, after the derivation of the corresponding likelihood function, an appropriate dataset is used for estimation. For a model that is designed for a comprehensive matched employer-employee panel dataset with fairly detailed information on wages, tenure, experience and a range of other covariates, it may be seen that the raw dataset may contain both completed and uncompleted job spells. A job duration might be incomplete because the beginning of the job spells is not observed, which is an incidence of left censoring (Bagger, 2005). For some further examples, one may refer to Balakrishnan (1989), Balakrishnan and Varadan (1991), Lee *et al.* (1980), etc. The rest of the paper is organized as follows. In Section 2 we derive the maximum likelihood estimators of $GE(\alpha,\, \lambda)$ in the presence of left censoring. In Section 3, we provide the complete enumeration of the Fisher Information matrix and discuss certain issues on the limiting Fisher information matrix. Simulation results and discussions are provided in Section 4.

2. Maximum Likelihood Estimation

In this section, maximum likelihood estimators of the $GE(\alpha,\, \lambda)$ are derived in presence of left censored observations. Let $X_{(r+1)}, \ldots,\, X_{(n)}$ be the last $n-r$ order statistics from a random sample of size n following $GE(\alpha,\, \lambda)$ distribution. Then the joint probability density function of $X_{(r+1)}, \ldots,\, X_{(n)}$ is given by

$$\begin{aligned} f\left(x_{(r+1)}, \ldots,\, x_{(n)};\, \alpha,\, \lambda\right) &= \frac{n!}{r!}\left(F(x_{(r+1)})\right)^r f\left(x_{(r+1)}\right)\ldots f\left(x_{(n)}\right) \\ &= \frac{n!}{r!}\left(1-e^{-\lambda\, x_{(r+1)}}\right)^{r\,\alpha}(\alpha\,\lambda)^{n-r}\, e^{-\lambda\sum_{i=r+1}^{n} x_{(i)}} \mathop{\pi}_{i=r+1}^{n}\left(1-e^{-\lambda\, x_{(i)}}\right)^{\alpha-1}. \end{aligned} \tag{2.1}$$

Then the log likelihood function denoted by $L\left(x_{(r+1)}, \ldots,\, x_{(n)};\, \alpha,\, \lambda\right)$ (or simply, $L\left(\alpha,\, \lambda\right)$) is

$$\begin{aligned} L\left(\alpha,\, \lambda\right) &= \ln n! - \ln r! + (n-r)\,\ln\alpha + (n-r)\,\ln\lambda + \alpha\, r\,\ln\left(1-e^{-\lambda\, x_{(r+1)}}\right) \\ &\quad + (\alpha-1)\sum\nolimits_{i=r+1}^{n}\ln\left(1-e^{-\lambda\, x_{(i)}}\right) - \lambda\sum\nolimits_{i=r+1}^{n} x_{(i)}. \end{aligned} \tag{2.2}$$

The normal equations for deriving the maximum likelihood estimators become

$$\frac{\partial L}{\partial \alpha}=\frac{n-r}{\alpha}+r\,\ln\left(1-e^{-\lambda\,x_{(r+1)}}\right)+\sum\nolimits_{i=r+1}^{n}\ln\left(1-e^{-\lambda\,x_{(i)}}\right)=0 \quad (2.3)$$

and
$$\frac{\partial L}{\partial \lambda}=\frac{n-r}{\lambda}+\frac{r\alpha}{1-e^{-\lambda x_{(r+1)}}}x_{(r+1)}e^{-\lambda\,x_{(r+1)}}+(\alpha-1)\sum\nolimits_{i=r+1}^{n}\frac{x_{(i)}e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}}-\sum\nolimits_{i=r+1}^{n}x_{(i)}=0. \quad (2.4)$$

From (2.3), we obtain the maximum likelihood estimator of αas a function of λ, say $\hat{\alpha}\left(\lambda\right)$, where

$$\hat{\alpha}\left(\lambda\right)=-\frac{n-r}{r\ln\left(1-e^{-\lambda x_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1-e^{-\lambda x_{(i)}}\right)} \quad (2.5)$$

Putting $\hat{\alpha}\left(\lambda\right)$in (2.2) we obtain the profile log-likelihood on λ as

$$\begin{aligned}L\left(\hat{\alpha}\left(\lambda\right),\lambda\right)=&\ln n!-\ln r!+(n-r)\ \ln\lambda+(n-r)\ \ln\left(-\frac{n-r}{r\ln\left(1-e^{-\lambda x_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1-e^{-\lambda x_{(i)}}\right)}\right)\\&-\frac{r\left(n-r\right)}{r\ln\left(1-e^{-\lambda x_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1-e^{-\lambda x_{(i)}}\right)}\ln\left(1-e^{-\lambda\,x_{(r+1)}}\right)\\&-\left(\frac{n-r}{r\ln\left(1-e^{-\lambda x_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1-e^{-\lambda x_{(i)}}\right)}+1\right)\sum\nolimits_{i=r+1}^{n}\ln\left(1-e^{-\lambda\,x_{(i)}}\right)-\lambda\sum\nolimits_{i=r+1}^{n}x_{(i)}\end{aligned}$$

i.e.

$$\begin{aligned}L\left(\hat{\alpha}\left(\lambda\right),\lambda\right)=&k+(n-r)\ \ln\lambda-\lambda\sum\nolimits_{i=r+1}^{n}x_{(i)}-\sum\nolimits_{i=r+1}^{n}\ln\left(1-e^{-\lambda\,x_{(i)}}\right)\\&-(n-r)\left(\ln\left(r\left(-\ln\left(1-e^{-\lambda\,x_{(r+1)}}\right)\right)\right)+\ln\left(\sum\nolimits_{i=r+1}^{n}-\ln\left(1-e^{-\lambda\,x_{(i)}}\right)\right)\right)\\&=g(\lambda),\ \text{say.}\end{aligned} \quad (2.6)$$

where, k in (2.6) is a constant independent of λ. Thus, the maximum likelihood estimator of λ, say $\hat{\lambda}_{MLE}$, can be obtained by maximizing (2.6) with respect to λ. The maximizing λ can be obtained (Gupta and Kundu; 1999b) from the fixed point solution of

$$h(\lambda)=\lambda, \quad (2.7)$$

where, $h(\lambda)$ is obtained from the fact that $\frac{\partial g(\lambda)}{\partial\lambda}=0$ and is given by

$$h\left(\lambda\right)=\left(\frac{1}{n-r}\sum\nolimits_{i=r+1}^{n}\frac{x_{(i)}}{1-e^{-\lambda x_{(i)}}}+\frac{\frac{rx_{(r+1)}e^{-\lambda x_{(r+1)}}}{1-e^{-\lambda x_{(r+1)}}}+\sum_{i=r+1}^{n}\frac{x_{(i)}e^{-\lambda x_{(i)}}}{1-e^{-\lambda x_{(i)}}}}{r\ln\left(1-e^{-\lambda x_{(r+1)}}\right)+\sum_{i=r+1}^{n}\ln\left(1-e^{-\lambda x_{(i)}}\right)}\right)^{-1}. \quad (2.8)$$

We apply iterative procedure to find the solution of (2.7). Once we obtain $\hat{\lambda}_{MLE}$, the maximum likelihood estimator of αsay $\hat{\alpha}$ can be obtained from (2.5) as $\hat{\alpha}_{MLE}=\hat{\alpha}\left(\hat{\lambda}_{MLE}\right)$.

3. Approximate and Limiting Fisher Information Matrices

3.1. *Approximate Fisher Information Matrix*

In this sub-section, we first obtain the approximate Fisher information matrix of the unknown parameters of GE distribution when the data are left censored, which can be used to construct asymptotic confidence intervals. The Fisher information matrix $I(\alpha,\lambda)$ can be written as follows;

$$I\left(\alpha,\lambda\right)=-\frac{1}{n}\left[\begin{array}{cc}E\left(\partial^2L/\partial\alpha^2\right)&E\left(\partial^2L/\partial\alpha\partial\lambda\right)\\E\left(\partial^2L/\partial\lambda\partial\alpha\right)&E\left(\partial^2L/\partial\lambda^2\right)\end{array}\right]. \quad (3.1)$$

Note that the elements of the Fisher Information matrix can be written as;

$$E\left(\frac{\partial^2 L}{\partial\alpha^2}\right) = -\left(\frac{n-r}{\alpha^2}\right),$$

$$E\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right) = E\left(\frac{rX_{(r+1)}e^{-\lambda X_{(r+1)}}}{1-e^{-\lambda X_{(r+1)}}} + \sum\nolimits_{i=r+1}^{n}\frac{X_{(i)}e^{-\lambda X_{(i)}}}{1-e^{-\lambda X_{(i)}}}\right) = E\left(\frac{\partial^2 L}{\partial\lambda\partial\alpha}\right) \tag{3.2}$$

and

$$E\left(\frac{\partial^2 L}{\partial\lambda^2}\right) = -E\left(\frac{n-r}{\lambda^2} + \frac{\alpha r X_{(r+1)}^2 e^{-\lambda X_{(r+1)}}}{\left(1-e^{-\lambda X_{(r+1)}}\right)^2} + (\alpha-1)\sum\nolimits_{i=r+1}^{n}\frac{X_{(i)}^2 e^{-\lambda X_{(i)}}}{\left(1-e^{-\lambda X_{(i)}}\right)^2}\right). \tag{3.3}$$

Thus to compute (3.2) and (3.3) we are required to obtain explicit expressions of expectations of the forms $E\left(\frac{X_{(i)}e^{-\lambda X_{(i)}}}{1-e^{-\lambda X_{(i)}}}\right)$ and $E\left(\frac{X_{(i)}^2 e^{-\lambda X_{(i)}}}{\left(1-e^{-\lambda X_{(i)}}\right)^2}\right)$ for $i = r+1, \ldots, n$. Note that the density of the i^{th} order statistic from a random sample of size n following the $GE(\alpha,\, \lambda)$ distribution is

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!}\left(1-e^{-\lambda\, x}\right)^{\alpha(i-1)}\left(1-\left(1-e^{-\lambda\, x}\right)^{\alpha}\right)^{n-i}\alpha\,\lambda\,\left(1-e^{-\lambda\, x}\right)^{\alpha-1}e^{-\lambda\, x};\quad x>0.$$

Then,

$$\begin{aligned}
E\left(\frac{X_{(i)}e^{-\lambda X_{(i)}}}{1-e^{-\lambda X_{(i)}}}\right) &= C_{n,\,i}\,\alpha\,\lambda\int_0^\infty x\,\left(1-e^{-\lambda,x}\right)^{\alpha\, i-2}e^{-2\lambda\, x}\left(1-\left(1-e^{-\lambda\, x}\right)^{\alpha}\right)^{n-i}dx\\
&= -\frac{C_{n,i}}{\lambda}\alpha\int_0^1 y\,\ln y\,(1-y)^{\alpha\, i-2}\left(1-(1-y)^{\alpha}\right)^{n-i}dy\\
&= -\frac{C_{n,i}}{\lambda}\alpha\sum\nolimits_{k=0}^{n-i}(-1)^{n-i-k}\binom{n-i}{k}\int_0^1 y\,\ln y\,(1-y)^{\alpha\, n-\alpha\, k-2}dy\\
&= -\frac{C_{n,i}}{\lambda}\alpha\sum\nolimits_{k=0}^{n-i}(-1)^{n-i-k}\binom{n-i}{k}\sum\nolimits_{l=0}^{\alpha\, n-\alpha\, k-2}(-1)^{\alpha\, n-\alpha\, k-l-2}\binom{\alpha\, n-\alpha\, k-2}{l}\\
&\times\int_0^1 y^{\alpha\, n-\alpha\, k-l-1}\ln y\,dy\\
&= \frac{C_{n,i}}{\lambda}\alpha\sum\nolimits_{k=0}^{n-i}(-1)^{(n-k)(\alpha+1)-(i+2)}\binom{n-i}{k}\sum\nolimits_{l=0}^{\alpha\, n-\alpha\, k-2}(-1)^{l}\binom{\alpha\, n-\alpha\, k-2}{l}\\
&\times\frac{1}{(\alpha n-\alpha k-l)^2};\ \text{for}\ \alpha\, n-\alpha\, k-l>0
\end{aligned} \tag{3.4}$$

where,

$$C_{n,i} = \frac{n!}{(i-1)!(n-i)!}.$$

Similarly,

$$\begin{aligned}
E\left(\frac{X_{(i)}^2 e^{-\lambda X_{(i)}}}{\left(1-e^{-\lambda X_{(i)}}\right)^2}\right) &= C_{n,i}\,\alpha\,\lambda\int_0^\infty x^2\left(1-e^{-\lambda\, x}\right)^{\alpha\, i-3}e^{-2\lambda\, x}\left(1-\left(1-e^{-\lambda\, x}\right)^{\alpha}\right)^{n-i}dx\\
&= \frac{C_{n,i}}{\lambda^2}\alpha\sum\nolimits_{k=0}^{n-i}(-1)^{(n-k)(\alpha+1)-(i+3)}\binom{n-i}{k}\sum\nolimits_{l=0}^{\alpha\, n-\alpha\, k-3}(-1)^{l}\binom{\alpha\, n-\alpha\, k-3}{l}\\
&\times\frac{2}{(\alpha n-\alpha k-l)^3};\ \text{for}\ \alpha\, n-\alpha\, k-l>0
\end{aligned} \tag{3.5}$$

(3.4) and (3.5) are obtained using the fact that

$$\int_0^1 y^m (\ln y)^n \, dy = \frac{(-1)^n n!}{(m+1)^{n+1}};\; m > n-1;\; n = 0, 1, 2....$$

For $i = n$,

$$E\left(\frac{X_{(i)} e^{-\lambda X_{(i)}}}{1 - e^{-\lambda X_{(i)}}}\right) = \frac{n\alpha}{\lambda} (-1)^{\alpha n - 2} \sum_{l=0}^{\alpha n - 2} (-1)^l \binom{\alpha n - 2}{l} \frac{1}{(\alpha n - l)^2}; \quad \alpha n - l > 0,$$

and

$$E\left(\frac{X_{(i)}^2 e^{-\lambda X_{(i)}}}{\left(1 - e^{-\lambda X_{(i)}}\right)^2}\right) = \frac{n\alpha}{\lambda^2} (-1)^{\alpha n - 3} \sum_{l=0}^{\alpha n - 3} (-1)^l \binom{\alpha n - 3}{l} \frac{2}{(\alpha n - l)^3}; \quad \alpha n - l > 1.$$

3.2. LIMITING FISHER INFORMATION MATRIX

In this sub-section we explore the asymptotic efficiency and hence attempt to obtain the limiting information matrix when $\frac{r}{n}$ converges to, say, p which lies in $(0, 1)$. For the left censored observations at the time point T, it has been observed by Gupta, Gupta and Sankaran (2004) that the limiting Fisher information matrix can be written as

$$I(\alpha, \lambda) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{3.6}$$

where

$$b_{ij} = \int_T^\infty \left(\frac{\partial}{\partial \theta_i} \ln r(x, \theta)\right) \left(\frac{\partial}{\partial \theta_j} \ln r(x, \theta)\right) f(x; \theta) \, dx,$$

and $\theta = (\alpha, \lambda)$, $r(x, \theta) = \frac{f(x;\theta)}{F(x;\theta)}$, the reversed hazard function. Moreover, it is also known, see Zheng and Gastwirth (2000), that for location and scale family, the Fisher information matrix for Type-I and Type-II (both for left and right censored data) are asymptotically equivalent. It is also mentioned by Zheng and Gastwirth (2000) that for general case (not for location and scale family) the results for Type-II censored data (both for left and right) of the asymptotic Fisher information matrices are not very easy to obtain. Unfortunately, the GE family does not belong to the location and scale family and we could not obtain the explicit expression for the limiting Fisher information matrix in this case. Numerically, we have studied the limiting behavior of the Fisher information matrix by taking $n = 5000$ (assuming it is very large) and compare them with the different small samples and different 'p' values. The numerical results are reported in Section 4.

4. Numerical Results and Discussions

In this section we report extensive simulation results for different sample sizes, for different parameter values and for different censored proportions. We mainly observe the performance of the proposed MLEs and the confidence intervals based on the asymptotic distribution of the MLEs. The performance of MLEs are based on their means squared errors (MSEs) and the performance of the confidence intervals are based on the coverage percentages (CPs). We begin with the generation of the $GE(\alpha, \lambda)$ random sample. Note that, if U is a random variable following an Uniform distribution in $[0, 1]$, then $X = \left(-\ln\left(1 - U^{1/\alpha}\right)\right)/\lambda$ follows$GE(\alpha, \lambda)$. Using the uniform random number generator, the generation of the GE random deviate is immediate. We consider different sample sizes ranging from small to large. Since λ is the scale parameter and the MLE is scale invariant, without loss of generality, we take $\lambda = 1$ in all our computations and consider different values of α. We report the average relative estimates and the average relative MSEs over 1000 replications for different cases. We compute the maximum likelihood estimates when both the parameters are unknown. $\hat{\lambda}$ can be obtained from the fixed point solution of (2.7) and $\hat{\alpha}(\lambda)$ can be obtained from (2.5). We consider the following sample sizes $n = 15, 20, 50, 100$, whereas α for different sample sizes are taken as $\alpha = 0.25, 0.5, 1.0, 2.0$ and 2.5. For left censoring, we leave out the first 10% and 20% of the data in each of the above cases of different combinations of n and α. Throughout, we consider $\lambda = 1$ and for each combination of n and

α generate a sample of size n from $GE\left(\alpha,1\right)$ and estimate α and λ in the case of left censoring of the given data of given order. We report the average values of $(\hat{\alpha}/\alpha)$, called the relative estimates, $\hat{\lambda}$ (also its relative estimate since the true parameter is $\lambda=1$)and also the corresponding average MSEs. All reported results are based on 1000 replications. Furthermore, using the asymptotic covariance matrix we obtain the average lower and the upper confidence limits of the estimates of both the shape and the scale parameters and also report the estimated coverage probability, computed as the proportion of the number of times, out of 1000 replications, the estimated confidence interval contains the true parameter value. The results corresponding to the shape parameter α, for various sample sizes are reported in Tables 1-4 and the results for the scale parameter λ are presented in Table 5-8.

Table 1. Average relative estimates, average relative MSEs, confidence limit and coverage probability of α when λ is unknown $(n=15)$

α	**No. of observations in left censoring**	**Average relative estimate**	**MSE**	**Average LCL**	**Average UCL**	**Coverage Probability**
0.25	3	1.2133	0.2896	0.1025	0.5041	0.9650
	2	1.2090	0.2816	0.1128	0.4917	0.9630
0.50	3	1.2833	0.5381	0.1563	1.1270	0.9750
	2	1.2591	0.4321	0.1863	1.0728	0.9750
1.00	3	1.3991	0.9235	0.1245	2.6736	0.9700
	2	1.3503	1.0461	0.2184	2.4822	0.9750
2.00	3	1.8132	2.2276	0.0000	8.5207	0.9580
	2	1.4857	1.4957	0.0000	6.0154	0.9680
2.50	3	1.7451	3.7330	0.0000	9.8770	0.9620
	2	1.4940	3.0090	0.0000	7.8995	0.9470

Table 2. Average relative estimates, average relative MSEs, confidence limit and coverage probability of α when λ is unknown $(n=20)$

α	**No. of observations in left censoring**	**Average relative estimate**	**MSE**	**Average LCL**	**Average UCL**	**Coverage Probability**
0.25	4	1.1341	0.1430	0.1233	0.4437	0.9740
	2	1.1239	0.1279	0.1342	0.4277	0.9570
0.50	4	1.2109	0.2763	0.2181	0.9928	0.9740
	2	1.1533	0.2165	0.2436	0.9096	0.9680
1.00	4	1.3051	0.4810	0.3125	2.2977	0.9760
	2	1.2113	0.3270	0.3995	2.0230	0.9620
2.00	4	1.4219	1.3320	0.1536	5.5341	0.9660
	2	1.3037	0.5870	0.5063	4.7085	0.9700
2.50	4	1.4463	1.8454	0.0000	7.2910	0.9520
	2	1.3013	0.5963	0.4804	6.0260	0.9740

From the simulations results, we observe that for a fixed level of left censoring, as sample size increases the biases and the average relative MSE of the estimates decrease quite rapidly. For example, for 10% left censored data and when $n=15$, the average relative MSE of the estimate of the shape parameter $\alpha=2$, is 1.4957 which reduces to 0.5870 for $n=20$, 0.0941 for $n=50$ and 0.0392 for $n=100$. A similar trend is observed for other levels of chosen shape parameter values and censoring levels (20%). This is indicative of the fact that the estimators are consistent and approaches the true parameter values as the sample size increases. Furthermore, for a fixed level of left censoring, as sample size increases the length of the confidence intervals also decrease significantly keeping the coverage probability around 0.95 to 0.98. For example, for a 10% censoring level and $n=15$, the average length of the confidence interval for $\alpha=0.25$ is 0.3789, this reduces to 0.2935 for $n=20$, 0.1696 for $n=50$ and 0.1179 for $n=100$, all with a coverage probability of 0.96. Note that in Tables 1, 2 and 5 for some α, LCL's are zero. Actually, they were negative, since $\alpha>0$, we forcefully truncated them at 0. We also observe that for a fixed sample size, the performance of the estimators deteriorate as the number of left censored observations increase, which is a natural

Table 3. Average relative estimates, average relative MSEs, confidence limit and coverage probability of α when λ is unknown $(n = 50)$

α	No. of observations in left censoring	Average relative estimate	MSE	Average LCL	Average UCL	Coverage Probability
0.25	10	1.0518	0.0425	0.1703	0.3556	0.9560
	5	1.0393	0.0334	0.1750	0.3446	0.9580
0.50	10	1.0640	0.0536	0.3245	0.7395	0.9500
	5	1.0566	0.0422	0.1710	0.3573	0.9510
1.00	10	1.0892	0.0884	0.5967	1.5817	0.9690
	5	1.0933	0.0727	0.6480	1.5385	0.9490
2.00	10	1.1090	0.1262	1.0229	3.4132	0.9570
	5	1.0857	0.0941	1.1374	3.2053	0.9710
2.50	10	1.1185	0.1466	1.1974	4.3951	0.9660
	5	1.1063	0.1127	1.3680	4.1633	0.9550

Table 4. Average relative estimates, average relative MSEs, confidence limit and coverage probability of α when λ is unknown $(n = 100)$

α	No. of observations in left censoring	Average relative estimate	MSE	Average LCL	Average UCL	Coverage Probability
0.25	20	1.0361	0.0190	0.1947	0.3233	0.9570
	10	1.0239	0.0156	0.1970	0.3149	0.9550
0.50	20	1.0342	0.0234	0.3756	0.6586	0.9570
	10	1.0327	0.0189	0.3869	0.6457	0.9550
1.00	20	1.0375	0.0315	0.7111	1.3640	0.9570
	10	1.0383	0.0260	0.7436	1.3330	0.9510
2.00	20	1.0558	0.0508	1.3230	2.9002	0.9540
	10	1.0501	0.0392	1.4028	2.7975	0.9570
2.50	20	1.0680	0.0510	1.6120	3.7282	0.9640
	10	1.0382	0.0354	1.6898	3.5012	0.9650

Table 5. Average relative estimates, average relative MSEs, confidence limit and coverage probability of λ when α is unknown $(n = 15)$

α	No. of observations in left censoring	Average relative estimate	MSE	Average LCL	Average UCL	Coverage Probability
0.25	3	1.4862	1.5186	0.0000	2.9978	0.9690
	2	1.5140	1.6061	0.0000	3.0353	0.9590
0.50	3	1.3227	0.5718	0.2549	2.3904	0.9570
	2	1.2930	0.6331	0.2701	2.3159	0.9500
1.00	3	1.2291	0.3111	0.3942	2.0640	0.9470
	2	1.2423	0.3049	0.4202	2.0644	0.9590
2.00	3	1.1790	0.2069	0.4675	1.8904	0.9500
	2	1.1770	0.1887	0.4945	1.8595	0.9530
2.50	3	1.2119	0.2086	0.5050	1.9188	0.9510
	2	1.1378	0.1701	0.4939	1.7817	0.9460

consequence of censoring. The degree of deterioration however is not significantly felt for moderate to high sample sizes (sample sizes 50 and 100). It is further observed that for a fixed sample size and a fixed level of censoring, the average relative MSE of the estimates and the length of the respective confidence intervals increase as the value of the shape parameter α increases. This indicates that estimation of the shape parameter under left censoring becomes difficult when the value of the shape parameter of the underlying GE distribution is large. It may indicate that the Fisher information contained in the left censored data may be a decreasing function of α.

Table 6. Average relative estimates, average relative MSEs, confidence limit and coverage probability of λ when α is unknown $(n = 20)$

α	No. of observations in left censoring	Average relative estimate	MSE	Average LCL	Average UCL	Coverage Probability
0.25	4	1.3296	0.7683	0.1322	2.5271	0.9590
	2	1.3473	0.8900	0.1519	2.5427	0.9560
0.50	4	1.2394	0.3360	0.3636	2.1152	0.9570
	2	1.1871	0.2686	0.3614	2.0129	0.9500
1.00	4	1.2025	0.2183	0.4902	1.9148	0.9530
	2	1.1393	0.1479	0.4845	1.7940	0.9570
2.00	4	1.1515	0.1399	0.5453	1.7577	0.9400
	2	1.1420	0.1254	0.5728	1.7112	0.9440
2.50	4	1.1333	0.1348	0.5537	1.7129	0.9410
	2	1.1093	0.0985	0.5727	1.6458	0.9470

Table 7. Average relative estimates, average relative MSEs, confidence limit and coverage probability of λ when α is unknown $(n = 50)$

α	No. of observations in left censoring	Average relative estimate	MSE	Average LCL	Average UCL	Coverage Probability
0.25	10	1.1088	0.1418	0.4609	1.7567	0.9570
	5	1.1129	0.1587	0.4699	1.7559	0.9550
0.50	10	1.0747	0.0729	0.5775	1.5720	0.9530
	5	1.1039	0.1380	0.4596	1.7482	0.9620
1.00	10	1.0613	0.0538	0.6507	1.4719	0.9470
	5	1.0787	0.0553	0.6802	1.4772	0.9500
2.00	10	1.0438	0.0374	0.6877	1.3999	0.9440
	5	1.0438	0.0346	0.7070	1.3805	0.9460
2.50	10	1.0483	0.0368	0.7021	1.3946	0.9510
	5	1.0466	0.0311	0.7214	1.3718	0.9480

Table 8. Average relative estimates, average relative MSEs, confidence limit and coverage probability of λ when α is unknown $(n = 100)$

α	No. of observations in left censoring	Average relative estimate	MSE	Average LCL	Average UCL	Coverage Probability
0.25	20	1.0622	0.0585	0.6207	1.5036	0.9560
	10	1.0618	0.0563	0.6253	1.4982	0.9510
0.50	20	1.0325	0.0319	0.6921	1.3729	0.9510
	10	1.0417	0.0315	0.7076	1.3758	0.9540
1.00	20	1.0294	0.0224	0.7451	1.3136	0.9530
	10	1.0301	0.0225	0.7579	1.3023	0.9460
2.00	20	1.0305	0.0190	0.7807	1.2804	0.9430
	10	1.0273	0.0170	0.7921	1.2625	0.9430
2.50	20	1.0304	0.0153	0.7887	1.2720	0.9620
	10	1.0188	0.0133	0.7935	1.2441	0.9550

Now we study the limiting behaviour of the Fisher information matrix as $n \to \infty$. Since it is not possible to compute it analytically, we take very large n ($n = 5000$) and compute $I(\alpha, \lambda)$ as defined in (3.1). We compute $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right)$ and $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\lambda^2}\right)$ for different n, r and α values and report the results in Table 9. Corresponding to each $n(r)$ and α values, the first and second figures represent $-\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right)$ and $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\lambda^2}\right)$, respectively. We mainly compare

$\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right)$and $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\lambda^2}\right)$ for $n = 15, 20, 50, 100$ with $n = 5000$ for different $p = \frac{r}{n}$ values. We do not report $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\alpha^2}\right)$, because for all n it is constant if $r = np$. The results are presented in the following table.

Table 9. Elements of the approximate and asymptotic Fisher information matrices

$n(r)$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 1.00$	$\alpha = 2.00$	$\alpha = 2.50$
15(3)	(0.2630, 0.5933)	(0.4417, 0.6312)	(0.6310, 0.9982)	(0.7828, 1.9628)	(0.8202, 2.4629)
20(4)	(0.2641, 0.5948)	(0.4432, 0.6318)	(0.6322, 0.9985)	(0.7834, 1.9630)	(0.8207, 2.4631)
50(10)	(0.2668, 0.5974)	(0.4455, 0.6328)	(0.6336, 0.9989)	(0.7843, 1.9632)	(0.8215, 2.4633)
100(20)	(0.2679, 0.5983)	(0.4464, 0.6333)	(0.6334, 0.9991)	(0.7843, 1.9630)	(0.8215, 2.4631)
5000(1000)	(0.2687, 0.5994)	(0.4467, 0.6339)	(0.6340, 0.9991)	(0.7842, 1.9629)	(0.8214, 2.4630)
15(2)	(0.2836, 0.6005)	(0.4552, 0.6340)	(0.6386, 0.9993)	(0.7866, 1.9632)	(0.8234, 2.4633)
20(2)	(0.2927, 0.6031)	(0.4609, 0.6350)	(0.6418, 0.9997)	(0.7885, 1.9635)	(0.8248, 2.4636)
50(5)	(0.2946, 0.6039)	(0.4622, 0.6352)	(0.6426, 0.9999)	(0.7889, 1.9636)	(0.8252, 2.4636)
100(10)	(0.2955, 0.6042)	(0.4626, 0.6355)	(0.6426, 0.9999)	(0.7888, 1.9634)	(0.8251, 2.4634)
5000(500)	(0.2958, 0.6047)	(0.4625, 0.6358)	(0.6423, 1.0000)	(0.7886, 1.9633)	(0.8249, 2.4633)

From the tabulated results it is clear that even for small sample sizes, viz., $n = 15$ or 20, $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right)$ and $\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\lambda^2}\right)$ match very well with $\lim_{n\to\infty}\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\alpha\partial\lambda}\right)$ and $\lim_{n\to\infty}\frac{1}{n}E\left(\frac{\partial^2 L}{\partial\lambda^2}\right)$, respectively. It justifies the use of approximate Fisher information matrix to draw inference of the unknown parameters of the generalized exponential distribution, when the data are left censored.

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Estimation of Copula density based on NSD variables via Wavelets

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Abstract

In this paper, we consider the wavelet estimation of copula density for negatively superadditive dependent. We propose and develop a new wavelet-based methodology for this problem. In particular, a BlockShrink estimator is constructed and we prove that it enjoys powerful mean integrated squared error properties over Besov balls. The main result is prepared to display the performance of the wavelet based estimator.

1. Introduction

The theory of wavelet and their applications in statistics and other sciences have become a significant technique. Since the problem of estimating copula density has only recently begun to receive attention in the literature, one can find extensive work on statistical estimation of different type of density functions using wavelets under non-complete data. For example [1] used wavelet methods for estimating regression function under biased data or [4] obtained an optimal rate for wavelet estimating of copula function through censored data.

One of the most applicable dependence concepts is that of negative superadditive dependence (NSD), which was introduced by [5]. the definition of NSD random variables is expressed on the basis of the superadditive functions. A function $f : R^n \to R$ is called superadditive if

$$f(\mathbf{x} \vee \mathbf{y}) + f(\mathbf{x} \wedge \mathbf{y}) \geq f(\mathbf{x}) + f(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in R^n$, where $\vee$ and $\wedge$ stand for componentwise maximum and minimum, respectively. Consequently, the NSD concept is proposed as follows. A random vector $(X_1, ..., X_n)$ is said to be NSD if

$$Ef(X_1, ..., X_n) \leq Ef(X_1^*, ..., X_n^*) \tag{1}$$

where $X_1^*, ..., X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i, and $f(.)$ is a superadditive function such that the expectations above exist. if $f(.)$ has continuous second partial derivatives, then the superadditive of $f(.)$ is equivalent to $\partial^2 f/\partial x_i \partial x_j \geq 0$, $1 \leq i \neq j \leq n$. Also, a sequence $\{X_n, n \geq 1\}$ of random variables is NSD if every finite subfamily is NSD.

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In this paper we consider the estimation of copula density under NSD random variables. It has been shown by [2] that the family of NSD sequences contains negatively associated (NA) random variables as a spacial case. So, the NSD assumptions are more general. [6] obtained an spacial rate as exponential convergence rate for kernel density estimation under NSD, whereas, our proposed estimator here achieve optimal rate in general by using block-thresholding wavelet methods.
The organisation of the article is as follows. In Section 2, we present some preliminaries of wavelet expansion of a function and Besov balls that are used later to introduce the estimator for copula density and subsequent analysis. The form of the estimator is given in Section 3 and main results are given in Section 4.

2. Wavelet and Besov balls

The term wavelet is used to refer to a set of orthonormal basis functions. Here $h_{j,k}(.)$ denotes the function $2^{j/2}h(2^j. - k)$ for $h(.)$ being either $\phi(.)$ or $\psi(.)$. The bivariate wavelet basis is as follows

$$\phi_{j,k}(x,y) = \phi_{j,k_1}(x)\phi_{j,k_2}(y),$$

$$\psi^{\epsilon}_{j,k}(x,y) = \Pi_{m=1}^{2}\phi^{1-\epsilon_m}_{j,k_m}(x)\psi^{\epsilon_m}_{j,k_m}(y),$$

for all $k = (k_1, k_2)$ and $\epsilon = (\epsilon_1, \epsilon_2) \in S_2 = \{(0,1),(1,0),(1,1)\}$. For any $j_0 \in \mathbb{N}$, the set $\{\phi_{j_0,k}, \psi^{\epsilon}_{j,l} | j \geq j_0, k \in \{0,...,2^{j_0}-1\}^2, l \in \{0,...,2^j-1\}^2, \epsilon \in S_2\}$ is an orthonormal basis of $L_2([0,1]^2)$. The expansion of function h ie given by

$$h_{k'}(x,y) = \sum_{k\in\{0,...,2^{j_0}-1\}^2} \alpha_{j_0,k}\phi_{j_0,k}(x,y) + \sum_{j\geq j_0}\sum_{k\in\{0,...,2^j-1\}^2}\sum_{\epsilon\in S_2}\beta^{\epsilon}_{j,k}\psi^{\epsilon}_{j,k}(x,y), \quad x,y\in[0,1]^2,$$

where the scaling coefficient $\alpha_{j_0,k}$ and the wavelet coefficient $\beta^{\epsilon}_{j,k}$ are given by

$$\alpha_{j_0,k} = \int_{(0,1)^2} h(x,y)\phi_{j_0,k}(x,y)dxdy, \quad and \quad \beta^{\epsilon}_{j,k} = \int_{(0,1)^2} h(x,y)\psi^{\epsilon}_{j,k}(x,y)dxdy.$$

To simplify the notation, we omit the range of k and ϵ in the summation from now on.
Since we deal with the wavelet method, it is very common to consider Besov spaces as functional spaces because they are characterized in term of wavelet coefficients as follows. Besov spaces depend on three parameters $s > 0, 1 < p < \infty$ and $1 < q < \infty$ and are denoted by B^s_{pq}. Let $f \in L_2(\mathbb{R}^2)$ and let s be smaller than r (wavelet regularity), define the sequence norm of the wavelet coefficients of function $f \in B^s_{pq}$ by

$$|f_{B^s_{pq}}| = (\sum_{k\in\mathbb{Z}^2}|\alpha_{j_0k}|^p)^{1/p} + (\sum_{j\geq j_0}[2^{j_0(s+d(1/2-1/p))}(\sum_{k\in\mathbb{Z}^2}|\beta_{j,k}|^p)^{1/p}]^q)^{1/q},$$

where $(|\beta_{j,k}|^p)^{1/p} = (\sum_{k\in\mathbb{Z}^2}\sum_{\epsilon\in S_2}|\beta^{\epsilon}_{j,k}|^p)^{1/p}$. We assume that the copula density c belongs to the Besov spaces.

3. Estimation procedures

Assuming that the copula density c belongs to $L_2([0,1]^2)$, we present wavelet procedure of its estimation. We introduce the new wavelet estimator for copula density under NSD assumptions. Let $d \in (0,\infty)$, j_1 and j_2 be the integers defined by $2^{j_1-1} < (\log(n-K'))^{1/2} \leq 2^{j_1}$ and $2^{j_2-1} < (n-k')^{1/2} \leq 2^{j_2}$. For any $j \in \{j_1,...,j_2\}$, set $L = \log(n-k')$ and $A_j = \{0,...L^{-1}(2^j-1)\}$. For any $K \in A_j$, we consider the set $U_{j,K} = \{k \in \{0,...2^j-1\}^2; (K-1)L \leq k \leq KL-1\}$. We define the BlockShrink estimator under NSD assumptions by

$$\hat{c}_{k'}(u,v) = \sum_{k}\hat{\alpha}_{j_0,k}\phi_{j_0k}(u,v) + \sum_{j=j_1}^{j_2}\sum_{K\in A_j}\sum_{k,\epsilon}\hat{\beta}^{\epsilon}_{j,k}1_{\{\hat{b}_{j,K}\geq d(n-k')^{-1/2}\}}\psi^{\epsilon}_{j,k}(u,v), \quad u,v\in[0,1].$$

Where $\hat{b}_{j,K} = \left(L^{-1} \sum_{k \in U_{j,K}} |\hat{\beta}^{\epsilon}_{j,k}|^2\right)^{1/2}$, and the natural estimator of $\alpha_{j_0 k}$ and $\beta^{\epsilon}_{j,k}$ are given by

$$\hat{\alpha}_{j_0,k} = \frac{1}{n-k'} \sum_{i=1}^{n-k'} \phi_{j_0,k}(F(X_i), G(X_{k'+i})), \qquad \hat{\beta}^{\epsilon}_{j,k} = \frac{1}{n-k'} \sum_{i=1}^{n-k'} \psi^{\epsilon}_{j,k}(F(X_i), G(X_{k'+i})).$$

for a given threshold level $\lambda_n = \sqrt{\frac{\log(n-k')}{n-k'}}$ and a set of indicies (j_1, j_2).
If the functions F and G are unknown, then the estimator of $\alpha_{j_0 k}$ and $\beta^{\epsilon}_{j,k}$ are given by

$$\tilde{\alpha}_{j_0,k} = \frac{1}{n-k'} \sum_{i=1}^{n-k'} \phi_{j_0,k}(F_n(X_i), G_n(X_{k'+i})), \qquad \tilde{\beta}^{\epsilon}_{j,k} = \frac{1}{n-k'} \sum_{i=1}^{n-k'} \psi^{\epsilon}_{j,k}(F_n(X_i), G_n(X_{k'+i})).$$

So the wavelet-based estimator for copula density is as

$$\tilde{c}_{k'}(u,v) = \sum_k \tilde{\alpha}_{j_0,k} \phi_{j_0 k}(u,v) + \sum_{j=j_1}^{j_2} \sum_{K \in A_j} \sum_{k,\epsilon} \tilde{\beta}^{\epsilon}_{j,k} 1_{\{\hat{b}_{j,K} \geq d(n-k')^{-1/2}\}} \psi^{\epsilon}_{j,k}(u,v), \quad u,v \in [0,1]. \tag{2}$$

where $\tilde{b}_{j,K} = \left(L^{-1} \sum_{k \in U_{j,K}} |\tilde{\beta}^{\epsilon}_{j,k}|^2\right)^{1/2}$.
The nonlinear procedure given in (2) deponds on the level indices (j_1, j_2) and on the threshold value $\lambda_n > 0$. Note that Φ is compactly supported, we get $\hat{\alpha}_{j_0 k} = 0$ except for a finite number of k indices. The above expressions are the NSD versions of the corresponding wavelet density estimators in the direct data copula density estimation context. In the next section, the performance of our procedure is measured and it is explained how to choose the parameters to achieve optimal rate.
In order to fix the notations, we assume that $supp(X_1, Y_1) = [0,1]^2$, (another interval of the form $[a,b]$ with $a > 0$ and $b > 0$ can be considered).
H1. The sequence $\{X_n, n \geq 1\}$ is an identically distributed sequence of NSD random variables with the bounded density function $h(.)$.
H2. The functions ϕ and ψ have bounded and continuous partial derivatives of first and second orders.
H3. There exists a constant $C > 0$ such that $\int_0^1 \int_0^1 \phi(x,y) dx dy \leq C$.

4. Main results

4.1. Upper Bounds

The purpose of this section is to study the behavior of $\hat{h}_{k'}(x,y)$ as an estimator of the bivariate density function under NSD assumptions. The mean integrated square error is defined as

$$MISE(\tilde{c}_{k'}, c_{k'}) = E\left[\int_0^1 \int_0^1 (\hat{c}_{k'}(u,v) - c_{k'}(u,v))^2 \, dx dy\right],$$
$$MISE(\tilde{c}_{k'}, c_{k'}) \leq 2MISE(\tilde{c}_{k'}, \hat{c}_{k'}) + 2MISE(\hat{c}_{k'}, c_{k'}).$$

Precisely, suppose that $h_{k'}$ belongs to the ball of radius For $M > 0$ in the Besov space $B^s_{p,q}(M)$ where the parameters s and p are $s > 0$ and $p \geq 2,\ 1 \leq q \leq \infty$ or $s > 2/p - 1$ and $p \in [1,2],\ 1 \leq q \leq \infty$ and also s^* could be defined as $s^* = s + 1 - 2/p$, if $p \in [1,2]$ and for the otherwise, $s^* = s$.

Theorem 4.1. *Let ϕ be a scalling function that mentioned in section 2.2 having m derivatives with $m > 1 + 1/s^*$ and for arbitrary $j_0 \in \mathbb{N}$, let $\hat{c}^{Lin}_{k'}$ be the wavelet linear estimator of $c_{k'}$ defined as $\hat{c}^{Lin}_{k'} = \sum_{k \in \{0,\ldots,2^{j_0}-1\}^2} \alpha_{j_0,k} \phi_{j_0,k}(u,v)$. Then there exists a constant $C_1 > o$ such that for all $M \in (0,\infty)$, $s > 2/p - 1$ and $p,q \in [1,\infty)$, if j_0 satisfies $2^{j_0} \simeq (n-k')^{1/(2+2s^*)}$, then*

$$\sup_{c_{k'} \in B^s_{p,q}(M)} MISE(\tilde{c}^{Lin}_{k'}, c_{k'}) \leq C_1 (n-k')^{-s^*/(1+s^*)}.$$

The proof of Theorem (4.1) uses a suitable decomposition of the MISE and Proposition (4.3) in the next subsection. Theorem (4.2) below determines the upper bound for the MISE of $\tilde{c}_{k'}$ (the BlockShrink estimator of c) over Besov balls.

Theorem 4.2. *Let $p \geq 2$ and $\tilde{c}_{k'}$ be the BlockShrink estimator defined by (2) with a large enough threshold constant λ. Then there exists a constant $C_2 > 0$ such that, all $M \in (0, \infty)$, $s > 2/p - 1$ and $p, q \in [1, \infty)$, if j_0 satisfies $2^{j_0} \simeq (n - k')^{1/(2+2s^*)}$, we have*

$$\sup_{c \in B^s_{p,q}(M)} MISE(\tilde{c}_{k'}, c_{k'}) \leq C_2 (n - k')^{-s^*/(1+s^*)}.$$

Note that $(n - k')^{-s^*/(1+s^*)}$ is the optimal rate of convergence (in the minimax sense) for the standard nonparametric bivariate density function.

4.2. Intermediate Results

Proposition 4.3. *Suppose that a constant $C > 0$ such that $\int\int \phi^2(x, y) dxdy \leq C \quad (x, y) \in [0, 1]$, then there exists a constant $C > 0$ such that for any $j \geq j_0$ and any $k = (k_1, k_2) \in \{0, ..., 2^j - 1\}^2$, ϵ,*

$$E\left(\sum_k (\hat{\alpha}_{j_0 k} - \alpha_{j_0 k})^2\right) \leq C(n - k')^{-s^*/(1+s^*)}.$$

for some constant $C > 0$ depending only on ϕ and either

$$||c||_2 = \int_{[0,1]^2} c_{k'}(u, v)^2 dudv \qquad or \qquad ||c||_\infty = \sup_{u,v \in [0,1]} |c_{k'}(u, v)|$$

Proposition 4.4. *Suppose that the assumptions of Theorem (4.2) are satisfied. Then there exists a constant $C > 0$ such that, for any $j_1 \leq j \leq j_2$, any $k = (k_1, k_2) \in \{0, ..., 2^j - 1\}^2$, ϵ,*

$$E\left((\hat{\beta}^\epsilon_{j,k} - \beta^\epsilon_{j,k})^4\right) \leq C(n - k')^{-2}.$$

The above inequality holds for $(\hat{\alpha}_{j,k}, \alpha_{j,k})$ instead of $(\hat{\beta}^\epsilon_{j,k}, \beta^\epsilon_{j,k})$.

Proposition 4.5. *Suppose that the assumptions of Theorem (4.2) are satisfied. Then there exists a constant $\kappa > 0$ such that, for any $j_1 \leq j \leq j_2$, any $K \in A_j$, ϵ,*

$$p\left(\left(\sum_{k \in U_{j,K}} (\hat{\beta}^\epsilon_{j,k} - \beta^\epsilon_{j,k})^2\right)^{1/2} \geq 2^{-1}\kappa(\log(n - k'))^{1/2}(n - k')^{-1/2}\right) \leq (n - k')^{-2}$$

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Ridge Counterpart of Buckley's Approach

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Abstract

In regression modeling, existence of collinearity in input variables leads to take some cares about the method of parameter estimation. On the other hand, there are many response variables that are fuzzy in nature. Although fuzzy modeling should be used but crisp values are provided for them. In fact, considering the non-fuzzy values for the output variable means transferring the consequent error of the ambiguity in this variable to the error term. So, we need spread values as vagueness of the response values, But in practice we cannot determine those values for response variables, because we do not have definite functions of them. Now to solving of this lack, since the $\alpha - cut$ of the membership function of the fuzzy numbers based on the own structure can be equivalent with the $(1-\gamma)\%$ in confidence interval of classic regression in this paper, we propose a ridge regression model in which not only fix the collinearity problem in variables, but also improve the performance of output variable with produce the fuzzy response. This will done, with using the various confidence intervals instead of α-cuts.

1. Introduction

Fuzzy regression model (FRM) was initiated by Tanaka [26], using a linear programming model in 1982 for the first time. This method, called the possibility regression, was further developed and improved by other researches. The primary model was defined based on non-fuzzy inputs, coefficients and output triangular fuzzy numbers with the aim of minimizing the overall ambiguity of the model. Subsequently, in the following years, the change in the objective function, the fuzzy input and the change in the membership function of the various components of the model, created several ways to improve the possibility model [11, 13, 15–17, 20, 25, 29]. On the other hand, Celmins [1] and Diamond [7] proposed the least-squares view, which minimizes the distance between the observed and estimated response variable derived from the classical regression method. Also, in [18, 19], the method of least absolute error has been used with conditions and different ideas to reduce the impact of remote locations [17, 29]. Combination methods such as [5] have been proposed and used to improve the performance of the models. In the process of developing fuzzy regression, innovative methods have been presented such as bootstrap [9, 28], semi-parametric [10, 12, 30], Neural Networks [6, 24], Support Vector Machines [27], Logistic regression [21], and etc. In 1998, Sanders et al. [23] examined Ridge's regression in dual variables. Hong et al. in [14] presented Ridge estimation with crisp inputs and fuzzy output (with Gaussian membership function) in a form relationship. Balasundarma and Kapil [2] estimated

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the weighted fuzzy ridge regression parameters with crisp inputs and triangular fuzzy output. Recently, Fuzzy ridge regression were studied in [22]. But, in [3], Buckley introduced a new approach to relationship between classical and fuzzy regression by confidence interval. This view was examined in [8] and its effectiveness for symmetric distributions was confirmed and proposed a improved method for variance confidence intervals based on Buckley approach. Now, we can be utilize this approach, when the dataset has an ambiguous response variable but was reported with crisp values.

In this paper, based on [3, 4, 8] we propose a new method to estimation of coefficients of regression model based on ridge regression (in classic regression) with crisp inputs and output. In fact, we utilize confidence intervals $(1-\gamma)\%$ instead of α-cuts to estimate the fuzzy coefficients and provide the fuzzy model (see [4] for more details). In this perspective, we will pursue two objectives: first, the impact of this estimate when there is collinearity between explanatory variables. Second, introducing a fuzzy model that has better performance than the classic model in prediction of response observations, when the input or output data are vagus nature. So using this method, we can generate a flexible estimate, which in addition to eliminate collinearity, does not hide the produced ambiguity by the response variable in the model error.

Accordingly, in Section 2, Buckley's approach is described. Using the previous section, our proposed method and fuzzy criterion is presented in the next section. In Section 4, we will show the effectiveness of this method in numerical examples. In the final section, we will describe as summaries the conclusion of using this method.

2. Buckley's Approach

Let $X_1, ..., X_n$ be a random sample from a distribution with probability density/mass function $f(\mathbf{X}; \theta)$, with observed values $x_1, ..., x_n$. Since, θ is a unknown parameter and must be estimated, so we obtain a point estimation with a classic method such as maximum likelihood. But we would never expect this point estimate to exactly equal θ. Therefore, as usual, is computed a $(1-\gamma)\%$ confidence interval for θ.

Buckley In [3, 4] introduced and developed that, we can make a $(1-\gamma)\%$ confidence interval for θ by any γ $(0 \leq \gamma \leq 1)$ and denote these confidence intervals as $[L_\gamma(\theta), U_\gamma(\theta)]$. By increases γ, the length of confidence intervals is decreases, and vice versa. So, we used the confidence intervals employed, reserved for α-cuts of fuzzy sets. Notice that, these intervals are nested and for $\gamma = 0$ and 1 we have maximum and minimum intervals respectively. Thus, it is similar to α-cut structure in fuzzy numbers and could be intended as a fuzzy number. In this way, Buckley also used his view to estimate regression coefficients .

3. Ridge estimation in fuzzy regression modeling

In this section, in order to better represent the method, we introduce the model factors as follows.

Definition 3.1. Consider the following regression model

$$Y = \mathbf{X}\beta + \epsilon, \tag{1}$$

where, Y and ϵ (as the components of response and error of the regression model) are $n \times 1$ vectors, $\mathbf{X}$ is $n \times (p+1)$ matrix and β is $p \times 1$ vector. Also, the prerequisites are establishment for the model error term.

The regression model can be stated by (1) So, the model prediction is

$$\widehat{Y} = \mathbf{X}\widehat{\beta} \tag{2}$$

where, $\widehat{Y} = (y_1, ..., y_n)^\top$ and $\widehat{\beta} = (\beta_1, ..., \beta_{p+1})^\top$ are estimated of response and parameter vectors and $\mathbf{X}$ is a matrix of constants.

Now, if has existed collinearity between input variables, it should be used the ridge regression. Because, variance of the estimations of obtain from least-squares method will be big and we will have various estimations for coefficients. So, the regression problem can be rewritten under an optimization model.

$$\min_{\beta}\Big(\big(\widehat{Y}-Y\big)^{\top}\big(\widehat{Y}-Y\big)+k\beta^{\top}\beta\Big)= \\ \min_{\beta}\Big(\big(\mathbf{X}\beta-Y\big)^{\top}\big(\mathbf{X}\beta-Y\big)+k\beta^{\top}\beta\Big), \tag{3}$$

where, $k>0$ is tuning parameter.
Now, we write the spectral decomposition of the positive definite design matrix $\mathbf{X}^{\top}\mathbf{X}$ to get $\mathbf{X}^{\top}\mathbf{X} = \square\square\square^{\top}$, where $\square_{\big((p+1)\times(p+1)\big)}$ is a column orthogonal matrix of eigenvectors and $\square = Diag(\lambda_1,\lambda_2,...,\lambda_(p+1))$ where, $\lambda_j > 0, j=1,2,...,p+1$ is the ordered eigenvalue matrix corresponding to $\mathbf{X}^{\top}\mathbf{X}$. Then, with deferential from (3) under the β,
we have:

$$\widehat{\beta}^{Ridge} = \big(\mathbf{X}^{\top}\mathbf{X}+k\mathbf{I}\big)^{-1}\mathbf{X}^{\top}Y. \tag{4}$$

So, the covariance matrix (COV) for $\widehat{\beta}^{Ridge}$ is

$$COV(\widehat{\beta}^{Ridge}) = COV\Big[\big(\mathbf{X}^{\top}\mathbf{X}+k\mathbf{I}\big)^{-1}\mathbf{X}^{\top}Y\Big] \\ = \big(\mathbf{X}^{\top}\mathbf{X}+k\mathbf{I}\big)^{-1}COV\big(\mathbf{X}^{\top}Y\big)\big(\mathbf{X}^{\top}\mathbf{X}+k\mathbf{I}\big)^{-1}. \tag{5}$$

Thus, according to the spectral decomposition structure the total variance (VAR) of $\widehat{\beta}^{Ridge}$ is given by

$$VAR(\widehat{\beta}^{Ridge}) = \sigma^2 tr\Big[\big(\square+k\mathbf{I}\big)^{-1}\square\big(\square+k\mathbf{I}\big)^{-1}\Big] \\ = \sigma^2\sum_{j=1}^{p+1}\Big[\frac{\lambda_j}{\big(\lambda_j+k\big)^2}\Big]. \tag{6}$$

Hence, for obtain the variance of any $\widehat{\beta}_j^{Ridge}$ we have

$$VAR\big(\widehat{\beta}_j^{Ridge}\big) = \sigma^2\Big[\frac{\lambda_j}{\big(\lambda_j+k\big)^2}\Big], \tag{7}$$

where,

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i-\widehat{y}_i)^2}{(n-(p+1))}.$$

Therefore, the $(1-\gamma)\%$ confidence interval for any coefficient of the regression model is given by

$$\Big[\widehat{\beta_j}^{Ridge} - t_{\frac{\gamma}{2}}\sqrt{VAR(\widehat{\beta}_j^{Ridge})}, \widehat{\beta_j}^{Ridge} + t_{\frac{\gamma}{2}}\sqrt{VAR(\widehat{\beta}_j^{Ridge})}\Big], \tag{8}$$

where, $t_{\frac{\gamma}{2}}$ is quantile of $(1-\gamma)\%$ from the t distribution with (n-p-1) degrees of freedom.
Now, we put the above $(1-\gamma)\%$ confidence intervals instead of α-cut by using the various values from the $0.01 \leq \gamma < 1$ rang.
Thus, based on Buckley's approach, the lower and upper bound of confidence interval for any γ (as one α-cut) can be considered as follows

$$\Big[L_{\gamma}\big(\widehat{\beta_j}^{Ridge}\big), U_{\gamma}\big(\widehat{\beta_j}^{Ridge}\big)\Big], \tag{9}$$

so, in (9), minimum and maximum values are provided a symmetric fuzzy number.
Accordingly, the estimated response in fitted model, is obtained as follows into the fuzzy number structure

$$\hat{\tilde{Y}}_i(\gamma) = \Big[L_{\gamma}(\hat{\tilde{Y}}_i), U_{\gamma}(\hat{\tilde{Y}}_i)\Big] = \Big[\sum_{j=1}^{p+1}x_{ij}L_{\gamma}\big(\widehat{\beta_j}^{Ridge}\big), \sum_{j=1}^{p+1}x_{ij}U_{\gamma}\big(\widehat{\beta_j}^{Ridge}\big)\Big]. \tag{10}$$

Remark 3.2. *In equivalent (10), we relocate the lower and upper of* $\widehat{\beta}_j^{\,Ridge}$ *limits together, for any* $x_{ij} < 0$.

3.1. Fuzzy criterion

In this section, we introduce fuzzy criteria to evaluate the results and demonstrate the efficiency of the proposed method.

Let Eq.(10), as a goodness of fit measure the length of confidence interval (LCI) is

$$LCI(\hat{\tilde{Y}}_i(\gamma)) = U_\gamma(\hat{\tilde{Y}}_i) - L_\gamma(\hat{\tilde{Y}}_i).$$

Lemma 3.3. *In estimated fuzzy regression models (based on this approach) for any* $0 < \gamma \leq 1$ *(as a* α*-cut), the LCI of* i^{th} *estimated response is given by*

$$LCI\big(\hat{\tilde{Y}}_i(\gamma)\big) = 2t_{\frac{\gamma}{2}} \sum_{j=1}^{p+1} x_{ij} \sqrt{VAR(\hat{\beta}_j)}. \tag{11}$$

Proof. for any $0 < \gamma \leq 1$ (as a α-cut) we have

$$LCI\big(\hat{\tilde{Y}}_i(\gamma)\big) = U_\gamma(\hat{\tilde{Y}}_i) - L_\gamma(\hat{\tilde{Y}}_i),$$

so, based on equation(10) estimated β_j from both of method

$$LCI\big(\hat{\tilde{Y}}_i(\gamma)\big) = \sum_{j=1}^{p+1} x_{ij} U_\gamma\big(\widehat{\beta}_j\big) - \sum_{j=1}^{p+1} x_{ij} L_\gamma\big(\widehat{\beta}_j\big),$$

Now, with put on of Eq.(8), could be written for $\widehat{\beta}_j$

$$LCI\big(\hat{\tilde{Y}}_i(\gamma)\big) = \sum_{j=1}^{p+1} x_{ij} \Big(\widehat{\beta}_j + t_{\frac{\gamma}{2}} \sqrt{VAR(\hat{\beta}_j)}\Big) - \sum_{j=1}^{p+1} x_{ij} \Big(\widehat{\beta}_j - t_{\frac{\gamma}{2}} \sqrt{VAR(\hat{\beta}_j)}\Big),$$

thus, for any $0 < \gamma \leq 1$

$$LCI\big(\hat{\tilde{Y}}_i(\gamma)\big) = 2t_{\frac{\gamma}{2}} \sum_{j=1}^{p+1} x_{ij} \sqrt{VAR(\hat{\beta}_j)}. \tag{12}$$

□

Hence, can be used as a criterion the value of $\frac{LCI\big(\hat{\tilde{Y}}_i^{M_1}(\gamma)\big)}{LCI\big(\hat{\tilde{Y}}_i^{M_2}(\gamma)\big)}$ ratio.

Therefore, the ratio of LCI between M_1 and M_2 methods and is denoted by $RLCI(M_1, M_2)$, obtaining

$$RLCI(M_1, M_2) = \frac{LCI\big(\hat{\tilde{Y}}_i^{M_1}(\gamma)\big)}{LCI\big(\hat{\tilde{Y}}_i^{M_2}(\gamma)\big)} = \frac{\sum_{j=1}^{p+1} x_{ij} \sqrt{VAR(\hat{\beta}_j^{\,M_1})}}{\sum_{j=1}^{p+1} x_{ij} \sqrt{VAR(\hat{\beta}_j^{\,M_2})}} \tag{13}$$

In this measure, the value of greater than 1 indicates growth amount of the length of the $\hat{\tilde{Y}}_i^{M_1}(\gamma)$ confidence interval from $\hat{\tilde{Y}}_i^{M_2}(\gamma)$. In addition to, this scale is independent to γ (as a α-cut).

4. Computation

In the following, shown the better performance of Ridge model by creating the 95% confidence intervals for each coefficient in both method. Also, these intervals is obtained for some samples of estimated response variables. The superiority of the Ridge model is presented by some shapes and criteria. This section included by two examples Simulated and real data respectively.

4.1. Simulation study

Based on 30 simulated data set of size $n = 20$, consider the following proposed multiple linear regression model (1): where

1. $X_{i1} = z_{i1} + \epsilon_{i1}, z_{i1} \sim N(4,1) \quad and \quad \epsilon_{i1} \sim N(0, 0.02)$,
   $X_{ij} = z_{ij} + \epsilon_{ij}, z_{ij} \sim N(6, \mathbf{\Sigma})$, $[\Sigma_{lk}] = Cov(X_{il}, X_{ik}) = 0.9^{|l-k|}$ for $j = 2, 3, 4$,
   and $\quad \epsilon_{ij} \sim N(0, 0.04)$, $\quad$ for $j = 2, 3, 4$

2. $\beta_0 = 3, \quad \beta_1 = 1.5 \quad \beta_2 = -2, \quad \beta_3 = 0.5, \quad \beta_4 = 2,$

3. $\epsilon_i \sim N(0, 2)$.

Based on the simulated data above, the mean of the values of the RLCI criterion for 30 simulated data in each one of the samples which are produced from least squares (LS) and ridge (R) methods have been computed. Also, this measure has been presented for the 12^{th} data-set (as the best result obtained in all simulated data) which has been obtained by the ridge model tuning parameter. Notice that, the value of response variable is vagus but has been registered by crisp value.

All of these comparing results have been shown in Table 1. In addition, effectiveness of 12^{th} data, from the simulated data set, in reduction of mean square error of produced model by ridge method has been presented in Figure 1.

Table 1. Mean of the RLCIs criterion for 30 simulated data and this measure for the 12^{th} data-set.

Sample	Mean of the 30 simulated data	12^{th} simulated data
	RLCI(LSM, RM)	RLCI(LSM, RM)
1	1.428	2.234
2	1.305	2.165
3	1.571	2.586
4	2.105	2.359
5	1.895	2.365
6	2.216	2.403
7	1.287	2.408
8	1.942	2.579
9	2.421	2.694
10	1.362	2.606
11	2.307	2.457
12	2.477	2.670
13	1.749	2.512
14	1.903	2.384
15	1.338	2.437
16	1.957	2.675
17	2.407	2.573
18	2.097	2.684
19	1.876	2.465
20	2.253	2.494
Tuning parameter (k)	—	**2.008**
Mean of RLCI(LSM, RM)	**1.995**	**2.477**

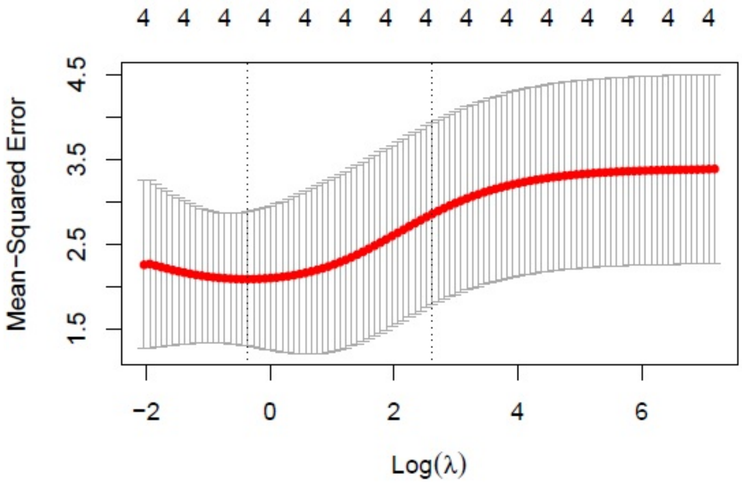

Fig. 1. Logarithm of the tuning parameter (k) in 12^{th} data set.

Despite the added ambiguity of the response variable by Buckley's approach, the value registered in mean of RLCI(LSM, RM) criterion for average of the 30 simulated data in Table 1, shows that the effectiveness of RM for each γ is 1.995 times than LSM. Also, in best result in 12^{th} from simulated data, this criterion has been obtain 2.477.
In next part, this performance has been shown in real data.

4.2. Real data (Cheese taste data)

These data produced in a real experiment from evaluate the impact of three variables on cheese tasting. In the data, acetic acid, hydrogen sulfide, and lactic acid are the three input variables, respectively. The output variable is also considered as the cheese flavor presented by an expert. The observations, introduce in Table 2.
In this real example, the output value have a vague nature but it was registered by crisp values. Therefore, the output should be reported in fuzzy values. Thus, the ambiguity resulting from the response variable is not added to error term.

Table 2. Data set 1 for quality of cheese taste.

No.	Acetic Acid	Hydrogen Sulfide	Lactic Acid	Cheese Taste
1	4.543	3.135	0.86	12.3
2	5.159	5.043	1.53	20.9
3	5.366	5.438	1.57	39.0
4	5.759	7.496	1.81	47.9
5	4.663	3.807	0.99	5.6
6	5.697	7.601	1.09	25.9
7	5.892	8.726	1.29	37.3
8	6.078	7.966	1.78	21.9
9	4.898	3.85	1.29	18.1
10	5.242	4.176	1.58	21.0
11	5.74	6.142	1.68	34.9
12	6.446	7.908	1.9	57.2
13	4.477	2.996	1.06	0.7
14	5.236	4.942	1.30	25.9
15	6.151	6.752	1.52	54.9

The data set is presented in Table 2. To distinguish whether the multicollinearity effect is present, we computed variation inflation factor (VIF) index and observed $VIF_1 = 12.064$, $VIF_2 = 7.914$, and $VIF_3 = 2.808$. In addition, the condition number is $\sqrt{\lambda_1/\lambda_3} = 13.67$, where λ_1 and λ_3 are the maximum and minimum eigenvalues of $\mathbf{X}^\top\mathbf{X}$, respectively.
But, there is collinearity between explanatory variables, so we use the ridge regression modeling to estimate the model parameters. Now based on ridge programming (with R package) the results of cheese taste data are shown in Figure 2.
According to the Figure 2, the value of ridge tuning parameter is $k = 0.705$ in which we have best situation versus of least squares method (LSM). Here, we have obtained the estimation of coefficients for both the least squares and ridge method (RM)(in best k) in Figure 3 using the Buckly's view point.

Table 3. Estimated coefficients and their variances of cheese taste data for both of methods.

Coefficient & variance	Method		Measure value in each sample	
	LSM	RM	Sample	RLCI(LSM, RM)
$\hat{\beta}_0$	-127.69	-43.18	1	2.527
			2	2.595
$VAR(\hat{\beta}_0)$	2225.95	1168.27	3	2.611
			4	2.614
$\hat{\beta}_1$	31.12	8.47	5	2.532
			6	2.556
$VAR(\hat{\beta}_1)$	203.06	14.71	7	2.561
			8	2.635
$\hat{\beta}_2$	-2.92	1.68	9	2.582
			10	2.634
$VAR(\hat{\beta}_2)$	13.37	20.72	11	2.642
			12	2.686
$\hat{\beta}_3$	2.76	11.21	13	2.538
			14	2.591
$VAR(\hat{\beta}_3)$	170.56	28.41	15	2.661
Tuning parameter (k)	—	**0.705**		
	Mean of RLCI(LSM, RM)			**2.598**

Also, the result of Table 3 from mean of the RLCI scale, show that this value is equal to 2.598 and it means that, the LCI in LSM for each γ is over 2.5 times than LCI in RM.
According to the Figure 2, the value of mean square error (MSE) for RM is less than LSM in $k = 0.705$. So, the performance of RM is better than LSM in prediction. Figure 3 introduce effectiveness of RM on reduce of estimated coefficients variance clearly. Therefore, the correspondent confidence intervals for parameters and their estimated response in RM, doubtless are smaller than LSM.
For illustrating the better performance of RM versus LSM in improvement of the estimated model, we obtained the these intervals for estimated response variables in samples of 4, 7, 11, and 15 from the observations and illustrated in Figure 4. Due to the this plot, superiority of the RM is shown clearly.

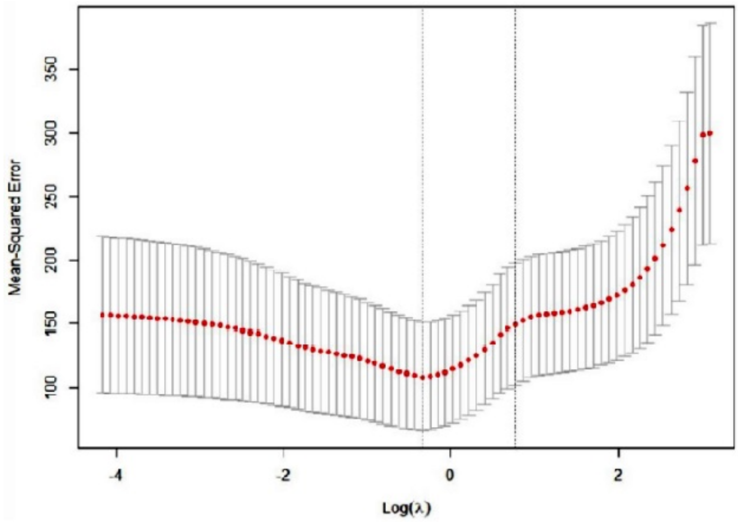

Fig. 2. Logarithm of the tuning parameter (k) in ridge model

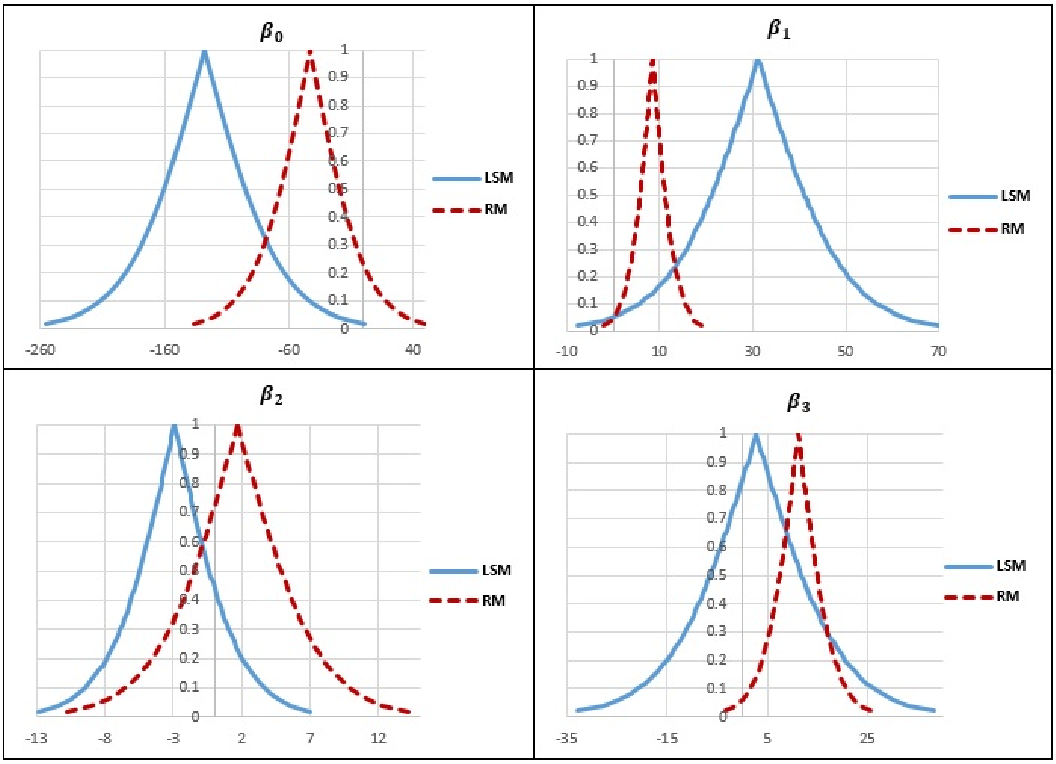

Fig. 3. Membership functions of estimated coefficients for Least squares(LS) and ridge(R) methods.

5. Conclusion

The nature of some natural phenomena are vagus. So, the value of them should be recorded as a fuzzy numbers. In such cases, if they are measured with crisp values, in fact, we will add a new error into the error of model, due to the ambiguity of nature of variable. Therefore, we increase the error generated in the estimated model. In other hand, existence of collinearity between explanatory variables in regression modeling cause be that the variance of estimated

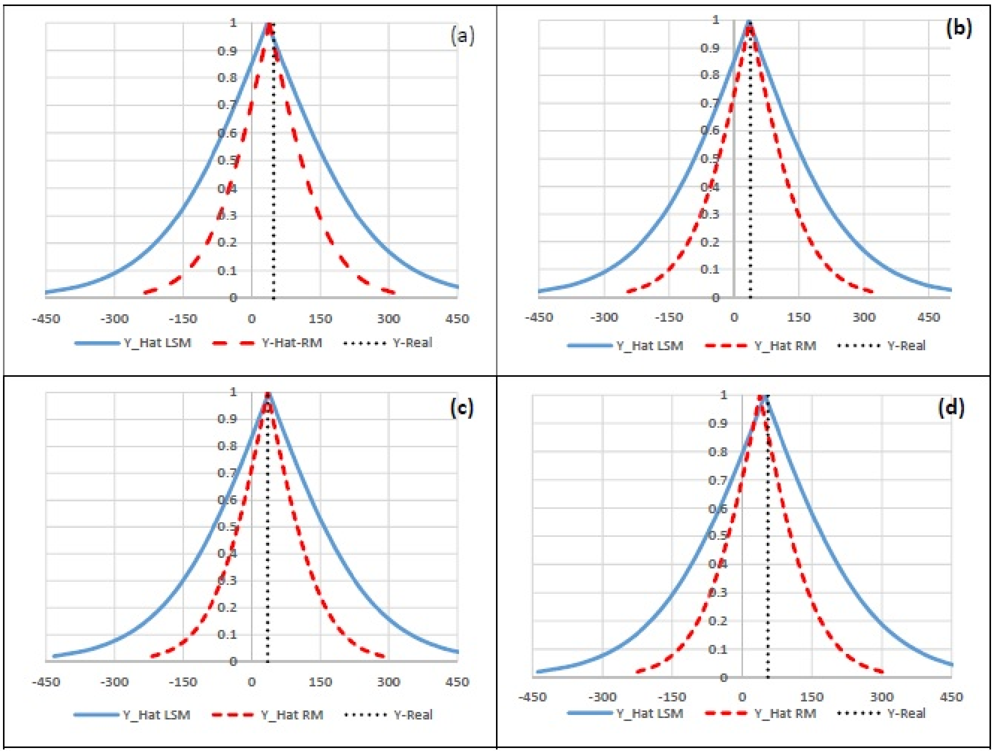

Fig. 4. Membership functions of estimated responses for Least squares(LS) and ridge(R) methods in some of the observations. (a) 4^{th} sample (b) 7^{th} sample (c) 11^{th} sample (d) 15^{th} sample.

coefficients of model have be large.
Thus, using the Buckley approach and utilization of the ridge method in its, we obtain the optimal model. So, not only is removed the collinearity effect, but the ambiguity of response variable is also taken from the error term. In this paper, has been illustrated the efficient of both method along with superiority of RM versus LSM (Buckley's approach) in real data.

5.1. Discussion

Usually in regression modeling, are tested existence or not the estimated coefficients. The results of these hypotheses identify the effective or inefficient input variables and present the final model. Hence, can be using by [4, 8], performed the respective hypotheses.
These tests are based on the t-distribution and their statistic structure for each coefficient of β_j^*, as follow.

$$\begin{cases} H_0 : \beta_j^* = 0 \\ H_1 : \beta_j^* \neq 0 \end{cases} \quad , \quad T - statistic = \frac{\hat{\beta}_j^*}{\sqrt{\bar{\sigma}^2(\hat{\beta}_j^*)}}$$

critical values of T-statistic based on confidence intervals is given by $[T_L, T_U]$ in which,

$$T_L = \frac{\hat{\beta}_j^*}{\sqrt{\bar{\sigma}_L^2(\hat{\beta}_j^*)}}, \quad T_U = \frac{\hat{\beta}_j^*}{\sqrt{\bar{\sigma}_U^2(\hat{\beta}_j^*)}}, \tag{14}$$

and

$$\bar{\sigma}_L^2 = \frac{(n-1)Var(\hat{\beta}_j^*)}{\chi^2_{n-1,\gamma\nu+1-\gamma}}, \quad \bar{\sigma}_U^2 = \frac{(n-1)Var(\hat{\beta}_j^*)}{\chi^2_{n-1,\gamma\nu}}, \tag{15}$$

where, the denominator of (15) fraction and $\nu = \chi^2_{n-1}(n-1)$ are values of density and cumulative probability of chi-square distribution respectively. Consequently, at first is provided the membership function of T-statistic to obtain the T_L and T_U then is made decision about the accept or reject of coefficients hypothesis tests.

Appendix

This section briefly reviews several concepts and terminologies related to fuzzy numbers and a distance between fuzzy numbers used throughout the paper. Let $\mathbb{X}$ be a universal set.

Definition 5.1. A fuzzy set of $\mathbb{X}$ is a mapping $\tilde{A} : \mathbb{X} \to [0,1]$, which assigns a degree of membership $0 \leq \tilde{A}(x) \leq 1$ to each $x \in \mathbb{X}$. The set $\tilde{A}_0$ is also defined as equal to the closure of $\{x \in \mathbb{R} | \tilde{A}(x) > 0\}$. Let $\mathbb{R}$ be the set of all real numbers.

Definition 5.2. The α-cut of a fuzzy number $\tilde{A}$ is a non-fuzzy set defined as

$$A_\alpha := \{x \in \mathbb{R}^n \,|\, \xi_{\tilde{A}}(x) \geq \alpha,\ 0 \leq \alpha \leq 1\}.$$

The set of all fuzzy numbers will be denoted by $\mathcal{F}_c(\mathbb{R})$. The α-cut A_α of $\tilde{A} \in \mathcal{F}_c(\mathbb{R})$ is a closed and bounded interval for each $\alpha \in [0,1]$. Hence a fuzzy number $\tilde{A} \in \mathcal{F}_c(\mathbb{R})$ is completely determined by the end points of the intervals $A_\alpha = \left[A^L_\alpha, A^U_\alpha\right]$. The arithmetic operations for two fuzzy numbers A_α and B_α are defined in the standard way, in terms of the α-cuts for $\alpha \in [0,1]$.
Addition: $A_\alpha + B_\alpha = \left[A^L_\alpha + B^L_\alpha, A^U_\alpha + B^U_\alpha\right]$.
Scalar multiplication: for given $k \in \mathbb{R}$,

$$k \cdot A_\alpha = \begin{cases} \left[kA^L_\alpha, kA^U_\alpha\right] & \text{if}\, k \geq 0 \\ \left[kA^U_\alpha, kA^L_\alpha\right] & \text{if}\, k < 0 \end{cases}$$

Scalar addition: $k + A_\alpha = \left[k + A^L_\alpha, k + A^U_\alpha\right]$.
The set of all fuzzy numbers with continuous membership functions is denoted by $\mathcal{F}(\mathbb{R})$. Notably, the most commonly used types of fuzzy numbers in $\mathcal{F}(\mathbb{R})$, is so-called LR-fuzzy numbers denoted by $\tilde{A} = (m_A, l_A, r_A)_{LR},\ l_A,\ r_A > 0$. The membership function of an LR-fuzzy number $\tilde{A}$ is defined by:

$$\tilde{A}(x) = \begin{cases} L\left(\frac{m_A - x}{l_A}\right), & x \leq m_A, \\ R\left(\frac{x - m_A}{r_A}\right), & x > m_A, \end{cases} \tag{16}$$

where L and R are strictly decreasing functions from $[0,1]$ to $[0,1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. A special type of LR-fuzzy number is the triangular fuzzy number (TFN) with the shape functions $L(x) = R(x) = \max\{0, 1 - |x|\}, \quad x \in \mathbb{R}$, where we denote by $\tilde{A} = (m_A, l_A, r_A)_T$. If $l_A = r_A = s_A$, then $\tilde{A}$ is called a symmetric triangular fuzzy number and it is denoted by $\tilde{A} = (m_A, s_A)_T$.

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GS-DRAZIN INVERSE

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Abstract

Let A be a Banach algebra. An element $a \in A$ has gs-Drazin inverse if there exists $b \in A$ such that

$$b = bab, ab = ba \; and \; a - ab \in A^{qnil}.$$

We study the gs-Drazin inverse of a 2×2 operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under several conditions and we investigate various perturbation conditions with spectral idempotent under which M has gs-Drazin invers.

1. Introduction

Throughout this article, A will denote a Banach algebra with an identity 1. The commutant of $a \in A$ is defined by $comm(a) = \{x \in A \mid xa = ax\}$. An element a in A has gs-Drazin inverse if and only if there exists $b \in A$ such that

$$b = bab, ab = ba, a - ab \in A^{qnil}.$$

Such b, if exists, is unique, and is denoted by $a^{''}$, where $a^{''}$ is called the gs-Drazin inverse of a. Here, $A^{qnil} = \{a \in A \mid 1 + ax \in U(A)\}$ for every $x \in comm(a)$ and is the set of quasinilpotent elements of A. As it is well known, for a Banach algebra A,

$$a \in A^{qnil} \iff \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0.$$

We always use $A^{''}$ to denote the set of all gs-Darazin invertible elements $a \in A$. An element $a \in A$ has generalized Drazin inverse, in case there is an element $b \in A$ such that

$$b = bab, ab = ba \; and \; a - a^2 b \in A^{qnil}.$$

Such b is called generalized Drazin inverse of a and denoted by a^d.

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In this paper, we consider the gs-Drazin inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad (*)$$

where $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have gs-Drazin inverses and X, Y are complex Banach spaces. Here, M is a bounded operator on $X \oplus Y$. We present the gs-Drazin inverse for a 2×2 oprator matrix M under a number of diffrent conditions, which generalizes [6, Theorem 2.1 and Theorem 2.2]. Now we will state auxiliary proposition and theorem.

Proposition 1.1. *Let A be a Banach algebra, and let $a, b, c \in A$. If $a, b \in A$ have gs-Drazin inverses, then $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(A)$ has gs-Drazin inverse.*

Theorem 1.2. *Let $a, b \in \mathcal{A}^{''}$. If $aba^2 = 0, abab = 0, ab^2a = 0$ and $ab^3 = 0$, then $a + b \in \mathcal{A}^{''}$ and*

$$\begin{aligned}
(a+b)^{''} &= \begin{pmatrix} a+b & ab+b^2 \end{pmatrix} M^{''} \begin{pmatrix} a \\ 1 \end{pmatrix}, \\
M^{''} &= F^{''} + (F^{''})^2 G, G^2 = 0, \\
F^{''} &= B^{''} + (B^{''})^2 A, A^2 = 0, \\
B^{''} &= (I - KK^{''}) \left[\sum_{n=0}^{\infty} K^n (H^{''})^n \right] H^{''} + K^{''} \left[\sum_{n=0}^{\infty} (K^{''})^n H^n \right] (I - HH^{''}), \\
H^{''} &= \begin{pmatrix} (a^{''})^3 & 0 \\ (a^{''})^4 + b(a^{''})^5 & 0 \end{pmatrix}, \quad K^{''} = \begin{pmatrix} 0 & 0 \\ (b^{''})^4 & (b^{''})^3 \end{pmatrix}
\end{aligned}$$

As an immediate consequence, we derive

Corollary 1.3. *Let $a, b \in \mathcal{A}^{''}$. If $abab = 0, aba^2 = 0$ and $b^2 = 0$, then $a + b \in \mathcal{A}^{''}$.*

2. Main results

Let $A \in \mathcal{L}(X), D \in \mathcal{L}(Y)$ have gs-Drazin inverse and M be given by $(*)$. In fact the explicit gs-Drazin invers of M could be computed by the formula in Theorem 1.2. Where M is a bounded liner operators on $X \oplus Y$.

Theorem 2.1. *If $BCA = 0, BCB = 0, ABD = 0$ and $CBD = 0$, then M has gs-Drazin inverse.*

Proof. Write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} := P + Q \ .$$

By Proposition 1.1, P, Q have gs-Drazin inverse. Then $Q^2 = 0, PQP^2 = 0$ and $PQPQ = 0$. Therefor we complete the proof by Corollary 1.3. □

Corollary 2.2. *If $BC = 0$ and $BD = 0$, then M has gs-Drazin inverse.*

Proof. This is obvious by Theorem 2.1. □

Theorem 2.3. *If $BCA = 0, DCA = 0, CBC = 0$ and $CBD = 0$, then M has gs-Drazin inverse.*

Proof. Write

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} := P + Q \ .$$

By Proposition 1.1, P, Q have gs-Drazin inverse. then $Q^2 = 0, PQP^2 = 0$ and $PQPQ = 0$. Therefor we complete the proof by Corollary 1.3. □

Corollary 2.4. *If $CA = 0$ and $CB = 0$, then M has gs-Drazin inverse.*

Let M be an operator matrix M given by $(*)$. It is of interest to consider the gs-Drazin inverse of M under generalized Schur condition $D = CA^{''}B$ (see [5]). We now investigate various perturbation conditions with spectral idempotent under which M has gs-Drazin invers.

Theorem 2.5. *Let $A \in \mathcal{L}(X)^{''}, D \in \mathcal{L}(Y)^{''}$ and M be given by $(*)$. If $CA^{\pi}BC = 0, BCA^{\pi}A^2 = 0, ABCA^{''} = BCAA^{''}$ and $D = CA^{''}B$, then $M \in \mathcal{L}(X \oplus Y)^{''}$.*

Proof. Clearly, we have

$$M = \begin{pmatrix} A & B \\ C & CA^{''}B \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} 0 & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} A & B \\ CAA^{''} & CA^{''}B \end{pmatrix}.$$

By assumption, we verify that $P^2 = 0, QPQ^2 = 0, QPQP = 0$. Clearly, P is nilpotent, and then it has gs-Drazin inverse. Moreover, we see that

$$Q = Q_1 + Q_2, Q_1 = \begin{pmatrix} A^2A^{''} & AA^{''}B \\ CAA^{''} & CA^{''}B \end{pmatrix}, \quad Q_2 = \begin{pmatrix} AA^{\pi} & A^{\pi}B \\ 0 & 0 \end{pmatrix}$$

and $Q_1Q_2 = 0$. Since $AA^{\pi} \in \mathcal{L}(X \oplus Y)^{qnil}$, it follows by Lemma 1.1 that $Q_2 \in \mathcal{L}(X \oplus Y)^{''}$. Moreover, we have $Q_1 = \begin{pmatrix} AA^{''} \\ CA^{''} \end{pmatrix} \begin{pmatrix} A & AA^{''}B \end{pmatrix}$. Clearly, we see that

$$\begin{pmatrix} A & AA^{''}B \end{pmatrix} \begin{pmatrix} AA^{''} \\ CA^{''} \end{pmatrix} = A^2A^{''} + AA^{''}BCA^{''}.$$

Since $A^2A^{''} + AA^{''}BCA^{''} = AA^{''}[A + BCA^{''}]$, we will suffice to prove $[A + BCA^{''}]AA^{''} = A^2A^{''} + BCA^{''}$ has gs-Drazin inverse, by means of Clines formula. Since $D = CA^{''}B$ has gs-Drazin inverse, it follows by [4, Corallary 2.4], that $BCA^{''}$ has gs-Drazin inverse. In view of [4, Corollary 3.3], $A^2A^{''} = A(AA^{''})$ has gs-Drazin inverse. Since $ABCA^{''} = BCAA^{''}$, we have $(A^2A^{''})(BCA^{''}) = A(AA^{''}BCA^{''}) = ABCA^{''} = BCAA^{''} = (BCA^{''})(A^2A^{''})$. In view of [4, Corollary 3.5], $A^2A^{''} + BCA^{''}$ has gs-Drazin inverse. By using Cline's formula, Q_1 has gs-Drazin inverse. Therefore Q has gs-Drazin inverse by [4, Crollary 3.5, Lemma 4.5]. According to Corollary 1.3, M has gs-Drazin inverse, as requird. ☐

Corollary 2.6. *Let $A \in \mathcal{L}(X)^{''}, D \in \mathcal{L}(Y)^{''}$ and M be given by $(*)$. If $CA^{\pi}BC = 0, BCA^{\pi}AB = 0, BCA^{\pi}A^2 = 0, A^2BCA = ABCA^2$ and $D = CA^{''}B$, then $M \in \mathcal{L}(X \oplus Y)^{''}$.*

Theorem 2.7. *Let $A \in \mathcal{L}(X)^{''}, D \in \mathcal{L}(Y)^{''}$ and M be given by $(*)$. If $A^{\pi}ABCA = 0, CA^{\pi}BCA = 0, BCA^{\pi}BC = 0, ABCA^{''} = BCAA^{''}$ and $D = CA^{''}B$, then $M \in \mathcal{L}(X \oplus Y)^{''}$.*

Proof. Clearly, we have $M = \begin{pmatrix} A & B \\ C & CA^{''}B \end{pmatrix} = P + Q$, where

$$P = \begin{pmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} A & AA^{''}B \\ C & CA^{''}B \end{pmatrix}$$

By assumption, we verify that $P^2 = 0, QPQ^2 = 0, QPQP = 0$. Clearly, P is nilpotent, and then it has gs-Drazin inverse. Moreover, we see that

$$Q = Q_1 + Q_2, Q_1 = \begin{pmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} A^2A^{''} & AA^{''}B \\ CAA^{''} & CA''B \end{pmatrix}$$

and $Q_1Q_2 = 0$. Clearly, Q_1 has gs-Drazin inverse by lemma 1.1. Moreover, we have $Q_2 = \begin{pmatrix} AA'' \\ CA'' \end{pmatrix} \begin{pmatrix} A & AA''B \end{pmatrix}$.
We see that $\begin{pmatrix} A & AA''B \end{pmatrix} \begin{pmatrix} AA'' \\ CA'' \end{pmatrix} = A^2A'' + AA''BCA''$. Since $A^{\pi}BCA^2 = 0$ and $ABCA'' = BCAA''$, as in the proof in Theorem 2.5, we see that Q_2 has gs-Drazin inverse. Therefore Q has gs-Drazin inverse. By using Corollary 1.3 again, M has gs-Drazin inverse, as required. □

Corollary 2.8. *Let $A \in \mathcal{L}(X)''$, $D \in \mathcal{L}(Y)''$ and M be given by $(*)$. If $A^{\pi}ABCA = 0, CA^{\pi}BCA = 0, BCA^{\pi}BC = 0, A^2BCA = ABCA^2$ and $D = CA''B$, then $M \in \mathcal{L}(X \oplus Y)''$.*

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An expansion of 2-Absorbing hyperideals

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Abstract

Let R be a muliplicative hyperring. In this paper, we introduce and study the concept of 2-absorbing ϕ-δ-primary hyperideals which is a extended class of 2-absorbing hyperideals. We give a number of main results to explain the general framework of these structures.

1. Introduction

The theory of hyperstructures has been introduced by Marty in 1934 at the 8th Congress of the Scandinavian Mathematicians [8]. He introduced hypergroups as a generalization of groups. In algebraic hyperstructures, the product of two elements is not an element but a set, while in classical algebraic structures, the binary operation of two elements of a set is an element of the set. In 1982, the notion of multiplicative hyperrings as an important class of algebraic hyperstructures was studied by R. Rota [10]. In the hyperrings, multiplication is a hyperoperation, while the addition is an operation. The concept of prime hyperideals in multiplicative hyperrings is introduced in brief by Procesi and Rota and it is further generalized by Dasgupta in [4]. Two extended classes of prime and 2-absorbing hyperideals which are called δ-primary and 2-absorbing δ-primary hyperideals, respectively, were presented by Ulucak in [7]. Badawi in [3] studied a generalization of prime ideals called 2-absorbing ideals and Ghiasvand generalized the idea in multiplicative hyperrings [6].
In this paper, we introduce and study the concept of 2-absorbing ϕ-δ-primary hyperideals which is a generalization of 2-absorbing hyperideals. We get some properties of such hyperideals.

2. Preliminaries

Recall first the basic terms and definitions from the hyperring theory. A hyperoperation " $\circ$ " on nonempty set G is a mapping of $G \times G$ into the family of all nonempty subsets of G. Let " $\circ$ " be a hyperoperation on G. Then $(G, \circ)$ is

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a hypergroupoid. We can extend the hyperoperation on G to subsets of G as follows. Let A and B be two nonempty subsets of G, then we denote $A \circ B = \bigcup_{x\in A,\ y\in B} x \circ y$, and if $r \in G$, then we denote $A \circ r = A \circ \{r\}$.
A semihypergroup is a hypergroupoid $(G, \circ)$, which is associative, that is for all $x, y, z \in G$, $(x \circ y) \circ z = x \circ (y \circ z)$.
A hypergroup is a semihypergroup $(G, \circ)$, that satisfies the reproduction axioms, that is $x \circ G = G = G \circ x$ for all $x \in G$.
A nonempty set R with two hyperoperations " $+$ " and " $\circ$ " is called hyperring hyperring if $(R, +)$ is a canonical hypergroup, $(R, \circ)$ is a semihypergroup with $x \circ 0 = 0 = 0 \circ x$ for all $x \in R$. The hyperoperation " $\circ$ " is distributive over " $+$ ", that is $x \circ (y + z) = x \circ y + x \circ z$ for all $x, y, z \in R$.
A *multiplicative hyperring* is an abelian group $(R, +)$ in which a hyperoperation $\circ$ is defined satisfying the following:

(1) for all $x, y, z \in R$, we have $x \circ (y \circ z) = (x \circ y) \circ z$,

(2) for all $x, y, z \in R$, we have $x \circ (y + z) \subseteq x \circ y + x \circ z$ and $(y + z) \circ x \subseteq y \circ x + z \circ x$,

(3) for all $x, y \in R$, we have $x \circ (-y) = (-x) \circ y = -(x \circ y)$.

If in (2) the equality holds then we say that the multiplicative hyperring is strongly distributive.
A non empty subset I of a multiplicative hyperring R is a *hyperideal* if it has the followings:

(1) $a - b \in I$ for each $a, b \in I$,

(2) $rox \subseteq I$ for each $x \in I$ and $r \in R$.

Definition 2.1. [4] A proper hyperideal P of R is called a *prime hyperideal* if $x \circ y \subseteq P$ for $x, y \in R$ implies that $x \in P$ or $y \in P$. The intersection of all prime hyperideals of R containing I is called the prime radical of I, being denoted by $\sqrt{I}$. If the multiplicative hyperring R does not have any prime hyperideal containing I, we define $\sqrt{I} = R$.

Definition 2.2. Let $\mathbf{C}$ be the class of all finite products of elements of R i.e. $\mathbf{C} = \{r_1 \circ r_2 \circ ... \circ r_n\ :\ r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a $\mathbf{C}$-hyperideal of R if, for any $A \in \mathbf{C}$, $A \cap I \neq \varnothing$ implies $A \subseteq I$.

Let I be a hyperideal of R. Then, $D \subseteq \sqrt{I}$ where $D = \{r \in R : r^n \subseteq I\ for\ some\ n \in \mathbb{N}\}$. The equality holds when I is a $\mathbf{C}$-hyperideal of R([**?**], proposition 3.2). In this paper, we assume that all hyperideals are $\mathbf{C}$-hyperideal. In this paper, we assume that all hyperideals are $\mathbf{C}$-hyperideal.

Definition 2.3. Let I, J be two hyperideals of R and $x \in R$. Then define:

$$(I : a) = \{r \in R : r \circ a \subseteq I\}$$

$$(I : J) = \{r \in R : r \circ J \subseteq I\}$$

Definition 2.4. [2] A function δ is called a hyperideal expansion of R if it assigns to each hyperideal I of R a hyperideal $\delta(I)$ such that $I \subseteq \delta(I)$ and if $I \subseteq J$ for any hyperideals I, J of R, then $\delta(I) \subseteq \delta(J)$.

For example, consider the hyperideal expansions δ_0, δ_1, δ_+ and δ_* of R defined with $\delta_0(I) = I$, $\delta_1(I) = \sqrt{I}$, $\delta_\star(I) = I + J$ (for some hyperideal J of R) and $\delta_*(I) = (I : K)$ (for some hyperideal K of R) for all hyperideals I of R, respectively. Also, let δ be a hyperideal expansion of R and I, J two hyperideals of R such that $I \subseteq J$. Let $\delta_q : R/I \longrightarrow R/I$ be defined by $\delta_q(J/I) = \delta(J)/I$. Then δ_q is a hyperideal expansion of R/I.

Definition 2.5. Let $L(R)$ be the set of all hyperideals of R. A function ϕ from $L(R)$ into $L(R) \cup \{\varnothing\}$ is called a hyperideal reduction of R if $\phi(I) \subseteq I$ and $I \subseteq I'$ implies that $\phi(I) \subseteq \phi(I')$ for every I and I' in $L(R)$.

3. main results

Theorem 3.1. *A proper hyperideal I of R is called 2-absorbing ϕ-δ-primary if whenever $x, y, z \in R$ with $x \circ y \circ z \subseteq I - \phi(I)$, then $x \circ y \subseteq I$ or $x \circ z \subseteq \delta(I)$ or $y \circ z \subseteq \delta(I)$.*

Example 3.2. *Let $(\mathbb{Z}, +, .)$ be the ring of integers. Consider multiplicative hyperring $(\mathbb{Z}, +, \circ)$ with a hyperoperation $x \circ y$ for all $x, y \in \mathbb{Z}$ and the hyperideal expansion $\bar{\delta}$ of $\mathbb{Z}$ with $\bar{\delta}(I) = I + p\mathbb{Z}$ for some prime integer p. Let q be a prime integer such that $p \neq q$ and $\phi(q\mathbb{Z}) = \varnothing$. Then the hyperideal $q\mathbb{Z}$ of $\mathbb{Z}$ is 2-absorbing ϕ-$\bar{\delta}$-primary.*

Theorem 3.3. *Let I be a proper hyperideal of R and $a \in R - I$ with $(\delta(I) : a) \subseteq \delta(I : a)$ and $(\phi(I) : a) \subseteq \phi(I : a)$. If I is a 2-absorbing ϕ-δ-primary hyperideal of R, then $(I : a)$ is so.*

Theorem 3.4. *Let I and I' be hyperideals of R such that $I' \nsubseteq I$. If I is 2-absorbing ϕ-δ-primary with $(\delta(I) : I') \subseteq \delta(I : I')$ and $(\phi(I) : I') \subseteq \phi(I : I')$, then $(I : I')$ is a 2-absorbing ϕ-δ-primary hyperideal of R.*

Theorem 3.5. *Let I is a 2-absorbing ϕ-δ-primary hyperideal of R. Then $(I : x \circ y) = (\delta(I) : x) \cup (\delta(I) : y) \cup (\phi(I) : x \circ y)$ for $x, y \in R$ with $x \circ y \nsubseteq I$.*

Theorem 3.6. *Let the hyperideal I of R be 2-absorbing ϕ-δ-primary with $\sqrt{\phi(I)} \subseteq \phi(\sqrt{I})$ and $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$. Then $\sqrt{I}$ is 2-absorbing ϕ-δ-primary.*

If the hyperideal expansion δ and the hyperideal reduction ϕ hold $\delta(I \cap I') = \delta(I) \cap \delta(I')$ and $\phi(I \cap I') = \phi(I) \cap \phi(I')$ for each hyperideals I,I' of R, they have the property of intersection preserving.

Theorem 3.7. *Let δ, ϕ have the property of intersection preserving. If $I_1, ..., I_n$ are 2-absorbing ϕ-δ-primary hyperideals such that $\delta(I_i) = \delta(I_j)$ and $\phi(I_i) = \phi(I_j)$ for every $1 \leq i, j \leq n$, then $\bigcap_{i=1}^{n} I_i$ is a 2-absorbing ϕ-δ-primary hyperideal of R.*

Theorem 3.8. *Let δ, γ be two hyperideal expansions of R such that $\delta(I) \subseteq \gamma(I)$ for all proper hyperideals I of R. Let ϕ, ψ be two hyperideal reductions of R such that $\phi(I) \subseteq \psi(I)$ for all hyperideals I of R. Then:*

(1) *Every 2-absorbing ϕ-δ-primary hyperideal of R is 2-absorbing ψ-δ-primary.*

(2) *Every 2-absorbing ϕ-δ-primary hyperideal of R is 2-absorbing ϕ-γ-primary.*

(3) *Every 2-absorbing ϕ-δ-primary hyperideal of R is 2-absorbing ψ-γ-primary.*

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings. A mapping from R_1 into R_2 is said to be a *good homomorphism* if for all $x, y \in R_1$, $\phi(x +_1 y) = \phi(x) +_2 \phi(y)$ and $\phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y)$ [5].
Let $h : R_1 \longrightarrow R_2$ be a good hyperring homomorphism, δ and γ hyperideal expansions of R_1 and R_2, respectively. Suppose that ϕ,ψ are hyperideal reductions of R_1 and R_2, respectively. Then h is called a (δ, ϕ)-(γ, ψ)-homomorphism if $\delta(h^{-1}(I_2) = h^{-1}(\gamma(I_2))$ and $\phi(h^{-1}(I_2) = h^{-1}(\psi(I_2))$ for each hyperideal I_2 of R_2. Moreover, If h is a (δ, ϕ)-(γ, ψ)-epimorphism and I_1 is a hyperideal of R_1 with $Kerh \subseteq I_1$, then $\gamma(h(I_1)) = h(\delta(I_1))$ and $\psi(h(I_1)) = h(\phi(I_1))$.

Theorem 3.9. *Let δ and γ be hyperideal expansions of R_1 and R_2, respectively. Let ϕ and ψ be hyperideal reductions of R_1 and R_2, respectively, such that $h : R_1 \longrightarrow R_2$ is a (δ, ϕ)-(γ, ψ)-homomorphism. Then:*

(1) *If I_2 is a 2-absorbing ψ-γ-primary hyperideal of R_2, then $h^{-1}(I_2)$ is a 2-absorbing ϕ-δ-primary hyperideal of R_1.*

(2) *Let h be an epimorphism. If I_1 is a 2-absorbing ϕ-δ-primary hyperideal of R_1 with $Kerh \subseteq I_1$, then $h(I_1)$ is a 2-absorbing ψ-γ-primary hyperideal of R_2.*

Theorem 3.10. *Let δ and ϕ be the hyperideal expansion and the hyperideal reduction of R, respectively. Suppose that I and I' be hyperideals of R with $I \subseteq \phi(I') \subseteq I'$. Then I' is a 2-absorbing ϕ-δ-primary hyperideal of R if and only if I'/I is a 2-absorbing ϕ_q-δ_q-primary hyperideal of R/I.*

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings with non zero identity. [7] Recall $(R_1 \times R_2, +, \circ)$ is a multiplicative hyperring with the operation $+$ and the hyperoperation $\circ$ are defined respectively as
$(x_1, x_2) + (y_1, y_2) = (x_1 +_1 y_1, x_2 +_2 y_2)$
$(x_1, x_2) \circ (y_1, y_2) = \{(x, y) \in R_1 \times R_2 \mid x \in x_1 \circ_1 y_1, y \in x_2 \circ_2 y_2\}$.
Let δ_1 and δ_2 be hyperideal expansions of R_1 and R_2, respectively. It is easy to see that $\delta_{R_1 \times R_2}(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ for every hyperideals I_1 of R_1 and I_2 of R_2 is a hyperideal expansion of $R_1 \times R_2$. Moreover, let ϕ_1 and ϕ_2 be hyperideal reductions of R_1 and R_2, respectively. It is seen that $\phi_{R_1 \times R_2}(I_1 \times I_2) = \phi_1(I_1) \times \phi_2(I_2)$ for every hyperideals I_1 of R_1 and I_2 of R_2 is a hyperideal reduction of $R_1 \times R_2$.

Theorem 3.11. *Let R_1 and R_2 be two multiplicative hyperrings and I_1 be a proper hyperideal of R_1. Suppose that δ_1,δ_2 are hyperideal expansions of R_1 and R_2, respectively, and ϕ_1,ϕ_2 are hyperideal reductions of R_1 and R_2, respectively. I_1 is a 2-absorbing δ_1-primary hyperideal of R_1 if and only if $I_1 \times R_2$ is a 2-absorbing $\phi_{R_1\times R_2}$-$\delta_{R_1\times R_2}$-primary hyperideal of $R_1 \times R_2$.*

Recall from [7] that a proper hyperideal I of R refers to a δ-primary hyperideal if for $x, y \in R$, $x \circ y \subseteq I$ implies either $x \in I$ or $y \in \delta(I)$. Also, a proper hyperideal I of R is called 2-absorbing δ-primary if for $x, y, z \in R$, $x \circ y \circ z \subseteq I$ implies $x \circ y \subseteq I$ or $y \circ z \in \delta(I)$ or $x \circ z \in \delta(I)$.

Theorem 3.12. *Let R_1 and R_2 be two multiplicative hyperrings and I_1, I_2 be two hyperideals of R_1 and R_2, respectively. Suppose that δ_1,δ_2 are hyperideal expansions of R_1 and R_2, respectively, and ϕ_1,ϕ_2 are hyperideal reductions of R_1 and R_2, respectively. If $I_1 \times I_2$ is a proper hyperideal of $R_1 \times R_2$ with $\delta_i(I_i) \neq R_i$ and $\phi_i(I_i) \neq R_i$ for each $i \in \{1, 2\}$, then the followings are equivalent.*

(1) $I_1 \times I_2$ is a 2-absorbing $\phi_{R_1\times R_2}$-$\delta_{R_1\times R_2}$-primary hyperideal of $R_1 \times R_2$.

(2) I_i is a δ_i-primary hyperideal of R_i for each $i \in \{1, 2\}$ or $I_1 = R_1$ and I_2 is a 2-absorbing δ_2-primary hyperideal of R_2 or $I_2 = R_2$ and I_1 is a 2-absorbing δ_1-primary hyperideal of R_1.

Theorem 3.13. *Let R_1 and R_2 be two multiplicative hyperrings and I_1 be a proper hyperideal of R_1. Assume that δ_1,δ_2 are hyperideal expansions of R_1 and R_2, respectively, and ϕ_1,ϕ_2 are hyperideal reductions of R_1 and R_2, respectively. Then the followings are equivalent.*

(1) I_1 is a 2-absorbing ϕ_1-δ_1-primary hyperideal of R_1 such that it is not 2-absorbing ϕ_1-δ_1-primary, $\phi_{R_1\times R_2}(I_1 \times R_2) \neq \varnothing$ and $\phi_2(R_2) = R_2$.

(2) $I_1 \times R_2$ is a 2-absorbing $\phi_{R_1\times R_2}$-$\delta_{R_1\times R_2}$-primary hyperideal of $R_1 \times R_2$ such that it is not 2-absorbing $\delta_{R_1\times R_2}$-primary hyperideal.

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weak prime ideals and weak zariski topology

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Abstract

An ideal Q of a duo ring R is said to be weak prime if for ideals I and J of R, the inclusion $I \cap J \subseteq Q$ implies that either $I \subseteq Q$ or $J \subseteq Q$. In this paper we study weak prime ideals in duo rings and a topology similar to the Zariski topology related to weak prime ideals. This topology is called weak Zariski topology and has the Zariski topology defined by prime ideals as one of its subspace topologies.

1. Preliminaries

Definition 1.1. A ring R is called a duo ring if each right or left ideal of R is a two sided ideal.

Example 1.2. *Commutative rings and division rings are duo ring.*

Example 1.3. *If D is a division ring, then $R = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in D\}$ is a duo ring.*

Proposition 1.4. *Let R be a ring. Then the following are equivalent:*

1. *R is a duo ring.*
2. *For every $x \in R$, $Rx = xR$.*
3. *If a and b are two elements of R, then there exist $x, y \in R$ such that, $ab = bx = ya$.*

Definition 1.5. Let R be a duo ring and I be an ideal of R. We denote by $\sqrt{I}$ the subset $\{r \in R | \exists n \in \mathbb{N}, r^n \in I\}$ of R.

Proposition 1.6. *Let R be a duo ring and P be a proper ideal of R. Then the following statements are equivalent:*

1. *P is prime ideal.*
2. *For every $x, y \in R$, if $xy \in P$ then $x \in P$ or $y \in P$.*

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Therefore, if P is a prime ideal of R and $x^n \in P$, for some $x \in R$ and $n \in \mathbb{N}$, then $x \in P$.

Corollary 1.7. *If P is a prime ideal of a duo ring R, then $\sqrt{P} = P$.*

Lemma 1.8. *Let R be a duo ring and I and J be ideals of R. Then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J} = \sqrt{IJ}$*

Proposition 1.9. *Let R be a duo ring and I be an ideal of R. Then $\sqrt{I}$ is an ideal of R.*

Proposition 1.10. *Let R be a duo ring. Then the ideals which are the power of prime ideals commute.*

Proposition 1.11. *Let R be a duo ring. Then $N(R) = \bigcap\limits_{P \in spec(R)} P$.*

Theorem 1.12. *Let $P_1, P_2, \ldots, P_n$ where $n \geqslant 2$, be ideals of duo ring R such that at most two of them are not prime. Let S be an additive subgroup of R which is closed under multiplication. Suppose that $S \subseteq \bigcup\limits_{i=1}^{n} P_i$. Then $S \subseteq P_j$ for some $i \in \{1, 2, \ldots, n\}$.*

Lemma 1.13. *Let b and c be elements in a duo ring R. Then $bR \cap cR = b(cR :_R bR) = c(bR :_R cR)$. Moreover; if I is an ideal in R such that $I \subseteq bR$; then $I = b(I :_R bR)$.*

2. Main Results

Definition 2.1. An ideal Q of duo ring R is weak prime if for ideals$I$ and J of R, the inclusion $I \cap J \subseteq Q$ implies that either $I \subseteq Q$ or $J \subseteq Q$.

Lemma 2.2. *Let Q be a weak prime ideal in a duo ring R. Then Q is irreducible.*

Corollary 2.3. *If Q is a waek prime ideal of Noetherian duo ring R, then Q is primary ideal.*

Lemma 2.4. *If Q is a prime ideal of duo ring R, then Q is weak prime.*

Lemma 2.5. *If Q is weak prime ideal of duo ring R and I is an ideal contained in Q, then $\frac{Q}{I}$ is weak prime in $\frac{R}{I}$ is weak prime in $\frac{R}{I}$.*

Proposition 2.6. *Let Q be an ideal of duo ring R. Then Q is weak prime if and only if, $Rx \cap Ry \subseteq Q$ implies that $x \in Q$ or $y \in Q$.*

Proposition 2.7. *Let Q be a weak prime ideal of duo ring R. Then Q is prime if and only if $\sqrt{Q} = Q$.*

Proposition 2.8. *For each proper ideal I of R, there is a minimal weak prime ideal over I.*

Proposition 2.9. *If Q is weak prime ideal of duo ring R and Q has primary decomposition, then Q is primary.*

Proposition 2.10. *If R be a duo ring, then every proper ideal of R is weak prime ideal if and only if, R is a uniserial duo ring.*

Proposition 2.11. *Let R be a local duo ring with maximal ideal M and let I be a weak prime M-primary ideal in R. Assume that $I \subsetneq (I :_R M)$. Then:*

1. *$(I :_R M)$ is a principal ideal.*
2. *$I = (I :_R M)M$.*
3. *For each ideal J in R either $J \subseteq I$ or $(I :_R M) \subseteq J$.*

If R is a duo ring, we consider $WSpec(R)$ to the set of all weak prime ideal of R. We call $WSpec(R)$, the weak prime spectrum of R. In this section, a topology on $WSpec(R)$ introduced. This topology is defined exactly similar to the Zariski topology defined by prime ideals and the set of prime ideals is a subspace topology of $WSpec(R)$. This topology is called weak Zariski topology.

Definition 2.12. If I is an ideal of R, variety of I, denoted by $WV(I)$, as also for each $a \in R$ and each ideal I of R, let $OP(I) = \{Q \in WSpec(R) | I \nsubseteq Q\}$, $OP(a) = OP(Ra)$. $WT(R) = \{OP(I) | I \ is \ an \ ideal \ of \ R\}$.

Proposition 2.13. *If R is a duo ring, then $WT(R)$ is a topology on $WSpec(R)$.*

Lemma 2.14. *Let x be an element of duo ring R. Then:*

1. *If $OP(x) = \emptyset$, then x is a nilpotent element of R.*
2. *$OP(x) = WSpec(R)$ if and only if, x is a unit element of R.*

Proposition 2.15. *If I and J are ideals of duo ring R and $OP(I) = OP(J)$, then $\sqrt{I} = \sqrt{J}$*

Proposition 2.16. *Let R be a duo ring. Then $WSpec(R)$ is a quasi-compact topological space.*

Proposition 2.17. *If R is a duo ring, then $WSpec(R)$ is a T_0 topological space.*

Proposition 2.18. *Let R be a Noetherian duo ring. Then closed sets of $WSpec(R)$ satisfy the descending chain condition.*

Definition 2.19. A nonempty closed set C of a topological space is said to be irreducible if C can not be written as the union of two distinct closed sets.

Lemma 2.20. *If I is an ideal of R such that $WV(I)$ is irreducible closed set. Then there eists an irreducible ideal J of R such that $WV(I) = WV(J)$.*

Proposition 2.21. *If I is an ideal of duo ring R and closed sets of $WSpec(R)$ satisfy the descending chain condition, then we have:*

1. *$WV(I)$ can be written as a finite union of irreducible closed sets $WV(I_k)$, $k = 1, 2, \ldots, n$ such that for each k, I_k is an irreducible ideal of R.*
2. *$WV(I)$ can be written as a finite union of irreducible closed sets $WV(I_k)$, $k = 1, 2, \ldots, n$ such that for each k, I_k is an irreducible ideal of R.*

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On Classification of 4-dimensional nilpotent complex Leibniz algebras

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Abstract

Leibniz algebras introduced by J. L. Loday (1993) are non-antisymmetric generalizations of Lie algebras. The classification problem of complex nilpotent Leibniz algebras was first studied by Loday himself. Recently, Albeverio, Omirov and Rakhimov obtained the classification of 4-dimensional complex nilpotent Leibniz algebras. In this short note, we show that one of the algebras should be omitted from their list, as it is isomorphic to another algebra in the list.

1. Introduction

Leibniz algebras was first introduced by Loday in [4] and [5] as a non-antisymmetric versions of Lie algebras. The classification problem of complex nilpotent Leibniz algebras was first studied by Loday. In [5] he gave a complete classification of complex nilpotent Leibniz algebras of dimension $n \leq 2$. Later Ayupov and Omirov classified $3-$dimensional complex nilpotent Leibniz algebras in [2]. Recently, Albeverio, Omirov and Rakhimov have obtained a classification of 4-dimensional complex nilpotent Leibniz algebras in [1].
One of the techniques to classify nilpotent Lie algebras was introduced by Skjelbred and Sund in [8]. Rakhimov and Langari used Skjelbred-Sund method in Leibniz algebras [3]. They also applied in [7] and [3] this technique to obtain the classification of complex nilpotent Leibniz algebras of dimension $n \leq 4$. Comparing the results of [3] and [6] with classification in [1] we realized that the Skjelbred-Sund method could be used to check the validity of the main result of [1]. In this part we give the basic definitions and properties of Leibniz algebras.

Definition 1.1. A Leibniz algebra L is a vector space over a field F equipped with a bilinear map

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad (x,y,z \in L).$$

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The first pure algebraic motivation of J.-L. Loday to introduce this class of algebras was the search for an "obstruction" to the periodicity in algebraic K-theory. Besides this purely algebraic motivation, some relationships with classical geometry have recently been discovered, which could lead to an investigation of the (co)homological theory of Leibniz algebras in view of concrete applications in non-commutative geometry and its physical interpretations. Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if

$$[x,x]=0, \quad (x\in L).$$

Let n be the dimension of Leibniz algebra L. Let $\{e_1,e_2,...,e_n\}$ be a basis in L. The structural constants of L are the numbers C_{ij}^k given by

$$[e_i,e_j]=\sum_{k=1}^{n} C_{ij}^k e_k \quad (i,j=1,...,n).$$

We can identify the Leibniz identity with its structural constants. These constants satisfy:

$$\sum_{l=1}^{n}(C_{jk}^l C_{il}^s - C_{ij}^l C_{lk}^s + C_{ik}^l C_{lj}^s)=0 \quad (i,j,k,s=1,...,n).$$

Definition 1.2. Let L is a Leibniz algebra. We define

$$L^1=L, L^k=\left[L^{k-1},L\right] \quad (k>1).$$

The series

$$L^1\supseteq L^2\supseteq L^3\supseteq ...$$

is called the descending central series of L. If the series terminates for some positive integer s, then the Leibniz algebra L is said to be nilpotent.

Theorem 1.3. *[1] The isomorphism classes of four-dimensional complex nilpotent Leibniz algebras are given by the following representatives.*

$$\begin{cases}
R_1\text{:} & [e_1,e_1]=e_2,[e_2,e_1]=e_3,[e_3,e_1]=e_4;\\
R_2\text{:} & [e_1,e_1]=e_3,[e_1,e_2]=e_4,[e_2,e_1]=e_3,[e_3,e_1]=e_4;\\
R_3\text{:} & [e_1,e_1]=e_3,[e_2,e_1]=e_3,[e_3,e_1]=e_4;\\
R_4(\alpha)\text{:} & [e_1,e_1]=e_3,[e_1,e_2]=\alpha e_4,[e_2,e_1]=e_3,[e_2,e_2]=e_4,[e_3,e_1]=e_4,\ \ \alpha\in\{0,1\};\\
R_5\text{:} & [e_1,e_1]=e_3,[e_1,e_2]=e_4,[e_3,e_1]=e_4;\\
R_6\text{:} & [e_1,e_1]=e_3,[e_2,e_2]=e_4,[e_3,e_1]=e_4;\\
R_7\text{:} & [e_1,e_1]=e_4,[e_2,e_1]=e_3,[e_3,e_1]=e_4,[e_1,e_2]=-e_3,[e_1,e_3]=-e_4;\\
R_8\text{:} & [e_1,e_1]=e_4,[e_2,e_1]=e_3,[e_3,e_1]=e_4,[e_1,e_2]=-e_3+e_4,[e_1,e_3]=-e_4;\\
R_9\text{:} & [e_1,e_1]=e_4,[e_2,e_1]=e_3,[e_2,e_2]=e_4,[e_3,e_1]=e_4,[e_1,e_2]=-e_3+2e_4,[e_1,e_3]=-e_4;\\
R_{10}\text{:} & [e_1,e_1]=e_4,[e_2,e_1]=e_3,[e_2,e_2]=e_4,[e_3,e_1]=e_4,[e_1,e_2]=-e_3,[e_1,e_3]=-e_4;\\
R_{11}\text{:} & [e_1,e_1]=e_4,[e_1,e_2]=e_3,[e_2,e_1]=-e_3,[e_2,e_2]=-2e_3+e_4;\\
R_{12}\text{:} & [e_1,e_2]=e_3,[e_2,e_1]=e_4,[e_2,e_2]=-e_3;\\
R_{13}(\alpha)\text{:} & [e_1,e_1]=e_3,[e_1,e_2]=e_4,[e_2,e_1]=-\alpha e_3,[e_2,e_2]=-e_4,\ \ \alpha\in\mathbb{C};\\
R_{14}(\alpha)\text{:} & [e_1,e_1]=e_4,[e_1,e_2]=\alpha e_4,[e_2,e_1]=-\alpha e_4,[e_2,e_2]=e_4,[e_3,e_3]=e_4,\ \ \alpha\in\mathbb{C};\\
R_{15}\text{:} & [e_1,e_2]=e_4,[e_1,e_3]=e_4,[e_2,e_1]=-e_4,[e_2,e_2]=e_4,[e_3,e_1]=e_4;\\
R_{16}\text{:} & [e_1,e_1]=e_4,[e_1,e_2]=e_4,[e_2,e_1]=-e_4,[e_3,e_3]=e_4;\\
R_{17}\text{:} & [e_1,e_2]=e_3,[e_2,e_1]=e_4;\\
R_{18}\text{:} & [e_1,e_2]=e_3,[e_2,e_1]=-e_3,[e_2,e_2]=e_4;\\
R_{19}\text{:} & [e_2,e_1]=e_4,[e_2,e_2]=e_3;\\
R_{20}(\alpha)\text{:} & [e_1,e_2]=e_4,[e_2,e_1]=\frac{1+\alpha}{1-\alpha}e_4,[e_2,e_2]=e_3,\ \ \alpha\in\mathbb{C}\setminus\{1\};\\
R_{21}\text{:} & [e_1,e_2]=e_4,[e_2,e_1]=-e_4,[e_3,e_3]=e_4.
\end{cases}$$

2. Main results

In this section we prove that $R_{13}(\alpha = 1) \cong \mathbb{R}_{19}$. Therefore, the Leibniz algebra R_{19} should be omitted from the list 1.3. Here we rewrite the relations in these algebras as follows:

$\mathrm{R}_{13}(\alpha = 1)$: $[e_1, e_1] = e_3, [e_1, e_2] = e_4, [e_2, e_1] = -e_3, [e_2, e_2] = -e_4;$

and

R_{19}: $[e'_2, e'_1] = e'_4, [e'_2, e'_2] = e'_3.$

The matrix A representing the linear transformation with respect to these bases is

$$A = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ -c_{22} & c_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{22}^2 & -c_{22}^2 \\ c_{41} & c_{42} & -c_{22}c_{11} & -c_{22}c_{11} \end{bmatrix};$$

where $\det(A) = -2c_{22}^4 c_{11} c_{12} - c_{22}^4 c_{11}^2 - c_{22}^4 c_{12}^2$.

We can easily get the following constrains for a matrix A such that $\det(A) \neq 0$:

$$c_{12} = c_{31} = c_{32} = c_{41} = c_{42} = 0,\ c_{22} = -1,\ c_{11} = 1.$$

Thus we get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Writing the elements of basis $R_{13}(\alpha = 1)$ in terms of the basis R_{19} we have

$$\begin{cases} e_1 = e'_1 + e'_2; \\ e_2 = -e'_1; \\ e_3 = e'_3 + e'_4; \\ e_4 = -e'_3. \end{cases}$$

It shows that $R_{13}(\alpha = 1) \cong \mathbb{R}_{19}$.

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Information systems and $\bigcap$-structures

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Abstract

In this paper, we first recall the concept of an information system and then introduce an algebraic closure operator on a set of some special subsets of the information system. Finally, we prove the class of information systems and $\bigcap$-structures are isomorphic.

1. First Section

2. introduction and preliminaries

Information systems have been introduced by D.S. Scott [2, 3] to provide an interpretation for states of knowledge in computational models. On the other hand, $\bigcap$-structures, that is, a family of subsets of a set which is closed under the intersection of any non-empty family, have an extensive application in order theory. More precisely, very many lattices in algebra are $\bigcap$-structure. Moreover, the relationship between closure operators and $\bigcap$-structure on a given set set is bijective one. In this paper, we first introduce some concepts from order theory and then that of information systems. Then we introduce a closure operator and prove the class information systems and $\bigcap$-structures are isomorphisms. First we recall from [1], some concepts that will be needed in the sequel.

Definition 2.1. Let P be an ordered set. A map $c\colon P \to P$ is called a *closure operator* (on P) if, for all $x, y \in P$,

(1) $x \leq c(x)$,
(2) $x \leq y \implies c(x) \leq c(y)$,
(3) $c(c(x)) = c(x)$.

An element $x \in P$ is called *closed* if $c(x) = x$. The set of all closed elements of P is denoted by P_c.

Proposition 2.2. *Let c be a closure operator on a complete lattice L. Then L_c is a complete lattice, under the order inherited from L, such that, for every subset S of L_c, $\bigwedge_{L_c} S = \bigwedge_L S$ and $\bigvee_{L_c} S = c(\bigvee_L S)$.*

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Definition 2.3. Let L be a complete lattice and let $k \in L$. k is called *finite* (in L) if, for every directed set D in L, that is for each pair $a, b \in D$ there is an element $c \in D$ with $a, b \leq c$,

$$k \leq \bigvee D \implies k \leq d \text{ for some } d \in D.$$

The set of all finite elements of L is denoted $F(L)$. A complete lattice L is said to be *algebraic* if, for each $a \in L$, $a = \bigvee\{k \in F(L) \mid k \leq a\}$.

Definition 2.4. Let c be a closure operator on a complete lattice L. We say that c is *algebraic* if, the complete lattice L_c is algebraic.

Definition 2.5. Let X be a set. The non-empty family $\{A_i\}_{i\in I}$ of subsets of X is said to be *directed* if for each pair of elements $i, j \in I$, there exists a $k \in I$ with $A_i, A_j \subseteq A_k$.

Definition 2.6. Let X be a set and $\mathfrak{L}$ a non-empty family of subsets of X. The family $\mathfrak{L}$ is said to be an $\bigcap$-*structure* if $\bigcap_{i\in I} A_i \in \mathfrak{L}$ for any non-empty family $\{A_i\}_{i\in I}$ in $\mathfrak{L}$. The $\bigcap$-structure $\mathfrak{L}$ is *algebraic* if $\bigcup_{i\in I} A_i \in \mathfrak{L}$ for any directed family $\{A_i\}_{i\in I}$ in $\mathfrak{L}$.

We denote the class of all $\bigcap$-structures by $\bigcap_{\mathbf{S}}$.
In the following we recall from [2], the definition of an information system.

Definition 2.7. An *information system* is a triple $\mathbf{A} = (A, Cons, \vdash)$ consisting of
(i) A set A of propositions or units of informations.
(ii) $Cons$ is a non-empty set (system) of finite subsets of A which satisfy
  (IS1) $Y \in Cons$ and $Z \subseteq Y$ implies $Z \in Cons$,
  (IS2) $a \in A$ implies $\{a\} \in Cons$,
(iii) $\vdash$ called *entailment relation* is a relation between member of $Cons$ and members of A (formally $\vdash$ is a subset of $Cons \times A$) satisfying
  (IS3) $Y \cup \{a\} \in Cons$ whenever $Y \in Con$, $a \in A$ and $Y \vdash a$,
  (IS4) $Y \in Cons$ and $a \in Y$ implies $Y \vdash a$,
  (IS5) $Y, Z \in Cons$ and $a \in A$ satisfy $Y \vdash b$ for all $b \in Z$, and $Z \vdash a$, then $Y \vdash a$.

Read $Y \vdash a$ as "Y entails a" or "a is deducible from Y".
We denote the class of all information systems by **INF**.

Definition 2.8. Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system. An arbitrary subset $X \subseteq A$ is said to be *consistent* if every finite subset of X is in $Cons$. We show that the set of all consistent subset of A by $Cons(A)$.

Remark 2.9. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system.*

(1) *In our notations, $Y \Subset A$ means that Y is a finite (possibly empty) subset of A.*
(2) *Considering* (IS1)*, every set in $Cons$ is a consistent subset of A, so the terminology $Cons$ is in fact an abbreviation for consistent subsets.*

3. Main results

In this section, we first prove some technical lemmas an theorems, and using them to prove that the class of information systems and $\bigcap$-structures are isomorphic. The following rules are frequently used in proofs:

Lemma 3.1. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system.*

(a) *If $Y \in Cons$, $Z \subseteq Y$ and $Z \vdash a$ then $Y \vdash a$.*
(b) *If $Y \in Con$, Z is finite and $Y \vdash a$ for every $a \in Z$, then $Z \in Cons$.*

Proof. (a): By (IS4), $Y \vdash z$ for every $z \in Z$. Having this and $Z \in Cons$, (IS5) implies $Y \vdash a$.
(b): First, we use induction on $|Z|$ to show that $Y \cup Z \in Cons$, then using (IS1), we conclude $Z \in Cons$, as required. If $|Z| = 1$, then IS2 implies $Z \in Cons$, as required. Let $|Z| = 2$ and take $Z = \{z_1, z_2\}$. By hypothesis, $Y \vdash z_1$ and $Y \vdash z_2$. So by (IS3), $Y \cup \{z_1\} \in Cons$. Having $Y \vdash z_2$, (a) implies $Y \cup \{z_1\} \vdash z_2$. Then (IS3) implies $(Y \cup \{z_1\}) \cup \{z_2\} \in Cons$. Having this and $\{z_1, z_2\} \subseteq Y \cup \{z_1, z_2\}$, (IS1) implies $Z = \{z_1, z_2\} \in Cons$, as required. Now let for every finite subset Z with $|Z| = n$, (b) holds. Take a finite subset $K = \{k_1, \cdots, k_{n+1}\}$ of A with $|K| = n+1$ such that $Y \vdash k$ for every $k \in K$. By induction hypothesis, $Y \cup K' \in Cons$ where $K' := \{k_1, \cdots, k_n\}$. Hence and by $Y \vdash z_{n+1}$, (a) implies $Y \cup K' \vdash z_{n+1}$. Thus (IS3) implies that $Y \cup K = (Y \cup K') \cup \{z_{n+1}\} \in Cons$. Now, by (IS1), $K \in Cons$, as required. □

Information systems have a strong connection with $\bigcap$-structures. To make this precise, we need more definitions.

Definition 3.2. Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system. A subset E of A is called an *element* of $\mathbf{A}$ if E is consistent and $\vdash$-closed, in the sense that $Y \in Cons$, $Y \Subset E$ and $Y \vdash a$ imply $a \in E$. The set of elements of $\mathbf{A}$ is denoted by $|\mathbf{A}|$.

Definition 3.3. For any consistent set X we define $\overline{X} := \{a \in A \mid (\exists\, Y \Subset X)\; Y \vdash a\}$, this may be interpreted as the set of informations deducible from X.

Lemma 3.4. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system and $X \in Cons$. Then $\overline{X} = \{a \in A \mid X \vdash a\}$.*

Proof. We have to show that $\{a \in A \mid (\exists\, Y \Subset X)\; Y \vdash a\} = \{a \in A \mid X \vdash a\}$. By Lemma 3.1$(a)$, the first is a subset of the latter. The inverse inclusion trivially holds. □

Lemma 3.5. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system and X a consistent subset of A. Then $\overline{X}$ is an element of $\mathbf{A}$.*

Proof. First we show that $\overline{X}$ is consistent. To prove this, we take $T \Subset \overline{X}$ and show that $T \in Cons$. For each $t \in T$, there exists $Z_t \Subset X$ with $Z_t \vdash t$. Moreover $\bigcup_{t \in T} Z_t \Subset X$ and so $\bigcup_{t \in T} Z_t \in Cons$, since X is consistent. Now, using Lemma 3.1 (a), $\bigcup_{t \in T} Z_t \vdash t$, for all $t \in T$. Hence and by Lemma 3.1 (b), $T \in Cons$, as required. Finally, using the definition of $\overline{X}$, it is easy to see that $\overline{X}$ is $\vdash$-closed. Consequently, $\overline{X}$ is an element. □

Lemma 3.6. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system and X a consistent subset of A. Then X is an element of $\mathbf{A}$ if and only if $\overline{X} = X$.*

Proof. Let $X = \overline{X}$, then by Lemma 3.5, X is an element of $\mathbf{A}$. To prove the other direction, let X be an element of $\mathbf{A}$, we show that $\overline{X} = X$. For all $x \in X$, $\{x\} \in Cons$ and $\{x\} \vdash x$ imply $x \in \overline{X}$. So $X \subseteq \overline{X}$. Now, take $a \in \overline{X}$, then there is $Y \Subset X$ with $Y \vdash a$. Thus, since X is $\vdash$-closed, $a \in X$. This gives $\overline{X} \subseteq X$. Consequently, $\overline{X} = X$, as required. □

Theorem 3.7. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system. Then the map $\overline{(-)}\colon Cons(A) \to Cons(A)$, $X \to \overline{X}$ is an algebraic closure operator.*

Proof. Let X be a consistent subset of A. Moreover, for all $x \in X$, $\{x\} \Subset X$, $\{x\} \in Cons$ and $\{x\} \vdash x$. So, $x \in \overline{X}$. This gives $X \subseteq \overline{X}$. Now, we show that if $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$. Take an element $a \in \overline{X}$, then there exists $Z \in Cons$ with $Z \Subset X$ and $Z \vdash a$. Thus $Z \Subset X \subseteq Y$. This gives that $a \in \overline{Y}$, as required. Finally, we show that for any consistent subset X of A, $\overline{\overline{X}} = \overline{X}$. By Lemma 3.5, $\overline{X}$ is an element and so by Lemma 3.6, $\overline{\overline{X}} = \overline{X}$, as required. Now, we show that $\overline{(-)}$ is algebraic. First notice that the ordered set $(Cons(A), \subseteq)$ is complete, because the intersection of any family of consistent subset is consistent. Second, by Lemma 3.6, the set of all closed elements of $(Cons(A), \subseteq)$ is $|\mathbf{A}|$ which is a complete lattice (by Theorem 2.2). It is easy to show that the finite elements of $|\mathbf{A}|$ are exactly the sets $\overline{Y}$ where $Y \in Cons$. Thus, using Theorem 3.8(ii), we obtain that $|\mathbf{A}|$ is algebraic. □

Theorem 3.8. *Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system and let $E \subseteq A$. Then the following are equivalent:*

(i) *E is an element of $\mathbf{A}$.*

(ii) $\{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\}$ *is directed and*

$$E = \bigcup\{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\}$$

.

(iii) $E = \overline{X}$*, for some consistent set* X.

Proof. (iii $\Longrightarrow$ i) Let $Z := \{x_1, \ldots, x_n\} \Subset E$. For each i, there exists $Y_i \Subset X$ with $Y_i \vdash x_i$. Then $Y := Y_1 \cup \ldots \cup Y_k \Subset X$ and so $Y \in Cons$, since X is consistent. By Rule (a) in 3.1, $Y \vdash x_i$ for each i. By Rule (b) in 3.1, $Z \in Cons$. Hence E is consistent. To show E is $\vdash$-closed, assume $Z \vdash a$. The set Y above is such that $Y \vdash b$ for each $b \in Z$. By (IS5), $Y \vdash a$, so $a \in \overline{X} = E$.
(i $\Longrightarrow$ ii) Let $\mathfrak{T} := \{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\}$. Take $\overline{Y}, \overline{Z} \in \mathfrak{T}$. We have $(Z \cup Y) \Subset E$ and since E is consistent, $(Z \cup Y) \in Cons$. Thus $\overline{Z \cup Y} \in \mathfrak{T}$ with $\overline{Y}, \overline{Z} \subseteq \overline{Z \cup Y}$ (notice that $\overline{(-)}$ is a closure operator). Consequently, $\mathfrak{T}$ is directed. To prove the second part, we have for all $Y \in Cons$ and $Y \Subset E$, $\overline{Y} \subseteq \overline{E} = E$ (the last equality is true by Lemma 3.6). Thus $\bigcup\{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\} \subseteq E$. To prove the reverse inclusion, take $e \in E$. Then $\{e\} \vdash e$ gives that $x \in \overline{\{e\}}$ and so $e \in \bigcup\{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\}$. Thus $E \subseteq \bigcup\{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\}$, as required.
(ii $\Longrightarrow$ iii) By Proposition 2.2,

$$\bigcup\{\overline{Y} \mid Y \in Cons \text{ and } Y \Subset E\} = \overline{\bigcup\{Y \mid Y \in Cons \text{ and } Y \Subset E\}}.$$

Hence $E = \overline{\bigcup\{Y \mid Y \in Cons \text{ and } Y \Subset E\}}$, as required. □

Theorem 3.9. *Let* $\mathbf{A} = (A, Cons, \vdash)$ *be an information system. Then* $|\mathbf{A}|$ *is an algebraic* $\bigcap$*-structure.*

Proof. We show that $|\mathbf{A}|$ is a non-empty family of sets closed under intersections and directed unions of non-empty subfamilies. Since $|\mathbf{A}|$ contains $\overline{\emptyset}$, it is non-empty. It is routine to show that if $\{E_i\}_{i\in I}$ is a non-empty subfamily of $|\mathbf{A}|$ then $\bigcap_{i\in I} E_i$ is consistent and $\vdash$-closed, and so is in $|\mathbf{A}|$. Finally, assume $\mathcal{D} = \{E_i\}_{i\in I}$ is a directed system in $|\mathbf{A}|$ and let $E = \bigcup_{i\in I} E_i$. Take $Y \Subset E$. Because $\mathcal{D}$ is directed, $Y \Subset E_i$ for some i. Since E_i is consistent, we have $Y \in Cons$. Assume also $Y \vdash a$. Then $a \in E_i$ since E_i is $\vdash$-closed. This completes the proof of the claim. The finite elements of $|\mathbf{A}|$ are exactly the sets $\overline{Y}$ where $Y \in Cons$. So by Theorem 3.8, $|\mathbf{A}|$ is algebraic. □

Theorem 3.10. *Let* $\mathfrak{L}$ *be an algebraic* $\bigcap$*-structure. Then* $\mathbf{IS}(\mathfrak{L}) = (A, Cons, \vdash)$ *is an information system where,*

(1) $A := \bigcup \mathfrak{L}$,
(2) $Cons := \{Y \mid (\exists U \in \mathfrak{L})\ Y \Subset U\}$,
(3) $Y \vdash a$ *if and only if* $a \in \bigcap\{U \in \mathfrak{L} \mid Y \Subset U\}$.

Proof. It is very easy to show that $\mathbf{IS}(\mathfrak{L})$ is an information system. □

Theorem 3.11. *The class of information systems and the class of algebraic* $\bigcap$*-structure are isomorphic.*

Proof. The maps $|-|\colon \mathbf{INF} \to \bigcap_{\mathbf{S}}$, $\mathbf{A} \mapsto |\mathbf{A}|$ and $\mathbf{IS}(-)\colon \bigcap_{\mathbf{S}} \to \mathbf{INF}$, $\mathfrak{L} \mapsto \mathbf{IS}(\mathfrak{L})$ are inverse. Let $\mathbf{A} = (A, Cons, \vdash)$ be an information system, we show that $\mathbf{A} = \mathbf{IS}((A, |\mathbf{A}|))$. We have:

(i) $A = \bigcup |\mathbf{A}|$ (by (IS2));
(ii) if $Y \Subset A$, then $Y \in Cons \Leftrightarrow (\exists E \in |\mathbf{A}|) Y \Subset E$ (for the forward implication note that by Theorem 3.8, $\overline{Y} \in |\mathbf{A}|$ and for the reverse recall that any $E \in |\mathbf{A}|$ is consistent);
(iii) if $Y \in Cons$ and $a \in A$, then $Y \vdash a \Leftrightarrow a \in \bigcap\{E \in |\mathbf{A}| \mid Y \Subset E\}$ (for the forward implication recall that any any $E \in |\mathbf{A}|$ is $\vdash$-closed and for he reverse use the fact that $Y \subseteq \overline{Y} \in |\mathbf{A}|$). Consequently, by looking at Theorem 3.9, we see that $\mathbf{A} = \mathbf{IS}((A, |\mathbf{A}|))$, as required. Finally, Let $\mathfrak{L}$ be an algebraic $\bigcap$-structure. We show that $|\mathbf{IS}(\mathfrak{L})| = \mathfrak{L}$. Using Theorem 3.8 and Theorem 3.9, for an element E of $|\mathbf{IS}(\mathfrak{L})$, we have

$$E = \bigcup\{\bigcap\{U \in \mathfrak{L} \mid U \supseteq Y\} \mid Y \Subset E\},$$

with the union taken over a directed set. Since $\mathfrak{L}$ is algebraic, we have $|\mathbf{IS}(\mathfrak{L})| \subseteq \mathfrak{L}$. Conversely, the definitions of consistency and entailment in $|\mathbf{IS}(\mathfrak{L})|$ imply that $\mathfrak{L} \subseteq |\mathbf{IS}(\mathfrak{L})|$. Consequently, $|\mathbf{IS}(\mathfrak{L})| = \mathfrak{L}$, as required.

□

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Some remarks on uniserial acts

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Abstract

In [3], uniserial acts are investigated. In this paper, more properties of these kinds of acts are studied and the relationship between them and some other concepts are investigated.

1. Introduction

Throughout this article S will denote a monoid and an S-act A_S (or A) is a right S-act. From [1] an S-act A is called $uniserial$ if the set of its subacts is linearly ordered by inclusion. Also an S-act A is said to be serial in case that it is a coproduct of uniserial acts. This paper is a continuation of [3] and we study more properties of uniserial and serial acts. Here it is necessary to recall some notions. From [4] a non-zero S-act A is called $uniform$ if every its non-zero subact is large in A i.e., for any non-zero subact B of A, any S-homomorphism $g : A \longrightarrow C$ such that $g|_B$ is a monomorphism is itself a monomorphism. We denote this situation by $B \subseteq' A$. Moreover an S-act A is called *injective(C-injective, F-injective)* if for any S-act B, any (cyclic, finitely generated) subact C of B and any homomorphism $f : C \longrightarrow A$, there exists a homomorphism $\bar{f} : B \longrightarrow A$ such that $\bar{f}\mid_C = f$. Also an S-act A is called *quasi-injective* if it is injective relative to all inclusions from its subacts (see [2], [6], [7]). We encourage the reader to see [2] for basic results and definitions relating to acts not defined here.

2. Main results

Proposition 2.1. *Suppose S is a commutative monoid and A is a uniserial S-act. If A (S_S) satisfies the descending chain condition on subacts (ideals), then A is quasi-injective.*

From [5] an S-act A is called torsion free if for any $a, b \in A$ and for any element $s \in S$, the equality $as = bs$ implies $a = b$.

Proposition 2.2. *Over a commutative monoid S any uniserial torsion free S-act is unifrom.*

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Corollary 2.3. *Let S be a commutative monoid and A be a torsion free uniserial S-act which contains a zero element. Then the following conditions are equivalent:*

(i) *A is C-injective.*

(ii) *A is F-injective.*

(iii) *A is injective.*

(iv) *A contains a C-injective subact.*

(v) *A contains a F-injective subact.*

(vi) *A contains an injective subact.*

For an S-act A an element $\theta \in A$ is called a zero element if $\theta s = \theta$ for every $s \in S$. Moreover the one element act is denoted by $\Theta = \{\theta\}$.

Corollary 2.4. *Suppose A is an S-act with the same conditions in the previous proposition. If A is C-injective, then for any weakly injective subact B of A, $A \cong B \sqcup \Theta$.*

Proposition 2.5. *Suppose S is a monoid with a left zero and $\{A_i\}_{i\in I}$ is a family of S-acts. If $\prod_{i\in I} A_i$ is uniserial, then for at most one $i \in I, A_i \neq \Theta$.*

Proposition 2.6. *The following conditions are equivalent over a monoid S:*

(i) *Every indecomposable S-act is uniserial.*

(ii) *Every composable S-act ic serial.*

(iii) *Every indecomposable S-act is uniform.*

(iv) *S is a group.*

Theorem 2.7. *Suppose S is a monoid and A is a uniserial (serial) S-act, then the following conditions are equivalent:*

(i) *A satisfies condition (E).*

(ii) *A satisfies condition (P).*

(iii) *A is strongly flat.*

(iv) *A is equalizer flat.*

(iv) *A is pullbacks flat.*

Suppse $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are S-homomorphisms. From [1], the sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called a $Rees\ short\ exact\ sequence$ if f is one-to-one, g is onto and $kerg = \rho_{Imf}$ where $\rho_{Imf} = (f(A) \times f(A)) \cup \Delta_B$.

Theorem 2.8. *Let S be a monoid and $A \xrightarrow{f} B \xrightarrow{g} C$ be a Rees short exact sequence. Then B is uniserial if and only if both A and C are uniserial.*

Proof. **Sufficiency**. Suppose A and C are uniserial and b_1, b_2 are two elements of B. Due to Proposition 1.2 of [3], we prove that $b_1 S \subseteq b_2 S$ or vice versa. For some $c_1, c_2 \in C, g(b_1) = c_1$ and $g(b_2) = c_2$. Since C is uniserial, we can suppose $c_1 = c_2 s$ for some $s \in S$. Hence $g(b_1) = c_2 s = g(b_2 s)$. Consequently $(b_1, b_2 s) \in kerg = \rho_{Imf}$ and so $b_1 = b_2 s$ or $b_1, b_2 s \in Imf$. In the first case the proof is completed. For the second case suppose $b_1 = f(a_1)$ and $b_2 s = f(a_2)$ for some $a_1, a_2 \in A$. Also by assumption we can suppose that $a_1 = a_2 t$ for some $t \in S$ and hence $b_1 = f(a_1) = f(a_2 t) = b_2 st$ which completes the proof.
Necessity. Note that factors and subacts of uniserial acts are also uniserial.

□

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$G(\mathsf{E})$ for graded division algebras

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Abstract

Let E be a graded division algebra finite-dimensional over its center F. Let E^* be the multiplicative group of E and denote by E' the commutator subgroup of E^*. Let $G(\mathsf{E}) = \mathsf{E}^*/\operatorname{Nrd}_{\mathsf{E}}(\mathsf{E}^*)\mathsf{E}'$ where $\operatorname{Nrd}_{\mathsf{E}}(\mathsf{E}^*)$ is the image of E^* under the reduced norm map. In this note we investigate the group $G(\mathsf{E})$.We then build a bridge between our results and the ungraded case. As a consequence, we recover some known formulas for the corresponding group in the level of valued division algebras.

1. Introduction

Let D be a finite-dimensional division algebra over its center F. Recall from Theorem 4 of [1, §7] that, as a vector space, $\dim_F D = n^2$ for a positive integer n. Such n is called the degree of D and is denoted by $\deg(D)$. Let $K_1(D)$ be the Whitehead group of D (for a background on K_1 we refer the reader to [1, §20]). Recall that $K_1(D) \cong D^*/D'$, where D^* and D' are the multiplicative group and commutator group of D, respectively. Let $G(D)$ denote the cokernel of the composite map

$$K_1(D) \to K_1(F) \to K_1(D),$$

where the left map is induced by the reduced norm $\operatorname{Nrd}_D : D^* \longrightarrow F^*$ (see [1, §22]) and the right map is the inclusion map. So that $G(D) = D^*/\operatorname{Nrd}_D(D^*)D'$. It is known that $G(D)$ is an abelian torsion group of bounded exponent n (see [2, Corollary 5.3]), and thus from the Prüfer-Baer Theorem we conclude that $G(D) \cong \oplus_i \mathbb{Z}_{n_i}$ for some n_i dividing n. In [3] and [4] some algebraic properties of $G(D)$ and its applications in realizing the group theoretic structure of D is investigated. It is also shown that how one can compute this group (up to isomorphism) for special classes of division algebras over Henselian fields. In this paper, following the method used in [2, §7], we investigate this group for graded division algebras. In the last decade, experiences have shown that working with K_1-group in the level of graded division algebras is much more easy and more transparent than in the non-graded setting. The reason is that when we work with graded structures, instead working with homomorphic images and quotient structures, we need

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only to work with some substructures. After giving some results concerining graded division algebras, we then build a bridge between our results and the ungraded case.

We now mention some of the terminology that will be used throughout the paper:

Let Γ be a totally ordered abelian group. Let R be a ring graded by Γ. This means that $R = \oplus_{\gamma\in\Gamma} R_\gamma$, where each R_γ is an additive subgroup of R and $R_\gamma R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Let $\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq 0\}$ be the grade set of R and $R^h = \bigcup_{\gamma\in\Gamma_R} R_\gamma$ be the set of all homogeneous elements of R. A subring S of R is called a graded subring if $S = \oplus_{\gamma\in\Gamma_R}(S \cap R_\gamma)$. For example, the center of R, which is denoted by $Z(R)$, is a graded subring of R. For a graded ring R, a graded left R-module M is a left R-module with a grading $M = \bigoplus_{\gamma\in\Gamma'}$ where each M_γ is an abelian group and Γ' is a totally ordered abelian group containing Γ, such that $R_\gamma \cdot M_\delta \subseteq M_{\gamma+\delta}$ for all $\gamma \in \Gamma_R, \delta \in \Gamma'$. Then, Γ_M and M^h are defined analogously to Γ_R and R^h. A graded left R-module M is said to be *graded free* if it has a base as a free R-module whose all elements are homogeneous. A graded ring $\mathsf{E} = \oplus_{\gamma\in\Gamma}\mathsf{E}_\gamma$ is called a *graded division ring* if every non-zero homogeneous element of E has a multiplicative inverse. It is clear from the definition that E_0 is a division ring itself. If T is a commutative graded division ring, then we call it a *graded field*. In particular, $Z(\mathsf{E})$, the center of a graded division ring E, is a graded field. It is also clear that E is a $Z(\mathsf{E})$-graded module. By an easy adaptation of the ungraded arguments, one can prove that E is $Z(\mathsf{E})$-graded free and the cardinality of every two homogenous basis are the same. We write $[\mathsf{E} : Z(\mathsf{E})]$ for the rank of E as graded left free $Z(\mathsf{E})$-module. Moreover, if $[\mathsf{E} : Z(\mathsf{E})] < \infty$ then E is called a *graded division algebra*. By Corollary 2.29 of [6] we have $[\mathsf{E} : Z(\mathsf{E})] = n^2$ for some positive integer n. Such a positive integer n is called the degree of E and is denoted by $\deg(\mathsf{E})$. E is called *unramified* if $\Gamma_\mathsf{E} = \Gamma_\mathsf{T}$. E is said to be *totally ramified* if $\mathsf{E}_0 = \mathsf{T}_0$. We also say that E is *semiramified* if E_0 is a field and $[\mathsf{E}_0 : \mathsf{T}_0] = |\Gamma_\mathsf{E} : \Gamma_\mathsf{T}| = \deg(\mathsf{E})$.

Let E be a graded division algebra over its center F. If L be a maximal graded subfield of E then $\mathsf{E} \otimes_\mathsf{F} \mathsf{L} \cong M_n(\mathsf{L})$ where $n = \deg(\mathsf{E})$. So for each $a \in \mathsf{E}^*$ one can consider $a \otimes 1$ as an element of $M_n(\mathsf{L})$. Now, the reduced norm of a is defined by $\mathrm{Nrd}_\mathsf{E}(a) = \det(a \otimes 1)$. It can be observe that $\mathrm{Nrd}_\mathsf{E}(a) \in \mathsf{F}^*$ (for these results see [2, §7.5]). Moreover, multiplicative property of determinant shows that $\mathrm{Nrd}_\mathsf{E} : \mathsf{E}^* \longrightarrow \mathsf{F}^*$ is a group homomorphism. Now, similar to ungraded setting we define $G(\mathsf{E}) = \mathsf{E}^*/\,\mathrm{Nrd}_\mathsf{E}(\mathsf{E})^*\mathsf{E}'$ where E' is the commutator subgroup of E^*. It is clear from the definition that $G(\mathsf{E})$ is an abelian group.

2. Main Results

We begin by the following lemma. For a proof see [2, Prop. 7.6]

Lemma 2.1. *Let E be a graded division algebra over its center F. Let $\deg(\mathsf{E}) = n$. If N is a normal subgroup of E^* then $N^n \subseteq \mathrm{Nrd}_\mathsf{E}(\mathsf{E}^*)[\mathsf{E}^*, N]$ where $[\mathsf{E}^*, N] = \langle aba^{-1}b^{-1} | a \in \mathsf{E}^*, b \in N\rangle$.*

It is clear that if in Lemma 2.1 we replace N by E^* then it follows that the group $G(\mathsf{E})$ is torsion of bounded exponent n. The following theorem is the main result of this note. Its proof is too long to include in this note. It will be appear in a forthcoming paper.

Theorem 2.2. *Let E be a graded division algebra over its center F. Let $\deg(\mathsf{E}) = n$. Then*

1. *If E is unramified, then we have the following short exact sequence*
$$1 \to G(\mathsf{E}_0) \to G(\mathsf{E}) \to \frac{\Gamma_\mathsf{E}}{n\Gamma_\mathsf{E}} \to 1.$$
2. *If E is totally ramified, then we have the following short exact sequence*
$$1 \to \frac{\mathsf{F}_0^*}{\mathsf{F}_0^{*n}\mu_e(\mathsf{F}_0)} \to G(\mathsf{E}) \to \frac{\Gamma_\mathsf{E}}{n\Gamma_\mathsf{E}} \to 1,$$
where e is the exponent of $\Gamma_\mathsf{E}/\Gamma_\mathsf{F}$ and $\mu_e(\mathsf{F}_0)$ is the group of all e-th roots of unity in F_0.
3. *If E is semiramified and $\mathsf{E}_0/\mathsf{F}_0$ is a cyclic extension then we have the following short exact sequence*
$$1 \to \frac{N_{\mathsf{E}_0/\mathsf{F}_0}(\mathsf{E}_0^*)}{N_{\mathsf{E}_0/\mathsf{F}_0}(\mathsf{E}_0^*)^n} \to G(\mathsf{E}) \to \frac{\Gamma_\mathsf{E}}{n\Gamma_\mathsf{E}} \to 1,$$
where $N_{\mathsf{E}_0/\mathsf{F}_0} : \mathsf{E}_0^* \to \mathsf{F}_0^*$ is the norm map of field extensions.

If in Theorem 2.2 we have $\Gamma_{\mathsf{E}} = \oplus_{i=1}^{k}\mathbb{Z}$ (here the order of $\oplus_{i=1}^{k}\mathbb{Z}$ is given by right-to-left lexicographical ordering) then $\Gamma_{\mathsf{E}}/n\Gamma_{\mathsf{E}} \cong \oplus_{i=1}^{k}\mathbb{Z}_n$ which is a free $\mathbb{Z}_n$-module. Since each exact sequence in Theorem 2.2 is in fact an exact sequence of $\mathbb{Z}_n$-modules, all of them split and we have the following corollary.

Corollary 2.3. *Let E be a graded division algebra over its center F. Let $\deg(\mathsf{E}) = n$ and $\Gamma_{\mathsf{E}} = \oplus_{i=1}^{k}\mathbb{Z}$. Then*

1. *If E is unramified, then*
$$G(\mathsf{E}) \cong G(\mathsf{E}_0) \oplus \left(\oplus_{i=1}^{k}\mathbb{Z}_n\right).$$
2. *If E is totally ramified, then*
$$G(\mathsf{E}) \cong \frac{\mathsf{F}_0^*}{\mathsf{F}_0^{*n}\mu_e(\mathsf{F}_0)} \oplus \left(\oplus_{i=1}^{k}\mathbb{Z}_n\right).$$
3. *If E is semiramified and $\mathsf{E}_0/\mathsf{F}_0$ is a cyclic extension, then*
$$G(\mathsf{E}) \cong \frac{N_{\mathsf{E}_0/\mathsf{F}_0}(\mathsf{E}_0^*)}{N_{\mathsf{E}_0/\mathsf{F}_0}(\mathsf{E}_0^*)^n} \oplus \left(\oplus_{i=1}^{k}\mathbb{Z}_n\right).$$

Example 2.4. *Let D be a division algebra over its center F of degree n. Let $\mathsf{E} = D[x, x^{-1}]$ where*

$$D[x, x^{-1}] = \{\sum_{i=0}^{k} d_i x^i | d_i \in D, n_i \in \mathbb{Z}, k \geq 0\}.$$

Since $\mathsf{E} = \oplus_{n\in\mathbb{Z}} Dx^n$ (as an abelian group), it follows that E is a $\mathbb{Z}$-graded division algebra with $\deg(\mathsf{E}) = \deg(D)$. Moreover, we have $Z(\mathsf{E}) = F[x, x^{-1}]$ which is a $\mathbb{Z}$-graded field itself. So E is an unramified graded division algebra and hence by Corollary 2.3 we obtain $G(\mathsf{E}) \cong G(D) \oplus \mathbb{Z}_n$.

Now, let D be a division algebra over its center F and let Γ be a totally ordered abelian group. By a valuation on D with values in Γ, we mean a map $v : D^* \to \Gamma$ satisfying, for all $a, b \in D^*$, $v(ab) = v(a) + v(b)$ and $v(a + b) \geq \min\{v(a), v(b)\}$. We write Γ_D for the value group of v, i.e., $\Gamma_D = v(D^*)$. We denote the valuation ring by $V_D = \{d \in D^* \mid v(d) \geq 0\} \cup \{0\}$. It is not hard to observe that this ring has a unique maximal ideal denoted by $M_D = \{d \in D^* \mid v(d) > 0\} \cup \{0\}$. We denote its residue division rin by $\overline{D} = V_D/M_D$. Let $U_D = \{d \in D^* \mid v(d) = 0\}$ and observe that $U_D = V_D^*$. When we restrict v to F^*, we obtain a valuation w on the field F. The objects for w corresponding to those for v are denoted by Γ_F, V_F, M_F, U_F and $\overline{F}$. Since $V_F \cap M_D = M_F$, it may be view the residue field $\overline{F}$ as a subalgebra of $\overline{D}$. In this setting, F is called *Henselian* if its valuation has a unique extension to any algebraic extension of F. By Theorem 1.4 of [6], when F is Henselian, its valuation has a unique extension to each finite dimensional F-division algebra. D is called *tame* if $Z(\overline{D})/\overline{F}$ is separable and char $\overline{F} \nmid n$. We say that D is *unramified* over F if $[\Gamma_D : \Gamma_F] = 1$. At the other extreme, D is said to be *totally ramified* if $[D : F] = [\Gamma_D : \Gamma_F]$. In a case in the middle, D is said to be *semiramified* if $\overline{D}$ is a field and $[\overline{D} : \overline{F}] = |\Gamma_D : \Gamma_F| = \deg(D)$. Moreover, If D has maximal subfelds L and K which are respectively unramifed and totally ramifed over F, then D is called *nicely semiramified*.
The following theorm will be used to produce bridge between graded and ungraded case (see [2, Th. 5.12]).

Theorem 2.5. *Let D be a tame division algebra over a Henselian field $F = Z(D)$, of index n. Then*

$$(1 + M_D) = (1 + M_F)[D^*, 1 + M_D].$$

Given a valued division algebra D, we associate to D a graded division algebra as follows: Given $\gamma \in \Gamma_D$, let

$D^{\geq\gamma} = \{d \in D \mid v(d) \geq \gamma\} \cup \{0\}$, an additive subgroup of D;

$D^{>\gamma} = \{d \in D \mid v(d) > \gamma\} \cup \{0\}$, a subgroup of $D^{\geq\gamma}$; and

$\mathrm{gr}(D)_\gamma = D^{\geq\gamma}/D^{>\gamma}$.
Then we define

$$\mathrm{gr}(D) = \bigoplus_{\gamma\in\Gamma_D} \mathrm{gr}(D)_\gamma.$$

Using Theorem 2.5, one can prove the following result.

Theorem 2.6. *Let D be a valued division algebra over its Henselian center F, of degree n. Then $G(D) \cong G(\mathrm{gr}(D))$.*

Combining Theorem 2.2 and 2.6 we can prove the next result which is in fact a generalization of [4, Cor. 4.27]. It also has been previosely appeared in [5].

Theorem 2.7. *Let D be a tame division algebra over its Henselian center F, of degree n. If F is Henselian, then*

1. *If D is unramified, then we have the following short exact sequence*

$$1 \to G(\overline{D}) \to G(D) \to \frac{\Gamma_D}{n\Gamma_D} \to 1.$$

2. *If D is totally ramified, then we have the following short exact sequence*

$$1 \to \frac{\overline{F}^*}{\overline{F}^{*n}\mu_e(\overline{F})} \to G(D) \to \frac{\Gamma_D}{n\Gamma_D} \to 1,$$

where e is the exponent of Γ_D/Γ_F and $\mu_e(\overline{F})$ is the group of all e-th roots of unity in $\overline{F}$.

3. *If E is semiramified and $\overline{D}/\overline{F}$ is a cyclic extension then we have the following short exact sequence*

$$1 \to \frac{N_{\overline{D}/\overline{F}}(\overline{D}^*)}{N_{\overline{D}/\overline{F}}(\overline{D}^*)^n} \to G(D) \to \frac{\Gamma_D}{n\Gamma_D} \to 1,$$

where $N_{\overline{D}/\overline{F}} : \overline{D}^* \to \overline{F}^*$ is the norm map of field extensions.

Proof. Since D is a strongly tame we observe that $Z(\mathrm{gr}(D)) = \mathrm{gr}(F)$ and $[\mathrm{gr}(D) : \mathrm{gr}(F)] = [D : F]$. Now, a simple argument shows that in either of the cases (1)-(3) here, $\mathrm{gr}(D)$ is in the corresponding case of Theorem 2.2. Thus, using the isomorphism $G(D) \cong G(\mathrm{gr}(D))$ given by Theorem 2.6 together with Theorem 2.2, we obtain the result. □

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Stanley's Conjecture on the path complexes of trees

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Abstract

A tree is called double starlike if it has exactly two vertices of degree greater than two. Let $H(p, n, q)$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the line graph Ln and q pendant vertices to the other pendant vertex of Ln. Also let $H(p, n)$ be graph obtained by attaching p pendant vertices to one pendant vertex of the line graph Ln. Let G be an undirected tree. It is shown that $\Delta_t(G)$ is partitionable for all $t \geq 2$ and Stanley's conjecture holds for $K[\Delta_t\,(G)]$, where $G = H(p, n, q)$ or $G = H(p, n)$.

1. Introduction

Let Δ be a simplicial complex on vertex set $[n] = \{1, \cdots, n\}$, i.e. Δ is a collection of subsets of $[n]$ with the the property that if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximum dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. Otherwise it is called non-pure. We denote the simplicial complex Δ with facets $F_1, \ldots, F_t$ by $\Delta = \langle F_1, \ldots, F_t\rangle$. A simplex is a simplicial complex with only one facet.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{F \cup G \ : \ F \in \Delta_1,\ G \in \Delta_2\}$.

For the face F in Δ, the link, deletion and star of F in Δ are respectively, denoted by $\mathrm{link}_\Delta F$, $\Delta \setminus F$ and $\mathrm{star}_\Delta F$ and are defined by $\mathrm{link}_\Delta F = \{G \in \Delta \ : \ F \cap G = \varnothing,\ F \cup G \in \Delta\}$ and $\Delta \setminus F = \{G \in \Delta \ : \ F \nsubseteq G\}$ and $\mathrm{star}_\Delta F = \langle F\rangle * \mathrm{link}_\Delta F$.

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over a field K. To a given simplicial complex Δ on the vertex set $[n]$, the Stanley–Reisner ideal is the squarefree monomial ideal whose generators correspond to the non-faces of Δ. we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

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We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ. Also we define the simplicial complex $\Delta_t(G)$ to be

$$\Delta_t(G) = \langle \{x_{i_1}, \ldots, x_{i_t}\} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length t in G } \rangle.$$

We say the simplicial complex Δ is Cohen–Macaulay if $K[x_1, \ldots, x_n]/I_\Delta$ is Cohen–Macaulay. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$- graded ring and M a $\mathbb{Z}^n$- graded R- module. Then Stanley [2] conjectured that

$$\text{depth}\,(M) \leq \text{sdepth}\,(M)$$

He also conjectured in [3] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [1] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. In this paper, for all $t \geq 2$ we show that $\Delta_t(G)$ is vertex decomposable if and only if $G = H(p, n, q)$ or $G = H(p, n)$. As a consequence we show that $\Delta_t(G)$ is partitionable for all $t \geq 2$ and Stanley's conjecture holds for $K[\Delta_t\,(G)]$, where $G = H(p, n, q)$ or $G = H(p, n)$.

2. Main Results

As the main result of this section, it is shown that $\Delta_t(G)$ is partitionable for all $t \geq 2$ and Stanley's conjecture holds for $K[\Delta_t\,(G)]$, where $G = H(p, n, q)$ or $G = H(p, n)$. For the proof we need the following lemmas and propositions:

Definition 2.1. A simplicial complex Δ is recursively defined to be *vertex decomposable*, if it is either a simplex, or else has some vertex v so that,

(a) Both $\Delta \setminus v$ and $\text{link}_\Delta(v)$ are vertex decomposable, and

(b) No face of $\text{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.

A vertex v which satisfies in condition (b) is called a *shedding vertex*.

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is connected.

Remark 2.2. *All Cohen-Macaulay simplicial complexes of positive dimension are connected.*

Lemma 2.3. *Let $\Delta_t(Ln)$ be a simplicial complex on the vertices $\{x_1, \ldots, x_n\}$ and $2 \leq t \leq n$. Then $\Delta_t(Ln)$ is vertex decomposable.*

Remark 2.4. *Let Ln be a line graph on the vertices $\{x_1, \ldots, x_n\}$ and $H(2, n)$ be a graph obtained by attaching two pendant vertices to pendant vertex x_n. Then $\Delta_t(H(2, n))$ is vertex decomposable for all $t \geq 2$.*

Proof. By lemma 2.3 proof is trivial. □

Proposition 2.5. *Let Ln be a line graph on the vertices $\{x_1, \ldots, x_n\}$ and $H(p, n)$ be a graph obtained by attaching p pendant vertices to pendant vertex x_n. Then $\Delta_t(H(p, n))$ is vertex decomposable for all $t \geq 2$.*

Lemma 2.6. *Let $p = 2$ and $q \geq 2$, Then $\Delta_t(H(2, n, q))$ is vertex decomposable for all $2 \leq t \leq n + 2$*

Proposition 2.7. *Let Q_1, Q_2 be two paths of maximum length k in tree G and y be a leaf of G such that $y \in Q_1 \cap Q_2$, $|Q_1 \cap Q_2| = L$. Then $\Delta_k(G)$ is not vertex decomposable.*

Proposition 2.8. *Let G be a double starlike tree such that $G = H(p, n, q)$. Then $\Delta_t(G)$ is vertex decomposable for all $2 \leq t \leq n + 2$.*

Now, we are ready that prove one of the main results of this paper.

Theorem 2.9. *Let G be a tree such that is not a path. Then $\Delta_t(G)$ is vertex decomposable for all $t \geq 2$ if and only if $G = H(p, n, q)$ or $G = H(p, n)$.*

Proof. ($\Longrightarrow$)We prove by contradiction. Suppose $G \neq H(p, n, q)$ and $G \neq H(p, n)$. So there exists two paths of maximum length k in G which contain L common vertices such that one of these vertices is a leaf. Therefore by proposition 2.7 $\Delta_k(G)$ is not vertex decomposable which is a contradiction. ($\Longleftarrow$) By proposition 2.5 and proposition 2.8 the proof is completed. $\square$

Stanley conjectured in [2] the upper bound for the depth of $K[\Delta]$ as the following:

$$\text{depth}\,(K[\Delta]) \leq \text{sdepth}\,(K[\Delta])$$

. Also we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1, \ldots, x_n\}$ with facets $G_1, \ldots, G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{t} [F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [3] and [4] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [1] proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1, \ldots, x_n\}$ we have that $\text{depth}\,(K[\Delta]) \leq \text{sdepth}\,(K[\Delta])$ if and only if Δ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results we obtain :

Corollary 2.10. *Let G be a tree such that is not a path. if $G = H(p, n, q)$ or $G = H(p, n)$ then $\Delta_t(G)$ is partitionable for all $t \geq 2$ and Stanley's conjecture holds for $K[\Delta_t\,(G)]$.*

Proof. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable.Therefore by theorem 2.9 proof is completed. $\square$

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Some notes on n-Lie algebras

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Abstract

An n-Lie algebra analogue of Schur's theorem and its converse as well as a Lie algebra analogue of Baer's theorem and its converse are presented. Also, it is shown that, an n-Lie algebra with finite dimensional derived subalgebra and finitely generated central factor is isoclinic to some finite dimensional n-Lie algebra.

Introduction

In 1969, Kurosh [5] introduced the Ω-algebras, that is, an algebra equipped with an n-ary n-linear product. He discussed skew-symmetric Ω-algebras and noted that they contain the class of Lie algebras. The first and most effective generalization of Lie algebras is given by Filippov in 1985. Filippov [2], introduced the concept of *n-Lie algebras*, as an n-ary multilinear and skew-symmetric operation $[x_1, \ldots, x_n]$, which satisfies the following generalized Jacobi identity

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]].$$

Clearly, such an algebra becomes a Lie algebra when $n = 2$. Analogous to Lie algebras, a *derivation* α of an n-Lie algebra L is defined as a linear transformation satisfying

$$\alpha([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, \alpha(x_i), x_{i+1}, \ldots, x_n]$$

for all $x_1, \ldots, x_n \in L$. The set of all derivations of L is denoted by $Der(L)$. Similarly, the *adjoint derivation* corresponding to $n-1$ elements $x_1, \ldots, x_{n-1} \in L$, is defined by

$$ad_{x_1,\ldots,x_{n-1}}(x) = [x, x_1, \ldots, x_{n-1}].$$

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The set of all adjoint derivations of L is denoted by $Ad(L)$. The set $Der(L)$ is a Lie algebra under the following commutator operator

$$(D_1, D_2) \mapsto [D_1, D_2] := D_1D_2 - D_2D_1$$

and $Ad(L)$ is its Lie ideal.
In 1986, Kasymov[4] introduced the notion of nilpotency of an n-Lie algebra as follows:
An n-Lie algebra L is *nilpotent* if $L^s = 0$ for some non-negative integer s, where L^i is defined inductively by $L^0 = L$ and $L^{i+1} = [L^i, L, \ldots, L]$. The ideal $L^1 = [L, \ldots, L]$ is called the *derived subalgebra* of L. The *center* of L is defined by

$$Z(L) = \{x \in L : [x, L, \ldots, L] = 0\}.$$

Let $Z_0(L) = \{0\}$. Then the ith center of L is defined inductively by

$$\frac{Z_i(L)}{Z_{i-1}(L)} = Z\left(\frac{L}{Z_{i-1}(L)}\right)$$

for all $i \geq 1$. Clearly, $Z_1(L) = Z(L)$.
In 1904, Schur [8] proved that if G is a group such that $G/Z(G)$ is finite, then G' is also finite. In 1952, Baer [1] generalized Schur's theorem and showed that if G is a group such that $G/Z_i(G)$ is finite, then so is $\gamma_{i+1}(G)$. In 1994, Hegarty [3] generalizes Schur's theorem and proves that if G is a group such that $G/L(G)$ is finite, then so is $K(G)$, where $L(G) = \{x \in G : x^{-1}x^{\alpha} = 1, \alpha \in Aut(G)\}$ is the absolute center of G and $K(G) =< \{x^{-1}x^{\alpha} : x \in G, \alpha \in Aut(G)\} >$ is the autocommutator subgroup of G. These results are extended to Lie algebras by Stitzinger and Turner [9] in terms of derivations of Lie algebras. They proved that if L is a Lie algebra such that $L/H(L)$ has finite dimension, then L^* has also finite dimension, where

$$H(L) = \bigcap_{\alpha \in Der(L)} Ker(\alpha)$$

and

$$L^* = \sum_{\alpha \in Der(L)} Im(\alpha).$$

Clearly, $H(L) \subseteq Z(L)$ and $L^1 \subseteq L^*$. We intend to generalize the works of Stitzinger and Turner to an arbitrary n-Lie algebra. Also, we shall extend the result of Niroomand [7] as a converse to Schur's theorem. Niroomand proves that if G is a group with finite derived subgroup such that $G/Z(G)$ is finitely generated, then

$$\left|\frac{G}{Z(G)}\right| \leq |G'|^{d\left(\frac{G}{Z(G)}\right)},$$

where $d(G/Z(G))$ is the minimum number of generators of $G/Z(G)$.
Using the notion of isoclinism, we shall prove that an n-Lie algebra L such that L^1 has finite dimension and $L/Z(L)$ is finitely generated is isoclinic to some finite dimensional n-Lie algebra.

1. Schur's theorem and its converses

We begin with providing a generalization of Stitzinger and Turner's Lie algebra version of Schur's theorem to an arbitrary n-Lie algebra. Our proof uses the same method applied by Stitzinger and Turner in [9].

Lemma 1.1. *Let L be an n-Lie algebra. If $\{x_1+H(L), \ldots, x_d+H(L)\}$ is a basis for $L/H(L)$, then a linear operator ϕ on $H(L)$ for which $\phi(m_{i_1,\ldots,i_n}) = 0$ for all $1 \leq j \leq n$ and $1 \leq i_j \leq d$ can be extended to a derivation of L, in which*

$$m_{i_1,\ldots,i_n} = [x_{i_1}, \ldots, x_{i_n}] - \sum_{r=1}^{d} \alpha^r_{i_1,\ldots,i_n} x_r$$

and $\alpha^r_{i_1,\ldots,i_n}$ are defined by

$$[x_{i_1} + H(L), \ldots, x_{i_n} + H(L)] = \sum_{r=1}^{d} \alpha^r_{i_1,\ldots,i_n}(x_r + H(L)).$$

Proof. If $l \in L$, then $l = \sum_{i=1}^{d} \alpha_i x_i + h_l$ for some $h_l \in H(L)$. Now, define the map $\varphi : L \longrightarrow L$ by $\varphi(l) = h_l$. We claim that φ is the required derivation. If $l_1, \ldots, l_n \in L$, then

$$\begin{aligned}
\varphi([l_1, \ldots, l_n]) &= \varphi([\sum_{i_1=1}^{d} \alpha_{i_1} x_{i_1} + h_{l_1}, \ldots, \sum_{i_n=1}^{d} \alpha_{i_n} x_{i_n} + h_{l_n}]) \\
&= \varphi(\sum_{i_1,\ldots,i_n} \alpha_{i_1} \ldots \alpha_{i_n} [x_{i_1}, \ldots, x_{i_n}]) \\
&= \sum_{i_1,\ldots,i_n} \alpha_{i_1} \ldots \alpha_{i_n} \varphi([x_{i_1}, \ldots, x_{i_n}]) \\
&= \sum_{i_1,\ldots,i_n} \alpha_{i_1} \ldots \alpha_{i_n} \varphi(m_{i_1,\ldots,i_n} + \sum_{r=1}^{d} \alpha^r_{i_1,\ldots,i_n} x_r) \\
&= \sum_{i_1,\ldots,i_n} \alpha_{i_1} \ldots \alpha_{i_n} \phi(m_{i_1,\ldots,i_n}) = 0
\end{aligned}$$

for some $h_{l_1}, \ldots, h_{l_n} \in H(L) \subseteq Z(L)$.
On the other hand,

$$[\varphi(l_1), l_2, \ldots, l_n] = [l_1, \varphi(l_2), l_3, \ldots, l_n] = \cdots = [l_1, l_2, \ldots, l_{n-1}, \varphi(l_n)] = 0,$$

which implies that φ is a derivation. □

Theorem 1.2. *Let L be an n-Lie algebra. If* $\dim(L/H(L)) = d$*, then* $\dim(L) \leq \binom{d}{n} + d$*.*

Proof. Let $S_n = m_{i_1,\ldots,i_n} : i_1, \ldots, i_n = 1, \ldots, d$. Then $S_n \subseteq H(L) \subseteq Z(L)$ and $\dim(S_n) \leq \binom{d}{n}$. If $H(L) \neq S_n$, then we may define a non-zero linear transformation on $H(L)$ which vanishes on S_n. Hence, by Lemma 1.1, we reach to a derivation of L, which contradicts the definition of $H(L)$. Thus $H(L) = S_n$, which implies that $\dim(H(L)) \leq \binom{d}{n}$. Therefore $\dim(L) \leq \binom{d}{n} + d$. The proof is complete. □

Corollary 1.3. *If L is an n-Lie algebra such that $L/H(L)$ has finite dimension, then L^* has finite dimension.*

Corollary 1.4. *If L is a Lie algebra and* $\dim(L/H(L)) = d$*, then* $\dim(L) \leq \frac{1}{2}d(d+1)$*.*

The following theorems provide converses to Corollary 1.4.

Theorem 1.5. *Let L be an n-Lie algebra. If $Der(L)$ is finitely generated and L^* has finite dimension, then $L/H(L)$ has finite dimension and*

$$\dim\left(\frac{L}{H(L)}\right) \leq d(Der(L)) \dim(L^*),$$

where $d(Der(L))$ is the minimum number of generators of $Der(L)$.

Proof. Let $d(Der(L)) = k$. Then, there exist $D_1, \ldots, D_k \in Der(L)$ such that $Der(L) =< D_1, \ldots, D_k >$. Let f be the map defined as follows

$$\begin{array}{rcl}
f : \frac{L}{H(L)} & \longrightarrow & L^* \oplus \cdots \oplus L^* \\
x + H(L) & \longmapsto & (D_1(x), \ldots, D_k(x)).
\end{array}$$

Clearly, f is an injective linear transformation. Hence

$$\dim\left(\frac{L}{H(L)}\right) \leq k \dim(L^*),$$

as required. □

By restriction to adjoint derivations instead of arbitrary derivations, we are able to give a result similar to Theorem 1.5.

Theorem 1.6. *Let L be an n-Lie algebra. If $L/Z(L)$ is finitely generated and L^1 has finite dimension, then $L/Z(L)$ has finite dimension and*

$$\dim\left(\frac{L}{Z(L)}\right) \le \binom{k}{n-1}\dim(L^1),$$

where $k = d\,(L/Z(L))$.

Proof. Let $L/Z(L) =< \{x_1 + Z(L), \dots, x_k + Z(L)\} >$ and $\mathcal{S} = \{S_1, \dots, S_t\}$ be the set of all $(n-1)$-subsets of $\{x_1, \dots, x_k\}$. Let f be the map defined as follows

$$\begin{array}{rcl} f : \frac{L}{Z(L)} & \longrightarrow & L^1 \oplus \cdots \oplus L^1 \\ x + Z(L) & \longmapsto & (S_1(x), \dots, S_t(x)). \end{array}$$

Then, f is an injective linear transformation, from which it follows that

$$\dim\left(\frac{L}{Z(L)}\right) \le |\mathcal{S}|\dim(L^1) = \binom{k}{n-1}\dim(L^1),$$

as required. □

The above theorem can be stated in a more general form for Lie algebras and gives a Lie algebra analogue of Baer's theorem and its converse.

Theorem 1.7. *Let L be a Lie algebra such that $L/Z(L)$ is finitely generated. Then L^i has finite dimension if and only if $L/Z_i(L)$ has finite dimension.*

Proof. If $i = 1$, then the result follows by [6,Lemma 14] and Theorem 1.6. Thus, we may assume that $i > 1$. Let $L/Z(L) =< \{x_1 + Z(L), \dots, x_k + Z(L)\} >$. First suppose that L^i has finite dimension. Since $L^i = [L^{i-1}, L]$, $L^{i-1}/C_{L^{i-1}}(x)$ has finite dimension as a vector space, for each $x \in L$. Thus, $L^{i-1}/(L^{i-1} \cap Z(L))$ has finite dimension, which implies that

$$\frac{L^{i-1} + Z(L)}{Z(L)} = \left(\frac{L}{Z(L)}\right)^{i-1}$$

has finite dimension. On the other hand, $(L/Z(L))/Z(L/Z(L))$ is finitely generated. Hence, by using induction, $(L/Z(L))/Z_{i-1}(L/Z(L))$ has finite dimension, which implies that $L/Z_i(L)$ has finite dimension.
Now we prove the converse. Suppose that the result holds for $i-1$. Since $L/Z_i(L)$ has finite dimension $(L/Z(L))/(Z_{i-1}(L/Z(L)))$ has finite dimension, which implies that $(L/Z(L))^{i-1}$ has finite dimension. Thus

$$\frac{L^{i-1}}{L^{i-1} \cap Z(L)} \cong \frac{L^{i-1} + Z(L)}{Z(L)} = \left(\frac{L}{Z(L)}\right)^{i-1}$$

has finite dimension, from which it follows that $L^{i-1}/C_{L^{i-1}}(x)$ or equivalently $[L^{i-1}, x]$ has finite dimension as a vector space, for each $x \in L$. Therefore,

$$L^i = [L^{i-1}, L] = \sum_{j=1}^{k} [L^{i-1}, x_j]$$

has finite dimension. The proof is complete. □

2. Isoclinisms of n-Lie algebras

In this section, we shall use the notion of isoclinism for n-Lie algebras and show that an n-Lie algebra with given finiteness conditions on its derived subalgebra and central factor is isoclinic to a finite dimensional n-Lie algebra. We begin with the formal definition of isoclinism between n-Lie algebras.

Definition 2.1. Let L_1 and L_2 be two n-Lie algebras. Then L_1 and L_2 are said to be *isoclinic*, denoted by $L_1 \sim L_2$, if there exist two isomorphisms $\alpha : L_1/Z(L_1) \to L_2/Z(L_2)$ and $\beta : L_1^1 \to L_2^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \dfrac{L_1}{Z(L_1)} \times \dfrac{L_1}{Z(L_1)} & \rightarrow & L_1^1 \\ \alpha \times \alpha \downarrow & & \downarrow \beta \\ \dfrac{L_2}{Z(L_2)} \times \dfrac{L_2}{Z(L_2)} & \rightarrow & L_2^1 \end{array}$$

where the horizontal maps are defined by $(x + Z(L_i), y + Z(L_i)) \mapsto [x, y]$.

Note that, isoclinism between n-Lie algebras is an equivalence relation. Hence, n-Lie algebras fall into isoclinism classes of non-isoclinic n-Lie algebras. The next lemma illustrates an important property of isoclinism classes of n-Lie algebras.

Definition 2.2. An n-Lie algebra L is a *stem* n-Lie algebra if $Z(L)$ is a subset of L^1.

Lemma 2.3. *Let $\mathcal{C}$ be an isoclinism class of n-Lie algebras. Then $\mathcal{C}$ contains a stem n-Lie algebra.*

Proof. The proof is essentially the same as in [6,Lemma 4(1)] and it is omitted. □

Proposition 2.4. *Let L be an n-Lie algebra such that $L/Z(L)$ is finitely generated. If L^1 has finite dimension, then L is isoclinic to a finite dimensional n-Lie algebra.*

Proof. Clearly, by Corollary 1.6, $L/Z(L)$ has finite dimension. By Lemma 2.3, L is isoclinic to a stem n-Lie algebra S. Since $Z(S) \leq S^1$ and $S^1 \cong L^1$ has finite dimension, it follows that $Z(S)$ has finite dimension too. On the other hand, $S/Z(S) \cong L/Z(L)$ has finite dimension, which implies that S has finite dimension. The proof is complete. □

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Second homology of Leibniz algebras

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Abstract

The concept of Some properties of the second homology and cover of Leibniz algebras are established. By constructing a stem cover, the second Leibniz homology and cover of abelian, Heisenberg Lie algebras and cyclic Leibniz algebras are described. Also, for the dimension of a non-cyclic nilpotent Leibniz algebra L, we obtain $dim(HL_2(L)) \geq 2$.

1. Introduction

All algebras considered in this paper are finite dimensional over a field of characteristic different from 2. The terminology and notations employed agree with the standard usage as in [3]. Leibniz algebras are non-antisymmetric generalizations of Lie algebras. Loday (see [5, 6]) propounded a new type of algebras satisfying only Leibniz relations when he tried to formulate the non-commutative homology of a Lie algebra which is defined by replacing $\otimes$ by $\wedge$ in the Chevalley-Eilenberg complex of a Lie algebra. Recently, the theory of Leibniz algebras has been studied in some articles and several results of Lie algebras have been developed to Leibniz algebras. An algebra L over a field K is called a (left) Leibniz algebra if for any $a \in L$ the left multiplication map $l_a : L \to L$ given by $l_a(x) = [a, x]$ is a derivation, i.e. for all $x, y, z \in L$

$$[x, [y, z]] = [[x, y]z] + [y, [x, z]].$$

Obviously, if $[x, x] = 0$ for all $x \in L$, then a Leibniz algebra is a Lie algebra and Leibniz identity becomes the classical Jacobi identity.
It is well known that for a Leibniz algebra L, the space spanned by squares of elements, $Leib(L) = span\{[x, x]; x \in L\}$, is an ideal of L contained in the left center of L. Moreover,$Leib(L)$ is the minimal ideal of L with respect to the property that the quotient algebra $L\big/Leib(L)$ is a Lie algebra.
For any Leibniz algebra L, there is a tensor complex associated to L:

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$$CL_*(L) : \cdots \to L^{\otimes n} \xrightarrow{d} L^{\otimes(n-1)} \xrightarrow{d} \cdots \xrightarrow{d} L \xrightarrow{0} K$$
$$d(x_1 \otimes \cdots \otimes x_n) := \sum_{1 \le i \le j \le n} (-1)^i (x_1 \otimes \cdots \otimes \hat{x_i} \otimes \cdots \otimes [x_i, x_j] \otimes x_{j+1} \otimes \cdots \otimes x_n)$$

The Leibniz homology (with trivial coeffcients) of L is defined as

$$HL_*(L) := H_*(CL_*(L), d).$$

The Leibniz homology of L can be interpreted as

$$HL_*(L) = Tor^*_{UL}(U(L/Leib(L)), K),$$

where $U(L/Leib(L))$ is the universal enveloping algebra of the quotient Lie algebra $L/Leib(L)$ and UL is the universal enveloping of the Leibniz algebra L. See [7] for more details. If L is a Leibniz algebra of dimension n, then the maximal possible dimension for $HL_i(L)$ is n^i which is met if and only if L is abelian. In the following proposition, we refine this inequality in the second step.

Proposition 1.1. *Let L be a n-dimensional Leibniz algebra. Then* $\dim(L^2) + \dim(HL_2(L)) \le n^2$.

2. Stem cover of Leibniz algebras

Wiegold (1965) obtained an estimate for the order of commutator subgroup of a p-group G in terms of the order of $G/Z(G)$. Later, Batten (1993) in her dissertation obtained a similar result for Lie algebras. We start by establishing a parallel result for Leibniz algebra

Lemma 2.1. *Let L be a Leibniz algebra such that* $\dim(L/Z(L)) = n$ *then* $\dim(L^2) \le n^2$.

Now, we go on to show that when equality holds in Lemma 2.1. We use the following notations through rest of the paper

$$Z^l = \{x \in L : [x, L] = 0\},$$
$$Z_2(L) = \{x \in L : [x, L], [L, x] \subseteq Z(L)\}.$$

Proposition 2.2. *Let L be a non-abelian nilpotent Leibniz algebra such that* $\dim(L/Z(L)) = n$ *and* $\dim(L^2) = n^2$ *then $L/Z(L)$ is a Lie algebra.*

Definition 2.3. For any integer n, let $L_n = span\{x_1, \cdots, x_n, x_{ij} : 1 \le i, j \le n\}$ be the $(n^2 + n)$-dimensional Leibniz algebra with $[x_i, x_j] = x_{ij}$ for all $1 \le i, j \le n$ and all other products of basis elements being zero.

Proposition 2.4. *Let L be a non-abelian nilpotent Leibniz algebra such that* $\dim(L/Z(L)) = n$ *and* $\dim(L^2) = n^2$. *Then there exists an integers n such that $L \cong L_n \oplus A$, where A is a finite-dimensional abelian Lie algebra.*

Definition 2.5. Let $(e) : 0 \to N \to K \xrightarrow{\pi} L \to 0$ be a central extension of Leibniz algebras, then (e) (or π according to the notations of category theory) is called a stem extension of L if the induced morphism $HL_1(\pi) : HL_1(K) \to HL_1(L)$ is an isomorphism. Furthermore, (e) is called a stem cover if $HL_2(\pi)$ is zero.

Remark 2.6. *If $(e) : 0 \to N \to K \xrightarrow{\pi} L \to 0$ is a stem extension of a finite-dimensional Leibniz algebra L then by Lemma 2.1, N and consequently K are also of finite dimensions. Similar to contexts of Lie algebras, (e) is called a maximal stem extension of L if* $\dim(K)$ *is maximal among all stem extensions of L.*

Proposition 2.7. *Let $(e) : 0 \to N \to K \to L \to 0$ be a central extension of Leibniz algebras, then*

(i) (e) is a stem extension if and only if $N \subseteq L^2$.

(ii) If (e) *is a stem cover, then* (e) *is isomorphic to (the unique class of) stem extension* $0 \to HL_2(L) \to L^{\circ} \to L \to 0$.

(iii) Every stem extension of L is an epimorphic image of some stem cover.

Corollary 2.8. . *Let L be a finite-dimensional Leibniz algebra, then* $(e) : 0 \to N \to L^{\circ} \to L \to 0$ *is a stem cover of L if and only if* L° *has the maximal dimension among all stem extensions of L.*

Remark 2.9. *Suppose L is a Lie algebra. The Lie algebra* L^* *is called a Lie cover of L if there exists an ideal* $A \subseteq (L^*)^2 \cap Z(L^*)$ *such that* $A \cong H_2(L)$ *and* $L^*/A \cong L$*, where* $H_2(L)$ *is the second Chevalley-Eilenberg homology of L. It is well known that* L^* *has maximal dimension among all stem extensions of L in the category of Lie algebras. Hence, besides Leibniz covers, we can think about Lie covers for a Lie algebra.*

3. The second homology of nilpotent Leibniz algebras

Let $L = \langle a \rangle$ be a cyclic Leibniz algebra of dimension n and suppose $\{a, a^2 = [a, a], \cdots, a^i = [a, a^{i-1}], \cdots, a^n\}$ is a basis for L. It can be easily checked that $[a, a^n] = \alpha_2 a^2 + \cdots + \alpha_n a^n$ for some $\alpha_2, \cdots, \alpha_n \in K$. Note that if L is a nilpotent Leibniz algebra, then we should have $[a, a^n] = 0$. In the following proposition, we compute the second homology of a cyclic Leibniz algebra.

Proposition 3.1. *Let L be a cyclic Leibniz algebra of dimension* n. *Then* $\dim(HL_2(L)) = 1$.

Theorem 3.2. *Let L be a nilpotent Leibniz algebra then* $HL_2(L)$ *is nontrivial. In particular, if L is a nilpotent non-cyclic Leibniz algebra then* $\dim(HL_2(L)) \geq 2$.

Corollary 3.3. . *Let L be a two-step nilpotent Lie algebra. Then*

$$\dim(L/Z(L)) \leq \dim(HL_2(L)).$$

Now, we present the following general result to compare the Lie cover and Leibniz cover of a Lie algebra

Theorem 3.4. *Let L be a finite-dimensional Lie algebra and* $L^{\square}$, L° *be the Lie cover and Leibniz cover of L, respectively. Then* $L^{\square} \cong L^{\circ}/Leib(L^{\circ})$.

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Local-Global principle of generalized local cohomology modules

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Abstract

Let R be a commutative Noetherian ring with non-zero identity and I be an ideal of R. Let M and N be two I-cofinite modules. The purpose of this paper is to show that for a positive integer t, the R-module $H^i_I(M,N)$ is finitely generated for $i < t$ if and only if $R_\mathfrak{p}$-module $H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is finitely generated for $i < t$ and also we show that $H^i_I(M,N)$ is finitely generated if and only if $I \subseteq \mathrm{Rad}(0 :_R H^i_I(M,N))$ for $i < t$.

1. Introduction

Throughout this paper, Let R denote a commutative Noetherian ring and I be an ideal of R. Let M and N be two finitely generated R-modules. The notion of generalized local cohomology was introduced by Herzog in [4]. The ith generalized local cohomology modules of M and N with respect to I is defined as

$$H^i_I(M,N) \cong \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(M/I^n M, N).$$

It is clear that $H^i_I(R,N)$ is just the ordinary local cohomology module $H^i_I(N)$. Generalized local cohomology modules have been studied by several authors (see for example [5] and [6]).
Hartshorn in [3] defined an R-module M to be I-cofinite, if $\mathrm{Supp}(M) \subseteq \mathrm{V}(I)$ and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated module for all $i \geq 0$.
Let M and N be finitely generated R-modules. As a generalisation of the J-finiteness dimension $f^J_I(N)$ of N with respect to I, defined

$$f^J_I(M,N) = \inf\left\{i \in \mathbb{N}_0 : \ J \nsubseteq \mathrm{Rad}(0 :_R H^i_I(M,N))\right\}$$

and denote $f^I_I(M,N)$ by $f_I(M,N)$.
The purpose of this paper is to generalize local-global principle for finiteness of generalized local cohomology modules to the class of all R-modules that are I-cofinite. More precisely, we shall prove:

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Theorem 1.1. *Let R be a Noetherian ring and I be an ideal of R. Suppose that t is an integer. Let M and N be I-cofinite modules. Then the folowing conditions are equivalent.*

1. *The R-module $H^i_I(M,N)$ is finitely generated for each $i < t$.*
2. *The $R_{\mathfrak{p}}$-module $H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is finitely generated for each $i < t$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.*
3. *$I \subseteq \operatorname{Rad}(0 :_R H^i_I(M,N))$ for each $i < t$.*

For any ideal I of R, we denote $\{\mathfrak{p} \in \operatorname{Spec}R : \mathfrak{p} \supseteq I\}$ by $\mathrm{V}(I)$. We refer the reader to [2] for any unexplained notion and terminology.

2. The results

The main purpose of this section is to prove Theorem 1.1. But first of all we need the following auxiliary lemma.

Lemma 2.1. *Let R be a Noetherian ring and I be an ideal of R and M, N be finitely generated R- modules. Let $E^\bullet$ be an injective resolution of N. Then*

$$H^0_I(M,N) \cong \Gamma_I(\operatorname{Hom}_R(M,N)) \cong \operatorname{Hom}_R(M, \Gamma_I(N)),$$

and

$$H^i_I(M,N) \cong H^i(\Gamma_I(\operatorname{Hom}_R(M,E^\bullet))) \cong H^i(\operatorname{Hom}_R(M,\Gamma_I(E^\bullet))).$$

□

Theorem 2.2. *Let R be a Noetherian ring and I be an ideal of R. Suppose that t is an integer. Let M and N be I-cofinite modules. Then the following conditions are equivalent.*

1. *The R-module $H^i_I(M,N)$ is finitely generated for each $i < t$.*
2. *The $R_{\mathfrak{p}}$-module $H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is finitely generated for each $i < t$ and $\mathfrak{p} \in \operatorname{Spec}(R)$.*
3. *$I \subseteq \operatorname{Rad}(0 :_R H^i_I(M,N))$ for each $i < t$.*

Proof. (i)$\Longrightarrow$(ii) Since for each $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$(H^i_I(M,N))_{\mathfrak{p}} \cong H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}),$$

so, the assertion holds.
(ii)$\Longrightarrow$(i) We use induction on t. If $= 1$, then we have the exact squence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0,$$

then we have

$$0 \longrightarrow \operatorname{Hom}_R(R/I, \Gamma_I(N)) \longrightarrow \operatorname{Hom}_R(R/I, N) \longrightarrow \operatorname{Hom}_R(R/I, N/\Gamma_I(N)).$$

Since

$$\begin{aligned}\operatorname{Hom}_R(R/I, N/\Gamma_I(N)) &= (0 :_{N/\Gamma_I(N)} I)\\ &\subseteq (0 :_{N/\Gamma_I(N)} I^n)\\ &= \Gamma_I(N/\Gamma_I(N)) = 0\end{aligned}$$

it follows that $\operatorname{Hom}_R(R/I, N/\Gamma_I(N)) = 0$. Therefore $\operatorname{Hom}_R(R/I, \Gamma_I(N)) \cong \operatorname{Hom}_R(R/I, N)$. Since N is I-cofinite it follows that $\operatorname{Hom}_R(R/I, N)$ is finitely generated. Hence $\operatorname{Hom}_R(R/I, \Gamma_I(N))$ is finitely generated so $\Gamma_I(N)$ is finitely generated and $H^0_I(M,N) \cong \operatorname{Hom}_R(N, \Gamma_I(N))$. Then $H^0_I(M,N)$ is finitely generated.
Now suppose inductively that $t > 1$ and the assertion holds for $t - 1$. The exact sequence

$$0 \longrightarrow \Gamma_I(N) \longrightarrow N \longrightarrow N/\Gamma_I(N) \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Ext}^i_R(R/I,N) \longrightarrow \mathrm{Ext}^i_R(R/I,N/\Gamma_I(N)) \longrightarrow \mathrm{Ext}^{i+1}_R(R/I,\Gamma_I(N)) \qquad (1)$$

Since $\Gamma_I(N) \leq N$, it follows that $\Gamma_I(N)$ is finitely generated. Hence by according the exact sequence (1), we can deduce that $\mathrm{Ext}^i_R(R/I,N/\Gamma_I(N))$ is finitely generated. Since

$$H^t_I(M,N) \cong H^t_I(M,N/\Gamma_I(N)).$$

Then $\Gamma_I(N) = 0$. Now, let $E_R(N)$ denote the injective hull of N. Since $\Gamma_I(N) = 0$, it follows that $E_R(\Gamma_I(N)) = 0$. We have the exact sequence

$$0 \longrightarrow N \longrightarrow E_R(N) \longrightarrow E_R(N)/N \longrightarrow 0.$$

Hence

$$H^{i+1}_I(M,N) \cong H^i_I(M,E_R(N)/N), \qquad (2)$$

for each $i > t$.
Also for each $i > t$, $\mathrm{Hom}_R(R/I,E_R(N)) = 0$. Hence $\mathrm{Ext}^i_R(R/I,E_R(N)/N) \cong \mathrm{Ext}^{i+1}_R(R/I,N)$ then $\mathrm{Ext}^i_R(R/I,E_R(N)/N)$ is finitely generated. Also $R_\mathfrak{p}$-module $(H^i_I(M,E_R(N)/N))_\mathfrak{p}$ is finitely generated then by inductive hypothesise, $H^i_I(M,E_R(N)/N)$ is finitely generated for each $i < t-1$. Consequently by (2), $H^i_I(M,N)$ is finitely generated for each $i < t$.
(i)$\Longrightarrow$(iii) Since $H^i_I(M,N)$ is finitely generated for each $i < t$, it follows that there is $n \in \mathbb{N}$ such that $I^n H^i_I(M,N) = 0$ which implies $I \subseteq \mathrm{Rad}(0 :_R H^i_I(M,N))$.
(iii)$\Longrightarrow$(i) We use induction on t. If $t = 1$, then $I \subseteq \mathrm{Rad}(0 :_R H^0_I(M,N))$ so there exists $n \in \mathbb{N}$ such that $I^n H^0_I(M,N) = 0$. Therefore

$$\begin{aligned}\mathrm{Hom}_R(R/I^n,H^0_I(M,N)) &= 0 :_{H^0_I(M,N)} I^n \\ &= H^0_I(M,N) \\ &= \mathrm{Hom}_R(M,\Gamma_I(N)).\end{aligned}$$

Since N is I-cofinite module, it follows that $\mathrm{Ext}^i_R(R/I,N)$ is finitely generated for each $i \geq 0$. Hence $\mathrm{Hom}_R(R/I,N)$ is finitely generated then N is finitely generated. Since $\Gamma_I(N) \subseteq N$, it follows that $\Gamma_I(N)$ is finitely geerated then $H^0_I(M,N) = \mathrm{Hom}_R(M,\Gamma_I(N))$ is finitely generated. Now, Assume that $t > 1$ and the assertion holds for $t-1$. By inductive hypothesis, $H^0_I(M,N)$, $H^1_I(M,N)$, ... and $H^{t-2}_I(M,N)$ is finitely generated. It is enough to prove $H^{t-1}_I(M,N)$ is finitely generated. Since $H^{t-2}_I(M,N)$ is finitely generted, it follows by [1, Theorem 1.2], that $\mathrm{Hom}_R(R/I,H^{t-1}_I(M,N))$ is finitely generated. Therefore

$$\mathrm{Hom}_R(R/I,H^{t-1}_I(M,N)) = 0 :_{H^{t-1}_I(M,N)} I^n = H^{t-1}_I(M,N).$$

Hence $H^{t-1}_I(M,N)$ is finitely generated. □

Corollary 2.3. *Let R be a Noetherian ring and I be an ideal of R. Let M and N be I-cofinite modules. Then*

$$f_I(M,N) = \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}\, R\}.$$

Corollary 2.4. *Let R be a Noetherian ring and I be an ideal of R. Let M and N be I-cofinite modules. Then*

$$f_I(M,N) = \inf\{i \in \mathbb{N}_0 : I \nsubseteq \mathrm{Rad}(0 :_R f_I(M,N))\}.$$

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IA-Commutator Series

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Abstract

In this paper, we first define a new series and two automorphisms on this series. Then we identify the relationships of the members of these series. Finally, we study the relationships of these two new automorphisms with IA(G), $Aut_c(G)$, Inn(G), and each other.

1. Introduction

The various series have many applications in algebra. In particular, they are necessary for important definitions such as nilpotency and solubility of groups. On the other hand, All kinds of automorphisms also have interesting properties. Hence, automorphisms have been the idea of many researchers articles.

Let G be a group and j be any positive integer. Let us denote by G', Z(G), Aut(G) and Inn(G), respectively the commutator subgroup, the centre, the full automorphism group and the inner automorphisms. Also,

$$Aut_c(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) = [g,\alpha] \in Z(G),\ \forall\, g \in G\right\}.$$

is the central automorphisms group.

Bachmuth [1] in 1965 defined an IA-automorphisms group as

$$IA(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in G',\ \forall\, g \in G\right\}.$$

For any group G, $Inn(G) \leqslant IA(G) \trianglelefteq Aut(G)$.

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Hegarty [5] in 1994 introduced the autocommutator subgroup as follows:

$$G^* = \langle [g, \alpha] \mid g \in G,\ \alpha \in Aut(G) \rangle.$$

Thereafter, the researchers named it K(G).
On the similar lines, Ghumde and Ghate [4] in 2015 introduced the subgroup

$$G^{**} = \langle [g, \alpha] \mid g \in G,\ \alpha \in IA(G) \rangle.$$

For any group G, $G' = G^{**} \leqslant K(G)$.

Bonanome et al. [3] in 2011 have studied the IA-group of a group G for which the upper central series stalls at some point. We [2] defined new automorphisms on the lower central series and the derived series and identified the relationships of these automorphisms with IA(G), $Aut_c(G)$, Inn(G), and each other. Also, according to the definitions of Bonanome et al. [3] on the upper central series, we present some results that generalize their work.

In this paper, according to Ghumde and Ghate [4] definition, we define a new series and study the automorphisms of a group G for which this series stalls at some point.

2. Main results

In this section, after some new definitions, we give our main results about the automorphisms on the IA-commutator series.

Definition 2.1. we define the IA-commutator series of G in the following way:

$$\cdots \subseteq G_n^{**} \subseteq \cdots \subseteq G_2^{**} \subseteq G_1^{**} = G^{**} = G' \subseteq G_0^{**} = G$$

where

$$\begin{aligned} G_n^{**} =& \langle [g, \alpha_1, \dots, \alpha_n] \mid g \in G,\ \alpha_1, \dots, \alpha_n \in IA(G) \rangle \\ =& [G_{n-1}^{**}, IA(G)]. \end{aligned}$$

Definition 2.2. A group G is called an G_j^{**}-group if the IA-commutator series stalls at some point. This means that there exists a least positive integer j for which $G_j^{**} = G_{j+1}^{**} = \cdots$.

For example, if G be an abelian group, then it is a G_1^{**}-group.

Definition 2.3. The kernel of the natural homomorphism from Aut(G) to $Aut(G/G_j^{**})$ is called the group of G_j^{**}-automorphism and denoted by $Aut_{G_j^{**}}$.

According to the above definition, A G_j^{**}-automorphism group acts as the identity on G modulo G_j^{**}, Thus:

$$Aut_{G_j^{**}}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in G_j^{**},\ \forall\ g \in G\} \trianglelefteq Aut(G).$$

Also, we have $Aut_{G^{**}}(G) = IA(G)$ and $Aut_{G_j^{**}}(G) \leqslant IA(G)$ for every $j \geq 2$.

Remark 2.4. *We use the notation $Aut_{AG_j^{**}}(G) = Aut_c(G) \cap Aut_{G_j^{**}}(G)$. Another definition of $Aut_{AG_j^{**}}(G)$ is given by*

$$Aut_{AG_j^{**}}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in Z(G) \cap G_j^{**},\ \forall\ g \in G\}.$$

Proposition 2.5. *For any group G,*

a) $\varphi \in Aut_{G_j^{**}}(G)$ *if and only if* $[\alpha, \varphi] \in Aut_{G_j^{**}}(G)$, *for every* $\alpha \in Aut(G)$.

b) $Aut_{AG_j^{**}}(G)$ *is a normal subgroup of* $Aut_c(G)$ *and we have* $\dfrac{Aut_c(G)}{Aut_{AG_j^{**}}(G)} \cong \dfrac{Aut_c(G)Aut_{G_j^{**}}(G)}{Aut_{G_j^{**}}(G)}$.

Proof. a) It is obvious by the normality of $Aut_{G_j^{**}}(G)$.

b) First, we prove that $G_j^{**} \overset{ch}{\leqslant} G$. Because G_j^{**} is a generated subgroup, so it is obvious that $G_j^{**} \leqslant G$. Let $\psi \in Aut(G)$ and $[g, \alpha_1, \dots, \alpha_j] \in G_j^{**}$. Then, one can write

$$\begin{aligned}
\psi([g, \alpha_1, \dots, \alpha_j]) &= \psi\big(g^{-1}\alpha_1 \cdots \alpha_j(g)\big) \\
&= \psi(g)^{-1}\psi\big(\alpha_1 \cdots \alpha_j(g)\big) \\
&= \big(\psi(g)\big)^{-1}\psi\alpha_1 \cdots \alpha_j\big(\psi^{-1}\psi(g)\big) \\
&= \big(\psi(g)\big)^{-1}(\psi\alpha_1 \cdots \alpha_j\psi^{-1})\big(\psi(g)\big) \\
&= [\underbrace{\psi(g)}_{\in G}, \underbrace{\psi\alpha_1 \cdots \alpha_j\psi^{-1}}_{\in IA(G)}] \in G_j^{**}.
\end{aligned}$$

Therefore, $\psi([g, \alpha_1, \dots, \alpha_j]) \in G_j^{**}$.

Now, let $\sigma \in Aut_c(G)$ and $\beta \in Aut_{AG_j^{**}}(G)$. We show that $\sigma^{-1}\beta\sigma \in Aut_{AG_j^{**}}(G)$. For every $g \in G$, we have

$$\begin{aligned}
g^{-1}\big(\sigma^{-1}\beta\sigma\big)(g) &= g^{-1}\sigma^{-1}\Big(\sigma(g)\big(\sigma(g)\big)^{-1}\beta\big(\sigma(g)\big)\Big) \\
&= g^{-1}g\sigma^{-1}\Big(\underbrace{\big(\sigma(g)\big)^{-1}\beta\big(\sigma(g)\big)}_{\in\, Z(G)\cap G_j^{**}}\Big).
\end{aligned}$$

Because the intersection of two characteristic subgroups is a characteristic subgroup, the first part is proved.

For the second part, the result follows from the definition of $Aut_{AG_j^{**}}(G)$ and the third isomorphism theorem. □

Corollary 2.6. *For any group G,* $[Aut(G), Aut_{G_j^{**}}(G)] \leqslant Aut_{G_j^{**}}(G)$.

Theorem 2.7. *Let G be a group. If* $Aut_c\big(G/G_j^{**}\big) = Inn\big(G/G_j^{**}\big)$*, then*

$$Aut_c(G) \leqslant Inn(G)Aut_{G_j^{**}}(G).$$

Proof. Let $\alpha \in Aut_c(G)$. By hypothesis, $Aut_c\big(G/G_j^{**}\big) = Inn\big(G/G_j^{**}\big)$, so there exists $g \in G$ such that for all $x \in G$,

$$\alpha(x)G_j^{**} = x^g G_j^{**}.$$

Hence,

$$\begin{aligned}
& x^{-g}\alpha(x) = \Big(x^{-1}\big(\alpha(x)\big)^{g^{-1}}\Big)^g \in G_j^{**} \\
\Longrightarrow\quad & x^{-1}\big(\alpha(x)\big)^{g^{-1}} \in G_j^{**} \\
\Longrightarrow\quad & x^{-1}g\big(\alpha(x)\big)g^{-1} \in G_j^{**} \\
\Longrightarrow\quad & x^{-1}\varphi_g^{-1}\alpha(x) \in G_j^{**}
\end{aligned}$$

where $\varphi_g \in Inn(G)$.

Consequently, $\varphi_g^{-1}\alpha \in Aut_{G_j^{**}}(G)$, i.e., $\alpha = \varphi_g\varphi_g^{-1}\alpha \in Inn(G)Aut_{G_j^{**}}(G)$. □

In the special case j=1, we have the following result

Corollary 2.8. *Let G be a group. If* $Aut_c\big(G/G'\big) = Inn\big(G/G'\big)$*, then*

$$Aut_c(G) \leqslant Inn(G)IA(G).$$

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Some Properties of Formal Local Cohomology Modules

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Abstract

Let $\mathfrak{a}$ be an ideal of local ring $(R, \mathfrak{m})$ and M a finitely generated R-module and $n \in \mathbb{N}$. This note related to some criteria in cominimaxness of formal local cohomology modules.

1. Introduction

Throughout this note, R is a commutative Noetherian ring with identity(non-zero), $\mathfrak{a}$ is an ideal of R and M is an R-module. Let $V(\mathfrak{a})$ be the set of prime ideals in R containing $\mathfrak{a}$. For an integer i, let $H^i_{\mathfrak{a}}(M)$ denote the i-th local cohomology module of M. We have the isomorphism of $H^i_{\mathfrak{a}}(M)$ to $\varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M)$ for every $i \in \mathbb{N}_0$, see [3] for more details.

Consider the family of local cohomology modules $\{H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every n there is a natural homomorphism $H^i_{\mathfrak{m}}(M/\mathfrak{a}^{n+1}M) \to H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. The projective limit $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$. Formal local cohomology modules were used by Peskine and Szpiro in [9] when R is a regular ring in order to solve a conjecture of Hartshorne in prime characteristic.

The basic properties of formal local cohomology modules are found in [10], [2]. One of important problems concerning formal local cohomology modules are finiteness results (see, e.g., [6]). Not much has been proven on this subject. But studies are being done on this. In [2], Asgharzadeh and Divani-Aazar have investigated some properties of formal local cohomology modules. For instance they showed that $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian for $d := \dim M$.(See [5, pro. 2.1])

Recall that a module M is a minimax module if there is a finitely generated submodule N of M such that the quotient module M/N is Artinian. An R-module M is an $\mathfrak{a}$-cominimax module if $Supp_R(M) \subseteq V(\mathfrak{a})$ and $Ext^i_R(R/\mathfrak{a}, M)$ is a minimax module for all $i \geq 0$. The class of cominimax modules includes all cofinite and all Artinian modules.

The notions of weakly Laskerian modules were introduced by Divaani-Aazar and Mafi in [4]. An R module M is said to be weakly Laskerian if the set of associated primes of any quotient module of M is finite. Moreover it is closed

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under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of R-modules.
In this paper we investigate some cominimaxness properties of formal local cohomology modules.

2. Cominimaxness of formal local cohomology modules

We begin with an example show that the class of cofinite modules with respect to an ideal is strictly contained in the class of cominimax modules with respect to the same ideal.

Example 2.1. *(see [1]) Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{p}$ a prime ideal of R such that $dim R/\mathfrak{p} = 1$. Then it is easy to see that the R-module $E(R/\mathfrak{p})$ is $\mathfrak{p}$-cominimax but not $\mathfrak{p}$-cofinite.*

The following lemma is used in the sequel.

Lemma 2.2. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M an minimax R-module such that $Supp_R(M) \subseteq V(\mathfrak{a})$. Then the following statements are equivalent:*

(a) *M is $\mathfrak{a}$-cominimax.*

(b) *The R-module $Hom_R(R/\mathfrak{a}, M)$ is minimax.*

Proof. We know by definitions that (b) follows from (a). Let N be a finite submodule of M such that M/N is Artinian and suppose the R-module $Hom_R(R/\mathfrak{a}, M)$ is minimax. The exactness of

$$0 \to Hom_R(R/\mathfrak{a}, N) \to Hom_R(R/\mathfrak{a}, M) \to Hom_R(R/\mathfrak{a}, M/N) \to Ext^1_R(R/\mathfrak{a}, N)$$

implies that $Hom_R(R/\mathfrak{a}, M/N)$ is minimax.
Since M/N is Artinian, it is easy to see that $Hom_R(R/\mathfrak{a}, M/N)$ is an Artinian R-module. As M/N is $\mathfrak{a}$-torsion, it follows by Melkersson's theorem that M/N is Artinian. Thus M is minimax. The $\mathfrak{a}$-torsionness of M imples that it is $\mathfrak{a}$-cominimax. □

Theorem 2.3. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M an R-module such that $dim M \leq 1$ and $Supp M \subseteq V(\mathfrak{a})$. Then the following statements are equivalent:*

(a) *M is $\mathfrak{a}$-cominimax.*

(b) *The R-modules $Hom_R(R/\mathfrak{a}, M)$ and $Ext^1_R(R/\mathfrak{a}, M)$ are minimax.*

Proof. The conclusion (b) follows from (a) is obvious. In order to prove $(b) \Rightarrow (a)$ using lemma 2.2, we may assume $dim M = 1$. Now use Lemma 2.2 instead of [8], Lemma 2.1, and the $\mathfrak{a}$ -cominimaxness instead of $\mathfrak{a}$-cofiniteness in the proof of [8], Theorem 2.3. □

Theorem 2.4. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M an weakly Laskerian R-module such that $Supp M \subseteq V(\mathfrak{a})$. Then the following statements are equivalent:*

(a) *M is $\mathfrak{a}$-cominimax.*

(b) *The R-modules $Hom_R(R/\mathfrak{a}, M)$ and $Ext^1_R(R/\mathfrak{a}, M)$ are minimax.*

Proof. The conclusion (b) follows from (a) is obvious. In order to prove (a) follows from (b), by definition there is a finitely generated submodule N of M such that $dim(M/N) \leq 1$ and $Supp M/N \subseteq V(\mathfrak{a})$. Also, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0(\star)$$

induces the exact sequence

$$0 \longrightarrow Hom_R(R/\mathfrak{a}, N) \longrightarrow Hom_R(R/\mathfrak{a}, M) \longrightarrow Hom_R(R/\mathfrak{a}, M/N) \longrightarrow Ext^1_R(R/\mathfrak{a}, N)$$
$$\longrightarrow Ext^1_R(R/\mathfrak{a}, M) \longrightarrow Ext^1_R(R/\mathfrak{a}, M/N) \longrightarrow Ext^2_R(R/\mathfrak{a}, N)$$

Hence, it follows that the R-modules $Hom_R(R/\mathfrak{a}, M/N)$ and $Ext^1_R(R/\mathfrak{a}, M/N)$ are finitely generated. Therefore, in view of lemma 2.2, the R-module M/N is $\mathfrak{a}$-cominimax. Now it follows from the exact sequence $(\star)$ that M is $\mathfrak{a}$-cominimax. □

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a is nonzero finitely generated R-module. Let $t \in \mathbf{N}_0$. Suppose that the R-module $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$, and the R-modules $Ext^t_R(R/\mathfrak{a}, M)$ and $Ext^{t+1}_R(R/\mathfrak{a}, M)$ are minimax. Then the R-modules $Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)$ and $Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)$ are minimax.*

Proof. We use induction on t.The exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma\mathfrak{a}(M) \longrightarrow 0 \qquad (\star)$$

induces the exact sequence:

$$0 \longrightarrow Hom_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow Hom_R(R/\mathfrak{a}, M) \longrightarrow Hom_R(R/\mathfrak{a}, M/\Gamma\mathfrak{a}(M))$$
$$\longrightarrow Ext^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow Ext^1_R(R/\mathfrak{a}, M)$$

Since $Hom_R(R/\mathfrak{a}, M/\Gamma\mathfrak{a}(M)) = 0$ so $Hom_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ and $Ext^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ are minimax. Assume inductively that $t > 0$ and that we have established the result for nonnegative integers smaller than t. By applying the functor $Hom_R(R/\mathfrak{a}, -)$ to the exact sequence $(\star)$, we can deduce that $Ext^j_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$ is minimax for $j = t, t+1$. On the other hand, $\mathfrak{F}^0_{\mathfrak{a}}(M/\Gamma\mathfrak{a}(M)) = 0$ and $\mathfrak{F}^j_{\mathfrak{a}}(M/\Gamma\mathfrak{a}(M)) \simeq \mathfrak{F}^j_{\mathfrak{a}}(M)$ for all $j > 0$. Therefore we may assume that $\Gamma\mathfrak{a}(M) = 0$. Let E be an injective hull of M and put $N = E/M$. Then $Hom_R(R/\mathfrak{a}, E) = 0 = \Gamma\mathfrak{a}(E)$. Hence $Ext^j_R(R/\mathfrak{a}, N) \simeq Ext^{j+1}_R(R/\mathfrak{a}, M)$ and $\mathfrak{F}^j_{\mathfrak{a}}(N) \simeq \mathfrak{F}^{j+1}_{\mathfrak{a}}(M)$ for all $j \geq 0$. Now, the induction hypothsis yields that $Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ and $Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ are minimax, as required. □

We are now ready to state and prove the main tteorem.

Theorem 2.6. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module, and $t \in \mathbf{N}_0$ such that $Ext^i_R(R/\mathfrak{a}, M)$ are minimax for all $i \leq t+1$. Let the R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ be weakly laskerian R-modules for all $i < t$. Then the following assertions hold:*

(a) *The R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax for all $i < t$.*

(b) *For all minimax submodules N of $\mathfrak{F}^i_{\mathfrak{a}}(M)$, the R-modules*

$$Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \quad , \quad Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N)$$

are minimax. In particular, the set $Ass_R(\mathfrak{F}^t_{\mathfrak{a}}(M)/N)$ is finite.

Proof.

(a) We proceed by induction on t. In the case $t = 0$ there is nothing to prove. So, let $t > 0$ and suppose the result has been proved for smaller values of t. By the inductive assumption, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for $i = 0, 1, \cdots, t-2$. Hence by theorem 2.5 and the assumption, $Hom_R(R/\mathfrak{a}, \mathfrak{F}^{t-1}_{\mathfrak{a}}(M))$ and $Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^{t-1}_{\mathfrak{a}}(M))$ are minimax. Therefore by Theorem 2.4, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i < t$. This completes the inductive step.

(b) In view of (a) and theorem 2.5, $Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ and $Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M))$ are minimax. On the other hand, N is $\mathfrak{a}$-cominimax. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^t_{\mathfrak{a}}(M)/N \longrightarrow 0$$

induces the exact sequence:

$$\longrightarrow Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)) \longrightarrow Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \longrightarrow Ext^1_R(R/\mathfrak{a}, N)$$
$$\longrightarrow Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)) \longrightarrow Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \longrightarrow Ext^2_R(R/\mathfrak{a}, N)$$

Consequently,

$$Hom_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N) \quad , \quad Ext^1_R(R/\mathfrak{a}, \mathfrak{F}^t_{\mathfrak{a}}(M)/N)$$

are minimax, as required. □

Corollary 2.7. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module such that $Ext^i_R(R/\mathfrak{a}, M)$ are minimax for all i and the R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ are weakly laskerian R-modules for all i. Then:*

(a) *The R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax for all i.*

(b) *For any $i \geq 0$ and for any minimax submodule N of $\mathfrak{F}^i_{\mathfrak{a}}(M)$, the R-module $\mathfrak{F}^i_{\mathfrak{a}}(M)/N$ is $\mathfrak{a}$-cominimax.*

Proof. (a) Clear. (b) In view of (a) the R-module $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all i. Hence the R-module $Hom_R(R/\mathfrak{a}, N)$ is minimax, and so it follows that N is $\mathfrak{a}$-cominimax. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M)/N \longrightarrow 0$$

imply that the R-module $\mathfrak{F}^i_{\mathfrak{a}}(M)/N$ is $\mathfrak{a}$-cominimax. □

Corollary 2.8. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module such that the R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ are weakly laskerian R-modules for all i. Then the following conditions are equivalent:*

(a) *The R-modules $Ext^i_R(R/\mathfrak{a}, M)$ are minimax for all i.*

(b) *The R-modules $\mathfrak{F}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cominimax for all i.*

Proof. (a) $\Rightarrow$ (b) follows by Corollary 2.7.
(a) $\Rightarrow$ (b) follows by [7], Proposition 3.9. □

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n-Isoclinism of regular Hom-Lie algebras

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Abstract

In this paper, we introduce the concept of n-isoclinism between two regular Hom-Lie algebras, and obtain some equivalent conditions under which Hom-Lie algebras are n-isoclinic. As a main result, we prove that two n-isoclinic regular Hom-Lie algebras can be isoclinically embedded into one Hom-Lie algebra.

1. Introduction

The notion of isoclinism was introduced by P. Hall in 1940 [1], which is an equivalence relation on the class of all groups. In 1994, Moneyhun [3] showed that isoclinism and isomorphism between Lie algebras with the same finite dimension are equivalent. The notion of n-isoclinism and characterizing n-isoclinism classes of Lie algebras were given by Salemkar in [5]. In [2], Hartwig, Larsson and Silvestrov introduced the notion of Hom-Lie algebras and in [4] isoclinism of regular Hom-Lie algebras was generalized.
In this article, we define n-isoclinism of Hom-Lie algebras and investigate some results on n-isoclinism of Lie algebras which can be extended to n-isoclinism of Hom-Lie algebras.

1.1. Basic definition

Throughout this paper we fix F as a ground field and all the vector spaces are considered over F and linear maps are F-linear maps. We begin by reviewing some basic concepts and recalling known facts which will be used in the article.

Definition 1.1. A *Hom-Lie algebra* is a triple $(V, [-,-], \varphi)$ consisting a vector space V, a bilinear map $[-,-]$: $V \times V \longrightarrow V$ and linear map $\varphi : V \longrightarrow V$ provided

$$(i) \quad [x, y] = -[y, x], \quad (skew - symmetry)$$
$$(ii) \quad [\varphi(x), [y, z]] + [\varphi(y), [z, x]] + [\varphi(z), [x, y]] = 0, \quad (Hom - Jacobi\ identity)$$

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for all $x, y, z \in V$.
A *Hom-Lie subalgebra* of (V, φ) is a vector subspace W of V, which is closed by bracket and φ, i.e. $[w, w'], \varphi(w) \in W$ for all $w, w' \in W$. A Hom-Lie subalgebra $(W, \varphi_|)$ is said to be an *ideal* if $[w, v] \in W$ for all $w \in W$, $v \in V$ in which $\varphi_|$ is the restriction of φ to W. For any ideal W of (V, φ), we can naturally define the quotient Hom-Lie algebra on the quotient vector space V/W with $\widetilde{\varphi}$: $V/W \longrightarrow V/W$ which induced naturally by φ.
In the whole paper, we assume that φ preserves the product which is called *multiplicative*, i.e. $\varphi([v_1, v_2]) = [\varphi(v_1), \varphi(v_2)]$ for all $v_1, v_2 \in V$. Taking $\varphi = id_V$, we exactly recover the Lie algebras. A vector space endowed with trivial bracket and any linear map is called an *abelian* Hom-Lie algebra. The *center* of (V, φ) is the vector space $Z(V) = \{x \in V : [x, v] = 0, \ \forall\ v \in V\}$. The upper central series of V defined inductively by $Z_0(V) = 0$ and $Z_{n+1}(V)/Z_n(V)$ is the center of $V/Z_n(V)$, for $n \geqslant 0$. A multiplicative Hom-Lie algebra (V, φ) is said to be *regular* if φ is bijective. It must be noted that $Z(V)$ is not always an ideal of (V, φ). When (V, φ) is regular, then $Z(V)$ will be an ideal. The nth term of the lower central series of V, is denoted by V^n and defined inductively by $V^1 = V$ and $V^{n+1} = [V^n, V]$, for $n \geqslant 1$.
Let (V, φ_1) and (W, φ_2) be two Hom-Lie algebras. A linear map $f : V \longrightarrow W$ is a *Hom-Lie algebra morphism*, if for all $v_1, v_2 \in V$, $f([v_1, v_2]) = [f(v_1), f(v_2)]$ and $f \circ \varphi_1 = \varphi_2 \circ f$. In other words, the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ V & \xrightarrow[f]{} & W \end{array}$$

Definition 1.2. Let (V, φ_1) and (W, φ_2) be two regular Hom-Lie algebras and $\alpha : V/Z_n(V) \longrightarrow W/Z_n(W)$ and $\beta : V^{n+1} \longrightarrow W^{n+1}$ be two Hom-Lie algebra morphisms such that the following diagram commutes

$$\begin{array}{ccc} \dfrac{V}{Z_n(V)} \oplus \cdots \oplus \dfrac{V}{Z_n(V)} & \longrightarrow & V^{n+1} \\ \alpha^{n+1} \downarrow & & \downarrow \beta \\ \dfrac{W}{Z_n(W)} \oplus \cdots \oplus \dfrac{W}{Z_n(W)} & \longrightarrow & W^{n+1}, \end{array}$$

where horizontal maps are defined by $(\overline{v}_1, \ldots, \overline{v}_{n+1}) \longmapsto [v_1, \ldots, v_{n+1}]$, for all $v_1, \ldots, v_{n+1} \in V$. In other words, $\beta([v_1, \ldots, v_{n+1}]) = [w_1, \ldots, w_{n+1}]$, whenever $v_i \in V$ and $w_i \in \alpha(v_i + Z_n(V))$ for $i = 1, \ldots, n+1$. Then the pair (α, β) is called *n-homoclinism* and if they are both isomorphisms, then (α, β) is *n-isoclinism* and in this case, we write $V \overset{n}{\sim} W$.

Note that for $n = 0$, the above definition yields isomorphism of Hom-Lie algebras, and for $n = 1$ it gives the isoclinism of Hom-Lie algebras.

2. Main Results

In this section, we present some equivalent conditions under which Hom-Lie algebras are n-isoclinic. Finally, we show that two n-isoclinic regular Hom-Lie algebras can be isoclinically embedded into a Hom-Lie algebra.

Lemma 2.1. *Let (V, φ) be a regular Hom-Lie algebra with a Hom-Lie subalgebra H and a Hom-ideal W. Then*

(i) $V \overset{n}{\sim} V \oplus W$, *for each nilpotent regular Hom-Lie algebra W of class at most n, ($n \in \mathbb{N}$).*

(ii) $H \overset{n}{\sim} H + Z_n(V)$. *In particular, if $V = H + Z_n(V)$, then $V \overset{n}{\sim} H$.*

(iii) $\dfrac{V}{W} \overset{n}{\sim} \dfrac{V}{W \cap V^{n+1}}$. *In particular, if $W \cap V^{n+1} = 0$, then $\dfrac{V}{W} \overset{n}{\sim} V$.*

Proof. (i) Since W is nilpotent of class at most n, we have $W^{n+1} = 0$ and $Z_n(W) = W$, hence $Z_n(V \oplus W) = Z_n(V) \oplus W$ and $(V \oplus W)^{n+1} \cong V^{n+1} \oplus W^{n+1} = V^{n+1} \oplus 0$. Now we define $\alpha : \dfrac{V}{Z_n(V)} \longrightarrow \dfrac{V \oplus W}{Z_n(V \oplus W)}$ such that $\alpha(v + Z_n(V)) = (v, 0) + Z_n(V \oplus W)$ and $\beta : V^{n+1} \longrightarrow (V \oplus W)^{n+1}$ by $\beta([v_1, \cdots, v_{n+1}]) = ([v_1, \cdots, v_{n+1}], 0)$. Trivially, α and β are desired isomorphisms, i.e. $\alpha([\overline{v_1}, \overline{v_2}]) = [\alpha(\overline{v_1}), \alpha(\overline{v_2})]$ and $\alpha \circ \varphi_1 = \varphi_2 \circ \alpha$ in the following diagram

$$\begin{array}{ccc} \dfrac{V}{Z_n(V)} & \xrightarrow{\alpha} & \dfrac{V \oplus W}{Z_n(V \oplus W)} \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ \dfrac{V}{Z_n(V)} & \xrightarrow[\alpha]{} & \dfrac{V \oplus W}{Z_n(V \oplus W)} \end{array} \qquad\qquad \begin{array}{ccc} \hat{v} & \xrightarrow{\alpha} & \widetilde{(v,0)} \\ \varphi_1 \downarrow & & \downarrow \varphi_2 \\ \widehat{\varphi(v)} & \xrightarrow{\alpha} & (\widetilde{\varphi(v), 0}) \end{array}$$

such that $\varphi_1(\hat{v}) = \widehat{\varphi(v)}$ and $\varphi_2(\widetilde{(v,w)}) = (\widetilde{\varphi(v), \varphi(w)})$, for each $v \in V, w \in W$. The same relations for β are obtained.

(ii) We know that $Z_n(H + Z_n(V)) = Z_n(H) + Z_n(V)$. Now, define

$$\begin{aligned} \alpha : \frac{H}{Z_n(H)} &\longrightarrow \frac{H + Z_n(V)}{Z_n(H + Z_n(V))} \\ h + Z_n(H) &\longmapsto h + Z_n(H + Z_n(V)) \end{aligned}$$

and $\beta : H^{n+1} \longrightarrow (H + Z_n(V))^{n+1} \cong H^{n+1}$, identically. One can easily check that α and β are both Hom-Lie algebra isomorphisms and we have the following commutative diagrams

$$\begin{array}{ccc} \dfrac{H}{Z_n(H)} & \xrightarrow{\alpha} & \dfrac{H + Z_n(V)}{Z_n(H + Z_n(V))} \\ \overline{\varphi}_1 \downarrow & & \downarrow \overline{\varphi}_2 \\ \dfrac{H}{Z_n(H)} & \xrightarrow[\alpha]{} & \dfrac{H + Z_n(V)}{Z_n(H + Z_n(V))} \end{array} \qquad \begin{array}{ccc} h + Z_n(H) & \xrightarrow{\alpha} & h + Z_n(H + Z_n(V)) \\ \overline{\varphi}_1 \downarrow & & \downarrow \overline{\varphi}_2 \\ \varphi(h) + Z_n(H) & \xrightarrow{\alpha} & \varphi(h) + Z_n(H + Z_n(V)) \end{array}$$

where $\overline{\varphi}_1(h + Z_n(H)) = \varphi(h) + Z_n(H)$ and $\overline{\varphi_2}(h + Z_n(H + Z_n(V))) = \varphi(h) + Z_n(H + Z_n(V))$, for each $h \in H$.

(iii) Let $\overline{V} = \dfrac{V}{W}$ and $\widetilde{V} = \dfrac{V}{W \cap V^{n+1}}$. Then $\overline{v}_0 \in Z_n(\overline{V})$ if and only if $\widetilde{v_0} \in Z_n(\widetilde{V})$. So the following maps are the required n-isoclinism pairs

$$\begin{aligned} \alpha : \frac{\overline{V}}{Z_n(\overline{V})} &\longrightarrow \frac{\widetilde{V}}{Z_n(\widetilde{V})} \\ \overline{v} + Z_n(\overline{V}) &\longmapsto \widetilde{v} + Z_n(\widetilde{V}) \end{aligned} \qquad\qquad \begin{aligned} \beta : \overline{V}^{n+1} &\longrightarrow \widetilde{V}^{n+1} \\ [\overline{v}_1, \ldots, \overline{v}_{n+1}] &\longmapsto [\widetilde{v}_1, \ldots, \widetilde{v}_{n+1}] \end{aligned}$$

and the following diagrams commute

$$\begin{array}{ccc} \dfrac{\overline{V}}{Z_n(\overline{V})} & \xrightarrow{\alpha} & \dfrac{\widetilde{V}}{Z_n(\widetilde{V})} \\ \overline{\varphi}_1 \downarrow & & \downarrow \widetilde{\varphi}_2 \\ \dfrac{\overline{V}}{Z_n(\overline{V})} & \xrightarrow[\alpha]{} & \dfrac{\widetilde{V}}{Z_n(\widetilde{V})} \end{array} \qquad\qquad \begin{array}{ccc} \overline{V}^{n+1} & \xrightarrow{\beta} & \widetilde{V}^{n+1} \\ \hat{\varphi}_1 \downarrow & & \downarrow \varphi_2^* \\ \overline{V}^{n+1} & \xrightarrow[\beta]{} & \widetilde{V}^{n+1} \end{array}$$

where $\overline{\varphi}_1(\overline{v} + Z_n(\overline{V})) = \varphi_1(\overline{v}) + Z_n(\overline{V}), \widetilde{\varphi}_2\ (\widetilde{v} + Z_n(\widetilde{V})) = \varphi_2(\widetilde{v}) + Z_n(\widetilde{V}), \hat{\varphi}_1([\overline{v}_1, \ldots, \overline{v}_{n+1}]) = [\varphi_1(\overline{v}_1), \ldots, \varphi_1(\overline{v}_{n+1})]$ and $\varphi_2^*([\widetilde{v}_1, \ldots, \widetilde{v}_{n+1}]) = [\varphi_2(\widetilde{v}_1), \ldots, \varphi_2(\widetilde{v}_{n+1})]$, for all $v, v_i \in V, 1 \leq i \leq n + 1$. □

Let (α, β) be an n-isoclinism pair between regular Hom-Lie algebras (V, φ_1) and (W, φ_2) and $T = \{(v, w) \in V \oplus W \mid \alpha(v + Z_n(V)) = w + Z_n(W)\}$, $Z_V = \{(v, 0) \mid v \in Z_n(V)\}$, $Z_W = \{(0, w) \mid w \in Z_n(W)\}$.
Clearly, T is a Hom-Lie subalgebra of $V \oplus W$ and Z_V, Z_W are Hom-ideals in T such that T^{n+1} is generated by the set $\{([v_1, \ldots, v_{n+1}], \beta([v_1, \ldots, v_{n+1}])) \mid v_i \in V, 1 \leq i \leq n+1\}$ and $Z_V \cap T^{n+1} = Z_W \cap T^{n+1} = 0$.

Proposition 2.2. *Let (α, β) be an n-isoclinism pair between regular Hom-Lie algebras (V, φ_1) and (W, φ_2). Then*

$$V \cong \frac{T}{Z_W} \overset{n}{\sim} \frac{T}{Z_W} \oplus \frac{T}{T^{n+1}} \overset{n}{\sim} T_W \cong T \cong T_V \overset{n}{\sim} \frac{T}{Z_V} \oplus \frac{T}{T^{n+1}} \overset{n}{\sim} \frac{T}{Z_V} \cong K$$

for some Hom-Lie subalgebras T_W of $\frac{T}{Z_W} \oplus \frac{T}{T^{n+1}}$ and T_V of $\frac{T}{Z_V} \oplus \frac{T}{T^{n+1}}$.

The next corollary indicates that an n-isoclinism between two regular Hom-Lie algebras yields some certain m-isoclinisms between their upper central factor Hom-Lie algebras and lower commutator subalgebras.

Corollary 2.3. *Let (V, φ_1) and (W, φ_2) be two n-isoclinic regular Hom-Lie algebras. Then*

(i) for all $0 \leq i \leq n$, $\frac{V}{Z_i(V)} \overset{n-i}{\sim} \frac{W}{Z_i(W)}$.

(ii) for all $0 \leq i \leq n$, $V^{i+1} \overset{n-i}{\sim} W^{i+1}$.

(iii) for all $m \geq n$, $V \overset{m}{\sim} W$.

To prove the final theorem, the following lemma is used.

Lemma 2.4. *Let (V, φ_1) and (W, φ_2) be two n-isoclinic regular Hom-Lie algebras. Then there exists a Hom-Lie algebra X containing Hom-Lie algebras X_1, X_2 and a nilpotent ideal Z_n of class at most n such that*

$$V \cong X_1 \overset{n}{\sim} X_1 \oplus Z_n = X = X_2 + Z_n(X) \overset{n}{\sim} X_2 \text{ and } X_2 \overset{n-1}{\sim} W.$$

The following theorem may be considered as a kind of dual to 2.2 which states that two n-isoclinic regular Hom-Lie algebras can be isoclinically embedded into a Hom-Lie algebra.

Theorem 2.5. *Let (V, φ_1) and (W, φ_2) be n-isoclinic regular Hom-Lie algebras. Then there exists a Hom-Lie algebra $\overline{T}$ containing subalgebras $\overline{T}_V, \overline{T}_W$ and a nilpotent ideal $\overline{N}$ of class at most n such that*

$$V \cong \overline{T}_V \overset{n}{\sim} \overline{T}_V \oplus \overline{N} = \overline{T} = \overline{T}_W + Z_n(\overline{T}) \overset{n}{\sim} \overline{T}_W \cong W.$$

Proof. Clearly the result holds for $n = 1$. So we assume that $n > 1$ and X, X_1, X_2, Z_n are the Hom-Lie algebras obtained in lemma 2.4 . Using induction on n, one may find a Hom-Lie algebra Y containing Hom-Lie subalgebras Y_1, Y_2 and some nilpotent ideal Z_{n-1} of class at most $n-1$ such that

$$K \cong Y_1 \overset{n-1}{\sim} Y_1 + Z_{n-1}(Y) = Y = Y_2 \oplus Z_{n-1} \overset{n-1}{\sim} Y_2 \cong X_2. \tag{1}$$

Let $\overline{T}$ be equal to the external direct sum of Hom-Lie algebras X_1, Z_n, Z_{n-1} and the maps $i_1 : X = X_1 \oplus Z_n \longrightarrow \overline{T}$ and $i_2 : Z_{n-1} \longrightarrow \overline{T}$ be the canonical Hom-Lie algebra monomorphisms.
Put $\overline{X}_1 = i_1(X_1), \overline{Z}_n = i_1(Z_n)$ and $\overline{Z}_{n-1} = i_2(Z_{n-1})$. Then it is easily seen that:

(i) $\overline{T} = \overline{X}_1 \oplus (\overline{Z}_n + \overline{Z}_{n-1})$, where $\overline{Z}_n + \overline{Z}_{n-1}$ is a nilpotent Hom-Lie algebra of class at most n.

(ii) $Z_n(\overline{T}) = Z_n(\overline{X}_1) + \overline{Z}_n + \overline{Z}_{n-1}$.

(iii) The composite map $Y_2 \xrightarrow{\tau} X_2 \xrightarrow{\subseteq} X \xrightarrow{i_1} \overline{T}$ is monomorphism, where τ is the isomorphism given in 1 .

Note that an element $y \in Y$ can be written uniquely as the form of $y = y_2 + z$ where $y_2 \in Y_2$ and $z \in Z_{n-1}$. The map $\lambda : Y \longrightarrow \overline{T}$, such that $\lambda(y) = i_1(\tau(y_2)) + i_2(z)$ is a monomorphism with $\lambda(Y) = i_1(\tau(Y_2)) + \overline{Z}_{n-1}$. By using 1, we have

$$\begin{aligned}\lambda(Y) &= \lambda(Y_1) + \lambda(Z_{n-1}(Y)) = \lambda(Y_1) + Z_{n-1}(\lambda(Y)) = \lambda(Y_1) + Z_{n-1}(i_1(X_2) + \overline{Z}_{n-1})\\ &= \lambda(Y_1) + Z_{n-1}(i_1(X_2)) + \overline{Z}_{n-1} \leqslant \lambda(Y_1) + Z_n(i_1(X_2))\end{aligned}$$

Now we prove that $\lambda(Y_1) + Z_n(\overline{T}) = \overline{T}$

$$\begin{aligned}\overline{T} &= \overline{X}_1 + \overline{Z}_n + \overline{Z}_{n-1} = i_1(\overline{X}_1) + i_1(\overline{Z}_n) + \overline{Z}_{n-1} = i_1(X) + \overline{Z}_{n-1}\\ &= i_1(X_2) + i_1(Z_n(X)) + \overline{Z}_{n-1} = i_1(X_2) + Z_n(i_1(X)) + \overline{Z}_{n-1}\\ &= i_1(\tau(Y_2)) + \overline{Z}_{n-1} + Z_n(i_1(X)) = \lambda(Y) + Z_n(\overline{X}_1 + \overline{Z}_n)\\ &\leq \lambda(Y_1) + Z_n(i_1(X_2)) + \overline{Z}_{n-1} + Z_n(\overline{X}_1) + \overline{Z}_n\\ &= \lambda(Y_1) + Z_n(i_1(X)) \cap i_1(X_2) + \overline{Z}_{n-1} + Z_n(\overline{X}_1) + \overline{Z}_n\\ &= \lambda(Y_1) + Z_n(\overline{X}_1 + \overline{Z}_n) \cap i_1(X_2) + \overline{Z}_{n-1} + Z_n(\overline{X}_1) + \overline{Z}_n\\ &\lambda(Y_1) + \overline{Z}_{n-1} + Z_n(\overline{X}_1) + \overline{Z}_n = \lambda(Y_1) + Z_n(\overline{T}) \leq \overline{T}.\end{aligned}$$

Consequently,

$$V \cong X_1 \overset{n}{\sim} X_1 \oplus (\overline{Z}_n \oplus \overline{Z}_{n-1}) = \overline{T} = \lambda(Y_1) + Z_n(\overline{T}) \overset{n}{\sim} \lambda(Y_1) \cong Y_1 \cong W.$$

Now, if we put $\overline{T}_V = X_1$ and $\overline{T}_W = \lambda(Y_1)$, then the proof is complete. □

The following corollary can be conclude immediately from theorem 2.5 which states some conditions for a regular Hom-Lie algebra to lie in an arbitrary n-isoclinism class.

Corollary 2.6. *If (V, φ_1) and (W, φ_2) are Hom-Lie algebras, then (W, φ_2) lies in the n-isoclinism class of $\{V\}$ if and only if one of the following conditions hold:*
(i) $W \cong V_1 \oplus Z_n$, where V_1 is a Hom-Lie algebra isomorphic to V, and Z_n is a nilpotent Hom-Lie algebra of class at most n;
(ii) W is a Hom-Lie subalgebra of Hom-Lie algebra M in $\{V\}$ with $M = W + Z_n(M)$;
(iii) There exists an epimorphism δ from a Hom-Lie algebra M in $\{V\}$ onto W such that $ker\delta \cap M^{n+1} = 0$.

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K-Theory and skew linear groups

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Abstract

Assume that D be an F-central division algebras and let the unit group $GL_n(D)$ of the full $n \times n$ matrix ring $M_n(D)$ with $n \geq 1$. For the most important results concerning the subgroups of this unit group, the skew linear groups, can be found in [7] as a good reference, also in as [8] for linear groups. Let D' the commutator subgroup of the multiplicative group D^*. We define $G(D) := D^*/RN(D^*)D'$, where $RN(D^*)$ is the image of D^* under the reduced norm of D to F. Let A be an F-central quaternion algebra . Then either $G(A) := A\square/F^*A' = \bigoplus \mathbb{Z}_2$ or $F^{*2} = RN_{D/F}(D^*)$.

1. Introduction

Let D be an division ring with centre F. Consider that D' be the commutator subgroup of the multiplicative group D^*. Denote by $G(D) := D^*/RN(D^*)D'$, where $RN(D^*)$ is the image of D^* under the reduced norm of D to F, is an abelian periodic group of a bounded exponent dividing the index of D over F. It is easily checked that this group is not trivial in general. For example, when D is the algebra of real quaternions, we have $G(D)$ is trivial whereas for rational quaternions $G(D)$ is isomorphic to a direct product of copies of $\mathbb{Z}_2$. When G(D) is not trivial, thus by Prufer-Baer Theorem, we conclude that G(D) is isomorphic to a direct product of $\mathbb{Z}_{r_i}$, when r_i divides the index of D over F. The structure of $GL_n(D)$ for $n \geq 1$ is generally unknown. In addition, we conclude that the existence of normal maximal subgroups of finite index in $D^\square$. Thus, when $G(D)$ is not trivial, then $D^\square$ contains maximal subgroups. For a given subgroup G of D^*, G is maximal in D^* if for any subgroup H of D^* such that $G \subseteq H$, we have $H = D^\square$. One way of looking into this problem is to investigate its maximal subgroups if they actually exist. For $n = 1$ the question of the existence of maximal subgroups has not been completely settled yet. But as an important class of skew linear groups, the structure of maximal subgroups of $GL_n(D)$ has been investigated recently by several authors.
For more information on these concepts, please refer to [2], [3], [4], [6],[7] and [8].

2. Main Results

In references [1], [3]and [4] various studies have been performed on the maximal subgroups of multiplicative subgroups of division algebras as well as the G (D) structure. Proven for example:

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Theorem A. Let D be an F-central division algebra of index p^e such that F contains a primitive p-th root of unity and $G(D) = 1$. Then D is a quaternion algebra.

Theorem B. Given an F-central division algebra D of index n, the following conditions are equivalent:
(1) $G(D) = 1$;
(2) $SK_1(D) = 1$ and $F^{*2} = F^{*2n}$;
(3) $G_0(D) = 1$ and $F^{*2} = F^{*2n}$;
(4) $D^*$is F_π-perfect where π is the set of all primes dividing $ind(D)$.

Also, examples show that $G(D)$ is not stable under the extension over formal Laurent series. We then have an analogue of the Lipnickii Theorem by showing that there exists a field F and F-central division algebra D of odd index such that $G(D)$ (or $G(D)$) can be any finite cyclic group.

For example , we know that there is a strong connection between the question of the existence of maximal subgroups in the unit group of a division algebra and the Albert's conjecture that concerning the cyclicity of division algebras of prime degree.If D be a finite dimensional F-central division algebra, every subgroup G of $GL_n(D)$ may be viewed as a linear group. Thus, by the Tits Alternative, G contains a noncyclic free subgroup or it is solvable-by-locally finite. Applying this Alternative, the structure of maximal subgroups of $GL_1(D)$ is investigated.

In this manner, we prove the following result:

Main Result. Let A be an F-central quaternion algebra . Then either $G(A) := A\square/F^*A' = \bigoplus \mathbb{Z}_2$ or $F^{*2} = RN_{D/F}(D^*)$.

Proof. Assume that A is a quaternion algebra over F. If A is not a division ring, then $A = M_2(F)$. By Theorem 5.7 of [5], $G(A) = A^*/F^*A' = F^*/F*^{\square 2}$. Thus, $G(A) = \bigoplus \mathbb{Z}_2$.

If A is a division algebra, there exist $\alpha, \beta \in F^*$ such that $A = (\frac{\alpha,\beta}{F})$. Therefore, $G(A) = RN_{A/F}(A^*)/F^{*2}$. By Prufer–Baer Theorem, we conclude that $G(A)$ is isomorphic to a direct product of copies $\mathbb{Z}_2$.

If the number of copies of $\mathbb{Z}_2$ is finite, we obtain $RN_{A/F}(A^*) = F^{*2} \bigcup F^{*2}a_1 \bigcup \cdots \bigcup F^{*2}a_m$, where, $a_1, ..., a_m \in F^*$.

For example when F is a local or global field, it can proved that the number of copies is infinite.

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Some of the graph energies of zero-divisor graphs of finite commutative rings

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Abstract

In this paper, we investigate some of the graph energies of the zero-divisor graph $\Gamma(R)$ of finite commutative rings R. Let $Z(R)$ be the set of zero-divisors of a commutative ring R with non-zero identity and $Z^*(R) = Z(R) \setminus \{0\}$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set in $Z^*(R)$ and two vertices u and v are adjacent if and only if $uv = vu = 0$.
We investigate some energies of $\Gamma(R)$ for the commutative rings $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where p, q the prime numbers.

1. Introduction

Assume that $G = (V, E)$ is a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The number of edges incident to vertex u in G is denoted $deg_G(u) = d(u)$. The isolated vertex and pendant vertex are the vertices with degrees zero and 1 in graph G, respectively.
The adjacency matrix of G, $A(G) = (a_{ij})$ is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E$ and $a_{ij} = 0$ otherwise. The eigenvalues of graph G are the eigenvalues of its adjacency matrix $A(G)$ [16]. The energy of a graph G was introduced in the 1970s as $E(G) = \sum_{i=1}^{n} |\lambda_j(G)|$ in which $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A(G)$ [12]. The edge energy of a graph G is defined as the sum of the absolute values of eigenvalues of $A(L_G)$ [6] in which L_G is the line graph of G. The line graph L_G of G is the graph that each vertex of it represents an edge of G and two vertices of L_G are adjacent if and only if their corresponding edges are incident in G [16].
Let $D(G)$ be the diagonal matrix of order n whose (i, i)-entry is the degree of the vertex v_i of the graph G. Then the matrices $L(G) = D(G) - A(G)$ and $L^+(G) = D(G) + A(G)$ are the Laplacian matrix and the signless Laplacian

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matrix, respectively, of the graph G. If $\mu_1, \mu_2, \ldots, \mu_n$ and $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ are, respectively, the eigenvalues of the matrices $L(G)$ and $L^+(G)$, then the Laplacian energy of G is defined as [13]

$$LE = LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|,$$

and the signless Laplacian energy is defined as follows [14]

$$LE^+ = LE^+(G) = \sum_{i=1}^{n} |\mu_i^+ - \frac{2m}{n}|.$$

Details on the properties and results of Laplacian and signless Laplacian energies and energy of a line graph can be found in [9, 10, 13, 14, 23].
A subset D of V is the dominating set of graph G if every vertex of $V \setminus D$ is adjacent to some vertices in D. Any dominating set with minimum cardinality is called a minimum dominating set [17]. The minimum dominating energy of graph G, denoted by $E_D(G)$, is introduced as the sum of the absolute values of eigenvalues of the minimum dominating matrix [24]. The minimum dominating matrix $A_D(G)$ is as following

$$A_D(G) := (a_{ij}) = \begin{cases} 1 & \text{if } v_iv_j \in E \\ 1 & \text{if } i = j \textit{ and } v_i \in D \\ 0 & otherwise \end{cases}$$

A set F of edges in G is the edge dominating set if every edge in $E \setminus F$ is adjacent to at least one edge in F. The edge domination number, denoted by γ', is the minimum the cardinalities of the edge dominating sets of G [11]. Note that F is the minimum edge dominating set of G or the minimum dominating set of L_G. The minimum edge dominating matrix of G is the $m \times m$ matrix defined by $A_F(G) := (a_{ij})$ in which

$$A_F(G) := (a_{ij}) = \begin{cases} 1 & \text{if } e_i \textit{ and } e_j \textit{ are adjacent}, \\ 1 & \text{if } i = j \textit{ and } e_i \in F, \\ 0 & otherwise. \end{cases}$$

The minimum edge dominating energy of G is introduced and studied in [3] as following

$$EE_F(G) = \sum_{i=1}^{m} |\lambda_i|,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $A_F(G)$. Note the minimum edge dominating energy of graph G is a minimum dominating energy for its line graph L_G. Details on the properties and results of the minimum dominating energy of a graph and its line graph can be found in [3, 18, 20–22, 24].
Let R be a ring and $Z(R)$ denotes the set of all zero-divisors of R. The zero-divisor graph of R is a simple graph $\Gamma(R)$ with vertex set $Z(R) \setminus \{0\}$ such that distinct vertices x and y are adjacent if and only if $xy = 0$ [4].
In this paper, we investigate graph energy, Laplacian energy, signless Laplacian energy, edge energy and the minimum edge dominating energy of $\Gamma(R)$ for the commutative rings $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where p, q the prime numbers.

2. Preliminaries

In this section, we state some previous results that will be used in the next section. First, we recall the definition of the Zagreb index of graph G. The Zagreb index $M(G)$ is defined as $M(G) = \sum_{i=1}^{n} d_i^2$ such that the vertices have the degree d_i for $i = 1, 2, \ldots, n$ [15].

Lemma 2.1. [20] *Let G be a graph with m edges. If F is the minimum edge dominating set of graph G, then*

$$EE_F(G) \leq M(G) - m,$$

where $M(G)$ is the Zagreb index of graph G.

Lemma 2.2. [20] *Let G be a graph of the order n with m edges. If F is the minimum edge dominating set of graph G with cardinality k, then*

$$EE_F(G) \leq 4m - 2n + k.$$

Lemma 2.3. [20] *Let G be a connected graph with n vertices and $m(\geq n)$ edges. Then*

$$EE_F(G) \geq 4(m - n + s) + 2p,$$

where p and s are the number of pendant vertices and isolated vertices in G.

Lemma 2.4. [14] *Let G be a graph of order n with m edges. Then*

$$\sqrt{2M(G) - 4m} \leq E(L_G) \leq M(G) - 2m,$$

where L_G and $M(G)$ are the line graph and the Zagreb index of graph G.

Lemma 2.5. [10] *Let G be a connected graph of order n. Then*

$$E(L_G) \geq 2\big(E(G) - 2v^+\big),$$

where v^+ is the number of positive eigenvalues.

Lemma 2.6. [14] *Let G be a graph with n vertices and m edges such that $m > n$. Then*

$$E(L_G) < LE^+(G) + 4(m - n).$$

Lemma 2.7. [8] *Let G be a graph of order n with $m \geq \frac{n}{2}$ edges and the maximum degree Δ. Then*

$$2\big(\Delta + 1 - \frac{2m}{n}\big) \leq LE(G) \leq 4m - 2\Delta - \frac{4m}{n} + 2.$$

Lemma 2.8. [20] *Let G be a simple graph and L_G the line graph of G. If F is the minimum edge dominating set with $|F| = k$, then*

$$EE_F(G) \leq EE(G) + k.$$

Lemma 2.9. [26, 28] *For a graph G with n vertices and m edges,*

$$\frac{4m}{n} \leq LE(G) \leq 4m\big(1 - \frac{1}{n}\big).$$

Lemma 2.10. [1] *Let G be a graph with n vertices and m edges. Then*

$$LE^+(G) \leq 4m\big(1 - \frac{1}{n}\big).$$

Lemma 2.11. [5] *Let G be a graph with n vertices and m edges. Then, $\gamma' \leq \lfloor \frac{n}{2} \rfloor$.*

3. Main Results

In this section, we study energies of the zero-divisor graphs $\Gamma(R)$ such as the edge energy, the minimum edge dominating energy, the Laplacian energy and the signless Laplacian energy of the commutative rings $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where p, q the prime numbers. Firstly, we investigate these energies of the zero divisor graph $\Gamma\big(\mathbb{Z}_{p^2} \times \mathbb{Z}_q\big)$ where is defined as follows.

For $x \in \mathbb{Z}_{p^2}$ and $y \in \mathbb{Z}_q$, $(x, y) \notin V\Big(\Gamma\big(\mathbb{Z}_{p^2} \times \mathbb{Z}_q\big)\Big)$ if and only if $x \neq p, 2p, \ldots, (p-1)p$ and $y \neq 0$. According to the structure of graph $\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)$, the number of vertices is equal to $p^2+pq-p-1$ [2]. Authors in [2], characterized the vertices of graph $\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)$ according to their degree as follows.

$$\begin{aligned}
&A=\big\{(0,y):y\in\{1,2,\dots,q-1\}\big\}\ ,\ |A|=q-1,\\
&B=\big\{(x,y):x=p,2p,\dots,(p-1)p\ and\ y\in\{1,2,\dots,q-1\}\big\}\ ,\ |B|=(p-1)(q-1),\\
&C=\big\{(x,0):x\in\mathbb{Z}_{p^2}\setminus\{0,p,2p,\dots,(p-1)p\}\big\}\ ,\ |C|=p-1,\\
&D=\big\{(x,0):x=p,2p,\dots,(p-1)p\big\}\ ,\ |D|=p(p-1).
\end{aligned}$$

Also, they obtained the degree sequence DS of this graph as follows.

$$DS\big(\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)\big)=\big\{(p^2-1)^{[q-1]},(p-1)^{[(p-1)(q-1)]},(pq-2)^{[p-1]},(q-1)^{[(p^2-p)]}\big\}. \tag{1}$$

Therefore, the number of edges in this graph is equal to $m=\frac{(p-1)(4pq-3p-2)}{2}$ [2].

Theorem 3.1. *Let $\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)$ for prime numbers $p,q>2$ be the zero-divisor graph of size m.*

i) If $p>q$, then $2\big(p^2-\alpha\big)\le LE\big(\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)\big)\le 4(m+1)-(2p^2+\alpha)$,

ii) If $p<q$, then $2\big(pq-\alpha-1\big)\le LE\big(\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)\big)\le 4(m+1)-2(pq+\alpha-1)$.

where $\alpha=\frac{2(p-1)(4pq-3p-2)}{p^2+pq-p-1}$.

Proof. Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_{p^2}\times\mathbb{Z}_q\big)$ for prime numbers $p,q>2$ such that the number of vertices and the number of edges are $n=p^2+p(q-1)-1$ and $m=\frac{(p-1)(4pq-3p-2)}{2}$, respectively. According to the sequence degree (1) of the zero-divisor graph G, we consider two following cases.
Case 1: If $p>q$, then the maximum degree of graph G is $\Delta=p^2-1$. Using Lemma 2.7, we have

$$\begin{aligned}
LE(G)&\le 4m-2\Delta-\frac{4m}{n}+2\\
&=4m-2(p^2-1)+2-\frac{4(p-1)(4pq-3p-2)}{2(p^2+p(q-1)-1)}\\
&=4m+4-2p^2-\frac{2(p-1)(4pq-3p-2)}{(p^2+p(q-1)-1)}.
\end{aligned}$$

With putting $\alpha=\frac{2(p-1)(4pq-3p-2)}{p^2+pq-p-1}$, the result holds for the upper bound.
For the lower bound, using Lemma 2.7, we have

$$\begin{aligned}
LE(G)&\ge 2\big(\Delta+1-\frac{2m}{n}\big)\\
&=2\Big(p^2-1+1-\frac{2(p-1)(4pq-3p-2)}{(p^2+p(q-1)-1)}\Big).
\end{aligned}$$

So, the result holds.
Case 2: If $p<q$, then $\Delta=pq-2$. Similar to the proof of case 1, the result completes. $\square$

Theorem 3.2. *Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R\simeq\mathbb{Z}_{p^2}\times\mathbb{Z}_q$ for prime numbers $p,q>2$. Then*

$$\sqrt{2(p-1)(\alpha-\beta)}\le E\big(L_{\Gamma(R)}\big)\le(p-1)(\alpha-\beta),$$

in which $\alpha=p(q-1)(q+p(p+2)-4)+pq-2$ and $\beta=4pq-3p-2$.

Proof. We suppose $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ for prime numbers $p, q > 2$ of order $n = p^2+p(q-1)-1$ and size $m = \frac{(p-1)(4pq-3p-2)}{2}$. Using the sequence degree (1) of graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ and the definition of the Zagreb index, we get

$$\begin{aligned} M(G) &= \sum_{i=1}^{p^2+p(q-1)-1} d_i^2 \\ &= (q-1)(p^2-1)^2 + (p-1)^3(q-1) + (pq-2)^2(p-1) + (q-1)^2(p^2-p) \\ &= (p-1)\big[p(q-1)\big(q+p(p+2)-4\big) + pq - 2\big]. \end{aligned}$$

Thus, using Lemma 2.4, we have

$$\begin{aligned} E\big(\Gamma(G)\big) &\leq M(G) - 2m \\ &= (p-1)\big[p(q-1)\big(q+p(p+2)-4\big) + pq-2\big] - 2\Big(\frac{(p-1)(4pq-3p-2)}{2}\Big) \\ &= (p-1)\Big(\big[p(q-1)\big(q+p(p+2)-4\big)+pq-2\big] - (4pq-3p-2)\Big). \end{aligned}$$

With setting $\alpha = p(q-1)(q+p(p+2)-4)+pq-2$ and $\beta = 4pq-3p-2$ in the above relation, the result holds for the upper bound.
By applying Lemma 2.4 and similar to the above discussion, the lower bound follows. □

Theorem 3.3. *Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ for prime numbers $p, q > 2$. If F is the minimum edge dominating set of $\Gamma(R)$, then*

$$EE_F\big(\Gamma(R)\big) \leq \frac{(p-1)(2\alpha-\beta)}{2},$$

in which $\alpha = p(q-1)(q+p(p+2)-4)+pq-2$ and $\beta = 4pq-3p-2$.

Proof. According to the proof of Theorem 3.2 and using Lemma 2.1, the result completes. □

In the following, we are interested to investigate some energies of the zero-divisor graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ and $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for a prime $p > 2$. To do this, we need the following known result.

Lemma 3.4. [7] *Let G be a simple graph of the order n and size m. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of graph G, then*

$$\sum_{i=1}^{n} \lambda_i^2 = 2m.$$

First, we consider the connected graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ of order $n = 3p(p-1)$. In the following theorem, we compute the energy of graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$.

Theorem 3.5. *Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then*

$$E\big(\Gamma(R)\big) = 2(p-1)\big(\sqrt{4p-3} + \sqrt{p}\big).$$

Proof. Suppose that $\Gamma(R)$ is the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ with the number of vertices $n = 3p(p-1)$. According to the structure of the zero-divisor graph $\Gamma(R)$ in [19], the spectrum of $\Gamma(R)$ is as follows.

$$\begin{aligned} Spec\big(\Gamma(R)\big) = \Big\{ &\frac{1}{2}\big((p-1)(\sqrt{4p-3}-1)\big)^{[2]}, \frac{1}{2}\big((1-p)(\sqrt{4p-3}+1)\big)^{[2]}, \\ &\big((p-1)(1+\sqrt{p})\big)^{[1]}, \big((p-1)(1-\sqrt{p})\big)^{[1]}, 0^{[3(p+1)(p-2)]} \Big\}. \end{aligned} \tag{2}$$

Therefore, the energy of graph $\Gamma(R)$ equals

$$
\begin{aligned}
E\big(\Gamma(R)\big) &= \sum_{i=1}^{3p(p-3)} |\lambda_i| \\
&= 2\Big|\frac{1}{2}\big((p-1)(\sqrt{4p-3}-1)\big)\Big| + 2\Big|\frac{1}{2}\big((1-p)(\sqrt{4p-3}+1)\big)\Big| \\
&+ \big|(p-1)(1+\sqrt{p})\big| + \big|(p-1)(1-\sqrt{p})\big| \\
&= (p-1)(\sqrt{4p-3}-1) + (p-1)(\sqrt{4p-3}+1) \\
&+ (p-1)(1+\sqrt{p}) + (p-1)(\sqrt{p}-1) \\
&= 2(p-1)\sqrt{4p-3} + 2(p-1)\sqrt{p} \\
&= 2(p-1)\big(\sqrt{4p-3}+\sqrt{p}\big).
\end{aligned}
$$

□

Theorem 3.6. *Let $\Gamma(R)$ be the zero-divisor graph of the commutative ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$4(p-1) \leq LE\big(\Gamma(R)\big) \leq 4\big(p-1\big)\big(3p^2-3p-1\big).$$

Proof. Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ of order $n = 3p(p-1)$ and size m. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of graph G. Using Lemma 3.4 and the spectrum of graph $\Gamma(R)$ in (2), we have

$$
\begin{aligned}
m &= \frac{\sum_{i=1}^{n} \lambda_i^2}{2} \\
&= \frac{6p(p-1)^2}{2} = 3p(p-1)^2.
\end{aligned}
$$

By applying Lemma 2.9, we have

$$
\begin{aligned}
LE(G) &\leq 4m - \frac{4m}{n} \\
&= 4\big(3p(p-1)^2\big) - \frac{4\big(3p(p-1)^2\big)}{3p(p-1)} \\
&= 12p(p-1)^2 - 4\big(p-1\big) \\
&= 4(p-1)\big(3p^2-3p-1\big).
\end{aligned}
$$

And for the lower bound,

$$
\begin{aligned}
LE(G) &\geq \frac{4m}{n} \\
&= \frac{4\big(3p(p-1)^2\big)}{3p(p-1)} \\
&= 4(p-1).
\end{aligned}
$$

□

The following result is obtained directly from Lemma 2.10 and the proof of Theorem 3.6.

Corollary 3.7. *For the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ where $p > 2$ is a prime,*

$$LE^{+}\Big(\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)\Big) \leq 4\big(p-1\big)\big(3p^2-3p-1\big).$$

Theorem 3.8. *Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$E\big(L_{\Gamma(R)}\big) < 4(p-1)\big(6p^2 - 9p - 1\big).$$

Proof. The size of graph $\Gamma(R)$ is $m = 3p(p-1)^2$. So using Lemmas 2.6 and 2.10,

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &< LE^+\big(\Gamma(R)\big) + 4(m-n) \\ &\leq 4m - \frac{4m}{n} + 4m - 4n \\ &= 24p(p-1)^2 - \frac{12p(p-1)^2}{3p(p-1)} - 12p(p-1) \\ &= 24p(p-1)^2 - 4(p-1) - 12p(p-1) \\ &= 4(p-1)\big(6p^2 - 9p - 1\big). \end{aligned}$$

□

Theorem 3.9. *Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then*

$$E\big(L_{\Gamma(R)}\big) \geq 4\Big(\big((p-1)\sqrt{4p-3} + \sqrt{p}\big) - 3\Big).$$

Proof. According to the spectrum of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ in the proof of Theorem 3.5, the number of positive eigenvalues of $\Gamma(R)$ is $v^+ = 3$. Therefore using Lemma 2.5 and Theorem 3.5, we get

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &\geq 2E\big(\Gamma(R)\big) - 4v^+ \\ &= 4(p-1)\Big(\sqrt{4p-3} + \sqrt{p}\Big) - 12 \\ &= 4\Big(\big((p-1)\sqrt{4p-3} + \sqrt{p}\big) - 3\Big). \end{aligned}$$

□

Theorem 3.10. *Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$4n\big(p - \frac{13}{8}\big) \leq EE_F\big(\Gamma(R)\big) \leq 4n(p-2).$$

Proof. Assume that G is the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ where $p > 2$ is a prime. Let F be the minimum edge dominating set of graph G. Since G is a connected graph of order $n = 3(p^2 - p)$ with $m = 3p(p-1)^2$ edges without any isolated and pendant vertex for $p > 2$, then using Lemma 2.3, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\geq 4(m - n + s) + 2p \\ &= 4\big((3p(p-1)^2) - n\big) \\ &= 4\big(n(p-1) - n\big) \\ &= 4n(p-2). \end{aligned}$$

Using Lemma 3.4, $\gamma' = |F| \leq \lfloor \frac{n}{2} \rfloor$. Then by applying Lemma 2.2, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\leq 4m - 2n + |F| \\ & 4(3p(p-1)^2) - 2n + \lfloor \frac{n}{2} \rfloor \\ &\leq 4n(p-1) - 2n + \frac{n}{2} \\ &= 4n\big(p - 1 - \frac{12}{8}\big) \\ &= 4n\big(p - \frac{13}{8}\big). \end{aligned}$$

Therefore, the result completes. □

Now, we consider the connected graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ where $p > 2$ is a prime. The graph $\Gamma(R)$ is a connected graph of order $n = 2(p-1)(2p^2 - p + 1)$ [19].

Theorem 3.11. *Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then*

$$E\big(\Gamma(R)\big) = 14p^2 - 21p + 8.$$

Proof. In [19], the spectrum of graph $\Gamma(R)$ where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ is obtained as follows.

$$\begin{aligned} Spec\big(\Gamma(R)\big) = \Big\{ &\Big((p-1)^2\Big)^{[5]}, (-p^2 + p - 1)^{[1]}, \frac{-1}{2}\Big((p-1)\big((2p-1) \pm \sqrt{4p-3}\big)\Big)^{[3]}, \\ &\frac{1}{2}\Big((p-1)\big((2p+1) \pm \sqrt{12p-3}\big)\Big)^{[1]}, 0^{[(p^3+p^2+5p+7)(p-2)]}\Big\}. \end{aligned}$$

Therefore, the energy of graph $\Gamma(R)$ equals

$$\begin{aligned} E\big(\Gamma(R)\big) = &\sum_{i=1}^{(2p-2)(2p^2-p+1)} |\lambda_i| \\ &= 5(p-1)^2 + (p^2 - p + 1) + \frac{3}{2}(p-1)\big((2p-1) + \sqrt{4p-3}\big) \\ &+ \frac{3}{2}(p-1)\big((2p-1) - \sqrt{4p-3}\big) + \frac{1}{2}(p-1)\big((2p+1) + \sqrt{12p-3}\big) \\ &+ \frac{1}{2}(p-1)\big((2p+1) - \sqrt{12p-3}\big). \end{aligned}$$

With the simplification of the above relation, the result follows. □

Theorem 3.12. *Let $\Gamma(R)$ be the zero-divisor graph where $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$\frac{4p\big(5n + \alpha\big) + 8}{n} \leq LE\big(\Gamma(R)\big) \leq \frac{\big(4p(5n + \alpha) + 8\big)(n-1)}{n},$$

where $\alpha = -6p^3 + p + 4$.

Proof. The zero-divisor graph $\Gamma(R)$ for $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ has $n = 2(p-1)(2p^2 - p + 1)$ vertices. Using Lemma 3.4 and the spectrum of zero-divisor graph $\Gamma(R)$ in the proof Theorem 3.11, the number of edges of graph

$\Gamma(R)$ is $m = 14p^4 - 30p^3 + 21p^2 - 6p + 2$. Using Lemma 2.9, we get

$$\begin{aligned} LE\big(\Gamma(R)\big) &\leq 4m\big(1 - \frac{1}{n}\big) \\ &= \frac{2(14p^4 - 30p^3 + 21p^2 - 6p + 2)\big(4p^3 - 6p^2 + 4p - 3\big)}{2p^3 - 3p^2 + 2p - 1} \\ &= \frac{2\big(p\big(5(4p^3 - 6p^2 + 4p - 2) - 6p^3 + p + 4\big) + 2\big)(n-1)}{\frac{n}{2}}. \end{aligned}$$

With putting $n = 4p^3 - 6p^2 + 4p - 2$ and $\alpha = -6p^3 + p + 4$, the upper bound for the Laplacian energy of $\Gamma(R)$ follows.
For the lower bound, we get

$$\begin{aligned} LE\big(\Gamma(R)\big) &\geq \frac{4m}{n} \\ &= \frac{4(14p^4 - 30p^3 + 21p^2 - 6p + 2)}{n} \\ &= \frac{4\big(p\big(5(4p^3 - 6p^2 + 4p - 2) - 6p^3 + p + 4\big) + 2\big)}{n} \\ &= \frac{4\big(p\big(5n - 6p^3 + p + 4\big) + 2\big)}{n}. \end{aligned}$$

With putting $\alpha = -6p^3 + p + 4$ in the above relation, the result completes. □

The following result is obtained directly from Lemma 2.10 and the proof of Theorem 3.12.

Corollary 3.13. *Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$LE^+\big(\Gamma(R)\big) \leq \frac{\big(4p(5n + \alpha) + 8\big)(n-1)}{n},$$

where $\alpha = -6p^3 + p + 4$.

Theorem 3.14. *Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$E\big(L_{\Gamma(R)}\big) < \frac{\big(4p(5n + \alpha) + 8\big)(2n-1) - 4n^2}{n},$$

where $\alpha = -6p^3 + p + 4$.

Proof. Since the number of edges of graph $\Gamma(R)$ is $m = 14p^4 - 30p^3 + 21p^2 - 6p + 2$, using Lemmas 2.6 and 2.10, we get

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &< LE^+\big(\Gamma(R)\big) + 4(m - n) \\ &\leq 4m - \frac{4m}{n} + 4m - 4n \\ &= 8(14p^4 - 30p^3 + 21p^2 - 6p + 2) - \frac{4(14p^4 - 30p^3 + 21p^2 - 6p + 2)}{n} - 4n. \end{aligned}$$

with considering $n = 4p^3 - 6p^2 + 4p - 2$ and $\alpha = -6p^3 + p + 4$ and the similar to the discussion in proof of Theorem 3.12, the result follows. □

Theorem 3.15. *Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$. Then*

$$E\big(L_{\Gamma(R)}\big) \geq 2\big(14p^2 - 21p - 4\big).$$

Proof. According to the spectrum of the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ in the proof of Theorem 3.11, the number of positive eigenvalues of $\Gamma(R)$ is $v^+ = 6$. Therefore using Lemma 2.5 and Theorem 3.11, we get

$$\begin{aligned} E\big(L_{\Gamma(R)}\big) &\geq 2E\big(\Gamma(R)\big) - 4v^+ \\ &= 2\big(14p^2 - 21p + 8\big) - 24 \\ &= 28p^2 - 42p - 8. \end{aligned}$$

□

Theorem 3.16. *Let $\Gamma(R)$ be the zero-divisor graph of the ring $R \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ for prime number $p > 2$ of order n. Then*

$$56p^4 - 136p^3 + 108p^2 - 56p + 16 \leq EE_F\big(\Gamma(R)\big) \leq 56p^4 - 126p^3 + 93p^2 - 36p + 11.$$

Proof. Let G be the zero-divisor graph $\Gamma\big(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p\big)$ where $p > 2$ is a prime. Let F be the minimum edge dominating set of graph G.
Since G is a connected graph of order $n = 4p^3 - 6p^2 + 8p - 2$ with $m = 14p^4 - 30p^3 + 21p^2 - 6p + 2$ edges without any isolated and pendant vertex for $p > 2$, then using Lemma 2.3, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\geq 4(m - n + s) + 2p \\ &= 4\big((14p^4 - 30p^3 + 21p^2 - 6p + 2) - (4p^3 - 6p^2 + 8p - 2)\big) \\ &= 4\big(14p^4 - 34p^3 + 27p^2 - 14p + 4\big). \end{aligned}$$

Using Lemma 3.4, $\gamma' = |F| \leq \lfloor \frac{n}{2} \rfloor$, and Lemma 2.2, we get

$$\begin{aligned} EE_F\big(\Gamma(R)\big) &\leq 4m - 2n + |F| \\ &\leq 4(14p^4 - 30p^3 + 21p^2 - 6p + 2) - 2n + \lfloor \frac{n}{2} \rfloor \\ &\leq 4(14p^4 - 30p^3 + 21p^2 - 6p + 2) - 2n + \frac{n}{2} \\ &= 4(14p^4 - 30p^3 + 21p^2 - 6p + 2) - \frac{3(4p^3 - 6p^2 + 8p - 2)}{2} \\ &= 56p^4 - 126p^3 + 93p^2 - 36p + 11. \end{aligned}$$

Therefore, the result completes. □

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Some of Topological Indices of S-R Double Join of Graphs

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Article Info

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Abstract

The S-R double join, $G_1{}_S\odot_R G_2$, of two disjoint graphs G_1 and G_2 is the graph obtained from $S(G_1)$ and $R(G_2)$ by joining every vertex of $V(G_1)$ to every vertex of $I(G_2)$ and, every vertex of $I(G_1)$ to every vertex of $V(G_2)$. In this paper we determine the generalized first Zagreb index, second Zagreb index and the forgotten topological index of the S-R double join of two graphs.

1. Introduction

Throughout this paper, we consider simple connected graphs. Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree of a vertex $x \in V(G)$ is the number of vertices adjacent to x and is denoted by $\deg_G(x)$. In mathematical chemistry and chemical graph theory, a topological index is a numerical parameter (a real number) that is measured based on the molecular graph of a chemical constitution [2].
Two important topological indices introduced about forty years ago by Ivan Gutman and Trinajstic [3] are the first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$ which are defined as:

$$M_1(G) = \sum_{v\in V(G)} (\deg_G(v))^2$$

$$M_2(G) = \sum_{uv\in E(G)} \deg(u)\deg(v).$$

Also the Forgotten topological index is defined as [1]:

$$F(G) = \sum_{v\in V(G)} (\deg_G(v))^3.$$

The generalized first Zagreb index of a graph G is defined as [4]:

$$Z^\alpha(G) = \sum_{x\in V(G)} \deg_G^\alpha(x) = \sum_{uv\in E(G)} \deg_G(u)^{\alpha-1} + \deg_G(v)^{\alpha-1},$$

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where $\alpha \in R$, $\alpha \neq 0$, $\alpha \neq 1$. If $\alpha = 3$, then generalized first Zagreb index becomes Forgotten index. For more details on these topological indices we refer the reader to [4] and [5]. For a connected graph G, two related graphs are defined as follows [1].
$S(G)$ is the graph obtained by inserting an additional vertex in each edge of G, in other words, each edge of G is replaced by a path of length two.
$R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge.

2. Main results

Definition 2.1. The S-R double join, $G_1 {}_S\odot_R G_2$, of two disjoint graphs G_1 and G_2 is the graph obtained from $S(G_1)$ and $R(G_2)$ by joining every vertex of $V(G_1)$ to every vertex of $I(G_2)$ and, every vertex of $I(G_1)$ to every vertex of $V(G_2)$, where $I(G_1)$ denotes the vertex set of the added new vertices in $S(G_1)$ and $I(G_2)$ denotes the vertex set of the added new vertices in $R(G_2)$.

For instance the figure below represents the S-R double join of two graphs P_2 and K_3.

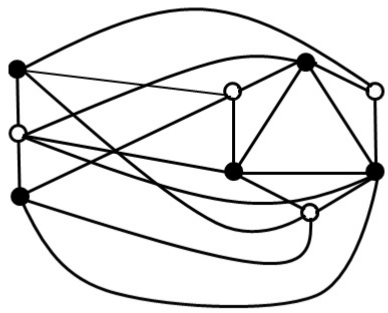

Fig. 1. $P_2 {}_S\odot_R K_3$.

Proposition 2.2. *The degree of the vertices of S-R double join graph are given by*

$$\deg_{G_1 {}_S\odot_R G_2}(x) = \begin{cases} \deg_{G_1}(x) + m_2 & x \in V(G_1) \\ 2 + n_2 & x \in I(G_1) \\ 2\deg_{G_2}(x) + m_1 & x \in V(G_2) \\ 2 + n_1 & x \in I(G_2) \end{cases}$$

Theorem 2.3. *Let α be positive integer. the generalized first Zagreb index of $G_1 {}_S\odot_R G_2$ is given by*

$$Z^{\alpha}(G) = m_1(2+n_2)^{\alpha} + m_2(2+n_1)^{\alpha} + \sum_{i=0}^{\alpha} \binom{\alpha}{i} (m_2)^{\alpha-i} Z^i(G_1) + \sum_{i=0}^{\alpha} 2^i \binom{\alpha}{i} (m_1)^{\alpha-i} Z^i(G_2)$$

Proof. By considering vertex degrees of the $G = G_1 {}_S\odot_R G_2$,as represented observation 2.2, and the binomial expansion in the generalized first Zagreb index is calculated as follows:

$$\begin{aligned}
Z^{\alpha}(G) &= \sum_{x\in V(G)} \deg_G(x)^{\alpha} \\
&= \sum_{x\in V(G_1)} \deg_G(x)^{\alpha} + \sum_{x\in I(G_1)} \deg_G(x)^{\alpha} + \sum_{x\in V(G_2)} \deg_G(x)^{\alpha} + \sum_{x\in I(G_2)} \deg_G(x)^{\alpha} \\
&= \sum_{x\in V(G_1)} (\deg_{G_1}(x)+m_2)^{\alpha} + \sum_{x\in I(G_1)} (2+n_2)^{\alpha} \\
&+ \sum_{x\in V(G_2)} (2\deg_{G_2}(x)+m_1)^{\alpha} + \sum_{x\in I(G_2)} (2+n_1)^{\alpha} \\
&= [\sum_{x\in V(G_1)} \sum_{i=0}^{\alpha} \binom{\alpha}{i} (\deg_{G_1}(x))^i (m_2)^{\alpha-i}] + [\sum_{x\in V(G_2)} \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2\deg_{G_2}(x))^i (m_1)^{\alpha-i}] \\
&+ m_1(2+n_2)^{\alpha} + m_2(2+n_1)^{\alpha} \\
&= \sum_{i=0}^{\alpha} [\binom{\alpha}{i} (m_2)^{\alpha-i} \sum_{x\in V(G_1)} (\deg_{G_1}(x))^i] + \sum_{i=0}^{\alpha} [\binom{\alpha}{i} (m_1)^{\alpha-i} \sum_{x\in V(G_2)} (2\deg_{G_2}(x))^i] \\
&+ m_1(2+n_2)^{\alpha} + m_2(2+n_1)^{\alpha} \\
&= m_1(2+n_2)^{\alpha} + m_2(2+n_1)^{\alpha} + \sum_{i=0}^{\alpha} \binom{\alpha}{i} (m_2)^{\alpha-i} Z^i(G_1) + \sum_{i=0}^{\alpha} 2^i \binom{\alpha}{i} (m_1)^{\alpha-i} Z^i(G_2).
\end{aligned}$$

□

Corollary 2.4. *For two arbitrary graphs G_1 and G_2 we have*

$$M_1(G_{1S}\odot_R G_2) = M_1(G_1) + 4M_1(G_2) + (2+n_2)^2 m_1 + (2+n_1)^2 m_2 + 12m_1m_2 + m_1^2 n_2 + m_2^2 n_1$$

Proof. It is sufficient to let $\alpha = 2$ in Theorem 2.3. □

Corollary 2.5. *For two arbitrary graphs G_1 and G_2 we have*

$$\begin{aligned}
F(G_{1S}\odot_R G_2) &= F(G_1) + 8F(G_2) + 3m_2M_1(G_1) + 12m_1M_1(G_2) + n_2m_1^3 + m_1(2+n_2)^3 \\
&+ m_2(2+n_1)^3 + 12m_1^2m_2 + 6m_1m_2^2 + m_2^3 n_1
\end{aligned}$$

Proof. It is sufficient to let $\alpha = 3$ in Theorem 2.3. □

Theorem 2.6. *The second Zagreb index of $G_{1S} \odot_R G_2$ is given by*

$$\begin{aligned}
M_2(G_{1S}\odot_R G_2) &= (2+n_2)[M_1(G_1) + 2m_1m_2] + [4M_2(G_2) + 2m_1M_1(G_2) + m_1^2 m_2] \\
&+ 2n_1[2M_1(G_2) + 2m_1m_2] + [m_2(2+n_1)(2m_1 + n_1m_2)] \\
&+ m_1(2+n_2)[4m_2 + m_1n_2].
\end{aligned}$$

Proof. According to observation 2.2, the second Zagreb index of is as follows.

$$\begin{aligned}
M_2(G_{1\,S}\!\odot_R G_2) &= \sum_{uv\in E(G_{1\,S}\odot_R G_2)} \deg(u)\deg(v)\\
&= \sum_{uv\in E(S(G_1))} (\deg_{G_1}(u)+m_2)(2+n_2)\\
&+ \sum_{uv\in E(G_2)} (2\deg_{G_2}(u)+m_1)(2\deg_{G_2}(v)+m_1)\\
&+ \sum_{uv\in E(S(G_2))} (2\deg_{G_2}(v)+m_1)(2+n_1)\\
&+ \sum_{u\in V(G_1)}\sum_{v\in I(G_2)} (2+n_1)(\deg_{G_1}(u)+m_2)\\
&+ \sum_{u\in I(G_1)}\sum_{v\in V(G_2)} (2+n_2)(2\deg_{G_2}(v)+m_1)\\
&= (2+n_2)[M_1(G_1)+2m_1m_2]+[4M_2(G_2)+2m_1M_1(G_2)+m_1^2m_2]\\
&+ 2n_1[2M_1(G_2)+2m_1m_2]+[m_2(2+n_1)(2m_1+n_1m_2)]\\
&+ m_1(2+n_2)[4m_2+m_1n_2].
\end{aligned}$$

□

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The generalized first Zagreb index of subdivision-vertex and subdivision-edge neighbourhood corona

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Abstract

The subdivision-vertex neighbourhood corona, $G_1 \odot G_2$, is the graph obtained from $S(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbours of the i-th vertex of n_1 to every vertex in the i-th copy of G_2, and the subdivision-edge neighbourhood corona $G_1 \ominus G_2$, is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbours of the i-th vertex of $I(G_1)$ to every vertex in the i-th copy of G_2.
In this paper we determine the generalized first Zagreb index, of the subdivision-vertex and subdivision-edge neighbourhood corona.

1. Introduction

Throughout this paper, we consider simple connected graphs. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$ so that the order and size of G is equal to n and m respectively. The degree of a vertex $x \in V(G)$ is the number of first neighbors of x, and is denoted by $\deg_G(x)$. In mathematical chemistry and chemical graph theory, a topological index is a numerical parameter (a real number) that is measured based on the molecular graph of a chemical constitution [2].
The F-index of a graph is defined as the sum of cubes of the vertex degrees of the graph [1]:

$$F(G) = \sum_{v \in V(G)} (\deg_G(v))^3.$$

The generalized first Zagreb index of a graph G is defined as [3]:

$$Z^k(G) = \sum_{x \in V(G)} \deg_G^k(x) = \sum_{uv \in E(G)} \deg_G(u)^{k-1} + \deg_G(v)^{k-1},$$

where $k \in R$, $k \neq 0$, $k \neq 1$. If $k = 3$, then generalized first Zagreb index becomes Forgotten index. For more details on these topological indices we refer the reader to [3] and [4]. For a connected graph G, two related graphs are defined as follows [1]:

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$S(G)$ is the graph obtained by inserting an additional vertex in each edge of G, in other words, each edge of G is replaced by a path of length two.

2. Main results

Definition 2.1. [5] The subdivision-vertex neighbourhood corona of G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from $S(G_1)$ and n_1 copies of G_2, all vertex-disjoint, and joining the neighbours of the i-th vertex of n_1 to every vertex in the i-th copy of G_2.

For instance, the figure 1 represents the subdivision-vertex neighbourhood corona of two graphs P_4 and P_2.

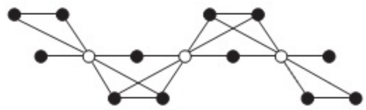

Fig. 1. $P_4 \odot P_2$.

Proposition 2.2. *The degree of the vertices of subdivision-vertex neighbourhood corona graph are given by*

$$\begin{aligned}
\deg_{G_1\odot G_2}(v_i) &= \deg_{G_1}(v_i), \quad i=1,2,...,n_1,\\
\deg_{G_1\odot G_2}(e_i) &= 2+2n_2, \quad i=1,2,...,m_1,\\
\deg_{G_1\odot G_2}(u_j^i) &= \deg_{G_2}(u_j)+\deg_{G_1}(v_i), \quad i=1,2,...,n_1, j=1,2,...,n_2.
\end{aligned}$$

Theorem 2.3. *Let k be positive integer the generalized first Zagreb index of $G_1 \odot G_2$ is given by*

$$Z^k(G_1\odot G_2) = Z^k(G_1)+m_1(2n_2+2)^k+\sum_{t=0}^{k}\binom{k}{t}Z^t(G_2)Z^{k-t}(G_1)$$

Proof. By considering the degree vertices of the $G = G_1 \odot G_2$, as represented observation 2.2, and the binomial expansion in the generalized first Zagreb index is calculated as follows:

$$\begin{aligned}
Z^k(G) &= \sum_{x\in V(G)} \deg_G(x)^k\\
&= \sum_{i=1}^{n_1}\deg_{G_1}(v_i)^k+\sum_{i=1}^{m_1}(2n_2+2)^k+\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}[\deg_{G_2}(u_j)+\deg_{G_1}(v_i)]^k\\
&= Z^k(G_1)+m_1(2n_2+2)^k+\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{t=0}^{k}\binom{k}{t}\deg_{G_2}(u_j)^t\deg_{G_1}(v_i)^{k-t}\\
&= Z^k(G_1)+m_1(2n_2+2)^k+\sum_{t=0}^{k}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\binom{k}{t}\deg_{G_2}(u_j)^t\deg_{G_1}(v_i)^{k-t}\\
&= Z^k(G_1)+m_1(2n_2+2)^k+\sum_{t=0}^{k}\binom{k}{t}Z^t(G_2)Z^{k-t}(G_1).
\end{aligned}$$

□

Corollary 2.4. *For two arbitrary graph G_1 and G_2 we have*

$$F(G_1\odot G_2) = F(G_1)+m_1(2n_2+2)^3+n_2F(G_1)+6m_2M_1(G_1)+6m_1M_1(G_2)$$

Proof. It is sufficient to let $k = 3$ in theorem 2.3. □

Definition 2.5. [5] The subdivision-edge neighbourhood corona of G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $S(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, and joining the neighbours of the i-th vertex of $I(G_1)$ to every vertex in the i-th copy of G_2.

For example, the figure 2 represents the subdivision-vertex neighbourhood corona of two graphs P_4 and P_2.

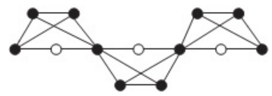

Fig. 2. $P_4 \ominus P_2$.

Proposition 2.6. *The degree of the vertices of subdivision-edge neighbourhood corona graph are given by*

$$\begin{aligned}
\deg_{G_1 \ominus G_2}(v_i) &= [1+n_2]\deg_{G_1}(v_i), \qquad i = 1, 2, ..., n_1, \\
\deg_{G_1 \ominus G_2}(e_i) &= 2, \quad i = 1, 2, ..., m_1, \\
\deg_{G_1 \ominus G_2}(u_j^i) &= \deg_{G_2}(u_j) + 2, \qquad i = 1, 2, ..., n_1, j = 1, 2, ..., n_2.
\end{aligned}$$

Theorem 2.7. *Let k be positive integer the generalized first Zagreb index of $G_1 \ominus G_2$ is given by*

$$Z^k(G_1 \ominus G_2) = [1+n_2]^k Z^k(G_1) + m_1 2^k + \sum_{t=0}^{k} \binom{k}{t} 2^t n_1 Z^{k-t}(G_2)$$

Proof. By considering the degree vertices of the $G = G_1 \ominus G_2$, as represented observation 2.6, and the binomial expansion in the generalized first Zagreb index is calculated as follows:

$$\begin{aligned}
Z^k(G) &= \sum_{x \in V(G)} \deg_G(x)^k \\
&= \sum_{i=1}^{n_1} [1+n_2]^k \deg_{G_1}(v_i)^k + \sum_{i=1}^{m_1} 2^k + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} [2 + \deg_{G_2}(u_j)]^k \\
&= [1+n_2]^k Z^k(G_1) + m_1 2^k + \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{t=0}^{k} \binom{k}{t} 2^t \deg_{G_2}(u_j)^{k-t} \\
&= [1+n_2]^k Z^k(G_1) + m_1 2^k + \sum_{t=0}^{k}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \binom{k}{t} 2^t \deg_{G_2}(u_j)^{k-t} \\
&= [1+n_2]^k Z^k(G_1) + m_1 2^k + \sum_{t=0}^{k} \binom{k}{t} 2^t \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \deg_{G_2}(u_j)^{k-t} \\
&= [1+n_2]^k Z^k(G_1) + m_1 2^k + \sum_{t=0}^{k} \binom{k}{t} 2^t \sum_{i=1}^{n_1} Z^{k-t}(G_2) \\
&= [1+n_2]^k Z^k(G_1) + m_1 2^k + \sum_{t=0}^{k} \binom{k}{t} 2^t n_1 Z^{k-t}(G_2).
\end{aligned}$$

□

Corollary 2.8. *For two arbitrary graph G_1 and G_2 we have*

$$F(G_1 \ominus G_2) = (1+n_2)^3 F(G_1) + 8m_1 + n_1 F(G_2) + 6n_1 M_1(G_2) + 24n_1 m_2 + 8n_1 n_2$$

Proof. It is sufficient to let $k = 3$ in theorem 2.7. □

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The game chromatic number of power two of Cartesian product of two paths

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Abstract

The graph G^2, power two of graph G is defined on vertex set $V(G)$ and two vertices are joined if their distance is at most 2 in graph G. In this paper, we determine the exact value of game chromatic number of power two of Cartesian product $P_2 \Box P_n$, where P_n is a path on n vertices.

1. Introduction

Let G be a simple graph and X be a color set of cardinality k. Consider two players, Alice and Bob. They alternately color a vertex of G with a color from X such that Alice starting first. In this coloring no two two adjacent vertices recieve the same color. In the end if all vertices are colored then Alice wins, otherwise Bob wins. The game chromatic number of G, denoted by $\chi_g(G)$, is the least number$k$ for which Alice has a strategy to win. The game coloring introduced by Bodlaender [4]. Bartnicki et al. [3] determined the exact value $\chi_g(K_2 \Box H)$, where H can be replaced by a path graph (P_n), a cycle graph (C_n) or a cpmplete graph (K_n) with n vertices. Also Sia [6] determined $\chi_g(S_m \Box H)$, where H is P_n or C_n and S_m is a star graph with $n+1$ vertices and $\chi_g(P_2 \Box H)$, where H can be replaced by a wheel graph (W_n) or a complete bipartite graph ($K_{m,n}$). There are some results for power graph, see [1, 2, 5]. The trivial bounds on the game chromatic number are:

$$\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1. \tag{1}$$

where $\chi(G)$ and $\Delta(G)$ are chromatic number and maximum degree of the graph G, respectively.
The dth power graph G^d of a graph G is given by $V(G^d) = V(G)$ and two vertices u and v are adjacent in G^d, if their distance (number of edges in a shortest uv-path) in G is at most d. The Cartesian product of graphs G and H, denoted by $G \Box H$, where two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$. In this paper we determine the exact value of $\chi_g((P_2 \Box P_n)^2)$, where P_n is a path with n vertices.

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2. Game chromatic number of $(P_2\square P_n)^2$

We denote vertices of copies P_n of graph $P_2\square P_n$ by $V_1 = \{v_1, ..., v_n\}$ and $V_2 = \{v'_1, ..., v'_n\}$.

Theorem 2.1. $\chi_g((P_2\square P_2)^2) = 4$

Proof. As $(P_2\square P_2)^2$ isomorphic to the complete graph K_4 and the result can be concluded from trivial bounds. □

Theorem 2.2. $\chi_g((P_2\square P_3)^2) = 5$

Proof. At first, we show that Bob has a winning strategy using 4 colors. For Alice first move consider following the cases:
Case 1: Alice colors a vertex of degree 4.
Without loss of generality, suppose Alice colors vertex v_1 with color 1. Then Bob replies with vertex v'_3 with color 2. In the next move, if Alice colors a vertex of V_1, she must use color 3 and Bob replies with vertex v'_1 with color 4 and Bob wins. Similarly, it is proved that if Alice color a vertex of V_2, Bob wins.
Case 2: Alice colors a vertex of degree 5.
Without loss of generality, suppose Alice colors vertex v_2 with color 1. Then Bob colors vertex v'_2 with color 2. Consider $S = V((P_2\square P_3)^2) - \{v_2, v'_2\}$. The induced subgraph on subset S is a cycle and Bob wins.
Now we give a winning strategy for Alice with 5 colors. Alice in the first move colors vertex v_2 with color 1. After Alice next move, vertex v'_2 is colored and uncolored vertices are of degree 4 and Alice wins.

□

Lemma 2.3. *Let $X = \{1, ..., k\}$ be a color set. Consider uncolored vertex x of degree at least k such that it has at least $k-1$ colored neighbors with $k-1$ colors. If Bob turns then he wins.*

Lemma 2.4. *Let $X = \{1, ..., k\}$ be a color set. Consider uncolored vertices x and y such that they are adjacent and each of them has at least $k-1$ colored adjacent vertices with $k-1$ different colors then Bob wins the game.*

Theorem 2.5. $\chi_g((P_2\square P_4)^2) \geq 6$

Proof. Let $X = \{1, ..., 5\}$ be a color set. We show that Bob has a winning strategy with color set X. For Alice first move consider the following cases:
Case 1: Alice starts of a vertex of degree 6.
Without loss of generality, suppose Alice colors vertex v_2 with color 1. Then Bob replies with vertex v'_4 with color 2.
Case 1.1: Alice colors a vertex from V_1 in her second move.
Now suppose Alice colors vertex v_1 with color 3. Then Bob colors vertex v'_3 with color 4 and by Lemma 2.4 Bob wins. If she colors vertex v_1 with color 2 then Bob replies with vertex v'_3 with color 3. Alice for next move needs color 4 and after her move vertices v_3 or v'_2 have at least 4 colored neighbors with 4 colors and by Lemma 2.3 Bob wins.
Consider Alice colors vertices v_3 or v_4, therefore she needs color 3 for coloring. Then Bob colors v'_1 or v'_2 with color 4 and by Lemma 2.4 Bob wins.
Case 1.2: Alice colors a vertex from V_2 in her second move.
Consider vertex v'_1 and suppose Alice colors it with color 3. Then Bob replies with vertex v_3 with color 4 and by Lemma 2.4 Bob wins. If she colors vertex v'_1 with color 2 then Bob colors vertex v_3 with color 3. After Alice third move, by Lemma 2.3 Bob wins.
Now suppose Alice colors verices v'_2 or v'_3 in second move. She needs color 3. Corresponding Alice' move, Bob colors v_4 or v_1 and by Lemma 2.4 Bob wins.
Case 2: Alice starts of a vertex of degree 4.
Without loss of generality, suppose Alice colors vertex v_1 with color 1. Then Bob replies with vertex v'_3 with color 2.
Case 2.1: Alice colors a vertex from V_1 in her second move.
If Alice color vertices v_2 or v_3 then she must use color 3 for coloring. Then corresponding Alice' move, Bob colors vertex v'_4 or v'_1 with color 4 and by Lemma 2.4 Bob wins. Now suppose she colors vertex v_4. If she uses color 3 then Bob colors vertex v'_2 with color 4 and by Lemma 2.4 Bob wins. Consider she colors with color 1 then Bob colors

vertex v_2' with color 3 and Alice needs color 4 for third move and by Lemma 2.3 Bob wins.
Case 2.2: Alice colors a vertex from V_2 in her second move.
If Alice color vertex v_1' or v_2' then she must use color 3 for coloring. Then corresponding Alice' move, Bob colors vertex v_3 or v_4 with color 4 and by Lemma 2.4 Bob wins. Suppose she colors vertex v_4'. If she uses color 3 then Bob colors vertex v_2 with color 4 and by Lemma 2.4 Bob wins. If Alice colors with color 1 then Bob colors vertex v_2 with color 3. Hence Alice needs color 4 for next move and by Lemma 2.3 Bob wins. $\square$

Theorem 2.6. $\chi_g((P_2 \Box P_4)^2) \leq 6$

Proof. Let $X = \{1, ..., 6\}$ be a color set. We show that Alice has a winning strategy with color set X. At first, Alice colors vertex v_2 with color 1. For Bob first move, consider the following cases:
Case 1: Bob colors a vertex of degree 6.
Bob colors one of the vertices v_3, v_2' or v_3' with color 2. Then Alice corresponding Bob' move, colors one of vertices v_1', v_4 or v_1 with color 2. Therefore every uncolored vertices of degree 6 have two colored neighbors with the same color and all uncolored vertices are colored with this color set.
Case 2: Bob colors a vertex of degree 4.
Bob colors one of vertices v_1, v_4 or v_1' with color 2. Then Alice corresponding Bob' move colors one of vertices v_3', v_2' or v_3 with color 2 and she wins.
If Bob colors vertex v_4' with color 2 then Alice in second move colors vertex v_1' with color 2. After Alice third move, vertex v_3 is colored. Thus every uncolored vertices of degree 6 have two colored neighbors with the same color and Alice wins. $\square$

Corollary 2.7. $\chi_g((P_2 \Box P_4)^2) = 6$

Theorem 2.8. *For every positive integer $n \geq 4$, we have $\chi_g((P_2 \Box P_n)^2) = 6$.*

Proof. As $(P_2 \Box P_4)^2$ is an induced subgraph of graph $(P_2 \Box P_n)^2$, $n \geq 4$, we have $\chi_g((P_2 \Box P_n)^2) \geq 6$. Let $X = \{1, ..., 6\}$ be a color set. We show that Alice has a winning strategy with color set X. At first, Alice colors a vertex of graph. In the following moves, if Bob colors a vertex v_i from V_1 with color j then in first priority Alice colors vertex v_{i+2}' or v_{i-2}' and in second priority she colors v_{i+3} or v_{i-3} with color j. If it is not possible, she colors a vertex among v_{i+4}, v_{i-4}, v_{i+3}' or v_{i-3}' with color j. Otherwise Alice colors another vertex or she uses new color. Similarly, if Bob colors a vertex v_i' from V_2 with color j then Alice colors a vertex among v_{i+2}, v_{i-2}, v_{i+3}' or v_{i-3}' with color j. If it is not possible, she colors a vertex among v_{i+3}, v_{i-3}, v_{i+4}' or v_{i-4}' with color j. If it is not possible, she colors any vertex with an available color. In this strategy any uncolored vertex is adjacent to colored neighbors with at most 5 distinict colors and Alice wins. $\square$

3. Conclusion

In this paper we have determined the exact value of the game chromatic number power two of Cartesian product $P_2 \Box P_n$ and the following results are obtained:

$$\chi_g((P_2 \Box P_n)^2) = \begin{cases} 4 & n = 2 \\ 5 & n = 3 \\ 6 & n \geq 4 \end{cases}$$

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A New Viewpoint Regarding Ordered and Unordered Factorization Of a Positive Integers

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Abstract

As a well-known enumerative problem, the number of solutions to the equation $m = m_1 + \ldots + m_k$ with $m_1 \leqslant \ldots \leqslant m_k$ in positive integers is $\Pi(m,k) = \sum_{i=0}^{k} \Pi(m-k,i)$ and Π is called the additive partition function. In this paper, we give a recursive formula for the number of solutions to the equation $m = m_1 \ldots m_k$ with $m_1 \leqslant \ldots \leqslant m_k$ in positive integers. In particular, using elementary techniques, we give an explicit formula for the cases $k = 1, 2, 3, 4$.

1. Introduction

Let $\mathcal{F}(n;k,\ell)$ be the number of unordered factorizations of a positive integer n to exactly k parts, such that each parts $\geqslant \ell$. We denote the number of all unordered factorizations of a positive integer n by $\mathcal{F}(n)$, that is the number of ways a positive integer n can be written as a product $n = n_1 \times n_2 \times \ldots \times n_k$, where $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k > 1$. The integers $n_1, n_2, \ldots, n_k$ are called the *factors* of the factorization, and it's clearly that $\mathcal{F}(n) = \sum_{k=1}^{n} \mathcal{F}(n;k,2)$. We call $\mathcal{F}(n)$ is *the unordered Factorization function* of n. For example $\mathcal{F}(12)$, corresponding to $2\times 6, 2\times 2\times 3, 3\times 4$ and 12. The sequence $\mathcal{F}(n)$ is listed in [1]. The Dirichlet generating function for $\mathcal{F}(n)$ is

$$\prod_{k=2}^{\infty} \frac{1}{1-k^{-s}} = \sum_{n=1}^{\infty} \frac{\mathcal{F}(n)}{n^s}.$$

For positive integers $\ell, k \geqslant 1$, we denote the number of *ordered factorization* of positive integer n in exactly k parts, such that each part $\geqslant \ell$ by $\mathcal{H}(n;k,\ell)$. We use $\mathcal{H}(n)$ to represent the number of all ordered factorization of the integer n (in analogy with compositions for sum), then $\mathcal{H}(n) = \sum_{k=1}^{n} \mathcal{H}(n;k,2)$. An additive partition of a positive integer n that denoted $p(n)$, is an integer k-tuple $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_k > 0$, for some k, such that $n = n_1 + n_2 + \ldots + n_k$ (in analogy with factorization function $\mathcal{F}(n)$ for product). The integers $n_1, n_2, \ldots, n_k$ are the *parts* of the partitions. For

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example $p(4)$ corresponding to, $1+1+1+1, 1+1+2, 1+3, 2+2$ and 4. It is important note that if $n = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$, where $p_1, p_2, \dots, p_k$ are distinct prime numbers and $\beta_i \in \mathbb{N}$ for $1 \leqslant i \leqslant k$, then $\mathcal{F}(n)$ and $\mathcal{H}(n)$ depend only to $\beta_1, \beta_2, \dots, \beta_k$. For instance, if a positive integer n is a prime power $n = p^k$, $k \geqslant 1$, then $\mathcal{F}(n) = p(k)$, and $\mathcal{H}(n) = 2^{k-1}$. Also, if a positive integer n is square free as $n = p_1 \times p_2 \times \dots \times p_k$ then $\mathcal{F}(n) = \sum_{i=1}^{k} \left\{ {k \atop i} \right\}$, where $\left\{ {n \atop k} \right\}$ is the *Stirling number of the second kind*, and $\mathcal{H}(n) = \sum_{i=1}^{k} i! \left\{ {k \atop i} \right\}$. Let $\mathcal{F}(n; \{\beta_1, \dots, \beta_r\}, \ell)$, be the number of unordered factorizations of a positive integer n as $n = n_1^{\beta_1} \times \dots \times n_r^{\beta_r}$, such that $\beta_1 + \dots + \beta_r = k$ and $\ell \leqslant n_1 < \dots < n_r$, also, $\mathcal{H}(n; \{\beta_1, \dots, \beta_r\}, \ell)$ be the number of ordered factorizations of a positive integer n as $n = n_1^{\beta_1} \times \dots \times n_r^{\beta_r}$, such that $n_i \geqslant \ell$, and $\{n_1, \dots n_k\} = \{n'_1, \dots, n'_r\}$ and $\beta_j = |\{i : n_i = n'_j\}|$, for each $1 \leqslant i, j \leqslant r$. For example, $\mathcal{F}(n; \{1, 1, 2\}, \ell)$ is the number of unordered factorization positive integer n as the form xyz^2, where x, y and z are different positive integers and $x > y > z \geqslant \ell$ and $\mathcal{H}(n; \{1, 1, 2\}, \ell) = 2! \mathcal{F}(n; \{1, 1, 2\}, \ell)$. It is easy to see that

$$\mathcal{F}(n; \{\beta_1, \dots, \beta_r\}, \ell) = \frac{(\beta_1 + \dots + \beta_r)!}{\beta_1! \dots \beta_r!} \mathcal{H}(n; \{\beta_1, \dots, \beta_r\}, \ell).$$

More on factorization partitions, including results on bounds and asymptotes of $\mathcal{F}(n)$ and algorithms for calculating the values, can be found in [2–4] and [5].

The goal of this paper is to give some recursive formula for $\mathcal{F}(n)$ and $\mathcal{H}(n)$ also we obtain $\mathcal{F}(n, k, \ell)$ and $\mathcal{H}(n, k, \ell)$ for cases $n = 2, 3, 4$, with elementary ways. Also, we give another proof of general formula for the number $\mathcal{H}_\ell(n, k)$ of ordered factorizations of a positive integer n in exactly k factors that each factor greater than 1, was found in 1893 by *MacMahon* [6]:

$$\mathcal{H}(n; k, 2) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{n} \binom{\beta_j + k - i - 1}{k - i - 1}.$$

At the end, we closed our paper by posting several propositions about additive partition function $p(n)$.

2. Results

In this section we give some recursive formula $\mathcal{F}(n; k, \ell)$ and $\mathcal{H}(n; k, \ell)$. Let $n = n_1^{\beta_1} \times \dots \times n_r^{\beta_r}$ be a positive integer, where $\beta_i \in \mathbb{N}$. By using of above notations, we can write

$$\mathcal{F}(n; k, \ell) = \sum_{\substack{\beta_1 + \dots + \beta_r = k; \\ \beta_1 < \dots < \beta_r,}} \mathcal{F}(n; \{\beta_1, \dots, \beta_r\}, \ell); \tag{1}$$

and

$$\mathcal{H}(n; k, \ell) = \sum_{\beta_1 + \dots + \beta_r = k} \mathcal{H}(n; \{\beta_1, \dots, \beta_r\}, \ell). \tag{2}$$

Theorem 2.1. *Let $n > 1$ and k, ℓ be positive integers. Suppose that ℓ^s divides n but ℓ^{s+1} does not divide n. Then*

$$\mathcal{F}(n; k, \ell) = \sum_{i=\max\{k-s, 1\}}^{\min\{k, s\}} \mathcal{F}(n; i, \ell + 1).$$

Corollary 2.2. *Let $n > 1$ and k, ℓ be positive integers. Then*

$$\mathcal{F}(n; k, 1) = \sum_{i=1}^{k} \mathcal{F}(n; i, 2).$$

Lemma 2.3. *Let n, k and ℓ be positive integers. Then*

$$\mathcal{F}(n; k, \ell) = \sum_{\ell \leqslant d | n} \mathcal{F}_d(\frac{n}{d}, k - 1).$$

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Graph of derangements

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Abstract

The graph of derangements on n elements is the graph whose vertex set is derangements on n elements. Also, two vertices are adjacent if one is a derangement of the other. In this paper, we observe that the graph of derangements is connected.

1. Introduction

Let G be a finite transitive group on the set Ω. An element $g \in G$ is called a derangement if g has no fixed points on Ω, that is, $\alpha^g \neq \alpha$ for each $\alpha \in \Omega$. The number of derangements is given by the recurrence relation,

$$D(n) = \begin{cases} 0 & \text{if } n = 1; \\ 1 & \text{if } n = 2; \\ (n-1)(D(n-1) + D(n-2)) & \text{if } n \geq 3\,. \end{cases}$$

Suppose that S_n is the symmetric group on the set $[n] = \{1, 2, \dots, n\}$. consider (1) the trivial permutation which fixes each element. Let D be the set of all permutations of the values $1, \dots, n$, which is a derangement of (1). The function A for $d_i, d_j \in D$ is defined as a following

$$A(d_i, d_j) = \begin{cases} 1 & \text{if } d_j \text{ is a derangement of } d_i; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $D(n) = \{d |\ d \in S_n, A((1), d) = 1\}$.

Definition 1.1. The graph of derangement $D_n = (V, E)$ is the graph whose vertices are derangements on n elements such that an edge connects two vertices $d_i, d_j \in V$ if and only if $A(d_i, d_j) = 1$.

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Example 1.2. *The $D_4 = (V, E)$ for four elements is the graph whose vertices an edges are as follows:*

$$V = \{2143, 4312, 3421, 3412, 4321, 3142, 2413, 2341, 4123\},$$

$$E = \{(a, b)|a, b \in K_i, i = 1, \dots, 4\}$$

where

$$K_1 = \{2143, 3421, 4312\}, K_2 = \{2143, 3412, 4321\},$$
$$K_3 = \{2413, 3142, 4321\}, K_4 = \{2341, 3412, 4123\}.$$

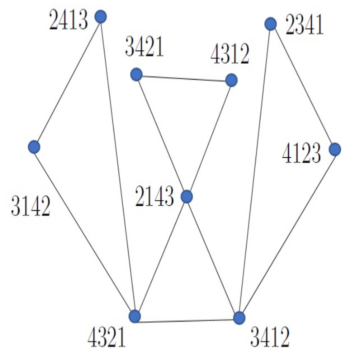

Fig. 1. (Graph D_4 with four 3-cycle)

Definition 1.3. Let A be a finite set, and let $\mathcal{A} = (A_1, A_2, \dots, A_n)$ be a collection of subsets of A. A system of representatives of $\mathcal{A}$ is a collection of elements $x_1, x_2, ..., x_n$ such that $x_i \in A_i$ for all $i \in \{1, 2, \dots, n\}$. A Distinct System of Representatives (SDR) of $\mathcal{A}$ is a collection of elements $x_1, x_2, \dots, x_n$ such that $x_i \neq x_j$ for all $i, j \in \{1, 2, \dots, n\}$ and $i \neq j$.

Example 1.4. *Let $A = \{x_1, x_2, x_3, x_4, x_5\}$. Also, consider the collection of subsets $\mathcal{A} = (A_1, A_2, A_3, A_4, A_5)$ defined to be $A_1 = \{x_1, x_2\}$, $A_2 = \{x_2, x_3\}$, $A_3 = \{x_3, x_4\}$, $A_4 = \{x_4, x_5\}$, and $A_5 = \{x_1, x_5\}$. If we let $x_1 \in A_1$ represent A_1, $x_2 \in A_2$ represent A_2, ..., $x_5 \in A_5$ to represent A_5, then the collection of elements x_1, x_2, x_3, x_4, x_5 is a SDR of $\mathcal{A}$.*

Theorem 1.5 ([1]). *Let $A_l, \dots, A_n$ be subsets of S such that, for some m,*

(i) $|A_i| = m$ for each $i \in [m]$

(ii) each element of S occurs in exactly m of the A_i.

Then $A_1, \dots, A_n$ possesses an SDR.

2. main theorem

Theorem 2.1. *Consider a D_n graph (V, E) for n elements. Then D_n is a connected graph*

Proof. First, suppose that $n = 4$. By Example 1.2, it is easy to see that graph D_4 is connected. Now, supposed that $n \geq 5$and $\alpha = (\alpha_1 \dots \alpha_n)$ and $\beta = (\beta_1 \dots \alpha_n)$ are two arbitrary vertex of V. Let $\alpha_i \neq \beta_j$ for $i, j \in [n]$. This means that there is one edge between two vertices α and β. Now, suppose that there exists at least one $i \in [n]$ such that

$\alpha_i = \beta_i$. With loss of generality, Assume that $\alpha_1 = \beta_1$. We have shown that there exists one path between vertices α and β. Consider the following reduced Latin rectangle $3 \times n$ on $[n]\backslash\{1\}$.

2	...	n
α_2	...	α_n
β_2	...	β_n

Let A_i denote the set of elements of $[n]\backslash\{1\}$ that do not occur in the ith column of Latin rectangle. Then by Theorem 1.5, set A_i possesses an SDR which can be extended to a $4 \times n$ Latin rectangle on $[n]\backslash\{1\}$. Set $\gamma' = (\gamma_2 \dots \gamma_n)$ as a 4th row of the required rectangle. Assume that $\gamma_1 \neq \alpha_1$ and $\gamma_1 \in [n]\backslash\{\gamma_2, \dots, \gamma_n\}$, Then it is easy to check that $\gamma = (\gamma_1 \dots \gamma_n)$ is a derangement on n vertex and so $\gamma \in V$. Since $\gamma_i \neq \beta_j$ for $i, j \in [n]$, then there exist an arc between α and γ. Similarly, there exist an arc between γ and β. Similar to the above argument in other cases, it can show that graph A is connected. □

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Total Domination Number of 4-regular Knödel Graphs

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Pigeonhole Principle

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Abstract

A subset D of vertices of a graph G is a *total dominating set* if for each $u \in V(G)$, u is adjacent to some vertex $v \in D$. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. For an even integer $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, a *Knödel graph* $W_{\Delta,n}$ is a Δ-regular bipartite graph of even order n, with vertices (i, j), for $i = 1, 2$ and $0 \leq j \leq \frac{n}{2} - 1$, where for every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod \frac{n}{2})$, for $k = 0, 1, \cdots, \Delta - 1$. In this paper, we determine the total domination number in 4-regular Knödel graphs $W_{4,n}$.

1. Introduction

For graph theory notation and terminology not given here, we refer to [15]. Let $G = (V, E)$ denote a simple graph of order $n = |V(G)|$ and size $m = |E(G)|$. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$. The *open neighborhood* of a vertex $u \in V(G)$ is denoted by $N(u) = \{v \in V(G) | uv \in E(G)\}$ and for a vertex set $S \subseteq V(G)$, $N(S) = \bigcup\limits_{u \in S} N(u)$. The cardinality of $N(u)$ is called the *degree* of u and is denoted by $\deg(u)$, (or $\deg_G(u)$ to refer it to G). The *maximum degree* and *minimum degree* among all vertices in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is a *bipartite graph* if its vertex set can be divide into two disjoint sets X and Y such that each edge in $E(G)$ connects a vertex in X with a vertex in Y. A set $D \subseteq V(G)$ is a *dominating set* if for each $u \in V(G) \setminus D$, u is adjacent to some vertex $v \in D$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. A set $D \subseteq V(G)$ is a *total dominating set* if for each $u \in V(G)$, u is adjacent to some vertex $v \in D$. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. The concept of domination theory is a widely studied concept in graph theory and for a comprehensive study see, for example [14, 15].
An interesting family of graphs namely *Knödel graphs* have been introduced about 1975 [17], and have been studied seriously by some authors since 2001, see for example [1–4, 7–9]. For an even integer $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, a *Knödel graph* $W_{\Delta,n}$ is a Δ-regular bipartite graph of even order n, with vertices (i, j), for $i = 1, 2$ and $0 \leq j \leq \frac{n}{2} - 1$, where for every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod \frac{n}{2})$, for $k = 0, 1, \cdots, \Delta - 1$ (see [23]). Knödel graphs, $W_{\Delta,n}$, are one of the three important families of graphs that they have good properties in terms of broadcasting and gossiping, see for example [5, 6, 10–13, 16]. It is worth-noting that any Knödel graph is a Cayley graph and so it is a vertex-transitive graph (see [3]).

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For simplicity, in this paper, we re-label the vertices of a Knödel graph as follows: we label $(1, i)$ by u_{i+1} for each $i = 0, 1, ..., \frac{n}{2} - 1$, and $(2, j)$ by v_{j+1} for $j = 0, 1, ..., \frac{n}{2} - 1$. Let $U = \{u_1, u_2, \cdots, u_{\frac{n}{2}}\}$ and $V = \{v_1, v_2, \cdots, v_{\frac{n}{2}}\}$. From now on, the vertex set of each Knödel graph $W_{\Delta,n}$ is $U \cup V$ such that U and V are the two partite sets of the graph. If S is a set of vertices of $W_{\Delta,n}$, then clearly, $S \cap U$ and $S \cap V$ partition S, $|S| = |S \cap U| + |S \cap V|$, $N(S \cap U) \subseteq V$ and $N(S \cap V) \subseteq U$. Note that two vertices u_i and v_j are adjacent if and only if $j \in \{i+2^0-1, i+2^1-1, \cdots, i+2^{\Delta-1}-1\}$, where the addition is taken in modulo $\frac{n}{2}$. Figure 1, shows new labeling of Knödel graphs $W_{3,14}$ and $W_{4,16}$.

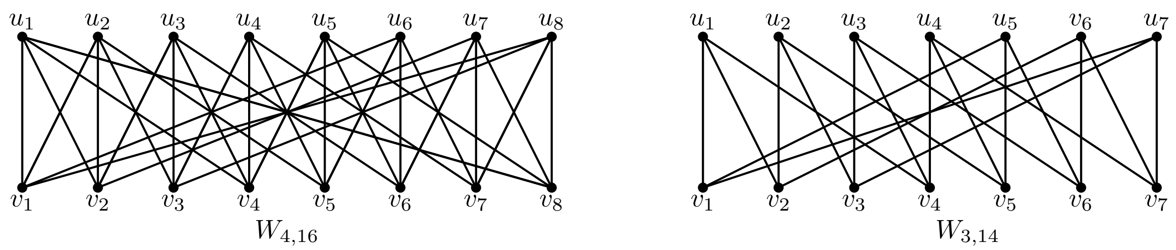

Fig. 1. New labeling of Knödel graphs $W_{4,16}$ and $W_{3,14}$.

Some domination parameters in Knödel graphs are studied in [22]. Xueliang et. al. [23] studied the domination number in 3-regular Knödel graphs $W_{3,n}$. They obtained the exact domination number for $W_{3,n}$. Mojdeh et. al. [19, 20] determined the exact total domination number for $W_{3,n}$ and the exact domination number for $W_{4,n}$. Domination critical and stable Knödel graphs are studied in [18] and the diameter of the general Knödel Graphs is discussed in [21]. In this paper, we determine the total domination number in 4-regular Knödel graphs $W_{4,n}$. Infact, we will prove the following:

$$\gamma_t(W_{4,n}) = 8\left\lfloor\frac{n-4}{26}\right\rfloor + 2\left\lfloor\frac{n-4}{8} - \frac{13}{4}\left\lfloor\frac{n-4}{26}\right\rfloor\right\rfloor + 4 = 8\left\lfloor\frac{n}{26}\right\rfloor + \begin{cases} 0 & n \equiv 0 \textit{ (mod 26)} \\ 2 & n \equiv 2 \textit{ (mod 26)} \\ 4 & n \equiv 4, 6, 8, 10 \textit{ (mod 26)} \\ 6 & n \equiv 12, 14, 16, 18 \textit{ (mod 26)} \\ 8 & n \equiv 20, 22, 24 \textit{ (mod 26)} \end{cases} \quad , \quad n \geq 16$$

In Section 2, we state some necessary lemmas, and in the Section 3 we prove our main result.

2. Necessary lemmas

In this section we state some necessary definitions and lemmas which we need for the proof of the main result. For any subset $\{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$, we correspond a sequence based on the differences of the indices of u_j, $j = i_1, ..., i_k$, as follows.

Definition 2.1. For any subset $A = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ we define a sequence $n_1, n_2, \cdots, n_k$, namely **cyclic-sequence**, where $n_j = i_{j+1} - i_j$ for $1 \leq j \leq k-1$ and $n_k = \frac{n}{2} + i_1 - i_k$. For two vertices $u_{i_j}, u_{i_{j'}} \in A$ we define **index-distance** of u_{i_j} and $u_{i_{j'}}$ by $id(u_{i_j}, u_{i_{j'}}) = min\{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$.

Observation 2.2. *[20] Let $A = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\} \subseteq U$ be a set such that $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ and let $n_1, n_2, \cdots, n_k$ be the corresponding cyclic-sequence of A. Then,*
(1) $n_1 + n_2 + \cdots + n_k = \frac{n}{2}$.
(2) If $u_{i_j}, u_{i_{j'}} \in A$, then $id(u_{i_j}, u_{i_{j'}})$ equals to sum of some consecutive elements of the cyclic-sequence of A and $\frac{n}{2} - id(u_{i_j}, u_{i_{j'}})$ is sum of the remaining elements of the cyclic-sequence. Furthermore, $\{id(u_{i_j}, u_{i_{j'}}), \frac{n}{2} - id(u_{i_j}, u_{i_{j'}})\} = \{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$.

We henceforth use the notation $\mathscr{M}_\Delta = \{2^a - 2^b : 0 \leq b < a < \Delta\}$ for $\Delta \geq 2$. For example, $\mathscr{M}_4 = \{1, 2, 3, 4, 6, 7\}$.

Lemma 2.3. *[20] In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for two distinct vertices u_i and u_j, $N(u_i) \cap N(u_j) \neq \emptyset$ if and only if $id(u_i, u_j) \in \mathscr{M}_\Delta$ or $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$.*

Lemma 2.4. *[20] In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for two distinct vertices u_i and u_j, $|N(u_i) \cap N(u_j)| = 2$ if and only if $id(u_i, u_j) \in \mathscr{M}_\Delta$ and $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$.*

Corollary 2.5. *[20] (i) In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for each $1 \leq i < j \leq \frac{n}{2}$, $|N(u_i) \cap N(u_j)| = 1$ if and only if precisely one of the values $id(u_i, u_j)$ and $\frac{n}{2} - id(u_i, u_j)$ belongs to $\mathscr{M}_\Delta$.*
(ii) In the Knödel graph $W_{\Delta,n}$, there exist distinct vertices with two common neighbors if and only if $n = 2^a - 2^b + 2^c - 2^d$ and $a > b \geq 1, c > d \geq 1$.

Corollary 2.6. *[20] Any three vertices in the Knödel graph $W_{\Delta,n}$ have at most one common neighbor. Indeed, any Knödel graph is a $K_{2,3}$-free graph.*

Lemma 2.7. *[20] In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$ and $\Delta < \log_2(\frac{n}{2} + 2)$, we have:*
(i) $|N(u_i) \cap N(u_j)| \leq 1$, $1 \leq i < j \leq \frac{n}{2}$.
(ii) $|N(u_i) \cap N(u_j)| = 1$ if and only if $id(u_i, u_j) \in \mathscr{M}_\Delta$.

Lemma 2.8. *[20] Let $W_{\Delta,n}$ be a Knödel graph with vertex set $U \cup V$. For any non-empty subset $A \subseteq U$:*
(i) $\sum\limits_{v \in N(A)} |N(v) \cap A| = \Delta|A|$.
(ii) The corresponding cyclic-sequence of A has at most $\Delta|A| - |N(A)|$ elements belonging to $\mathscr{M}_\Delta$.

We remark that one can define the cyclic-sequence and index-distance for any subset of V in a similar way, and thus the Observation 2.2, Lemmas 2.3 and 2.4 and corollaries 2.5 and 2.6 are valid for cyclic-sequence and index-distance on subsets of V as well. Therefore, we have:

Lemma 2.9. *The subset $D = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ dominates V if and only if the subset $D^* = \{v_{\frac{n}{2}+1-i_1}, v_{\frac{n}{2}+1-i_2}, \cdots, v_{\frac{n}{2}+1-i_k}\}$ of V dominates U.*

Proof. It is obvious by the definition or by vertex-transitivity of Knödel graphs. □

Remark 2.10. *We name D^* as dual of D. By Lemma 2.9, if D dominates V, then $D \cup D^*$ is a total dominating set of the Knödel graph. In section 3, we find a subset $D \subseteq U$ with minimum number of vertices that dominates V.*

3. Total domination number of $W_{4,n}$

Before starting the proof of the theorem, we explain some observations about the cyclic-sequnce of a subset $A \subseteq U$ in the case $\Delta = 4$.

Observation 3.1. *In a Knödel graph $W_{4,n}$, let $A = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ be a subset of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ and $n_1, n_2, \cdots, n_k$ be the cyclic-sequence of A. If A dominates V, we have:*
i) $1 \leqslant n_i \leqslant 8$, $i = 1, 2, \cdots, k$.
ii) If $n_i = 8$, then $n_{i-3} = n_{i-2} = n_{i-1} = 1$.
iii) If $n_i = n_{i+1}$ for some i, then $n_i \leqslant 4$.
iv) If $n_i = 5$, then $n_{i-2}, n_{i-1}, n_{i+1}, n_{i+2} \in \{1, 2, 3, 4\}$.
v) If $n_i = n_{i+3} = 5$, then $n_{i+1} = 2$ and $n_{i+2} = 1$.
vi)) If $n_i = n_{i+4} = 5$, then $n_{i+1}, n_{i+2}, n_{i+3} \in \{1, 2, 3, 4\}$.

We are now ready to determine the total domination number of $W_{4,n}$. Clearly $n \geq 16$ is an even integer by the definition of $W_{4,n}$. We will prove that for each even integer $n \geq 16$,

$$\gamma_t(W_{4,n}) = 8\left\lfloor \frac{n}{26} \right\rfloor + \begin{cases} 0 & n \equiv 0 \text{ (mod 26)} \\ 2 & n \equiv 2 \text{ (mod 26)} \\ 4 & n \equiv 4, 6, 8, 10 \text{ (mod 26)} \\ 6 & n \equiv 12, 14, 16, 18 \text{ (mod 26)} \\ 8 & n \equiv 20, 22, 24 \text{ (mod 26)} \end{cases}$$

Observation 3.2. *In Table 1, we show a minimum dominating set of V for all Knödel graphs $W_{4,n}$, $n \in \{16, 18, \cdots, 40\}$.*

Table 1.

A minimum dminating set of V	Knödel graph	Total domination number
$D_8 = \{u_0, u_3, u_4\}$	$W_{4,16}$	6
$D_9 = \{u_0, u_4, u_5\}$	$W_{4,18}$	6
$D_{10} = \{u_0, u_3, u_5, u_8\}$	$W_{4,20}$	8
$D_{11} = \{u_0, u_4, u_6, u_7\}$	$W_{4,22}$	8
$D_{12} = \{u_0, u_4, u_6, u_7\}$	$W_{4,24}$	8
$D_{13} = \{u_0, u_3, u_5, u_8\}$	$W_{4,26}$	8
$D_{14} = \{u_0, u_3, u_5, u_9, u_{10}\}$	$W_{4,28}$	10
$D_{15} = \{u_0, u_1, u_2, u_3, u_{10}, u_{11}\}$	$W_{4,30}$	12
$D_{16} = \{u_0, u_1, u_2, u_3, u_{11}, u_{12}\}$	$W_{4,32}$	12
$D_{17} = \{u_0, u_3, u_5, u_9, u_{11}, u_{12}\}$	$W_{4,34}$	12
$D_{18} = \{u_0, u_3, u_5, u_8, u_{13}, u_{16}\}$	$W_{4,36}$	12
$D_{19} = \{u_0, u_3, u_5, u_8, u_{11}, u_{13}, u_{14}\}$	$W_{4,38}$	14
$D_{20} = \{u_0, u_3, u_5, u_6, u_{11}, u_{15}, u_{16}\}$	$W_{4,40}$	14

Observation 3.3. *In the Knödel graph $W_{4,n}$, if $n \geqslant 42$ and $n = 26k + 2t$ where $k \geqslant 1$ and $8 \leqslant t \leqslant 20$, then*

$$D = D_t \cup \{u_{t+13i}, u_{t+3+13i}, u_{t+5+13i}, u_{t+8+13i} : i = 0, 1, \cdots, k-1\}$$

dominates V and $|D| = |D_t| + 4k$.

Corollary 3.4. *By Observations 3.2 and 3.3, we have:*

$$\gamma_t(W_{4,n}) \leqslant 8\left\lfloor \frac{n}{26} \right\rfloor + \begin{cases} 0 & n \equiv 0 \pmod{26} \\ 2 & n \equiv 2 \pmod{26} \\ 4 & n \equiv 4, 6, 8, 10 \pmod{26} \\ 6 & n \equiv 12, 14, 16, 18 \pmod{26} \\ 8 & n \equiv 20, 22, 24 \pmod{26} \end{cases}$$

Now, we have an upper bound for the total domination number of all 4-regular Knödel Graphs.

Lemma 3.5. *If $W_{4,n}$ be a Knödel Graph with $n = 26k + 2t$, where $k \geqslant 1$ and $t \in \{8, 10, \cdots, 40\}$, then $\gamma_t(W_{4,n}) = 8k + 2|D_t|$.*

Proof. On the contrary, suppose that $\gamma_t(W_{4,n}) < 8k + 2|D_t|$ and $W_{4,n}$ has a total dominating set $D = A \cup A^*$ with $8k + 2(|D_t| - 1)$ vertices. Now, we have $\frac{26k+2t}{8k+2(|D_t|-1)} = 3 + \frac{2k+2(t-3|D_t|+3)}{8k+2(|D_t|-1)} \geqslant 3$, that means each vertex in dominating set dominates 3 vertices in average. According to Lemma 2.8, the number of 5's or 8's in the cyclic-sequnece of a minimum total dominating set should be as large as possible.
By Observation 3.1(ii), for each 8 in cyclic-sequence, there exists three consequent 1 in it. Hence, there is four vertices that dominate only 11 vertices, less than avsrage 3, a contradiction.
By Observation 3.1(v), If we see the string $\cdots, 5, 2, 1, 5, 2, 1, \cdots$ in cyclic-sequence, then there is three vertices that dominate only 8 vertices, less than avsrage 3, a contradiction.
Finaly, we consider the case that introduced in part (vi) of Observation 3.2. In this case, the number of 5's in the cyclic-sequence of $A(\subseteq U)$ is at most $\frac{1}{4}|A|$. Between two 5's there exist three elements less than 5. The string $\cdots, 5, 3, 2, 3, 5, 3, 2, 3, \cdots$ in a cyclic-sequence is the best choice such that each 4 vertices of A can dominate 13 vertices in V. We have $|N(A)| \leqslant \frac{13}{4}(4k + |D_t| - 1) = 13k + \frac{13}{4}(|D_t| - 1) < 13k + t$, and so A does not dominate V, a contradiction.

□

Theorem 3.6. *For each even integer $n \geq 16$, we have:*

$$\gamma_t(W_{4,n}) = 8\left\lfloor \frac{n-4}{26} \right\rfloor + 2\left\lfloor \frac{n-4}{8} - \frac{13}{4}\left\lfloor \frac{n-4}{26} \right\rfloor \right\rfloor + 4$$
$$= 8\left\lfloor \frac{n}{26} \right\rfloor + \begin{cases} 0 & n \equiv 0 \pmod{26} \\ 2 & n \equiv 2 \pmod{26} \\ 4 & n \equiv 4, 6, 8, 10 \pmod{26} \\ 6 & n \equiv 12, 14, 16, 18 \pmod{26} \\ 8 & n \equiv 20, 22, 24 \pmod{26} \end{cases}$$

Proof. See Observations 3.2 and 3.3, Corollary 3.4 and Lemma 3.5. □

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on domination number of middle of special family of graphs

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Abstract

In this paper, we study the domination number of middle graphs. Indeed, we obtain tight bounds for this number in terms of the order of the graph G. We also compute the domination number of some families of graphs such as star graphs, double start graphs, path graphs, cycle graphs, wheel graphs, complete graphs, complete bipartite graphs and friendship graphs, explicitly.

1. Introduction

The notion of domination and its many generalizations have been intensively studied in graph theory and the literature on this subject is vast, see for example [2], [3], [4], [6], [7] and [5]. Throughout this paper, we use standard notation for graphs and we assume that each graph is non-empty, finite, undirected and simple. We refer to [1] as a general reference on graph theory.
Let G be a graph with vertex set $V(G)$ of *order* n and edge set $E(G)$ of *size* m. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and, similarly, the *closed neighborhood* of a vertex $v \in V(G)$ is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is defined as $d_G(v) = |N_G(v)|$. The *distance* $d_G(v, w)$ in G of two vertices $v, w \in V(G)$ is the length of the shortest path connecting v and w. The *diameter* of G, denoted $diam(G)$, is the shortest distance between any two vertices in G.

Definition 1.1. A *dominating set*, briefly DS, of a graph G is a set $S \subseteq V(G)$ such that $N_G[v] \cap S \neq \emptyset$, for any vertex $v \in V(G)$. The *domination number* of G is the minimum cardinality of a DS of G and it is denoted by $\gamma(G)$.

For any non-empty $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced on the vertex set S. For any $v \in V(G)$, we denote by $G \setminus v$ the subgraph of G induced on the vertex set $V(G) \setminus \{v\}$. Given two graphs G and H with distinct vertices, we can construct a new graph $G \cup H$ by imposing $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$.

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Given a graph G, its *complement*, denoted by $\overline{G}$, is a graph with vertex set $V(G)$ such that for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$.
The *line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$, where vertices x and y are adjacent in $L(G)$ if and only if the corresponding edges x and y share a common vertex in G.
The concept of middle graph $M(G)$ of a graph G was introduced by Hamada and Yoshimura in [**?**] as an intersection graph on the vertex set of G.

Definition 1.2. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one the following holds

1. $x, y \in E(G)$ and x, y are adjacent in G;
2. $x \in V(G)$, $y \in E(G)$, and x, y are incident in G.

In other words, the middle graph $M(G)$ of a graph G of order n and size m is a graph of order $n + m$ and size $2m + |E(L(G))|$ which is obtained by subdividing each edge of G exactly once and joining all the adjacent edges of G in $M(G)$. It is obvious that $M(G)$ always contains the line graph $L(G)$ as an induced subgraphs.
In order to avoid confusion throughout the paper, we fix a "standard" notation for the vertex set and the edge set of the middle graph $M(G)$. Assume $V(G) = \{v_1, v_2, \ldots, v_n\}$, then we set $V(M(G)) = V(G) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{ij} \mid v_iv_j \in E(G)\}$ and $E(M(G)) = \{v_im_{ij}, v_jm_{ij} \mid v_iv_j \in E(G)\} \cup E(L(G))$.
The paper proceeds as follows. In Section 2, first we present some upper and lower bounds for $\gamma(M(G))$ in terms of the order of the graph G. In Section 3, we compute explicitly $\gamma(M(G))$ for several known families of graphs: star graphs, double star graphs, path graphs, cycle graphs, wheel graphs, complete graphs, complete bipartite graphs, corona graphs, 2-corona graphs, join of graphs and friendship graphs.

2. General Bounds

We start our study of domination numbers of middle graph with two key Lemmas.

Lemma 2.1. *Let G be a graph of order $n \geq 2$ without isolated vertices and S a dominating set of $M(G)$. Then there exists $S' \subseteq E(G)$ a dominating set of $M(G)$ with $|S'| \leq |S|$.*

Proof. If $S \subseteq E(G)$, then take $S' = S$. On the other hand, assume that there exists $v \in S \cap V(G)$. If all incident edges to v are already in S, then take $S_1 = S \setminus \{v\}$. Otherwise, let $e \in E(G) \setminus S$ an edge incident to v. Then consider $S_1 = (S \cup \{e\}) \setminus \{v\}$. By construction, S_1 is again a dominating set of $M(G)$. Since S is finite, then this process terminates after a finite number of steps, and hence we obtain the described S'.

□

Lemma 2.2. *Let G be a graph of order $n \geq 2$ and $v \in V(G)$. Then*

$$\gamma(M(G \setminus v)) \leq \gamma(M(G)) \leq \gamma(M(G \setminus v)) + 1.$$

Proof. For any dominating set S of $M(G \setminus v)$, we have that $S \cup \{v\}$ is a dominating set of $M(G)$, and hence $\gamma(M(G)) \leq \gamma(M(G \setminus v)) + 1$.
On the other hand, let S be a minimal dominating set of $M(G)$. If v is an isolated vertex, then $v \in S$ and $S \setminus \{v\}$ is a minimal dominating set of $M(G \setminus v)$. This implies that in this case $\gamma(M(G)) = \gamma(M(G \setminus v)) + 1$. Assume that G has no isolated vertices. By Lemma 2.1, we can assume that $S \subseteq E(G)$. Consider $S_v = N_{M(G)}(v) \cap S$. Since S is a dominating set, $|S_v| \geq 1$. Assume $S_v = \{e_1, \ldots, e_k\}$. For any $1 \leq i \leq k$, e_i is an edge of G of the form w_iv. Define $S' = (S \setminus S_v) \cup \{w_1, \ldots, w_k\}$. By construction S' is a dominating set of $M(G \setminus v)$ with $|S'| = |S|$, and hence $\gamma(M(G \setminus v)) \leq \gamma(M(G))$.

□

We start our study of the domination number by describing a lower and an upper bound for the domination number of the middle graph of a tree.

Theorem 2.3. *Let T be a tree with $n \geq 2$ vertices. Then*

$$\lceil \frac{n}{2} \rceil \leq \gamma(M(T)) \leq n-1.$$

If we denote by $leaf(T) = \{v \in V(T) \mid d_T(v) = 1\}$ the set of leaves of a tree T, then we have the following result.

Proposition 2.4. *Let T be a tree with $n \geq 2$ vertices. Then*

$$\gamma(M(T)) \geq |leaf(T)|.$$

If we specialize the class of trees that we are considering, we obtain an exact value for the domination number.

Theorem 2.5. *Let T be a tree of order $n \geq 4$ with $diam(T) = 3$. Then*

$$\gamma(M(T)) = n-2.$$

Since path graphs are special type of trees, in general for a tree T, $\gamma(M(T)) = n-2$ does not imply $diam(T) = 3$, as the next example shows.

Example 2.6. *Consider the path graph P_5. Then $diam(P_5) = 4$ and $\gamma(M(P_5)) = 3 = n-2$.*

Next we describe a lower and an upper bound for the domination number of the middle graph of an arbitrary graph.

Theorem 2.7. *Let G be a graph with $n \geq 2$ vertices. Assume G has no isolated vertices, then*

$$\lceil \frac{n}{2} \rceil \leq \gamma(M(G)) \leq n-1.$$

3. Middle graph of special family of graphs

In this section, we obtain the domination number of the middle graph of some special families of graphs.

Proposition 3.1. *For any star graph $K_{1,n}$ on $n+1 \geq 2$ vertices, we have*

$$\gamma(M(K_{1,n})) = n.$$

Theorem 3.2. *Let G be a connected graph of order $n \geq 4$. Then*

$$G = K_{1,n-1} \quad \text{if and only if} \quad \gamma(M(G)) = n-1.$$

Putting together Theorems 2.7 and 3.2, we obtain the following result.

Corollary 3.3. *Let G be a connected graph of order $n \geq 4$. Assume that $G \neq K_{1,n-1}$, then*

$$\gamma(M(G)) \leq n-2.$$

Next we calculate the domination number of double star graph $S_{1,n,n}$. Notice that the graph $S_{1,n,n}$ is important because it is an example of a non-complete bipartite graph.

Definition 3.4. A double star graph $S_{1,n,n}$ is obtained from the star graph $K_{1,n}$ by replacing every edge with a path of length 2.

Proposition 3.5. *For any double star graph $S_{1,n,n}$ on $2n+1$ vertices, with $n \geq 2$, we have*

$$\gamma(M(S_{1,n,n})) = n+1.$$

Proposition 3.6. *For any path P_n of order $n \geq 2$, we have*

$$\gamma(M(P_n)) = \lceil \frac{n}{2} \rceil.$$

Proof. To fix the notation, assume $V(P_n) = \{v_1, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. Then $V(M(P_n)) = V(P_n) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{i(i+1)} \mid 1 \leq i \leq n-1\}$.
Assume that n is even and consider $S = \{m_{12}, m_{34}, \dots, m_{(n-1)n}\}$. Then S is a dominating set for $M(P_n)$ with $|S| = \frac{n}{2}$. Similarly, if n is odd, consider $S = \{m_{12}, m_{34}, \dots, m_{(n-2)(n-1)}, m_{(n-1)n}\}$. Then S is a dominating set for $M(P_n)$ with $|S| = \frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$. This shows that $\gamma(M(P_n)) \leq \lceil \frac{n}{2} \rceil$. On the other hand, by Theorem 2.3, $\gamma(M(P_n)) \geq \lceil \frac{n}{2} \rceil$.

□

Remark 3.7. *Since the star graphs and the path graphs are examples of trees, by Propositions 3.1 and 3.6, the inequalities of Theorems 2.3 and 2.7 are all sharp.*

Using Theorem 2.7 and Proposition 3.6 we obtain the following result.

Theorem 3.8. *Let G be a graph with $n \geq 2$ vertices. If G has a subgraph isomorphic to P_n, then*

$$\gamma(M(G)) = \lceil \frac{n}{2} \rceil.$$

Notice that in general the opposite implication of Theorem 3.8 is false, in fact we have the following example.

Example 3.9. *Let G be the graph with vertex set $V(G) = \{v_0, v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{v_0v_1, v_0v_2, v_0v_3, v_3v_4\}$. Then a direct computation shows that $\gamma(M(G)) = 3 = \lceil \frac{5}{2} \rceil$, but G has no subgraphs isomorphic to P_5.*

Since any cycle graph C_n, any wheel graph W_n and any complete graph K_n contain a subgraph isomorphic to P_n, Theorem 3.8 gives us the following result.

Corollary 3.10. *Let $n \geq 3$. Then*

$$\gamma(M(C_n)) = \gamma(M(W_n)) = \gamma(M(K_n)) = \lceil \frac{n}{2} \rceil.$$

Proposition 3.11. *Let K_{n_1,n_2} be the complete bipartite graph with $n_2 \geq n_1 \geq 1$. Then*

$$\gamma(M(K_{n_1,n_2})) = n_2.$$

Notice that if we consider the case when $n_1 = 1$, the previous result gives us Proposition 3.1.

Definition 3.12. The *corona* $G \circ K_1$ (also denoted by $cor(G)$) of a graph G is the graph of order $2|V(G)|$ obtained from G by adding a pendant edge to each vertex of G. The 2-*corona* $G \circ P_2$ of G is the graph of order $3|V(G)|$ obtained from G by attaching a path of length 2 to each vertex of G so that the resulting paths are vertex-disjoint.

Theorem 3.13. *For any connected graph G of order $n \geq 2$,*

$$\gamma(M(G \circ K_1)) = n.$$

Theorem 3.14. *For any connected graph G of order $n \geq 2$,*

$$\gamma(M(G \circ P_2)) = n + \gamma(M(G)).$$

Definition 3.15. The *friendship* graph F_n of order $2n+1$ is obtained by joining n copies of the cycle graph C_3 with a common vertex.

Proposition 3.16. *Let F_n be the friendship graph with $n \geq 2$. Then*

$$\gamma(M(F_n)) = n + 1.$$

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Some results on total restrained geodetic number of graphs

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Abstract

A geodetic set S in a graph G is called a total restrained geodetic set if the induced subgraphs $G[S]$ and $G[V - S]$ have no isolated vertex. The minimum cardinality of a total restrained geodetic set in G is the total restrained geodetic number and is denoted by $g_{tr}(G)$. In this paper, we continue the study of the total restrained geodetic number in graphs.

1. introduction

Through this paper, all graphs G are assumed to be non-trivial, simple and connected with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E, respectively). For notation and terminology not presented here we refer the reader to [7]. For a vertex $v \in V$ the open neighborhood of v is $N(v) = \{u \in V|uv \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The maximum and minimum degree among the vertices of G is denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex of degree one is called an end-vertex (or leaf if the graph is a tree), and its unique neighbor is called a stem. The sets of all end-vertices and all stems are denoted by $L(G)$ and $Stem(G)$, respectively. We remark that in K_2 a vertex is both an end-vertex and a stem. A vertex of G is called simplicial if the subgraph induced by its neighborhood is complete. In particular every end-vertex is a simplicial vertex. The set of all simplicial vertices of a graph G is denoted by $Ext(G)$.

A cycle on n vertices is denoted by C_n and a path on m vertices by P_m. The girth of a graph G, denoted by $girth(G)$, is the length of its shortest cycle. The join $H \vee K$ of two disjoint graphs H and K is the graph obtained from their union by adding new edges joining each vertex of $V(H)$ to every vertex of $V(K)$. Let G_1 and G_2 be two vertex-disjoint graphs, and $u_i \in V(G_i)$ for $i = 1, 2$. The coalescence $G_1 \overset{u}{\circ} G_2$ of G_1 and G_2 on u is the graph obtained from the union of these two graphs by identifying the vertices u_1 and u_2.

The distance $d_G(x, y)$ between two vertices x and y in a connected graph G is the length of a shortest $x - y$ path in G. For a vertex x of G, the eccentricity $e_G(x)$ of x is the distance between x and a vertex farthest from x. The maximum eccentricity among the vertices of G is the diameter of G and is denoted by $diam(G)$. An $x - y$ path of length $d_G(x, y)$

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is called an $x-y$ geodesic. The geodetic interval $I[x,y]$ consists of x,y and all vertices lying in some $x-y$ geodesic of G, and for a nonempty subset S of $V(G)$, we define $I[S] = \cup_{x,y\in S} I[x,y]$.
A subset S of vertices of G is a geodetic set if $I[S] = V$. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set of G. A g-set of G is a geodetic set of G of size $g(G)$ (see [2, 4–6]). A geodetic set S in a graph G is a total geodetic set if the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total geodetic set is the total geodetic number and is denoted by $g_t(G)$. A geodetic set $S \subseteq V(G)$ is a restrained geodetic set if the subgraph $G[V-S]$ has no isolated vertex. The minimum cardinality of a restrained geodetic set is the restrained geodetic number (see [3]).
In this paper, we continue the study of the total restarained geodetic number in graphs. A geodetic set S of a graph G is a total restrained geodetic set,(or just TRGS), if neither of the induced subgraphs $G[S]$ and $G[V-S]$ have no isolated vertex. The minimum cardinality of a total restrained geodetic set is the total restrained geodetic number and is denoted by $g_{tr}(G)$. A total restrained geodetic set of cardinality $g_{tr}(G)$ is called a g_{tr}-set .
It follows from the definitions that for any connected graph G,

$$g_{tr}(G) \geq max\{g_r(G), g_t(G)\}.$$

We make use of the following results:

Observation 1.1. *([1]) Let G be a graph of order $n \geq 2$ and S be an arbitrary total restrained geodetic set of G. Then*

(i) $Ext(G) \cup Stem(G) \subseteq S$.

(ii) $2 \leq g(G) \leq g_t(G) \leq g_{tr}(G) \leq n$. *Further* $g_{tr}(G) \neq n-1$.

(iii) $g_{tr}(G) = 2$ *if and only if* $G = K_2$.

Proposition 1.2. *([1])*

(i) For $n \geq 2$, $g_{tr}(K_n) = n$.

(ii) For $3 \leq m \leq n$, $g_{tr}(K_{m,n}) = 4$ *and* $g_{tr}(K_{m,n}) = m+n$ *otherwise.*

(iii) Let T be a nontrivial tree and $M = \{v|N(v) \subseteq Stem(T)\}$. Then $Stem(T) \cup M$ is the unique g_{tr}-set of T and $g_{tr}(T) = |Stem(T)| + |M|$.

(iv) $g_{tr}(P_n) = 4$ *when* $n \geq 6$, *and* $g_{tr}(P_n) = n$ *when* $2 \leq n \leq 5$.

(v) $g_{tr}(C_n) = n$ *when* $n \in \{3,4,5\}$, $g_{tr}(C_n) = n-2$ *when* $n \in \{6,7\}$ *and* $g_{tr}(C_n) = 4$ *otherwise.*

2. Results

Lemma 2.1. *Let G be a connected graph of order $n \geq 5$. If $g_{tr}(G) = n-2$, then the induced subgraph on $V-S$ is K_2, where S is a g_{tr}-set.*

Proof. Let S be a g_{tr}-set of G. Since $g_{tr}(G) = n-2$, $|V-S| = 2$. Now, by the Definition of TRGS of any graph G, the induced subgraph on $V-S$ has no isolated vertex. From this, we conclude that $G[V-S] = K_2$. □

Lemma 2.2. *Let T be a nontrivial tree of order $n \geq 6$. Then $g_{tr}(T) = n-2$ if and only if T consists of exactly two adjacent vertices, which are not belong to $L(T) \cup Stem(T)$.*

Proof. Assume $g_{tr}(T) = n-2$ and S is a g_{tr}-set of T. Therefore, we have two vertices, say u and v, which does not belong to $L(T) \cup Stem(T)$(by Observation 1.1-(i)). Lemma 2.1 shows that these are adjacent.
Conversely, assume that T is a tree which has exactly two adjacent vertices those are not belong to $L(T) \cup Stem(T)$. It is easy to see that the remaining vertices are in $L(T)$ or $Stem(T)$. By using Observation 1.1-(i), we imply that $g_{tr}(G) = n-2$. □

Let $\mathcal{G}$ be the family of graphs which obtained from K_n by removing $n-k-1>0$ edges with $3 \le k \le n-2$, which are incident at one vertex of K_n (see Fig. 1).

Theorem 2.3. *If $G \in \mathcal{G}$, then $g_{tr}(G) = n-k+1$.*

Proof. Let G be a graph in $\mathcal{G}$. It is easy to check that G having a vertex, say v, of degree k ($deg(v) = n-1-(n-k-1) = k$). On the other hand, $deg(u) = n-2$ for each vertex $u \notin N(v)$, and the remaining vertices have degree $n-1$.
We first want to prove $g_{tr}(G) \le n-k+1$. Assume S is a g_{tr}-set of G. Since $deg(u) = n-1$ when $u \in N(v)$, so all the neighbors of the vertex of v are adjacent to all vertices of the graph. Therefore, the vertex v is a simplicial vertex, so by Observation 1.1, we imply that $v \in S$.
To obtain the g_{tr}-set we make a geodetic set and say S'. Consider v than the its farest vertices of the graph (to get the minimum cardinality of the g-set). Suppose $N(v) = \{v_2, \dots, v_r\}$ and $v_{r+1}, \dots, v_n \notin N(v)$. Clearly the length of the shortest path between v and vertices that are not neighbors of vertex v is 2. Therefore $S' = \{v, v_{r+1}, \dots, v_n\}$ and $G - S' = \{v_2, \dots, v_r\} (= N(v))$. Since every vertices of $S' - \{v\}$ is not in the neighbor of v, therefore v is an isolated vertex in $G[S']$. So add one of the vertices of $G - S'$ to S', and call it S. Clearly neither of the induced subgraphs $G[S]$ and $G[V(G) - S]$ has an isolated vertex. Therefore

$$\begin{aligned} g_{tr}(G) &\le |S| \\ &\le |S'| + 1 \\ &\le |V(G)| - |N(v)| + 1 \\ &\le n-k+1. \end{aligned}$$

Now we want to prove $g_{tr}(G) = n-k+1$. Assume $g_{tr}(G) < n-k+1$. So $g_{tr}(G)$ can be one of the numbers $2, \dots, n-k-1, n-k$. Suppose S'' is a g_{tr}-set for G of order $n-k$. Clearly $|G - S| = n-(n-k+1) = k-1$ and $|G - S''| = n-(n-k) = k$. Therefore the set $V(G - S'')$ has one more vertex than the set $V(G - S)$, call this vertex u_1. There are two positions for the vertex u_1 occured. (i) $u_1 \in N(v)$ or (ii) $u_1 \notin N(v)$.
Case (i): If $u_1 \in N(v)$, then $deg(v) = k$ and $k-1$ vertices adjacent to v are in $G - S$, So by adding the vertex u_1, there are k neighbors of v in $G - S''$. Therefore the vertex v in $G[S'']$ is an isolated vertex, which contradicts the assumption.
Case (ii): If $u_1 \notin N(v)$ then there exists a vertex u_2 that $u_1 \in I[v, u_2]$ (clearly $u_1 \ne v, u_2$, because $v, u_2 \in S$). Since the length of the shortest path between v and vertices such as u_2, which is not adjacent to v, is 2. So u_1 is adjacent to v, which contradicts the assumption. Similarly for $g_{tr}(G) = 2, \dots, n-k-1$ contradicts the assumption. Therefore $g_{tr}(G) = n-k+1$. $\square$

Example 2.4. *According to Theorem 5 if $n = 7$, $deg(e) = k = 3$ then $g_{tr}(G) = n-k+1 = 5$.*

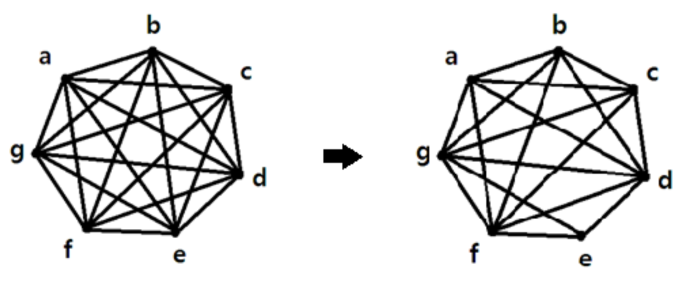

Fig. 1. An example of the family of $\mathcal{G}$

Theorem 2.5. *Let G be a connected graph of order $n \ge 5$ and let $G = K_t \vee \overline{K_{n-t}}$ with $n-t \ge 2$ and $t \ge 3$. Then $g_{tr}(G) = n-t+1$.*

Proof. Assume S is a $g_{tr}(G)$-set for graph G. Clearly all vertices of the subgraph $\overline{K_{n-t}}$ are simplicial and are in S according to Observation 1.1. On the other hand, since the induced subgraph on $V(\overline{K_{n-t}})$ are isolated vertices, we conclude that $g_{tr}(G) \geq n - t + 1$. Now, let $S = V(K_{n-t}) \cup \{u\}$ where $u \in V(K_t)$. It is easy to check that S is a g_{tr}-set of G. Hence the result. □

Corollary 2.6. *According to Theorem 5, if* $G = K_3 \vee \overline{K_{n-3}}(n - 3 \geq 2)$*, then* $g_{tr}(G) = n - 2$.

Let the family of $\mathcal{H}_1$ be obtained from $((K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t}) \cup (K_{n_{t+1}} \cup \cdots \cup K_{n_r})) \vee \{u\}$ and let the family of $\mathcal{H}_2$ be obtained from $((K_{n_{t+1}} \cup \cdots \cup K_{n_r}) \cup (K_{n_{r+1}} \cup \cdots \cup K_{n_l})) \vee \{v\}$. Let the family of $\mathcal{H}$ be obtained from a graph of $\mathcal{H}_1$ and a graph from $\mathcal{H}_2$ by joining u and v (see Fig. 2).

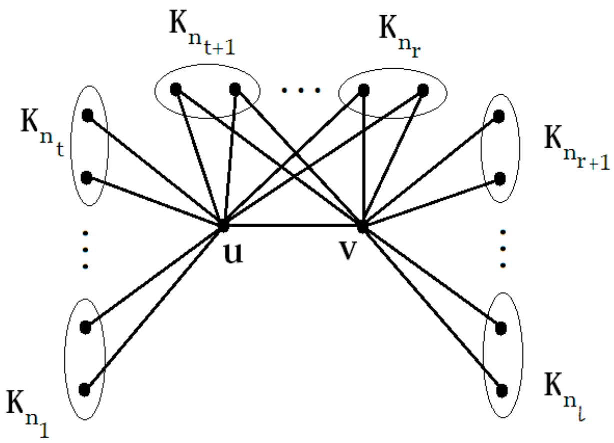

Fig. 2. The family of $\mathcal{H}$

Theorem 2.7. $g_{tr}(G) = n - 2$*, when* $G \in \mathcal{H}$.

Proof. It is easy to check that each vertex of $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t} \cup K_{n_{t+1}} \cup \ldots K_{n_r} \cup K_{n_{r+1}} \cup \cdots \cup K_{n_l}$ is a simplicial vertex, which we imply that $g_{tr}(G) \geq n - 2$. On the other hand, $V(G) - \{u, v\}$ is the unique g_{tr}-set of G. Hence $g_{tr}(G) = n - 2$. □

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Conformally flat with non-degenerate Ricci operator non-Walker

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Abstract

We study the Walker structures over the conformally flat four-dimensional homogeneous manifolds with non-degenerate Ricci operator. We prove this space is not Walker manifold.

1. Introduction

In the pseudo-Riemannian setting the problem is more complicated and of course interesting. In dimension three, the conformally flat examples were classified independently in [4], where contrary to the Riemannian case they showed the existence of non-symmetric examples. By expanding the results of [4], the same authors solved the classification problem for the Lorentzian manifolds of any dimension with diagonalizable Ricci operator. For homogeneous spaces, the classification problem has been completely solved for both Lorentzian and neutral signatures in dimension four [3]. A fundamental step for this classification was to determine the forms (*Segre types*) of the Ricci operator. Homogeneous spaces are the subject of many interesting research projects in the pseudo-Riemannian framework. Recently, the author studied the wave equation on Lorentzian conformally flat spaces [1, 2].

A pseudo-Riemannian manifold which admits a parallel degenerate distribution is called a *Walker* manifold. Walker spaces were introduced by Arthur Geoffrey Walker in 1949. The existence of such structures causes many interesting properties for the manifold with no Riemannian counterpart. Walker also determined a standard local coordinates for these kind of manifolds [5].

In this paper, which is based on the study of conformally flat spaces in [3], we have determined invariant Walker structures in the case of four-dimensional conformally flat homogeneous manifolds. Conformally flat homogeneous spaces have been studied classically in pseudo-Riemannian geometry. As it is known, the existence of Walker structures on a manifold can be responsible for the existence of non-symmetric examples. So we analyze the conformally flat homogeneous pseudo-Riemannian Walker four-manifolds.

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2. Preliminaries

Let (M, g) be a pseudo-Riemannian manifold of dimension $n \geq 3$ and ∇ its Levi-Civita connection. We use the curvature tensor with the sign convention $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for all vector fields X, Y on M. The Ricci tensor is given by the identity

$$\varrho(X,Y) = \sum_{i=1}^{4} \varepsilon_i g(R(e_i, X)Y, e_i), \tag{1}$$

for all $X, Y \in T_pM$, where $\{e_1, e_2, e_3, e_4\}$ is a pseudo-orthonormal basis for the tangent space T_pM. We denote the Ricci operator and the scalar curvature by Q and τ respectively. Let p be a point of M and $\{e_1, ..., e_n\}$ an orthonormal basis of the tangent space T_pM. It is well-known that for a conformally flat space the curvature tensor can be completely determined using the Ricci tensor by the identity

$$\begin{aligned} R_{ijkh} = &\tfrac{1}{n-2}(g_{ih}\varrho_{jk} + g_{jk}\varrho_{ih} - g_{ik}\varrho_{jh} - g_{jh}\varrho_{ik}) \\ &-\tfrac{\tau}{(n-1)(n-2)}(g_{ih}\varrho_{jk} - g_{ik}\varrho_{jh}). \end{aligned} \tag{2}$$

Moreover, the Equation (2) characterizes conformally flat pseudo-Riemannian manifolds of dimension $n \geq 4$, while it is trivially satisfied by any three-dimensional manifold. Conversely, the condition

$$\nabla_i \varrho_{jk} - \nabla_j \varrho_{ik} = \tfrac{1}{2(n-2)}(g_{jk}\nabla_i\tau - g_{ik}\nabla_j\tau), \tag{3}$$

which characterizes three-dimensional conformally flat spaces, is trivially satisfied by any conformally flat Riemannian manifold of dimension greater than three.
Now, let (M, g) be a locally homogeneous pseudo-Riemannian manifold. Then, for each pair of points $p, p' \in M$, there exists a local isometry ϕ between neighbourhoods of p and p', such that $\phi(p) = p'$. In particular, for any choice of an index k, $\phi^* : T_{p'}M \to T_pM$ satisfies $\phi^*(\nabla^i R_{p'}) = \nabla^i R_p$ for all $i = 0, .., k$. Consequently, chosen a pseudo-orthonormal basis $\{e_i\}_p$ for T_pM, by means of the isometries between p and any other point $p' \in M$, one can build a pseudo-orthonormal frame field $\{e_i\}$ on M, with respect to which the components of the curvature tensor and its covariant derivatives up to order k are globally constant on M.
In the special case when (M, g) is conformally flat, this is equivalent to determining a pseudo-orthonormal frame field $\{e_i\}$ on (M, g), such that the components of the Ricci tensor ϱ and its covariant derivatives $\nabla^i\varrho$, for $i = 1, \ldots, k$, are constant globally on M. To note that in particular, with respect to $\{e_i\}$, the components of the Ricci operator Q are constant. Specially the Segre type of the Ricci operator stays constant on the whole space.
By the result of [4], for a conformally flat homogeneous manifold of dimension four with digonalizable Ricci operator, the problem of study Walker structures reduces to the well known space forms.

Theorem 2.1. *[4] Let M_q^n be an $n(\geq 3)$-dimensional conformally flat homogeneous pseudo-Riemannian manifold with diagonalizable Ricci operator. Then, M_q^n is locally isometric to one of the following:*

(i) A pseudo-Riemannian space form;

(ii) A product manifold of a m-dimensional space form of constant curvature $k \neq 0$ and a $(n-m)$-dimensional pseudo-Riemannian manifold of constant curvature $-k$, where $2 \leq m \leq n-2$;

(iii) A product manifold of a $(n-1)$-dimensional pseudo-Riemannian manifold of index $q-1$ of constant curvature $k \neq 0$ and an one-dimensional Lorentzian manifold, or a product of a $(n-1)$-dimensional pseudo-Riemannian manifold of index q of constant curvature $k \neq 0$ and an one-dimensional Riemannian manifold.

It is obvious from the last theorem that if (M, g) have digonalizable Ricci operator then the Ricci operator is degenerate. So the study of cases with non-degenerate Ricci operator restricts to the not diagonalizable ones.

3. Conformally flat with non-degenerate Ricci operator

A conformally flat (locally) homogeneous Riemannian manifold is (locally) symmetric. Let (M, g) be a conformally flat homogeneous four dimensional manifold with non-degenerate Ricci operator. For any point $p \in M$, we have that $g(0, p) = \{0\}$ if and only if Q_p is non-degenerate. Therefore, (M, g) is locally isometric to a Lie group equipped with a left-invariant pseudo-Riemannian metric and the Ricci operator of conformally flat homogeneous pseudo-Riemannian four-manifolds can only be of Segre type $[1, 11\bar{1}]$ if g is neutral, or $[11, 1\bar{1}]$ if g is Lorentzian [3]. The Lie group structure of the mentioned types could be realized by the following theorems.

Theorem 3.1. *[3] Let (M, g) be a conformally flat homogeneous four-dimensional manifold with the Ricci operator of Segre type $[1, 11\bar{1}]$. Then, (M, g) is locally isometric to one of the unsolvable Lie groups $SU(2) \times \mathbb{R}$ or $SL(2, \mathbb{R}) \times \mathbb{R}$, equipped with a left-invariant neutral metric, admitting a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for their Lie algebra, such that the Lie brackets take one of the following forms:*

$$\begin{array}{llll} \text{i)} & [e_1, e_2] = \varepsilon\alpha e_3, & [e_1, e_3] = -\varepsilon\alpha e_2, & [e_2, e_3] = 2\alpha(e_1 + \varepsilon e_4), \\ & [e_2, e_4] = -\alpha e_3, & [e_3, e_4] = \alpha e_2, & \end{array}$$

$$\begin{array}{llll} \text{ii)} & [e_1, e_2] = -\varepsilon\alpha e_1, & [e_1, e_3] = \alpha e_1, & [e_1, e_4] = 2\alpha(\varepsilon e_2 - e_3), \\ & [e_2, e_4] = -\varepsilon\alpha e_4, & [e_3, e_4] = \alpha e_4, & \end{array}$$

where $\alpha \neq 0$ is a real constant and $\varepsilon = \pm 1$.

And for the Lorentzian signature we have:

Theorem 3.2. *[3] Let (M, g) be a conformally flat homogeneous Lorentzian four-manifold with the Ricci operator of Segre type $[11, 1\bar{1}]$. Then, (M, g) is locally isometric to one of the unsolvable Lie groups $SU(2) \times \mathbb{R}$ or $SL(2, \mathbb{R}) \times \mathbb{R}$, equipped with a left invariant Lorentzian metric, admitting a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the Lie algebra, such that the Lie brackets take one of the following forms:*

$$\begin{array}{llll} \text{i)} & [e_1, e_2] = -2\alpha(\varepsilon e_3 + e_4), & [e_1, e_3] = \varepsilon\alpha e_2, & [e_1, e_4] = \alpha e_2, \\ & [e_2, e_3] = \varepsilon\alpha e_1, & [e_2, e_4] = \alpha e_1, & \end{array}$$

$$\begin{array}{llll} \text{ii)} & [e_1, e_2] = 2\alpha(\varepsilon e_3 + e_4), & [e_1, e_3] = \varepsilon\alpha e_2, & [e_1, e_4] = \alpha e_2, \\ & [e_2, e_3] = \varepsilon\alpha e_1, & [e_2, e_4] = \alpha e_1, & \end{array}$$

where $\alpha \neq 0$ is a real constant and $\varepsilon = \pm 1$.

By using the above classification theorems we have enough tools to study Walker structures. The result is the following theorem.

Theorem 3.3. *Let (M, g) be a conformally flat homogeneous four-dimensional manifold with non-degenerate Ricci operator. Then (M, g) does not admit any left-invariant Walker structure.*

Proof. Since the Ricci operator of (M, g) in non-degenerate, according to the Theorem 3.1 for signature $(2, 2)$ and Theorem 3.2 for Lorentzian signature, we have the explicit description of Lie groups and their Lie algebras. We prove that the existence of a left invariant parallel null distribution in any possible case leads to a contradiction. We report the calculations for the case (i) of signature $(2, 2)$. Choose the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ and suppose there exists a two-dimensional parallel distribution $\bar{\mathcal{D}} = \mathrm{span}(v, w)$, where $v = \sum_{i=1}^{4} v_i e_i$ and $w = \sum_{i=1}^{4} w_i e_i$ are linearly independent and $g(v, v) = g(w, w) = g(w, v) = 0$ for arbitrary parameters v_i and w_i. Setting $\Lambda_i = \nabla_{e_i}$, the components of the Levi-Civita connection are calculated using the well known *Koszul* formula and are

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon\alpha & 0 \\ 0 & -\varepsilon\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_2 = \begin{pmatrix} 0 & 0 & \alpha(1-\varepsilon) & 0 \\ 0 & 0 & 0 & 0 \\ \alpha(1-\varepsilon) & 0 & 0 & -\alpha(1+\varepsilon) \\ 0 & 0 & \alpha(1+\varepsilon) & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} 0 & 0 & -\alpha(1+\varepsilon) & 0 \\ \alpha(1+\varepsilon) & 0 & 0 & \alpha(1-\varepsilon) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha(1-\varepsilon) & 0 \end{pmatrix}.$$

Being parallel of $\mathcal{D}$ is expressed by the equations

$$\begin{array}{ll} \nabla_{e_1} v = a_1 v + b_1 w, & \nabla_{e_1} w = c_1 v + d_1 w, \\ \nabla_{e_2} v = a_2 v + b_2 w, & \nabla_{e_2} w = c_2 v + d_2 w, \\ \nabla_{e_3} v = a_3 v + b_3 w, & \nabla_{e_3} w = c_3 v + d_3 w, \\ \nabla_{e_4} v = a_4 v + b_4 w, & \nabla_{e_4} w = c_4 v + d_4 w, \end{array} \tag{4}$$

for some parameters $\{a_i, b_i, c_i, d_i\}_{i=1}^4$. From $g(v,v) = g(w,w) = g(w,v) = 0$ and the equations $\nabla_{e_1} v = a_1 v + b_1 w$ and $\nabla_{e_2} v = a_2 v + b_2 w$ we have:

$$\begin{array}{l} v_1^2 + v_2^2 - v_3^2 - v_4^2 = 0, \quad w_1^2 + w_2^2 - w_3^2 - w_4^2 = 0, \\ v_1 w_1 + v_2 w_2 - v_3 w_3 - w_4 v_4 = 0, \quad b_1 w_1 + a_1 v_1 = 0, \\ b_1 w_4 + a_1 v_4 = 0, \quad b_1 w_2 + a_1 v_2 - \alpha v_3 = 0, \\ b_1 w_3 + a_1 v_3 - \alpha v_2 = 0, \quad b_2 w_2 + a_2 v_2 = 0, \\ b_2 w_1 + a_2 v_1 - \alpha v_3 (1-\varepsilon) = 0, \\ b_2 w_4 + a_2 v_4 - \alpha v_3 (1+\varepsilon) = 0, \quad b_2 w_3 + a_2 v_3 + \alpha (v_1 + v_4)(1-\varepsilon) = 0. \end{array}$$

These equations yield that the vector v must vanish which contradicts the linear independence of v, w. By similar argument we suppose that $\mathcal{D} = \text{span}(x)$ is a null parallel line field, where $x = \sum_{i=1}^4 x_i e_i$ for arbitrary parameters x_i. Thus, the following equations must be satisfied for some parameters ω_i and x_i,

$$\begin{array}{l} x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0, \\ \omega_1 x_1 = 0, \quad \omega_1 x_4 = 0, \quad \omega_1 x_2 - \alpha x_3 = 0, \quad \omega_1 x_3 - \alpha x_2 = 0, \\ \omega_2 x_2 = 0, \quad \omega_2 x_1 + \alpha x_3 (\varepsilon - 1) = 0, \quad \omega_2 x_4 - \alpha x_3 (\varepsilon + 1) = 0, \\ \omega_2 x_3 + \alpha x_4 (\varepsilon + 1) + \alpha x_1 (\varepsilon - 1) = 0, \\ \omega_3 x_3 = 0, \quad \omega_3 x_1 + \alpha x_2 (\varepsilon + 1) = 0, \quad \omega_3 x_4 + \alpha x_2 (\varepsilon - 1) = 0, \\ \omega_3 x_2 + \alpha x_4 (\varepsilon - 1) - \alpha x_1 (\varepsilon + 1) = 0, \\ \omega_4 x_1 = 0, \quad \omega_4 x_4 = 0, \quad \omega_4 x_2 + \alpha x_3 \varepsilon = 0, \quad \omega_4 x_3 + \alpha x_2 \varepsilon = 0. \end{array}$$

This system of equations yields that $x = 0$ which is a contradiction. This shows that no left-invariant parallel null line field exists in this case and this matter finishes the proof. □

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The signed Bowen-Franks group matrix

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Abstract

The invariant introduced by Franks is the signed Bowen-Franks group, which is an augmentation of the Bowen-Franks group.

1. Introduction

The invariant introduced by Franks is the signed Bowen-Franks group, which is an augmentation of the Bowen-Franks group[2].

Definition 1.1. Let A be an $n \times n$ integer matrix with non-negative entries. The Bowen-Franks group of A is given by the quotient

$$BF(A) = \mathbb{Z}^n/(I - A)\mathbb{Z}^n$$

The Bowen-Franks group can be computed quite easily with elementary linear algebra. For the square integer matrix A, we consider the elementary operations over $\mathbb{Z}$ on A to be the following[1]

1. Exchanging two rows or columns of A.
2. Multiplying a row or column of A by -1.
3. Adding an integer multiple of a row or column of A to another row or column, respectively, of A.

Every such elementary operation over $\mathbb{Z}$ has an elementary matrix representing it. An elementary matrix is a square integer matrix, which performs an elementary operation on a matrix if multiplied from either the left or the right. Since every elementary operation has an inverse elementary operation, all elementary matrices are invertible over $\mathbb{Z}$.

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We recall that the Smith normal form of A is the unique diagonal matrix, reachable from A by elementary operations over $\mathbb{Z}$, of the form

$$D = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix} \tag{1}$$

where $d_i \geq 0$ and $d_i | d_{i+1}$ for all i with the convention that every integer divides zero and no positive integer divides zero. It turns out that we can derive the Bowen-Franks group of a matrix from its Smith normal form.

Lemma 1.2. *Let B be an $n \times n$ integer matrix and E an elementary matrix. Then*

$$\mathbb{Z}^n = B\mathbb{Z}^n \simeq \mathbb{Z}^n/(BE\mathbb{Z}^n) \simeq \mathbb{Z}^n/(EB\mathbb{Z}^n).$$

Proposition 1.3. *If $I - A$ has Smith normal form D written as (1), then*

$$BF(A) \simeq \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n},$$

where $\mathbb{Z}_0 = \mathbb{Z}$ and $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z},\ n \neq 0$.

Sadly, the Bowen-Franks group does not constitute a complete invariant for flow equivalence, so we introduce an additional component.

Definition 1.4. The signed *Bowen-Franks group* is given by the pair

$$BF_+(A) = (sgn\, det(I - A), BF(A)).$$

We write $BF_+(A) \simeq BF_+(B)$ if $sgn\, det(I - A) = sgn\, det(I - B)$ and $BF(A) \simeq BF(B)$.

Actually, the complete invariant is given by the determinant $det(I - A)$ and the Bowen-Franks group $BF(A)$, but $|det(I - A)|$ can be extracted from $BF(A)$ since$|det(I - A)| = |det(D)|$, where D is the Smith normal form of $I - A$. Thus, only the sign of the determinant is necessary for the complete invariant[3, 4].

2. Main results

We say that the matrix A is irreducible if the shift space X_A is irreducible. The result by Franks now reads.

Theorem 2.1. *(Franks [2]). Suppose that A and B are non-negative irreducible integer matrices such that neither X_A nor X_B is a single orbit. Then $X_A \sim_{FE} X_B$ if and only if $BF_+(A) \simeq BF_+(B)$.*

Example 2.2. *Let $r > 1$ be an integer. The full $r-$shift $X_{[r]}$ has matrix representation $A_r = (r)$, we have $det(I - A_r) = 1 - r$, and $(r - 1)$ is the Smith normal form of $I - A_r$. So the signed Bowen-Franks group of A_r is*

$$BF_+(A_r) \simeq (-, \mathbb{Z}_{r-1})$$

It follows that for two different integers $r, s > 1, X_{[r]} \nsim_{FE} X_{[s]}$. Thus, two full shifts are flow equivalent if and only if they are conjugate.

As the above example illustrates, the signed Bowen-Franks group is a very convenient invariant as it is easily computed given two matrices. We will now show the necessity of the invariant.

Lemma 2.3. *For a non-negative integer matrix $A = (a_{ij})$ with $a_{kl} > 0$ we define*

$$\bar{A} = \begin{pmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & a_{11} & \cdots & a_{1l} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & a_{1k} & \cdots & a_{kl} - 1 & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{nl} & \cdots & a_{nn} \end{pmatrix} \tag{2}$$

Write $X \sim_{SE} Y$ if $X = \bar{Y}$ or $Y = \bar{X}$. Flow equivalence is generated by the relations $\sim_{SE}$ and $\approx$.

Proposition 2.4. *If A, B are non-negative integer matrices with $X_A \sim_{FE} X_B$, then $BF(A) \simeq BF(B)$.*

Lemma 2.5. *(Sylvester's theorem [5]). Let A be an $m \times n$ and B an $n \times m$ matrix. Then $det(I_m + AB) = det(I_n + BA)$.*

Proof. Define the block matrix

$$M = \begin{pmatrix} I_m & -A \\ B & I_n \end{pmatrix}$$

and see that we can decompose M as

$$M = \begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix} \begin{pmatrix} I_m & -A \\ 0 & I_n + BA \end{pmatrix} \text{ and } M = \begin{pmatrix} I_m + AB & -A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_m & 0 \\ B & I_n \end{pmatrix}$$

Taking determinants we get $det(I_n + BA) = det(M) = det(I_m + AB)$. □

Theorem 2.6. *If A, B are non-negative integer matrices with $X_A \sim_{FE} X_B$ then $det(I - A) = det(I - B)$.*

Proof. We show invariance separately for elementary equivalences and for symbol expansions. As seen in Lemma 2.3 these relations generate flow equivalence and the result follows. Let C and D be elementary equivalent non-negative integer matrices with $C = RS$ and $D = RS$ for non-negative integer matrices S, R. Then it follows by Sylvester's theorem that

$$det(I - C) = det(I + (-R)S) = det(I + S(-R)) = det(I - D).$$

For symbol expansion, let A be an integer matrix. Then by adding rows, we get

$$\begin{aligned} det(I - \bar{A}) &= det \begin{pmatrix} 1 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 1 - a_{11} & \cdots & -a_{1l} & \cdots & -a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -1 & -a_{k1} & \cdots & -a_{kl} + 1 & \cdots & -a_{kn} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & -a_{n1} & \cdots & -a_{nl} & \cdots & 1 - a_{nn} \end{pmatrix} \\ &= det \begin{pmatrix} 1 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 1 - a_{11} & \cdots & -a_{1l} & \cdots & -a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & -a_{k1} & \cdots & -a_{kl} & \cdots & -a_{kn} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & -a_{n1} & \cdots & -a_{nl} & \cdots & 1 - a_{nn} \end{pmatrix} \\ &= det \begin{pmatrix} 1 - a_{11} & \cdots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & 1 - a_{nn} \end{pmatrix} \\ &= det(I - A) \end{aligned}$$

where the last equality follows by expansion of the lth column. Note, again, that the above representations of matrices are not accurate when $k = l$, but the calculation is still the same. □

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Ends of groups with compatible asymptotic resemblance relations

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Abstract

We generalize the concept of ends of finitely generated groups to all asymptotic resemblance spaces. We show that our new notion defines a new large scale property. We also investigate the space of ends of groups with compatible asymptotic resemblance relations and we show that some of well know properties of the space of ends of finitely generated groups can be generalized to groups with compatible asymptotic resemblance relations.

1. Introduction

Suppose that G is a finitely generated group and S is a finite set of generators of G such that $S = S^{-1}$. The Cayley graph of G associated to S is a graph that has all elements of G as vertices and two elements $g, h \in G$ are connected by an edge, if $g^{-1}h \in S$. Consider each edge of the Cayley graph as an isometric copy of the interval $[0, 1]$. We can define the distance between two elements g and h of G to be equal to the infimum of lengths of all paths in the Cayley graph joining g and h. In this way, we obtain a left invariant proper metric on G. Clearly, this metric depends on the choice of the generating set S. Fortunately, it can be shown that associated metrics to different finite generating sets are quasi-isometric. Those properties of metric spaces which are invariant under quasi-isometries are called large scale properties. The main goal of geometric group theory is to study large scale properties of finitely generated groups. The space of ends of finitely generated groups is one of the earliest large scale properties introduced by H. Freudenthal ([2]).

Definition 1.1. Let (X, d) be a metric space. A continuous map $r : [0, +\infty) \to X$ is said to be a proper ray in X if $r^{-1}(K)$ is compact for all compact $K \subseteq X$. Two proper rays $r_1, r_2 : [0, +\infty) \to X$ are said to have the same end if for each compact subset K of X there exists some $N \in \mathbb{N}$ such that $r_1([N, +\infty))$ and $r_2([N, +\infty))$ are contained in the same path component of $X \setminus K$. This defines an equivalence relation on the family of all proper rays in X and for each proper ray r in X we denote the equivalence class that contains r by end(r). In addition define

$$\mathrm{Ends}(X) = \{\mathrm{end}(r) \mid r : [0, +\infty) \to X \textit{ is a proper ray in } X\}$$

If Ends(X) has $n \in \mathbb{N} \cup \{0\}$ members we say X has n ends.

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It is known that two quasi-isometric metric spaces have homeomorphic spaces of ends and hence we are facing a large scale property of metric spaces. By the space of ends of a finitely generated group G we simply mean the space of ends of its Cayley graph associated to some generating set. Clearly, the space of ends of finitely generated groups is well defined and there exists no ambiguity here. It is known that each finitely generated group has $0, 1, 2$ or infinitely many ends. The family of all finitely generated groups with 0 ends is equal to the family of all finite groups, and a finitely generated group has 2 ends, if and only if, it is virtually cyclic (it has a finite indexed infinite cyclic subgroup). We recommend the reader to see [1] for more detailed arguments. A famous result due to Stalling completely characterizes finitely generated groups with infinitely many ends ([7]).
There are many ways for defining large scale structures on sets. For example we can mention coarse structures ([6]) and asymptotic resemblance relations ([4]) as two known large scale structures. We can use these large scale structures for expanding the domain of geometric group theory to much more general types of groups. Let us recall the definition of asymptotic resemblance spaces.

Definition 1.2. Let X be a nonempty set. An equivalence relation λ on X is said to be an *asymptotic resemblance* (an AS.R) on X if,
i) $A_1\lambda B_1$ and $A_2\lambda B_2$ then $(A_1 \cup A_2)\lambda(B_1 \cup B_2)$, for all $A_1, A_2, B_1, B_2 \subseteq X$.
ii) Suppose that $A_1, A_2 \neq \emptyset$ and $(A_1 \cup A_2)\lambda B$ then there are non empty subsets B_1 and B_2 of B such that $B = B_1 \cup B_2$ and $A_i\lambda B_i$ for $i = 1, 2$.
In this case the pair (X, λ) is called an asymptotic resemblance space (AS.R space).

Now we recall the definition of AS.R mappings.

Definition 1.3. Let (X, λ) be an AS.R space. A subset D of X is called bounded if it is empty or $D\lambda\{x\}$ for some $x \in X$. An AS.R space (X, λ) is called to be *connected* if $\{x\}\lambda\{y\}$, for all $x, y \in X$. It can be shown that the union of two bounded subsets of the connected AS.R space (X, λ) is bounded and each subset of a bounded set in (X, λ) is also bounded ([4]). A map $f : X \to Y$ from the AS.R space (X, λ) to the AS.R space (Y, λ') is called an AS.R mapping if the inverse image of each bounded subset of Y is bounded in X and $f(A)\lambda' f(B)$ if $A\lambda B$, for all $A, B \subseteq X$. The AS.R mapping $f : X \to Y$ is called an asymptotic equivalence if there exists an AS.R mapping $g : Y \to X$ such that $g \circ f(A)\lambda A$ and $f \circ g(B)\lambda' B$, for all $A \subseteq X$ and $B \subseteq Y$. In this case g is called an asymptotic inverse of f and AS.R spaces (X, λ) and (Y, λ') are called asymptotically equivalent.

For defining compatible AS.Rs on groups we use the notion of generating family ([5]).

Definition 1.4. A family $\mathcal{F}$ of subsets of the group G is called a *generating family* on G if $\mathcal{F}$ contains a nonempty element and for each $A, B \in \mathcal{F}$ it satisfies the following properties,
i) $A^{-1}, AB, A \cup B \in \mathcal{F}$,
ii) If $C \subseteq A$ then $C \in \mathcal{F}$,
where $AB = \{ab \mid a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} \mid a \in A\}$.

Let $\mathcal{F}$ be a generating family on the group G. For two subsets A and B of G define $A\lambda_{\mathcal{F}}B$ if $A \subseteq BK$ and $B \subseteq AK$, for some $K \in \mathcal{F}$. It can be shown that $\lambda_{\mathcal{F}}$ is an AS.R on G and it is compatible with the group structure of G, i.e. if $A\lambda_{\mathcal{F}}B$ then $gA\lambda_{\mathcal{F}}gB$, for all $g \in G$.

Example 1.5. *Suppose that $\mathcal{F}$ denotes the family of all finite subsets of the group G. Then $\mathcal{F}$ is a generating family on G. The family of all relatively compact subsets of the topological group G is a generating family on G.*

The concept of a large scale continuum in an AS.R space plays the most important role in our definition of space of ends. This concept has been first appeared and investigated in [3]. We show that somehow this notion can fill the place of proper ray in Definition 1.1.

Definition 1.6. Let (X, λ) be an AS.R space. An unbounded subset D of X is called large scale continuum in (X, λ), if $D = D_1 \cup D_2$ for two asymptotically disjoint subsets D_1 and D_2 of X then D_1 is bounded or D_2 is bounded. Recall that two subsets A and B of the AS.R space (X, λ) are called asymptotically disjoint if they do not contain asymptotically alike unbounded subsets.

2. Main Results

Now we can make clear what we mean by the space of ends of an AS.R space.

Definition 2.1. Let (X, λ) be an AS.R space. We say two large scale continuums D_1 and D_2 in X do not have the same end, if there are asymptotically disjoint subsets X_1 and X_2 of X such that X_1 and X_2 are asymptotically disjoint from D_1 and D_2 respectively and $X = X_1 \cup X_2$. We write $\mathrm{end}_\lambda(D_1) = \mathrm{end}_\lambda(D_2)$, if D_1 and D_2 are two large scale continuums in (X, λ) with the same end. We denote the family of all ends of the AS.R space (X, λ) by $\mathrm{Ends}_\lambda(X)$, i.e.

$$\mathrm{Ends}_\lambda(X) = \{\mathrm{end}_\lambda(D) \mid D \text{ is a large scale continuum in } (X, \lambda)\}$$

The cardinality of the set $\mathrm{Ends}_\lambda(X)$ is called the *number of ends* of (X, λ). If $\mathrm{Ends}_\lambda(X) = \emptyset$ we say that X is 0-ended.

In the following definition we show that how we can topologize the space of ends of an AS.R space.

Definition 2.2. Assume that (X, λ) is an AS.R space. Suppose that $\mathcal{D}_\lambda(X)$ denotes the family of all subsets Y of X such that Y and Y^c are asymptotically disjoint. Let $\mathrm{end}_\lambda(D) \in \mathrm{Ends}_\lambda(X)$ and $\Gamma \subseteq \mathrm{Ends}_\lambda(X)$. We say that $\mathrm{end}_\lambda(D) \notin \overline{\Gamma}$ if there exists some $Y \in \mathcal{D}_\lambda(X)$ such that,
i) $D \subseteq Y$,
ii) $D' \cap Y$ is bounded for all $\mathrm{end}_\lambda(D') \in \Gamma$.

It can be shown that Definition 2.2 offers a Kuratowski closure operator and thus we have a topology on spaces of ends of AS.R spaces. Definitions 2.1 and 2.2 give us a large scale property for AS.R spaces.

Theorem 2.3. *Two asymptotic equivalent AS.R spaces have homeomorphic spaces of ends.*

Let (X, d) be a metric space. For two subsets A and B of X, we say $A\lambda_d B$ if A and B have finite Hausdoff distance. It is easy to see that λ_d is an AS.R relation on X.

Proposition 2.4. *Let (X, d) be a proper geodesic metric space. Then the space of ends of the AS.R space (X, λ_d) is homeomorphic to the space of ends of the metric space (X, d).*

Let G be a group and let $\mathcal{F}$ denote a generating family on G. We denote the space of ends of the AS.R space $(X, \lambda_\mathcal{F})$ by $\mathrm{Ends}_\mathcal{F}(G)$.

Proposition 2.5. *Suppose that G is a locally compact compactly generated topological group. Let $\mathcal{K}$ denotes the family of all relatively compact subsets of G. Then $\mathrm{Ends}_\mathcal{K}(G) = \emptyset$, if and only if, G is compact.*

Proof. If G is compact then each subset of G is bounded in (X, λ), so clearly it does not contain any large scale continuums. Proving the rest of this proposition needs more technical results and we do not mention it here. □

If $\mathcal{K}$ denotes the familly of all relatively compact subsets of the topological group G, then we call the AS.R $\lambda_\mathcal{K}$, the group compact AS.R on G.

Proposition 2.6. *Let G be a locally compact topological group. Assume that λ is the group compact AS.R on G. If (G, λ) is 0-ended then every compactly generated subgroup of G is relatively compact.*

Proof. Assume that K is a compact subset of G. Since G is locally compact, there exists a relatively compact open neighborhood U of the neutral element of G such that $K \subseteq U$. Suppose that H denotes the subgroup of G generated by $\overline{U}$. Since U is open, H is simultaneously open and closed in G. Since H is closed and G is locally compact, H is also locally compact. So Proposition 2.5 shows that H is compact. □

Corollary 2.7. *Let G be a group and suppose that $\mathcal{F}$ denotes the family of all finite subsets of G. If $(G, \lambda_\mathcal{F})$ is 0-ended then G is locally finite.*

Proof. If we consider G with the discrete topology then this corollary is a direct consequence of Proposition 2.6. □

Many known results about the space of ends of finitely generated groups can be generalized to more general cases by using our definition. Theorem 2.8 is a good example.

Theorem 2.8. *Let G be a locally compact hemicompact topological group. If $\mathcal{K}$ denotes the family of all relatively compact subsets of G then $\mathrm{Ends}_\mathcal{K}(G)$ has $0, 1, 2$ or infinitely many members.*

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Classification for a kind of surface of revolution under Gauss map and Cheng-Yau operator

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Abstract

Our main aim in this paper is to improve some method in classifying surfaces using Gauss map and the sequence of L_k-operators specially L_1 which is known as a famous Cheng-Yau operator, for a class of surfaces of revolution in 3-Euclidean space.

1. Introduction

Giving classification of surfaces and hypersurfaces is one of the hot topics in Riemannian and differential geometry. As an example, Lopéz has classified before all minimal translation surfaces in $\mathbb{R}^3$ (see [8]). Also he determined all translation surfaces in Euclidean space with constant Gaussian curvature in [9]. Some similar works have been done about Wingarten surfaces. Also there are some other techniques for classifying surfaces and other submanifolds.
It was studied before that associated to the eigenvalues of Laplacian which can be seen as coordinate functions of n-dimensional submanifolds isometrically immersed into the Euclidean space $\mathbb{R}^{n+m}$, are minimal submanifolds in $\mathbb{R}^{n+m}$ or minimal submanifolds in a round hypersphere $\mathbb{S}^{n+m-1}(r) \subset \mathbb{R}^{n+m}$ of radius r (for more details see [14]). Particularly, Takahashi theorem established that in the case of the codimension is $m = 1$, if $x : M^n \to \mathbb{R}^{n+1}$ is an immersed hypersurface in Euclidean space and Δ denotes its Laplacian operator, then x satisfies

$$\Delta x + \lambda x = 0,$$

for real λ if and only if $\lambda = 0$ and M is minimal in $\mathbb{R}^{n+1}$ or $\lambda > 0$ and M is an open piece of a round hypersphere of radius $\sqrt{\frac{n}{\lambda}}$ centered at the origin of $\mathbb{R}^{n+1}$.
There are also some extensions of Takahashi theorem. Specially, Garay studied hypersurfaces in $\mathbb{R}^{n+1}$ satisfying

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$\Delta x = Ax$, where $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is a constant diagonal matrix, and he established that the only hypersurfaces satisfying in the mentioned equality are minimal hypersurfaces in $\mathbb{R}^{n+1}$ and open pieces of round hypersurfaces or spherical cylinders (see [7]).
By the similar way, Dillen et al. in [6] considered surfaces in $\mathbb{R}^3$ whose immersion satisfies the condition

$$\Delta x = Ax + b,$$

where $A \in \mathbb{R}^{3\times 3}$ is a constant matrix and $b \in \mathbb{R}^3$ is a constant vector. They established that the only surfaces which satisfy the above equation are minimal surfaces and open pieces of round spheres and circular cylinders. These results were extended later by Hasanis, Vlachos, Chen and Petrovic for hypersurfaces in $\mathbb{R}^{n+1}$ (for more details see [2] and [10]).

2. Preliminaries

Let $x : M^n \to \mathbb{R}^{n+1}$ be a connected orientable hypersurface immersed into the Euclidean space and G denotes the Gauss map of M. For $X, Y \in \chi(M)$, $S : \chi(M) \to \chi(M)$ is the famous shape operator of M. It is known that S defines a self-adjoint linear operator on each tangent plane T_pM for arbitrary $p \in M$, also its eigenvalues $\kappa_1(p), ..., \kappa_n(p)$ are the principal curvatures of the hypersurface. Through these eigenvalues, there are n algebraic invariants given by

$$s_k(p) = \sigma_k(\kappa_1(p), ..., \kappa_n(p)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(x_1, ..., x_n) = \sum_{i_1<...<i_k} x_{i_1}...x_{i_k}.$$

The kth mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = s_k, \quad 0 \leq k \leq n.$$

As an example, for $k = 1$,

$$H_1 = \frac{1}{n}\sum_{i=1}^{n} \kappa_i = \frac{1}{n} tr(S) = H,$$

is the mean curvature of M.
The classical Newton transformation

$$P_k : \chi(M) \to \chi(M),$$

are defined by shape operator as

$$P_0 = I, \quad P_k = s_k I - S \circ P_{k-1} = \binom{n}{k} H_k I - S \circ P_{k-1},$$

where I is identity in $\chi(M)$. It was known from Cayley-Hamilton theorem that $P_n = 0$ also $P_k(p)$ is a self-adjoint linear operator on each tangent plane T_pM which commutes with $S(p)$. It can be easily seen that

$$tr(P_k) = c_k H_k, \quad tr(S \circ P_k) = c_k H_{k+1},$$

and

$$tr\left(S^2 \circ P_k\right) = \binom{n}{k+1} \left(nH_1H_{k+1} - (n-k-1)H_{k+2}\right),$$

where

$$c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}.$$

Respect to the Newton transformations P_k, we define the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ as

$$L_k(f) = tr\left(P_k \circ \nabla^2 f\right),$$

where $\nabla^2 f : \chi(M) \to \chi(M)$ is self-adjoint linear operator given by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle,$$

for $X, Y \in \chi(M)$. Also it can be rewrite as

$$L_k(f) = div\left(P_k(\nabla f)\right),$$

which is the divergence form differential operator on M.
The Laplace operator of a hypersurface which is immersed into $\mathbb{R}^{n+1}$ is second order linear differential operator which can be naturally seen as a first variation of the mean curvature. From this point of view, Δ can be seen as the first one of the sequence of n operators $L_0 = \Delta, L_1, ..., L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)$-th mean curvature and also L_1 is known as a famous Cheng-Yau operator (see [3]).
The notion of finite type immersion of submanifolds of a Euclidean space has been used in classifying and characterizing well-known Riemannian submanifolds. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. A Euclidean submanifold is said to be of Chen finite type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian Δ (see [4]).
Suppose that M is a surface in the Euclidean space $\mathbb{E}^3$ and $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{E}^3$ centered at the origin. The map

$$G : M \longrightarrow \mathbb{S}^2 \subset \mathbb{E}^3,$$

which sends each point p of M to the unit normal vector $G(p)$ to M at the point p is called the Gauss map of the surface M. It is well-known that M has constant mean curvature if and only if $\Delta G = ||dG||^2 G$ (see [13]). As an example it was proved before that the only surfaces with Gauss map G which is an eigenfunction of Laplacian, that is, $\Delta G = \lambda G$ for some constant $\lambda \in \mathbb{R}$ are the planes, circular cylinders and spheres (see [5]).
The following lemma was established by [1] which so important in this area.

Lemma 2.1. *Let $x : M^n \to \mathbb{E}^{n+1}$ be a connected orientable hypersurface immersed into the Euclidean space with Gauss map G. Then G satisfies*

$$-L_k G = \binom{n}{k+1}\nabla H_{k+1} + \binom{n}{k+1}\left(nH_1H_{k+1} - (n-k-1)H_{k+2}\right)G.$$

In a special case, for $k=1$ and $n=2$ we get

$$-L_1 G = \nabla K + 2HKG,$$

where L_1 is known as Cheng-Yau operator and K is the Gauss curvature of the M.
As a special example, let $x : M \to \mathbb{E}^3$ be a translation surface in $\mathbb{E}^3$ where M is parameterized by

$$x(t_1, t_2) = (t_1, t_2, f(t_1) + g(t_2)), \quad (t_1, t_2) \in I \times J, \tag{1}$$

for smooth functions f and g. Also the natural frame $\{x_{t_1}, x_{t_2}\}$ is given by

$$x_{t_1} = \frac{\partial x}{\partial t_1} = (1, 0, f'), \quad x_{t_2} = \frac{\partial x}{\partial t_2} = (0, 1, g').$$

This surface was studied before in [12]. They established that

$$K=\frac{f'(t_1)\,g'(t_2)}{Q^4},\quad 2H=\frac{\left(1+f^2\right)g'(t_2)+\left(1+g^2\right)f'(t_1)}{Q^3},$$

where K and H are Gauss and mean curvatures of M respectively, also

$$Q=|x_{t_1}\times x_{t_2}|=\sqrt{1+f'^2+g'^2}.$$

They classified all translation surfaces parameterized by (1) satisfying

$$L_1G=AG,$$

where G is Gauss map of M and $A\in\mathbb{R}^{3\times 3}$.

Theorem 2.2 (Kim and Kim [12]). *Let M be a translation surface in the Euclidean 3-space $\mathbb{E}^3$. Then the only translation surfaces with Gauss map G satisfying $L_1G=AG$ for some nonzero 3×3 matrix A are the cylindrical surfaces.*

Consider $\mathbb{E}^3$ as a Euclidean 3-space with the scalar product given by

$$\langle,\rangle=di^2+dj^2+dk^2,$$

where (i,j,k) is a rectangular coordinate system of $\mathbb{E}^3$, then the norm of the vector $V\in\mathbb{E}^3$ is given by

$$||V||=\sqrt{\langle V,V\rangle}.$$

It is known that if $V=(v_1,v_2,v_3)$ and $W=(v_1',v_2',v_3')$ are arbitrary vectors in $\mathbb{E}^3$, the vector product of V and W is given by

$$V\wedge W=\left(v_2v_3'-v_3v_2',v_3v_1'-v_1v_3',v_1v_2'-v_2v_1'\right).$$

In this paper we are going to study a class of surfaces of revolution parametrized by

$$R(u,v)=\left(f(u)\sinh v,f(u)\cosh v,g(u)\right),\tag{2}$$

where f and g are smooth non-constant functions.
In the general case for a surface parameterized by $x(u,v)$, the unit normal vector field can be defined by

$$G=\frac{x_u\wedge x_v}{||x_u\wedge x_v||}.$$

The first fundamental form I of the surface is

$$I=E_Idu^2+2F_Idudv+G_Idv^2,$$

where

$$E_I=\langle x_u,x_u\rangle,\quad G_I=\langle x_v,x_v\rangle,\quad F_I=\langle x_u,x_v\rangle,$$

also the second fundamental form II of the surface is given by

$$II=L_{II}du^2+2M_{II}dudv+N_{II}dv^2,$$

with the coefficients

$$L_{II}=\langle x_{uu},G\rangle,\quad N_{II}=\langle x_{vv},G\rangle,\quad M_{II}=\langle x_{uv},G\rangle.$$

Under these parametrization, the Gaussian curvature K and the mean curvature H are given by

$$K=\frac{L_{II}N_{II}-M_{II}^2}{E_IG_I-F_I^2},\quad H=\frac{E_IN_{II}+G_IL_{II}-2F_IM_{II}}{2\left(E_IG_I-F_I^2\right)}.$$

In this paper by improving the previous methods we are going to study the class of surfaces of revolution given by (2) satisfying the condition $L_1G=AG$, where G is the Gauss map of the surface, L_1 is known Cheng-Yau operator and A is non-zero 3×3 matrix. We are going to prove following theorems.

Theorem 2.3. *Consider M as a surface of revolution parametrized by (2). If L_1 and G denote the Cheng-Yau operator and the Gauss map of the M respectively, then G satisfies $L_1G=AG$ for $A\in\mathbb{R}^{3\times3}$ if and only if M is flat.*

Theorem 2.4. *Let M be a surface of revolution parametrized by (2). If L_1 and G denote the known Cheng-Yau operator and the Gauss map of M respectively, then for a vector (a_1,a_2,a_3) orthogonal to* $\ker(A)$ *and $A\in\mathbb{R}^{3\times3}$, G satisfies $L_1G=AG$ if and only if the following relation holds between f and g.*

$$g(u)=\int\frac{C_1\cosh(v)\,k-a_3\left(\frac{d}{du}f(u)\right)f(u)}{kf(u)\cosh(v)}du+C_2,$$

where C_1 and C_2 are constants and

$$k=\frac{(a_2)^2-(a_1)^2}{a_2}.$$

3. Proofs and Main results

Proof of Theorem 2.3. Consider M as a surface of revolution parametrized by (2). From fundamental forms the Gauss map of the surface M is given by

$$G=\frac{1}{Q}\left(-g'f\sinh v,g'f\cosh v,ff'\right),$$

where $Q=\sqrt{(g')^2f'\cosh 2v+f^2(f')^2}$. Also the Gaussian curvature and the mean curvature of the surface are given respectively as

$$K=\frac{1}{Q^2}\frac{g'f^2\left(g'ff''+ff'g''\right)}{(f')^2f^2+f^2(g')^2\cosh 2v},$$

and

$$2H=\frac{1}{Q^3}\left[\cosh 2v\left(g'f^2(f')^2+f^3g'f''+f^3f'g''\right)+(g')^3f^2\right].$$

If we put $e_1=L_{II}\frac{\partial}{\partial u}$ and $e_2=N_{II}\frac{\partial}{\partial v}$, then the gradient of the Gaussian curvature ∇K can be computed as

$$\begin{aligned}\nabla K&=e_1(K)e_1+e_2(K)e_2\\&=Q^{-6}f^2\Big[Q^2g''\left(g'ff''+ff'g''\right)+2f'g'\left(g'f''+f'g''\right)\\&+g'\left(2g''ff''+g'f'f''+g'ff'''+ff'g'''\right)\\&-4\left(g'ff''+ff'g''\right)\left[\cosh 2v\left(g'g''f^2+ff'(g')^2\right)\right.\\&\left.+f(f')^3+f'f''f^2\right]\Big]e_1\\&-Q^{-6}\left(g'f^2\left(g'ff''+ff'g''\right)\right)e_2.\end{aligned}$$

The last term can be rewrite as

$$\begin{aligned}\nabla K &= Q^{-7} f^2 \omega X\left(f'\right) e_1 \\ &\quad - Q^{-7}\left(g'\right)^2 f^4 \omega e_2,\end{aligned}$$

where

$$\omega = g' f f'' + f f' g'',$$

and $X\left(f'\right)$ is the polynomial respect to f'. These together with Lemma 2.1 conclude that

$$\begin{aligned}AG = Q^{-7} f^2 \omega \Big[& X\left(f'\right) - \left(g'\right)^2 f^2 + \cosh 2v \left(g' f^2 \left(f'\right)^2 + f^3 g' f'' + f^3 f' g'' \right) \\ & + \left(g'\right)^3 f^2 \Big] g' G.\end{aligned} \tag{3}$$

By considering on degree of f' from both sides of equation, we get that

$$\begin{aligned}0 = Q^{-7} f^2 \omega \Big[& X\left(f'\right) - \left(g'\right)^2 f^2 + \cosh 2v \left(g' f^2 \left(f'\right)^2 + f^3 g' f'' + f^3 f' g'' \right) \\ & + \left(g'\right)^3 f^2 \Big] g' G,\end{aligned}$$

it follows that ω vanishes identically, therefore M is flat. One may consider that M is non-flat and A is zero matrix hence we have $L_1 G = 0$, which together with Lemma 2.1 implies that $K = 0$. This contradiction shows that M is flat anyway. □

Proof of Theorem 2.4. Suppose that $L_1 G = AG$ for some non-zero 3×3 matrix A. Then from the previous theorem, it follows that the surface M is flat, so we have $L_1 G = AG = 0$. In this case if $\ker(A)$ denotes the kernel space of the matrix A,

$$\ker(A) = \{X \in \mathbb{E}^3 | AX = 0\},$$

then the image of the Gauss map G lies in the $\ker(A)$. Since A is non-zero $\ker(A)$ is of at most 2-dimensional. Hence there exists a unit vector $a = (a_1, a_2, a_3)$ which is orthogonal to $\ker(A)$. Since

$$G = \frac{1}{Q}\left(-g' f \sinh v, g' f \cosh v, f f'\right),$$

is the Gauss map of the surface M we obtain

$$-a_1 g' f \sinh v + a_2 g' f \cosh v + a_3 f f' = 0.$$

Now by taking derivative from both sides respect to u and solve the differential equation by $g(u)$ we get what we were looking for. □

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On the Boundary of Convex Subsets in Complete Riemannian Manifolds

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Abstract

We generalize the Krein-Milman theorem to complete Riemannian manifolds of nonpositive sectional curvature.

1. Introduction

Convexity is an important tool in the study of Riemannian manifolds with nonpositive sectional curvature. It is very interesting to describe a convex subset K of a Riemannian manifolds by its boundary points and get global properties of K. We are going to show that a compact convex subset of a complete Riemannian manifold with nonpositive sectional curvature is equal to convex hull of its extreme points. Let M be a Riemannian manifold and $A \subseteq M$, a point $p \in A$ is called an extreme point of A, if p is not a relative interior point of any segment in A. For example, the unit circle is the set of extreme points of the closed unit disk in the Eucledean space $\mathbb{R}^2$. The exterme points are important in convex subsets. For instance Beltagy and Shenawy in [2] proved that every closed convex subset in Eucledean space is the convex hull of its extreme points. In fact, they refined the well-known Krein-Milman theorem which asserted every compact convex subset in a Hausdorff locally convex topological vector space is equal to the convex hull of its extreme points. Shenawy in [6] generlized the Krien-Milman theorem to complete Riemannian manifold without conjugate points.

In this paper, we characterize the extreme points by the second fundamental form, for a closed convex submanifold which its boundary is an immersed submanifold. Thus we can describe the extreme points of an immersed submanifold by an extrinsic invariant of the submanifold. We also show that the Krein-Milman theorem is valid on Riemannian manifolds with nonpositive sectional curvature by using the properties of covering map on manifolds.

Throughout this paper, $intA$, $\overline{A}$, A^c and ∂A will denote the interior, closure, complement and boundary of A and we adopt the definition and notation used in the book [3].

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2. Result

Let (M, g) be a complete Riemannian manifold and $d : M \times M \to \mathbb{R}$ be the metric induced by g. Let p be a fixed point of M. Then the function $f : M \to \mathbb{R}$ given by $f(q) = d(p, q)$ is continuous and if A is a subset of M, then the restriction $f_{|_A}$ is also continuous.

Remark 2.1. *Let (M, g) be a complete Riemannian manifold and K be a compact subsets of M. For each $p \in int(K)$, there exists a point $q \in \partial K$, such that q is the nearest boundary point of K to p.*

Definition 2.2. A subset A of M is called (strongly) convex, if for each two arbitrary points $p, q \in A$, all points of each (minimizing) geodesic segment joining p to q is contained in A.

If $B \subset M$, then the (strong) convex hull of B, which we denote it by $(C_s(B))$ $C(B)$ is by definition the smallest (strongly) convex set containing B.

Remark 2.3. *Let B be a subset of Riemannian manifold M. Put*

$$C_1(B) = \{\gamma(t) \mid \gamma\ is\ a\ geodesic\ joining\ two\ points\ of\ B\}$$
and
$$C_{i+1}(B) = C_1(C_i(B)),$$

then $C(B) = \cup_{i \in \mathbb{N}} C_i(B)$.
If we replace "geodesic" by minimizing geodesic, in the definition of $C_1(B)$, then we obtain strongly the definition of convex hull.

Proposition 2.4. *Let M be a complete Riemannian manifold. If $K \subset M$ is a closed and bounded convex subset of M such that K has at least two points, then $K = C(\partial K)$.*

Proof. Since K is a closed subset, $\partial K \subset K$ and it implies $C(\partial K) \subset C(K)$. By convexity of K, we get $C(\partial K) \subseteq K$. Conversely, suppose that p is a point in K, we are now going to show that $p \in C(\partial K)$. If p is a point in ∂K, then $p \in C(\partial K)$. Let $p \notin \partial K$. By the assumption, $p \in int(K)$. By Remark 2.1, there exists a $q \in \partial K$ such that q is the nearest point in ∂K to p. There exists a $\delta > 0$ such that $K \subset exp_q(\overline{B_\delta(0)})$ (since K is bounded). If $d(p, q) = r$, then we can find a $v \in T_qM$ such that $|v| = 1$ and $\gamma : [0, r] \to M$ with $\gamma(t) = exp_q(tv)$ is a minimizing geodesic segment joining q to p. We extend γ on interval $[0, \infty)$ by $\gamma(t) = exp_q(tv)$ (where $t \in [0, \infty)$), and denote it again by γ. Consider the set

$$B = \{t \in [0, \infty) \mid \gamma(t) \cap K \neq \emptyset\}.$$

B has a supremum. Let $sup(B) = s$. We show that $\gamma(s) \in \partial K$. Let $\{t_n\}$ be a pairwise disjoint increasing sequence of B, so that $\{t_n\}$ converges to s. Since γ is a continuous function and K is compact, $\{\gamma(t_n)\}$ is convergent to $\gamma(s)$ and $\gamma(s) \in K$. $\gamma(s)$ is a boundary point of K (because if not then there exists a $\varepsilon > 0$ such that $\gamma(s - \varepsilon, s + \varepsilon) \subset K$ that is a contradiction by the choise of s). Thus, $p \in C_1(\partial K)$ and therefore $p \in C(K)$.

□

The compactness condition in this theorem is necessary. For example boundary of $A = \{(x, y) \in \mathbb{E}^2 \mid x^2 + y^2 \geq 1\}$ is circle in $\mathbb{R}^2$, but $C(S^1) \neq A$.

Remark 2.5. *Let M be a Riemannian manifold and A be a subset of M. Then,*
A) The set of all extreme points of A is called the extreme A and is denoted by $E(A)$.
B) If A is a subset of a Riemannian manifold M, then $E(A) \subseteq \partial A$.

Theorem 2.6. *Let $\tilde{M}$ be a complete Riemannian manifold, M be the boundary of an open subset of $\tilde{M}$. If M is an immersed submanifold of $\tilde{M}$ and $p \in M$, then $p \in M$ is an extreme point of M if and only if for all $x \in T_pM$ and all $\eta \in (T_pM)^\perp$, $\prod_\eta(x) \neq 0$, where $\prod$ is the second fundamental form.*

Proof. Let $\overline{\bigtriangledown}$ be the Riemannian connection of $\tilde{M}$ and $\bigtriangledown$ be the induced connection of $\overline{\bigtriangledown}$ on M, and $f : M \to \overline{M}$ be the isometric immersion.
Suppose that p is an extreme point of M and $\prod_\eta(x) = 0$, for a $x \in T_pM$ and $\eta \in (T_pM)^\perp$. There exists a $\varepsilon > 0$ such that $\gamma : (-\varepsilon, \varepsilon) \to M$ is a geodesic passing through p with velocity x. Let X be a local extension of $\gamma'(t)$ to a tangent vector field on M and N be a local extension normal to M of η. By definition of the second fundamental form, we have

$$\prod_\eta(x) = H_\eta(x, x) =< B(x, x), \eta >=$$
$$< \overline{\bigtriangledown}_X X - \bigtriangledown_X X, N > (p) =< \overline{\bigtriangledown}_X X, N > (p),$$

By the assumption $\prod_\eta(x) = 0$, then $\overline{\bigtriangledown}_X X$ does not have a normal component. Since γ is a geodesic on M, then $\bigtriangledown_X X = 0$. Therefore, γ is a geodesic in $\overline{M}$ such that $\gamma(-\varepsilon, \varepsilon) \subset M$. This is a contradiction.
Conversely, assume that p is not an extreme point which leads to a contradiction. If p is not an extreme point, then there is a geodesic $\gamma : (-\varepsilon, \varepsilon) \to \overline{M}$ such that passes through p and $\gamma(-\varepsilon, \varepsilon) \subset M$. We consider the curve $\alpha : (-\varepsilon, \varepsilon) \to M$, with the property that $f \circ \alpha = \gamma$. Thus, we can show that α is a geodesic in M and is also in $\overline{M}$. Let $\eta \in (T_pM)^\perp$. By a suitable extention of $\alpha'(0)$ and η, we conclude $\prod_\eta(\alpha'(0)) = 0$, which contradicts the hypothesis. ☐

Remark 2.7. *A geodesic loop in a Riemannian manifold M is a curve $\alpha : [0, 1] \to M$ such that $\alpha(0) = \alpha(1)$ and α is geodesic on interior points of its domain (in $(0, 1)$). Note that a closed geodesic is a geodesic loop.*

Lemma 2.8. *Let M be a complete Riemannian manifold with nonpositive sectional curvature, $\tilde{M}$ be its universal covering manifold and $\pi : \tilde{M} \to M$ be the covering map. If $\tilde{B} \subset \tilde{M}$ and $\pi(C(\tilde{B}))$ has no geodesic loop, then*

$$\pi(C(\tilde{B})) = C(\pi(\tilde{B})).$$

Proof. By the definition of convex hull, we have

$$B \subset C_1(B) \subset C_2(B) ... \subset C(B).$$

Thus

$$\pi(B) \subset \pi(C_1(B)) \subset \pi(C_2(B)) \subset ... \subset \pi(C(B)).$$

$\pi(C(B))$ is a convex subset containing $\pi(B)$ and by the definition of convex hull, we have $C(\pi(B)) \subset \pi(C(B))$.
Conversely, we show that $\pi(C_1(B)) \subset C_1(\pi(B))$. Let $a \in \pi(C_1(B))$. There is a $\tilde{a} \in C_1(B)$ such that $\pi(\tilde{a}) = a$ and also there is a geodesic $\tilde{\alpha}$ in $\tilde{M}$ with endpoints in B such that $\tilde{\alpha}(t) = \tilde{a}$ for some $t \in [0, 1]$. Let $\gamma : [0, 1] \to M$ be a minimizing geodesic joining $\pi(\tilde{\alpha}(0))$ to $\pi(\tilde{\alpha}(1))$. If $\tilde{\gamma}$ is the lift of γ to the point $\tilde{\alpha}(0)$, then $\tilde{\gamma}(1) = \tilde{\alpha}(1)$ (since if not, $\pi(C(B))$ has a loop). Thus $\tilde{\gamma} = \tilde{\alpha}$ and $a \in C_1(\pi(B))$. We have

$$\pi(C_2(B)) = \pi(C_1(C_1(B)) \subset C_1(\pi(C_1(B)) \subset C_1(C_1(\pi(B))) = C_2(\pi(B)).$$

Hence, $\pi(C_2(B)) \subset C_2(\pi(B))$. In the similar way, for all i, we have $\pi(C_i(B)) \subset C_i(\pi(B))$. Therefore, $\pi(C(B)) \subset C(\pi(B))$. ☐

Theorem 2.9. *Let M be a complete Riemannian manifold with nonpositive sectional curvature. If $B \subset M$ is a closed strongly convex subset without geodesic loop, then $C(E(B)) = B$.*

Proof. Since M is a complete Riemannian manifold with nonpositive sectional curvature, then for every $p \in B$, the exponential map $exp_p : T_pM \to M$ is a covering map. Let a be another point in B. Consider $\tilde{B}$ as a convex component in T_pM containing 0 in $exp_p^{-1}(B)$.
We show that $exp(\tilde{B}) = B$.
Clearly $\tilde{B} \subset exp_p^{-1}(B)$, so $exp_p(\tilde{B}) \subseteq B$. For every b in B, there exists a minimizing geodesic γ from p to b. Suppose that $\tilde{\gamma}$ is the lift of γ to the point 0. Since $\tilde{\gamma}([0, 1])$ is a convex subset of $exp_p^{-1}(B)$, then $\tilde{\gamma}([0, 1]) \subseteq \tilde{B}$. Therefore, $B \subseteq exp_p(\tilde{B})$.
By Gauss‘s Lemma, we can show that $exp_pE(\tilde{B}) = E(B)$.
Since T_pM is a vector space, then the Krien-Milman theorem is valid in T_pM, thus $\tilde{B} = C(E(\tilde{B}))$. We have

$$B = exp_p(\tilde{B}) = exp_p(C(E(\tilde{B}))) = C(exp_p(E(\tilde{B})) = C(E(B)).$$

□

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Periodicity in IFS over arbitrary shift spaces

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Abstract

The orbit of a point $x \in X$ in a classical iterated function system (IFS) is defined as $\{f_u(x) = f_{u_n} \circ \cdots \circ f_{u_1}(x) : u = u_1 \cdots u_n$ is a word of a full shift on finite symbols$\}$. In other words, an IFS is parameterized over the full shift. Here, we parameterize over an arbitrary shift space Σ. We associate to $\sigma \in \Sigma$ a non-autonomous system $(X,\ f_\sigma)$ where trajectory of $x \in X$ is defined as $x,\ f_{\sigma_1}(x),\ f_{\sigma_1\sigma_2}(x), \ldots$. we show that if $\mathcal{I}$ is periodic along a transitive point $t \in \Sigma$ then $\mathcal{I}$ is periodic along any orbit of Σ.

1. Introduction

In a classical dynamical system, here called *conventional dynamical system*, we have a phase space and a unique map where the trajectories of points are obtained by iterating this map. However, in various problems, including applied ones, one may have some finite sequence of maps in place of a single map acting on the same phase space. For instance, in Physics by two or more maps have appeared in [1, 12], Economy in [14] and Biology in [4]. In Mathematics, this has been studied either by non-autonomous systems in many literature such as [9] or as iterated function system (IFS) for constructing and studying some fractals in [5, 8] or for investigating dynamical properties in many places such as [2, 3, 6, 7].
In a "classical" IFS, a compact metric space X and a set of some k finite continuous functions $\{f_0, \cdots, f_{k-1}\}$ on X are assumed and the trajectory of a point $x \in X$ is considered to be the action on x of the sequence of freely combination of those maps, or action on x of combination of those maps over the words of a full shift: just write

$$f_u = f_{u_1} \circ \cdots \circ f_{u_m} \tag{1}$$

where $u = u_1 \cdots u_m$ is a word of the full shift over k symbols. Hence no limitation is applied as in our aforesaid example on the robotic arms where there words were forbidden to have 11 as a subword. The limitation applied on the shift space would transfer to some limitations on the system. For instance, a system may be transitive in classical IFS but not in our case, i.e., when the full shift is replaced with a more general subshift.
Thus one may look at X as a phase space and the subshift Σ as a parameter space showing how the maps must be combined. In other words, there are some words that one cannot perform (2). This is the case where a subshift instead of the full shift must be considered and it is of our interest.

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2. Preliminaries

2.1. Iterated function systems

Throughout the paper, X will be a compact metric space. The *classical* iterated function system (IFS) consists of finitely many continuous self maps $\mathcal{F} = \{f_0, \dots, f_{k-1}\}$ on X. The *forward orbit* of a point $x \in X$, denoted by $\mathcal{O}^+(x)$, is the set of all values of finite possible combinations of f_i's at x. We need the following equivalent statement: Let Σ_F be the full shift on k symbols and let $\mathcal{L}(\Sigma_F)$ called the *language of* Σ_F be the set of words. Define $f_u(x) : X \to X$ by

$$f_{u_n} \circ \cdots \circ f_{u_1}(x), \quad u = u_1 \cdots u_n \in \mathcal{L}(\Sigma_F). \tag{2}$$

Then $\mathcal{O}^+(x) = \{f_u(x) : \ u \in \mathcal{L}(\Sigma_F)\}$. Such iterated function systems, here called *classical IFS*, have been the subject of study for quite a long time.
Here we define an IFS to be

$$\mathcal{I} = (X, \mathcal{F} = \{f_0, \dots, f_{k-1}\}, \Sigma). \tag{3}$$

where f_i is continuous and Σ is an arbitrary subshift on k symbols, not necessarily the full shift Σ_F as in the classical IFS. For review of symbolic dynamics see [10]. By this setting, Σ_F above will be replaced with Σ and thus $\mathcal{O}^+(x) = \{f_u(x) : \ u \in \mathcal{L}(\Sigma)\}$ is the forward orbit of x. In particular, $f_u(f_v(x)) = f_{vu}(x)$ whenever vu is admissible or equivalently $vu \in \mathcal{L}(\Sigma)$. Let $u = u_1 \cdots u_n \in \mathcal{L}(\Sigma)$ and set $u^{-1} := u_n \cdots u_1$. Then for $A \subseteq X$,

$$\begin{aligned} (f_u)^{-1}(A) &= (f_{u_n} \circ \cdots \circ f_{u_1})^{-1}(A) \\ &= f_{u_1}^{-1} \circ \cdots \circ f_{u_n}^{-1}(A) \\ &= f_{u^{-1}}^{-1}(A), \end{aligned}$$

where for the last equality, we used (2). Also

$$\begin{aligned} f_{u^{-1}}^{-1}(f_{v^{-1}}^{-1}(A)) &= f_{v^{-1}u^{-1}}^{-1}(A) = f_{(uv)^{-1}}^{-1}(A) \\ &= (f_{uv})^{-1}(A). \end{aligned}$$

Thus the backward orbit and the (full) orbit of a point $x \in X$ are $\mathcal{O}_-(x) = \{f_{u^{-1}}^{-1}(x) : \ u \in \mathcal{L}(\Sigma)\}$ and $\mathcal{O}(x) = \mathcal{O}_-^+(x) = \mathcal{O}^+(x) \cup \mathcal{O}_-(x)$ respectively.
When all f_i's are homeomorphisms, the backward, forward and full trajectory of x is defined.

2.2. Symbolic dynamics

A brief recall of the symbolic dynamics is given here. Notations are borrowed from [10] and the proofs of the claims can be found there. Let $\mathcal{A}$ be a non-empty finite set and let $\Sigma_F = \mathcal{A}^{\mathbb{Z}}$ (resp. $\mathcal{A}^{\mathbb{N}}$) be the collection of all bi-infinite (resp. right-infinite) sequences of symbols from $\mathcal{A}$. The shift map on Σ_F is the map τ where $\tau(\sigma) = \sigma'$ is defined by $\sigma'_i = \sigma_{i+1}$. The pair (Σ_F, τ) is the *full shift* and any closed invariant subset Σ of Σ_F is called a *subshift* or a *shift space*. A *word* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. Denote by $\mathcal{L}_n(\Sigma)$ the set of all admissible n-words and call $\mathcal{L}(\Sigma) := \bigcup_{n=0}^{\infty} \mathcal{L}_n(\Sigma)$ the *language* of Σ. For $u \in \mathcal{L}_k(\Sigma)$, let the cylinder ${}_\ell[u]_{\ell+k-1} = {}_\ell[u_\ell \cdots u_{\ell+k-1}]_{\ell+k-1}$ be the set $\{\sigma = \sigma_0\sigma_1 \cdots \in \Sigma : \sigma_\ell \cdots \sigma_{\ell+k-1} = u\}$. If $\ell = 0$, we drop the subscripts and we just write $[u]$.
A shift space Σ is *irreducible* if for every ordered pair of words $u, v \in \mathcal{L}(\Sigma)$ there is a word $w \in \mathcal{L}(\Sigma)$ so that $uwv \in \mathcal{L}(\Sigma)$. A point $\sigma \in \Sigma$ is *transitive* if every word in Σ appears in σ infinitely many often. A subshift Σ is irreducible iff Σ has a transitive point.
Shift spaces described by a finite set of forbidden blocks are called *shifts of finite type* (SFT) and their factors are called *sofic*. A word $w \in \mathcal{L}(\Sigma)$ is called *synchronizing* if $uwv \in \mathcal{L}(\Sigma)$ whenever $uw, wv \in \mathcal{L}(\Sigma)$. A *synchronized system* is an irreducible shift which has a synchronizing word. Any sofic is synchronized.

3. Periodicity in IFS

The notion of a periodic point in the case of a conventional dynamical system $(X,\ f)$ is very natural and intuitive. This is not the case for an IFS or a non-autonomous system. One may check [13] where a survey of the periodic points for a non-autonomous system is offered. It turns out that one way is to define a periodic point along an orbit in an IFS as in the conventional dynamical system. By that we mean that $x \in X$ is periodic of period p along $\sigma \in \Sigma$ if there is a word u, $|u| = p$ such that $f_u(x) = x$ and $\sigma = u^\infty = \sigma_1\sigma_2\cdots\sigma_p\sigma_{p+1}\cdots$ is periodic in Σ. Thus $\sigma_{\ell p+i} = u_i$ for $\ell \in \mathbb{N}$ and

$$f_{\sigma_1\cdots\sigma_{\ell p+i}}(x) = f_{\sigma_1\cdots\sigma_i}(x), \quad 1 \leq i < p. \tag{4}$$

This is not the case for a general IFS, for if $f_u(x) = x$, then one cannot guarantee that u^n is an admissible word for $n \in \mathbb{N}$.

Definition 3.1. Let $\mathcal{I} = (X,\ \mathcal{F},\ \Sigma)$ be an IFS. A point $x \in X$ is periodic of period p along $\sigma = \sigma_1\sigma_2\cdots \in \Sigma$ if for any $\ell \in \mathbb{N}$, $f_{\sigma_1\cdots\sigma_{\ell p}}(x) = x$.

Lemma 3.2. *Let $\mathcal{I} = (X,\ \mathcal{F},\ \Sigma)$ be an IFS where Σ is an M-step SFT. Assume x is periodic of period ℓ along a $\sigma \in \Sigma$ which is not necessarily periodic. Then there is a periodic σ' so that x is periodic of period ℓ' along σ' and $\ell' = q\ell$ for some $q \in \mathbb{N}$.*

Proposition 3.3. *Let $\mathcal{I}$ and $x \in X$ be as in Lemma 3.2, but Σ an irreducible sofic. Then the conclusion of that lemma is valid.*

Recall that SFT's $\subsetneq$ sofics $\subsetneq$ synchronized systems.

Example 3.4. *The conclusion of Proposition 3.3 is not valid for the case when Σ is a non-sofic synchronized system.*

Proof. Let Σ be the synchronized subshift whose cover is presented in Figure 1. In this cover, $m = m_0m_1\cdots$ is a fixed point of the Morse substitution and thus $\Sigma_m = \overline{\mathcal{O}^+(m)}$ is a minimal subshift [11]. Let us briefly remind the Morse substitution. Set

$$\varrho(v) = \begin{cases} 01, & \text{if } v = 0, \\ 10, & \text{if } v = 1, \end{cases}$$

to be the substitution map which means for any word $u = u_0\cdots u_k$ in $\{0,\ 1\}^{\mathbb{N}}$, $\varrho(u) = \varrho(u_0)\cdots\varrho(u_k)$. This gives a primitive substitution with two fixed points

$$\begin{aligned} m &= 0 \mapsto 01 \mapsto 0110 \mapsto 01101001 \mapsto \cdots \\ m' &= 1 \mapsto 10 \mapsto 1001 \mapsto 10010110 \mapsto \cdots \end{aligned}$$

Now, the closure of orbits of either of these fixed points under the shift map gives a minimal subshift.
Let $\mathcal{I} = (S^1,\ \mathcal{F} = \{f_0,\ f_1,\ f_2\},\ \Sigma)$ where $f_0 : S^1 \to S^1$ is an irrational rotation, $f_1 = f_0^{-1}$ and f_2 another irrational rotation say $f_2 = f_0 \circ f_0$.

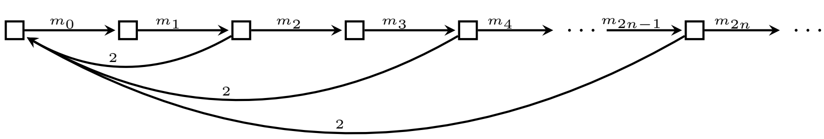

Fig. 1. The cover for Σ. Here, $m = m_0m_1\cdots$ is a fixed point of the Morse substitution.

$\square$

Theorem 3.5. *Let $\mathcal{I} = (X, \mathcal{F}, \Sigma)$ be an IFS. If $x \in X$ is periodic of period p along a transitive $t = t_1 t_2 \cdots \in \Sigma$, then for any $\sigma \in \Sigma$ there is a $1 \leq k < p$ such that x is periodic along $\sigma' = \tau^k(\sigma)$ where τ is our shift map.*

As an application of Proposition 3.5, consider $\mathcal{I} = (S^1, \{f_0, f_1\}, \Sigma_F)$ where f_0 is an irrational rotation and $f_1(z) = z^2$ and Σ_F the full shift and hence sofic. Here, $\mathcal{I}$ has a set of dense periodic points along $\sigma = 1^\infty \in \Sigma$ and no periodic points along $0^\infty \in \Sigma$ and so no periodic points along any transitive $t \in \Sigma$.

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Numerical Solution for Delayed and Fractional Order Logistic Equation

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Abstract

In this study an efficient semi-analytical method is proposed for numerical approximation of fractional order logistic equation (FOLEs) with two constant rational delays. Due to the structure of fractional delay differential equations (FDDEs) with non-equal delays utilizing of multi-step methods are noticeable selection. In the proposed approach the method of steps is combined with generalized differential transformation method (GDTM) to obtain numerical solution for FOLE with delay. Compared to the current numerical methods the proposed algorithm seems to be very reliable, effective and more convenient technique. Numerical application is shown with an example.

1. Introduction

Delayed logistic equation which can be used to model the population growth was proposed and investigated by Hutchinson [1], in the following form:

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{k}\right), \tag{1}$$

where $\tau > 0$ is a maturation time. For describing a situation where the several processes affecting the population occur with different time delays, Gyori [2], introduced and studied equation as follows:

$$N'(t) = N(t)\left(a - \sum\nolimits_{j=1}^{n} b_j N(t-\tau_j)\right). \tag{2}$$

Another form of neutral logistic equation were studied by Gyori and Ladas [3] was

$$N'(t) = N(t)\left\{r\left(1 - \frac{N(t-\delta)}{k}\right) + c\frac{N'(t-\tau)}{N(t-\tau)}\right\}, \tag{3}$$

where the term $c\frac{N'(t-\tau)}{N(t-\tau)}$ is related to the growth rate of the population at time $t-\tau$. Rebenda and Smarda [4] used a combination of the method of steps and differential transform method (DTM) to propose a numerical solution for Eq.

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(3). Delayed and natural type logistic equation with two different delays in fractional order (FOLE) can be formulated in the following form:

$$ {}_{0}^{c}D_t^{\alpha} u(t) = u(t) \left\{ r \left[1 - \frac{u(t-\sigma)}{\kappa} \right] + c \frac{{}_{0}^{c}D_t^{\alpha}(u(t-\tau))}{u(t-\tau)} \right\} 0 < \alpha \leq 1, \tag{4} $$

when initial function $u(t) = \phi(t), t\epsilon[-\gamma, 0], \gamma = \max\{\sigma, \tau\}$ and σ , τ are positive rational delays. Fractional derivative in Eq. (4) is in Caputo sense [5], because Caputo fractional derivatives have the advantages of defining integer order initial conditions for fractional order differential equations that is suitable for problems which arise in true world physical phenomena. Ahmed M. A. et al. [5] used Adams-type predictor-corrector method to obtain numerical solution for fractional-order logistic equation (FOLE) with two different delays [6]. N. H. Sweilam et al. [7] applied Chebyshev approximations to obtain numerical solution for FOLE with delays.
The main idea of the present work is based on apply the method of steps and generalized differential transform method to achieve numerical solution for Eq. (4). Comparison with the other numerical methods reveals that the proposed technique seems to be easy and more applicable. The remaining of this paper is organized as follows. First some basic definitions of fractional calculus are presented, then generalized differential transform method with some properties are introduced. Subsequently the implementation of fractional differential transform method to FOLE with two different delays is investigated. Furthermore, an example is examined to show the relevancy of the proposed method.

2. Fractional calculus definitions

In this section, we mainly recall some definitions which will be used in this study.
Definition 1 Let $\mu\epsilon\mathbb{R}$and $m\epsilon\mathbb{N}$. A function $f : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathbb{C}_\mu$ if there exists $k\epsilon\mathbb{R}, k > \mu$ and gϵC $[0, \infty)$ such that $f(x) = x^k g(x), \forall x\epsilon\mathbb{R}^+$. Moreover, $f\epsilon\mathbb{C}_\mu^m$ if $f^{(m)}\epsilon\mathbb{C}_\mu$.
Definition 2 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t)\,\epsilon C_\mu, \mu \geq -1$ is defined as:
$\begin{cases} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0, \\ J^0 f(t) = f(t), \end{cases}$
where $\Gamma(\mathrm{z}) = \int_0^\infty \mathrm{e}^{-\mathrm{t}} \mathrm{t}^{\mathrm{z}-1} \mathrm{dt}, \mathrm{z} \in \mathrm{C}$. For $f\epsilon C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$, the operator J^α satisfies the following properties:

$$ (1)\, J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t) = J^\beta J^\alpha f(t), $$
$$ (2)\, J^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} t^{\alpha+\gamma}. $$

The Caputo fractional differentiation operator D^β defined as:
$D^{\boldsymbol{\beta}} f(t) = \begin{cases} J^{m-\beta} \frac{d^m}{dt^m} f(t), m-1 < \beta < m, \\ \frac{d^m}{dt^m} f(t), \beta = m. \end{cases}$
Moreover, the operator D^β satisfies the following properties:
Let $m - 1 < \beta \leq m, m\epsilon N$ and $f\epsilon C_\mu^m, \mu \geq -1$, and $\gamma > \beta - 1$, then

1. $D^\beta J^\beta f(t) = f(t)$,
2. $J^\beta \left[D^\beta f(t)\right] = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}, t > 0,$
3. $D^\beta t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\beta)} t^{\gamma-\beta}$
4. $D^\beta \mathrm{c} = 0$, where c is constant.

For more details about fractional calculus we refer the reader to [5].

3. Overview of the generalized differential transform method (GDTM)

In this section GDTM is recalled. Suppose that $u(t)$ is an analytic function so it can be expanded in terms of a fractional power series in the form

$$u\left(t\right)=\sum\nolimits_{k=0}^{\infty}U\left(k\right)\left(t-t_0\right)^{\frac{k}{\theta}}, \tag{5}$$

where θ is the order of fraction to be selected and $U(k)$ is the fractional differential transform of $u(t)$. Since the initial conditions are implemented by integer-order derivatives for practical applications, the transformation of the initial conditions is defines as follows [8].

$$U\left(k\right)=\begin{cases}\frac{1}{\left(\frac{k}{\theta}\right)!}\left[\frac{d^{\frac{k}{\theta}}u(t)}{dt^{\frac{k}{\theta}}}\right]_{t=t_0} & \frac{k}{\theta}\epsilon Z^{+}\\ 0 & \frac{k}{\theta}\notin Z^{+}.\end{cases} \tag{6}$$

Where $k=0,1,2,\ldots,(\beta\theta-1)$ and β is the order of the fractional differential equation being considered so θ should be chosen such that $\beta\theta$ is a positive integer.
Some basic properties of GDTM are accessible in [8, 9].
Theorem 1 Let $u(t)$, $v\left(t\right)$ and $w(t)$ be functions of time t and $U\left(k\right)$, $V\left(k\right)$ and $W\left(k\right)$ are their related fractional transform, then the following relations are hold,
1. If $u\left(t\right)=v\left(t\right)+w\left(t\right)$, then $U\left(k\right)=V\left(k\right)+W\left(k\right)$.
2. If $u\left(t\right)=cv\left(t\right)$, then $U\left(k\right)=cV\left(k\right)$, where c is constant.
3. If $u\left(t\right)=v\left(t\right)w\left(t\right)$, then $U\left(k\right)=\sum\nolimits_{l=0}^{k}V\left(l\right)W\left(k-l\right)$.
4. If $u\left(t\right)=(t-t_0)^{p}$ where p is constant., then $U\left(k\right)=\delta\left(k-\theta p\right)$, while $\delta\left(k\right)=\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$
5. If $u\left(t\right)=D^{\gamma}v\left(t\right)$, then $U\left(k\right)=\frac{\Gamma\left(\frac{k}{\theta}+\gamma+1\right)}{\Gamma\left(\frac{k}{\theta}+1\right)}V\left(k+\gamma\theta\right)$.
The method and proofs of theorem 1 well addressed in literature [10].

4. Implementation of GDTM for FOLE with two different Delays

The main goal of this study is the combination of the generalized differential transformation method with the method of steps [11] to find a numerical solution for fractional order logistic equations with two different delays. In this process first the terms of involving delays are replaced with initial conditions and its derivatives, so the fractional-order delay logistic equation is reduced to a fractional-order logistic equation. Then the generalized differential transform method is applied to convert the fractional-order logistic equation to a system of algebraic recurrence relations. Recall Eq. (4)

$${}_{0}^{c}D_{t}^{\alpha}u\left(t\right)=u\left(t\right)\left\{r\left[1-\frac{u\left(t-\tau_1\right)}{\kappa}\right]+c\frac{{}_{0}^{c}D_{t}^{\alpha}(u\left(t-\tau_2\right))}{u\left(t-\tau_2\right)}\right\}0<\alpha\leq 1. \tag{7}$$

Subject to the initial condition $u\left(t\right)=\phi\left(t\right), t\epsilon\left[-\gamma,t_0\right], \gamma=\max\{\tau_1,\tau_2\}$ such that $u\left(t_0\right)=\phi\left(t_0\right), {}_{0}^{c}D_{t}^{\alpha}u\left(t_0\right)={}_{0}^{c}D_{t}^{\alpha}\phi\left(t_0\right)$.
The idea of how to solve Eq. (7) generally on the interval $(t_0,\gamma],\gamma>t_0$ and $\gamma=max\{\sigma,\tau\}$ is as follows:
First we apply the method of steps on the interval $(t_0,\tau_{min}]$ where $\tau_{min}=\min\left\{\sigma,\tau\right\}$ .We replaces the initial function $\phi(t)$ and its derivatives in all places where unknown function and its derivatives in delayed form are appeared. Then the Eq. (7) changes to the fractional logistic differential equation:

$${}_{0}^{c}D_{t}^{\alpha}u\left(t\right)=u\left(t\right)\left\{r\left[1-\frac{\phi\left(t-\tau_1\right)}{\kappa}\right]+c\frac{{}_{0}^{c}D_{t}^{\alpha}(\phi\left(t-\tau_2\right))}{\phi\left(t-\tau_2\right)}\right\}0<\alpha\leq 1. \tag{8}$$

Applying GDTM we get the recurrence relation as follows

$$\frac{\Gamma\left(\alpha(k+1)+1\right)}{\Gamma\left(\alpha k+1\right)}U_1\left(k+1\right)=U_1\left(k\right)\left\{r\left[1-\frac{\varnothing(k)}{\kappa}\right]+c\frac{\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}\left(\psi\left(k\right)\right)}{\psi\left(k\right)}\right\}. \tag{9}$$

While $\varnothing(k)$ and $\psi(k)$ are generalized differential transform of $\phi\left(t-\tau_i\right), i=1,2$ respectively.
After calculating $U_1\left(k\right), k=1,2,\ldots$, from (9) with the inverse transformation, we find the solution of Eq. (7) on the interval $(t_0, \tau_{min}]$ in the form of generalized Taylor series

$$u_1\left(t\right)=\sum\nolimits_{k=0}^{\infty}U_1\left(k\right)\left(t-t_0\right)^{\frac{k}{\alpha}}. \tag{10}$$

When we find an approximation solution of $u_1(t)$ in the interval $(t_0, \tau_{min}]$ as aforementioned, then we need to find a numerical solution for $u_2(t)$ in the interval $(\tau_{min}, \gamma]$. Rewrite Eq. (7) as follows

$${}_0^cD_t^\alpha u\left(t\right)=u\left(t\right)\left\{r\left[1-\frac{\phi\left(t-\tau_1\right)}{\kappa}\right]+c\frac{{}_0^cD_t^\alpha(\phi\left(t-\tau_2\right))}{\phi\left(t-\tau_2\right)}\right\}0<\alpha\leq 1 \tag{11}$$

Where $\phi\left(t-\tau_i\right) are \begin{cases}\phi\left(\mathrm{t}-\tau_{\mathrm{i}}\right)=\mathrm{u}_1\left(t-\tau_i\right) & t-\tau_i\in\left(t_0,\gamma\right],\\ \phi\left(t-\tau_i\right)=\phi\left(t-\tau_i\right), & t-\tau_i\in[t_0-\gamma,t_0].\end{cases}$ (12)

Then we apply GDTM to Eq. (11) which leads us to the recurrence relation as

$$\frac{\Gamma\left(\alpha(k+1)+1\right)}{\Gamma\left(\alpha k+1\right)}U_2\left(k+1\right)=U_2\left(k\right)\left\{r\left[1-\frac{\varnothing(k)}{\kappa}\right]+c\frac{\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}\left(\psi\left(k\right)\right)}{\psi\left(k\right)}\right\}, \tag{13}$$

while $\varnothing(k)$ and $\psi(k)$ are generalized differential transform of $\phi\left(t-\tau_i\right), i=1,2$ respectively. After calculation $U_2\left(k\right) k=1,2,\ldots$ from (13) we obtain the solution $u_2\left(t\right)$ using the inverse transformation in the form

$$u_2\left(t\right)=\sum\nolimits_{k=0}^{\infty}U_2\left(k\right)\left(t-t_0\right)^{\frac{k}{\alpha}}, t\epsilon\left(\tau_{min},\gamma\right]. \tag{14}$$

Thus the numerical solution of Eq. (7) on the interval $[t_0, \gamma]$ is achieved in the form

$$u\left(t\right)=\begin{cases}u_1\left(t\right), & t\epsilon(t_0,\tau_{min}]\\ u_2\left(t\right), & t\epsilon\left(\tau_{min},\gamma\right].\end{cases} \tag{15}$$

5. Numerical example

In this section, an example is presented to show the effectiveness and accuracy of the proposed method.
Example : Consider the neutral type logistic equation of fractional order

$${}_0^cD_t^\alpha u\left(t\right)=u\left(t\right)\left\{r\left[1-\frac{u\left(t-\sigma\right)}{\kappa}\right]+c\frac{{}_0^cD_t^\alpha(u\left(t-\tau\right))}{u\left(t-\tau\right)}\right\}0<\alpha\leq 1, \tag{16}$$

with initial function

$$u\left(t\right)=\phi\left(t\right), t\epsilon\left[-\gamma,t_0\right], \gamma=\max\left\{\sigma,\tau\right\}.$$

Equation (16) was studied by Rebenda and Šmarda in case $\alpha=1$ [4].
Let $\kappa=3, r=0.45, \sigma=2, c=0.3, \tau=1 and \phi\left(t\right)=2.3 for t\epsilon\left[-2,0\right]$. In this case Eq. (16) becomes

$${}_0^cD_t^\alpha u\left(t\right)=u\left(t\right)\left\{0.45\left[1-\frac{u\left(t-2\right)}{3}\right]+0.3\frac{{}_0^cD_t^\alpha(u\left(t-1\right))}{u\left(t-1\right)}\right\}t\epsilon\left(0,2\right], \tag{17}$$

subject to the initial condition

$$u\left(t\right)=\phi\left(t\right)=2.3, t\epsilon\left[-2,0\right].$$

Since there are two constant delays $\sigma=\tau_1=2, \tau=\tau_2=1$, so we can follow the algorithm described in this study. $t_0=0, \gamma=2$ with the aforementioned algorithm we calculate $\tau_{min=}1$. On the interval $(0,1]=(t_0,\tau_{min}]$ we use the method of steps for Eq. (17) to obtain

$${}_0^cD_t^\alpha u_1\left(t\right)=u_1\left(t\right)\left\{0.45\left[1-\frac{2.3}{3}\right]+0.3\frac{{}_0^cD_t^\alpha\left(2.3\right)}{2.3}\right\}. \tag{18}$$

Using property (4) from Caputo fractional definition equation (18) is simplified to

$$ {}_0^cD_t^\alpha u_1(t) = 0.105u_1(t). \tag{19} $$

With applying GDTM for Eq. (19) we have the recurrence relation as

$$ U_1(k+1) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)} 0.105 U_1(k). \tag{20} $$

Since$U_1(0) = 2.3$, thus we compute

$$ \begin{aligned} U_1(1) &= \frac{\Gamma(1)}{\Gamma(\alpha+1)} 2.3.0.105, \\ U_1(2) &= \frac{\Gamma(1)}{\Gamma(2\alpha+1)} 2.3.(0.105)^2, \\ U_1(3) &= \frac{\Gamma(1)}{\Gamma(2\alpha+1)} 2.3.(0.105)^3, \\ &\vdots \\ U_1(n) &= \frac{\Gamma(1)}{\Gamma(2\alpha+1)} 2.3.(0.105)^n \end{aligned} \tag{21} $$

Using the inverse GDTM the solution will be in the form of

$$ u_1(t) = \sum_{k=0}^{\infty} U_1(k)(t-t_0)^{k\alpha} = 2.3+2.3.\frac{0.105.}{\Gamma(\alpha+1)} t^\alpha + 2.300.\frac{0.105^2}{\Gamma(2\alpha+1)} t^{2\alpha} + \ldots (22) = 2.3E_\alpha(0.105t^\alpha)\, t \in (0,1], $$

where $E_\alpha(t)$ is a one-parameter Mittag Leffler function which is defined as $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha+1)}$ with the property ${}_0^cD_t^\alpha(E_\alpha(\lambda z)) = \lambda {}_0^cD_t^\alpha(E_\alpha(z))$.
It is required to calculate $u_2(t)$ on the interval (1, 2]. In this case Eq. (17) has the initial conditions with the form of:

$$ u(t) = \begin{cases} 2.3, & t\epsilon[-2,0] \\ 2.3E_\alpha(0.105t^\alpha), & t\epsilon(0,1], \end{cases} \tag{23} $$

with this conditions and using method of steps Eq .(17) changes to

$$ {}_0^cD_t^\alpha u_2(t) = u_2(t) \left\{ 0.45 \left[1 - \frac{2.3}{3} \right] + 0.3 \frac{{}_0^cD_t^\alpha(E_\alpha(0.105(t-1)^\alpha))}{E_\alpha(0.105(t-1)^\alpha)} \right\} = 1.3.0.105u_2(t). \tag{24} $$

Now with using GDTM the recurrence relation of Eq. (24) is:

$$ \begin{aligned} U_2(k+1) &= \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)} 1.3.0.105 U_2(k), \\ U_2(0) &= u_1(1) = 2.3E_\alpha(0.105). \end{aligned} \tag{25} $$

It is easy to obtain

$$ \begin{aligned} U_2(1) &= \frac{1.3.0.105}{\Gamma(\alpha+1)} 2.3E_\alpha(0.105), \\ U_2(2) &= \frac{(1.3.0.105)^2}{\Gamma(2\alpha+1)} 2.3E_\alpha(0.105), \\ U_2(3) &= \frac{(1.3.0.105)^3}{\Gamma(3\alpha+1)} 2.3E_\alpha(0.105), \\ &\vdots \end{aligned} \tag{26} $$

Thus the solution is as follows:

$$u_2(t) = \sum_{k=0}^{\infty} U_2(k)(t-1)^{k\alpha} = 2.3E_\alpha(0.105)$$
$$+ 2.3\frac{1.3.0.1051}{\Gamma(\alpha+1)}E_\alpha(0.105)(t-1)^\alpha + 2.3\frac{(1.3.0.1051)^2}{\Gamma(2\alpha+1)}E_\alpha(0.105)(t-1)^{2\alpha} + \dots$$
$$= 2.3E_\alpha(0.105)E_\alpha(1.3.0.105(t-1)^\alpha)\, t \in (1,2]. \quad (27)$$

In case $\alpha = 1$ the results in (27) are similar to the results of Rebenda et al. [4].

6. Conclusion

In this study it was tried to combine two powerful and effective methods such as method of steps and generalized differential transform method to find an approximation solution for fractional delayed logistic differential equations with two several delays, while the fractional derivatives were in Caputo sense. The numerical result was presented in terms of power series. The results confirmed that the described method need low calculations and it was very convenient in calculation. The test example was proposed to illustrate the efficiency and reliability of the proposed method.

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Modified Legendre functions for solving the telegraph equation on an unbounded domain

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Abstract

In this paper, a numerical method is developed for solving the telegraph equation on an unbounded domain with vanishing boundary conditions. To this end, two classes of the basis functions are introduced and employed. First, the shifted Legendre polynomials are utilized for approximating solution of the problem in time direction. Then, a new class of basis functions called the modified Legendre functions is generated and used for approximating solution of the problem in unbounded space direction. The method converts the equation under consideration into a system of algebraic equations that its numerical solution can be easily obtained. The accuracy of the method is examined by solving a test problem.

1. Introduction

The main objective of this study is to describe an appropriate numerical scheme for numerical solution of the following class of the telegraph equations defined over an unbounded domain:

$$\begin{cases} u_{tt}(x,t) + \alpha^2 u_t(x,t) + 2\beta u(x,t) = \gamma u_{xx}(x,t) + f(x,t), \quad x \in \mathbb{R}, \quad t \in [0,T], \\ \lim\limits_{|x|\to\infty} u(x,t) = 0, \\ u(x,0) = h_1(x), \\ u_t(x,0) = h_2(x), \end{cases} \tag{1}$$

where α and β are given constants (with $\alpha > \beta > 0$), f, h_1 and h_2 are known functions and u is the unknown solution. This kind of linear partial differential equation is used to model the voltage on a very small piece of telegraph wire which consists of a resistor and coil of inductance [1]. Many problems in the fields of physics and engineering lead

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to the solution of differential equations in unbounded domains. During the past decades, researchers have devoted particular attention to solving this kind of equations on unbounded domain. Since, as far as we know, there has not been any study dealing with the numerical solution of the telegraph equation on an unbounded domain, it motivates our interest to develop an appropriate numerical method for solving such problems. In this paper, we use the Legendre polynomials to generate two useful classes of the basis functions known as the shifted Legendre polynomials and the modified Legendre functions for solving the problem introduced in Eq. (1). More precisely, the shifted Legendre polynomials are used in the time direction, while the modified Legendre functions together with their operational matrices of derivative are utilized in the space direction to convert the problem under consideration into an algebraic system of equations.

2. Basis functions

2.1. The shifted Legendre polynomials (LPs)

Let $L_j(\tau)$ is the Legendre polynomial of degree $j \in \mathbb{N} \cup \{0\}$ defined over $[-1, 1]$, which satisfies the following recurrence relation [2]:

$$(j+1)L_{j+1}(\tau) = (2j+1)\tau L_j(\tau) - jL_{j-1}(\tau), \qquad j = 1, 2, \cdots, \tag{2}$$

with $L_0(\tau) = 1$ and $L_1(\tau) = \tau$.
The shifted LPs can be defined over $[0, T]$ by using the change of variable $\tau = \frac{2t}{T} - 1$ and renaming $L_n\left(\frac{2t}{T} - 1\right)$ by $P_n(t)$ as follows

$$(n+1)P_{n+1}(t) = (2n+1)\left(\frac{2t}{T} - 1\right)P_n(t) - nP_{n-1}(t), \qquad n = 1, 2, \cdots, \tag{3}$$

where $P_0(t) = 1$ and $P_1(t) = \frac{2t}{T} - 1$. We remind that the orthogonality of these polynomials are as follows

$$\int_0^T P_k(t)P_{k'}(t)dt = \frac{T}{2k+1}\delta_{kk'}.$$

We can use the shifted LPs for approximating an square integrable function $u(t)$ defined over $[0, T]$ as follows

$$u(t) \simeq \sum_{j=0}^{n} c_j P_j(t) \triangleq C^T \Psi_n(t) \tag{4}$$

where

$$c_j = \frac{2j+1}{T}\int_0^T u(t)P_j(t)dt, \qquad \Psi_n(t) = [P_0(t)\ P_1(t)\ \ldots\ P_n(t)]^T. \tag{5}$$

The derivative of the above vector can be written as follows [3]

$$\frac{d\Psi_n(t)}{dt} = \mathbf{D}_n^{(1)}\Psi_n(t), \tag{6}$$

where $\mathbf{D}_n^{(1)}$ is an $(n+1) \times (n+1)$ matrix with entries

$$\left[\mathbf{D}_n^{(1)}\right]_{ij} = \begin{cases} \dfrac{2(\sqrt{(2i-1)(2j-1)})}{T}, & i = 2, \cdots, n+1,\ j = 1, \cdots, i-1 \ \ and \ \ i+j \ \ odd, \\ 0, & otherwise. \end{cases}$$

Generally, for $k = 2, 3, \ldots$, we have

$$\frac{d^k\Psi_n(t)}{dt^k} = \mathbf{D}_n^{(k)}\Psi_n(t), \tag{7}$$

where

$$\mathbf{D}_n^{(k)} = \underbrace{\mathbf{D}_n^{(1)} \times \mathbf{D}_n^{(1)} \times \ldots \times \mathbf{D}_n^{(1)}}_{k\ times}.$$

2.2. The modified Legendre functions (LFs)

The modified LFs can be defined over $[-1,1]$ by the following formula:

$$\tilde{P}_i(x) := \sqrt{\left(i+\frac{1}{2}\right)(1-x^2)}\, L_i(x), \qquad i=0,1,2,\dots, \tag{8}$$

where $L_i(x)$ is the ith Legendre polynomial. The set of modified LFs $\left\{\tilde{P}_i(x)\right\}_{i=0}^{\infty}$ forms an orthonormal system with respect to the weight function $w(x)=\frac{1}{1-x^2}$ on the domain $[-1,1]$. So, an square integrable function $u \in L^2_w([-1,1])$ with $u(-1)=u(1)=0$ can be approximated by the modified LFs as follows

$$u(x) \simeq \sum_{i=0}^{m} \tilde{c}_i \tilde{P}_i(x) \triangleq \tilde{C}^T \Phi_m(x), \tag{9}$$

where

$$\tilde{c}_i = \int_{-1}^{1} u(x)\tilde{P}_j(x)w(x)dx, \qquad \Phi_m(x)=\left[\tilde{P}_0(x)\ \tilde{P}_1(x)\ \dots\ \tilde{P}_m(x)\right]^T. \tag{10}$$

The derivative of the above vector can be presented as follows

$$\frac{d\Phi_m(x)}{dx} = \mathbf{Q}_m^{(1)}\Phi_m(x), \tag{11}$$

where $\mathbf{Q}_m^{(1)}$ is an $(m+1)\times(m+1)$ square matrix with components

$$\left[\mathbf{Q}_m^{(1)}\right]_{ij} = \int_{-1}^{1} \frac{d\tilde{P}_{i-1}(x)}{dx}\tilde{P}_{j-1}(x)w(x)dx, \qquad i,j=1,2,\dots,m+1. \tag{12}$$

Generally, for $k=2,3,\dots$, one has

$$\frac{d^k\Phi_m(x)}{dx^k} = \mathbf{Q}_m^{(k)}\Phi_m(x), \tag{13}$$

where

$$\mathbf{Q}_m^{(k)} = \underbrace{\mathbf{Q}_m^{(1)} \times \mathbf{Q}_m^{(1)} \times \dots \times \mathbf{Q}_m^{(1)}}_{k\ times}.$$

Remark 2.1. *The integral expressed in Eq. (12) can be numerically calculated via an M-point Gauss-Legendre quadrature rule [2] with the nodal points ξ_r and the corresponding weights ω_r. So, we have*

$$\left[\mathbf{Q}_m^{(1)}\right]_{ij} \simeq \sum_{r=1}^{M} \omega_r \tilde{P}'_{i-1}(\xi_r)\tilde{P}_j(\xi_r)w(\xi_r), \qquad i,j=1,2,\dots,m+1.$$

Through the paper, we put $M=25$ for numerical integration.

3. The proposed method

3.1. Transforming the problem into a bounded domain

Let $\varphi:\mathbb{R}\longrightarrow\mathbb{R}$ is a two times differentiable function satisfying $\varphi(-\infty)=\varphi(\infty)=0$. By using the change of variable $x=tanh^{-1}(\upsilon)$ where $\upsilon\in[-1,1]$, we can define the function $\phi(\upsilon)=\varphi(tanh^{-1}(\upsilon))$ over $[-1,1]$ with $\phi(-1)=\phi(1)=0$. Moreover, for the first and second derivatives, we have

$$\frac{d\varphi(x)}{dx} = \frac{d\phi(\upsilon)}{d\upsilon}\frac{d\upsilon}{dx} = (1-\upsilon^2)\frac{d\phi(\upsilon)}{d\upsilon}, \tag{14}$$

and

$$\frac{d^2\varphi(x)}{dx^2} = (1-\upsilon^2)^2\frac{d^2\phi(\upsilon)}{d\upsilon^2} - 2\upsilon(1-\upsilon^2)\frac{d\phi(\upsilon)}{d\upsilon}. \tag{15}$$

Using Eqs. (14) and (15), the problem expressed in Eq. (1) can be rewritten in the following equivalent form:

$$\begin{cases} w_{tt}(\upsilon,t)+\alpha^2 w_t(\upsilon,t)+2\beta w(\upsilon,t)=\gamma\left(\left(1-\upsilon^2\right)^2 w_{\upsilon\upsilon}(\upsilon,t)-2\upsilon\left(1-\upsilon^2\right)w_\upsilon(\upsilon,t)\right)+\tilde{f}(\upsilon,t),\ \ \upsilon\in[-1,1],\ \ t\in[0,T],\\ w(-1,t)=w(1,t)=0,\\ w(\upsilon,0)=\tilde{h}_1(\upsilon),\\ w_t(\upsilon,0)=\tilde{h}_2(\upsilon), \end{cases} \tag{16}$$

where $w(\upsilon,t)=u(tanh^{-1}(\upsilon),t)$, $\tilde{f}(\upsilon,t)=f(tanh^{-1}(\upsilon),t)$, $\tilde{h}_1(\upsilon)=h_1(tanh^{-1}(\upsilon))$ and $\tilde{h}_2(\upsilon)=h_2(tanh^{-1}(\upsilon))$.

3.2. Approximation by the basis functions

In order to solve Eq. (16), we assume

$$w(\upsilon,t)\simeq\Phi_m(\upsilon)^T\mathbf{W}\Psi_n(t), \tag{17}$$

where $\mathbf{W}$ is an $(m+1)\times(n+1)$ unknown matrix and $\Phi_m(\upsilon)$ and $\Psi_n(t)$ are already defined in Eqs. (10) and (5), respectively. From Eq. (6), we have

$$w_t(\upsilon,t)\simeq\Phi_m(\upsilon)^T\mathbf{W}\mathbf{D}_n^{(1)}\Psi_n(t),\qquad w_{tt}(\upsilon,t)\simeq\Phi_m(\upsilon)^T\mathbf{W}\mathbf{D}_n^{(2)}\Psi_n(t). \tag{18}$$

Moreover, using Eq. (11), we have

$$w_\upsilon(\upsilon,t)\simeq\Phi_m(\upsilon)^T\left(\mathbf{Q}_m^{(1)}\right)^T\mathbf{W}\Psi_n(t),\quad w_{\upsilon\upsilon}(\upsilon,t)\simeq\Phi_m(\upsilon)^T\left(\mathbf{Q}_m^{(2)}\right)^T\mathbf{W}\Psi_n(t). \tag{19}$$

Now, by substituting Eqs. (17)-(19) into Eq. (16), we introduce the following residual function:

$$\begin{aligned} R(\upsilon,t)\triangleq&\Phi_m(\upsilon)^T\left(\mathbf{W}\mathbf{D}_n^{(2)}+\alpha^2\mathbf{W}\mathbf{D}_n^{(1)}+2\beta\mathbf{W}-\gamma\left(1-\upsilon^2\right)^2\left(\mathbf{Q}_m^{(2)}\right)^T\mathbf{W}+2\gamma\upsilon\left(1-\upsilon^2\right)\left(\mathbf{Q}_m^{(1)}\right)^T\mathbf{W}\right)\Psi_n(t)\\ &-\tilde{f}(\upsilon,t)\simeq 0 \end{aligned} \tag{20}$$

Meanwhile, by using the initial conditions expressed in Eq. (16), we define

$$\begin{aligned} \Lambda_1(\upsilon)&\triangleq\Phi_m(\upsilon)^T\mathbf{W}\Psi_n(0)-\tilde{h}_1(\upsilon)\simeq 0,\\ \Lambda_2(\upsilon)&\triangleq\Phi_m(\upsilon)^T\mathbf{W}\mathbf{D}_n^{(1)}\Psi_n(0)-\tilde{h}_2(\upsilon)\simeq 0. \end{aligned} \tag{21}$$

By putting the collocation points $\upsilon_i=-\cos\left(\frac{(2i-1)\pi}{2(m+1)}\right)$ and $t_j=\frac{T}{2}\left(1-\cos\left(\frac{(2j-1)\pi}{2(n+1)}\right)\right)$ into Eqs. (20) and (21), we generate the following system of $(m+1)(n+1)$ algebraic equations:

$$\begin{cases} R(\upsilon_i,t_j)=0, & i=1,2,\dots,m+1,\ j=3,4,\dots,n+1,\\ \Lambda_r(\upsilon_i)=0, & r=1,2,\ i=1,2,\dots,m+1. \end{cases} \tag{22}$$

By solving the above system and determining $\mathbf{W}$, an approximate solution is obtained for Eq. (16) by using Eq. (17). Eventually, the approximate solution of Eq. (1) is computed as $u(x,t)=w(\tanh(x),t)$ for $(x,t)\in\mathbb{R}\times[0,T]$.

4. Numerical example

Example 4.1. *Consider problem (1) with $\alpha=\sqrt{2}$, $\beta=\frac{1}{2}$ and $\gamma=1$ which the analytic solution $u(x,t)=t^3\sin(t)\exp\left(-10x^2\right)$. The right side function (RSF) and the associated initial conditions (ICs) can be extracted by the exact solution. The results obtained by the presented scheme are reported in Table 1 and Fig. 1. The reported results show that the algorithm is very accuracy in solving this example.*

Table 1. Maximum absolute error in Example 4.1.

m	n	T=1	T=2	T=3
7	7	0.18E+00	2.31E+00	3.65E+00
13	13	1.14E-03	1.02E-02	1.32E-02
19	19	2.04E-06	1.78E-05	2.32E-05
25	25	6.86E-09	7.32E-08	1.13E-07

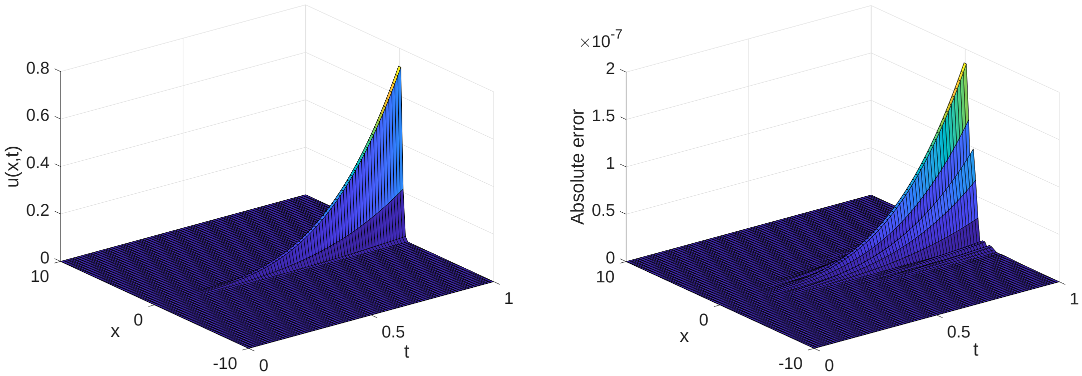

Fig. 1. Numerical results in Example 4.1 with $(m = 23, n = 21)$.

Example 4.2. *Consider problem (1) with* $\alpha = \sqrt{2}$, $\beta = \dfrac{1}{2}$ *and* $\gamma = 1$ *and the analytic solution* $u(x,t) = \frac{t^2}{(1+x^2)^{10}}$. *The RSF and the ICs can be computed by the exact solution. The results obtained by the presented method are numerically and graphically reported in Table 2 and Fig. 2. These results confirm the accuracy and efficiency of the method for this example.*

Table 2. Maximum absolute error in Example 4.2.

m	n	T=1	T=2	T=3
7	7	0.30E+00	1.61E+00	4.18E+00
13	13	6.34E-03	2.58E-02	5.86E-02
19	19	6.07E-05	2.43E-04	5.48E-04
25	25	3.08E-07	1.24E-06	2.81E-06

Example 4.3. *Consider problem (1) with* $\alpha = \sqrt{2}$, $\beta = \dfrac{1}{2}$ *and* $\gamma = 1$ *and the analytic solution* $u(x,t) = \cos(t)\exp\left(-10x^2\right)$. *The RSF and the ICs can be extracted from the analytic solution. The results obtained by the expressed approach are numerically and graphically shown in Table 3 and Fig. 3.*

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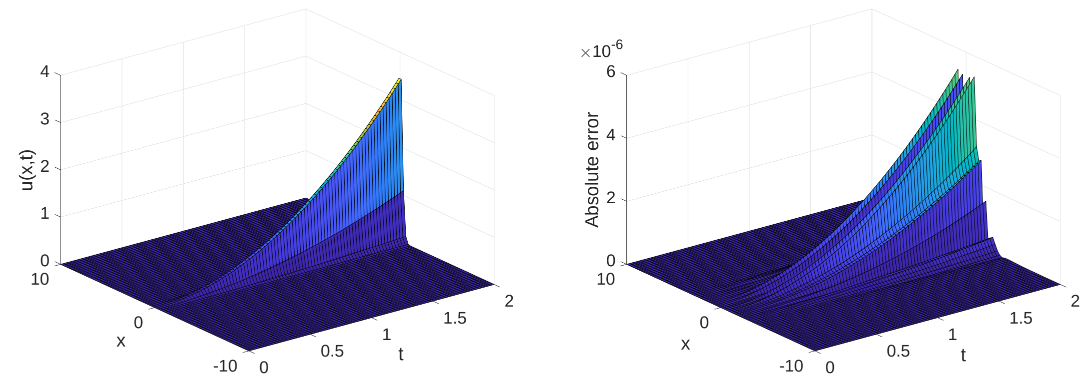

Fig. 2. Numerical results in Example 4.2 with $(m = 22, n = 21)$.

Table 3. Maximum absolute error in Example 4.3.

m	n	T=1	T=2	T=3
7	7	0.37E+00	0.22E+00	0.33E+00
13	13	1.76E-03	1.20E-03	3.18E-04
19	19	2.23E-06	2.38E-06	6.60E-07
25	25	1.94E-08	9.63E-08	2.30E-09

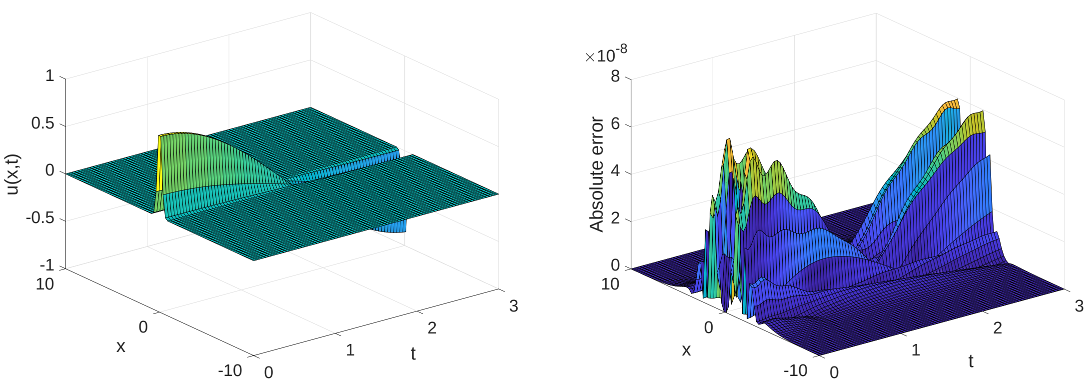

Fig. 3. Numerical results in Example 4.1 with $(m = 24, n = 20)$.

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Study of qualitative a nonstandard finite difference scheme for a class of stochastic partial differential equations

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Abstract

The basic idea of this paper is to construct a nonstandard finite difference scheme (NSFD) for the numerical solutions of stochastic partial differential equations (SPDEs) of Itô type. The main properties of the NSFD scheme, i.e., consistency, stability and convergence are proved. In order to the efficiency and accuracy of the proposed NSFD scheme, some numerical simulations are presented.

1. Introduction

In recent years, SPDEs arise in many branches of applied sciences, medical and engineering. Most of the SPDEs can not be solved by well-known analytical techniques. Hence, various stochastic numerical methods have been designed to solve such equations. In this work, we are going to construct a NSFD scheme for the following SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial^2 u}{\partial x^2} + b\dfrac{\partial u}{\partial x} + cu + \gamma u\dot{W}(t) = 0, \quad x \in [0,1], \quad t \in [0,1], \\ u(x,0) = u_0(x), \quad u(0,t) = u_1(t), \quad u(1,t) = u_2(t), \end{cases} \tag{1}$$

where $a,\ b,\ c$ and γ are constants and $\xi(t)$ indicates a standard Wiener process. The paper aims are organized as follows. In Section 1 the basic properties of the NSFD schemes are given. Section 2 is devoted to analyze consistency, stability and convergency the proposed NSFD scheme. At the end, numerical simulations are given in Section 3.

2. NSFD schemes for ordinary differential equations

The NSFD scheme was first introduced by R. Mickens. These schemes preserve properties like consistency, stability and convergency. Moreover, these schemes can also preserve essential properties of the continuous systems, such as

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positivity and boundedness. In order to describe the characteristic NSFD schemes for an ordinary differential equations system, consider the following autonomous form

$$y'(t) = f(y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T],$$

where f may be a nonlinear vector function and y is a vector function. Let $t_n = t_0 + nh$ denote mesh points where h is called stepsize, hence the discretized version of continuous differential equation (1) can be discretized as follows

$$\mathcal{D}_n y_n = \mathcal{F}_n(f, y_n, y_{n+1}, \ldots),$$

where $\mathcal{D}_n y_n$ denotes the descritized version of $y'(t_n)$ and $\mathcal{F}_n(f, y_n, y_{n+1}, \ldots)$ approximates $f(y)$ at mesh points $t_n,\ t_{n+1}$ and so on. A finite difference scheme is called a NSFD scheme if at least one of the following conditions is achieved [3].

1. In the descritized version $\mathcal{D}_n y_n$, denominator function instead of h can be replaced by an increasing and non-negative function $\phi(h)$ such that fulfills $\phi(h) = h + O(h^2)$ as $h \to 0$.
2. The nonlinear and linear terms in $f(y(t))$ can be approximated in a nonlocal way in several points of the mesh. For example, the terms y and y^2 can be approximated as follows

$$y(t_k) \approx \frac{1}{2}(y_k + y_{k+1}), \quad y^2(t_k) \approx \frac{1}{2} y_k (y_k + y_{k+1}),$$

where y_{k+j} denotes an approximation for $y(t_{k+j})$.

3. A stochastic NSFD scheme

In this section, we analyze qualitative behaviour a NSFD scheme for the SPDE (1). In order to construct a NSFD scheme, we approximate the space and time derivatives in the SPDE (1) by the following finite difference approximations

$$u_t(k\Delta x, n\Delta t) = \frac{u_k^{n+1} - u_k^n}{\Delta t}, \quad u_x(k\Delta x, n\Delta t) \approx \frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x},$$
$$u_{xx}(k\Delta x, n\Delta t) \approx \frac{1}{(\Delta x)^2}\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right),$$

where u_k^n is an approximation for $u(k\Delta x, n\Delta t)$, and Δx and Δt are considered as the space and time stepsizes, respectively. Substituting the above approximations into (1), we can find

$$u_k^{n+1} = \left(1 + \frac{5}{2}r - \frac{c\Delta t}{2}\right)u_k^n + \frac{r}{12}u_{k-2}^n + \frac{r}{12}u_{k+2}^n + \left(s - \frac{4}{3}r\right)u_{k-1}^n - \left(\frac{4}{3}r + s + \frac{c\Delta t}{2}\right)u_{k+1}^n - \gamma u_{k+1}^n \Delta W_n, \quad (2)$$

where $r = \frac{a\Delta t}{\Delta x^2}$, $s = \frac{b\Delta t}{2\Delta x}$ and $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$ is a Gaussian distribution with zero mean and variance Δt [2]. In the reminder of this paper we assume that in the scheme (2), the Wiener process increments are independent on the state u_k^n. Basically, the convergence of the stochastic difference scheme to the SPDE solution is important. To do this, suppose a SPDE in the form of $Lu = f$ is given, where L represents the differential operator. Let the stochastic variable u_k^n be a solution which is approximated by a stochastic difference scheme indicated by L_k^n. By using the stochastic difference scheme to the SPDE, we get $L_k^n u_k^n = f_k^n$, where f_k^n is the approximation of f. In order to access the consistency, stability and convergence results, a norm is needed. Because of this, for the sequence $\{u_k^n\}_{k\in\mathbb{Z}}$, we define the sup–norm as $\|u^n\|_\infty = \sqrt{\sup_{k\in\mathbb{Z}} |u_k^n|^2}$. For more concern in the concepts of consistency, stability and convergence see [1].

Definition 3.1. A stochastic difference scheme $L_k^n u_k^n = f_k^n$ is said to be pointwise consistent in mean square with SPDE $Lu = f$ at point (x, t), if for any continuously differentiable function $u = u(x, t)$ of this equation, we have $\mathbb{E}\|(Lu - f)_k^m - (L_k^m u_k^m - f_k^m)\|^2 \to 0$, as $(\Delta x, \Delta t) \to (0, 0)$ and $(k\Delta x, (m+1)\Delta t) \to (x, t)$.

Theorem 3.2. *The numerical scheme (2) in the sense of mean square is pointwise consistent.*

Proof. For the smooth function $\phi = \phi(x,t)$, we have

$$L(\phi)|_k^n = \phi(k\Delta x,(n+1)\Delta t) - \phi(k\Delta x, n\Delta t) + a\int_{n\Delta t}^{(n+1)\Delta t} \phi_{xx}(k\Delta x, s)\,\mathrm{d}s + b\int_{n\Delta t}^{(n+1)\Delta t} \phi_x(k\Delta x, s)\,\mathrm{d}s + c\int_{n\Delta t}^{(n+1)\Delta t} \phi(k\Delta x, s)\,\mathrm{d}s + \gamma\int_{n\Delta t}^{(n+1)\Delta t} \phi(k\Delta x, s)\,\mathrm{d}W(s),$$

$$L_k^n\phi = \phi(k\Delta x,(n+1)\Delta t)-\phi(k\Delta x,n\Delta t)+\frac{a\Delta t}{\Delta x^2}\Big(-\frac{1}{12}\phi((k-2)\Delta x,(n+1)\Delta t)+\frac{4}{3}\phi((k-1)\Delta x,n\Delta t)-\frac{5}{2}\phi(k\Delta x,n\Delta t) + \frac{4}{3}\phi((k+1)\Delta x,n\Delta t) - \frac{1}{12}\phi((k+2)\Delta x,n\Delta t)\Big) + \frac{b\Delta t}{2\Delta x}\Big(\phi((k+1)\Delta x,n\Delta t) - \phi((k-1)\Delta x,n\Delta t)\Big) + \frac{c\Delta t}{2}\Big(\phi(k\Delta x,n\Delta t)+\phi((k+1)\Delta x,n\Delta t)\Big) + \gamma\phi((k+1)\Delta x,n\Delta t)\Delta W_n.$$

Accordingly

$$\mathbb{E}\big|L(\phi)|_k^n - L_k^n\phi\big|^2 \le 2a^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big[\phi_{xx}(k\Delta x,s) - \frac{1}{\Delta x^2}\Big(-\frac{1}{12}\phi((k-2)\Delta x,n\Delta t)+\frac{4}{3}\phi((k-1)\Delta x,n\Delta t) - \frac{5}{2}\phi(k\Delta x,n\Delta t)+\frac{4}{3}\phi((k+1)\Delta x,n\Delta t) - \frac{1}{12}\phi((k+2)\Delta x,n\Delta t)\Big)\Big]\mathrm{d}s\Big|^2 + 4b^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big[\phi_x(k\Delta x,s) - \frac{1}{2\Delta x}\Big(\phi((k+1)\Delta x,n\Delta t) - \phi((k-1)\Delta x,n\Delta t)\Big)\Big]\mathrm{d}s\Big|^2 +8c^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big[\phi(k\Delta x,s)-\frac{1}{2}\Big(\phi((k+1)\Delta x,n\Delta t)+\phi(k\Delta x,n\Delta t)\Big)\Big]\mathrm{d}s\Big|^2 +8\gamma^2\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}\Big|\phi(k\Delta x,s)-\phi((k+1)\Delta x,n\Delta t)\Big|^2\mathrm{d}s$$

Inasmuch as $\phi(x,t)$ is a deterministic function, $\mathbb{E}|L(\phi)|_k^n - L_k^n\phi|^2 \to 0$ as $k,n\to+\infty$. This shows that, the numerical scheme (2) is consistent with the SPDE (1). $\square$

Definition 3.3. A stochastic finite difference $L_k^m u_k^m = f_k^m$ is said to be stable in mean square with SPDE $Lu = f$ if there exist some positive constants Δx^*, Δt^*, K and β such that $\mathbb{E}\|u^{n+1}\|^2 \le Ke^{\beta t}\mathbb{E}\|u^0\|^2$, for all $t=(n+1)\Delta t$, $0<\Delta x<\Delta x^*$ and $0<\Delta t<\Delta t^*$, where $u^{n+1} = (\ldots,u_{k-2}^{n+1},u_{k-1}^{n+1},u_k^{n+1},u_{k+1}^{n+1},u_{k+2}^{n+1},\ldots)^T$.

The following stability result is important to prove the convergence of the stochastic NSFD scheme (2).

Theorem 3.4. *The stochastic finite difference scheme (2) in mean square is unconditionally stable.*

Proof. The stochastic finite difference scheme is given by

$$u_k^{n+1} = \left(1+\frac{5}{2}r - \frac{c\Delta t}{2}\right)u_k^n + \frac{r}{12}u_{k-2}^n + \frac{r}{12}u_{k+2}^n + \left(s-\frac{4}{3}r\right)u_{k-1}^n - \left(\frac{4}{3}r+s+\frac{c\Delta t}{2}\right)u_{k+1}^n - \gamma u_{k+1}^n\Delta W_n.$$

Applying $\mathbb{E}|.|^2$ to the above difference scheme we have

$$\mathbb{E}|u_k^{n+1}|^2 = \mathbb{E}\left|\left(1+\frac{5}{2}r-\frac{c\Delta t}{2}\right)u_k^n + \frac{r}{12}u_{k-2}^n + \frac{r}{12}u_{k+2}^n + \left(s-\frac{4}{3}r\right)u_{k-1}^n - \left(\frac{4}{3}r+s+\frac{c\Delta t}{2}\right)u_{k+1}^n\right|^2 +\gamma^2\Delta t\mathbb{E}|u_{k+1}^n|^2 \le \left[\left(1+\frac{5}{2}r-\frac{c\Delta t}{2}\right)+\frac{r}{12}+\frac{r}{12}+\left(s-\frac{4}{3}r\right)-\left(\frac{4}{3}r+s+\frac{c\Delta t}{2}\right)\right]^2\sup_k\mathbb{E}|u_k^n|^2+\gamma^2\Delta t\sup_k\mathbb{E}|u_k^n|^2,$$

and so

$$\begin{aligned}\mathbb{E}|u_k^{n+1}|^2 &\leq \left((1-c\Delta t)^2+\gamma^2\Delta t\right)\sup_k \mathbb{E}|u_k^n|^2\\ &\leq \left(1+\gamma^2\Delta t\right)\sup_k \mathbb{E}|u_k^n|^2,\end{aligned}$$

and the usage of supposition $\Delta t = \frac{t}{n+1}$, one concludes that

$$\mathbb{E}\|u^{n+1}\|_\infty^2 \leq \left(1+\frac{\gamma^2 t}{n+1}\right)^{n+1}\mathbb{E}\|u^0\|_\infty^2 \leq e^{\gamma^2 t}\mathbb{E}\|u^0\|_\infty^2.$$

□

Definition 3.5. The stochastic difference scheme $L_k^n u_k^n = f_k^n$, which approximates the SPDE $Lu = f$, is convergent in mean square at time t, if $\mathbb{E}\|v^{n+1}-u^{n+1}\|^2 \to 0$ as $\Delta x \to 0$ and $\Delta t \to 0$, for $t = (n+1)\Delta t$.

Theorem 3.6. *The stochastic difference scheme (2) is convergent in mean square with respect to* $\|.\|_\infty = \sqrt{\sup_k |.|^2}$ *and* $t = (n+1)\Delta t$.

Proof. From stochastic version of the Lax–Richtmyer theorem the convergence of the scheme (2) is also concluded. □

4. Numerical experiments

In the present section, to demonstrate the efficiency and accuracy of the proposed scheme, one example is solved.

Example 4.1. *Consider SPDE in the following form*

$$u_t(x,t) + au_{xx}(x,t) + \gamma u(x,t)\dot{W}(t) = 0, \tag{3}$$

subject to the following initial condition

$$u(x,0) = \exp\left(\frac{(x-0.2)^2}{a}\right), \quad x \in [0,1]$$

and the boundary conditions

$$\begin{aligned}u(0,t) &= \frac{1}{\sqrt{4t+1}}\exp\left(\frac{0.04}{a(4t+1)}\right), \quad t \in [0,1],\\ u(1,t) &= \frac{1}{\sqrt{4t+1}}\exp\left(\frac{0.64}{a(4t+1)}\right), \quad t \in [0,1].\end{aligned}$$

The exact solution in the absence of stochastic term can be expressed as

$$u(x,t) = \frac{1}{\sqrt{4t+1}}\exp\left(\frac{(x-0.2)^2}{a(4t+1)}\right).$$

The stochastic finite difference scheme is given by

$$u_k^{n+1} = \left(1+\frac{5}{2}r\right)u_k^n + \frac{r}{12}u_{k-2}^n + \frac{r}{12}u_{k+2}^n - \frac{4}{3}ru_{k-1}^n - \frac{4}{3}ru_{k+1}^n - \gamma u_{k+1}^n\Delta W_n, \tag{4}$$

where $r = \dfrac{a\Delta t}{\Delta x^2}$. *Let M and N be the total numbers of grid points for the space and time discretization, respectively. In figures 1–4 the approximation solutions of SPDE (3) for the different values of parameters using the stochastic finite difference scheme (4) are shown.*

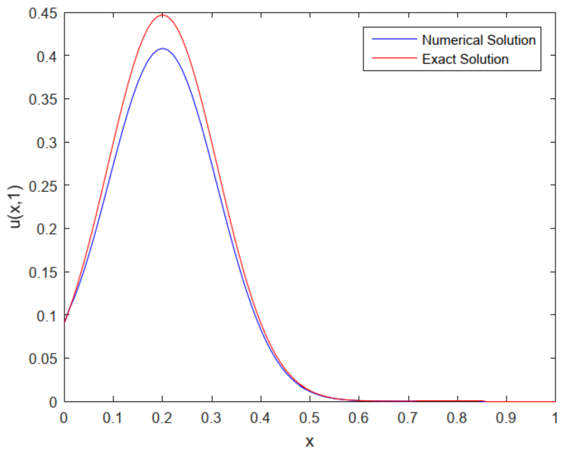

Fig. 1. Comparison between the deterministic and stochastic numerical solution of (3) with $a = 0.005,\ \gamma = 1,\ M = 100$ and $N = 500$.

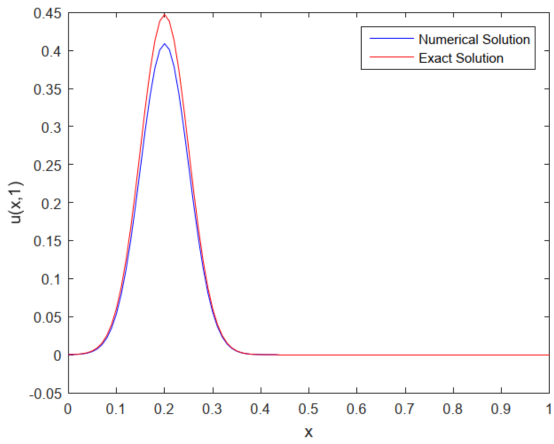

Fig. 2. Comparison between the deterministic and stochastic numerical solution of (3) with $a = 0.001,\ \gamma = 1,\ M = 100$ and $N = 500$.

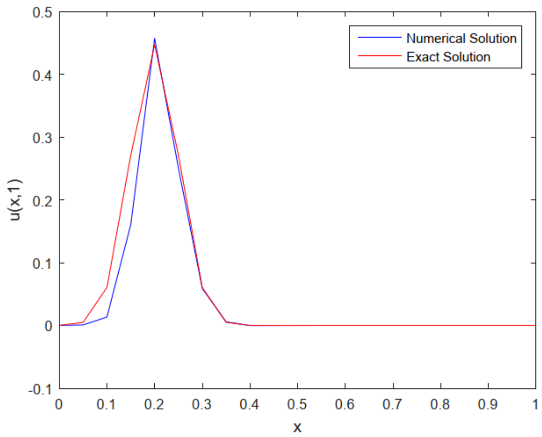

Fig. 3. Comparison between the deterministic and stochastic numerical solution of (3) with $a = 0.005,\ \gamma = 1$ and $M = N = 10$.

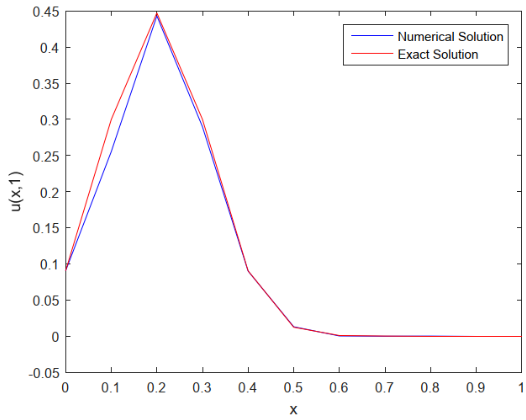

Fig. 4. Comparison between the deterministic and stochastic numerical solution of (3) with $a = 0.005,\ \gamma = 0.5$ and $M = N = 10$.

5. Conclusion

This paper presented a nonstandard finite difference scheme applied to the solution of SPDE. Mathematical analyses of the proposed scheme were provided. To confirm the accuracy and efficacy of the proposed scheme, one test problem is presented, and the associated numerical results were compared with the exact solution.

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Lyapunov-type inequality for conformable fractional differential equations with different types of boundary conditions

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Abstract

Lyapunov's inequality is a powerful tool in the study of differential equations. This inequality helps us to give a general interpretation for solutions of differential equations. Recently, the concept of conformable fractional derivative and conformable fractional differential equations introduced by Abdeljawad [Journal of Computational and Applied Mathematics, 279 (2015) 57–66] and Khalil *et al.* [Journal of Computational and Applied Mathematics 264 (2014) 65–70]. In fractional differential equations, one thing that seems missing is Lyapunov's inequality for conformable fractional differential equations. In this paper, we study Lyapunov-type inequality for conformable fractional differential equations with different types of boundary conditions.

1. Introduction

Lyapunov's inequality is very important in the study of differential equations. This inequality was first introduced by Lyapunov in [5]. Since then several papers have been devoted to the study of Lyapunov's inequality, mainly due to its applications.

Theorem 1.1. *[5] If the following boundary value problem*

$$y''(t) + q(t)\, y(t) = 0,\ a < t < b,$$
$$y(a) = 0 = y(b),$$

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has a nontrivial solution, where q is a real and continuous function in $[a, b]$*, then*

$$\int_a^b |q(s)|\, ds \geq \frac{4}{b-a}. \tag{1}$$

The main propose of investigate Lyapunov's inequality was to give a general interpretation for solutions of differential equations.

Fractional Lyapunov's inequality based on a Riemann-Liouville fractional derivative was first studied in [2]. Then fractional Lyapunov's inequality with respect to a Caputo fractional derivative was first considered by Ferreira [3] in 2014. Based on these results, Ferreira also obtained the nonexistence of real zeros of a certain Mittag–Leffler function [6].

However, Riemann-Liouville fractional derivative and the Caputo fractional derivative seemed complicated and lost some of important algebraic properties of fractional order differentiation such as the product rule and the chain rule. For these important reasons, a new well-behaved simple fractional derivative called "the conformable fractional derivative", depending on the basic limit definition of the derivative, was introduced in [1, 4].

The study of conformable fractional calculus has led to interesting applications in many fields [7–13]. For example, in 2016, new conformable fractional derivative for converting fractional coupled nonlinear Schrodinger equations into the ordinary differential equations was proposed by Eslami in [8]. In 2017, Hosseini *et al.* [9] suggested modified Kudryashov method for solving the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities. In nonlinear differential equation, an existence of solution for a local fractional nonlinear differential equation with initial condition was proposed by Bayour *et al.* [7]. Then Unal *et al.*[10] obtained an operator method for local fractional linear differential equations. In nonlinear partial differential equations, Cenesiz *et al.* studied new exact solutions of Burgers' type equations with conformable derivative [11]. In physics and engineering, a class of new fractional derivative named general conformable fractional derivative to describe the physical world was introduced by Zhao and Luo [12].

One thing that seems missing is the developments of Lyapunov inequality for the conformable fractional derivative. In this paper, we study Lyapunov-type inequalities for the conformable fractional derivative.

Let us recall some notations of conformable fractional derivative used in the subsequent section of this paper [1].

Definition 1.2. (i) The (left) fractional derivative starting from a of a function $f : [a, \infty) \longrightarrow R$ of order $0 < \alpha \leq 1$ is defined by

$$(T_a^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon (t-a)^{1-\alpha}\right) - f(t)}{\epsilon}.$$

When $a = 0$ we write T^α. If $(T_a^\alpha f)(t)$ exist on (a, b) then $(T_a^\alpha f)(a) = \lim_{t \to a^+} (T_a^\alpha f)(t)$.

(ii) The (right) fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$\left({}_{b}^{\alpha}Tf\right)(t) = -\lim_{\epsilon\to 0}\frac{f\left(t+\epsilon\left(b-t\right)^{1-\alpha}\right)-f\left(t\right)}{\epsilon}.$$

If $\left({}_{b}T^{\alpha}f\right)(t)$ exist on (a,b) then $\left({}_{b}T^{\alpha}f\right)(b) = \lim_{t\to b^{-}}\left({}_{b}T^{\alpha}f\right)(t)$.

Definition 1.3. Let $\alpha \in (n, n+1]$ and $\beta = \alpha - n$. Then, the (left) conformable fractional derivative of order α for the function $f : [a, +\infty) \to R$, is defined by

$$\left(\mathbb{T}_{a}^{\alpha}f\right)(t) = (T_{a}^{\beta}f^{(n)})(t)$$

When $a = 0$ we write $\mathbb{T}^{\alpha}$. The (right) conformable fractional derivative of order α terminating at b from f is defined by

$$\left({}_{b}^{\alpha}\mathbb{T}f\right)(t) = (-1)^{n+1}\left({}_{b}^{\beta}Tf^{(n)}\right)(t)$$

Some basic properties of conformable fractional derivative were summarized in [1], we cite some of them:

Proposition 1.4. *Let $\alpha \in (0,1]$ and $f(t), h(t)$ be conformable fractional differentiable of order α functions. Then*

(1) $\mathbb{T}_{a}^{\alpha}(kf(t)+h(t)) = k\mathbb{T}_{a}^{\alpha}(f(t)) + \mathbb{T}_{a}^{\alpha}(h(t))$ for all $k \in R$;

(2) $\mathbb{T}_{a}^{\alpha}(f(t)\,h(t)) = f(t)\mathbb{T}_{a}^{\alpha}(h(t)) + h(t)\mathbb{T}_{a}^{\alpha}(f(t))$;

(3) $\mathbb{T}_{a}^{\alpha}(C) = 0$ for all constants $C \in R$;

(4) $\mathbb{T}_{a}^{\alpha}\left(t^{\beta}\right) = \beta t^{\beta-\alpha}$ for all constants $\beta \in R$.

The problem is whether Lyapunov's inequality can be proved for the conformable fractional derivative? Next section answers this question.

2. Lyapunov-type inequality for conformable fractional differential equations

In this section, we give Lyapunov-type inequality for conformable fractional differential equations with different types of boundary conditions.

Theorem 2.1. *If the following boundary value problem*

$$\mathbb{T}_{a}^{\alpha}y\left(t\right) + q\left(t\right)y\left(t\right) = 0,\ a < t < b, 1 < \alpha \leq 2, \tag{2}$$

under the boundary conditions

$$y\left(a\right) = 0, \quad y\left(b\right) = 0, \tag{3}$$

or

$$y\left(a\right) = 0, \quad y'\left(b\right) = 0, \tag{4}$$

or

$$y'(a) = 0, \quad y(b) = 0, \tag{5}$$

has a nontrivial solution, where q is a real and continuous function in $[a, b]$*, then for* (2)-(3)

$$\int_a^b |q(s)|\, ds \geq \frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}},$$

and for (2)-(4)

$$\int_a^b |q(s)|\, ds \geq (b-a)^{1-\alpha},$$

also for (2)-(5)

$$\int_a^b (s-a)^{\alpha-2} |q(s)|\, ds \geq \frac{1}{b-a}.$$

Remark 2.2. *If* $\alpha = 2$ *in Theorem 2.1, then we have the classical Lyapunov inequality 1.*

For proof of Theorem 2.1, we need the following lemmas.

Lemma 2.3. $y \in C[a, b]$ *is a solution*

i) for (2)-(3) *if and only if y satisfies the integral equation*

$$y(t) = \int_a^b G_1(t, s)\, q(s)\, y(s)\, ds,$$

where

$$G_1(t, s) = \begin{cases} \frac{(s-a)^\alpha (b-t)}{(b-a)(s-a)}, & a \leq s \leq t \leq b, \\ \frac{(b-s)(t-a)}{(b-a)(s-a)^{2-\alpha}}, & a \leq t \leq s \leq b. \end{cases}$$

ii) for (2)-(4) *if and only if y satisfies the integral equation*

$$y(t) = \int_a^b G_2(t, s)\, q(s)\, y(s)\, ds,$$

where

$$G_2(t, s) = \begin{cases} (s-a)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)}{(s-a)^{2-\alpha}}, & a \leq t \leq s \leq b. \end{cases}$$

iii) for (2)-(5) *if and only if y satisfies the integral equation*

$$y(t) = \int_a^b G_3(t, s)\, q(s)\, y(s)\, ds,$$

where

$$G_3(t, s) = \begin{cases} \frac{(b-t)}{(s-a)^{2-\alpha}}, & a \leq s \leq t \leq b, \\ \frac{(b-s)}{(s-a)^{2-\alpha}}, & a \leq t \leq s \leq b. \end{cases}$$

Proof. $y(t)$ is a solution to (2) if and only if

$$y(t) = y(a) + y'(a)(t-a) - \mathbb{I}_a^\alpha (q(t)\, y(t)).$$

i) Since $y(a) = y(b) = 0$, we get

$$y(t) = \frac{t-a}{b-a}\int_a^b \frac{(b-s)\, q(s)\, y(s)\, ds}{(s-a)^{2-\alpha}} - \int_a^t \frac{(t-s)\, q(s)\, y(s)\, ds}{(s-a)^{2-\alpha}},$$

ii) by $y(a) = 0$, $y'(b) = 0$, we have

$$y(t) = (t-a)\int_a^b \frac{q(s)\, y(s)\, ds}{(s-a)^{2-\alpha}} - \int_a^t \frac{(t-s)\, q(s)\, y(s)\, ds}{(s-a)^{2-\alpha}},$$

iii) from $y'(a) = 0$, $y(b) = 0$, we obtain

$$y(t) = \int_a^b \frac{(b-s)\, q(s)\, y(s)\, ds}{(s-a)^{2-\alpha}} - \int_a^t \frac{(t-s)\, q(s)\, y(s)\, ds}{(s-a)^{2-\alpha}},$$

which completes the proof (i)-(iii).

Lemma 2.4. *We have the following properties:*

$$\begin{aligned} i) \quad \max_{(t,s)\in[a,b]\times[a,b]} G_1(t,s) &= \frac{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}{\alpha^\alpha}, \\ ii) \quad \max_{(t,s)\in[a,b]\times[a,b]} G_2(t,s) &= (b-a)^{\alpha-1}, \\ iii) \quad \max_{t\in[s,b]} G_3(t,s) &= \frac{(b-s)}{(s-a)^{2-\alpha}}. \end{aligned}$$

Proof. For **(i)** Define

$$\begin{aligned} g_1(t,s) &= \frac{(s-a)^\alpha (b-t)}{(b-a)(s-a)}, \quad a \le s \le t \le b, \\ g_2(t,s) &= \frac{(b-s)(t-a)}{(b-a)(s-a)^{2-\alpha}}, \quad a \le t \le s \le b. \end{aligned}$$

It is easy to see that $g_2(t,s) \ge 0$ and $g_1(t,s) \ge 0$. Obviously, $g_2(t,s)$ is an increasing function and $g_1(t,s)$ is a decreasing with respect to t for every fixed s. So,

$$\max_{t\in[s,b]} g_1(t,s) = g_1(s,s) \quad \text{and} \quad \max_{t\in[a,s]} g_2(t,s) = g_2(s,s).$$

Therefore,

$$\max_{t\in[a,b]} G_1(t,s) = G_1(s,s) = \frac{(s-a)^{\alpha-1}(b-s)}{b-a}, \quad s \in [a,b].$$

Then, for $s \in (a,b)$, we have

$$\frac{\partial}{\partial s}\left(G\left(s,s\right)\right)=\frac{1}{b-a}\left(s-a\right)^{\alpha-2}\left(a-b+b\alpha-s\alpha\right).$$

Thus,

$$\frac{\partial}{\partial s}\left(G\left(s,s\right)\right)>0 \quad \text{if } s<\tfrac{1}{\alpha}\left(a-b+b\alpha\right),$$
$$\frac{\partial}{\partial s}\left(G\left(s,s\right)\right)=0 \quad \text{if } s=\tfrac{1}{\alpha}\left(a-b+b\alpha\right),$$
$$\frac{\partial}{\partial s}\left(G\left(s,s\right)\right)<0 \quad \text{if } s>\tfrac{1}{\alpha}\left(a-b+b\alpha\right).$$

So,

$$\begin{aligned}\max_{s\in[a,b]} G\left(s,s\right)&=\frac{1}{b-a}\left(\frac{1}{\alpha}\left(a-b+b\alpha\right)-a\right)^{\alpha-1}\left(b-\frac{1}{\alpha}\left(a-b+b\alpha\right)\right)\\&=\frac{1}{\alpha^{\alpha}}\left(\alpha-1\right)^{\alpha-1}\left(b-a\right)^{\alpha-1}.\end{aligned}$$

Also for (ii) and (iii), they are clear which complete the proof. □

Proof of Theorem 2.1. Let $\mathcal{B}=C[a,b]$ be the Banach space endowed with norm $\|y\|=\sup_{t\in[a,b]}|y(t)|$. Lemma 2.3 implies that solutions for (2)-(3), (2)-(4) and (2)-(5) satisfiy the following integral equations, respectively

$$y\left(t\right)=\int_a^b G_i\left(t,s\right)q\left(s\right)y\left(s\right)ds, \quad i=1,2,3.$$

Hence,

$$\begin{aligned}\|y\|&\leq\|y\|\int_a^b\left|G_i\left(t,s\right)q\left(s\right)\right|ds, \quad i=1,2,3\\ \Longrightarrow\quad 1&\leq\int_a^b\left|G_i\left(t,s\right)q\left(s\right)\right|ds, \quad i=1,2,3.\end{aligned}$$

Now by Lemma 2.4 the proof is completed. □

Conclusions

In this work, we have extended the Lyapunov-type inequality for conformable fractional differential equations with different types of boundary conditions. The obtained results are so good. Also we have seen that the Lyapunov-type inequalities with conformable fractional derivative are coincident to ones that order of derivative is equal 2, as expected.

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On a Type of Limit-Shadowing Property

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Abstract

In this study, we consider a type of limit shadowing, exponential limit shadowing property, which has been recently introduced. By giving examples, it is shown that this type of limit shadowing is different from the other shadowings. Furthermore, we extend this type of shadowing property to positively expansive maps with the shadowing property.

1. Introduction and Preliminaries

The shadowing property is an important subject in dynamical systems. Given $\delta > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset X$ with the property

$$d(f(x_i), x_{i+1}) < \delta, \quad i \in \mathbb{Z}, \tag{1}$$

is called a δ-*pseudo-orbit*. Often pseudo-orbits are obtained as result of the numerical studies of dynamical systems. The dynamical system (X, f) has the *shadowing property* (or *POTP*, for short) on a set $Y \subset X$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that for a given δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset Y$ there is some points $p \in X$ with the property that

$$d(f^i(p), x_i) < \epsilon, \quad i \in \mathbb{Z}. \tag{2}$$

If this holds on $Y = X$, then it is said that f has the POTP. It is well known that a diffeomorphism has the POTP on a neighborhood of its hyperbolic set.

We say that the dynamical system f has the *Lipschitz shadowing property* (LpSP) on Y if there exist constants $L, \delta_0 > 0$ such that for every δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset Y$ with $0 < \delta < \delta_0$, there exists some point $p \in X$ so that $d(f^i(p), x_i) < L\delta$, $i \in \mathbb{Z}$. Indeed, the LpSP is stronger than the POTP. Also, it is proved that the LpSP holds on a neighborhood of hyperbolic set [5]. Note that Pilyugin et al. in [7] interestingly, showed that the LpSP implies structural stability (and therefore, the LpSP is equivalent to structural stability). In addition, they proved that Anosov systems are equivalent to expansive systems that have the LpSP [7, Corollary 3].

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Another kind of shadowing property is the *limit shadowing property* (*LmSP*)[2]. Precisely, we say that the dynamical system (X, f) has the limit shadowing property if for any sequence $\xi = \{x_i\}_{i\geq 0} \subset X$ with the property

$$d(f(x_i), x_{i+1}) \to 0, \quad i \to \infty \tag{3}$$

there is a point $p \in X$ such that

$$d(f^i(p), x_i) \to 0, \quad i \to \infty. \tag{4}$$

From the numerical point of view, this property means that if we apply a numerical method that approximates the orbits of f with improving accuracy so that one-step errors go to zero as time goes to infinity, then the numerically obtained trajectories tend to the real ones.
Lee and Sakai in [4], proved that expansive systems with the shadowing property have limit shadowing property. More recently, Kulczycki et al. considered the converse case. In [3], they proved that in compact dynamical systems, chain transitivity together with limit shadowing property implies the shadowing property and transitivity. Therefore, in transitive expansive systems, the shadowing and the limit shadowing are equivalent. See [3, Corollary 7.5].
Ahmadi and Molaei in [1] introduced a new type of limit shadowing such that one-step errors tend to zero with exponential rate. Their definition is as follows :

Definition 1.1. [1] The dynamical system f has the *strong exponential limit shadowing (SELmSP, for short)* on M if there exist constants $L > 0$ and $\lambda \in (0, 1)$ such that for every sequence $\xi = \{x_k\}_{k\geq 0}$ with

$$d(f(x_k), x_{k+1}) \leq \lambda^k, \quad k \geq k_1,\ k_1 \in \mathbb{N} \tag{5}$$

there exists a point $p \in M$ and $k_2 \in \mathbb{N}$ such that

$$d(f^k(p), x_k) \leq L\lambda^k, \quad k \geq k_2,\ k_2 \in \mathbb{N}. \tag{6}$$

In [1] the authors studied a weaker form of the strong exponential limit shadowing, named the *exponential limit shadowing property* (ELmSP, for short). Indeed, their definition replaces the exponential term λ^k by $\lambda^{\frac{k}{2}}$ in relation 6. They proved that the ELmSP holds on a neighborhood of a hyperbolic set.

2. The SELmSP property

Note that it is easy to show that the strong exponential limit shadowing property is invariant of topological conjugacy. In fact, suppose (X, f) and (Y, g) are two conjugate systems, i.e., $hof = goh$, where h is a conjugacy.
Assume that f has the SELmSP with constants L and λ, and $\xi = \{y_k\}_{k\geq 0}$ is a sequence such that $d_Y(g(y_k), (y_{k+1})) \leq \lambda^{\frac{k}{2}}$ for $k \geq k_1$. Fix i, k big enough ($i \geq k \geq k_1$). By uniform continuity of h^{-1} choose $\delta > 0$ corresponding to $\epsilon = \lambda^i$. Note that, if necessary, by increasing k_1 we can assume that $\lambda^{\frac{k}{2}} < \delta$. Now $d_X\big(fh^{-1}(y_k), h^{-1}(y_{k+1})\big) = d_X\big(h^{-1}g(y_k), h^{-1}(y_{k+1})\big) < \lambda^i < \lambda^k,\ k \geq k_1$. So, $h^{-1}(\xi)$ satisfies relation 5 for f. Therefore, there exists $z \in X$ and $k_2 \in \mathbb{N}$ so that $d_X(f^k(z), h^{-1}(y_k)) < L\lambda^k$, for $k \geq k_2$. Again, fix i, k big enough ($i \geq k \geq k_2$). By uniform continuity of h choose $\eta > 0$ corresponding to $\epsilon = L\lambda^i$. Note that, if necessary, by increasing k_2 we can assume that $L\lambda^k < \eta$. By choosing η and k_2 we have $d_Y(g^k h(z), y_k) = d_Y(hf^k(z), hh^{-1}(y_k)) < L\lambda^i < L\lambda^{\frac{k}{2}}$, for $k \geq k_2$ hence $h(z)$ is the required point and g has the SELmSP with constants L and $\lambda^{\frac{1}{2}}$. The remaining part is similar.

Proposition 2.1. *If f is a surjection that has the SELmSP, then so does f^n for all $n > 0$.*

Proof. Fix $n > 0$ and suppose $\lambda \in (0, 1)$ and $L > 0$ are the constants in definition of the SELmSP for f. Let $\xi = \{x_i\}_{i=0}^{\infty}$ be a sequence which satisfies the relation 5 with constant λ^n in place of λ for the map f^n, define the sequence $\eta = \{y_i\}_{i=0}^{\infty}$ with

$$y_k = \begin{cases} x_0 & k = 0, \\ f^{k-nq-1}(x_{q+1}) & nq < k \leq n(q+1). \end{cases}$$

Indeed,

$$\begin{aligned}\eta = &\{x_0, x_1, f(x_1), f^2(x_1), \ldots, f^{n-1}(x_1), x_2, f(x_2), f^2(x_2), \ldots, \\ &f^{n-1}(x_2), x_3, f(x_3), f^2(x_3), \ldots, f^{n-1}(x_3), \ldots\}\end{aligned}$$

then for $k = nq$ we have

$d(f(y_k), y_{k+1}) = d(f^n(x_q), x_{q+1}) < (\lambda^n)^q = \lambda^k$. So, $\eta = \{y_i\}_{i=0}^{\infty}$ satisfies the relation 5 for the map f. Hence there exists $z \in X$ such that $d(f^k(z), y_k) < L\lambda^k$. If we put $k = nq - (n-1)$ in the last inequality we get $y_k = x_q$, and $d\big((f^n)^q(p), x_q\big) < L\lambda^{nq-(n-1)} = L\lambda^{-(n-1)}(\lambda^n)^q$, which $p = f^{-(n-1)}(z)$. Therefore f^n has the SELmSP with constants λ^n and $L_0 = L\lambda^{-(n-1)}$ □

In the following, we are going to show that the SELmSP holds for a class of non-homeomorphisms. Precisely, we show that the SELmSP holds for positively expansive maps having the POTP. In [1, Corollary 3.1], the author deduced that expansive homeomorphisms on a compact metric space with the POTP also have the exponential limit shadowing property. Here, we are going to extend this result for positively expansive maps.

Theorem 2.2. *Let $f : X \to X$ be a positively expansive map on compact metric space X having the LpSP. Then f has also the SELmSP.*

Proof. □

It is well-known that for positively expansive maps on a compact metric space, being an open map, the standard shadowing property, and the Lipschitz shadowing are all equivalent [8, Theorem 1]. So, readily we get the following result.

Corollary 2.3. *Suppose that $f : X \to X$ is a positively expansive map on a compact metric space. If f has the POTP (or equivalently is an open map), then it has the SELmSP.*

3. Examples

The following example shows that not every system has the SELmSP property.

Example 3.1. *Let $X = \{0, 1, \frac{1}{2}, \dots\}$ be a metric space with the usual metric on $\mathbb{R}$, and define $f(0) = 1$, $f(\frac{1}{n}) = \frac{1}{n+1}$. Now, take $(x_i)_{i\in\mathbb{N}} = \{1, \frac{1}{2}, \dots\}$ so we have $d(f(x_i), x_{i+1}) = 0$. It can be easily shown that $f^i(z) = \frac{z}{iz+1}$ for every $i > 0$. Hence for every $L > 0$ and $\lambda \in (0, 1)$, there is no points $z \in X$ such that $d(f^i(z), x_i) \le L\lambda^i$.*

The next example, which is called permutation of two points, is clearly an open positively expansive map on a compact metric space that does not have the two-sided limit shadowing. However, we observe that by Corollary 2.3, it has the strong exponential limit-shadowing. Therefore, the SELmSP is different from the two-sided limit shadowing.

Example 3.2. *Let $X = \{a, b\}$ and define $f(a) = b$ and $f(b) = a$, so f is a homeomorphism on X. Fix $0 < \lambda < 1$, for each exponentially-limit pseudo-orbit $\{x_n\}_{n\ge0}$ (relation 5) there exists $N \in \mathbb{N}$ such that $x_{N+k} = a$, if k is even or $x_{N+k} = b$, if k is odd. In other words, each exponentially limit pseudo-orbit is, by neglecting a finite beginning terms, a periodic sequence of the form $\overline{ab}$ or $\overline{ba}$. Then either $z = a$ or b, exponentially limit shadows the sequence $\{x_n\}_{n\ge0}$, i.e., the relation 6 holds with constants $\lambda \in (0, 1)$ and $L = 1$.*

Remark 3.3. *It can be easily shown that the above example is transitive, but not topologically mixing (and since it has the POTP, so equivalently not chain mixing). Therefore, the chain mixing (and topologically mixing) is not necessary for the SELmSP.*

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A numerical way for optimal control problems with pantograph delay via Boubaker polynomials

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Abstract

In this paper, we focus on Boubaker polynomials in optimal control problems (OCPs) with pantograph delays. In fact, the functions of the problem are approximated by Boubaker polynomials with unknown coeffcients in the constraint equations, performance index and conditions. Numerical result is given for a test example to demonstrate the applicability and effciency of the method.

1. Introduction

OCPs have an important role in some areas including engineering economics and finance. A computational strategy for solving optimal control problem (OCP) is developed by Wu, et al. [8] which is obtained by a switched dynamical system with time delay. Kharatishidi [4] approached this problem by extending the Pontryagin's maximum principle to time delay systems (TDS). The actual solution involves a two-point boundary value problem in which advances and delays are stated. In addition, this solution does not yield a feedback controller. OCP with time delay has been considered by Oguztoreli [7] who achieving several findings concerning bang-bang controls which are parallel to those of LaSalle [6] for non delay systems. For a time invariant system with an infinite upper limit in the performance measure, Krasovskii [5] developed the forms of the controller and the performance measure. An optimal regulator for a linear system with multiple states, input delays and a quadratic criterion is presented in [1]. The optimal regulator equations were achieving by reducing original problem to the linear quadratic regulator design for a system without delays (see [1] and [2]).
Here, we state Boubaker polynomials for solving OCPs with pantograph delays. The outline of this paper is as follows: In Section 2, Boubaker polynomials is introduced. an example is given in Section 3. Section 4 is dedicated the conclusion.

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2. Boubaker polynomials

In this section, Boubaker polynomials (BPs), which are used for solving OCPs with pantograph delay, are reviewed briefy. The BPs were established for the first time by Boubaker et al.[3] to solve heat equation inside a physical model. The first monomial dfinition of the Boubaker polynomials on $[0,1]$ is as follows:

$$B_k(t) = \sum_{r=0}^{\lfloor i/2 \rfloor} (-1)^r \binom{i-r}{r} \frac{i-4r}{i-r} t^{i-2r}, i \geq 2,$$

where the $\lfloor . \rfloor$ is the floor function. Also we have

$$\begin{aligned}
&B_0(t) = 1,\\
&B_1(t) = t,\\
&B_2(t) = t^2 + 2,\\
&B_3(t) = t^3 + t,\\
&B_m(t) = tB_{m-1}(t) - B_{m-2}(t), \quad m > 2,
\end{aligned}$$

In this paper, the state and cotrol variables are approximated as follows:

$$\begin{aligned}
x_n(t) &= \sum_{k=0}^{n} a_k B_k(t), \quad n = 1, 2, 3, \ldots,\\
u_n(t) &= \sum_{k=0}^{n} b_k B_k(t), \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Note: For function approximation, one may approximate $x(t)$ and $u(t)$ by Boubaker functions, and subsititute in OCPs. Therefore one may solve a new problem for finding the unknown variables a_k and b_k.

2.1. Function approximation

If $Y = span\{B_0(t), B_1(t), \ldots, B_m(t)\}$ is a finite dimensional and closed subspace of the Hilbert spcae $H = L^2[0,1]$, then there is a unique best approximation out of Y such as $\tilde{x} \in Y$ for each $x \in H$, that is

$$\forall y \in Y, \|x - \tilde{x}\| \leq \|x - y\|.$$

Since $\tilde{x} \in Y$, there exist unique coefficients $c_0, c_1, \ldots, c_n$ such that

$$\begin{aligned}
&x(t) \approx \tilde{x}(t) = \sum_{j=0}^{n} c_j B_j(t) = C^T \psi(t),\\
&C = [c_0, c_1, \ldots, c_n]^T,\\
&\psi(t) = [B_0(t), B_1(t), \ldots, B_n(t)]^T.
\end{aligned}$$

where T indicate transposition.

3. Numerical example

Here, we solve an example for showing the validity of the method. Also, it can be shown that if the number of Boubaker basis functions is increased this method is convergent.

Example 3.1. *Presently, consider the following OCPs with pantograph delay*

$$\begin{aligned}
\min\ J \quad &= \quad \frac{1}{2}\int_0^4 x^2(t) + u^2(t)\ dt,\\
&s.t.\ \frac{dx(t)}{dt} = x(0.5t) + 4u(t),\\
&x(0) = 1,
\end{aligned}$$

where the obtained cost function is $J = 0.181347551889203373$. *The Fig.1 is obtained by proposed method.*

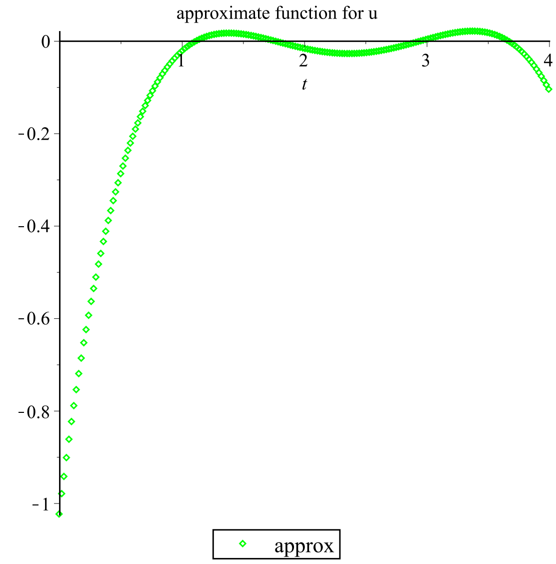

Fig. 1. The graph of approximated solution for Example 3.1

4. Conclusions

This paper presents a numerical technique for solving OCPs with pantpgraph delays via Boubaker polynomials. Also, we solved an example for showing the validity of the method. Also, it can be shown that if the number of Boubaker basis functions is increased this method is convergent.

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Distance Based Critical Node in Symmetric Travelling Salesman Problem

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Abstract

The travelling salesman problem one of the well-known NP-hard problems, and there are various models with respect to its different specifications of the travalling salesman problem. Specially, the symmetric travelling salesman problem is one of the most studied models, which has been applied for routing models. The critical node detection problem has received increasing attention throughout the routing models. It is defined as the node its deletion from the network results in the largest decrease of the optimal cost. The 2-opt heuristic is applied by the critical node in the symmetric traveling salesman problem and the the iterations are reduced significantly. Then, the pseudo-critical node is detected in the approximate solution, whose removal results in the largest decrease of the approximate cost. So, the 2-opt heuristic is applied by the pseudo-critical node and the optimal or nearby optimal solution is obtained.

1. Introduction

The critical node is an important issue for decision makers in the optimization models. There are various definition of the critical node according to the applications; however, the most addressed critical (or important) node is related to the connectivity of the network with respect to a connectivity criteria [2, 8, 10, 11]. The critical node is detected as the node its removal results in the maximum decrease of the optimal cost. So, let $C^*(n)$ be the optimal cost of network $G = (N, A)$ and $C^*(n-1)$ be the optimal cost of the reduced network $\bar{G} = (\bar{N}, \bar{A})$ by removing the critical node v_c from G, then $\bar{G} = G \setminus v_c$.
In the general case the critical node problem (CNP) is NP-complete [3] and it was shown for trees the problem is NP-complete, too [6]. Since the present definition of the critical node is based on the optimal solution, then it remains NP-complete, too. However, there are polynomial time algorithms for the travelling salesman problem (TSP) where the costs are satisfying the triangle inequality [5, 7].
Jiang et al. [8] applied a nonconvex quadratically constrained quadratic programming model instead of integer linear programming model to formulate the critical node detection problem (CNDP); they determined approximate solutions by semidefinite programming technique semidefinite programming technique. Santos et al. [11] studied p critical

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nodes problem according to the connectivity of the network and the solution computationally were improved. Veremyev et al. [13] considered two deterministic and probabilistic versions of CNDP; so, they studied a mixed integer linear programming for the deterministic one, and based on Markov chain process a scenario based formulation was presented for the probabilistic one. Aringhieri et al. [2] studied some classes of CNP where the objective function is impacted by the distances of the node pairs; they showed in the general cases the problem is NP-complete. Li et al. [9] studied a bi-objective critical node detection problem based on the psychology of decision makers and the pairwise connectivity of the network, and they proved the problem is NP-Hard in the general case.
Chen et al. [4] considered the negative CNP where the larger edge weights demonstrate the weaker relationship between nodes; their objective was simultaneously minimization of pairwise connectivity and maximization of the weights between the nodes. Alozie et al. [1] developed an algorithm for distance based CNDP for separating the problem by breadth first search tree generation. Zhou et al. [14] considered node weighted critical node problem, and they applied a local search procedure and a late acceptance strategy to find a local optimal solution. Shukla [12] developed an algorithm to solve a three dimensional CNDP.

2. Distance critical node detection problem formulation

In this paper, the critical node is defined according to the most decreasing in the optimal cost of TSP (D-CNDP). The network $G = (N, A)$ is a complete network with symmetric arc costs, so $c_{ij} = c_{ji}$ for all $(i, j) \in A$. Thus, it is assumed the arc costs c_{ij} are satisfying the triangle inequality, and there are some approximate solution in polynomial time [5, 7].
Consider the following TSP formulation to obtain minimum Hamiltonian path in the network

$$min \sum_{i=1}^{n} \sum_{j\neq i, j=1}^{n} c_{ij}x_{ij} \tag{1}$$

$$\sum_{i=1, i\neq j}^{n} x_{ij} = 1, j = 1, 2, ..., n \tag{2}$$

$$\sum_{j=1, j\neq i}^{n} x_{ij} = 1, i = 1, 2, ..., n \tag{3}$$

$$\sum_{i\in V} \sum_{j\neq i, j\in V} x_{ij} \leq |V| - 1, \forall V \subsetneq \{1, 2, ..., n\}, |V| \geq 2 \tag{4}$$

The objective function 1 determines a minimum length Hamiltonian cycle; the constraints 2 and 3 imply any node to traverse exactly once, and the last constraint 4 implies any pure subset of the nodes constructs a path. To determine the critical node in the network the following formulation is presented the above problem should be solved for any node $i \in N$, and in the reduced network $G \setminus \{i\}$, in the general form.

Fig. 1. The O-SSP algorithm

Input the optimal tour P^*
Let $C*$ be the optimal cost of P^*
Let $\bar{C}^* = \infty$
for $v_i = 1$ to $n = |P^*|$ **do**
  $\Delta_{v_i} = c_{v_{i-1},v_i} + c_{v_i,v_{i+1}} - c_{v_{i-1},v_{i+1}}$
  Delete node v_i for P^*
  $\bar{C}_{v_i} = C^* - \Delta_{v_i}$
  if $\bar{C}_{v_i} < \bar{C}^*$ **then**
    $\bar{C}^* = \bar{C}_{v_i}$
    $\bar{v} = v_i$

Theorem 2.1. *The critical node for the original network $G = (N, A)$ is the critical node for the optimal tour P^* in G.*

By Theorem 2.1, the optimal solution P^* contains n nodes and the D-CNDP could be solved in $O(n)$ (see Figure 1). let $\bar{v}$ be the critical node in G, then

$$\bar{v} = argmax_{\forall v_i \in N}\{C^* - (c_{v_{i-1},v_i} + c_{v_i,v_{i+1}}) + c_{v_{i-1},v_{i+1}}\}$$

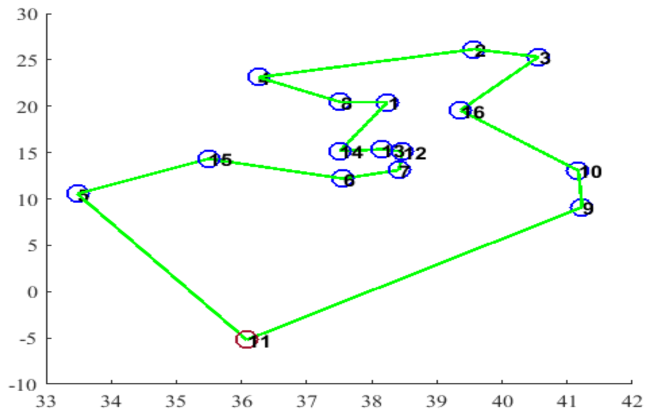

Fig. 2. The instance network ulysses16 and the optimal tour

For the instance network ulysses16 (see Figure 2), there 16 nodes in the network with the symmetric and geometric lengths for the arc lengths; the critical node is detected as node 11 and the length of the optimal tour is 6859 and the optimal cost by the removal of the critical node 11 is obtained 5216.

Theorem 2.2. *The optimal solution P^* of symmetric traveling salesman problem (S-TSP) in the original network $G = (N, A)$ is reduced to the optimal solution $\bar{P}^*$ in the reduced network $\bar{G} = (\bar{N}, \bar{A})$, where $\bar{N} = N \setminus \{i\}$ and $\bar{A} = A \setminus \{(\bar{v}, i) \colon \bar{v}$ is the critical node in the network $G\}$.*

By Theorem 2.2, the optimal solution for network G is transformed into the optimal solution of the reduced network $\bar{G}$ by removal the critical node from the optimal solution P^*. So, in the instance network ulysses16, the optimal solution 1,14,13,12,7,6,15,5,11,9,10,16,3,2,4,8 is transformed into the optimal solution 1,14,13,12,7,6,15,5,9,10,16,3,2,4,8, for the reduced network

2.1. Pseudo critical node in the approximation solution

According to the complexity of S-TSP, the approximate solution for the problem is respected in the polynomial time. So, the pseudo-critical node is defined similarly to the critical node, but it is related to the approximate solution. Thus, $\tilde{v}$ is the pseudo-critical node with respect to the approximate solution $\tilde{P}$, if its removal results the largest decrease in the length of the approximate solution.
The 2-opt algorithm applies two point exchanges in a feasible tour for TSP, and it attempts to improve the tour lengths alliteratively. We apply the 2-opt algorithm by the pseudo-critical node and it results in fewer iterations against the general algorithm. For the instance network ulysses16, the obtained approximate solution by Christofides' algorithm [5] is shown in Figure 3. The node 11 is detected as the pseudo-critical node. After 4 iterations of the 2-opt algorithm (iterations 3 and 4 are the same), the approximate solution 1,8,4,2,3,16,13,12,7,6,10,9,11,14,15,5 with length 7788 is transformed into the optimal solution (see Figure 4).

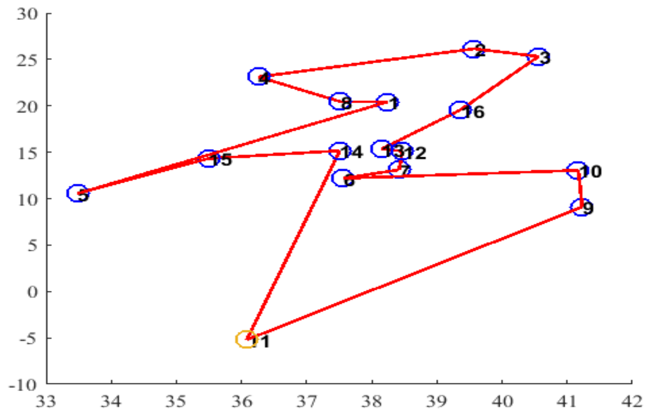

Fig. 3. The approximate solution for the instance network ulysses16

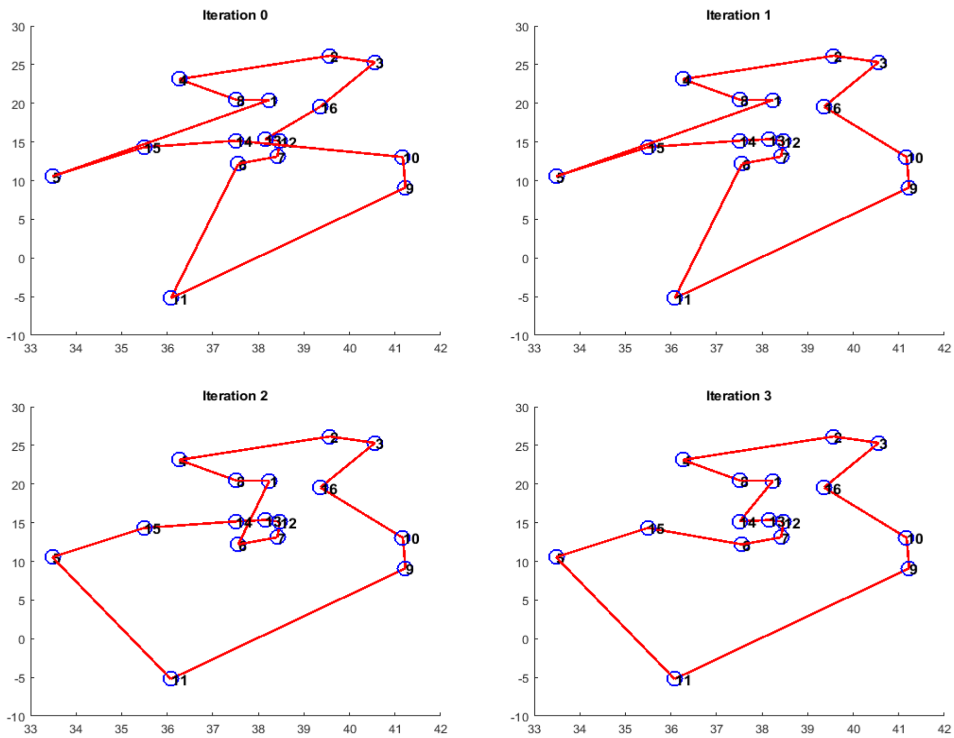

Fig. 4. The instance network ulysses16 and the iterations of the 2-opt algorithm

3. Conclution

The S-TSP is considered with the arc lengths satisfying the triangle inequality. So, a distance critical node is defined as the node its removal results in the largest decrease in the optimal tour. It is shown, by removing the critical node from the optimal solution, the optimal solution is obtained for the reduced network. Then, the pseudo-critical node is detected in an approximate solution, the 2-opt algorithm is improved to find the optimal solution (nearby optimal solution) in a fewer iterations against the general algorithm.

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Anti fuzzy SU-subalgebra under conorms

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Abstract

In this paper, anti fuzzy SU-subalgebra under conorms will be defined and will be investigated their properties. Also union and direct sum of them will be introduced and will be obtained some new results about them. Finally, will be investigated them under SU-algebra homomorphisms.

1. Introduction

Since several years ago, the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of this theory can be found, for example, in artificial intelligence, computer science, control engineering, decision theory, expert systems, logic, management science, operations research, pattern recognition, and robotics. Theoretical advances have been made in many directions. In fact it is extremely difficult for a newcomer to the field or for somebody who wants to apply fuzzy set theory to his problems to recognize properly the present "state of the art." Therefore, many applications use fuzzy set theory on a much more elementary level than appropriate and necessary. On the other hand, theoretical publications are already so specialized and assume such a background in fuzzy set theory that they are hard to understand. Fuzzy sets were introduced independently by Zadeh in 1965 as an extension of the classical notion of set [34]. Keawrahun and Leerawat [3] introduced new structured algebra called SU-Algebra. The First author by using norms, investigated some properties of fuzzy algebraic structures[5-33]. In this paper, by using t-conorms, we define anti fuzzy SU-subalgebra and direct sum of them and obtain some results about them. Next we define union of them and investigate their properties. Later, we consider SU-subalgebra homomorphisms over them and prove some properties of them.

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2. Preliminaries

This section contains some basic definitions and preliminary results which will be needed in the sequal. For more details we refer to [1-7].

Definition 2.1. A SU-algebra is a non-empty set X with a consonant 0 and a single binary operation $*$ (denoted by $(X, *, 0)$) satisfying the following axioms for any $x, y, z \in X$:
(1) $((x * y) * (x * z)) * (y * z) = 0$,
(2) $x * 0 = x$,
(3) if $x * y = 0$, then $x = y$.

Definition 2.2. A non-empty subset S of a SU-algebra X is said to be a subalgebra if $x * y \in S$ for all $x, y \in S$.

Definition 2.3. A function $f : X \to Y$ of SU-algebras X and Y is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 2.4. A function $f : X \to Y$ of SU-algebras X and Y is called anti homomorphism if $f(x*y) = f(y)*f(x)$ for all $x, y \in X$.

Definition 2.5. A fuzzy subset of a set X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$.

Definition 2.6. Let $\varphi : X \to Y$ be a function such that $\mu \in [0, 1]^X$ and $\nu \in [0, 1]^Y$. For all $x \in X, y \in Y$, we define

$$\varphi(\mu)(y) = \begin{cases} \inf\{\mu(x) \mid x \in X, \varphi(x) = y\} & \text{if } \varphi^{-1}(y) \neq \emptyset \\ 0 & \text{if } \varphi^{-1}(y) = \emptyset \end{cases}$$

also $\varphi^{-1}(\nu)(x) = \varphi^{-1}(\nu)(x) = \nu(\varphi(x))$.

Definition 2.7. A t-conorm C is a function $C : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(C1) $C(x, 0) = x$
(C2) $C(x, y) \leq C(x, z)$ if $y \leq z$
(C3) $C(x, y) = C(y, x)$
(C4) $C(x, C(y, z)) = C(C(x, y), z)$,
for all $x, y, z \in [0, 1]$.

Corollary 2.8. *Let C be a C-conorm. Then for all $x \in [0, 1]$*
(1) $C(x, 1) = 1$.
(2) $C(0, 0) = 0$.

Example 2.9. *(1) Standard union t-conorm $C_m(x, y) = \max\{x, y\}$.*
(2) Bounded sum t-conorm $C_b(x, y) = \min\{1, x + y\}$.
(3) Algebraic sum t-conorm $C_p(x, y) = x + y - xy$.
(4) Drastic T-conorm

$$C_D(x, y) = \begin{cases} y & \text{if } x = 0 \\ x & \text{if } y = 0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic T-norm.
(5) Nilpotent maximum T-conorm, dual to the nilpotent minimum T-norm:

$$C_{nM}(x, y) = \begin{cases} \max\{x, y\} & \text{if } x + y < 1 \\ 1 & \text{otherwise.} \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x, y) = \dfrac{x + y}{1 + xy}$ is a dual to one of the Hamacher t-norms. Note that all t-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x, y) \leq C(x, y) \leq C_D(x, y)$ for any t-conorm C and all $x, y \in [0, 1]$.

Recall that t-norm T(t-conorm C) is idempotent if for all $x \in [0,1]$, $T(x,x) = x(C(x,x) = x)$.

Lemma 2.10. *Let C be a t-conorm. Then*

$$C(C(x,y), C(w,z)) = C(C(x,w), C(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.11. Let $\mu, \nu \in [0,1]^X$ and define unon μ and νB as $(\mu \cup \nu)(x) = C(\mu(x), \nu(x))$ for all $x \in X$.

Definition 2.12. Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. By using t-conorm C, define $\mu \oplus \nu \in [0,1]^{X \oplus Y}$ as direct sum of μ and ν such that $(\mu \oplus \nu)(x,y) = C(\mu(x), \nu(y))$ for all $x \in X$ and $y \in Y$.

3. Conorms over anti fuzzy SU-subalgebras

Definition 3.1. A fuzzy subset $\mu : (X, *, 0) \to [0,1]$ in a SU-algebra $(X, *, 0)$ is said to be an anti fuzzy SU-subalgebra of X under t-conorm C if $\mu(x * y) \leq C(\mu(x), \mu(y))$ for all $x, y \in X$. We denote the set of all anti fuzzy SU-subalgebras of X under t-conorm C by $AFSUC(X)$.

Example 3.2. *Let $X = \{0, 1, 2, 3\}$ be a set with the following table:*

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a SU-algebra. Define fuzzy subset $\mu : (X, *, 0) \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.3 & \textit{if } x = 0 \\ 0.4 & \textit{if } x = 1 \\ 0.5 & \textit{if } x = 2 \\ 0.55 & \textit{if } x = 3 \end{cases}$$

and let $C_p(a,b) = a + b - ab$ for all $a, b \in [0,1]$. Then $\mu \in AFSUC(X)$.

Proposition 3.3. *Let $\mu \in AFSUC(X)$. If C be idempotent, then $\mu(0) \leq \mu(x)$ for all $x \in X$.*

Proof. Let $x \in X$. Then

$$\mu(0) = \mu(x * x) \leq C(\mu(x), \mu(x)) = \mu(x)$$

and thus $\mu(0) \leq \mu(x)$. □

Proposition 3.4. *Let $\mu, \nu \in AFSUC(X)$. Then $\mu \cup \nu \in AFSUC(X)$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} &(\mu \cup \nu)(x * y) \\ &= C(\mu(x * y), \nu(x * y)) \\ &\leq C(C(\mu(x), \mu(y)), C(\nu(x), \nu(y))) \\ &= C(C(\mu(x), \nu(x)), C(\mu(y), \nu(y))) \\ &= C((\mu \cup \nu)(x), (\mu \cup \nu)(y)) \end{aligned}$$

and then

$$(\mu \cup \nu)(x * y) \leq C((\mu \cup \nu)(x), (\mu \cup \nu)(y)).$$

Therefore

$$\mu \cup \nu \in AFSUC(X).$$

□

Proposition 3.5. *Let φ be a homomorphism from SU-algebra of X into SU-algebra of Y and $\mu \in AFSUC(X)$. Then $\varphi(\mu) \in AFSUC(Y)$.*

Proof. Let $y_1, y_2 \in Y$. Then

$$\varphi(\mu)(y_1 * y_2) = \inf\{\mu(x_1 * x_2) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$\leq \inf\{C(\mu(x_1), \mu(x_2)) \mid x_1, x_2 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$
$$\leq C(\inf\{\mu(x_1) \mid x_1 \in X, \varphi(x_1) = y_1\}, \inf\{\mu(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\})$$
$$= C(\varphi(\mu)(y_1), \varphi(\mu)(y_2))$$

thus

$$\varphi(\mu)(y_1 * y_2) \leq C(\varphi(\mu)(y_1), \varphi(\mu)(y_2))$$

then $\varphi(\mu) \in AFSUC(Y)$. □

Proposition 3.6. *Let φ be an anti homomorphism from SU-algebra of X into SU-algebra of Y and $\mu \in AFSUC(X)$. Then $\varphi(\mu) \in AFSUC(Y)$.*

Proof. The proof is similar to proof of Proposition 3.5. □

Proposition 3.7. *Let φ be a homomorphism from SU-algebra of X into SU-algebra of Y and $\nu \in AFSUC(Y)$. Then $\varphi^{-1}(\nu) \in AFSUC(X)$.*

Proof. Let $x_1, x_2 \in X$. Then

$$\varphi^{-1}(\nu)(x_1 * x_2) = \nu(\varphi)(x_1 * x_2) = \nu(\varphi(x_1) * \varphi(x_2))$$
$$\leq C(\nu(\varphi(x_1)), \nu(\varphi(x_2))) = C(\varphi^{-1}(\nu)(x_1), \varphi^{-1}(\nu)(x_2))$$

and thus

$$\varphi^{-1}(\nu)(x_1 * x_2) \leq C(\varphi^{-1}(\nu)(x_1), \varphi^{-1}(\nu)(x_2)).$$

Therefore $\varphi^{-1}(\nu) \in AFSUC(X)$. □

Proposition 3.8. *Let φ be an anti homomorphism from SU-algebra of X into SU-algebra of Y and $\nu \in AFSUC(Y)$. Then $\varphi^{-1}(\nu) \in AFSUC(X)$.*

Proof. The proof is similar to proof of Proposition 3.7. □

Proposition 3.9. *Let $\mu \in AFSUC(X)$ and $\nu \in AFSUC(Y)$. Then $\mu \oplus \nu \in AFSUC(X \oplus Y)$ for every SU-algebra of X and SU-algebra of Y.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Thus

$$(\mu \oplus \nu)((x_1, y_1) * (x_2, y_2))$$
$$= (\mu \oplus \nu)(x_1 * x_2, y_1 * y_2)$$
$$= C(\mu(x_1 * x_2), \nu(y_1 * y_2))$$
$$\leq C(C(\mu(x_1), \mu(x_2)), C(\nu(y_1), \nu(y_2)))$$
$$= C(C(\mu(x_1), \nu(y_1)), C(\mu(x_2), \nu(y_2)))$$
$$= C((\mu \oplus \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2))$$

then

$$(\mu \oplus \nu)((x_1, y_1) * (x_2, y_2)) \leq C((\mu \oplus \nu)(x_1, y_1), (\mu \oplus \nu)(x_2, y_2))$$

so $\mu \oplus \nu \in AFSUC(X \oplus Y)$. □

Proposition 3.10. *Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. If $\mu \oplus \nu \in AFSUC(X \oplus Y)$, then at least one of the following statements hold:*
(1) $\mu(0) \geq \nu(y)$ for all $y \in Y$,
(2) $\nu(0) \geq \mu(x)$ for all $x \in X$.

Proof. Let none of the statements holds, then we can find $x \in X$ and $y \in Y$ such that $\mu(0) < \nu(y)$ and $\nu(0) < \mu(x)$. Therefore

$$(\mu \oplus \nu)(x,y) = C(\mu(x), \nu(y)) < C(\nu(0), \mu(0)) = C(\mu(0), \nu(0)) = (\mu \oplus \nu)(0,0)$$

and this is contradiction with Proposition 3.3 and then at least one of the statements hold. □

Proposition 3.11. *Let $\nu \in [0,1]^Y$. If $\nu \oplus \nu \in AFSUC(Y \oplus Y)$ and $\nu(x) \geq \nu(0)$ for all $x \in Y$. Then $\nu \in AFSUC(Y)$.*

Proof. Since $\nu(x) \geq \nu(0)$ for all $x \in Y$ so $\nu(x * y) \geq \nu(0) = \nu(0 * 0)$ for all $x, y \in Y$. Thus

$$\nu(x * y) = C(\nu(x * y), \nu(0 * 0))$$

$$= (\nu \oplus \nu)(x * y, 0 * 0)$$
$$= (\nu \oplus \nu)((x,0) * (y,0))$$
$$\leq C((\nu \oplus \nu)(x,0), (\nu \oplus \nu)(y,0))$$
$$= C(C(\nu(x), \nu(0)), C(\nu(y), \nu(0)))$$
$$= C(\nu(x), \nu(y))$$

therefore

$$\nu(x * y) \leq C(\nu(x), \nu(y))$$

and then $\nu \in AFSUC(Y)$. □

4. Open Problem

In this study, using t-norms, we defined and investigated fuzzy SU-Algebras. Now One can this work for B-Algebras and TM-Algebras and this can be as an open problem.
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S-norms over anti fuzzy implicative ideals, anti fuzzy positive implicative ideals in BCK-algebras

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Abstract

In this work, as using S-norms, we introduce anti fuzzy implicative ideals, anti fuzzy positive implicative ideals in BCK-algebras. Also we obtain the links between them and investigate properties of them. Finally, we consider them under union, direct sum and homomorphisms(image and pre image) and we investigate related properties.

1. Introduction

Imai and Iseki introduced the notion of BCK-algebra[4]. The concept of uncertainty has gone through a paradigmatic change in the last few decades and now it is not only an unavoidable plague to science and mathematics but it has, in fact, a great utility. Zadeh[58] introduced the concept of fuzzy sets. Many authors[2, 6, 8, 9, 11, 12, 13, 16, 59] considered the fuzzification of ideals and subalgebras in BCK-algebras. S-norms are operations which generalize the logical conjunction and logical disjunction to fuzzy logic. The First author by using norms, investigated some properties of fuzzy algebraic structures[17-56]. In this study we define anti fuzzy implicative ideals, anti fuzzy positive implicative ideals in BCK-algebras under s-norms. Next we link them with subalgebras, ideals, implicative ideals, positive implicative ideals in BCK-algebras. Later we consider them under union, direct sum and homomorphisms(image and pre image) and we consider related properties of them.

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2. Preliminaries

In this section we cite the fundamental definitions and results that will be used in the sequel. For more details we refer readers to [1, 3, 5, 7, 10, 14, 15, 16, 31, 35, 57].

Definition 2.1. By a BCK-algebra we mean a nonempty set X with a binary operation $*$ and a constant 0 satisfying the axioms:
(1) $((x * y) * (x * z)) \leq (z * y)$,
(2) $(x * (x * y)) \leq y$,
(3) $x \leq x$,
(4) $x \leq y$ and $y \leq x$ imply that $x = y$,
(5) $0 \leq x$
for all $x, y, z \in X$.
A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. In any BCK-algebra X the following holds:
(6) $x * 0 = x$,
(7) $x * y \leq x$,
(8) $(x * y) * z = (x * z) * y$,
(9) $(x * z) * (y * z) \leq x * y$,
(10) $x * (x * (x * y)) = x * y$,
(11) if $x \leq y$, then $x * z \leq y * z$ and $z * y \leq z * x$
for all $x, y, z \in X$.

Definition 2.2. A BCK-algebra X is said to be implicative if $x = x * (y * x)$, for all $x, y \in X$.

Definition 2.3. A BCK-algebra X is said to be positive implicative if $(x*y)*z = (x*z)*(y*z)$ for all $x, y, z \in X$.

Definition 2.4. A non-empty subset I of a BCK-algebra X is called an implicative ideal of X if
(1) $0 \in I$,
(2) $(x * (y * x)) * z \in I$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in X$.

Definition 2.5. A non-empty subset I of a BCK-algebra X is called a positive implicative ideal of X if
(1) $0 \in I$,
(2) $(x * y) * z \in I$ and $y * z \in I$ imply that $x * z \in I$ for all $x, y, z \in X$.

Definition 2.6. A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

Definition 2.7. Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0, 1]$. The set of all fuzzy subsets of X is called the $[0, 1]$-power set of X and is denoted $[0, 1]^X$. For a fixed $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is called a lower level of μ.

Definition 2.8. Let φ be a function from set X into set Y such that $\mu \in [0, 1]^X$ and $\nu \in [0, 1]^Y$. For all $x \in X, y \in Y$, we define

$$\varphi(\mu)(y) = \inf\{\mu(x) \mid x \in X, \varphi(x) = y\}$$

and

$$\varphi^{-1}(\nu)(x) = \nu(\varphi(x)).$$

Definition 2.9. An S-norm S is a function $S : [0, 1] \times [0, 1] \to [0, 1]$ having the following four properties:
(1) $S(x, 0) = x$,
(2) $S(x, y) \leq S(x, z)$ if $y \leq z$,
(3) $S(x, y) = S(y, x)$,
(4) $S(x, S(y, z)) = S(S(x, y), z)$,
for all $x, y, z \in [0, 1]$.

We say that S is idempotent if for all $x \in [0, 1]$,$S(x, x) = x$.

Example 2.10. *The basic S-norms are $S_m(x, y) = \max\{x, y\}$,$S_b(x, y) = \min\{1, x + y\}$ and $S_p(x, y) = x + y - xy$ for all $x, y \in [0, 1]$.*
S_m is standard union, S_b is bounded sum, S_p is algebraic sum.

We say that S be idempotent if for all $x \in [0, 1]$ we have $S(x, x) = x$.

Definition 2.11. Let $\mu, \nu \in [0, 1]^X$ and define the union of μ and ν is denoted by $\mu \cup \nu \in [0, 1]^X$ as

$$(\mu \cup \nu)(x) = S(\mu(x), \nu(x))$$

for all $x \in X$.

Definition 2.12. Let $\mu \in [0, 1]^X$ and $\nu \in [0, 1]^Y$. Define the direct sum of μ and ν is denoted by $\mu \oplus \nu \in [0, 1]^{X \oplus Y}$ as

$$(\mu \oplus \nu)(x, y) = S(\mu(x), \nu(y))$$

for all $(x, y) \in X \oplus Y$.

Lemma 2.13. *Let S be a s-norm. Then*

$$S(S(x, y), S(w, z)) = S(S(x, w), S(y, z))$$

for all $x, y, w, z \in [0, 1]$.

3. S-norms over anti fuzzy: implicative ideals and positive implicative ideals in BCK-algebras

Throughout this paper, X, Y always mean two BCK-algebras unless otherwise specified.

Definition 3.1. Define $\mu \in [0, 1]^X$ is an anti fuzzy implicative ideal of X under s-norm S if it satisfies the following inequalities:
(1) $\mu(0) \leq \mu(x)$,
(2) $\mu(x) \leq S(\mu(x * (y * x)), \mu(z))$,
for all $x, y, z \in X$.
Denote by $AFIIS(X)$, the set of all anti fuzzy implicative ideals of X under s-norm S.

Proposition 3.2. *Let $\mu \in [0, 1]^X$ and S be idempotent. Then $\mu \in AFIIS(X)$ if and only if the set $\mu_t = \{x \in X : \mu(x) \leq t\}$ is either empty or an implicative ideal of X for every $t \in [0, 1]$.*

Proof. Let $\mu \in AFIIS(X)$ and $x, y \in X$. Thus $\mu(0) \leq \mu(x) \leq t$ so $0 \in \mu_t$.
Also let $(x * (y * x)) * z \in \mu_t$ and $z \in \mu_t$. Then

$$\mu(x) \leq S(\mu((x * (y * x)) * z), \mu(z)) \geq S(t, t) = t$$

thus $x \in \mu_t$. Then μ_t will be an implicative ideal of X for every $t \in [0, 1]$.
Conversely, let μ_t is either empty or an implicative ideal of X for every $t \in [0, 1]$.
Let $t = S(\mu((x * (y * x)) * z), \mu(z))$ with $(x * (y * x)) * z \in \mu_t$ and $z \in \mu_t$. Then $x \in \mu_t$ thus

$$\mu(x) \leq t = S(\mu((x * (y * x)) * z), \mu(z))$$

so $\mu \in AFIIS(X)$. □

Definition 3.3. Define $\mu \in [0, 1]^X$ is an anti fuzzy positive implicative ideal of X under s-norm S if it satisfies the following inequalities:
(1) $\mu(0) \leq \mu(x)$,
(2) $\mu(x * z) \leq S(\mu((x * y) * z), \mu(y * z))$,
for all $x, y, z \in X$.
Denote by $AFPIIS(X)$, the set of all anti fuzzy positive implicative ideals of X under s-norm S.

Proposition 3.4. *Let $\mu \in [0,1]^X$ and S be idempotent. Then $\mu \in AFPIIS(X)$ if and only if the set*

$$\mu_t = \{x \in X : \mu(x) \leq t\}$$

is either empty or a positive implicative ideal of X for every $t \in [0,1]$.

Proof. Let $\mu \in AFPIIS(X)$ and $x, y \in X$. Then $\mu(0) \leq \mu(x) \geq t$ and and then $0 \in \mu_s$.
Also let $(x * y) * z \in A_{s,t}$ and $y * z \in \mu_s$. Then

$$\mu(x * z) \leq S(\mu((x * y) * z), \mu(y * z)) \leq S(t,t) = t$$

thus $x \in \mu_t$. Then μ_t is a posive implicative ideal of X for every $t \in [0,1]$.
Conversely, let μ_t is either empty or a positive implicative ideal of X for every $t \in [0,1]$.
Let $t = S(\mu((x * y) * z), \mu(y * z))$ with $(x * y) * z \in \mu_t$ and $y * z \in \mu_t$. Then $x \in \mu_t$ thus

$$\mu(x) \leq t = S(\mu((x * (y * x)) * z), \mu(z))$$

so $\mu \in AFPIIS(X)$. □

Proposition 3.5. *Let $\mu \in AFPIIS(X)$ and $x, y, z, a, b \in X$.*
(1) If $((x * y) * y) * a \leq b$, then

$$\mu(x * y) \leq S(\mu(a), \mu(b)).$$

(2) If $((x * y) * z) * a \leq b$, then

$$\mu((x * z) * (y * z)) \leq S(\mu(a), \mu(b)).$$

Proof. Let $\mu \in AFPIIS(X)$ and $x, y, z, a, b \in X$.
(1) Let $((x * y) * y) * a \leq b$ then we get that $\mu((x * y) * y) \leq S(\mu(a), \mu(b))$. Thus

$$\mu(x * y) \leq S(\mu((x * y) * y), \mu(y * y)) = S(\mu((x * y) * y), \mu(0))$$
$$= \mu((x * y) * y) \leq S(\mu(a), \mu(b))$$

then

$$\mu(x * y) \leq S(\mu(a), \mu(b)).$$

(2) Let $((x * y) * z) * a \leq b$, so we get that

$$\mu((x * z) * (y * z)) \leq \mu((x * y) * z) \leq S(\mu(a), \mu(b)).$$

□

Proposition 3.6. *Let $\mu \in [0,1]^X$ and $((x * y) * y) * a \leq b$ for all $x, y, a, b \in X$. If $\mu(x * y) \leq S(\mu(a), \mu(b))$, then $\mu \in AFPIIS(X)$.*

Proof. Let $x, y, z \in X$ such that $x*y \leq z$. Definition 2.1 properties (1) gives us that $((x*0)*0)*y*z = (x*y)*z = 0$ thus $((x * 0) * 0) * y \leq z$. Put $y = 0, a = y, b = z$ in hypothesis then $\mu(x) = \mu(x * 0) \leq S(\mu(y), \mu(z))$. Thus we get that $\mu \in AFIS(X)$. As $(((x * y) * y) * ((x * y) * y)) * 0 = 0$ so $(((x * y) * y) * ((x * y) * y)) \leq 0$ for all $x, y \in X$. Using hypothesis will give us $\mu(x*y) \leq S(\mu((x*y)*y), \mu(0)) = \mu((x*y)*y)$. Therefore $\mu \in AFPIIS(X)$. □

Proposition 3.7. *Let $\mu \in [0,1]^X$ and $((x * y) * z) * a \leq b$ for all $x, y, z, a, b \in X$. If*

$$\mu((x * y) * (y * z)) \leq S(\mu(a), \mu(b))$$

then $\mu \in AFPIIS(X)$.

Proof. Let $((x * y) * z) * a \leq b$ for all $x, y, z, a, b \in X$. Then $(((x * y) * z) * a) * b = 0$. Now

$$\mu(x * y) = \mu((x * y) * 0) = \mu((x * y) * (y * y)) \leq S(\mu(a), \mu(b))$$

and as Proposition 3.5 we will have that $\mu \in AFPIIS(X)$. □

4. Union, direct sum and homomorphisms over introduced conceps

Proposition 4.1. *If $\mu, \nu \in AFIIS(X)$, then $\mu \cup \nu \in AFIIS(X)$.*

Proof. Let $x, y, z \in X$. Then
(1)
$$(\mu \cup \nu)(0) = S(\mu(0), \nu(0)) \leq S(\mu(x), \nu(x)) = (\mu \cup \nu)(x)$$
thus
$$(\mu \cup \nu)(0) \leq (\mu \cup \nu)(x).$$
(2)
$$\begin{aligned}(\mu \cup \nu)(x) = S(\mu(x), \nu_B(x)) &\leq S(S(\mu((x * (y * x)) * z), \mu(z)), S(\nu((x * (y * x)) * z), \nu(z)))\\ &= S(S(\mu((x * (y * x)) * z), \nu((x * (y * x)) * z), S(\mu(z), \nu(z))) \text{ (Lemma 2.15)}\\ &= S((\mu \cup \nu)((x * (y * x)) * z)), (\mu \cup \nu)(z))\end{aligned}$$
so
$$(\mu \cup \nu)(x) \leq S((\mu \cup \nu)((x * (y * x)) * z)), (\mu \cup \nu)(z)).$$
Then $\mu \cup \nu \in AFIIS(X)$. □

Proposition 4.2. *Let $\mu, \nu \in AFPIIS(X)$. Then $\mu \cup \nu \in AFPIIS(X)$.*

Proof. Let $x, y, z \in X$. Then
(1)
$$(\mu \cup \nu)(0) = S(\mu(0), \nu(0)) \leq S(\mu(x), \nu(x)) = (\mu \cup \nu)(x)$$
thus
$$(\mu \cup \nu)(0) \leq (\mu \cup \nu)(x).$$
(2)
$$\begin{aligned}(\mu \cup \nu)(x * z) = S(\mu(x * z), \nu(x * z)) &\leq S(S(\mu((x * y) * z), \mu(y * z)), S(\nu((x * y) * z), \nu(y * z)))\\ &= S(S(\nu((x * y) * z), \nu((x * y) * z)), S(\mu(y * z), \nu(y * z))) \text{ (Lemma 2.15)}\\ &= S((\mu \cup \nu)((x * y) * z)), (\mu \cup \nu)(y * z))\end{aligned}$$
so
$$(\mu \cup \nu)(x * z) \leq S((\mu \cup \nu)((x * y) * z))), (\mu \cup \nu)(y * z)).$$
Therefore $\mu \cup \nu \in AFPIIS(X)$. □

Proposition 4.3. *Let $\mu \in AFIIS(X)$ and $\nu \in AFIIS(Y)$. Then $\mu \oplus \nu \in AFIIS(X \oplus Y)$.*

Proof. Let $(x, y) \in X \oplus Y$. Then
$$(\mu \oplus \nu)(0, 0) = S(\mu(0), \nu(0)) \leq S(\mu(x), \nu(y)) = (\mu \oplus \nu)(x, y)$$
thus $(\mu \oplus \nu)(0, 0) \leq (\mu \oplus \nu)(x, y)$.
Also let $x_i \in X$ and $y_i \in Y$ for $i = 1, 2, 3$. Now
$$\begin{aligned}(\mu \oplus \nu)(x_1, y_1) = S(\mu(x_1), \nu(y_1)) &\leq S(S(\mu(x_1 * (x_2 * x_1)), \mu(x_3)), S(\nu(y_1 * (y_2 * y_1)), \nu(y_3)))\\ &= S(S(\mu(x_1 * (x_2 * x_1)), \nu(y_1 * (y_2 * y_1))), S(\mu(x_3), \nu(y_3))) \text{ (Lemma 2.15)}\\ &= S((\mu \times \nu)(x_1 * (x_2 * x_1), y_1 * (y_2 * y_1)), (\mu \oplus \nu)(x_3, y_3))\\ &= S((\mu \oplus \nu)((x_1, y_1) * ((x_2, y_2) * (x_1, y_1)), (\mu \oplus \nu)(x_3, y_3))\end{aligned}$$
thus
$$(\mu \oplus \nu)(x_1, y_1) \leq S((\mu \oplus \nu)((x_1, y_1) * ((x_2, y_2) * (x_1, y_1)), (\mu \oplus \nu)(x_3, y_3)).$$
Then $\mu \oplus \nu \in AFIIS(X \oplus Y)$. □

Proposition 4.4. *Let $\mu \in AFPIIS(X)$ and $\nu \in AFPIIS(Y)$. Then $\mu \oplus \nu \in AFPIIS(X \oplus Y)$.*

Proof. Let $(x, y) \in X \oplus Y$. Then

$$(\mu \oplus \nu)(0,0) = S(\mu(0), \nu(0)) \leq S(\mu(x), \nu(y)) = (\mu \oplus \nu)(x, y)$$

thus $(\mu \oplus \nu)(0,0) \leq (\mu \oplus \nu)(x, y)$.
Also let $x_i \in X$ and $y_i \in Y$ for $i = 1, 2, 3$. Then

$$\begin{aligned}(\mu \oplus \nu)((x_1, y_1) * (x_3, y_3)) &= (\mu \oplus \nu)(x_1 * x_3, y_1 * y_3) = S(\mu(x_1 * x_3), \nu(y_1 * y_3)) \\ &\leq S(S(\mu((x_1 * x_2) * x_3), \mu(x_2 * x_3)), S(\nu((y_1 * y_2) * y_3), \nu(y_2 * y_3))) \\ &= S(S(\mu((x_1 * x_2) * x_3), \nu((y_1 * y_2) * y_3)), S(\mu(x_2 * x_3), \nu(y_2 * y_3))) \text{ (Lemma 2.15)} \\ &= S((\mu \times \nu)((x_1 * x_2) * x_3, (y_1 * y_2) * y_3), (\mu \oplus \nu)(x_2 * x_3, y_2 * y_3)) \\ &= S((\mu \oplus \nu)((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), (\mu \times \nu)((x_2, y_2) * (x_3, y_3)))\end{aligned}$$

and so

$$(\mu \oplus \nu)((x_1, y_1) * (x_3, y_3)) \leq S((\mu \oplus \nu)((x_1, y_1) * (x_2, y_2)) * (x_3, y_3)), (\mu \oplus \nu)((x_2, y_2) * (x_3, y_3))).$$

Then $\mu \oplus \nu \in AFPIIS(X \oplus Y)$. □

Proposition 4.5. *If $\mu \in AFIIS(X)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi(\mu) \in AFIIS(Y)$.*

Proof. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu)(0) = \inf\{\mu(0) \mid 0 \in X, \varphi(0) = 0\} \leq \inf\{\mu(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu)(y)$$

thus $\varphi(\mu)(0) \leq \varphi(\mu)(y)$.
Also let $x, x_1, x_2 \in X$ such that $\varphi(x) = y, \varphi(x_1) = y_1, \varphi(x_2) = y_2$. Then

$$\begin{aligned}\varphi(\mu)(y) &= \inf\{\mu(x) \mid x \in X, \varphi(x) = y\} \\ &\leq \inf\{S(\mu(x * (x_1 * x)), \mu(x_2)) \mid x, x_1, x_2 \in X, \varphi(x) = y, \varphi(x_1) = y_1, \varphi(x_2) = y_2\} \\ &= S(\inf\{\mu(x * (x_1 * x)) \mid x, x_1 \in X, \varphi(x) = y, \varphi(x_1) = y_1\}, \inf\{\mu(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\}) \\ &= S(\inf\{\mu(x * (x_1 * x)) \mid x, x_1 \in X, \varphi(x * (x_1 * x)) = y * (y_1 * y)\}, \inf\{\mu(x_2) \mid x_2 \in X, \varphi(x_2) = y_2\} \\ &= S(\varphi(\mu)(y * (y_1 * y)), \varphi(\mu)(y_2))\end{aligned}$$

therefore

$$\varphi(\mu)(y) \leq S(\varphi(\mu)(y * (y_1 * y)), \varphi(\mu)(y_2)).$$

Therefore $\varphi(\mu) \in AFIIS(Y)$. □

Proposition 4.6. *If $\nu \in AFIIS(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi^{-1}(\nu) \in AFIIS(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \leq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x)$$

As

$$\begin{aligned}\varphi^{-1}(\nu)(x) &= \nu(\varphi(x)) \leq S(\nu(\varphi(x) * (\varphi(x_1) * \varphi(x)), \nu(\varphi(x_2))) \\ &= S(\nu(\varphi(x * (x_1 * x))), \nu(\varphi(x_2))) = S(\varphi^{-1}(\nu)(x * (x_1 * x)), \varphi^{-1}(\nu)(x_2))\end{aligned}$$

so

$$\varphi^{-1}(\nu)(x) \leq S(\varphi^{-1}(\nu)(x * (x_1 * x)), \varphi^{-1}(\nu)(x_2)).$$

Therefore $\varphi^{-1}(\nu) \in AFIIS(X)$. □

Proposition 4.7. *If $\mu \in AFPIIS(X)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi(\mu) \in AFPIIS(Y)$.*

Proof. Let $x \in X$ and $y \in Y$ with $\varphi(x) = y$. Now

$$\varphi(\mu)(0) = \inf\{\mu(0) \mid 0 \in X, \varphi(0) = 0\} \geq \inf\{\mu(x) \mid x \in X, \varphi(x) = y\} = \varphi(\mu)(y)$$

thus

$$\varphi(\mu)(0) \leq \varphi(\mu)(y).$$

Also let $x_1, x_2, x_3 \in X$ such that $\varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(x_3) = y_3$. Then

$$\varphi(\mu)(y_1 * y_3) = \inf\{\mu(x_1 * x_3) \mid x_1, x_3 \in X, \varphi(x_1) = y_1, \varphi(x_3) = y_3\}$$

$$\leq \inf\{S(\mu((x_1 * x_2) * x_3), \mu(x_2 * x_3)) \mid x_1, x_2, x_3 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(x_3) = y_3\}$$
$$= S(\inf\{\mu((x_1 * x_2) * x_3)) \mid x_1, x_2, x_3 \in X, \varphi(x_1) = y_1, \varphi(x_2) = y_2, \varphi(x_3) = y_3\}$$
$$, \inf\{\mu(x_2 * x_3) \mid x_2, x_3 \in X, \varphi(x_2) = y_2, \varphi(x_3) = y_3\})$$
$$= S(\inf\{\mu((x_1 * x_2) * x_3)) \mid x_1, x_2, x_3 \in X, \varphi((x_1 * x_2) * x_3) = (y_1 * y_2) * y_3\}$$
$$, \inf\{\mu(x_2 * x_3) \mid x_2, x_3 \in X, \varphi(x_2 * x_3) = y_2 * y_3\}$$
$$= S(\varphi(\mu)((y_1 * y_2) * y_3), \varphi(\mu_A)(y_2 * y_3))$$

therefore

$$\varphi(\mu)(y_1 * y_3) \leq S(\varphi(\mu)((y_1 * y_2) * y_3), \varphi(\mu)(y_2 * y_3)).$$

Therefore $\varphi(\mu) \in AFPIIS(Y)$. □

Proposition 4.8. *If $\nu \in AFPIIS(Y)$ and $\varphi : X \to Y$ be a homomorphism of BCK-algebras, then $\varphi^{-1}(\nu) \in AFPIIS(X)$.*

Proof. Let $x \in X$. Then

$$\varphi^{-1}(\nu)(0) = \nu(\varphi(0)) \leq \nu(\varphi(x)) = \varphi^{-1}(\nu)(x).$$

Let $x_1, x_2, x_3 \in X$. As

$$\varphi^{-1}(\nu)(x_1 * x_3) = \nu(\varphi(x_1 * x_3)) = \nu(\varphi(x_1) * \varphi(x_3))$$
$$\leq S(\nu((\varphi(x_1) * \varphi(x_2)) * \varphi(x_3)), \nu(\varphi(x_2) * \varphi(x_3)))$$
$$= S(\nu(\varphi(x_1 * x_2) * x_3), \nu(\varphi(x_2 * x_3))) = S(\varphi^{-1}(\nu)((x_1 * x_2) * x_3), \varphi^{-1}(\nu)(x_2 * x_3))$$

so

$$\varphi^{-1}(\nu)(x_1 * x_3) \leq S(\varphi^{-1}(\nu)((x_1 * x_2) * x_3), \varphi^{-1}(\nu)(x_2 * x_3)).$$

Therefore $\varphi^{-1}(\nu) \in AFPIIS(X)$. □

5. Open Problem

In this study, using s-norms, we introduced anti fuzzy subalgebras and anti fuzzy ideals in BCK-algebras. Now One can this work for BCH-Algebras and Q-Algebras and this can be as an open proble

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The currency pairs correlation matrix in Forex

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Abstract

Assume that A is an $n \times n$ real matrix. Then, necessary and sufficient condition for this matrix to be a correlation matrix is that A be symmetric, positive semi-definite and all entries of A are between -1 and 1, with 1 along the main diagonal.

1. Introduction

Problems of relations between finance and Mathematics have been discussed since classical times in a some of sources from Platoís Dialogues. Louis Bachelier who, in the year 1900, published the now famous memoir entitled ìTheorie de la spÈculationî. Bachelierís achievements are remarkable in several respects. Before Einstein and others, he developed the first theory of Brownian motion which he used in order to quantify the evolution of stock prices. In other word, Bachelier assumed that the stock price follows an arithmetic Brownian motion and, consequently, is distributed normally at any given time. He derived the pricing formulas for call and put options on such stocks. so, since Bachelierís theory predicted that stock prices can become negative and because of the sheer complexity of its mathematical apparatus, the theory was neglected by the mainstream economists for more than fifty years. Fundamental contributions to modern relations between finance and Mathematics were made in the 1950's by several authors. Arrow (1953) and Debreu (1959) extended the existing economic models by incorporating uncertainty and showed how to solve the corresponding asset allocation problem. Modigliani and Miller (1958) proved that the financial structure of the firm, i.e., the firmís choice between equity and debt financing, does not affect its value. The method of financial arbitrage they used turned out to be even more useful than the theorem itself and became the method of choice for generations of financial engineers.

A particularly interesting feature of financial markets data is the, not yet fully explored, complex interdependent dynamics of the prices between various assets, which is most often modeled through the empirical correlations. This

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issue has been of specific value to the econophysics community and its members have provided numerous contributions addressing the relationships between stocks, market indexes and currencies.

The currency pairs correlation matrix is one of the new mathematical tools that is widely used in the Forex financial market. The currency pairs correlation matrix shows the statistical correlation existing between any currency pair selected by a trader, on his chosen time-frame. Correlation is a statistical index which tells how a currency pair follows the same movement of another currency pair and how strong this co-movement is. The currency pairs correlation matrix is an useful tool for Forex traders who wish to detect all the possible combinations of trades and their related risks.Correlation is measured on a scale of +100 to -100, or +1.0 percent to -1.0 percent, in a percentage scale. A value of +1.0 percent (perfectly positive correlation) means that two currency pairs move perfectly identically. A value of -1.0 percent (perfectly negative correlation) means the opposite. A value of 0.0 percent (absence of correlation) means that two currency pairs have moved completely independently. When two currency pairs show a strong positive correlation, this means that we don't expect a high trading risk. For example, if the GBPUSD and the CADUSD show a strong correlation, say 0.8 percent, this means that trading both pairs will have the same effects as placing one big single trade, instead of two separate trades. If the correlation is strongly negative, say -0.8 percent, it is like not to have trade at all. Since the currency pairs move in the opposite directions, the profit obtained on one trade is cancelled out by the loss obtained on the other trade. In order to plan a profitable strategy, a trader should always focus on pair combinations showing weak correlation, in order to obtain a basket of independent trades. For further reading in this regard, dear readers, refer to the references [1], [2], [3], [4] and [5].

2. Main Results

In statistical modelling, correlation matrices representing the relationships between variables are categorized into different correlation structures, which are distinguished by factors such as the number of parameters required to estimate them. For example, in an exchangeable correlation matrix, all pairs of variables are modeled as having the same correlation, so all non-diagonal elements of the matrix are equal to each other. On the other hand, an autoregressive matrix is often used when variables represent a time series, since correlations are likely to be greater when measurements are closer in time. Other examples include independent, unstructured, M-dependent, and Toeplitz. The correlation matrix of n random variables $X_1, \ldots, X_n$ is the $n \times n$ matrix whose (i, j) entry is corr(X_i, X_j), when corr$(X_i, X_j) = \frac{Cov(X_i,X_j)}{\sigma(X_i)\sigma(X_j)}$. Thus the diagonal entries are all identically unity. If the measures of correlation used are product-moment coefficients, the correlation matrix is the same as the covariance matrix of the standardized random variables $X_i/\sigma(X_i)$ for $i = 1, \ldots, n$. This applies both to the matrix of population correlations (in which case σ is the population standard deviation, and to the matrix of sample correlations (in which case σ denotes the sample standard deviation. Consequently, each is necessarily a positive semi-definite matrix. Moreover, the correlation matrix is strictly positive definite if no variable can have all its values exactly generated as a linear function of the values of the others. The correlation matrix is symmetric because the correlation between X_i and X_j is the same as the correlation between X_j and X_i. Based on what has been said so far, we will continue to obtain information about this matrix using mathematical tools.We try to give a new proof for one of the theorems on correlation matrices.

Theorem. Assume that A is an $n \times n$ real matrix. Then, necessary and sufficient condition for this matrix to be a correlation matrix is that A be symmetric, positive semi-definite and all entries of A are between -1 and 1, with 1 along the main diagonal.

Proof. First, assume that A is an $n \times n$ correlation matrix. It is clear that A be symmetric and all entries of $A = [a_{ij}]$ are between -1 and 1 and for all i we have $a_{ii} = 1$. We must prove that A positive semi-definite. It is enough to prove that $\sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j a_{ij} \geq 0$, for any arbitrary real number b_i, b_j. Since A is a correlation matrix, there exist n random variables $X_1, \ldots, X_n$ such that $a_{ij} = $ corr$(X_i, X_j) = \frac{Cov(X_i,X_j)}{\sigma(X_i)\sigma(X_j)}$. Therefore, $\sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j a_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i b_j \frac{Cov(X_i,X_j)}{\sigma(X_i)\sigma(X_j)} = Var(\sum_{i=1}^{n} b_i \frac{X_i}{\sigma(X_i)})$.This phrase is always non-negative. Therefore A positive semi-definite. Now, assume that A be symmetric, positive semi-definite and all entries of A are between -1 and 1. we prove that A is a correlation matrix. We must find n random variables $X_1, \ldots, X_n$ such that $a_{ij} = $ corr$(X_i, X_j) =$

$\frac{Cov(X_i,X_j)}{\sigma(X_i)\sigma(X_j)}$. A real matrix A is symmetric if and only if A can be diagonalized by an orthogonal matrix. Without loss of generality, we may assume that A is a diagonal matrix with non-negative eigenvalues $\lambda_1, \cdots, \lambda_n$. Consider that $Y_1, \cdots, Y_n$ are n arbitrary independent random variable. Set $X_i = \sqrt{\lambda_i} Y_i$, and so by a simple calculation, we obtain the conclusion.

The purpose of this article is to get acquainted with the currency pairs correlation matrix. In the Forex market, the relationship between different currencies is very important. Sometimes minor changes in one cause changes in other world currencies. Especially today with the advent of digital currencies that cause huge price shocks in financial markets, finding linear relationships between such currencies is very important. In fact, the currency pairs correlation matrix allows us to do this. A very important point that comes to mind here is that In appearance, the type of shocks in digital currencies such as Bitcoin is not at all comparable to currency fluctuations such as Dollars. Certainly, one should not expect at first glance that the currency pairs correlation matrix will have a favorable result on such currency pairs. The way it seems here is to neutralize the kind of inflationary effect of such currencies first. In fact, by changing the graph size of a currency using moving average curves, one can somehow approach this goal.
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On necessary and sufficient conditions to have a contraction C_0-semigroup

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Abstract

This work is devoted to a study of strongly continuous semigroups or equivalently C_0-semigroups and their associated operator known as infinitesimal generator. Here, our attention is focused on contraction semigroups i.e. those semigroups whose norms are less than or equal to 1. We are going to prove a theorem which gives necessary and sufficient conditions for an arbitrary semigroup to be contranction.

1. Introduction

Control theory is a significant branch of systems theory (Brogan 1990) that has attracted considerable attention in recent years. In order to put control theory into practice, one can consider the mathematical model of the phenomenon under study. Mathematical modeling is a tool that puts control theory into practice. By incorporating basic physical laws and considering interconnections and interactions between systems components a mathematically analytical model achieves which is in the form of a partial or ordinary differential equation. Those phenomena such as electrodynamics, chemical processes, diffusion, etc. that are modeled by partial differential equations are called infinite-dimensional systems and other systems modeled by ordinary differential equations are called finite-dimensional systems. Most of the studies on control theory problems have considered a finite dimensional control model which is represented by a first order differential equation known as state space form (see (Dumont et al. 2016; Jun et al. 2015; Isfahani et al. 2017; Jahromi et al. 2017)) and the references there in). In order to develop the achieved methods and results to infinite dimensional systems, incorporating C0-semigroups and their associated operators is useful. To see a few recent works that on control systems that incorporated C0-semigroups, one can refer to (Baudouin et al. 2018; Jacob et al. 2017; Yang 2018; Raposo et al. 2019).
A strongly continious semigroup or equvalently a C_0-semigroup, is a generalization of exponential function that provids solutions of linear constant coefficient ordinary differential equations in Banach spaces [4]. Recently, the theory of strongly continuous semigroups has been extensively developed by many scientists and achieved results are incorporated in many branches of science and sometimes can facilitate analysis of physical engineering systems; for

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example, in the theory of control systems one faces both ”finite dimensional system” (whose mathematical model is in the form of an ordinary differential equation) and ”infinite dimensional system” (whose mathematical model is in the form of a partial differential equation) while most of the studies are focused on finite dimensional systems[1]. In this case, C_0-semigroups are good tools for developing achieved methods and results to infinite dimensional systems. In this paper, after introducing some necessary concepts and preliminaries, we will prove a theorem that provides necessary and sufficient conditions for a C_0-semigroup to be contraction.

2. Main Results

Throughout this paper, we denote by X a real Hilbert space with inner product $< .,. >_X$ and norm $||.||_X = \sqrt{< .,. >_X}$.

Definition 2.1. ([2]) In a Hilbert space X, $(T(t))_{t\geq 0}$ is called a strongly continuous semigroup or C_0-semigroup if the following holds:

- For all $t \geq 0$, $T(t)$ is a bounded linear operator on X, i.e., $T(t) \in \mathcal{L}(X)$;
- $T(0) = I$;
- $T(t+\tau) = T(t)T(\tau)$ for all $t, \tau \geq 0$;
- For all $x_0 \in X$, we have that $\|T(t)x_0 - x_0\|_X$ converges to zero, when $t \downarrow 0$, i.e., $t \mapsto T(t)$ is strongly continuous at zero.

As an example of strongly continuous semigroup, one can consider the exponential of a matrix which is the easiest example of this concept.
We can associate an operator A to every C_0-semigroup $(T(t))_{t\geqslant 0}$ as follows:

Definition 2.2. ([2]) Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on the Hilbert space X, if the following limit exists:

$$\lim_{t\to 0} \frac{T(t)x_0 - x_0}{t}$$

then we say that x_0 is an element of the domain of A, i.e. $x_0 \in D(A)$, and we define Ax_0 as:

$$Ax_0 = \lim_{t\to 0} \frac{T(t)x_0 - x_0}{t}$$

we call A the infinitesimal generator of the strongly continuous semigroup $(T(t))_{t\geq 0}$.
It can be seen that for every $x_0 \in D(A)$ the function $t \vdash T(t)x_0$ is differentiable. Actually we have the following lemma:

Lemma 2.3. *([3]) Let $(T(t))_{\geqslant t_0}$ be a strongly continuous semigroup on a Hilbert space X with infinitesimal generator A. Then the following results hold:*

- *For $x_0 \in D(A)$ and $t \geqslant 0$ we have $T(t)x_0 \in D(A)$;*
- *$\frac{d}{dt}(T(t)x_0) = AT(t)x_0 = T(t)Ax_0$ for $x_0 \in D(A), t \geqslant 0$;*
- *$\frac{d^n}{dt^n}(T(t)x_0) = A^n T(t)x_0 = T(t)A^n x_0$ for $x_0 \in D(A), t \geqslant 0$.*

Lemma 2.4. *([4]) Suppose that $(T(t))_{t\geqslant 0}$ be a strongly continuous semigroup on the Hilbert space X and let $\omega_0 = inf(\frac{1}{t}log \parallel T(t) \parallel)$. Then for every $\omega > \omega_0$, there exists a constant M_ω such that for every $t \geqslant 0$ we have $\parallel T(t) \parallel \leqslant M_\omega e^\omega$.*

If the upper bound of $\parallel T \parallel$ in the above lemma equals to 1, then the smigroup is called a contraction semigroup:

Definition 2.5. ([3]) Let $(T(t))_{\geqslant t_0}$ be a C_0-semigroup on a Hilbert space $X.(T(t))_{\geqslant t_0}$ is called contraction semigroup if $\| T(t) \| \leqslant 1$ for every $t \geqslant 0$.

Now everything is ready for proposing the main theorem of this paper:

Theorem 2.6. *Let A be the infinitesimal generator of the C_0-semigroup $(T(t))_{t\geqslant 0}$ on the Hilbert space X, then $(T(t))_{t\geqslant 0}$ is a contraction semigroup if and only if:*

$$< Ax, x > + < x, Ax > \leqslant 0$$

for every $x \in D(A)$.

Proof. Let $\Phi(t) = \| T(t)x \|^2$ for every $x \in D(A)$. By differentiating we have:

$$\dot{\Phi}(t) = \frac{d}{dt} \| T(t)x \|^2 = \frac{d}{dt} < T(t)x, T(t)x > = < AT(t)x, T(t)x > + < T(t)x, AT(t)x > \quad (*)$$

Now suppose that $(T(t))_{t\geqslant 0}$ is a contraction semigroup. Then $\Phi(t) \leqslant \Phi(0)$ implies $\dot{\Phi}(0) = 0$ for every $t \geqslant 0$. By choosing $t = 0$ and substituting in $(*)$ we have:

$$< Ax, x > + < x, Ax > \leqslant 0, \quad \forall x \in D(A)$$

conversely, suppose that:

$$< Ax, x > + < x, Ax > \leqslant 0, \quad \forall x \in D(A)$$

then, as $T(t)$ is a mapping from $D(A)$ into $D(A)$ we can substitute x by $T(t)x$. Due to $(*)$ we have $\dot{\Phi} \leqslant 0$ for every $t \geqslant 0$. So, $t \longmapsto \Phi(t)$ is nonincreasing which implies $\Phi(t) \leqslant \Phi(0)$ and results $\| T(t)x \| \leqslant \| x \|$, for every $t \geqslant 0$, which means that $(T(t))_{t\geqslant 0}$ is a contraction semigroup.

□

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Fibonacci retracement and biological population growth

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Abstract

Today, the basics of technical analysis are widely used to study trends that have oscillating movements. For example, economics, financial markets, etc. are examples of these trends. One of the most important and widely used tools in technical analysis is Fibonacci ratios. In fact, Fibonacci ratios are used to determine the support and resistance points in predicting the future trend of a chart. Due to the very good behaviors that have been observed from these ratios in predicting the trend so far, we expect their application to be seen in natural and biological processes as well. In this article, we examined the birth trend in the last one hundred years in Iran. We tried to find the maximum and minimum points of birth in this process. We also tried to examine how Fibonacci ratios work on the birth rate in the last hundred years in Iran. To examine this, three main time periods were examined. 1320 to 1359, 1359 to 1379 and 1379 to 1394. Eventually as expected, It was observed that Fibonacci ratios had a significant effect on determining the points of support and resistance in the birth chart recorded in the last one hundred years in Iran.

1. Introduction

Population growth is also related to the Fibonacci series. In 1202, Leonardo Fibonacci investigated the question of how fast rabbits could breed under ideal circumstances. Here is the question that he posed: Suppose a newborn pair of rabbits, one male and one female, is put in the wild. The rabbits mate at the age of one month. At the end of its second month, a female can produce another pair of rabbits. Suppose that the rabbits never die and that each female always produces one new pair, with one male and one female, every month from the second month on. How many pairs will there be in one year?
Fibonacci (c. 1170 – c. 1240–50) was an Italian mathematician from the Republic of Pisa, considered to be "the most talented Western mathematician of the Middle Ages". Fibonacci popularized the Hindu–Arabic numeral system in the Western world primarily through his composition in 1202 of Liber Abaci (Book of Calculation). He also introduced Europe to the sequence of Fibonacci numbers, which he used as an example in Liber Abaci. In mathematics, the Fibonacci numbers form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1.
The beginning of the sequence is thus $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$.

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In 1969, Parberry posed and solved an interesting problem in population growth analogous to the rabbit problem considered by Fibonacci. Liber Abaci posed and solved a problem involving the growth of a population of rabbits based on idealized assumptions. The solution, generation by generation, was a sequence of numbers later known as Fibonacci numbers. Although Fibonacci's Liber Abaci contains the earliest known description of the sequence outside of India, the sequence had been described by Indian mathematicians as early as the sixth century.
For more result, see [1], [2]), [3], [4]), [5] and [6] .
N. Rivier and his collaborators ([10]) modelized natural phyllotaxis by the tiling by Voronoi cells of spiral lattices formed by points placed regularly on a generative spiral. Locally, neighboring cells are organized as three whorls or parastichies, labeled with successive Fibonacci numbers. The structure is encoded as the sequence of the shapes (number of sides) of the successive Voronoi cells on the generative spiral. Fibonacci spiral patterns were produced artificially ([4,5]) by manipulating the stress on inorganic microstructures made of a silver core and a silicon dioxide shell. It was found that an elastically mismatched bi-layer structure may cause stress patterns that give rise to Fibonacci spirals. Results suggest that plant patterns might be modeled by mutually repulsive entities for both spherical and conical surfaces. It is conjectured that Fibonacci spirals are the least energy configuration on conical supports.

2. Main Results

Fibonacci numbers can be used to study models of population growth that always increase over time. In this model, we start with a pair of rabbits. If we pay attention, the human population also begins with a pair of people created by Alloh SWT, namely Adam and Eve. Both of them also always gave birth to a pair of twins, male and female. However, the process of increasing human population is not the same as the process of increasing Fibonacci rabbits. The last descendants of Adam and Eve were single, not twin. From these rules, a pair of rabbits will have offspring in the third time unit. In the fourth time unit, the pair has 2 pairs of offspring. Thus, in units of time , the number of pairs of offspring is $2 - i$. The numbers on Fibonacci sequences indicate the number of rabbit pairs with a pair of rabbits meaning male and female. It is assumed at the beginning of the experiment (the first month) we have a pair of rabbits who are still babies. In the second month, the couple grew to become juvenils. In the third month, the rabbit has a pair of children (male and female). Each pair of rabbits will grow in the same pattern, namely infants, juvenils and adults. Each phase is passed in the same time interval, for example one month. By the time the couple becomes adults, each month will give birth to a pair of children. That is, a pair of rabbits that have entered the adult phase will give birth continuously every month. So, the baby bunny pair will give birth to a pair of children in the third, fourth, fifth month and so on. Thus, we have a population of Fibonacci rabbits consisting of three age groups according to the phases of rabbit growth, namely babies, juveniles and adults. The Fibonacci sequence, and its "quantum" extension, can be found in genetic codes, including amino acids and codons ([9]). Deoxyribonucleic acid (DNA) in biological systems replicates with the aid of proteins. However, Kim et al. ([8]) have designed a controllable self-replicating system that does not require proteins. The self-assembly process into rings continues through two different replication pathways: one grows exponentially, the other grows according to Fibonacci's sequence.
Here we must mention an important point. Today, many charts are constantly growing and declining have a very complex relationship with Fibonacci numbers. In other words, the growth and decline of these graphs follow the Fibonacci ratios. Fibonacci ratios somehow determine the next step of a chart. There are many examples of this in economics, biology, financial markets, and so on.
The Fibonacci "ratios" are 23.6, 38.2, 50, 61.8, and 100. These ratios show the mathematical relationship between the number sequences and are important to check the future trend of a chart. For reasons that remain a mystery, Fibonacci ratios often display the points at which a trend of a chart reverses its current position. Today, most of the trends that deal with growth and decline are closely related to Fibonacci inclusions. In a movement of growth and decline, you are always dealing with peaks and valleys. In fact, Fibonacci ratios are used to approximate these points. They can somehow limit the process of a chart. Population growth is one of the issues is always associated with growth and decline due to various factors. In fact, the nature of population issues has a structure of the problem of hunting and hunter. This means that you are not always dealing with growth alone when considering the population of a species. Various factors in nature will sometimes lead to a decrease in even your sample population. The chart below shows the number of people infected with the corona virus in several different countries over a period of time. As shown in the figure, due to various factors, the number of patients has never had a constant trend. Now the question is that: Is it possible to use Fibonacci ratios that actually act as support and resistance lines on a chart to predict the future of a

demographic trend.? For more result, see [7, 11, 12].

Here we intend to examine the birth rate chart in Iran from 1338 to 1398. The statistics in the following tables are taken from the sites of official institutions. It is necessary to mention one point here. The official statistics on the birth rate are up to 1338. That means we do not have official statistics for that before. Maximum and minimum numbers are very important in examining Fibonacci ratios on a trend. It seems that the minimum birth rate in Iran is related to before 1338. This will definitely cause some confusion in our calculations. Because moving from one minimum point to the next maximum point will practically be the main criterion for the next calculations. However, even if you set a minimum local point in your process, again, calculations will be largely reliable.

Here is an important point. To use Fibonacci ratios, a chart must have accurate access to the maximum and minimum points. As can be seen from Table 1, in 1359 and 1394, the number of people born in Iran is at a maximum. Also in 1379, we have a minimum point in the chart process. But we must also be able to get the minimum number of people born before 1338. There are no official statistics on this. But it seems that the number of people born in 1338 is not a minimum. To find the minimum number of people born before 1338, we try to refer to historical evidence. In the last century, the outbreak of World War II has had many negative effects on the economic situation of the Iranian people. The living conditions of a family as well as the economic situation as well as health and treatment are very important issues that have a great impact on the birth process. Between 1320 and 1324, Iran was involved in the aftermath of World War II. Over these years, Iran has been plagued by famine, high inflation, and military strikes by foreign forces. Therefore, most likely, these years can be a reference for measuring the minimum point of birth rate. Certainly, given the military fires in Iran, one should not look for official birth census statistics. Therefore, in terms of time period, we consider the years 1320 to 1324 as the period of the lowest birth. Even if we make a mistake in this approximation, the mistake does notseems to make a big mistake in terms of numerical value. Life expectancy among Iranians in recent years has been between 72 and 75 years. Now it is enough to pay attention to the death statistics in the last ten years. According to official statistics, from 1390 to 1395, the death rate was almost constant at an average of 300,000. That is, we can consider the lowest number of births in the last century to be about the same number. Therefore, we consider the number of people born before 1338 to be approximately 300,000. On the other hand, the maximum number of births between 1320 and 1359 is related to 1359. Approximately 2,450,000 people have been born this year. The difference between the two numbers is approximately 2,150,000. On the other hand, in 1379, when the lowest birth rate occurred in the years 1359 to 1379, the birth rate is approximately equal to 1100,000 people. As a result, the rate of birth change between 1359 and 1379 is approximately equal to 1350000 people. According to the rules known in technical analysis for Fibonacci ratios, there is always a major correction in an uptrend. One of the important ratios in a corrective move is 0.618.
$2150000 \times 0.618 = 1330000$ Therefore, the upward movement of birth in the years 1320 to 1359, has a corrective trend with the main ratio of Fibonacci (0.618). This downward movement occurred between 1359 and 1379. Now, if we consider the period of 1359 to 1379 as the main trend, the rate of birth change in these years has been about 1350000. After a downward movement between the years 1359 to 1379, in the years 1379 to 1394 we are facing an upward movement. The rate of birth change in the years 1379 to 1394 is approximately equal to 490,000 people.
$1350000 \times 0.38 = 513000.$
Therefore, the birth trend in the years 1379 to 1394 with a ratio of 0.38, which is one of the main Fibo ratios, can be considered as a correction of the downward trend from 1359 to 1379.
Now a question is formed in the mind to conclude. If we consider the years 1379 to 1394 as an upward trend in birth, how many corrections will this move make? In fact, the minimum birth rate in recent years can be what in a year. The birth rate change between 1379 and 1394 is about 490000 people. Given that the main Fibonacci correction ratios are usually one of the numbers 0.38, 0.5 and 0.618, the following situations will occur.
$490000 \times 0.38 = 186000.$ $490000 \times 0.50 = 245000.$ $490000 \times 0.618 = 303000.$
But according to official statistics, the birth rate in 1399 was approximately equal to 1,100,000 people. Therefore, in terms of technical analysis, all three Fibonacci ratios are broken as trend support numbers. nd now we are at the 100 Fibo support point, which is usually one of the most important support factors in Fibonacci ratios. It seems to be a little hard to break. In this article, we tried to show how the basics of technical analysis, which are widely used in the world of economics and financial markets today, can affect biological phenomena and nature.

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on matrices whose elements are integers with given determinant

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Abstract

For matrices with large positive integer elements with a small determinant is an interesting question in a linear algebra course. In this paper, we investigate matrices of order 3 with large positive integer elements and having a small determinant.

1. Introduction

In this paper, we consider a kind of finding conditional matrices, under certain given conditions. Obtaining such conditional matrices seem to be useful in specific problems when we wish to get expected results.
A linear Diophantine equation (in three variables) is an equation of the general form $ax + by + cz = d$ where a, b, c are given integers and x, y, z are unknown integers. This Diophantine equation has a solution (where x , y, and z are integers) if and only if d is a multiple of the greatest common divisor of a, b, and c (refer to[1] for more detail).
In [2], the author explains the method for finding an infinite family of matrices with a large positive integer of order 2 with having a small determinant. In this paper, we investigate matrices of order 3 with large positive integer elements and having a small determinant. typical example is

$$A = \begin{pmatrix} 6103159677781 & 6103159677782 & 19785404578332 \\ 6103159677782 & 6103159677783 & 19785404578687 \\ 12206319355563 & 12206319355563 & 21864070901909 41 \end{pmatrix}, \quad \det(A) = 1.$$

Theorem 1.1. *Given positive integers d and M, there exists many infinitely matrices $A = [a_{ij}]_{1\le i,j\le 3}$ with integer elements satisfying $a_{ij} \ge M$ and $\det(A) = d$.*

2. Proof of Theorem 1.1

Assume that

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

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Now, assume that

$$a_{11} = M,\ a_{12} = M+1,\ a_{21} = M+1,\ a_{22} = M+2,\ a_{31} = 2M+1 \text{ and } a_{32} = 2M+2.$$

Also, suppose that $a_{13} = x$, $a_{23} = y$ and $a_{33} = z$. By the above assumption $\det(A) = -Mx + (M+1)y - z$. Then we have the following linear Diophantine equation in three variables

$$-Mx + (M+1)y - z = d. \tag{1}$$

Since gcd $(\mathrm{M}, \mathrm{M}+1, 1) = 1$, then equation (1) has infinitely many solutions, and anyone's solution can be used to generate all the other ones. we can solve equation (1) by reducing it to a tow variable equation. Let

$$(M+1)y - z = W.$$

By (1) we have the following linear Diophantine equation in two variables

$$-Mx + W = d \tag{2}$$
$$(M+1)y - z = W \tag{3}$$

We observe that $x_0 = 0$, $W_0 = d$ is a particular integer solution of (2). Then all integer solutions, of the equation (2), are of the form

$$x = t, \qquad W = d + Mt, \qquad t \in \mathbb{Z}. \tag{4}$$

From (3) and (4) we have

$$(M+1)y - z = d + Mt.$$

Since gcd $(\mathrm{M}+1, 1) = 1$, then this equation has infinitely solutions for all integers M. It is easy to check that $y_0 = t$ and $z_0 = t - d$ is a particular integer solution and so all integer solutions to the equation are of the form

$$y = t - k, \qquad z = t - d - (M+1)k, \qquad t, k \in \mathbb{Z}.$$

Finally, we obtain

$$x = t, \quad y = t - k, \quad z = t - d - (M+1)k, \quad t, k \in \mathbb{Z}$$

as a general solution of (1). Let t be a fixed arbitrary integer number. For obtaining conditional matrices of the theorem, It sufficient $k < \lfloor \frac{t-d-M}{M+1} \rfloor$.

Remark 2.1. *Let p, q and r be different prime numbers greater than M. Now, Assume that*

$$a_{11} = p,\ a_{12} = p,\ a_{21} = q,\ a_{22} = q+1,\ a_{31} = r \text{ and } a_{32} = r.$$

Also, suppose that $a_{13} = x$, $a_{23} = y$ and $a_{33} = z$. Then, the equality $\det(A) = d$ reads as the following Diophantine equation:

$$pz - rx = d \tag{5}$$

Since $\gcd(p, r) = 1$, to make sure that existing many infinite solutions for the related Diophantine equation (5). Therefore, the problem can be investigated with several Diophantine equations.

Example 2.2. *Consider three prime numbers $15485863, 32452843$, and 49979687 greater than 15000000. Let*

$$a_{11} = a_{12} = 15485863$$
$$a_{21} = a_{22} - 1 = 32452843$$
$$a_{31} = a_{32} = 49979687$$

By Remark 2.1, we obtain

$$15485863z - 49979687x = 2 \quad \text{with} \quad M = 15000000 \quad \text{and} \quad d = 2.$$

It is easy to check that

$$x_0 = 13228138 \quad \text{and} \quad z_0 = 42693016$$

is a particular integer solution and so all integer solutions to the equation are of the form

$$x = 13228138 + 15485863t, \qquad y = k, \qquad z = 42693016 + 49979687t, \qquad t, k \in \mathbb{Z}.$$

We take $k = 160481183$*, which is a prime number, and* $t = 1$ *then*

$$A = \begin{pmatrix} 15485863 & 15485863 & 28714001 \\ 32452843 & 32452844 & 160481183 \\ 49979687 & 49979687 & 92672703 \end{pmatrix}, \quad \det(A) = 2.$$

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The properties of subsemicovering map

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Abstract

In this paper, by reviewing the concept of semicovering map and subsemicovering map, we present the properties of subsemicovering map. Also we define strong unique path lifting property and we prove every subsemicovering map has strong unique path lifting property.

1. Introduction

Assume that X and $\tilde{X}$ are topological spaces and that $p : \tilde{X} \to X$ is a continuous map. Let $f : (Y, y_0) \to (X, x_0)$ be a continuous map and let $\tilde{x}_0 \in p^{-1}(x_0)$. If there exists a continuous map $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a *lifting* of f. The map p has *path lifting property* (PLP for short) if for every path f in X, there exists a lifting $\tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x}_0)$ of f. Also, the map p has *unique path lifting property* (UPLP for short) if for every path f in X, there is at most one lifting $\tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x}_0)$ of f (see [5]).
Brazas [1, Definition 3.1] generalized the concept of covering map by the phrase *"A semicovering map is a local homeomorphism with continuous lifting of paths and homotopies"*. Note that a map $p : Y \to X$ has *continuous lifting of paths* if $\rho_p : (\rho Y)_y \to (\rho X)_{p(y)}$ defined by $\rho_p(\alpha) = p \circ \alpha$ is a homeomorphism, for all $y \in Y$, where $(\rho Y)_y = \{\alpha : I = [0, 1] \to Y | \alpha(0) = y\}$. Also, a map $p : Y \to X$ *has continuous lifting of homotopies* if $\Phi_p : (\Phi Y)_y \to (\Phi X)_{p(y)}$ defined by $\Phi_p(\phi) = p \circ \phi$ is a homeomorphism, for all $y \in Y$, where elements of $(\Phi Y)_y$ are endpoint preserving homotopies of paths starting at y. He also simplified the definition of semicovering maps by showing that having continuous lifting of paths implies having continuous lifting of homotopies (see [2, Remark 2.5]).

Definition 1.1. ([5]). Let $\widetilde{X}$ and X be topological spaces and let $p : \widetilde{X} \to X$ be continuous. An open set U in X is **evenly covered** by p if $p^{-1}(U)$ is a disjoint union of open sets S_i in $\widetilde{X}$, called **sheets**, with $p|_{S_i} : S_i \to U$ a homeomorphism for every i.

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Definition 1.2. ([5]). If X is a topological space, then an ordered pair $(\widetilde{X}, p)$ is a **covering space** of X if:

1. $\widetilde{X}$ is a path connected topological space;
2. $p : \widetilde{X} \to X$ is continuous;
3. each $x \in X$ has an open neighborhood $U = U_x$ that is evenly covered by p.

The following theorem can be concluded from [4, Theorem 2.4].

Theorem 1.3. *A map $p : \tilde{X} \to X$ is a semicovering map if and only if it is a local homeomorphism with UPLP and PLP.*

Lemma 1.4. *(see [4, Lemma 3.2]).*
Let $p : \tilde{X} \to X$ be a local homeomorphism with UPLP, let f be an arbitrary path in X and let $\tilde{x}_0 \in p^{-1}(f(0))$ such that there is no lifting of f starting at $\tilde{x}_0$. Then, using notation of the previous lemma, there exists a unique continuous map $\tilde{f}_\alpha : A_f = [0, \alpha) \to \tilde{X}$ such that $p \circ \tilde{f}_\alpha = f|_{[0,\alpha)}$. We call $\tilde{f}_\alpha$ the incomplete lifting of f by p starting at $\tilde{x}_0$.

2. Main results

Let $p : \tilde{X} \to X$ be a local homeomorphism. We are interested in finding some conditions on p or $\tilde{X}$ under which the map p can be extended to a semicovering map $q : \tilde{Y} \to X$. We recall that Steinberg [6, Section 4.2] defined a map $p : \tilde{X} \to X$ of locally path connected and semilocally simply connected spaces as a *subcovering map (and $\tilde{X}$ a subcover)* if there exist a covering map $p' : \tilde{Y} \to X$ and a topological embedding $i : \tilde{X} \to \tilde{Y}$ such that $p' \circ i = p$. We are going to extend this definition as follows.

Definition 2.1. Let $p : \tilde{X} \to X$ be a local homeomorphism. We say that *p can be extended to a local homeomorphism $q : \tilde{Y} \to X$*, if there exists an embedding map $\varphi : \tilde{X} \hookrightarrow \tilde{Y}$ such that $q \circ \varphi = p$. In particular, if q is a covering map, then p is called a *subcovering map* (see [6, Section 4.2]) and if q is a semicovering map, then we call the map p a *subsemicovering map*. Moreover, if $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$, then we call the map p *full subcovering* and *full subsemicovering*, respectively.

Note that since every covering map is a semicovering map, every subcovering map is a subsemicovering map. Also, if $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ can be extended to $q : (\tilde{Y}, \tilde{y}_0) \to (X, x_0)$ via $\varphi : (\tilde{X}, \tilde{x}_0) \to (\tilde{Y}, \tilde{y}_0)$, then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a subgroup of $q_*(\pi_1(\tilde{Y}, \tilde{y}_0))$.
The following example shows that there exists a subsemicovering map that is not a full subsemicovering map.

Example 2.2. *Let $X = \bigcup_{n\in\mathbb{N}}\{(x, y) \in \mathbb{R}^2 | (x-\frac{1}{n})^2+y^2 = \frac{1}{n^2}\}$ be the Hawaiian Earring space. Brazas [1, Example 3.8] introduced a connected semicovering $p : \tilde{X} \to X$ with discrete fibers, which is not a covering map. Put $\hat{X} = p^{-1}(X \setminus ((0, 1]) \times \{0\})$, then $\hat{X}$ is path connected. It is easy to see that every loop in $\hat{X}$ is null homotopic. Also, $q = p|_{\hat{X}} : \hat{X} \to X$ is a local homeomorphism with $q_*(\pi_1(\hat{X}, \hat{x}_0)) = \{1\} \leq \pi_1(X)$. Calcut and McCarthy [3, Theorem 1] proved that for a connected and locally path connected space X, semilocally simply connectedness of X is equivalent to openness of the trivial subgroup in $\pi_1^{qtop}(X)$. Hence the trivial subgroup is not open in $\pi_1^{qtop}(\mathbb{HE})$ since $\mathbb{HE}$ is not semilocally simply connected at the point $(0, 0)$. This implies that $q_*(\pi_1(\hat{X}, \hat{x}_0))$ is not open in $\pi_1^{qtop}(\mathbb{HE})$. Since q can be extended to the semicovering map p, q is a subsemicovering map. Note that q is not a full subsemicovering map since otherwise there exists a semicovering map $r : \tilde{Y} \to \mathbb{HE}$ such that $r_*(\pi_1(\tilde{Y}, \tilde{y}_0)) = q_*(\pi_1(\hat{X}, \hat{x}_0))$. $r_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ is open in $\pi_1^{qtop}(\mathbb{HE})$ but $q_*(\pi_1(\hat{X}, \hat{x}_0))$ is not open in $\pi_1^{qtop}(\mathbb{HE})$, which is a contradiction.*

In the following, we define a strong version of the unique path lifting property in order to find a necessary condition for a local homeomorphism to be subsemicovering.

Definition 2.3. Let $p : \tilde{X} \to X$ be a local homeomorphism, let $f : [0, \alpha) \to X$ be an arbitrary continuous map, and let $\tilde{f} : [0, \alpha) \to \tilde{X}$ be the incomplete lifting of f defined in Lemma 1.4 with starting point $\tilde{x}_0 \in p^{-1}(f(0))$. Then, we say that p has the *strong unique path lifting property* (strong UPLP for short) if there exist $\varepsilon_{(f,\tilde{x}_0)} > 0$ and an open set $U_{(f,\tilde{x}_0)} \subseteq \tilde{X}$ such that $\tilde{f}(\alpha - \varepsilon_{(f,\tilde{x}_0)}, \alpha) \subseteq U_{(f,\tilde{x}_0)}$ and $p|_{U_{(f,\tilde{x}_0)}} : U_{(f,\tilde{x}_0)} \to p(U_{(f,\tilde{x}_0)})$ is one-to-one. Note that $p|_{U_{(f,\tilde{x}_0)}}$ is a homeomorphism since it is open. We call $U_{(f,\tilde{x}_0)}$ a strong neighborhood.

In the following lemma, we show that every local homeomorphism with strong UPLP has UPLP.

Lemma 2.4. *If a local homeomorphism has strong UPLP, then it has UPLP.*

There exists a local homeomorphism with UPLP that does not have strong UPLP.

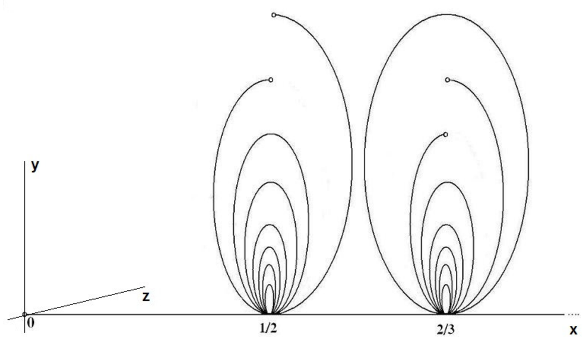

Fig. 1. $\tilde{X}$

Example 2.5. *Let* $X = \mathbb{HE} = \bigcup_{n\in\mathbb{N}}\{(x,y) \in \mathbb{R}^2|(x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ *be the Hawaiian Earring space. Put* $W_i = \bigcup_{n\in\{\mathbb{N}\setminus\{i,i+1\}\}}\{(y,z) \in \mathbb{R}^2|(y-\frac{1}{n})^2 + z^2 = \frac{1}{n^2}\}$ *and*

$$\begin{aligned} S_i =& \{(y,z)|(y-(1-\frac{1}{i}))^2 + z^2 = (\frac{1}{i})^2, z > 0\} \\ &\bigcup\{(y,z)|(y-(1-\frac{1}{i+1}))^2 + z^2 = (\frac{1}{(i+1)})^2, z < 0\} \end{aligned}$$

for every $i \in \mathbb{N}$*. Let* $\tilde{X} = ((0,1)\times\{0\}\times\{0\})\bigcup_{i=1}^{\infty}(\{1-\frac{1}{i+1}\}\times(W_i\cup S_i))$ *be a subset of* $\mathbb{R}^3$ *(see Figure 1). We define* $p : \tilde{X} \to X$ *by*

$$p(x,y,z) = \begin{cases} (y,z), & x = 1-\frac{1}{i+1},\ i\in\mathbb{N}, \\ \\ \frac{1}{i}(1+\cos(\frac{2\pi}{1-x}), \sin(\frac{2\pi}{1-x})), & 1-\frac{1}{i} < x < 1-\frac{1}{i+1},\ i \in \mathbb{N}. \end{cases}$$

It is routine to check that p *is a local homeomorphism that has UPLP. Let* $\alpha : I \to X$ *be a loop defined by*

$$\alpha(t) = \begin{cases} (0,0), & t \in [0,\frac{1}{2}]\cup\{1\}, \\ \frac{1}{i}(1+\cos(\frac{2\pi}{1-t}), \sin(\frac{2\pi}{1-t})), & 1-\frac{1}{i} \le t \le 1-\frac{1}{i+1},\ i\in\mathbb{N}\setminus\{1\}. \end{cases}$$

The loop α *has no lifting with starting point* $(\frac{1}{2},0,0)$ *and the incomplete lifting of* α *with starting point* $(\frac{1}{2},0,0)$ *is* $\tilde{\alpha} : [0,1) \to \tilde{X}$ *defined by*

$$\tilde{\alpha}(t) = \begin{cases} (\frac{1}{2},0,0), & t\in[0,\frac{1}{2}], \\ (t,0,0), & t \in [\frac{1}{2},1). \end{cases}$$

Thus $\tilde{\alpha}$ *does not have any strong neighborhood. Therefore* p *does not have strong UPLP.*

In the following theorem, we show that the strong UPLP is a necessary condition for a local homeomorphism to be a subsemicovering map.

Theorem 2.6. *If p is a subsemicovering map, then p has strong UPLP.*

Proof. If $p : \tilde{X} \to X$ is a semicovering map, then it is easy to check that p has strong UPLP. Suppose that p is subsemicovering which is not a semicovering map. So there exists a semicovering map $q : \tilde{Y} \to X$ with an embedding map $\varphi : \tilde{X} \to \tilde{Y}$ such that $q \circ \varphi = p$. Since p is not semicovering, there exists a path f in X with no lifting. By Lemma 1.4, there exists $\tilde{f} : [0, \alpha) \to \tilde{X}$ with starting point $\tilde{x}_0 \in p^{-1}(f(0))$ such that $p \circ \tilde{f} = f$. Also, since q is a semicovering map, the map q has PLP. Thus there exists a lifting $\hat{f}$ of f in $\tilde{Y}$ with starting point $\varphi(\tilde{x}_0)$ and $\varphi(\tilde{f}([0, \alpha))) = \hat{f}|_{[0,\alpha)}$. Since q is a semicovering map, there exists an open neighborhood U at $\hat{f}(\alpha)$ such that $p|_U : U \to p(U)$ is a homeomorphism. Put $U_{(f,\tilde{x})} = \varphi^{-1}(U) \cap \tilde{X}$, then there exists $\epsilon > 0$ such that $\tilde{f}(\alpha - \epsilon, \alpha) \subseteq U_{(f,\tilde{x})}$. Also, $p : U_{(f,\tilde{x})} \to p(U_{(f,\tilde{x})})$ is one-to-one since $q : U \to q(U)$ is a homeomorphism. □

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Some Results on p-biharmonic maps between Riemannian manifolds

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Abstract

In this paper, first, we consider the variational formulas of p-biharmonic maps with some new techniques. Then, some properties of this type of biharmonic maps are investigated and an example is presented.

1. introduction

Harmonic maps between Riemannian manifolds were introduced by Eells and Sampson in 1964. They showed that any map from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature can be deformed into a harmonic map. This result is known as fundamental existence thorem for harmonic maps. Harmonic maps play a key role in mathematical physics, [3].
A smooth map $\psi : (P, \ell) \longrightarrow (K, \rho)$ between two Riemannian manifolds is considered. The function

$$E(\psi) = \frac{1}{2}\int_U \mid d\psi \mid^2 dV_\ell, \tag{1}$$

where $U \subset M$ is a compact subset of M, is said to be the energy functional of ψ. Noting that ψ is harmonic if it is a critical point of the energy functional $E(\psi)$ for all compact subsets$U \subset M$. Setting $\tau(\psi) = Tr_g \nabla d\psi$ which is called the tension field of ψ. According to the definition of tension field of ψ, the Euler-Lagrange equation associated to (1) is obtained by the vanishing of the tension field of ψ

$$\tau(\psi) = Tr_g \nabla d\psi = 0, \tag{2}$$

more details are given in [4, 5].
p-harmonic maps are considered as an extension of harmonic maps. In mathematical physics and robatics, p-harmonic

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maps are invesigated in image processing for denoising color images. Furtheremore this type of harmonic maps have been extensively investigated by scholars who have done research on image processing [6]. Moreover, p-harmonic maps, play a key role in physical cosmology for depicting the phenomenon of the quintessence, [1].
In 1983, Eells and Lemaire introduced biharmonic maps. They are critical points of the bi-energy functional, E_2, which is defined as follows

$$E_2(\psi) = \int_P \mid \tau(\psi) \mid^2 dV_\ell. \tag{3}$$

Since the tension field of harmonic maps vanishes, it can be shown that harmonic maps are biharmonic maps and even more, minimal points of the bi-energy functional. The first and second variation formulas of the bienergy functional E_2 were first calculated by Jiang, [2]. Recently, many authors have done research on these type of harmonic maps extensively,[1, 2, 6].
For any real number $p > 2$, the notion of p-biharmonic maps, as an extension of biharmonic maps have an important role in depicting of reproducing the inflation in physical cosmology, [1]. For any smooth map ψ, the $p-$bienergy functional is defined as follows,

$$E_{p,2}(\psi) = \int_P \mid \tau(\psi) \mid^{\frac{p}{2}} dV_\ell. \tag{4}$$

The Euler-Lagrange equation associated to $E_{p,2}$ is obtained as follows

$$\begin{aligned} 0 &= 4p \mid \tau(\psi) \mid^{p-2} \tau_2(\psi) + (p(p-2)(p-4) \mid \tau(\psi) \mid^{p-6} \mid grad \mid \tau(\psi) \mid^2 \mid^2 \\ &- 2p(p-2) \mid \tau(\psi) \mid^{p-4} \Delta \mid \tau(\psi) \mid^2) \tau(\psi) + 4p(p-2) \mid \tau(\psi) \mid^{p-4} \nabla^{\psi}_{grad \mid \tau(\psi) \mid^2} \tau(\psi) \end{aligned} \tag{5}$$

where $\tau_2(\psi)$ is the bi-tension field of E_2. In this paper, first, we compute the second variational formula of $E_{p,2}$ with the new techniques. Then the stability of p-biharmonic maps between Riemannian manifolds is studied and an example is given.

2. Main Results

A smooth map $\psi : (P, \ell) \longrightarrow (K, \rho)$ between Riemannian manifolds are considered. Choose an arbitrary smooth variation $\{\psi_t\}_{t=0}$ of $\psi_0 = \psi$. The second variation formula of $E_{p,2}$ is calculated as follows

$$\begin{aligned} &\frac{d^2}{dt^2}\Big|_{t=0} E_{p,2}(\psi_t) \\ &= \int_P \rho\Big(W, 4p(p-1)H_\psi(\tau) + K_{p,\psi}(W)\tau_2(\psi) - 2p(p-2)\Delta \mid \tau(\psi) \mid^{p-2} J^\psi(W) - \Delta K_{p,\psi}(W)\tau(\psi) \\ &+ 2\nabla^{\psi}_{grad\ (K_{p,\psi}(W))}\tau(\psi) + R^K(d\psi(grad\ \mid \tau(\psi) \mid^{p-4}, W)\tau(\psi)\Big) dV_\ell \end{aligned} \tag{6}$$

where $W = \frac{\partial \psi_t}{\partial t} \mid_{t=0}$, $K_{p,\psi}(W) = p(p-2)\frac{d}{dt}_{t=0} \mid \tau(\psi_t) \mid^{p-4}$ and

$$H_\psi(\tau) = \int_P p \mid \tau(\psi) \mid^{p-2} \Big| - \Delta^\psi W - trace_g R^K(d\psi, W)d\psi\Big|^2 dV_\ell \tag{7}$$

Definition 2.1. A smooth map $\psi : (P, \ell) \longrightarrow (K, \rho)$ between Riemannian manifolds are considered. Choose an arbitrary smooth variation $\{\psi_t\}_{t=0}$ of $\psi_0 = \psi$. Suppose that $V = \frac{d\psi_t}{dt} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2}\Big|_{t=0} E_{p,2}(\psi_t). \tag{8}$$

the smooth map ψ is stable $p-$biharmonic map if $I(V) > 0$ for any vector field V along ψ.

Example 2.2. *Let (S^m, g) be the standard unit m-dimensional sphere. Consider a Riemannain metric ρ on $K = S^m \times R^3 - \{0\}$ as follows $\rho = g_{eucl} + L^2 g$ where $L \in C^\infty(R^3 - \{0\})$ is defined by $L(y) = \sqrt{\mid y \mid}$ for $y \in R^3 - \{0\}$. It can be shown that $\mid gradL^2 \mid^2 = 1$. For any $y_0 \in R^3 - \{0\}$ the tension and bitension field of*

$$\begin{aligned} \psi : S^m &\longrightarrow K \\ x &\longrightarrow (x, y_0) \end{aligned} \tag{9}$$

is obtained as follows

$$\tau(\psi) = -\frac{m}{2}(0, gradL^2) \circ \psi \qquad \tau_2(\psi) = -\frac{m^3}{m+4}(0, grad \mid gradL^2 \mid^2) \circ \psi \tag{10}$$

By (6), it can shown that ψ is a stable p-biharmonic map.

According to (6), the following theorem can be obtained.

Theorem 2.3. *Let $\psi : (P, \ell) \longrightarrow (K, \rho)$ be a non-trivial p-biharmonic map from a constant scalar curvature (P, ℓ) to a constant positive Ricci curvature (K, ρ). Suppose that $div(\mid grad \mid d\psi \mid^2 \mid^2 \ \ell) \neq 0$. Then ψ is unstable if $grad(\mid \nabla d\psi \mid^2)$ is constant.*

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Liouville-type theorems for exponential harmonic maps with potential

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Abstract

In the present paper, exponential harmonic maps with potential from a complete Riemannian manifold of constant sectional curvature to a non-negative Ricci curvature manifold are investigated. First, the variational formulas for this type of harmonic maps are obtained. Then, a Liouville-type theorem for exponential harmonic maps with potential is given .

1. introduction

In 1964, Eells and Sampson introduced Harmonic maps between Riemannian manifolds. They proved that any map $\psi : (P, \ell) \longrightarrow (K, \varrho)$ from any compact Riemannian manifold (P, ℓ) into a Riemannian manifold (K, ϱ) with non-positive sectional curvature can be deformed into a harmonic maps. This is well-known as the fundamental existence theorem for harmonic maps. In view of mechanics, these maps have been studied in many branch of mechanics, such as liquid crystal, ferromagnetic material, super conductor, etc., [6].
The notion of harmonic maps with potential, was initially studied by Ratto in [4]. Recently, these type of harmonic maps was developed by several authors :Y. Chu [1], A. Fardoun and all [2], V. Branding [5] and others.
Let $\psi : (P, \ell) \longrightarrow (K, \varrho)$ be a smooth map, and let H be a smooth function on K. The $H-$energy function of ψ is defined as follows

$$E_H(\psi) = \int_P [e(\psi) - H(\psi)] dV_\ell, \tag{1}$$

where dV_ℓ is the volume element of (P, ℓ) and $e(\psi)$ is the energy density of ψ which is defined by $e(\psi) := \frac{1}{2} \mid d\psi \mid^2$.
A smooth map ψ is said to be harmonic with potential H if ψ is a critical point of the energy functional of E_H.
Eells and Lemaire [3] extended the concept of harmonic maps to exponential harmonic maps, and considered the

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instability of these maps. The exponential energy functional of a smooth map $\psi : (P, \ell) \longrightarrow (K, \varrho)$ is defined as follows:

$$E_{_e}(\psi) = \int_P exp(\frac{| d\psi |^2}{2})dV_\ell. \tag{2}$$

A smooth map ψ is said to be a *exponential harmonic map* if ψ is a critical point of the exponential energy functional. In terms of the Euler-Lagrange equation, ψ is exponential harmonic if ϕ satisfies the following equation

$$\tau_e(\psi) = \tau(\psi) + d\psi(grad\, exp(e(\psi))) = 0. \tag{3}$$

The section $\tau_{_e}(\psi) \in \Gamma(\psi^{-1}TK)$ is said to be the exponential tension field of ψ, [3].
In this manuscript, first, the variational formulas for exponential harmonic maps with potential are derived. Then, a Liouville-type theorem for exponential harmonic maps with potential from a complete Riemannian manifold of constant sectional curvature to a non-negative Ricci curvature manifold is given.

2. Main Results

In this part, the variation formulas of exponential energy functional with potential H is calculated. Then a Liouville-type theorem for exponential harmonic maps with potential is given.

Let $\psi : (P, \ell) \longrightarrow (K, \varrho)$ be a C^3 map. Denote the Levi-Civita connection of P, K and $\psi^{-1}TK$ by ${}^P\nabla, {}^K\nabla$ and $\hat{\nabla}$, respectively. consider that the induced connection $\hat{\nabla}$ on $\psi^{-1}TK$ defined by $\hat{\nabla}_W T = {}^K \nabla_{d\psi(W)}T$, where $W \in \chi(M)$ and $T \in \Gamma(\psi^{-1}TK)$.

Definition 2.1. Let $\psi : (P, \ell) \longrightarrow (K, \varrho)$ be a smooth map, and let H be a smooth function on K. The exponential energy functional of ψ with potential H is defined as follows

$$E_{e,H}(\psi) = \int_P [e(\psi) - H(\psi)]dV_\ell, \tag{4}$$

where dV_ℓ is the volume element of (P, ℓ) and $e(\psi)$ is defined as follows $e(\psi) := \frac{1}{2} \mid d\psi \mid^2$. The smooth map ψ is said to be an exponential harmonic with potential H if ψ is a critical point of the exponential energy functional $E_{e,H}$.

By considering a local orthonormal frame field $\{e_i\}$ on P, the exponential tension field of ψ with potential H, $\tau_{e,H}(\psi)$, is defined as follows

$$\tau_{e,H}(\psi) = exp(\frac{| d\psi |^2}{2})\tau(\psi) + d\psi(grad\; exp(\frac{| d\psi |^2}{2})) +^K \nabla H \circ \psi, \tag{5}$$

here $\tau(\psi) = \sum_{i=1}^m \{\hat{\nabla}_{e_i} d\psi(e_i) - d\psi({}^P\nabla_{e_i} e_i)\}$ is the tension field of ψ. Based on the above notations we have

Lemma 2.2. *Let$\psi : (P, \ell) \longrightarrow (K, \varrho)$ be a smooth map. Then*

$$\frac{d}{dt} E_{e,H}(\psi_t) \mid_{t=0} = -\int_P h(\tau_{e,H}(\psi), V)dV_\ell, \tag{6}$$

where $V = \frac{d\psi_t}{dt} \mid_{t=0}$.

A smooth map ψ is called an exponential harmonic with potential H if $\tau_{e,H}(\psi) = 0$.

Definition 2.3. Let $\psi : (P, \ell) \longrightarrow (K, \varrho)$ be an exponetial harmonic map with potential H, and let $\{\psi_t : P \longrightarrow K\}$ be a variation of ψ such that $\psi_0 = \psi$ and $V = \dfrac{d\psi_t}{dt} \mid_{t=0}$. Setting

$$I(V) = \frac{d^2}{dt^2} E_{e,H}(\psi_t) \mid_{t=0}$$

The smooth map ψ is said to be stable if $I(V) \geq 0$ for any vector field V along ψ.

Let $Y, Z \in \chi(P)$ such that

$$\ell(Y, W) = exp(\frac{\mid d\psi \mid^2}{2}) < \hat{\nabla} V, d\psi > .\varrho(d\phi(W), V),$$
$$\ell(Z, W) = exp(\frac{\mid d\psi \mid^2}{2})\varrho(\hat{\nabla}_W V, V), \tag{7}$$

for every vector fields W on P, respectively. By (6), Green's Theorem and noting the divergence of X and Z, the following theorem is obtained.

Theorem 2.4. *Let $\psi : (P, \ell) \longrightarrow (K, \varrho)$ be an exponential harmonic map with potential H, and let $\{\psi_t : P \longrightarrow K\}$ be a variation of ψ such that $\psi_0 = \psi$. Then*

$$I(V) = \int_P exp(\frac{\mid d\psi \mid^2}{2})\langle \hat{\nabla} V, d\psi \rangle^2 - \varrho(trace_\ell\, {}^K R(V, d\psi)d\psi - ({}^K\nabla_V grad^K H) \circ \psi, V)\Big\} dV_\ell \tag{8}$$

where $\mid \hat{\nabla} V \mid$ denotes the Hilbert-Schmidt norm of the $\hat{\nabla} V \in \Gamma(T^*M \times \phi^{-1}TN)$ and $V = \frac{\partial \phi_t}{\partial t} \mid_{t=0}$.

Theorem 2.5. *Let $\psi : (P, \ell) \longrightarrow (K, \varrho)$ be a p-harmonic map with potential H from a complete Riemannian manifold of constant sectional curvature to a non-negative Ricci curvature manifold. Assume that*

$$\Delta grad \mid e^{|d\psi|^2} \mid^2 + \nabla grad \mid e^{|d\psi|^2} \mid^2 = 0, \qquad grad \mid^K \nabla H \mid^2 = 0. \tag{9}$$

Then ψ is a harmonic map.

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The cubic trigonometric B-spline collocation method for the time-fractional stochastic Advection-Diffusion equation

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Abstract

The ultimate goals of this perform study is to provide a proposed scheme for solving the time-fractional stochastic advection-diffusion equation (TFSADE) of order $\alpha(0 \leq \alpha < 1)$. In this proposed scheme, we utilize an approach based on cubic trigonometric B-spline collocation methods (CTBSCM). In this study, we replace the existing fractional derivative with the fractional Caputo derivative for time discretization and then replace the first and second derivatives of the equation using cubic trigonometric B-spline functions for spatial discretization. Applying this proposed scheme to TFSADE causes the equation to reduce to the linear system. In the end, the examples show that the order of convergence of the proposed method is $O(\tau^2 + h^2)$ where h and τ are the spatial and time step lengths, respectively.

1. Introduction

Recently, finding a solution for a class of fractional differential equations involving Brawnian motion is highly important, because this type of equation is rarely be solved due to randomness, and the analysis of differential equations involving random coefficients gives us more details of the phenomenon behavior. Mathematical models play a key role in the fields of science and industry. As a result, most scientists deal with stochastic differential equations. Mathematical models in most fields include coefficients that are not completely known and have types of random environmental disturbances and noise.
We originally intend to obtain the numerical solution of the following stochastic equation:

$$\begin{aligned} &D_t^{\alpha} u(x,t) + Lu(x,t) = f(x,t), \quad x \in (a,b), \quad t \in (0,T], \\ &Lu(x,t) = \sigma_1 \frac{\partial u(x,t)}{\partial x} - (\sigma_2 + \sigma_3 \dot{B}(t)) \frac{\partial^2 u(x,t)}{\partial x^2}, \end{aligned} \tag{1}$$

with the following initial and boundary conditions

$$u(x,0) = g(x), \quad x \in [a,b], \tag{2}$$

$$u(a,t) = u(b,t) = 0, \quad t \in (0,T], \tag{3}$$

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where σ_1 is the coefficient of advection, and σ_2 and σ_3 are the coefficients of diffusion terms. $g(x)$ is a continuous function. The source function $f(x,t)$ is a sufficiently smooth function. Here, $L$is a linier spatial derivative operator and $0 \leq \alpha < 1$ is order fractional coefficient of the equation. Also, the phrase $\dot{B}(t) = \frac{dB(t)}{dt}$ is white noise where $B(t)$ is a Brawnian motion. For discretization of $B(t)$, we set $t = t_j$ and let $B_j = B(t_j)$. $B(t)$ definition on the full probability space (Ω, F, P) with the filter $\{F_t\}_{t\geq 0}$ where Ω as a summary of possible events of an experiment with the member $\omega \in \Omega$, and F family is a subset of Ω that has the following properties:
1. $\phi \in F$, Where ϕ is an empty set.
2. If $A \in F$, then $A^c \in F$ where $A^c = \Omega - A$ is the same as the A supplement in Ω.
3. If $\{A_i\}_{i\geq 1} \subseteq F$, then $\bigcup_{i=1}^{+\infty} A_i \subseteq F$ and $P : F \to [0,1]$ is a probability function such that
1. $P(\Omega) = 1$.
2. For each sequence $\{A_i\}_{i\geq 1} \subset F$ that for each $i \neq j$, $A_i \cap A_j = \phi$ we have:

$$P(\bigcap_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

The classical books by Professor Mao; including Stability of Stochastic Differential Equations with Quasi-Martingale Ratio [8], Exponential Stability of Stochastic Differential Equations [9], Stochastic Differential Equations, and Applications first edited [10]. And the book Stochastic Differential Equations with Markov Motion [11] are excellent references for stochastic equations as well as in-depth presentations of various techniques and applications in computational. In 2000, Metzler et al. focused on the random walk's guide to anomalous diffusion [1]. In 2006, Kilbas worked on the theory and applications of fractional differential equations [7]. The article by Guang-an Zou (2018) concentrated on a Galerkin finite element method for time-fractional stochastic heat equation [12] and in 2019, Huang et al. prove Carleman estimates for the generalized time-fractional advection-diffusion equations [4] and Amirat et al. worked on Asymptotic analysis of an advection-diffusion equation and application [5]. The article by Babaei and et al. (2020) focused on the Chebyshev collocation methods on time-fractional stochastic heat equation [13] and Mirzaee et al. provide the Cubic B-spline approximation for the linear stochastic integrodifferential equation of fractional order [14]. In this work, in section 2, Our main focus is to provide the numerical scheme for solving TFSADEs. The convergence analysis of this numerical approach investigated in section 3. In section 4, we present some examples related to solving fractional stochastic equations.

2. Numerical Scheme

First we consider two arbitrary constants $M, N \in \mathbb{N}$. We assume

$$\begin{aligned} &a = x_0 < x_1 < \cdots < x_M = b, \quad x_i = a + i(\frac{b-a}{M}), (i = 0,1,2,\cdots,M) \\ &0 = t_0 < t_1 < \cdots < t_N = T, \quad t_k = k(\frac{T}{N}), (k = 0,1,2,\cdots,N). \end{aligned}$$

are uniform partition in the solution domain $[a,b]$ and $[0,T]$, respectively. Now let $TB_m(x)$ for $m = -1, \cdots, M+1$ be the cubic trigonometric B-spline function in the uniform partition on $[a,b]$ that can be defined as follows

$$TB_m^3(x) = \frac{1}{w}\begin{cases} p^3(x_m), & x \in [x_{m-2}, x_{m-1}], \\ p(x_m)[p(x_m)q(x_{m+2}) + q(x_{m+3})p(x_{m+1})] & \\ +q(x_{m+4})p^2(x_{m+1}), & x \in [x_{m-1}, x_m], \\ q(x_{m+4})[p(x_{m+1})q(x_{m+3}) + q(x_{m+4})p(x_{m+2})] & \\ +p(x_m)q^2(x_{m+3}), & x \in [x_m, x_{m+1}], \\ q^3(x_{m+4}), & x \in [x_{m+1}, x_{m+2}], \\ 0, & o.w. \end{cases} \tag{4}$$

where

$$p(x_m) = \sin\left(\frac{x - x_m}{2}\right), \quad p(x_m) = \sin\left(\frac{x_m - x}{2}\right),$$
$$w = \sin\left(\frac{h}{2}\right).\sin(h).\sin\left(\frac{3h}{2}\right),$$

It is obvious that the support of the cubic trigonometric B-spline $TB_m(x)$ and its derivative is $[x_{m-2}, x_{m+2}]$. Let $u(x,t)$ and $U(x,t)$ are the analytical and numerical solutions of the differential equation (1), respectively. According to the collocation method, the numerical solution can be approximated as

$$u(x,t) \simeq U(x,t) = \sum_{m=-1}^{M+1} \Upsilon_m(t) TB_m^3(x), \tag{5}$$

and the coefficients $\Upsilon_m(t)$ are to be determined by the numerical scheme proposed in this paper.
Given the bases of cubic trigonometric B-spline (4) and the numerical solution in (5), we present an approximation for the discretization of the first and second derivatives of equation (1) as follows

$$\begin{cases} U(x_m,t) = a_1\Upsilon_{m-1}(t) + a_2\Upsilon_m(t) + a_1\Upsilon_{m+1}(t), \\ \frac{\partial U(x_m,t)}{\partial x} = -a_3\Upsilon_{m-1}(t) + a_3\Upsilon_{m+1}(t), \\ \frac{\partial^2 U(x_m,t)}{\partial x^2} = a_4\Upsilon_{m-1}(t) - a_5\Upsilon_m(t) + a_4\Upsilon_{m+1}(t), \end{cases} \tag{6}$$

where

$$a_1 = csc(h).csc\left(\frac{3h}{2}\right).\sin^2\left(\frac{h}{2}\right),$$
$$a_2 = \frac{2}{1+2\cos(h)},$$
$$a_3 = \frac{3}{4}csc\left(\frac{3h}{2}\right),$$
$$a_4 = \frac{3+9\cos(h)}{4\cos\left(\frac{h}{2}\right) - 4\cos\left(\frac{5h}{2}\right)},$$
$$a_5 = -\frac{3cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}.$$

The Caputo's time fractional derivative of order $\alpha \in (0,1]$ is given by [15]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u}{\partial \tau}(x,\tau)\frac{d\tau}{(t-\tau)^\alpha}. \tag{7}$$

where Γ is the Gamma function.
Using forward difference formulation, for $k = 0, \cdots, N-1$ the equation (7) can be reshaped as follows

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha}(x, t_{k+1}) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-\tau)^{-\alpha} u_\tau(x,\tau) d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u^{j+1}-u^j}{\tau} \int_{t_j}^{t_{j+1}} (t_{k+1}-\tau)^{-\alpha} d\tau + O(\tau^2) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} [(k-j+1)^{1-\alpha} - (k-j)^{1-\alpha}](u^{j+1}-u^j) + O(\tau^2) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} [(j+1)^{1-\alpha} - j^{1-\alpha}](u^{k-j+1}-u^{k-j}) + O(\tau^2), \end{aligned} \tag{8}$$

where $\tau = \frac{T}{N}$.
By applying the difference form of the time derivative in (8), the fractional advection-diffusion equation (1), for $i = 1, \cdots, M-1,\ k = 0, \cdots N-1$ can be written as

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j \left[U_i^{k-j+1} - U_i^{k-j} \right] + \sigma_1 (U_x)_i^{k+1} = \theta_j (U_{xx})_i^{k+1} + f_i^{k+1} \tag{9}$$

where $U(x_j, t_k) := U_j^k$, and $f(x_j, t_k) := f_j^k$, and $\dot{B} \simeq \frac{B(t_j)-B(t_{j-1})}{\tau} := \zeta_j$ for $j = 1, \cdots, N$, and $\sigma_2 + \sigma_3 \zeta_j := \theta_j$. Then using the collocation method and substituting (6) in (9) for $i = 1, \cdots, M-1$ and $k = 1, \cdots, N$, leads to the following recurrence difference formula corresponding to the parameters Υ_m^k,

$$\mathcal{Z}_1 \Upsilon_{i-1}^{k+1} + \mathcal{Z}_2 \Upsilon_i^{k+1} + \mathcal{Z}_3 \Upsilon_{i+1}^{k+1} = f_i^{k+1} + r(\Upsilon_i^k - \sum_{j=0}^{k-1} b_j [\Upsilon_i^{k-j+1} - \Upsilon_i^{k-j}]), \tag{10}$$

Note that $b_0 = 1$ and

$$\begin{aligned} r &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}, \\ \mathcal{Z}_1 &= -a_3\sigma_1 - a_4\theta_j, \\ \mathcal{Z}_2 &= a_5\theta_j + rb_0, \\ \mathcal{Z}_3 &= a_3\sigma_1 - a_4\theta_j. \end{aligned}$$

and the matrix $\mathcal{Z}$ is

$$\mathcal{Z} = \begin{pmatrix} \mathcal{Z}_2 & \mathcal{Z}_3 & & & & & \\ \mathcal{Z}_1 & \mathcal{Z}_2 & \mathcal{Z}_3 & & & & \\ & \mathcal{Z}_1 & \mathcal{Z}_2 & \mathcal{Z}_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & \mathcal{Z}_1 & \mathcal{Z}_2 & \mathcal{Z}_3 & & \\ & & & \mathcal{Z}_1 & \mathcal{Z}_2 & \mathcal{Z}_3 & \\ & & & & \mathcal{Z}_1 & \mathcal{Z}_2 & \end{pmatrix},$$

where $\Upsilon_m^k = \Upsilon_m(t_k)$. The matrix form of Equation (10) is as follows

$$\mathcal{Z}\Upsilon^{k+1} = f_i^{k+1} + r(b_0\Upsilon_i^k - \sum_{j=0}^{k-1} b_j [\Upsilon_i^{k-j+1} - \Upsilon_i^{k-j}]), \quad k = 1, \cdots, N, \tag{11}$$

For $k = 1$ we have

$$\begin{cases} \Upsilon^{k+1} = \mathcal{Z}^{-1}(f_i^{k+1} + r\Upsilon_i^k), & k = 1, \\ \Upsilon^{k+1} = \mathcal{Z}^{-1}(f_i^{k+1} + r(\Upsilon_i^k - \sum_{j=0}^{k-1} b_j [\Upsilon_i^{k-j+1} - \Upsilon_i^{k-j}])), & k = 2, \cdots, N. \end{cases} \tag{12}$$

Note tha $\Upsilon^k = [\Upsilon_{-1}^k, \Upsilon_0^k, \Upsilon_1^k, \cdots, \Upsilon_M^k, \Upsilon_{N+1}^k]^T$ is the unknown parameters, A is the coefficients matrices and $f^k = [f_0^k, \cdots, f_M^k]^T$. For solve the system (12) with matrix in $(N+1) \times (N+3)$ dimensions, by using the boundary conditions of problem (2), the unknown parameters Υ_{-1}^k and Υ_{M+1}^k may be eliminated from the system as follows;

Let $i = 0$ and $i = M$, by using the conditions (3) and relation (6) we have

$$\begin{cases} U(x_0 = a, t) = a_1\Upsilon_{-1}(t) + a_2\Upsilon_0(t) + a_1\Upsilon_1(t) = 0, \\ U(x_M = b, t) = a_1\Upsilon_{M-1}(t) + a_2\Upsilon_M(t) + a_1\Upsilon_{M+1}(t) = 0. \end{cases}$$

Thus, for every k:

$$\begin{cases} \Upsilon_{-1}^k = -a_2\Upsilon_0^k - a_1\Upsilon_1^k, \\ \Upsilon_{M+1}^k = -a_2\Upsilon_M^k - a_1\Upsilon_{M-1}^k. \end{cases} \tag{13}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & \\ a_1 r & a_2 r & a_1 & 0 & & & & \\ 0 & a_1 r & a_2 r & a_1 r & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & a_1 r & a_2 r & a_1 r & 0 \\ & & & & 0 & a_1 r & a_2 r & a_1 r \\ & & & & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f^{k+1} = \begin{pmatrix} 0 \\ \vdots \\ f^{k+1} \\ \vdots \\ 0 \end{pmatrix}.$$

Having the initial vector Υ^0, the system (12) has a unique solution. The starting vector $\Upsilon^0 = [\Upsilon_{-1}^0, \Upsilon_0^0, \cdots, \Upsilon_{M+1}^0]^T$ can be determined by (6) and initial conditions of the problem, as the following forms

$$U(x_m, 0) = a_1\Upsilon_{i-1}^0 + a_2\Upsilon_i^0 + a_1\Upsilon_{i+1}^0 = g(x_m), \ i = 0, 1, \cdots, M,$$

Therefore, the initial vector Υ^0 is determined from the following matrix equation;

$$\begin{pmatrix} a_1 & a_2 & a_1 & & & & & \\ & a_1 & a_2 & a_1 & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & a_1 & a_2 & a_1 & \\ & & & & & a_1 & a_2 & a_1 \end{pmatrix}_{(M+1)\times(M+3)} \begin{pmatrix} \Upsilon_{-1}^0 \\ \Upsilon_0^0 \\ \vdots \\ \Upsilon_M^0 \\ \Upsilon_{M+1}^0 \end{pmatrix}_{(M+3)\times 1} = \begin{pmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_{M-1}) \\ g(x_M) \end{pmatrix}_{(M+1)\times 1}$$

Corresponding to the time fractional derivative discretization, and as regards the matrix $\mathcal{Z}$ is positive definite, then the proposed numerical scheme (12) is consistent to the differential equation (1)-(2) [16].

3. Numerical examples

To check the accuracy of the present scheme (12), numerical study of test examples is presented. The error norm L_2 and rate are calculated. The numerical results obtained from CBSCM are compared with given exact solutions and the numerical methods available in literature. Here, an example have been considered to verify the validity of proposed numerical algorithm (12).
Example 1: Consider the following time-fractional stochastic advection-diffusion equation:

$$\begin{cases} u_t^{(\alpha)}(x,t) + \sigma_1 u_x(x,t) = (\sigma_2 + \sigma_3\dot{B}(t)u_{xx}(x,t) + f(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 \le x \le 1, \end{cases} \tag{14}$$

where $0 < \alpha \le 1$. Let $\sigma_3 = \sigma_1 = 1$ and $\sigma = \frac{1}{\pi^2}$ the exact solution is $u(x,t) = t^2 \sin(\pi x)$, is the exact solution of the equation (15). So we have

$$f(x,t) = \frac{2t^{2-\alpha}\sin(\pi x)}{\Gamma(3-\alpha)} + (\frac{1}{\pi^2} + 1)\pi^2 t^2 sin(\pi x) - \pi t^2 \cos(\pi x).$$

In Table ??, we show the result of applying scheme (12) for solving the equation (15)

Table 1. Absolute error L_2 and experimental order of convergence of time-fractional stochastic advection-diffusion equations for Example 1 and $M = N = 1000$

(x_i, t_i)	Exact solution	Scheme (12)		
		$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
$(0.0, 0.0)$	0.000000000	0.000000000	0.000000000	0.000000000
$(0.1, 0.1)$	0.003090169	0.003083545	0.003057546	0.002911341
$(0.2, 0.2)$	0.023511410	0.023504785	0.023478786	0.023332581
$(0.3, 0.3)$	0.072811529	0.072804905	0.072778905	0.072632700
$(0.4, 0.4)$	0.152169042	0.152162416	0.152136418	0.151990213
$(0.5, 0.5)$	0.250000000	0.249993375	0.249967376	0.249821171
$(0.6, 0.6)$	0.342380345	0.342373721	0.342347722	0.342201517
$(0.7, 0.7)$	0.396418327	0.396411702	0.396385703	0.396239498
$(0.8, 0.8)$	0.376182561	0.376175937	0.376149937	0.376003732
$(0.9, 0.9)$	0.250303765	0.250297141	0.250271141	0.250124936
$(1.0, 1.0)$	0.000000000	0.000000000	0.000000000	0.0000000000

The Brawian code in python is the following

```
class Brawnian():
    """
    A Brawnian motion class constructor
    """
    def __init__(self,x0=0):
        """
        Init class
        """
        assert (type(x0)==float or type(x0)==int or x0 is None),
            "Expect a float or None for the initial value"

        self.x0 = float(x0)

    def gen_random_walk(self,n_step=100):
        """
        Generate motion by random walk

        Arguments:
            n_step: Number of steps

        Returns:
            A NumPy array with `n_steps` points
        """
        # Warning about the small number of steps
        if n_step < 1:
            print("WARNING! The number of steps is small.
            It may not generate a good
            stochastic process sequence!")

        w = np.ones(n_step)*self.x0

        for i in range(1,n_step):
            # Sampling from the Normal distribution
             with probability 1/2
            yi = np.random.choice([1,-1])
```

```
            # Weiner process
            w[i] = w[i-1]+(yi/np.sqrt(n_step))

        return w

M=100
b = Brawnian()

mybrawnian = list(b.gen_random_walk(M+2))

plt.plot(np.linspace(0,1,M+2),mybrawnian,marker = '*')
plt.show()
```

As a result of executing the above code, the random process will be as follows

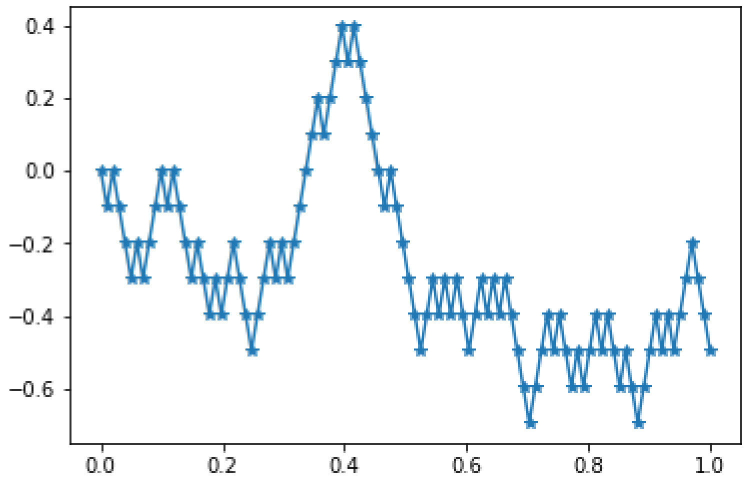
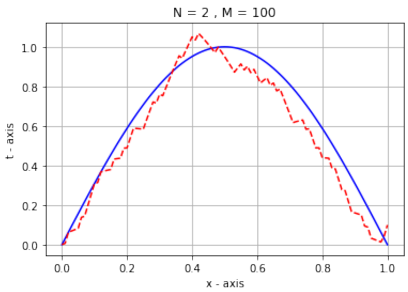
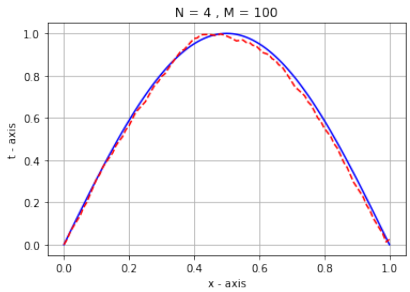


Fig. 1. points in Brawnian

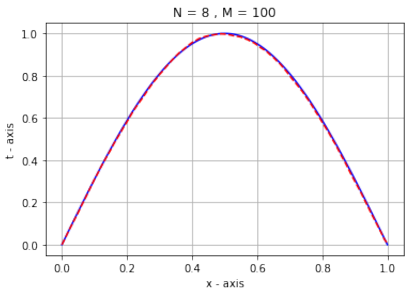
In Figure 2, as you can see, for the different M and N, proportional to the random process obtained as shown in Figure 1, we provide an exact and approximate answer. It is clear from the figure that as we increase M and N, the answer obtained by using Scheme (12) on the numerical example 1, the approximate answer gets closer and closer to the exact answer.



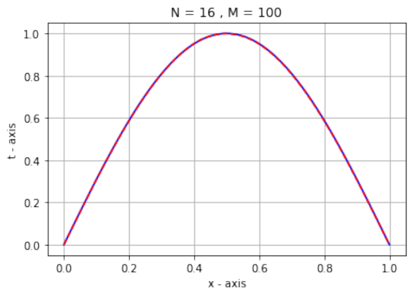
(a) $M = N = 2$ and $t = 0.4$ at different time level



(b) $M = N = 4$ and $t = 0.6$ at different time level



(c) $M = N = 8$ and $t = 0.4$ at different time level



(d) $M = N = 16$ and $t = 0.6$ at different time level

Fig. 2. Exact and numerical solutions in example 1 for $\alpha = 1.05$.

Example 2: Consider the following time-fractional stochastic advection-diffusion equation:

$$\begin{cases} u_t^{(\alpha)}(x,t) + \sigma_1 u_x(x,t) = (\sigma_2 + \sigma_3 \dot{B}(t) u_{xx}(x,t) + f(x,t), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 0, & 0 \leq x \leq 1, \end{cases} \tag{15}$$

where $0 < \alpha \leq 1$ Let $\sigma_3 = 1, \sigma_1 = -3$ and $\sigma_2 = 1$ the exact solution is $u(x,t) = e^{2x-2t}$, is the exact solution of the equation (15). So

$$f(x,t) = -2e^{2x-2t}[1 + 2(1 + \dot{B}(t)) - 3].$$

By re-executing the Python code, the random points from the random process are as follows

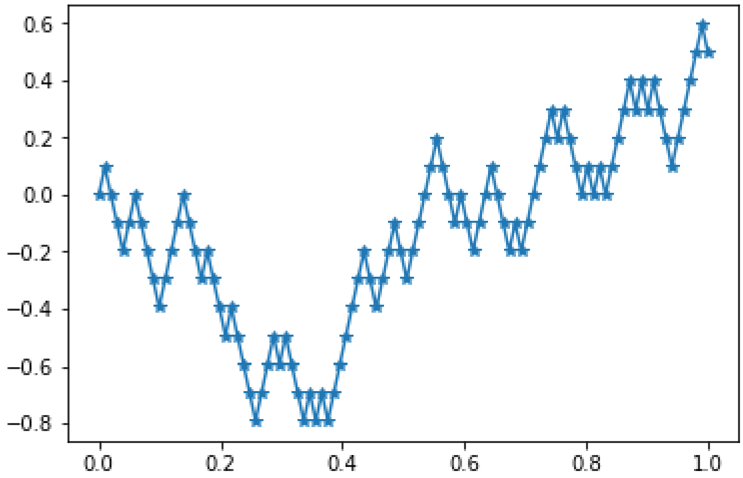


Fig. 3. points in Brawnian for exp

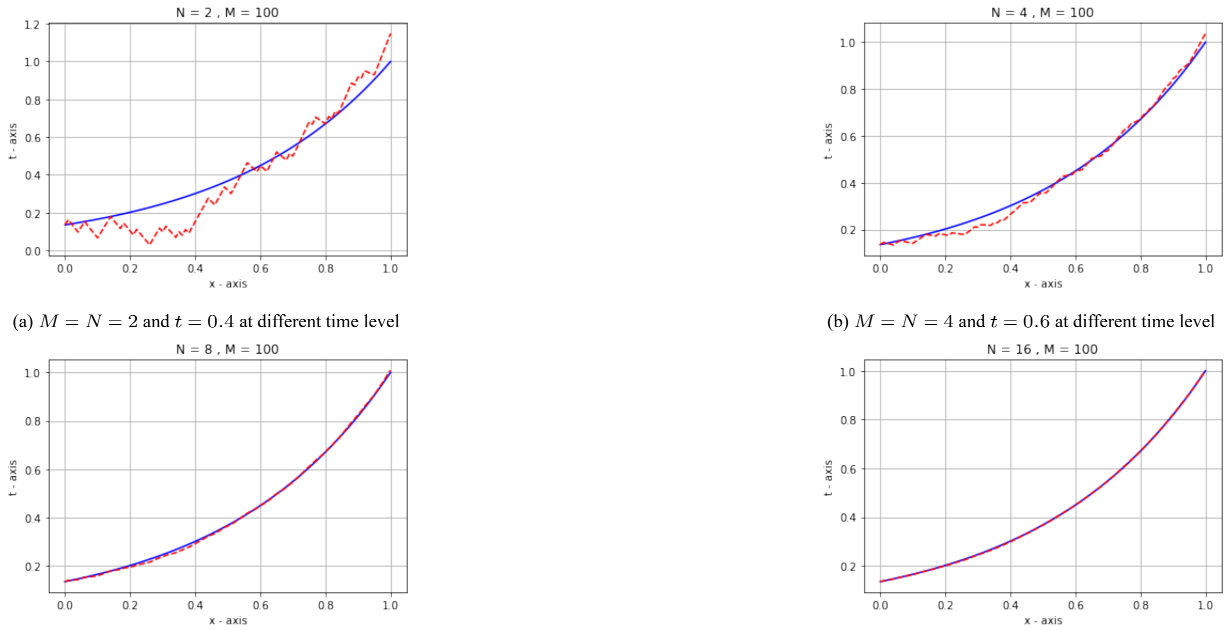
In Table 2, we consider the M to be constant and equal to 100, and for the various values of N, we find the order of convergence obtained using Method (12) on the equation of Example 2. Once, we set the N constant and equal to 100 and present the order of convergence in Table 3. As shown in Tables 2 and 3, the concordance order is 2.

α	N	τ	L_2-norm	$L_\infty-norm$	$Rate$
$\alpha=0.25$	2	0.5	0.8582072	0.198029	–
	4	0.25	0.2145514	0.049507	1.99
	8	0.125	0.0536379	0.0123768	2.00
	16	0.0625	0.0134094	0.0030942	2.00
	32	0.03125	0.0033523	0.0007735	2.00
	64	0.015625	0.0008380	0.0008381	2.00
$\alpha=0.5$	2	0.5	0.9121994	0.1732758	–
	4	0.25	0.2280498	0.0433189	1.99
	8	0.125	0.0570124	0.0108297	2.00
	16	0.0625	0.0014253	0.0027074	2.00
	32	0.03125	0.0035632	0.0006768	2.00
	64	0.015625	0.0008908	0.0001692	2.00
$\alpha=0.75$	2	0.5	0.9121994	0.1732758	–
	4	0.25	0.2280498	0.0433189	1.99
	8	0.125	0.0570124	0.0108297	2.00
	16	0.0625	0.0014253	0.0027074	2.00
	32	0.03125	0.0035632	0.0006768	2.00
	64	0.015625	0.0008908	0.0001692	2.00

Table 2. Comparison of the errors of approximate solutionsand rate when $h = 0.01$ and $t = 1$.

M	h	L_2-norm	$L_\infty-norm$	$Rate$
2	0.5	0.8582072	0.198029	–
4	0.25	0.2145514	0.049507	1.99
8	0.125	0.0536379	0.0123768	2.00
16	0.0625	0.0134094	0.0030942	2.00
32	0.03125	0.0033523	0.0007735	2.00
64	0.015625	0.0008380	0.0008381	2.00

Table 3. Comparison of the errors of approximate solutionsand rate when $\tau = 0.01$ and $t = 1$.



(a) $M = N = 2$ and $t = 0.4$ at different time level

(b) $M = N = 4$ and $t = 0.6$ at different time level

(c) $M = N = 8$ and $t = 0.4$ at different time level

(d) $M = N = 16$ and $t = 0.6$ at different time level

Fig. 4. Exact and numerical solutions in example 2 for $\alpha = 1.05$.

4. Results and Discussion

This article included four sections as follows: In section 1, we have described the problem statements. The numerical relations spline technique are developed for time-fractional stochastic advection-diffusion equations in Section 2 and the matrix representation for the numerical solution is also developed in this section. In this numerical scheme, we utilized the Caputo sense and a backward difference formula for discretization in time. Finally, two examples were provided to clearly demonstrate the applicability of the method. In Section 3 the $O(h^2 + \tau^2)$ order convergence of the presented algorithm has been discussed. The numerical investigations and discussion is given in Section 4 and the computational outcomes are found to be conformable with theoretical expectations.

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On prime and semiprime submodules of multiplication modules

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Abstract

In this paper we state some basic properties of submodules of multiplication modules. Then, since any submodule of a multiplication module is form of IM where I is an ideal of R, we state some results about ideals of ring R. Also, after recalling the definitions of prime and semiprime submodules, we state some results of these submodules for multiplication modules.

1. Introduction

In this paper all rings are commutative with identity and all modules over rings are unitary. Let M be an R-module. If K and N are submodules of an R-module M, we recall that $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R. A proper submodule N of an R-module M is said to be prime if for every $r \in R\,,\ x \in M\ ;\ rx \in N$ implies that $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p-prime. The set of all prime submodules of M is denoted by $Spec(M)$ and for a submodule N of M, $rad(N) = \bigcap_{L \in Spec(M), N \subseteq L} L$. If no prime submodule of M contains N, we write $rad(N) = M$. Also the set of all maximal submodules of M is denoted by $Max(M)$ and $RadM = \bigcap_{P \in Max(M)} P$. Also we recall that if I is an ideal of a ring R, then radical of I, i.e., $rad(I) = r(I)$ is defined as $\{r \in R \mid \exists k \in \mathbb{N}\ ;\ r^k \in I\}$. Now, let $\underline{a}$ be an ideal of a ring R and $\underline{a} = \bigcap_{i=1}^{l} q_i$, where $rad(q_i) = p_i$, is a normal primary decomposition of $\underline{a}$, then $ass(\underline{a}) = \{p_1, ..., p_l\}$.

2. Definitions and Results

Definition 2.1. An R-module M is called a multiplication module, if for any submodule N of M we have $N = IM$, where I is an ideal of R.

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Proposition 2.2. *Let M be a multiplication R-module. Then for every submodule IM of M, if $IM \subseteq pM$ where $p \in Spec(R)$, then $I \subseteq p$.*

Proof. Let $IM \subseteq pM$ for $I \trianglelefteq R$ and $p \in Spec(R)$. Since $I \subseteq (IM : M) \subseteq (pM : M)$, then by Lemma 2.2 of [5], $I \subseteq p$. □

Lemma 2.3. *Let M be a Noetherian multiplication R-module. Then R satisfies the ascending chain condition on prime ideals.*

Proof. Let $p_1 \subseteq p_2 \subseteq p_3 \subseteq ...$ be an ascending chain of prime ideals of R. Then $p_1M \subseteq p_2M \subseteq p_3M \subseteq ....$ But, M is a Noetherian R-module, hence there exists submodule by Theorem 2.5 part (i) of [2], specially a maximal submodule N of M such that $p_1M \subseteq p_2M \subseteq p_3M \subseteq ... \subseteq N$. But M is a multiplication R-module, hence by Theorem 2.5 part (ii) of [2], there exists a maximal ideal $\underline{m}$ of R such that $N = \underline{m}M$. So we have $p_1M \subseteq p_2M \subseteq p_3M \subseteq ... \subseteq \underline{m}M$ and hence $(p_1M : M) \subseteq (p_2M : M) \subseteq (p_3M : M) \subseteq ... \subseteq (\underline{m}M : M)$. Now by Lemma 2.2 of [5], $p_1 \subseteq p_2 \subseteq p_3 \subseteq ... \subseteq \underline{m}$. The proof is now completed. □

Corollary 2.4. *Let R be an arbitrary ring and let M be a multiplication R-module. Then $Ann_R(M) \subseteq p$ for each $(0) \neq p \in Spec(R)$.*

Proof. By the Lemma 2.2 of [5], $pM \in Spec(M)$ for every $(0) \neq p \in Spec(R)$. Therefore by Corollary 2.11 part (i), (iii) of [2], $Ann_R(M) \subseteq p$. □

Lemma 2.5. *Let R be a ring and M a multiplication R-module. Then $\bigcap_{\lambda\in\Lambda}(p_\lambda M) = (\bigcap_{\lambda\in\Lambda} p_\lambda)M$ for any non-empty collection of non-zero prime ideals p_λ $(\lambda \in \Lambda)$ of R. Also if R is a ring which is not an integral domain then $\bigcap_{0\neq P\in Spec(M)} P = \underline{n}_R M$ and $RadM = J(R)M$.*

Proof. Let M be a multiplication R-module and let p_λ $(\lambda \in \Lambda)$ be any non-empty collection of non-zero prime ideals of R. By Corollary 1.7 of [2], $\bigcap_{\lambda\in\Lambda}(p_\lambda M) = (\bigcap_{\lambda\in\Lambda}[p_\lambda + Ann_R(M)])M$. But by Corollary 2.4, $\bigcap_{\lambda\in\Lambda}(p_\lambda M) = (\bigcap_{\lambda\in\Lambda} p_\lambda)M$.
By Lemma 2.2 of [5], $\bigcap_{0\neq P\in Spec(M)} P = \bigcap_{(0)\neq p\in Spec(R)}(pM)$ and also by above we have $\bigcap_{(0)\neq p\in Spec(R)}(pM) = \underline{n}_R M$. So $\bigcap_{0\neq P\in Spec(M)} P = \underline{n}_R M$. Also by Lemma 2.2 of [5], $RadM = \bigcap_{\underline{m}\in Max(R)}(\underline{m}M)$ and by above $\bigcap_{\underline{m}\in Max(R)} \underline{m}M = J(R)M$. Hence $RadM = J(R)M$. □

Lemma 2.6. *Let R be a ring and M a multiplication R-module. Let IM be an arbitrary non-zero proper submodule of M for some ideal I of R. Then $rad(IM) = (radI)M$ and $(rad(IM) : M) = radI$, where $radI = r(I)$.*

Proof. It is easy to show that $rad(IM) = \bigcap_{p\in v(I)}(pM)$ (we recall that $v(I) = \{p \in Spec(R) | \ I \subseteq p\}$). By Lemma 2.5, $rad(IM) = (radI)M$ and consequently $(rad(IM) : M) = radI$. □

We recall the following definition from [4].

Definition 2.7. A proper submodule N of an R-module M is said to be *semiprime* in M, if for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Note that since the ring R is an R-module by itself, a proper ideal I of R is semiprime if for every ideals J and K of R, $J^2K \subseteq I$ implies that $JK \subseteq I$.

Definition 2.8. There exists another definition of semiprime submodules in [3] as follows:
A proper submodule N of the R-module M is *semiprime* if whenever $r^k m \in N$ for some $r \in R$, $m \in M$ and positive integer k, then $rm \in N$.

By Remark 2.6 of [6], we see that this definition is equivalent to Definition 2.7.

Definition 2.9. Let M be an R-module and $N \leq M$. The *envelope* of the submodule N is denoted by $E_M(N)$ or simply by $E(N)$ and is defined as $E(N) = \{x \in M \,|\, \exists r \in R,\ a \in M;\ x = ra \; and \; r^n a \in N \; for \; some \; positive \; integer \; n\}$.

The envelope of a submodule is not a submodule in general.
Let M be an R-module and $N \leq M$. If there exists a semiprime submodule of M which contains N, then the intersection of all semiprime submodules containing N is called *semi-radical* of N and is denoted by $S - rad_M N$, or simply $S - radN$. If there is no semiprime submodule containing N, then we define $S - radN = M$, in particular $S - radM = M$.
We say that M *satisfies the radical formula*, or M (s.t.r.f) if for every $N \leq M$, $radN = \langle E(N) \rangle$. Also we say that M *satisfies the semi-radical formula*, or M (s.t.s.r.f) if for every $N \leq M$, $S-radN = \langle E(N) \rangle$. Now let $x \in E(N)$ and P be a semiprime submodule of M containing N. Then $x = ra$ for some $r \in R,\ a \in M$ and for some positive integer n, $r^n a \in N$. But $r^n a \in P$ and since P is semiprime we have $ra \in P$. Hence $E(N) \subseteq P$. We see that $E(N) \subseteq \bigcap P$ (P is a semiprime submodule containing N). So $\langle E(N) \rangle \subseteq S - radN$. On the other hand, since every prime submodule of M is clearly semiprime, we have $S - radN \subseteq radN$. We conclude that $\langle E(N) \rangle \subseteq S - radN \subseteq radN$ and as a result if M (s.t.r.f) then it is also (s.t.s.r.f).

Lemma 2.10. *Let R be a ring and let M be a multiplication R-module. Then every proper submodule of M is a radical submodule, i.e., $radN = N$.*

Proof. By Theorem 2.12 of [2], $radN = rad(N : M)M$. But $rad(N : M)M \subseteq \langle E(N) \rangle \subseteq radN$, hence M (s.t.r.f) and so (s.t.s.r.f). Then $\langle E(N) \rangle = S - radN = radN$ for every proper submodule N of M. But by Proposition 4.1 of [4], $S - radN = N$ and therefore $radN = N$. □

Corollary 2.11. *Let R and M and IM be as in Lemma 2.6. Then $IM = (radI)M$.*

Proof. Let M be a multiplication R-module and IM be an arbitrary non-zero proper submodule of M for some ideal I of R. By Lemma 2.6, $rad(IM) = (radI)M$ and by Lemma 2.10, $rad(IM) = IM$. Therefore $IM = (radI)M$. □

Theorem 2.12. *Let R be a ring and let M be a multiplication R-module. Then N is a primary submodule of M if and only if it is a prime submodule of M.*

Proof. $\Longleftarrow$. It is clear.
$\Longrightarrow$. Let M be a multiplication R-module and let N be an arbitrary primary submodule of M. Then by Corollary 2 of [1], there exists a primary ideal q $(radq = p)$ of R such that $N = qM$.
But by Lemma 2.10 and Corollary 2.11, $qM = (radq)M = pM$. Therefore the proof is now completed. □

Theorem 2.13. *Let R be a ring which satisfies ascending chain condition on semiprime ideals and let M be a multiplication R-module. Then M is a Noetherian R-module.*

Proof. Let M be a multiplication R-module. Then M (s.t.r.f) and hence (s.t.s.r.f). Thus by Proposition 4.1 of [4], every proper submodule of M is a semiprime submodule of M. Now, let $I_1M \subseteq I_2M \subseteq I_3M \subseteq$ ... where I_i are ideals of R be ascending chain of submodules of M. Then $(I_1M : M) \subseteq (I_2M : M) \subseteq (I_3M : M) \subseteq$ .... But by Proposition 2.3(ii) of [4], $(N : M)$ is a semiprime ideal of R for any semiprime submodule N of M, hence by assumption there exists $n \in \mathbb{N}$ such that $(I_nM : M) = (I_{n+k}M : M)$ for each $k \in \mathbb{N}$. But then $(I_nM : M)M = (I_{n+k}M : M)M$ and so $I_nM = I_{n+k}M$. Therefore M is a Noetherian R-module.

□

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On weak multiplication modules

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Abstract

Let R be a commutative ring and M be a unitary R-module. In this work we recall definitions of multiplication and weak multiplication modules. Then we consider some basic properties such as prime and maximal submodules of weak multiplication modules in some different conditions.

1. Introduction

For the first time, in 1995, Abu-Saymeh in [1], defined the weak multiplication modules and discussed the properties of these modules. Then in 2000, Azizi in [2], examined the more properties of these modules. Throughout this paper all rings are commutative with identity and all modules over rings are unitary. Let M be an R-module. If K and N are submodules of an R-module M, we recall that $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R. A proper submodule N of an R-module M is said to be prime if for every $r \in R$, $x \in M$; $rx \in N$ implies that $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p-prime. The set of all prime submodules of M is denoted by $Spec(M)$. Also we recall that if I is an ideal of a ring R, then radical of I, i.e., $r(I)$ is defined as $\{r \in R \mid \exists k \in \mathbb{N} \,;\, r^k \in I\}$. Now, let $\underline{a}$ be an ideal of a ring R and $\underline{a} = \bigcap_{i=1}^{l} q_i$, where $r(q_i) = p_i$, is a normal primary decomposition of $\underline{a}$, then $ass(\underline{a}) = \{p_1, ..., p_l\}$.
An R-module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$. It can be shown that $N = (N : M)M$.

2. Definitions and Results

Definition 2.1. An R-module M is called *weak multiplication* if $Spec(M) = \emptyset$ or for every prime submodule N of M we have $N = IM$, where I is an ideal of R.

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It is clear that every multiplication module is weak multiplication. Also if N is a p-prime submodule of a weak multiplication module M it can be shown that $N = pM$.

Lemma 2.2. *Let M be a weak multiplication R-module such that for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$ and $N = p_1M$, $K = p_2M$ be prime submodules of M, where $p_1, p_2 \in Spec(R)$. Then $(N : K)M = (p_1 : p_2)M$.*

Proof. Let $y \in (p_1 : p_2)M$ be an arbitrary element. Then $y = \sum_{f.s.} s_i m_i$, where $s_i \in (p_1 : p_2)$ and $m_i \in M$. So $s_i p_2 \subseteq p_1$ which implies $(s_i p_2)M \subseteq p_1M$, that is, $s_iK \subseteq N$. Hence $s_i \in (N : K)$ and then $y \in (N : K)M$. Now, let $x \in (N : K)M$ be arbitrary, then $x = \sum_{f.s.} l_i m_i$, where $l_i \in (N : K) = (p_1M : p_2M)$ and $m_i \in M$. Since $l_i \in (p_1M : p_2M)$, then $l_i(p_2M) \subseteq p_1M$ and so $(l_i p_2)M \subseteq p_1M$. Therefore $l_i p_2 \subseteq (p_1M : M) = p_1$, and hence $l_i \in (p_1 : p_2)$. Therefore $x = \sum_{f.s.} l_i m_i \in (p_1 : p_2)M$ and so $(N : K)M \subseteq (p_1 : p_2)M$. We conclude that, $(N : K)M = (p_1 : p_2)M$. □

Proposition 2.3. *Let M be a weak multiplication R-module, where R is a Noetherian ring. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$. Then the number of minimal prime submodule of M is finite.*

Proof. Let $0 = Q_1 \cap ... \cap Q_n$ be a normal primary decomposition of the zero ideal, where Q_i is p_i-primary $(1 \leq i \leq n)$. Then all the minimal prime ideals of R can be found in the set $\{p_1, p_2, ..., p_n\}$. Let $\{p_1, p_2, ..., p_k\}$, where $k \leq n$, be the set of minimal prime ideals of R. We know that there is a one-to-one inclusion preserving correspondence between prime ideals of R and prime submodules of M in such a way that if $p \in Spec(R)$ corresponds to $N \in Spec(M)$ then $N = pM$ and $p = (N : M)$. This implies that $\{p_1M, ..., p_kM\}$ is the set of all minimal prime submodules of M. □

Proposition 2.4. *Let R be a non-trivial ring and $M \neq 0$ be an weak multiplication R-module. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$. Then M has a maximal submodule.*

Proof. We know that R has a maximal ideal $\underline{m}$-say. But $\underline{m} \in Spec(R)$ implies that $\underline{m}M \in Spec(M)$ and $(\underline{m}M : M) = \underline{m}$. Let a submodule H of M be such that $\underline{m}M \subseteq H \subsetneq M$. By Proposition 3 of [4], H is an $\underline{m}$-prime submodule of M. Now, by the construction of M, $H = \underline{m}M$ and so $\underline{m}M$ is a maximal submodule of M. □

Lemma 2.5. *Let M and R be as in above proposition. Now, let $I \trianglelefteq R$ such that $I \subseteq J(R)$, where $J(R)$ is the Jacobson radical of R. If $M = IM$ then $M = 0$.*

Proof. Let $M \neq 0$, then by the above proposition M has a maximal submodule L-say. Hence there exists an ideal $\underline{h} \in Max(R)$ such that $L = \underline{h}M$. So $M = IM \subseteq \underline{h}M$ and hence $M = \underline{h}M$, a contradiction. Therefore we have $M = 0$. □

In the above lemma we can remove the condition that $I \subseteq J(R)$, as it is seen in the following.

Corollary 2.6. *Let M and R be as in above lemma and let I be a proper ideal of a ring R. If $IM = M$ then $M = 0$.*

Proof. clear. □

Lemma 2.7. *Let R be a integral domain and M be an weak multiplication R-module. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$. Then M is torsion-free.*

Proof. Let $T(M) \neq 0$ so there exists a non-zero element $x \in T(M)$. Since $Ann(x) \neq 0$ there exists $c \in R$, $c \neq 0$ such that $cx = 0$. We know that $(0) \in Spec(R)$ and so $(0)M = 0 \in Spec(M)$. Now $cx = 0$ implies that $x \in (0)M = 0$ or $c \in ((0)M : M) = Ann_R(M) = (0)$. But $c \neq 0, x \neq 0$, a contradiction. Therefore $T(M) = 0$, that is, M is torsion-free. □

Corollary 2.8. *Let R be a integral domain and M be an weak multiplication R-module. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$. Then every direct summand of M is prime. Hence M is indecomposable.*

Proof. By the preceding proposition M is torsion-free and by Result of [4], every direct summand of M is a prime submodule. □

Lemma 2.9. *Let R be a non-trivial ring and M be a weak multiplication R-module. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$. Also, let every prime submodule of M is finitely generated. Then M is a Noetherian module.*

Proof. We assume that $M \neq 0$. By Proposition 2.4, M has a maximal submodule L-say. Since $L \subsetneq M$ there exists $x \in M \setminus L$ and by the maximal property of L we have $M = L + Rx$. By Proposition 4 of [4], L is a prime submodule of M and as a result finitely generated. Therefore $M = L + Rx$ is also finitely generated. Now by Theorem 2.7 of [2], M is a multiplication R-module. The result follows by Theorem 3.2 of [3]. □

Theorem 2.10. *Let R be a non-trivial ring and M be a weak multiplication R-module. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$ and let M' be an R-module. Let $\phi : M \longrightarrow M'$ be an epimorphism such that* $\ker \phi$ *is contained in every prime submodules of M. Then M' is an weak multiplication R-module such that for every $p \in Spec(R)$, pM' is a prime submodule of M' and $(pM' : M') = p$.*

Proof. First, let L' be an arbitrary prime submodules of M'. Then there exists a prime submodule L of M such that $\phi(L) = L'$ and so $\phi^{-1}(L') = L$. By the hypothesis of the theorem, thus there exists an ideal $p \in Spec(R)$ such that $pM = L$. Hence $L = pM = \phi^{-1}(L')$ implies that $\phi(pM) = L'$, that is, $p\phi(M) = L'$ which means $pM' = L'$. Therefore M' is a weak multiplication R-module.

Second, let $p \in Spec(R)$ be an arbitrary prime ideal, we must prove that $pM' \in Spec(M')$ and $(pM' : M') = p$. But $pM' = p\phi(M) = \phi(pM) \leq M'$, then $pM \in Spec(M)$ and so $pM' = \phi(pM) \in Spec(M')$. Now we must prove that $(pM' : M') = p$. Obviously, $p \subseteq (pM' : M') \quad (1)$.

We show that $(pM' : M') \subseteq p$. But $(pM' : M') = (p\phi(M) : \phi(M)) = (\phi(pM) : \phi(M))$. Let $r \in (pM' : M') = (\phi(pM) : \phi(M))$, so $r\phi(M) \subseteq \phi(pM)$, that is, $\phi(rM) \subseteq \phi(pM)$. Since $rM \subseteq \phi^{-1}(\phi(rM)) = \phi^{-1}(\phi(pM)) = \phi^{-1}(p\phi(M)) = \phi^{-1}(pM') = p\phi^{-1}(M') = pM$, then $rM \subseteq pM$ and so $r \in (pM : M) = p$. Therefore $(pM' : M') \subseteq p \quad (2)$.

By (1) and (2), $(pM' : M') = p$ and so M' is a weak multiplication R-module such for every $p \in Spec(R)$, pM' is a prime submodule of M' and $(pM' : M') = p$. □

Corollary 2.11. *Let R be a non-trivial ring and M be a weak multiplication R-module. Also, let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$ and N be a submodule of M such that N is contained in every prime submodule of M. Then $\frac{M}{N}$ is a weak multiplication R-module such that for every $p \in Spec(R)$, $p\frac{M}{N}$ is a prime submodule of $\frac{M}{N}$ and $(p\frac{M}{N} : \frac{M}{N}) = p$.*

Proof. The proof is clear by the above theorem. □

Corollary 2.12. *Let $M_\lambda (\lambda \in \Lambda)$ be a collection of R-modules. If $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a weak multiplication R-module, then for every $\lambda \in \Lambda$, M_λ is a weak multiplication R-module.*

Proof. We define the map ϕ as follows:

$$\phi : M = \bigoplus_{i=1}^{n} M_i \longrightarrow M_i \ , \ (\forall i = 1, ..., n) \ \ by$$

$$\phi(m_1, \ldots, m_n) = m_i \ , \ \ \forall (m_1, \ldots, m_n) \in \bigoplus_{i=1}^{n} M_i.$$

Since ϕ is an epimorphism, the result follows by the first part of the proof of Theorem 2.10. □

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On radical of submodules of weak multiplication modules

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Abstract

Let R be a commutative ring and M be a unitary R-module. In this work we recall definitions of multiplication and weak multiplication modules and then we state some results about prime and maximal submodules of weak multiplication modules in some different conditions. Also, we study some properties of radical submodules of some weak multiplication modules and we study the correctness of the radical formula for some weak multiplication modules.

1. Some Basic Definitions

Throughout this paper, all rings are commutative with identity and all modules over rings are unitary.

Definition 1.1. By [4], A proper submodule N of an R-module M is said to be prime if for every $r \in R$, $x \in M$; $rx \in N$ implies that $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p-prime. The set of all prime submodules of M is denoted by $Spec(M)$.

By [2], we recall that if M is an R-module and K and N are submodules of an R-module M, then $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R.

We have the following remark by [5]:

Remark 1.2. *Let M be an R-module and N be a submodule of M. Then the radical of N is denoted by $rad(N) = \bigcap_{L\in Spec(M),N\subseteq L} L$. If no prime submodule of M contains N, we write $rad(N) = M$. Also the set of all maximal submodules of M is denoted by $Max(M)$ and $RadM = \bigcap_{P\in Max(M)} P$.*

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Remark 1.3. *We recall that if I is an ideal of a ring R, then radical of I, i.e., $r(I)$ is defined as $\{r \in R \mid \exists k \in \mathbb{N}\,;\, r^k \in I\}$. Now, let $\underline{a}$ be an ideal of a ring R and $\underline{a} = \bigcap_{i=1}^{l} q_i$, where $r(q_i) = p_i$, is a normal primary decomposition of $\underline{a}$, then $ass(\underline{a}) = \{p_1, ..., p_l\}$.*

Definition 1.4. An R-module M is called a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$. It can be shown that $N = (N : M)M$.

Definition 1.5. By [1], An R-module M is called *weak multiplication* if $Spec(M) = \emptyset$ or for every prime submodule N of M we have $N = IM$, where I is an ideal of R.

It is clear that every multiplication module is weak multiplication. Also if N is a p-prime submodule of a weak multiplication module M it can be shown that $N = pM$.

Definition 1.6. Let M be an R-module and N be a submodule of M. We recall that the *envelope* of N, denoted by $E_M(N)$ or simply $E(N)$, is defined as

$$E(N) = \{x \in M \mid \exists r \in R,\ a \in M \ \ such\, that\ \ x = ra\ \ and\ \ r^n a \in N\ \ for\ some\ \ n \in \mathbb{Z}^+\}.$$

In general, $E(N)$ is not a submodule of M but there are some special cases that $E(N)$ can be a submodule of M, for example if N is a prime or semiprime submodule of M. Also we recall that an R-module M *satisfies the radical formula* if for every submodule N of M, $radN = \langle E(N) \rangle$, that is, the radical of N is equal to the submodule of M generated by $E(N)$. We can easily prove that for any submodule N of M

$$(rad(N : M))M \subseteq \langle E(N) \rangle \subseteq radN.$$

In Theorem 2.12 of [3], it is proved that for any proper submodule N of a multiplication R-module M, $radN = (rad(N : M))M$. Therefore $radN = \langle E(N) \rangle$ and we conclude that M satisfies the radical formula whenever M is a multiplication module.

2. Main Results

Lemma 2.1. *Let M be a free weak multiplication R-module. Then $\bigcap_{P \in Spec(M)} P = (\bigcap_{p \in Spec(R)} p)M$.*

Proof. Let $P \in Spec(M)$ then $P = pM$, where $p = (P : M) \in Spec(R)$.Hence

$$(\bigcap_{p \in Spec(R)} p)M \subseteq \bigcap_{p \in Spec(R)} pM = \bigcap_{P \in SpecM} P.$$

Next we show that $\bigcap_{p \in Spec(R)} pM \subseteq (\bigcap_{p \in Spec(R)} p)M$. Let $B = \{b_i\}_{i \in I}$ be a basis for M, and $Spec(R) = \{p_\lambda\}_{\lambda \in \Lambda}$. Let $x \in \bigcap_{\lambda \in \Lambda}(p_\lambda M)$. Thus for an arbitrary $\lambda \in \Lambda$, $x = \sum_{f.s.} s_i b_i$, where $s_i \in p_\lambda$, $b_i \in B$.
But for every other $\lambda' \in \Lambda$, $x = \sum_{f.s.} s_i b_i \in p_{\lambda'} M$ and since the representation of an element in M is unique we have $s_i \in p_{\lambda'}$. Therefore $x = \sum_{f.s.} s_i b_i \in (\bigcap_{\lambda \in \Lambda} p_\lambda)M$ and the result follows. □

Corollary 2.2. *Let M be a free weak multiplication R-module, then:*

$$\bigcap_{L \in Max(M)} L = RadM = J(R)M,$$

where $J(R)$ is the Jacobson radical of R.

Proof. It is easy to see that every maximal submodule N of M is a prime submodule and $\underline{m} = (N : M)$ is a maximal ideal of R. Hence by the preceding lemma,

$$RadM = \bigcap_{L \in Max(M)} L = \bigcap_{\underline{m} \in Max(R)} (\underline{m}M) = (\bigcap_{\underline{m} \in Max(R)} \underline{m})M = J(R)M.$$

□

Proposition 2.3. *Let M be a weak multiplication R-module and let for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$. Then there exists a bijection between $Spec(R)$ and $Spec(M)$.*

Proof. We define ψ as follows:

$$\psi : Spec(M) \longrightarrow Spec(R),\ P \longmapsto (P : M)$$

Obviously ψ is a surjection. Let $P_1, P_2 \in Spec(M)$ and $\psi(P_1) = \psi(P_2)$. We have $P_1 = p_1 M$ and $P_2 = p_2 M$, where $p_1, p_2 \in Spec(R)$ and $(p_1 M : M) = p_1$, $(p_2 M : M) = p_2$. But $\psi(P_1) = \psi(P_2)$ means $(P_1 : M) = (P_2 : M)$, that is, $p_1 = p_2$ and this implies that $P_1 = p_1 M = p_2 M = P_2$. Therefore ψ is a bijection. □

Proposition 2.4. *Let M be an R-module and let I be an ideal of R. Then $rad(IM) = \bigcap_{p \in V(I)} pM$ in the following cases:*
(i) M is a weak multiplication R-module such that for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$;
(ii) M is a free weak multiplication R-module.

Proof. clear. □

Remark 2.5. *We recall that if I is any ideal of R then $V(I)$ is the set of all prime ideals containing I.*

Lemma 2.6. *Let M be an R-module and I be an ideal of R, then:*
(1) $(rad(IM) : M) = radI$ in each of the following cases:
(i) M is a weak multiplication R-module such that for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$;
(ii) M is a free weak multiplication R-module.
Also if M is a free weak multiplication R-module then:
(2) $rad(IM) = (radI)M$;
(3) $(rad(IM) : M)M = rad(IM)$.

Proof. (1). By Proposition 2.4, $rad(IM) = \bigcap_{p \in V(I)} pM$ and since $(rad(IM) : M) = (\bigcap_{p \in V(I)} pM : M) = \bigcap_{p \in V(I)}(pM : M) = \bigcap_{p \in V(I)} p$, we have $(rad(IM) : M) = radI$.
(2). Since M is free weak multiplication $rad(IM) = \bigcap_{p \in V(I)} pM$ and by Lemma 2.1, $\bigcap_{p \in V(I)} pM = (\bigcap_{p \in V(I)} p)M = (radI)M$, hence $rad(IM) = (radI)M$.
(3). This clearly follows from (1) and (2). □

Proposition 2.7. *Let M be an R-module and let N be a submodule of M. If $radN \neq M$ then $(rad(N) : M) = rad(N : M)$ in the following cases:*
(i) M is a weak multiplication R-module such that for every $p \in Spec(R)$, pM is a prime submodule of M and $(pM : M) = p$;
(ii) M is a free weak multiplication R-module.

Proof. We have

$$(rad(N) : M) = (\bigcap_{P \in Spec(M), N \subseteq P} P : M) = (\bigcap_{p \in Spec(R), N \subseteq pM} pM : M) = \bigcap_{p \in Spec(R), N \subseteq pM} (pM : M).$$

Now $(pM : M) = p$ for every $p \in V((N : M))$ and therefore $(rad(N) : M) = rad(N : M)$. □

Corollary 2.8. *Let M be a free weak multiplication R-module and let N be a submodule of M such that $radN \neq M$. Then $radN = (rad(N : M))M$.*

Proof. clear. □

Proposition 2.9. *Let M be a free weak multiplication R-module and let N be a primary submodule of M such that $radN \neq M$. Then $radN$ is a prime submodule of M.*

Proof. Since N is a primary submodule, $(N : M)$ is a primary ideal of R and hence $rad(N : M)$ is prime. By Corollary 2.8, $radN = (rad(N : M))M$ is a prime submodule of M. □

Proposition 2.10. *Let M be a free weak multiplication R-module. Then M satisfies the radical formula.*

Proof. We know that for every R-module M and any submodule $N \leq M$

$$(rad(N : M))M \subseteq \langle E(N) \rangle \subseteq radN.$$

By Corollary 2.8, $radN = (rad(N : M))M$ and so $radN = \langle E(N) \rangle$. Since the submodule N was taken arbitrary we find that M satisfies the radical formula. □

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S-acts with pseudo-finite congruences

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Abstract

A right congruence θ is called *pseudo-finite* with respect to H if there exists $n \in \mathbb{N}$ such that for any $(a, b) \in \theta$, there is an H-sequence from a to b of length at most n. An S-act A is called pseudo-finite with respect to a finite subset $H \subseteq A \times A$ if $A \times A$ is pseudo-finite with respect to H. Also θ is said to be pseudo-finite, if it is pseudo-finite with respect to some finite subset H of $A \times A$. An S-act A is called pseudo-finite, if $A \times A$ is pseudo-finite.
In this manuscript we investigate some relationship between pseudo finite and finitely generated S-acts.

1. finiteness condition

A finitary condition for a class of algebras is a condition that is satisfied by at least all finite members of the class. Finitary conditions was introduced and developed by Noether and Artin in their seminal work. Subsequently, finitary conditions have been of enormous importance in understanding the structure and behaviour of rings, groups, semigroups and many other kinds of algebras. Algebras with Ascending(descending) chain condition on congruences or ideals are commonly known as Noetherian(Artinian) algebras, have closely related by finitely generation of congruences or ideals. It is clear that all of these properties are finiteness conditions. The study of right Noetherian semigroups was initiated by Hotzel in [2]. He showed that if a semigroup S is right Noetherian, then it contains only finitely many right ideals and are finitely generated. Dandan. Y et.all in [1], introduced another related notion is that when the universal right congruence being finitely generated. The stronger condition that every right congruence of finite index is finitely generated (where index means the number of classes) was introduced and studied in [6]. Also some authors such as [4, 5, 7, 8] work on another finitness conditions with respect to semigroups.
Another finiteness condition is the pseudo-finite, which we consider it for the semigroups and S-acts. Every Noetherian S-act is pseudo-finite and each pseudo-finite S-act is finitly generated. So it is important to fined some relationships between them. Here we give some conditions under which the finitely generated S-acts and pseudo-finite S-acts are equal. The notion of being pseudo-finite was introduced in [9] in the language of ancestry. Theorem 1.7 of [9] shows that for a monoid M the augmentation ideal $l_0^1(M)$ is finitely generated if and only if M is pseudo-finite. The work in [9] was motivated by the Dales-Zelazko conjecture, which states that a unital Banach algebra in which every maximal left ideal is finitely generated is necessarily finite dimensional.

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Throughout the paper S will be denoted by the semigroup with or without identity. We take $\mathcal{A}$ = **Act-S** to be the category of right acts over a semigroup and S-act homomorphisms between them. Let us first recall the definition and some ingredients of the category **Act-S** needed in the sequel. For more information and the notions not mentioned here see [3].
A set A is said to be an S-*act* if there is a, so called, *action* $\mu : A \times S \to A$ such that, denoting $\mu(a, s) := as$, $a(st) = (as)t$ and if S is a monoid with 1, $a1 = a$. Each semigroup S can be considered as an S-act with the action given by its multiplication. Notice that, adjoining an external left identity 1 to a semigroup S an S-act $S^1 := S \cup \{1\}$ is obtained.

The definitions of a *homomorphism of* S-*acts* or S-*maps*, *subact* A of B, written as $A \leq B$, an *extension* of A are all clear. An element $a \in A$ is called *fixed* element if $as = a$ for all $s \in S$. All fixed elements of an S-act A is a subact of A and denoted by $Fix(A)$. The S-act $A \cup \{0\}$ with a fixed adjoined to A is denoted by A^0. A fixed element of a Semigroup S is called a left zero element. All left zero elements of a Semigroup S is a right ideal of S and denoted by $Z(S)$.

An equivalence relation θ on A is called a congruence on A, if $as\theta bs$ whenever $a\theta b$, for each $a, b \in A$ and $s \in S$. For $H \subseteq A \times A$ we use the notation $\bar{H} = H \cup \{(x, y) \mid (y, x) \in H\}$. For $a, b \in A$, an H-sequence connecting a and b is any sequence $a = p_1s_1, q_1s_1 = p_2s_2, q_2s_2 = p_3s_3, ..., q_ns_n = b$ where $(p_i, q_i) \in \bar{H}$ and $s_i \in S$ for $1 \leq i \leq n$. For $H \subseteq A \times A$, the *congruence generated by* H, that is the smallest congruence on A containing H, is denoted by $\rho(H)$. Let $H \subseteq A \times A$ and $\rho = \rho(H)$. Then, for $a, b \in A$, one has $a\rho b$ if and only if either $a = b$ or there exist an H-sequence connecting a and b.

Definition 1.1. Let $\theta \in Con(A)$ being generated by a finite subset $H \subseteq \theta$. We say that θ is *pseudo-finite* with respect to H if there exists $n \in \mathbb{N}$ such that for any $(a, b) \in \theta$, there is an H-sequence from a to b of length at most n. An S-act A is called pseudo-finite with respect to a finite subset $H \subseteq A \times A(X \subseteq A)$ if $A \times A$ is pseudo-finite with respect to $H(X^2 = X \times X)$. Also θ is said to be pseudo-finite, if it is pseudo-finite with respect to some finite subset H of $A \times A$. An S-act A is called pseudo-finite, if $A \times A$ is pseudo-finite.

Clearly $\theta \in Con(A)$ is a finitely generated sub S-act of $A \times A$, then it is pseudo-finite of length 1.

Remark 1.2. *Since each congruence θ on A is a subact of $A \times A$, there is a subset $X \subseteq \theta$ such that X generates θ as a subact of $A \times A$ which is denoted as usual by $\theta = XS$. Now consider $(x, y) \in \theta$. So there exists $s \in S$ and $(a, b) \in X$ such that $(x, y) = (a, b)s$, which means θ is pseudo-finite of length 1 and $(x, y) \in \rho(a, b) \subseteq \rho(X)$. Thus $X \subseteq \theta \subseteq \rho(X)$ and since $\rho(X)$ is the smallest congruence containing X, $\theta = \rho(X)$.*
In particular if θ is finitely generated as a subact of $A \times A$, then θ is finitely generated congruence and if $\theta = (a, b)S$ is a cyclic subact of $A \times A$, then $\theta = \rho(a, b)$ is a monogenic congruence. In the following we show that, the converse of this fact is not in generally true.

Lemma 1.3. *An S-act A is pseudo-finite of length* 1 *if and only if $A \times A$ is a finitely generated S-act.*

Lemma 1.4. *For an S-act A,* (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (iii).
(i) *$A \times A$ is a finitely generated S-act.*
(ii) *$A \times A$ is a finitely generated congruence.*
(iii) *A is a finitely generated S-act.*

Proof. (i) $\Rightarrow$ (ii) By Remark 1.2.
(ii) $\Rightarrow$ (iii) Let $A \times A = \rho(H)$ in which $H = \{(a_1, b_1), \cdots, (a_n, b_n)\}$. Consider $X = \{a_1, \cdots, a_n, b_1, \cdots, b_n\}$. It is not difficult to check that $A = XS$. □

The converse of the implications in the above lemma does not generally holds. Indeed,
(ii) $\not\Rightarrow$(i) Consider the monoid $S = T \cup \{1, 0\}$ in which T is an infinite left zero semigroup, 1 an identity element and 0 a zero element. Then $S \times S$ is not finitely generated S-act. But $S \times S = \rho(\{(0, 1)\})$ is a monogenic congruence. Indeed, for each $(a, b) \in S$, we have $a = 1.a, 0.a = 0 = 0.b, 1.b = b$.

(iii) $\not\Rightarrow$(ii) The monoid $S = (\mathbb{N}, max)$ is a cyclic S-act which generates by 1. Let $S \times S = \rho(H)$ in which $H = \{(a_1, b_1), (a_2, b_2), \cdots, (a_k, b_k)\}$ and $m = max\{a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k\}$. Since $(1, m+1) \in A \times A$, there is a chain $1 = p_1s_1, q_1s_1 = p_2s_2, \cdots, q_ns_n = m+1$. Thus $1 = p_1 = s_1 \leq m$ and hence $q_1, p_2, s_2 \leq m$. Inductively we infer that for each $1 \leq i \leq n$, $p_i, q_i \leq m$ and in particular $q_ns_n \leq m < m+1$ which is a contradiction. Thus $S \times S$ is not finitely generated congruence.
For the converse of Lemma 1.4 we have,

Lemma 1.5. *Let A be an S-act with a fixed element $\{0\}$ and $\theta \in Con(A)$. The following are equivalent:*
(i) *θ is pseudo-finite.*
(ii) *θ is a finitely generated congruence.*

Proof. (ii) $\Rightarrow$ (i) Let θ be a finitely generated congruence which generates by a finite subset $H \subsetneq A \times A$. Consider $\mathcal{C}_H = \{x : \exists y \in A \ s.t \ (x, y) \in H \cup H^{-1}\}$. Now it is not difficult to check that θ is pseudo-finite of length 2 with respect to the finite set $\{0\} \times \mathcal{C}_H$. □

Corollary 1.6. *For an S-act A with a fixed element $\{0\}$, the following are equivalent:*
(i) *A is pseudo-finite.*
(ii) *$A \times A$ is a finitely generated congruence.*
(iii) *A is a finitely generated S-act.*

Proof. (iii) $\Rightarrow$ (i) Let A be a finitely generated S-act which generates by a finite subset X of A. For each $a, b \in A$ there exist $p, q \in X$ and $s, t \in S$ such that $(a, b) = (ps, qt)$. Thus $a = ps, 0s = 0t, qt = b$. So A is pseudo-finite of length 2 with respect to the finite set $\{0\} \times X$. □

Lemma 1.7. (i) *Let B be a finitely generated subact of an S-act A which generates by a subset $Y = \{y_1, y_2, \cdots, y_n\}$. If $X \subseteq A$ in which $B = YS = XS$, then there is a finite subset X' of X such that $B = X'S$ and $| X' | \leq | Y |$.*
(ii) *Let $Y = \{(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)\}$ and X be two subsets of an S-act $A \times A$ in which $\theta = \rho(Y) = \rho(X)$. Then there is a finite subset X' of X such that $\theta = \rho(X')$.*
Also, if θ is pseudo-finite with respect to Y of length $n \in \mathbb{N}$, then it is pseudo-finite with respect to X' of length $m \in \mathbb{N}$.

Proof. (i) We are done using the fact that, for each $y_i \in Y$, there exists $x_i \in X$ and $s_i \in S$ such that $y_i = x_is_i$.
(ii) For each $1 \leq i \leq n$, there is a sequence $a_i = p_{i1}s_{i1}, q_{i1}s_{i1} = p_{i2}s_{i2}, \cdots, q_{in_i}s_{in_i} = b_i$ such that $(p_{ij}, q_{ij}) \in X \cup X^{-1}$ and $s_i \in S$. This implies
$\theta = \rho(\cup_{i=1}^{n}\{(p_{i1}, q_{i1}), \cdots, (p_{in_1}, q_{in_1})\})$ which $X' = \cup_{i=1}^{n}\{(p_{i1}, q_{i1}), \cdots, (p_{in_1}, q_{in_1})\}$ is a finite subset of X.
For the second part, consider $(a, b) \in \theta$. Then there exists a Y-sequence from a to b of length at most n, such as,

$$a = p_1s_1, q_1s_1 = p_2s_2, \cdots, q_ks_k = b$$

, where $(p_i, q_i) \in Y \cup Y^{-1}$, $s_i \in S$ and $k \leq n$. Since $\theta = \rho(X')$, for each $(p_i, q_i) \in Y \cup Y^{-1}$, there exists an X'-sequence of length at most n_i connecting p_is_i to q_is_i. Consider $m = n \times \max\{n_i \mid 1 \leq i \leq k\}$. So there is an X'-sequence from a to b of length at most m. □

Lemma 1.8. *Let A be a right S-act and $\theta \in Con(A)$. The following are equivalent:*
(i) *There exists a finite subset B of A such that $\theta = \rho(B^2)$.*
(ii) *There exists a finite subset B of A such that for any $b \in B, \theta = \rho(\{b\} \times B)$.*

Proof. (i)$\Rightarrow$ (ii) Let $a, b \in A$ such that $b \in B$ and $a\theta b$. By (i), there exist $s_1, \cdots, s_n \in S$ and $(p_1, q_1), \cdots, (p_n, q_n) \in B^2$ such that $a = p_1s_1, q_1s_1 = p_2s_2, \cdots, q_ns_n = b$ and hence $a = p_1s_1, bs_1 = bs_1, q_1s_1 = p_2s_2, bs_2 = bs_2, \cdots, q_ns_n = b$ in which $(p_i, b) \in B \times \{b\}$ and $(b, q_i) \in \{b\} \times B$.
(ii)$\Rightarrow$(i) For each $c_1, c_2 \in C$, we have $c_1 = c_1.1, c.1 = c = c.1, c_2.1 = c_2$. So $(c_1, c_2) \in \theta$ and hence $\theta = \rho(C^2)$. □

Proposition 1.9. *Let A be a right S-act. The following are equivalent:*
(i) *$A \times A$ is a finitely generated congruence on A.*
(ii) *There exists a finite subset B of A such that $A \times A = \rho(B^2)$.*
(iii) *There exists a finite subset B of A such that for any $b \in B, A \times A = \rho(\{b\} \times B)$.*
(iv) *For each $c \in A$ there exists a finite subset C of A such that $c \in C$ and $A \times A = (\{c\} \times C)S \times S = \rho(\{c\} \times C)$.*

Proof. (i)⇒ (ii) By (i), $A \times A = \rho(\{(a_1, b_1), \cdots, (a_n, b_n)\})$. Consider $B = \{a_1, \cdots, a_n, b_1, \cdots, b_n\}$, so $\{(a_1, b_1), \cdots, (a_n, b_n)\} \subseteq B^2$ and hence $A \times A \subseteq \rho(B^2) \subseteq A \times A$.
(ii)⇔ (iii) By Lemma 1.8.
(iii)⇒(iv) Consider $C = B \cup \{c\}$. By (iii), for each $x, y \in A$, there exist $s_1, \cdots, s_n \in S$ and $(p_1, q_1), \cdots, (p_n, q_n) \in (\{b\} \times B) \cup (B \times \{b\})$ such that $x = p_1 s_1, q_1 s_1 = p_2 s_2, \cdots, q_n s_n = y$. So $x = p_1 s_1, cs_1 = cs_1, q_1 s_1 = p_2 s_2, cs_2 = cs_2, \cdots, q_n s_n = y$ in which $(p_i, c) \in C \times \{c\}$ and $(c, q_i) \in \{c\} \times C$. Thus $A \times A = \rho(\{c\} \times B)$.
(iv)⇒(i) Is clear. □

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Vertex Covering in Cubic Bipolar Fuzzy Graphs

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Abstract

Vertex covering have been well-studied concepts in graph theory .These concepts have also been defined in cubic fuzzy graph . In this paper we discuss the concept of vertex covering in cubic bipolar fuzzy graph . In the present study we introduced the strong vertex covering in cubic bipolar fuzzy and described about cardinarity vertex covering and the number of $\alpha - set$ in this graph.

1. Introduction

In 1965, Zadeh introduced the notion of a fuzzy subset of a set. In 1994, Zhang initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. The study of cubic set (CS) has been started in 2012 when Jun et al proposed this concept. Basically, CS is a combination of a fuzzy set (FS) and an interval-valued fuzzy set (IVFS). The recent developments in FSs have shown some good improvements among which the concept of bipolar fuzzy set (BFS) is a prominent one. A BFS improved the concept of FS by enlarging its range.The concept of BFS leads to the development of bipolar fuzzy graph (BFG) . Motivated by the developments of FGs, IVFGs and BFGs, this article aims to provide the concept of cubic bipolar fuzzy graphs(CBFGs) and covering vertex in CBFGs. The article is based on three sections. In section one, some history is recalled and some remarkable work on FG, IVFG, BFG and CFG have been discussed. In section two, some basic definitions are presented. In section three, we introduce a new notion, called a vertex covering in cubic bipolar fuzzy graph . The covering vertex in CBFG is introduced as a generalization of FG, IVFG and BFG and related terms are discussed.

2. Preliminaries

In this section, we present some preliminary results which will be used throughout the paper.

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Definition 2.1. A graph is an ordered pair $G^* = (V, E)$ where V is the set of vertices of G^* and E is the set of edges of G^*.

Definition 2.2. A fuzzy graph with an underlying set V is defined to be a pair $G = (\mu, \nu)$ where μ is a fuzzy function in V and ν is a fuzzy function in $V \times V$,such that $\nu(xy) \leq min\,\{\mu(x), \mu(y)\}$ for all $xy \in V \times V$.

Definition 2.3. A pair $G = (A, B)$ of graph $G = (V, E)$ is said to be interval value fuzzy graph (IVFG) where $A = \left\{[\mu_A^l, \mu_A^u]\right\}$ is an IVFS and $B = \left\{[\mu_B^l, \mu_B^u]\right\}$ is the IVF relation on E satisfying the following condition:
1)$V = \{v_1, v_2, ..., v_n\}$ such that $\mu_A^l : V \to [0, 1]$
$\mu_A^u : V \to [0, 1]$, represents the degree of membership of the element $v \in V$.

2)The function $\mu_B^l : V \times V \to [0, 1]$, $\mu_B^u : V \times V \to [0, 1]$

such that $\mu_B^l \leq min\left\{(\mu_A^l(x), \mu_A^l(y))\right\}$

$\mu_B^u \leq min\,\{(\mu_A^u(x), \mu_A^u(y))\}$ for all $xy \in E$.

Definition 2.4. Let X be a non-empty set .Then mapping $A = (\mu_A^P, \mu_A^N) : X \times X \to [0, 1] \times [-1, 0]$ is a bipolar fuzzy relation on X , such that $\mu_A^P(x, y) \in [0, 1]$, $\mu_A^N(x, y) \in [-1, 0]$.

Definition 2.5. A pair $G = (A, B)$ with underlying graph $G^* = (V, E)$ is said to be bipolar fuzzy graph (BFG) where $A = \left\{[\mu_A^P, \mu_A^N]\right\}$ is a bipolar fuzzy set (BFS) on V and $B = \left\{[\mu_B^P, \mu_B^N]\right\}$ is a bipolar fuzzy relation on E ,such that

$\mu_B^P(xy) \leq min\left\{(\mu_A^P(x), \mu_A^P(y))\right\}$

$\mu_B^N(xy) \geq max\left\{(\mu_A^N(x), \mu_A^N(y))\right\}$ for all $xy \in E$.

Definition 2.6. A cubic set(CS) in V is a structural as $A = \left\{[\mu^l(x), \mu^u(x)], \sigma(x) | x \in V\right\}$ where $[\mu^l(x), \mu^u(x)]$ is an interval-valued fuzzy membership degree and $\sigma(x)$ is fuzzy membership degree of x in A.

Definition 2.7. A pair $G = (A, B)$ of a graph $G = (V, E)$ is said to cubic fuzzy graph (CFG) where $A = \left\{[\mu_A^l(x), \mu_A^u(x)], \sigma_A(x) | x \in V\right\}$ is a CFS and $B = \left\{[\mu_B^l(x), \mu_B^u(x)], \sigma_B(x)\right\}$ is the cubic fuzzy relation on E satisfying the following condithion:

1) $V = \{v_1, v_2, ..., v_n\}$ such that $\mu_A^l : V \to [0, 1]$, $\mu_A^u : V \to [0, 1]$ and $\sigma_A : V \to [0, 1]$, represents the degree of membership of the element $v \in V$.

2)The function $\mu_B^l : V \times V \to [0, 1]$, $\mu_B^u : V \times V \to [0, 1]$ and
$\sigma_B : V \times V \to [0, 1]$
are such that $\mu_B^l(xy) \leq min\left\{(\mu_A^l(x), \mu_A^l(y))\right\}$

$\mu_B^u(xy) \leq min\,\{(\mu_A^u(x), \mu_A^u(y))\}$ and

$\sigma_B(xy) \leq min\,\{(\sigma_A(x), \sigma_A(y))\}$ for all $xy \in E$

the underlying graph of a CG is $G^* = (V^*, E^*)$ where $V^* = \left\{x | [\mu_A^l(x), \mu_A^u(x)] > 0, \sigma_A(x) > 0\right\}$,

$E^* = \left\{xy | [\mu_B^l(xy), \mu_B^u(xy)] > 0, \sigma_B(xy) > 0\right\}$, for all $xy \in V^*$.

Definition 2.8. A pair $G = (A, B)$ with underlying graph $G^* = (V, E)$ is said to be an IVBFG where $A = \left\{[\mu_A^{Pl}, \mu_A^{Pu}][\mu_A^{Nl}, \mu_A^{Nu}]\right\}$ is an IVBFS and $B = \left\{[\mu_B^{Pl}, \mu_B^{Pu}][\mu_B^{Nl}, \mu_B^{Nu}]\right\}$ is an interval-value bipolar fuzzy relation on E such that

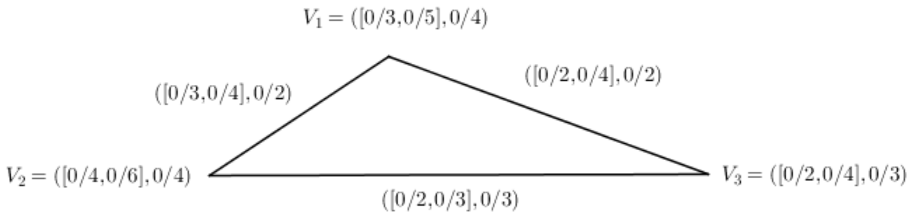

Fig. 1. cubic fuzzy graph

$\mu_B^{Pl}(xy) \leq min\left\{(\mu_A^{Pl}(x), \mu_A^{Pl}(y))\right\}$ and $\mu_B^{Pu}(xy) \leq min\left\{(\mu_A^{Pu}(x), \mu_A^{Pu}(y))\right\}$

$\mu_B^{Nl}(xy) \geq max\left\{(\mu_A^{Nl}(x), \mu_A^{Nl}(y))\right\}$ and $\mu_B^{Nu}(xy) \geq max\left\{(\mu_A^{Nu}(x), \mu_A^{Nu}(y))\right\}$

Definition 2.9. A pair $G = (A, B)$ with underlying graph $G^* = (V, E)$ is said to be an CBFG where $A = \left\{[\mu_A^{Pl}, \mu_A^{Pu}][\mu_A^{Nl}, \mu_A^{Nu}], \sigma_A^P, \sigma_A^N\right\}$ CBFS and $B = \left\{[\mu_B^{Pl}, \mu_B^{Pu}][\mu_B^{Nl}, \mu_B^{Nu}], \sigma_B^P, \sigma_B^N\right\}$ is a cubic bipolar fuzzy relation on E such that
$\mu_B^{Pl}(xy) \leq min\left\{(\mu_A^{Pl}(x), \mu_A^{Pl}(y))\right\}$ and $\mu_B^{Pu}(xy) \leq min\left\{(\mu_A^{Pu}(x), \mu_A^{Pu}(y))\right\}$
$\mu_B^{Nl}(xy) \geq max\left\{(\mu_A^{Nl}(x), \mu_A^{Nl}(y))\right\}$ and $\mu_B^{Nu}(xy) \geq max\left\{(\mu_A^{Nu}(x), \mu_A^{Nu}(y))\right\}$
$\sigma_B^P(xy) \leq min\left\{(\sigma_A^P(x), \sigma_A^P(y))\right\}$ and $\sigma_B^N(xy) \geq max\left\{(\sigma_A^N(x), \sigma_A^N(y))\right\}$ for all $xy \in E$.

Definition 2.10. An edge xy in CBFG is named cubic bipolar strong edge if

$$[\mu_B^{Pl}(xy), \mu_B^{Pu}(xy)] \geq [\mu_B^{\infty Pl}(xy), \mu_B^{\infty Pu}(xy)]$$

$$[\mu_B^{Nl}(xy), \mu_B^{Nu}(xy)] \leq [\mu_B^{\infty Nl}(xy), \mu_B^{\infty Nu}(xy)]$$

$\sigma_B^P(xy) \geq \sigma_B^{\infty P}(xy)$, $\sigma_B^N(xy) \leq \sigma_B^{\infty N}(xy)$

Definition 2.11. The strength of connectedness between x and y is shown by $\left\{[\mu_B^{\infty Pl}(xy), \mu_B^{\infty Pu}(xy)][\mu_B^{\infty Nl}(xy), \mu_B^{\infty Nu}(xy)], \sigma_B^{\infty P}(x\right.$ and it is the maximum of the strengths of all cubic bipolar paths between x, y.

Definition 2.12. The degree of a vertex in a CBFG $G^* = (V, E)$ is denoted and defined by

$$deg(x) = ([d^{Pl}(x), d^{Pu}(x)][d^{Nl}(x), d^{Nu}(x)], d^P(x), d^N(x))$$

where ,

$$d^{Pl}(x) = \sum \mu_B^{Pl}(xy), d^{Pu}(x) = \sum \mu_B^{Pu}(xy)$$

$$d^{Nu}(x) = \sum \mu_B^{Nu}(xy), d^{Nl}(x) = \sum \mu_B^{Nl}(xy)$$

$$d^P(x) = \sum \mu_B^P(xy), d^N(x) = \sum \mu_B^N(xy)$$

Definition 2.13. If $S \subseteq V$ in CBG is called
$S = (\mu_{ul}^P, \mu_{ul}^N) = (\sum_{x \in S} \dfrac{1 - \mu^{Pl} + \mu^{Pu} + \sigma^P}{3}, \sum_{x \in S} \dfrac{-1 - \mu^{Nl} + \mu^{Nu} + \sigma^N}{3})$

Definition 2.14. The cardinality of $S \subseteq V$ in CBG is described as

$$|S| = \frac{1 + \mu_{ul}^P + \mu_{ul}^N}{2}$$

Theorem 2.15. *Cubic bipolar fuzzy graph generalizes bipolar fuzzy graph and interval valued bipolar fuzzy graph.*

3. Vertex covering in cubic bipolar fuzzy graph

In this section the vertex covering is discussed in the cubic bipolar fuzzy graph and some of its properties is studied.

Definition 3.1. A vertex covering of a graph G is a set $S \subset V(G)$ that contains at least one endpoint of every edge. The vertices in S cover $E(G)$. A vertex covering set with minimum cardinality is called $\alpha - set$ of G. The cardinality of a minimum vertex covering set of G is called the vertex covering number of G and it is denoted as $\alpha(G)$.

Definition 3.2. Let $G = (A, B)$ has cubic bipolar fuzzy graph, a vertex and a cubic bipolar strong edge incident to it are named strong cover each other.

Definition 3.3. A strong vertex covering set in G is the set of vertices so that each cubic bipolar strong edge in G is incident with at least one vertex in C. The subset C is called the minimal strong vertex covering set of the cubic bipolar fuzzy graph whenever $C - \{v\}$ is not an strong vertex covering set. The minimum cardinality among all the minimal strong vertex covering sets of G is named the strong vertex covering set number of G and it is shown by α. A strong vertex covering set with minimum cardinality in cubic bipolar fuzzy graph G is named the minimum strong vertex covering set and it is denoted as $\alpha - set$.

Example 3.4. *Let $G = (V, E)$ be a cubic bipolar fuzzy graph in figure (2) . the cubic bipolar set edge are v_1v_4, v_3v_5, v_2v_6 the minimal strong vertex covering set in figure (2) are as follows:*

$C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_1, v_2, v_5\}$, $C_3 = \{v_1, v_3, v_6\}$

$C_4 = \{v_1, v_5, v_6\}$, $C_5 = \{v_2, v_3, v_4\}$, $C_6 = \{v_2, v_4, v_5\}$

$C_7 = \{v_3, v_4, v_6\}$, $C_8 = \{v_4, v_5, v_6\}$
From above strong vertex covering set , we have :

$C_1 = (1.46, -1.29)$, $C_2 = (1.52, -1.22)$, $C_3 = (1.5, -1.23)$

$C_4 = (1.56, -1.16)$, $C_5 = (1.46, -1.29)$, $C_6 = (1.52, -1.29)$

$C_7 = (1.5, -1.23)$, $C_8 = (1.56, -1.16)$

With definition 2.12 the cardinarity of the above strong vertex covering set, we have:

$|C_1| = 0.585$, $|C_2| = 0.65$, $|C_3| = 0.635$, $|C_4| = 0.7$

$|C_5| = 0.585$, $|C_6| = 0.615$, $|C_7| = 0.635$, $|C_8| = 0.7$

This clear that C_1 and C_5 have the minimum cardinality among other strong vertex covering set. Therefore , $\alpha = 0.585$, so C_1 and C_5 are the $\alpha - set$ of G.

Theorem 3.5. *If G be a cubic bipolar fuzzy graph and $v \in V$ then $\alpha(G - v) \leq \alpha(G)$*

Theorem 3.6. *If G be a cubic bipolar fuzzy graph and complete , and x be a vertex with the maximum cardinality of the vertices in G ,then $\alpha(G) = n - |\{x\}|$.*

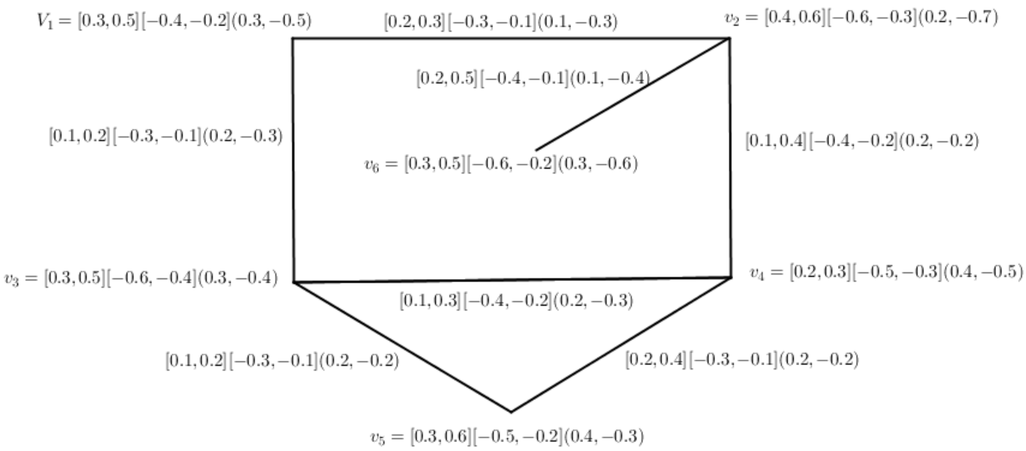

Fig. 2. cubic bipolar fuzzy graph

Theorem 3.7. *If $G = (A, B)$ is a cubic bipolar fuzzy graph without isolated vertex , then ,* $\alpha(G) \leq \dfrac{p}{2}$.

Theorem 3.8. *In a cubic bipolar fuzzy graph G,* $\alpha(G) \geq \delta(G)$.

Corollary 3.9. *The above results show that in cubic bipolar fuzzy graph , it should be* $\delta(G) \leq \alpha(G) \leq \dfrac{p}{2}$.

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Model structure and Gorenstein derived category

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Abstract

In this paper, we construct an exact model structure on certain exact category which has as its homotopy category is the Gorenstein derived category of R-modules.

1. Introduction and Preliminaries

A model category (sometimes called a Quillen model category) is a context for doing homotopy theory. Quillen in [6] developed the definition of a model category to formalize the similarities between homotopy theory and homological algebra: the key examples which motivated his definition were the category of topological spaces, the category of simplicial sets, and the category of chain complexes. The basic problem that model categories solve is the following. Given a category, one often has certain maps (weak equivalences) that are not isomorphisms, but one would like to consider them to be isomorphisms. One can always formally invert the weak equivalences, but in this case one loses control of the morphisms in the quotient category. If the weak equivalences are part of a model structure, however, then the morphisms in the quotient category from X to Y are simply homotopy classes of maps from a cofibrant replacement of X to a fibrant replacement of Y. Because this idea of inverting weak equivalences is so central in mathematics, model categories are extremely important.
In this section we introduce the concept of a model category, and derived some basic results.

Definition 1.1. Suppose $\mathcal{A}$ is a category. An object X of a category $\mathcal{A}$ is said to be a *retract* of an object Y if there exist morphism $i : X \to Y$ and $r : Y \to X$ such that $ri = id_X$. A map f in $\mathfrak{C}$ is a retract of a map g if f is a retract of g as objects of $\mathrm{Mor}(\mathfrak{C})$. That is, f is retract of g if and only if there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

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where the horizontal composites are identities.

Definition 1.2. Suppose $i : A \to B$ and $p : X \to Y$ are morphisms in a category $\mathcal{A}$. Then we say that i has the *left lifting property with respect to* p and p has the *right lifting property with respect to* i if, for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

there is a lift $h : B \to X$ such that $hi = f$ and $ph = g$.

Definition 1.3. A model structure on category $\mathcal{A}$ is three subcategories of $\mathrm{Mor}(\mathcal{A})$ called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying the following properties:

1. **(2-out-of-3)** If f and g are morphisms of $\mathfrak{C}$ such that gf is defined and two of f and g are weak equivalences, then so is the third.
2. **(Retract)** If f and g are morphisms of $\mathcal{A}$ such that f is a retract of g and g is a weak equivalence,cofibration, or fibration, then so is f.
3. **(Lifting)** Define a map to be a trivial cofibration if it is both a cofibration and weak equivalence . Similarly, define a map to be trivial fibration if it is both a fibration and weak equivalence. Then trivial cofibrations have the left lifting property with respect to fibrations, and trivial fibrations have the right lifting property with respect to cofibrations.
4. **(Factorization)** Any morphism f can be factored in two ways: (i)$f = pi$, where i is a cofibration and p is a trivial fibration, and (ii)$f = pi$, where i is a trivial cofibration and p is a fibration.

Definition 1.4. A *model category* is a category $\mathfrak{C}$ with all small limits and colimits together with a model structure on $\mathfrak{C}$.

If $\mathfrak{C}$ is a model category, then it has an initial object $\emptyset$, the colimit of empty diagram, and terminal object $*$, the limit of the empty diagram. the initial and terminal object allows us to define cofibrant and fibrant objects of $\mathfrak{C}$.

Definition 1.5. An object $A \in \mathfrak{C}$ is said to be a *cofibrant(trivially cofibrant)* if $\emptyset \to A$ is a cofibration(trivially fibration, resp.)Dually $B \in \mathfrak{C}$ is *fibrant(trivially fibrant)* if $B \to *$ is fibration(trivial fibration,resp.)
An object $X \in \mathfrak{C}$ is *trivial* if the map $\emptyset \to X$ is a weak equivalence.

Hovey's correspondence. Hovey defines an abelian model categories and characterizes them in terms of cotorsion pairs as we now describe. So in fact one could even take the cotorsion pairs given in the correspondence below as the definition of an abelian model category. First, we need some definitions.

Definition 1.6. An *abelian model category* is an complete and cocomplete abelian category $\mathfrak{A}$ equipped with a model structure such that

(1) A map is a cofibration if and only if it is a monomorphism with cofibrantn cokernel.

(2) A map is a fibration if and only if it is an epimorphism with fibrant kernel.

Definition 1.7. A *thick subcategory* of an abelian category $\mathfrak{A}$ is a class of objects $\mathcal{W}$ which is closed under direct summands and such that if two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

Theorem 1.8. *Let $\mathcal{A}$ be an abelian category with an abelian model structure. Let $\mathcal{C}$ be the class of cofibrant objects, $\mathcal{F}$ the class of fibrant objects and $\mathcal{W}$ the class of trivial objects. then $\mathcal{W}$ is a thick subcategory of $\mathcal{A}$ and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$. Conversely, given a thick subcategory $\mathcal{W}$ and classes $\mathcal{C}$ and $\mathcal{F}$ making $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ each complete cotorsion pairs, then there is an abelian model structure on $\mathcal{A}$ where $\mathcal{C}$ are the cofibrant objects, $\mathcal{F}$ are the fibrant objects and $\mathcal{W}$ are the trivial objects.*

Proof. See [5, Theorem 2.2]. $\square$

We point out that the abelian model structure on $\mathfrak{A}$ is then completely determined by the classes $\mathcal{C}$,$\mathcal{W}$ and $\mathcal{F}$. Indeed the cofibrations(resp. trivial cofibrations) are the monomorphisms with cokernel in $\mathcal{C}$(resp. $\mathcal{C} \cap \mathcal{W}$) and the fibrations (resp. trivial fibration) are the epimorphisms with kernel in $\mathcal{F}$(resp. $\mathcal{W} \cap \mathcal{F}$). The weak equivalences are he maps which factor as a trivial cofibration followed by a trivial fibration. We call $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ a *Hovey triple*.

Definition 1.9. An exact category is a pair $(\mathcal{A}, \mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is a distinguished class of diagrams of the form $A \xrightarrow{i} B \xrightarrow{d} C$. such that i is a kernel of d (called inflation) and d is cokernel of i(called deflation) which is closed under isomorphism and satisfies the certain axioms, see [3]

A map such as i is necessarily a monomorphism and in the language of exact categories is called *admissible monomorphism(or inflation)* while d is called an *admissible epimorphism(or deflation)*.
Cotorsion pairs in exact categories. Let $(\mathfrak{A}, \mathcal{E})$ be an exact category. The Yoneda bifunctor $\mathrm{Ext}^1_{\mathcal{E}}(X, Y)$ is the abelian group of equivalence classes of short exact sequences $Y \to Z \to X$. In particular, we get that $\mathrm{Ext}^1_{\mathcal{E}}(X, Y) = 0$ if and only if every short exact sequence $Y \to Z \to X$ isomorphic to the split exact sequence $Y \to Y \oplus X \to X$.

Definition 1.10. Let $(\mathfrak{A}, \mathcal{E})$ be an exact category. For a class $\mathcal{S}$ of objects of $\mathfrak{A}$ we define

$$\mathcal{S}^{\perp} = \{B \in \mathfrak{A} \mid \mathrm{Ext}^1_{\mathcal{E}}(S, B) = 0 \text{ for all } S \in \mathcal{S}\}$$

$$^{\perp}\mathcal{S} = \{A \in \mathfrak{A} \mid \mathrm{Ext}^1_{\mathcal{E}}(A, S) = 0 \text{ for all } S \in \mathcal{S}\}$$

A pair $(\mathcal{F}, \mathcal{C})$ of full subcategory of $\mathfrak{A}$ is called a *cotorsion pair* provided that

$$\mathcal{F} = {}^{\perp}\mathcal{C} \qquad \text{and} \qquad \mathcal{F}^{\perp} = \mathcal{C}$$

A cotorsion pair is said to have *enough projective* if for any $X \in \mathfrak{A}$ there is a short exact sequence $C \to F \to X$ where $C \in \mathcal{C}$ and $F \in \mathcal{F}$. We say that it has *enough injective* if it satisfies the dual statement. If both of these hold we say the cotorsion pair is *complete*. If the cotorsion pair has enough projectives in a way that is functorial with respect to X then we say the cotorsion pair has *enough functorial projective*. Similarly, we have the terms *enough functorial injective* and *functorial complete*.
Gillespie followed Hovey's theorem and focused on exact categories with model structure compatible with the exact structure. He saw that Hovey's correspondence between abelian model structures and cotorsion pairs naturally carries over to a correspondence between exact model structures and cotorsion pairs.

Theorem 1.11. *Let $(\mathcal{A}, \mathcal{E})$ be an exact category with an exact model structure. Let $\mathcal{C}$ be the class of cofibrant objects, $\mathcal{F}$ the class of fibrant objects and $\mathcal{W}$ the class of trivial objects. then $\mathcal{W}$ is a thick subcategory of $\mathcal{A}$ and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$. If we assume $(\mathcal{A}, \mathcal{E})$ is weakly idempotent complete then the converse holds. That is, given two compatible cotorsion pairs $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, each complete and with $\mathcal{W}$ a thick subcategory, then there is an exact model structure on $\mathcal{A}$ where $\mathcal{C}$ are the cofibrant objects, $\mathcal{F}$ are the fibrant objects and $\mathcal{W}$ are the trivial objects.*

Proof. See [4, Theorem 3.3] □

2. Model structure and Gorenstein derived category

In this section we consider an exact category on $\mathrm{Ch}(R)$, and then we construct a cotorsion pair on it. With some condition on R this cotorsion pair cogenerate by a set, and hence it is complete. We Then construct an exact model structure on $\mathrm{Ch}(R)$ such that its homotopy category is Gorenstein drived category.

Definition 2.1. A short exact sequence $\mathbf{S}_{\bullet} : 0 \to X \to Y \to Z \to 0$ is called $\mathcal{GP}$-proper if $\mathrm{Hom}(G, \mathbf{S}_{\bullet})$ is exact for all $G \in$ GPrj-R.

Now let $\mathcal{E}_{\mathcal{GP}}$ be a class of short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathrm{Ch}(R)$ such that $0 \to X_i \to Y_i \to Z_i \to 0$ is $\mathcal{GP}$-proper,for all $i \in \mathbb{Z}$. Then $(\mathrm{Ch}(R), \mathcal{E}_{\mathcal{GP}})$ is a weakly idempotent complete exact category.
It is shown in [2] that $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete cotorsion pair, whenever R is a Gorenstein ring or it is a (left)coherent ring in which all flat modules have finite projective dimension.
Throughout, we always suppose that R is a ring such that $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a complete cotorsion pair.

Definition 2.2. Define $\mathcal{W}_{\mathcal{GP}}$ a class of $\mathcal{GP}$-proper complexes in Ch(R), that is

$$\mathbf{X} \in \mathcal{W}_{\mathcal{GP}}, \text{ whenever Hom}(G, \mathbf{X}) \text{ is an exact complex, for all } G \in \text{GPrj-}R$$

Definition 2.3. A complex $\mathbf{G} \in C(\text{GPrj-}R)$ is called *DG-Gorenstein projective* if Hom$(\mathbf{G}, \mathbf{W})$ is exact for all $\mathbf{W} \in \mathcal{W}_{\mathcal{GP}}$. We will use the symbol $DG\text{-}\mathcal{GP}$ to denote the class of DG-Gorenstein projective complexes.

Proposition 2.4. *$(DG\text{-}\mathcal{GP} \cap \mathcal{W}_{\mathcal{GP}}, Ch(R))$ is a complete cotorsion pair in $(Ch(R), \mathcal{E}_{\mathcal{GP}})$.*

Proof. The proof is in the same manner of the proof of Proposition 3.3 of [1] □

Now we want to show that $(DG\text{-}\mathcal{GP}, \mathcal{W}_{\mathcal{GP}})$ is a cotorsion pair in exact category (Ch$(R), \mathcal{E}_{\mathcal{GP}})$. We have the following useful lemma.

Lemma 2.5. *Suppose $W \in \mathcal{GP}^{\perp}$. Then for any $X \in Mod(R)$, we have*

$$Ext^1_R(X, W) = Ext^1_{\mathcal{E}_{\mathcal{GP}}}(X, W)$$

Proof. " $\supseteq$ " is trivial. To prove the reverse inclusion let $0 \to W \to Y \to X \to 0$ be in Ext$^1_R(X, W)$. Applying Hom$(G, -)$ to the above short exact sequence we have

$$0 \to \text{Hom}(G, W) \to \text{Hom}(G, Y) \to \text{Hom}(G, X) \to \text{Ext}^1_R(G, W) \to \cdots$$

But Ext$^1_R(G, W) = 0$, since $W \in \mathcal{GP}^{\perp}$. Therefore $0 \to W \to Y \to X \to 0$ is a $\mathcal{GP}$-proper in each degree. □

Proposition 2.6. *$^{\perp}\mathcal{W}_{\mathcal{GP}}$ is the class of DG-Gorenstein projective complexes.*

Proof. Let $\mathbf{X} \in {}^{\perp}\mathcal{W}_{\mathcal{GP}}$ and $W \in \mathcal{GP}^{\perp}$. By previous lemma we have

$$Ext^1_R(X_i, W) = Ext^1_{\mathcal{E}_{\mathcal{GP}}}(X_i, W) = Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\mathbf{X}, e^i_\rho(W)) \qquad \text{for all } i \in \mathbb{Z}$$

But $e^i_\rho(W) \in \mathcal{W}_{\mathcal{GP}}$, hence $Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\mathbf{X}, e^i_\rho(W)) = 0$, since $\mathbf{X} \in {}^{\perp}\mathcal{W}_{\mathcal{GP}}$. Therefore $Ext^1_R(X_i, W) = 0$, and hence $X_i \in$ GPrj-R, since $(\mathcal{GP}, \mathcal{GP}^{\perp})$ is a cotorsion pair.
Now it remains to show that Hom$(\mathbf{X}, \mathbf{Y})$ is an exact complex for any $\mathbf{Y} \in \mathcal{W}_{\mathcal{GP}}$. We know that

$$\text{Hom}(\mathbf{X}, \mathbf{Y}) \text{ is exact} \quad \Longleftrightarrow \quad \text{H}^i\text{Hom}(\mathbf{X}, \mathbf{Y}) \quad \Longleftrightarrow \quad \text{Hom}_{\text{K}(R)}(\mathbf{X}, \mathbf{Y}[i]) = 0$$

But $f : \mathbf{X} \to \mathbf{Y}[i]$ is homotopic to 0 if and only if the exact sequence $0 \to \mathbf{Y}[i] \to M(f) \to \mathbf{X}[1] \to 0$ associated with the mapping cone $M(f)$ splits in Ch(R). So it is enough to show that $Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\mathbf{X}[1], \mathbf{Y}[i]) = 0$, and it is clear, since $\mathcal{W}_{\mathcal{GP}}$ is closed under shifting. □

Proposition 2.7. $({}^{\perp}\mathcal{W}_{\mathcal{GP}})^{\perp} = \mathcal{W}_{\mathcal{GP}}$

Proof. Clearly $\mathcal{W}_{\mathcal{GP}} \subseteq ({}^{\perp}\mathcal{W}_{\mathcal{GP}})^{\perp}$. So let $\mathbf{W} \in ({}^{\perp}\mathcal{W}_{\mathcal{GP}})^{\perp}$. We show that $\mathbf{W}$ is a $\mathcal{GP}$-proper complex. Suppose $G \in$ GPrj-R, and let $\underline{G}$ be a complex of Gorenstein projective module concentrated at 0, i.e. $G_n = 0$ if $n \neq 0$. Then clearly $\underline{G}$ (and so also any $\underline{G}[n], n \in \mathbb{Z}$) is DG- Gorenstein projective. But by proof of the previous proposition, Hom$(\underline{G}, textbfW)$ is exact, since $Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\underline{G}, \mathbf{W}) = 0$. □

Our intention now is to show that $(DG\text{-}\mathcal{GP}, \mathcal{W}_{\mathcal{GP}})$ is a complete cotorsion pair, whenever K(GPrj-R) is compactly generated.

Proposition 2.8. *Suppose $\mathbf{X} \in Ch(R)$. Then we have*

$$Ext^1_{\mathcal{E}_{\mathcal{GP}}}(G[i], \mathbf{X}) = H^i Hom(G, \mathbf{X}) \qquad \textit{for all } G \in \textit{GPrj-}R$$

Proof. Suppose $G \in$ GPrj-R. Let $\overline{G}$ be the complex

$$\cdots \longrightarrow 0 \longrightarrow G \xrightarrow{id} G \longrightarrow 0 \longrightarrow \cdots$$

with two G's in the 1st and 0th places. We let $\underline{G}$ be the complex with G in the 0th places. Then $\underline{G}$ is a subcomplex of $\overline{G}$, and we have $\overline{G}/\underline{G} = \underline{G}[1]$. So we have short exact sequence $\mathbf{S}_\bullet : 0 \to \underline{G} \to \overline{G} \to \underline{G}[1] \to 0$. Applying $\mathrm{Hom}(-, \mathbf{X})$ to the $\mathbf{S}_\bullet$, we have

$$\mathrm{Hom}(\overline{G}, \mathbf{X}) \xrightarrow{\varphi} \mathrm{Hom}(\underline{G}, \mathbf{X}) \xrightarrow{\psi} Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\underline{G}[1], \mathbf{X}) \longrightarrow Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\overline{G}, \mathbf{X})$$

But $Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\overline{G}, \mathbf{X}) = 0$, since every $0 \to \mathbf{X} \to \mathbf{Y} \to \overline{G} \to 0$ is split. Therefore we can say that

$$Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\underline{G}[1], \mathbf{X}) \cong \frac{\mathrm{Hom}(\underline{G}, X)}{\mathrm{Ker}\psi = \mathrm{Im}\varphi}$$

Now consider

$$\cdots \longrightarrow \mathrm{Hom}(G, X_1) \xrightarrow{d_1^*} \mathrm{Hom}(G, X_0) \xrightarrow{d_0^*} \mathrm{Hom}(G, X_{-1}) \longrightarrow \cdots$$

We have a map

$$\eta : \mathrm{Hom}(\underline{G}, \mathbf{X}) \longrightarrow \frac{\mathrm{Ker} d_0^*}{\mathrm{Im} d_1^*}$$

defined by the formula $f \longmapsto f_0$ for $f = (f_n)_{n \in \mathbb{Z}} \in \mathrm{Hom}(\underline{G}, \mathbf{X})$. clearly, $\mathrm{Ker}\eta = \mathrm{Im}\varphi$. This gives that $Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\underline{G}[1], \mathbf{X}) \cong \mathrm{H}_0(\mathrm{Hom}(G, \mathbf{X}))$.
More generally we have that

$$Ext^1_{\mathcal{E}_{\mathcal{GP}}}(\underline{G}[k+1], \mathbf{X}) \cong \mathrm{H}_k(\mathrm{Hom}(G, \mathbf{X})) \qquad \text{for any } k \in \mathbb{Z}$$

□

Proposition 2.9. $(DG\text{-}\mathcal{GP}, \mathcal{W}_{\mathcal{GP}})$ *is a complete cotorsion pair, whenever* $K(GPrj\text{-}R)$ *is compactly generated.*

Proof. It is well known that if $\mathcal{B} =$ GPrj-R, then the following are equivalent:

$$\mathrm{K}(\mathrm{GPrj}\text{-}R) \quad \text{is well generated} \quad \Longleftrightarrow \quad \mathrm{GPrj}\text{-}R \subseteq \mathrm{Add}S \quad \text{for some set} \quad S \subseteq \mathrm{GPrj}\text{-}R$$

By proposition 2.8

$$Ext^1_{\mathcal{E}_{\mathcal{GP}}}(G[i], \mathbf{X}) = \mathrm{H}^i\mathrm{Hom}(G, \mathbf{X}) \qquad \text{for all } G \in \mathrm{GPrj}\text{-}R$$

Hence a complex $textbfX$ is $\mathcal{GP}$-proper if and only if $Ext^1_{\mathcal{E}_{\mathcal{GP}}}(G[i], \mathbf{X}) = 0$. On the other hand K(GPrj-R) is a well generated, since it is compactly generated. Thus K(GPrj-R) $\subseteq$ Add$S$ whenever $S \subseteq$ GPrj-R. Now if consider $T = \{G[i] | G \in S\}$ then T is a set and $T^{\perp} = \mathcal{W}_{\mathcal{GP}}$. □

Now we find a Hovey pair so we can use Theorem 1.11 and construct a model structure on the category of complexes which its homotopy is exactly Gorenstein derive category.

Theorem 2.10. *According to the above assumptions, there is an exact model structure on the exact category* $(Ch(R), \mathcal{E}_{\mathcal{GP}})$*. In this model structure,* $DG\text{-}\mathcal{GP}$ *is the class of cofibrant objects,* $Ch(R)$ *is the class of fibrant objects and* $\mathcal{W}_{\mathcal{GP}}$ *is the class of trivial objects. As usual, we denote this model structure by the triple* $(DG\text{-}\mathcal{GP}; \mathcal{W}_{\mathcal{GP}}; Ch(R))$*. The homotopy category* $HoCh(R)$ *is equivalent to* $D_{\mathcal{GP}}(R)$ *as triangulated categories.*

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