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# Investigate a hybrid fractional Langevin inclusion involving q-derivative and variable coefficient

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**Abstract**

In this manuscript, we investigate a class of hybrid Langevin inclusion along with variable coefficient involving the generalized q-derivative. In fact, based on the famous fractional q-derivative that depends on other function $\lambda$, namely $\lambda$-Caputo fractional q-derivative, the proposed inclusion containing a variable coefficient is generalized. According to instructions of Dhage's fixed point theorem, we examine the required conditions for the existence of a solution for the hybrid q-differential Langevin inclusion, which includes checking variable coefficients. Presenting some practical examples at the end, together with the conclusion, will stimulate the theoretical achievements.

## 1. Introduction

It has been suggested to all researchers that fractional calculus is a better simulation for solving phenomena problems such as industrial engineering, environmental and biological studies [2, 6, 10, 11, 13]. The fractional equation of Langevin type ($\mathbb{FLE}$) generalizes the public $\mathbb{LE}$ by fractional derivative and integral. This characterization useful for describing more complex problems, which are not well defined by the general $\mathbb{LE}$. In 1908, Langevin introduced an integer-order equation $l_1\mathbb{y}''(\rho) = -l_2\mathbb{y}'(\rho) + \hbar(\rho)$ with parameters the mass of a Brownian particle $l_1$, the viscous force $-l_2\mathbb{y}'(\rho)$ with coefficient and the fluctuating force $\hbar(\rho)$. The general form $\mathbb{FLE}$ along with necessary parameters, is given as

$$l_1 \frac{\mathrm{d}^{\wp_1}\mathbb{y}(\rho)}{\mathrm{d}\rho^{\wp_1}} = -l_2 \frac{\mathrm{d}^{\wp_2}\mathbb{y}(\rho)}{\mathrm{d}\rho^{q}} + \hbar(\rho) + \mathcal{W}(\mathbb{y}(\rho)).$$

It is natural, having variable coefficients in the $\mathbb{LE}$, It will increase the range of investigation of complex equations of Langevin type [8, 13, 17].

As a valuable generalization with q-fractional differential inclusions (q-$\mathbb{FDI}$s) involving Caputo derivative of order $\alpha > 0$, ${}^{\mathrm{C}}_{\mathrm{q}}\mathbb{D}^{\wp}\mathbb{y}(\rho) \in \mathcal{W}(\rho, \mathbb{y}(\rho))$ where $\mathcal{W}(\rho, \mathbb{y}(\rho))$ is a set-valued map, we aim to study the following hybrid q-$\mathbb{FDI}$,

$$\begin{cases} {}^{\mathrm{C}}_{\mathrm{q}}\mathbb{D}^{r,\lambda}_{\mathrm{l}_1}\left({}^{\mathrm{C}}_{\mathrm{q}}\mathbb{D}^{\wp,\lambda}_{\mathrm{l}_1}\left(\frac{\mathbb{y}(\rho)}{\chi(\rho,\mathbb{y}(\rho))}\right) + \hbar(\rho)\mathbb{y}(\rho)\right) \in \mathcal{W}(\rho, \mathbb{y}(\rho)), & 0 < r, \wp < 1, \\ \mathbb{y}(\mathrm{l}_1) = 0, \quad \mathbb{y}(\mathrm{l}_2) = \sum_{\iota=1}^{\aleph} \ni_\iota \, {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{s_\iota,\lambda}\mathbb{y}(\eta_\iota), & s_\iota > 0, \end{cases} \tag{1}$$

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for $\rho \in E := [l_1, l_2]$, with a set-valued map $\mathcal{W} : E \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R}) = \{A \subset \mathbb{R} : A \neq \emptyset\}$, $\hbar \in C(\mathbb{R}^+)$, and $\chi \in C(E \times \mathbb{R}, \mathbb{R} \setminus \{0\})$.

## 2. Auxiliary notations and lemmas

The objects $[\wp]_q$ (q-number) and $(\wp - qv)^{(j)}$ (q-factorial function), are expressed for $\wp, v \in \mathbb{R}$, by $\frac{1-q^\wp}{1-q}$ and

$$(\wp - qv)^{(\mathring{\jmath})} = \begin{cases} \prod_{l=0}^{\mathring{\jmath}-1}(\wp - q^l v), & \mathring{\jmath} \in \mathbb{N}, \\ 1, & \mathring{\jmath} = 0, \end{cases} \tag{2}$$

respectively [1, 9]. Clearly, $(1-q)^{(\mathring{\jmath})} = \prod_{l=0}^{\mathring{\jmath}-1}(1-q^{l+1})$ for $\mathring{\jmath} \in \mathbb{N}$ whenever $\wp = 1$, $v = q$. For check related algorithms, see [16]. Further,

$$(\wp - qv)^{(\varrho)} = \wp^\varrho \prod_{l=0}^{\infty} \frac{\wp - q^l v}{\wp - q^{\varrho+l} v}, \qquad \varrho \in [0, \infty),\ \wp, v \in \mathbb{R}, \tag{3}$$

and $\wp^{(\varrho)} = \wp^\varrho$ whenever $v = 0$ [7]. The fractional q-integral and q-derivative of RL type for the function $\mathbb{t} \in C(\mathbb{R}^{>0})$ are expressed as [3, 14],

$${}_q\mathbb{I}_0^\wp \mathbb{t}(\varrho) = \begin{cases} \int_0^\varrho (\varrho - qr)^{(\wp-1)} \frac{\mathbb{t}(r)}{\Gamma_q(\wp)} \, \mathrm{d}_q r, & \wp > 0, \\ \mathbb{t}(\varrho), & \wp = 0, \end{cases} \tag{4}$$

and

$${}_q\mathbb{D}_0^\wp \mathbb{t}(\varrho) = \int_0^\varrho (\varrho - qr)^{(\aleph-\wp-1)} \frac{\mathbb{t}(r)}{\Gamma_q(\aleph-\wp)} \, \mathrm{d}_q r, \qquad \aleph - 1 < \wp < \aleph, \tag{5}$$

respectively, s.t. $\aleph = [\wp]$ along with the famous q-Gamma function, $\Gamma_q(\wp) = \frac{(1-q)^{(\wp-1)}}{(1-q)^{\wp-1}}$, $\forall \wp \in \mathbb{R} \setminus \{\cdots, -2, -1, 0\}$, and the relation $\Gamma_q(\wp + 1) = [\wp]_q \Gamma_q(\wp)$ holds [7, 9]. In the sense of Caputo, the fractional q-derivative is given for $\varrho \in \mathsf{W}_\tau$ as [14],

$${}^{\mathrm{C}}_q\mathbb{D}_0^\wp \mathbb{t}(\varrho) = \begin{cases} {}_q\mathbb{I}_0^{[\wp]-\wp} \left[ {}_q\mathbb{D}_0^{[\wp]} \mathbb{t}(\varrho) \right], & \wp > 0, \\ \mathbb{t}(\varrho), & \wp = 0. \end{cases} \tag{6}$$

**Lemma 2.1** ([3, 14]). *For function* $\mathbb{t} : \mathsf{W}_\tau \to \mathbb{R}$ *and* $\wp_1, \wp_2 \in \mathbb{R}^{\geq 0}$, *we have* ${}^{\mathrm{C}}_q\mathbb{D}_0^{\wp_1} \left[ {}_q\mathbb{I}_0^{\wp_1} \mathbb{t}(\varrho) \right] = \mathbb{t}(\varrho)$, ${}_q\mathbb{I}_0^{\wp_1} \left[ {}_q\mathbb{I}_0^{\wp_2} \mathbb{t}(\varrho) \right] = {}_q\mathbb{I}_0^{\wp_1+\wp_2} \mathbb{t}(\varrho)$, *and*

$$\begin{aligned} {}_q\mathbb{I}_0^{\wp_1} \left[ {}^{\mathrm{C}}_q\mathbb{D}_0^{\wp_1} \mathbb{t}(\varrho) \right] &= \mathbb{t}(\varrho) - \sum_{\mathring{\jmath}=0}^{[\wp_1]-1} \frac{\varrho^{\mathring{\jmath}}}{\Gamma_q(\mathring{\jmath}+1)} \, {}_q\mathbb{D}_0^{\mathring{\jmath}} \mathbb{t}(\varrho)\,(0)\,, \\ {}_q\mathbb{I}_0^{\wp_1} \left[ {}_q\mathbb{D}_0^{\wp_2} \mathbb{t}(\varrho) \right] &= {}_q\mathbb{D}_0^{\wp_2} \left[ {}_q\mathbb{I}_0^{\wp_1} \mathbb{t}(\varrho) \right] - \sum_{\mathring{\jmath}=0}^{[\wp_2]-1} \frac{\varrho^{\wp_1-\wp_2+\mathring{\jmath}}}{\Gamma_q(\wp_1+\mathring{\jmath}-\wp_2+1)} \, {}_q\mathbb{D}_0^{\mathring{\jmath}} \mathbb{t}(0). \end{aligned} \tag{7}$$

The RL type fractional q-integral for function $\mathbb{t}$ is get by [16],

$$\int_0^\varrho (\varrho - qr)^{(\wp-1)} \frac{\mathbb{t}(r)}{\Gamma_q(\wp)} \, \mathrm{d}_q r = \frac{\varrho^\wp (1-q)}{\Gamma_q(\wp)} \sum_{\mathring{\jmath}=0}^{\infty} q^{\mathring{\jmath}} \prod_{\mathring{\imath}=0}^{\infty} \frac{1-q^{\mathring{\jmath}+\mathring{\imath}}}{1-q^{\wp+\mathring{\jmath}+\mathring{\imath}-1}} \mathbb{t}\left(\varrho q^{\mathring{\jmath}}\right). \tag{8}$$

See [16, Algorithm 2], to can calculate fractional q-integral (8). For $\wp \geq 0$ and $\varkappa > -1$, we have

$$\left({}_q\mathbb{D}_0^\wp \left[(\varrho - \kappa)^{(\varkappa)}\right]\right)(\varrho) = \frac{\Gamma_q(\varkappa+1)}{\Gamma_q(\wp+\varkappa+1)} (\varrho - \kappa)^{(\wp+\varkappa)}, \qquad 0 < \kappa < \varrho, \tag{9}$$

and in particular $\left({}_q\mathbb{D}_0^\wp 1\right)(\varrho) = \frac{1}{\Gamma_q(\wp+1)} \varrho^{(\wp)}$ [15].

**Theorem 2.2** ([4, 5]). *Assume that $Y$ be a Banach algebra and consider the single-valued and multi-valued operators $\mathfrak{O}_{sv} : Y \to Y$ and $\mathfrak{O}_{mv} : Y \to \mathcal{P}_{cl,cv}(Y)$, with $\mathcal{P}_{cl,cv}(Y)$ as the family of subset $A \in \mathcal{P}(Y)$ s.t., $A$ is closed, bounded, and convex, is satisfied* I) $\mathfrak{O}_{sv}$ *is Lipschitz;* II) $\mathfrak{O}_{mv}$ *is compact and upper semicontinuous;* III) $2\mathfrak{u}\mathfrak{N} < 1$ *with* $\mathfrak{u} := \|\mathfrak{O}_{sv}(Y)\|$ *and* $\mathfrak{N} := \|\mathfrak{O}_{mv}(Y)\|$. *Then either* a) *the operator inclusion* $\mathfrak{y} \in \mathfrak{O}_{sv}\mathfrak{y}\mathfrak{O}_{mv}\mathfrak{y}$ *has a solution; or* b) $\Sigma = \{\mathfrak{y} \in Y : \gamma\mathfrak{y} \in \mathfrak{O}_{sv}\mathfrak{y}\,\mathfrak{O}_{mv}\mathfrak{y},\ 1 < \gamma\}$ *is unbounded.*

## 3. Main results

First, we consider the below mentioned key lemma.

**Lemma 3.1.** *The function* $\mathfrak{y}(\rho)$ *can be considered as a solution of the hybrid* q-$\mathbb{FDI}$

$$ {}^{C}_{q}\mathbb{D}^{r,\lambda}_{l_1}\left(\frac{\mathfrak{y}(\rho)}{\chi(\rho,\mathfrak{y}(\rho))}\right) + \hbar(\rho)\mathfrak{y}(\rho)\Big) = \varpi(\rho), \tag{10} $$

*for* $\rho \in E$, *under conditions* $\mathfrak{y}(l_1) = 0$ *and* $\mathfrak{y}(l_2) = \sum_{i=1}^{\aleph} \ni_i\, {}^{RL}_{q}\mathbb{I}^{s_i,\lambda}\mathfrak{y}(\eta_i)$, *if and only if*

$$ \begin{aligned} \mathfrak{y}(\rho) = \chi(\rho,\mathfrak{y}(\rho))\Bigg\{ & {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathfrak{y}(\rho) + \frac{(\lambda_{l_1}(\rho))^{\wp}}{\Omega\Gamma_q(\wp+1)}\Big[\chi(l_2,\mathfrak{y}(l_2))\, {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(l_2) \\ & - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\chi(l_2,\mathfrak{y}(l_2))\,\hbar(l_2)\,\mathfrak{y}(l_2) - \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i,\mathfrak{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi(\eta_i) \\ & + \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i,\mathfrak{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{\wp+s_i,\lambda}\hbar(\eta_i)\Big]\Bigg\}. \end{aligned} \tag{11} $$

*where* $\lambda_v(z) := \lambda(z) - \lambda(v)$ *and*

$$ \Omega := \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i,\mathfrak{y}(\eta_i))\frac{(\lambda_{l_1}(\eta_i))^{\wp+s_i}}{\Gamma_q(\wp+s_i+1)} + \chi(l_2,\mathfrak{y}(l_2))\frac{(\lambda_{l_1}(l_2))^{\wp}}{\Gamma_q(\wp+1)}. \tag{12} $$

*Proof.* Employing to Lemma 2.1 and taking ${}^{RL}_{q}\mathbb{I}^{r,\lambda}$ on hybrid q-$\mathbb{FDI}$ (10), for $\rho \in E$, imply that,

$$ {}^{C}_{q}\mathbb{D}^{\wp,\lambda}_{l_1}\left(\frac{\mathfrak{y}(\rho)}{\chi(\rho,\mathfrak{y}(\rho))}\right) + \hbar(\rho)\mathfrak{y}(\rho) = {}^{RL}_{q}\mathbb{I}^{r,\lambda}\varpi(\rho) + \tilde{k}_0, \qquad \tilde{k}_0 \in \mathbb{R}. \tag{13} $$

Again, due to the same two previous instructions, with ${}^{RL}_{q}\mathbb{I}^{\wp,\lambda}$ on Eq. (13), we get,

$$ \frac{\mathfrak{y}(\rho)}{\chi(\rho,\mathfrak{y}(\rho))} = {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathfrak{y}(\rho) + \frac{\tilde{k}_0}{\Gamma_q(r+1)}\left(\lambda_{l_1}(\rho)\right)^{\wp} + \tilde{k}_1, \qquad \tilde{k}_1 \in \mathbb{R}. \tag{14} $$

Thanks to condition $\mathfrak{y}(l_1) = 0$, Eq. (14) implies that $\tilde{k}_1 = 0$. Thus, we can write,

$$ \mathfrak{y}(\rho) = \chi(\rho,\mathfrak{y}(\rho))\left\{ {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathfrak{y}(\rho) + \frac{\tilde{k}_0}{\Gamma_q(r+1)}\left(\lambda_{l_1}(\rho)\right)^{\wp}\right\}. \tag{15} $$

Now, by examining the condition $\mathfrak{y}(l_2) = \sum_{i=1}^{\aleph} \ni_i\, {}^{RL}_{q}\mathbb{I}^{s_i,\lambda}\mathfrak{y}(\eta_i)$, we obtain

$$ \begin{aligned} \tilde{k}_0 = \frac{1}{\Omega}\Bigg\{ & {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathfrak{y}(\rho) - \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i,\mathfrak{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi(\eta_i) \\ & + \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i,\mathfrak{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{\wp+s_i,\lambda}\hbar(\eta_i)\Bigg\}. \end{aligned} $$

The obtained $\tilde{k}_0$ in Eq. (15) yields the solution which is given by Eq. (11). □

**Definition 3.2.** We say an element $\mathbb{y} \in C(E)$ can be a solution of hybrid q-$\mathbb{FDI}$ (1), whenever $\mathbb{y}(\mathrm{l}_1) = 0$, $\mathbb{y}(\mathrm{l}_2) = \sum_{\ell=1}^{\aleph} \ni_\ell {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{s_\ell,\lambda}\mathbb{y}(\eta_\ell)$, and exists $\varpi \in L^1(E)$ belongs to $\mathcal{W}(\rho, \mathbb{y}(\rho))$ a.e. on $E$ s.t.,

$$\begin{aligned}\mathbb{y}(\rho) = \chi(\rho,\mathbb{y}(\rho))\Bigg\{ & {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Big[\chi(\mathrm{l}_2,\mathbb{y}(\mathrm{l}_2)) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\mathrm{l}_2) \\ & - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\chi(\mathrm{l}_2,\mathbb{y}(\mathrm{l}_2))\hbar(\mathrm{l}_2)\mathbb{y}(\mathrm{l}_2) - \sum_{\ell=1}^{\aleph} \ni_\ell \chi(\eta_\ell,\mathbb{y}(\eta_\ell)) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp+s_\ell,\lambda}\varpi(\eta_\ell) \\ & + \sum_{\ell=1}^{\aleph} \ni_\ell \chi(\eta_\ell,\mathbb{y}(\eta_\ell)) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_\ell,\lambda}\hbar(\eta_\ell)\Big]\Bigg\}.\end{aligned}$$

For $\mathbb{y} \in C(E)$, we consider the below mentioned set of selections of $\mathcal{W}$,

$$\mathbb{S}_{\mathcal{W},\mathbb{y}} := \Big\{\varpi \in L^1(E) \,:\, \varpi \in \mathcal{W}(\rho,\mathbb{y}(\rho)),\ \text{for a.e } \mathbb{y}(\rho) \in E\Big\}. \tag{16}$$

**Lemma 3.3** ([12]). *For a linear continuous function $\Psi : L^1(E,\mathcal{G}) \to C(E,\mathcal{G})$, the composite function $\Psi \circ \mathbb{S}_{\mathcal{W},\mathbb{y}} : C(E,\mathcal{G}) \to \mathcal{P}_{\mathrm{cl,bd}}(C(E,\mathcal{G}))$, is a closed graph operator in $C(E,\mathcal{G}) \times C(E,\mathcal{G})$.*

**Theorem 3.4.** *Consider the below mentions assumptions:*

P1) $\chi \in C\big(E \times \mathbb{R}, \mathbb{R} \setminus \{0\}\big)$, *the bounded function* $\alpha(\rho) > 0$ *for* $\rho \in E$ *and for* $\mathbb{y}_1, \mathbb{y}_2 \in \mathbb{R}$

$$|\chi(\rho,\mathbb{y}_1(\rho)) - \chi(\rho,\mathbb{y}_2(\rho))| \le \alpha(\rho)|\mathbb{y}_1(\rho) - \mathbb{y}_2(\rho)|.$$

P2) $\mathcal{W} \in C : E \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is $L^1$-Carathéodory and possesses nonempty convex values, and for every* $\mathbb{y} \in C(E)$, $\mathbb{S}_{\mathcal{W},\mathbb{y}} \neq \emptyset$ *is convex.*

P3) $|\mathcal{W}(\rho,\mathbb{y})| := \sup\{|\varpi| \,:\, \varpi \in \mathcal{W}(\rho,\mathbb{y})\} \le \beta(\rho)(\mathbb{y})$ *for each* $\rho \in E$ *and* $\mathbb{y} \in C(E)$, *s.t.,* $\beta \in L^1(E,\mathbb{R}^+)$ *and* $\nabla : \mathbb{R}^+ \to [0,+\infty)$ *is a continuous, bounded, and nondecreasing function.*

*Then, hybrid* q-$\mathbb{FDI}$ (1) *possesses at least one solutions on $E$ whenever* $\Delta := \|\alpha\|\,(\|\beta\|\nabla(\|\mathbb{y}\|)\Upsilon + \Upsilon_\hbar) < \frac{1}{2}$, *where*

$$\begin{aligned}\Upsilon &= \frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{r+\wp}}{\Gamma_{\mathrm{q}}(r+\wp+1)} + \frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{\wp}}{|\Omega|\Gamma_{\mathrm{q}}(\wp+1)}\Big[\hat{\chi}\frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{r+\wp}}{\Gamma_{\mathrm{q}}(r+\wp+1)} + \hat{\chi}\sum_{\ell=1}^{\aleph}|\ni_\ell|\,\frac{(\lambda_{\mathrm{l}_1}(\eta_\ell))^{r+\wp+s_\ell}}{\Gamma_{\mathrm{q}}(r+\wp+s_\ell+1)}\Big], \\ \Upsilon_\hbar &= {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\big|\hbar(\mathrm{l}_2)\big| + \frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{\wp}}{|\Omega|\Gamma_{\mathrm{q}}(\wp+1)}\Big[\hat{\chi}\,{}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r,\lambda}|\hbar(\mathrm{l}_2)| + \hat{\chi}\sum_{\ell=1}^{\aleph}|\ni_\ell|\,{}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+s_\ell,\lambda}\big|\hbar(\eta_\ell)\big|\Big],\end{aligned} \tag{17}$$

*and* $\hat{\chi} = \max\{|\chi(\mathrm{l}_2,\mathbb{y}(\mathrm{l}_2))|, |\chi(\eta_\ell,\mathbb{y}(\eta_\ell))|\}$, $\ell = 1, 2, \dots, \aleph$.

*Proof.* We aim to convert the hybrid q-$\mathbb{FDI}$ (1) into a fixed point problem. In this case, we introduce the multivalued map $\Psi : C(E) \to \mathcal{P}(C(E))$, by

$$\Psi\mathbb{y} = \left\{\begin{array}{l}\mathrm{t} \in C(E) : \\ \mathrm{t}(\rho) = \left\{\begin{array}{l}\chi(\rho,\mathbb{y}(\rho))\Big\{ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) \\ \quad + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Big[ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\chi(\mathrm{l}_2,\mathbb{y}(\mathrm{l}_2))\varpi(\mathrm{l}_2) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\chi(\mathrm{l}_2,\mathbb{y}(\mathrm{l}_2))\hbar(\mathrm{l}_2)\mathbb{y}(\mathrm{l}_2) \\ \quad - \sum_{\ell=1}^{\aleph} \ni_\ell \chi(\eta_\ell,\mathbb{y}(\eta_\ell)) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp+s_\ell,\lambda}\varpi(\eta_\ell) \\ \quad + \sum_{\ell=1}^{\aleph} \ni_\ell {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_\ell,\lambda}\chi(\eta_\ell,\mathbb{y}(\eta_\ell))\hbar(\eta_\ell)\mathbb{y}(\eta_\ell)\Big]\Big\}, \quad \rho \in E,\ \varpi \in \mathbb{S}_{\mathcal{W},\mathbb{y}}\end{array}\right\}\end{array}\right\}. \tag{18}$$

Further, we define two operators $\mathfrak{O}_{sv} :: C(E) \to C(E)$ and $\mathfrak{O}_{mv} : C(\mathcal{O}) \to \mathcal{P}(C(E))$ with following definitions: $\mathfrak{O}_{sv}\mathbb{y}(\rho) = \chi\big(\rho, \mathbb{y}(\rho)\big)$, and

$$\mathfrak{O}_{mv}(\mathbb{y}) = \Big\{ \mathfrak{t} \in C(E) : \mathfrak{t}(\rho) = {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho)$$
$$+ \frac{(\lambda_{l_1}(\rho))^{\wp}}{\Omega\Gamma_q(\wp+1)}\Big[ {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\varpi(l_2) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\hbar(l_2)\,\mathbb{y}(l_2)$$
$$- \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i, \mathbb{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi(\eta_i) + \sum_{i=1}^{\aleph} \ni_i\, {}^{RL}_{q}\mathbb{I}^{\wp+s_i,\lambda}\chi(\eta_i, \mathbb{y}(\eta_i))\,\hbar(\eta_i)\,\mathbb{y}(\eta_i)\Big],\ \varpi \in \mathbb{S}_{\mathcal{W},\mathbb{y}}\Big\}.$$

Now, we establish the operator $\Psi\mathbb{y} = \mathfrak{O}_{sv}\mathbb{y}\,\mathfrak{O}_{mv}\mathbb{y}$ and check Theorem 2.2 for these operators. To do this, we continue the proof process in several steps.

*Step 1.* Let $\mathbb{y}_1, \mathbb{y}_2 \in C(E)$, $\rho \in E$ and utilizing (P1) in Theorem 3.4, we have

$$\big|\mathfrak{O}_{sv}\mathbb{y}_1(\rho) - \mathfrak{O}_{sv}\mathbb{y}_2(\rho)\big| \le |\chi(\rho, \mathbb{y}_1(\rho)) - \chi(\rho, \mathbb{y}_2(\rho))| \le |\alpha(\rho)|\,|\mathbb{y}_1(\rho) - \mathbb{y}_2(\rho)| \le \alpha^*|\mathbb{y}_1 - \mathbb{y}_2|, \tag{19}$$

where $\alpha^* = \sup_{\rho\in E}|\alpha(\rho)|$. Thus, $\big\|\mathfrak{O}_{sv}\mathbb{y}_1(\rho) - \mathfrak{O}_{sv}\mathbb{y}_2(\rho)\big\| \le \alpha^*\|\mathbb{y}_1 - \mathbb{y}_2\|$ which implies that $\mathfrak{O}_{sv}$ is Lipschitzian.

*Step 2.* For $\mathfrak{t}_j \in \mathfrak{O}_{mv}\mathbb{y}$, there is $\varpi_j \in \mathbb{S}_{\mathcal{W},\mathbb{y}}$, $j = 1, 2$, s.t.,

$$\mathfrak{t}_j(\rho) = {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi_j(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) + \frac{(\lambda_{l_1}(\rho))^{\wp}}{\Omega\Gamma_q(\wp+1)}\Big[ {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\varpi_j(l_2)$$
$$- {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\hbar(l_2)\,\mathbb{y}(l_2) - \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i, \mathbb{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi_j(\eta_i)$$
$$+ \sum_{i=1}^{\aleph} \ni_i\, {}^{RL}_{q}\mathbb{I}^{\wp+s_i,\lambda}\chi(\eta_i, \mathbb{y}(\eta_i))\,\hbar(\eta_i)\,\mathbb{y}(\eta_i)\Big], \quad \forall \rho \in E.$$

Hence, by assuming $0 \le \ell \le 1$, we obtain

$$\ell\mathfrak{t}_1(\rho) - (1-\ell)\mathfrak{t}_2(\rho) = {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}(\ell\varpi_1(\rho) - (1-\ell)\varpi_2(\rho)) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho)$$
$$+ \frac{(\lambda_{l_1}(\rho))^{\wp}}{\Omega\Gamma_q(\wp+1)}\Big[ {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,(\ell\varpi_1(l_2) - (1-\ell)\varpi_2(l_2)) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\hbar(l_2)\,\mathbb{y}(l_2)$$
$$- \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i, \mathbb{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{r+\wp+s_i,\lambda}(\ell\varpi_1(\eta_i) - (1-\ell)\varpi_2(\eta_i))$$
$$+ \sum_{i=1}^{\aleph} \ni_i\, {}^{RL}_{q}\mathbb{I}^{\wp+s_i,\lambda}\chi(\eta_i, \mathbb{y}(\eta_i))\,\hbar(\eta_i)\,\mathbb{y}(\eta_i)\Big], \quad \forall \rho \in E.$$

Thanks to the convexity $\mathcal{W}$, $\ell\mathfrak{t}_1(\rho) - (1-\ell)\mathfrak{t}_2(\rho) \in \mathfrak{O}_{mv}(\mathbb{y})$.

*Step 3.* Consider the bounded ball $B_p = \{\mathbb{y} \in C(E) : \|\mathbb{y}\| \le p\}$, $p > 0$. Then $\forall\, \mathfrak{t} \in \mathfrak{O}_{mv}(\mathbb{y})$ and $\mathbb{y} \in B_p$, there is $\varpi \in \mathbb{S}_{\mathcal{W},\mathbb{y}}$ in which

$$\mathfrak{t}(\rho) = {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) + \frac{(\lambda_{l_1}(\rho))^{\wp}}{\Omega\Gamma_q(\wp+1)}\Big[ {}^{RL}_{q}\mathbb{I}^{r+\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\varpi(l_2)$$
$$- {}^{RL}_{q}\mathbb{I}^{\wp,\lambda}\chi(l_2, \mathbb{y}(l_2))\,\hbar(l_2)\,\mathbb{y}(l_2) - \sum_{i=1}^{\aleph} \ni_i \chi(\eta_i, \mathbb{y}(\eta_i))\, {}^{RL}_{q}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi(\eta_i)$$
$$+ \sum_{i=1}^{\aleph} \ni_i\, {}^{RL}_{q}\mathbb{I}^{\wp+s_i,\lambda}\chi(\eta_i, \mathbb{y}(\eta_i))\,\hbar(\eta_i)\,\mathbb{y}(\eta_i)\Big].$$

It means that,

$$
\begin{aligned}
|\mathfrak{t}(\rho)| \leq \sup_{\rho\in E}\Bigg| & {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Bigg[ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\chi\left(\mathrm{l}_2,\mathbb{y}\left(\mathrm{l}_2\right)\right)\varpi\left(\mathrm{l}_2\right) \\
& - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\chi\left(\mathrm{l}_2,\mathbb{y}\left(\mathrm{l}_2\right)\right)\hbar\left(\mathrm{l}_2\right)\mathbb{y}\left(\mathrm{l}_2\right) - \sum_{i=1}^{\aleph}\ni_i\,\chi\left(\eta_i,\mathbb{y}\left(\eta_i\right)\right) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi\left(\eta_i\right) \\
& + \sum_{i=1}^{\aleph}\ni_i\ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_i,\lambda}\chi\left(\eta_i,\mathbb{y}\left(\eta_i\right)\right)\hbar\left(\eta_i\right)\mathbb{y}\left(\eta_i\right)\Bigg]\Bigg| \\
\leq\ & \|\beta\|\,\nabla\left(\|\mathbb{y}\|\right)\left\{\frac{\left(\lambda_{\mathrm{l}_1}(\mathrm{l}_2)\right)^{r+\wp}}{\Gamma_{\mathrm{q}}(r+\wp+1)} + \frac{\left(\lambda_{\mathrm{l}_1}(\mathrm{l}_2)\right)^{\wp}}{|\Omega|\Gamma_{\mathrm{q}}(\wp+1)}\left[\hat{\chi}\frac{\left(\lambda_{\mathrm{l}_1}(\mathrm{l}_2)\right)^{r+\wp}}{\Gamma_{\mathrm{q}}(r+\wp+1)} + \hat{\chi}\sum_{i=1}^{\aleph}|\ni_i|\frac{\left(\lambda_{\mathrm{l}_1}(\eta_i)\right)^{r+\wp+s_i}}{\Gamma_{\mathrm{q}}(r+\wp+s_i+1)}\right]\right\} \\
& + \|\mathbb{y}\|\left\{ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\left|\hbar(\rho)\mathbb{y}(\rho)\right| + \frac{\left(\lambda_{\mathrm{l}_1}(\mathrm{l}_2)\right)^{r}}{|\Omega|\Gamma_{\mathrm{q}}(\wp+1)}\left[\hat{\chi}\ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\left|\hbar(\eta_i)\mathbb{y}(\eta_i)\right| + \hat{\chi}\sum_{i=1}^{\aleph} {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_i,\lambda}|\ni_i|\ \left|\hbar(\eta_i)\mathbb{y}(\eta_i)\right|\right]\right\} \\
\leq\ & \|\beta\|\nabla(\|\mathbb{y}\|)\Upsilon + \|\mathbb{y}\|\Upsilon_{\hbar} \leq \|\beta\|\nabla(p)\Upsilon + \beta\Upsilon_{\hbar}.
\end{aligned}
$$

Indeed $\mathfrak{O}_{\mathsf{mv}}$ is bounded.

*Step 4.* For given arbitrary $\rho_1,\rho_2\in E$ with $\rho_1<\rho_2$, $\mathbb{y}\in B_p$, $\mathfrak{t}\in\mathfrak{O}_{\mathsf{mv}}\mathbb{y}$, there is $\varpi\in\mathbb{S}_{\mathcal{W},\mathbb{y}}$ s.t.,

$$
\begin{aligned}
\left|\mathfrak{t}(\rho_2)-\mathfrak{t}(\rho_1)\right| \leq\ & \frac{1}{\Gamma_{\mathrm{q}}(r+\wp)}\Bigg|\int_{\mathrm{l}_1}^{\rho_1}\lambda'(\xi)\Big[\left(\lambda_\xi(\rho_2)\right)^{r+\wp-1}-\left(\lambda_\xi(\rho_1)\right)^{r+\wp-1}\Big]\varpi(\xi)\,\mathrm{d}\xi \\
& + \int_{\rho_1}^{\rho_2}\lambda'(\xi)\left(\lambda_\xi(\rho_2)\right)^{r+\wp-1}\varpi(\xi)\,\mathrm{d}\xi\Bigg| + \frac{1}{\Gamma_{\mathrm{q}}(\wp)}\Bigg|\int_{\mathrm{l}_1}^{\rho_1}\lambda'(\xi)\Big[\left(\lambda_\xi(\rho_2)\right)^{\wp-1} \\
& - \left(\lambda_\xi(\rho_1)\right)^{\wp-1}\Big]\hbar(\xi)\mathbb{y}(\xi)\,\mathrm{d}\xi + \int_{\rho_1}^{\rho_2}\lambda'(\xi)\left(\lambda_\xi(\rho_2)\right)^{\wp-1}\hbar(\xi)\mathbb{y}(\xi)\,\mathrm{d}\xi\Bigg| \\
& + \frac{\left|\left(\lambda_{\mathrm{l}_1}(\rho_2)\right)^{\wp}-\left(\lambda_{\mathrm{l}_1}(\rho_1)\right)^{\wp}\right|}{|\Omega|\Gamma_{\mathrm{q}}(\wp+1)}\Bigg[\|\varpi(\xi)\|\hat{\chi}\frac{\left(\lambda_{\mathrm{l}_1}(\mathrm{l}_2)\right)^{r+\wp}}{\Gamma_{\mathrm{q}}(r+\wp+1)} + \|\mathbb{y}(\mathrm{l}_2)\|\ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hat{\chi}|\hbar(\mathrm{l}_2)| \\
& + \|\varpi(\xi)\|\hat{\chi}\sum_{i=1}^{\aleph}|\ni_i|\ \frac{\left(\lambda_{\mathrm{l}_1}(\eta_i)\right)^{r+\wp+s_i}}{\Gamma_{\mathrm{q}}(r+\wp+s_i+1)} + \|\mathbb{y}(\eta_i)\|\hat{\chi}\sum_{i=1}^{\aleph}|\ni_i|\ \frac{\left(\lambda_{\mathrm{l}_1}(\eta_i)\right)^{\wp+s_i}}{\Gamma_{\mathrm{q}}(\wp+s_i+1)}\ \left|\hbar(\eta_i)\right|\Bigg] \\
\leq\ & \frac{\|\beta\|\nabla(p)}{\Gamma_{\mathrm{q}}(r+\wp)}\Bigg|\int_{\mathrm{l}_1}^{\rho_1}\lambda'(\xi)\Big[\left(\lambda_\xi(\rho_2)\right)^{r+\wp-1}-\left(\lambda_\xi(\rho_1)\right)^{r+\wp-1}\Big]\,\mathrm{d}\xi \\
& + \int_{\rho_1}^{\rho_2}\lambda'(\xi)\left(\lambda_\xi(\rho_2)\right)^{r+\wp-1}\,\mathrm{d}\xi\Bigg| + \frac{p}{\Gamma_{\mathrm{q}}(\wp)}\Bigg|\int_{\mathrm{l}_1}^{\rho_1}\lambda'(\xi)\Big[\left(\lambda_\xi(\rho_2)\right)^{\wp-1} \\
& - \left(\lambda_\xi(\rho_1)\right)^{\wp-1}\Big]\hbar(\xi)\,\mathrm{d}\xi + \int_{\rho_1}^{\rho_2}\lambda'(\xi)\left(\lambda_\xi(\rho_2)\right)^{\wp-1}\hbar(\xi)\,\mathrm{d}\xi\Bigg| \\
& + \frac{\left|\left(\lambda_{\mathrm{l}_1}(\rho_2)\right)^{\wp}-\left(\lambda_{\mathrm{l}_1}(\rho_1)\right)^{\wp}\right|}{|\Omega|\Gamma_{\mathrm{q}}(\wp+1)}\Bigg[\hat{\chi}\|\beta\|\nabla(p)\frac{\left(\lambda_{\mathrm{l}_1}(\mathrm{l}_2)\right)^{r+\wp}}{\Gamma_{\mathrm{q}}(r+\wp+1)} \\
& + p\hat{\chi}\ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}|\hbar(\mathrm{l}_2)|\ \hat{\chi}\|\beta\|\nabla(p)\sum_{i=1}^{\aleph}|\ni_i|\frac{\left(\lambda_{\mathrm{l}_1}(\eta_i)\right)^{r+\wp+s_i}}{\Gamma_{\mathrm{q}}(r+\wp+s_i+1)} + p\hat{\chi}\sum_{i=1}^{\aleph}|\ni_i|\frac{\left(\lambda_{\mathrm{l}_1}(\rho_j)\right)^{\wp+s_i}}{\Gamma_{\mathrm{q}}(\wp+s_i+1)}|\hbar(\eta_i)|\Bigg].
\end{aligned}
$$

The obtained inequalities imply $\mathbb{y}\in B_p$ approach to zero when $\rho_1$ tends to $\rho_2$ which confirms equicontinuity of $\mathfrak{O}_{\mathsf{mv}}$.

*Step 5.* Consider converges sequences $\mathbb{y}_n\to\mathbb{y}_\circ$ and $\mathfrak{t}_n\to\mathfrak{t}_\circ$, $\mathfrak{t}_n\in\mathfrak{O}_{\mathsf{mv}}(\mathbb{y}_\circ)$. For $\mathfrak{t}_n\in\mathfrak{O}_{\mathsf{mv}}(\mathbb{y}_n)$, there is $\varpi_n\in\mathbb{S}_{\mathcal{W},\mathbb{y}_n}$

s.t., $\forall\, \rho \in E$,

$$\begin{aligned}\mathfrak{t}_n(\rho) = {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi_n(\rho) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}_n(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Bigg[ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\chi\left(\mathrm{l}_2, \mathbb{y}_n\left(\mathrm{l}_2\right)\right)\varpi_n\left(\mathrm{l}_2\right)\\ - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\chi\left(\mathrm{l}_2, \mathbb{y}_n\left(\mathrm{l}_2\right)\right)\hbar\left(\mathrm{l}_2\right)\mathbb{y}_n\left(\mathrm{l}_2\right) - \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\,\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}_n\left(\eta_{\mathfrak{l}}\right)\right) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp+s_{\mathfrak{l}},\lambda}\varpi_n\left(\eta_{\mathfrak{l}}\right)\\ + \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_{\mathfrak{l}},\lambda}\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}_n\left(\eta_{\mathfrak{l}}\right)\right)\hbar\left(\eta_{\mathfrak{l}}\right)\mathbb{y}_n\left(\eta_{\mathfrak{l}}\right)\Bigg].\end{aligned}$$

Now, we define an operator $T : L^1(E) \to C(E)$, $\varpi \to T(\varpi)(\rho)$ as

$$T(\varpi)(\rho) = {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Bigg[ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\chi\left(\mathrm{l}_2, \mathbb{y}\left(\mathrm{l}_2\right)\right)\varpi\left(\mathrm{l}_2\right) - \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\,\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}\left(\eta_{\mathfrak{l}}\right)\right)\varpi\left(\eta_{\mathfrak{l}}\right)\Bigg].$$

Lemma **??** implies that $T \circ \mathbb{S}_{\mathcal{W},\mathbb{y}}$ is a closed graph operator and so, $\mathfrak{t}_n \in T(\mathbb{S}_{\mathcal{W},\mathbb{y}_n})$. From $\mathbb{y}_n \to \mathbb{y}_\circ$ and $\mathfrak{t}_n \to \mathfrak{t}_\circ$, one can conclude there is $\varpi_\circ \in \mathbb{S}_{\mathcal{W},\mathbb{y}_\circ}$ s.t., for each $\rho \in E$,

$$\begin{aligned}\mathfrak{t}_\circ(\rho) = {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi_\circ(\rho) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}_\circ(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Bigg[ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\chi\left(\mathrm{l}_2, \mathbb{y}_\circ\left(\mathrm{l}_2\right)\right)\varpi_n\left(\mathrm{l}_2\right)\\ - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\chi\left(\mathrm{l}_2, \mathbb{y}_\circ\left(\mathrm{l}_2\right)\right)\hbar\left(\mathrm{l}_2\right)\mathbb{y}_\circ\left(\mathrm{l}_2\right) - \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\,\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}_\circ\left(\eta_{\mathfrak{l}}\right)\right) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp+s_{\mathfrak{l}},\lambda}\varpi_\circ\left(\eta_{\mathfrak{l}}\right)\\ + \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_{\mathfrak{l}},\lambda}\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}_\circ\left(\eta_{\mathfrak{l}}\right)\right)\hbar\left(\eta_{\mathfrak{l}}\right)\mathbb{y}_\circ\left(\eta_{\mathfrak{l}}\right)\Bigg].\end{aligned}$$

Consequently, $\mathfrak{O}_{\mathsf{mv}}$ possesses the upper semicontinuity and compactness properties.
*Step 6.* In this level, we show that $2\mathfrak{u}\mathfrak{R} < 1$. From *Step 3*, we have

$$\mathfrak{R} = \|\mathfrak{O}_{\mathsf{mv}}(C(E))\| = \sup\left\{|\mathfrak{O}_{\mathsf{mv}}(\mathbb{y})| \,:\, \mathbb{y} \in C(E)\right\} \leq \|\beta\|\nabla(p)\Upsilon + p\Upsilon_{\hbar}. \tag{20}$$

Thanks to (P4), we get $\|\alpha\|(\|\beta\|\nabla(p)\Upsilon + p\Upsilon_{\hbar}) < \frac{1}{2}$, which implies that $2\mathfrak{u}\mathfrak{R} < 1$, where $\mathfrak{u} = \|\alpha\|$.
*Step 7.* Consider the set $\Sigma = \left\{\mathbb{y} \in C(E) \,:\, \theta\mathbb{y} \in \mathfrak{O}_{\mathsf{sv}}\mathbb{y}\,\mathfrak{O}_{\mathsf{mv}}\mathbb{y},\ \theta > 1\right\}$ and $\mathbb{y} \in \Sigma$. The, $\theta\mathbb{y} \in \mathfrak{O}_{\mathsf{sv}}\mathbb{y}\,\mathfrak{O}_{\mathsf{mv}}\mathbb{y}$ and so there is $\varpi \in \mathbb{S}_{\mathcal{W},\mathbb{y}}$, s.t.,

$$\begin{aligned}\mathbb{y}(\rho) = \theta^{-1}\chi(\rho, \mathbb{y}(\rho))\Bigg\{ {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathrm{q}}(\wp+1)}\Big[\chi\left(\mathrm{l}_2, \mathbb{y}\left(\mathrm{l}_2\right)\right) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp,\lambda}\varpi\left(\mathrm{l}_2\right)\\ - {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp,\lambda}\chi\left(\mathrm{l}_2, \mathbb{y}\left(\mathrm{l}_2\right)\right)\hbar\left(\mathrm{l}_2\right)\mathbb{y}\left(\mathrm{l}_2\right) - \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\,\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}\left(\eta_{\mathfrak{l}}\right)\right) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{r+\wp+s_{\mathfrak{l}},\lambda}\varpi\left(\eta_{\mathfrak{l}}\right)\\ + \sum_{\mathfrak{l}=1}^{\aleph}\mathfrak{z}_{\mathfrak{l}}\,\chi\left(\eta_{\mathfrak{l}}, \mathbb{y}\left(\eta_{\mathfrak{l}}\right)\right) {}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{\wp+s_{\mathfrak{l}},\lambda}\hbar\left(\eta_{\mathfrak{l}}\right)\Big]\Bigg\}.\end{aligned}$$

Hence,

$$
\begin{aligned}
|\mathbb{y}(\rho)| \leq \theta^{-1}|\chi(\rho,\mathbb{y}(\rho))|\Big| & {}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{r+\wp,\lambda}\varpi(\rho) - {}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{\wp,\lambda}\hbar(\rho)\mathbb{y}(\rho) + \frac{(\lambda_{\mathrm{l}_1}(\rho))^{\wp}}{\Omega\Gamma_{\mathfrak{q}}(\wp+1)}\Big[\chi\left(\mathrm{l}_2,\mathbb{y}\left(\mathrm{l}_2\right)\right){}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{r+\wp,\lambda}\varpi\left(\mathrm{l}_2\right) \\
& - {}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{\wp,\lambda}\chi\left(\mathrm{l}_2,\mathbb{y}\left(\mathrm{l}_2\right)\right)\hbar\left(\mathrm{l}_2\right)\mathbb{y}\left(\mathrm{l}_2\right) - \sum_{i=1}^{\aleph}\mathfrak{z}_i\,\chi\left(\eta_i,\mathbb{y}\left(\eta_i\right)\right){}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{r+\wp+s_i,\lambda}\varpi\left(\eta_i\right) \\
& + \sum_{i=1}^{\aleph}\mathfrak{z}_i\,\chi\left(\eta_i,\mathbb{y}\left(\eta_i\right)\right){}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{\wp+s_i,\lambda}\hbar\left(\eta_i\right)\Big]\Big| \\
\leq & \Big[\|\alpha\|\,\|\mathbb{y}\| + \chi^*\Big]\|\beta\|\nabla(\|\mathbb{y}\|)\Bigg\{\frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{r+\wp}}{\Gamma_{\mathfrak{q}}(r+\wp+1)} + \frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{\wp}}{|\Omega|\Gamma_{\mathfrak{q}}(\wp+1)}\Bigg[\hat{\chi}\frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{r+\wp}}{\Gamma_{\mathfrak{q}}(r+\wp+1)} \\
& + \hat{\chi}\sum_{i=1}^{\aleph}|\mathfrak{z}_i|\frac{(\lambda_{\mathrm{l}_1}(\eta_i))^{r+\wp+s_i}}{\Gamma_{\mathfrak{q}}(r+\wp+s_i+1)}\Bigg]\Bigg\} + \|\mathbb{y}\|\Bigg\{{}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{\wp,\lambda}\left|\hbar(\rho)\mathbb{y}(\rho)\right| \\
& + \frac{(\lambda_{\mathrm{l}_1}(\mathrm{l}_2))^{r}}{|\Omega|\Gamma_{\mathfrak{q}}(\wp+1)}\Bigg[\hat{\chi}\,{}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{\wp,\lambda}\left|\hbar(\eta_i)\mathbb{y}(\eta_i)\right| + \hat{\chi}\sum_{i=1}^{\aleph}{}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{\wp+s_i,\lambda}|\mathfrak{z}_i|\,\left|\hbar(\eta_i)\mathbb{y}(\eta_i)\right|\Bigg]\Bigg\} \\
\leq & \Big[\|\alpha\|\,\|\mathbb{y}\| + \chi^*\Big]\|\beta\|\nabla(\|\mathbb{y}\|)\Upsilon + \|\mathbb{y}\|\Upsilon_{\hbar},
\end{aligned}
$$

where $\chi^* = \sup_{\rho\in E}|\chi(\rho,0)| > 0$. Indeed, $\|\mathbb{y}\| \leq \frac{\chi^*\|\beta\|\nabla(\|\mathbb{y}\|)\Upsilon}{1-\|\alpha\|\,\|\beta\|\nabla(\|\mathbb{y}\|)\Upsilon-\Upsilon_{\hbar}}$. We examine $\Sigma$ is bounded in $C(E)$. Suppose it is not so. Since, $\nabla$ is bounded, i.e. exists $k > 0$ s.t., $\nabla(\|\mathbb{y}\|) < k$, we obtain,

$$\lim_{\|\mathbb{y}\|\to 0}\frac{\chi^*\|\beta\|k\Upsilon}{\|\mathbb{y}\|(1-\|\alpha\|\,\|\beta\|k\Upsilon-\Upsilon_{\hbar})} \geq 1,$$

which is contradiction. Thus, $\Sigma$ is bounded. By summarizing the seven steps, $\Psi$ has at least one fixed point according to Theorem 2.2 which is follows that the Caputo hybrid $\mathfrak{q}$-$\mathbb{FDI}$ (1) admits a solution. □

## 4. Relative examples

**Example 4.1.** In view of hybrid $\mathfrak{q}$-$\mathbb{FDI}$ (1) consider

$$
\begin{cases}
{}^{\mathrm{C}}_{\mathfrak{q}}\mathbb{D}_0^{1/2,e^{\rho}/2}\left({}^{\mathrm{C}}_{\mathfrak{q}}\mathbb{D}_0^{1/5,e^{\rho}/2}\left(\frac{\mathbb{y}(\rho)}{\frac{\rho}{20}\sin\mathbb{y}(\rho)+\sqrt{30}}\right) + \frac{\rho^2}{2}\mathbb{y}(\rho)\right) \in \left[\frac{|\mathbb{y}(\rho)|}{2(\mathbb{y}(\rho)^2+\sqrt{5})} + \frac{\rho}{3}, \frac{|\mathbb{y}(\rho)|}{|\mathbb{y}(\rho)|+\sqrt{7}} + \frac{\rho}{5}\right], \\
\mathbb{y}(0) = 0, \quad \mathbb{y}(1) = \frac{1}{2}\,{}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{2/3,e^{\rho}/2}\mathbb{y}\left(\frac{1}{4}\right) + \frac{1}{3}\,{}^{\mathrm{RL}}_{\mathfrak{q}}\mathbb{I}^{6/7,e^{\rho}/2}\mathbb{y}\left(\frac{1}{9}\right),
\end{cases}
\tag{21}
$$

for $\rho \in E = [\mathrm{l}_1,\mathrm{l}_1]$, $\mathrm{l}_1 = 0$, $\mathrm{l}_2 = 1$ and three values of $\mathfrak{q} \in \left\{\frac{1}{8},\frac{1}{3},\frac{1}{2}\right\} \subseteq (0,1)$, where $r = \frac{1}{2}$, $\wp = \frac{1}{5}$, $\lambda(\rho) = \frac{e^{\rho}}{2}$, $\aleph = 2, \eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{9}, \mathfrak{z}_1 = \frac{1}{2}, \mathfrak{z}_2 = \frac{1}{3}, s_1 = \frac{2}{3}, s_2 = \frac{6}{7}$, and $\hbar(\rho) = \frac{\rho^2}{2}$. Take multi-valued map $\mathcal{W} : E\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$(\rho,\mathbb{y}) \to \mathcal{W}(\rho,\mathbb{y}) = \left[\frac{|\mathbb{y}(\rho)|}{2\left((\mathbb{y}(\rho)^2+\sqrt{5})\right)} + \frac{\rho}{3}, \frac{|\mathbb{y}(\rho)|}{|\mathbb{y}(\rho)|+\sqrt{7}} + \frac{\rho}{5}\right].$$

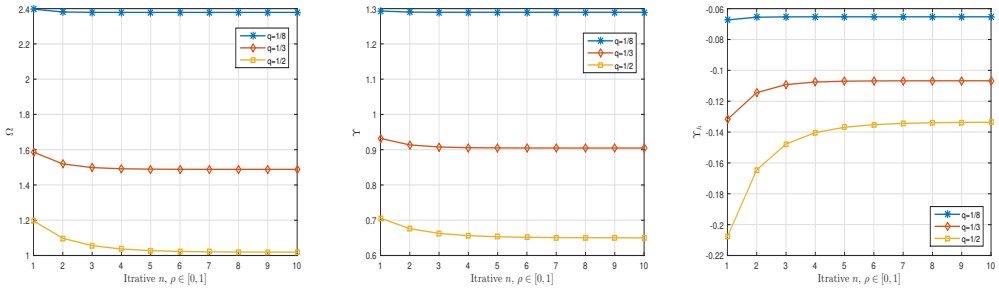
Then, $|\varpi| \leq \max\left\{\frac{|\mathbb{y}(\rho)|}{2\left((\mathbb{y}(\rho)^2+\sqrt{5})\right)}+\frac{\rho}{3}, \frac{|\mathbb{y}(\rho)|}{|\mathbb{y}(\rho)|+\sqrt{7}}+\frac{\rho}{5}\right\} \leq \frac{6}{5}$, for $\varpi \in \mathcal{W}(\rho,\mathbb{y})$, and $\hat{\chi} \simeq 5.5192$. Therefore, $\|\mathcal{W}(\rho,\mathbb{y})\| = \sup\{|\varpi| : \varpi \in \mathcal{W}(\rho,\mathbb{y})\} \leq \frac{6}{5} = \beta(\rho)\nabla(\|\mathbb{y}\|)$ for $\mathbb{y} \in \mathbb{R}$, where $\beta(\rho) = 1$ and $\nabla(\|\mathbb{y}\|) = \frac{6}{5}$. If we take $\chi(\rho,\mathbb{y}) = \frac{\rho}{20}\sin\mathbb{y}(\rho) + \sqrt{30}$, then for any $\mathbb{y}_1,\mathbb{y}_2 \in \mathbb{R}$, we obtain,

$$|\chi(\rho,\mathbb{y}_1) - \chi(\rho,\mathbb{y}_2)| = \left|\frac{\rho}{20}\sin\mathbb{y}_1(\rho) + \sqrt{30} - \left(\frac{\rho}{20}\sin\mathbb{y}_2(\rho) + \sqrt{30}\right)\right| < \frac{1}{20}|\mathbb{y}_1 - \mathbb{y}_2| = \alpha(\rho)|\mathbb{y}_1 - \mathbb{y}_2|. \tag{22}$$

Now, by using these data, from Eqs. (12) and (17), we can calculate $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ as follow:

$$\Omega \simeq \begin{cases} 2.379, & \mathrm{q}=\frac{1}{8}, \\ 1.489, & \mathrm{q}=\frac{1}{3}, \\ 1.019, & \mathrm{q}=\frac{1}{2}, \end{cases} \quad \Upsilon \simeq \begin{cases} 1.290, & \mathrm{q}=\frac{1}{8}, \\ 0.905, & \mathrm{q}=\frac{1}{3}, \\ 0.650, & \mathrm{q}=\frac{1}{2}, \end{cases} \quad \Upsilon_\hbar \simeq \begin{cases} -0.065, & \mathrm{q}=\frac{1}{8}, \\ -0.107, & \mathrm{q}=\frac{1}{3}, \\ -0.134, & \mathrm{q}=\frac{1}{2}. \end{cases}$$

These date are shown in Table 1, for $\rho \in E$ and iterative $n$. Further, the curves in Figs. 1a, 1b and 1c show the status of $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ for three values of q. As q increases to $\frac{1}{2}$, all three values $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ decrease. Then $\Delta =$




(a) $\Omega$ (b) $\Upsilon$ (c) $\Upsilon_\hbar$

Fig. 1. $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ for hybrid q-$\mathbb{FDI}$ (21) in Example 4.1.

$\|\alpha\|\,(\|\beta\|\nabla(\|\mathrm{y}\|)\Upsilon + \Upsilon_\hbar) \simeq 0.074, 0.049, 0.032 < \frac{1}{2}$ for $\mathrm{q} = \frac{1}{8}, \frac{1}{3}, \frac{1}{2}$, respectively. Thus, all the conditions of

Table 1. Numerical values of $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ for hybrid q-$\mathbb{FDI}$ (23) in Example 4.1.

| $n$ | $\mathrm{q}=\frac{1}{8}$ | | | | $\mathrm{q}=\frac{1}{3}$ | | | | $\mathrm{q}=\frac{1}{2}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\Omega$ | $\Upsilon$ | $\Upsilon_\hbar$ | $\Delta$ | $\Omega$ | $\Upsilon$ | $\Upsilon_\hbar$ | $\Delta$ | $\Omega$ | $\Upsilon$ | $\Upsilon_\hbar$ | $\Delta$ |
| 1 | 2.397 | 1.293 | −0.067 | 0.074 | 1.586 | 0.932 | −0.132 | 0.049 | 1.197 | 0.706 | −0.208 | 0.032 |
| 2 | 2.381 | 1.290 | −0.066 | 0.074 | 1.519 | 0.914 | −0.114 | 0.049 | 1.097 | 0.676 | −0.165 | 0.032 |
| 3 | 2.379 | 1.290 | −0.065 | 0.074 | 1.499 | 0.908 | −0.109 | 0.049 | 1.056 | 0.663 | −0.148 | 0.032 |
| 4 | 2.379 | 1.290 | −0.065 | 0.074 | 1.492 | 0.906 | −0.108 | 0.049 | 1.037 | 0.656 | −0.140 | 0.032 |
| 5 | 2.379 | 1.290 | −0.065 | 0.074 | 1.490 | 0.905 | −0.107 | 0.049 | 1.028 | 0.653 | −0.137 | 0.032 |
| 6 | 2.379 | 1.290 | −0.065 | 0.074 | 1.489 | 0.905 | −0.107 | 0.049 | 1.023 | 0.652 | −0.135 | 0.032 |
| 7 | 2.379 | 1.290 | −0.065 | 0.074 | 1.489 | 0.905 | −0.107 | 0.049 | 1.021 | 0.651 | −0.134 | 0.032 |
| 8 | 2.379 | 1.290 | −0.065 | 0.074 | 1.489 | 0.905 | −0.107 | 0.049 | 1.020 | 0.650 | −0.134 | 0.032 |
| 9 | 2.379 | 1.290 | −0.065 | 0.074 | 1.489 | 0.905 | −0.107 | 0.049 | 1.019 | 0.650 | −0.134 | 0.032 |
| 10 | 2.379 | 1.290 | −0.065 | 0.074 | 1.489 | 0.905 | −0.107 | 0.049 | 1.019 | 0.650 | −0.134 | 0.032 |

Theorem 3.4 hold which one can confirm that the hybrid q-$\mathbb{FDI}$ (21) has at least one solution on $E$.

**Example 4.2.** Consider next q-$\mathbb{FDI}$ (1) as form

$$\begin{cases} {}^{\mathrm{C}}_{\mathrm{q}}\mathbb{D}_0^{2/5,e^\rho}\left({}^{\mathrm{C}}_{\mathrm{q}}\mathbb{D}_0^{1/5,e^\rho}\left(\frac{\mathrm{y}(\rho)}{\frac{\rho}{2}\sin \mathrm{y}(\rho)+3}\right)+\rho\mathrm{y}(\rho)\right) \in \left[\frac{|\mathrm{y}(\rho)|}{6(|\mathrm{y}(\rho)|+1)}+\frac{\rho}{3}, \frac{|\cos \mathrm{y}(\rho)|}{6(|\cos \mathrm{y}(\rho)|+1)}+\frac{\rho}{6}\right], \\ \mathrm{y}(0)=0, \quad \mathrm{y}(1)=\frac{1}{3}\,{}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{3/4,e^\rho}\mathrm{y}\left(\frac{1}{4}\right)+\frac{2}{3}\,{}^{\mathrm{RL}}_{\mathrm{q}}\mathbb{I}^{3/7,e^\rho 2}\mathrm{y}\left(\frac{2}{7}\right), \end{cases} \tag{23}$$

for $\rho \in E = [\mathrm{l}_1, \mathrm{l}_1]$, $\mathrm{l}_1 = 0$, $\mathrm{l}_2 = 1$ and three values of $\mathrm{q} \in \left\{\frac{1}{2}, \frac{2}{3}, \frac{6}{7}\right\} \subseteq (0,1)$, where $r = \frac{2}{5}$, $\wp = \frac{1}{5}$, $\lambda(\rho) = e^\rho$, $\aleph = 2$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{2}{7}$, $\ni_1 = \frac{1}{3}$, $\ni_2 = \frac{2}{3}$, $s_1 = \frac{3}{4}$, $s_2 = \frac{3}{7}$, and $\hbar(\rho) = \rho$. Take multi-valued map $\mathcal{W} : E \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$(\rho, \mathrm{y}) \to \mathcal{W}(\rho, \mathrm{y}) = \left[\frac{|\mathrm{y}(\rho)|}{6(|\mathrm{y}(\rho)|+1)}+\frac{\rho}{3}, \frac{|\cos \mathrm{y}(\rho)|}{6(|\cos \mathrm{y}(\rho)|+1)}+\frac{\rho}{6}\right].$$

Then, $|\varpi| \le \max\left\{\frac{|\mathrm{y}(\rho)|}{6(|\mathrm{y}(\rho)|+1)}+\frac{\rho}{3}, \frac{|\cos \mathrm{y}(\rho)|}{6(|\cos \mathrm{y}(\rho)|+1)}+\frac{\rho}{6}\right\} \le \frac{1}{3}$, for $\varpi \in \mathcal{W}(\rho, \mathrm{y})$, and $\hat{\chi} \simeq 0.2091$. Therefore, by taking $\beta(\rho) = 1$ and $\nabla(\|\mathrm{y}\|) = \frac{2}{3}$, we have $\|\mathcal{W}(\rho, \mathrm{y})\| = \sup\{|\varpi| : \varpi \in \mathcal{W}(\rho, \mathrm{y})\} \le \frac{1}{3} = \beta(\rho)\nabla(\|\mathrm{y}\|)$ for $\mathrm{y} \in \mathbb{R}$. If we

take $\chi(\rho, \mathbb{y}) = \frac{\rho}{2}\sin \mathbb{y}(\rho) + 3$, then for any $\mathbb{y}_1, \mathbb{y}_2 \in \mathbb{R}$, we obtain,

$$\left|\chi(\rho, \mathbb{y}_1) - \chi(\rho, \mathbb{y}_2)\right| = \left|\frac{\rho}{2}\sin \mathbb{y}_1(\rho) + 3 - \left(\frac{\rho}{2}\sin \mathbb{y}_2(\rho) + 3\right)\right| < \alpha(\rho)|\mathbb{y}_1 - \mathbb{y}_2|, \qquad \alpha(\rho) = \frac{1}{2}. \tag{24}$$

Now, by using the se data, from Eqs. (12) and (17), we can calculate $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ as follow:

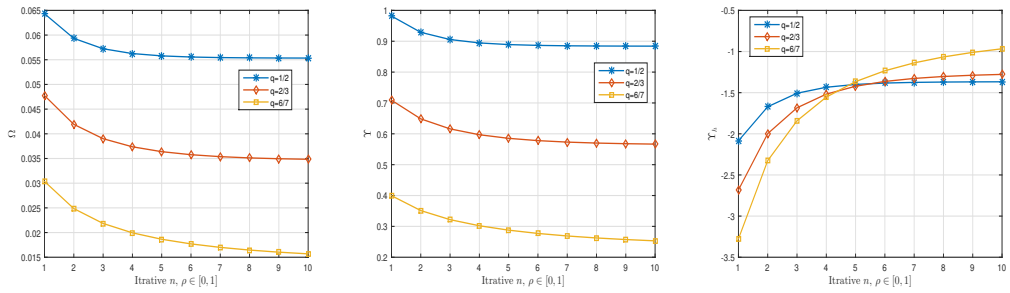
$$\Omega \simeq \begin{cases} 0.055, & \mathrm{q} = \frac{1}{2}, \\ 0.035, & \mathrm{q} = \frac{2}{3}, \\ 0.014, & \mathrm{q} = \frac{6}{7}, \end{cases} \qquad \Upsilon \simeq \begin{cases} 0.881, & \mathrm{q} = \frac{1}{2}, \\ 0.564, & \mathrm{q} = \frac{2}{3}, \\ 0.650, & \mathrm{q} = \frac{6}{7}, \end{cases} \qquad \Upsilon_\hbar \simeq \begin{cases} -1.367, & \mathrm{q} = \frac{1}{2}, \\ -1.258, & \mathrm{q} = \frac{2}{3}, \\ -0.763, & \mathrm{q} = \frac{6}{7}. \end{cases}$$

These date are shown in Table 2, for $\rho \in E$ and iterative $n$. Further, the curves in Figs. 2a, 2b and 2c show the status

Table 2. Numerical values of $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ for hybrid q-$\mathbb{FDI}$ (23) in Example 4.2.

| $n$ | $\mathrm{q} = \frac{1}{2}$ | | | | $\mathrm{q} = \frac{2}{3}$ | | | | $\mathrm{q} = \frac{6}{7}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\Omega$ | $\Upsilon$ | $\Upsilon_\hbar$ | $\Delta$ | $\Omega$ | $\Upsilon$ | $\Upsilon_\hbar$ | $\Delta$ | $\Omega$ | $\Upsilon$ | $\Upsilon_\hbar$ | $\Delta$ |
| 1 | 0.064 | 0.982 | −2.085 | 0.715 | 0.048 | 0.709 | −2.683 | 1.105 | 0.030 | 0.400 | −3.278 | 1.506 |
| 2 | 0.059 | 0.929 | −1.671 | 0.526 | 0.042 | 0.648 | −1.998 | 0.783 | 0.025 | 0.351 | −2.326 | 1.046 |
| 3 | 0.057 | 0.906 | −1.507 | 0.452 | 0.039 | 0.616 | −1.684 | 0.637 | 0.022 | 0.322 | −1.842 | 0.814 |
| 4 | 0.056 | 0.895 | −1.434 | 0.419 | 0.037 | 0.597 | −1.518 | 0.560 | 0.020 | 0.302 | −1.554 | 0.676 |
| 5 | 0.056 | 0.889 | −1.400 | 0.404 | 0.036 | 0.585 | −1.422 | 0.516 | 0.019 | 0.288 | −1.365 | 0.587 |
| 6 | 0.056 | 0.887 | −1.383 | 0.396 | 0.036 | 0.578 | −1.363 | 0.489 | 0.018 | 0.277 | −1.234 | 0.525 |
| 7 | 0.055 | 0.885 | −1.375 | 0.392 | 0.035 | 0.573 | −1.327 | 0.472 | 0.017 | 0.269 | −1.138 | 0.479 |
| 8 | 0.055 | 0.885 | −1.371 | 0.391 | 0.035 | 0.570 | −1.303 | 0.462 | 0.016 | 0.262 | −1.066 | 0.445 |
| 9 | 0.055 | 0.884 | −1.369 | 0.390 | 0.035 | 0.568 | −1.288 | 0.455 | 0.016 | 0.257 | −1.010 | 0.419 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 44 | 0.055 | 0.884 | −1.367 | 0.389 | 0.035 | 0.564 | −1.258 | 0.441 | 0.014 | 0.232 | −0.764 | 0.304 |
| 45 | 0.055 | 0.884 | −1.367 | 0.389 | 0.035 | 0.564 | −1.258 | 0.441 | 0.014 | 0.232 | −0.763 | 0.304 |
| 46 | 0.055 | 0.884 | −1.367 | 0.389 | 0.035 | 0.564 | −1.258 | 0.441 | 0.014 | 0.232 | −0.763 | 0.304 |

of $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ for three values of q. Then $|\Delta| \simeq 0.389,\ 0.441,\ 0.304 < \frac{1}{2}$ for $\mathrm{q} = \frac{1}{2}, \frac{2}{3}, \frac{6}{7}$, respectively. Thus, all the




(a) $\Omega$ (b) $\Upsilon$ (c) $\Upsilon_\hbar$

Fig. 2. $\Omega$, $\Upsilon$ and $\Upsilon_\hbar$ for hybrid q-$\mathbb{FDI}$ (23) in Example 4.2.

conditions of Theorem 3.4 hold which one can confirm that the hybrid q-$\mathbb{FDI}$ (23) has at least one solution on $E$.

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# Some results on Hopfian and co-Hopfian acts

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**Abstract**

In this paper we will study some properties of Hopfian and co-Hopfian acts over a monoid $S$. First recalling the concept of Hopfian and co-Hopfian acts, several conditions are given for an $S$-act to be Hopfian (co-Hopfian). Also, by considering the properties that are preserved under isomorphism, we study conditions under which an $S$-act is Hopfian (co-Hopfian).

## 1. Introduction and Preliminaries

In this paper, $S$ is a monoid and an $S$-act $A_S$ (or $A$) is a unitary right $S$-act. From [1], an $S$-act $A$ is called $Hopfian$ ($co\text{-}Hopfian$) if any epimorphism (monomorphism) of $A$ is an isomorphism. The study of Hopfian and co-Hopfian acts was initiated by Farsad et al ([1]) and continued in some other papers. In [4] some properties of these concepts are studied over a group $S$ and in [5] their interrelationship with some other concepts is investigated so that this study provided several equivalent conditions for a quasi-projective (quasi-injective) act to be Hopfian (co-Hopfian). The purpose of this paper is to investigate some additional important properties of these acts. First let us review some concepts which are needed in the sequel. An equivalence relation $\rho$ on an $S$-act $A$ is called a $congruence$ on $A$ if $x\ \rho\ y$ implies $(xs)\ \rho\ (ys)$ for every $x, y \in A, s \in S$. If $B$ is a subact of $A$, then the congruence $(B \times B) \cup \Delta_A$ on $A$ is called the $Rees\ congruence$ by the subact $B$ and is denoted by $\rho_B$. The set of all congruences on $A$ is denoted by $\mathrm{Con}(A)$. Clearly $\nabla_A = A \times A$ and $\Delta_A = \{(a, a) |\ a \in A\}$ are trivial elements of $\mathrm{Con}(A)$. Also by $E(A)$ we mean the injective envelope of an $S$-act $A$. We encourage the reader to see [3] for basic results and definitions relating to acts over monoids.

## 2. Main Results

**Proposition 2.1.** *Suppopse $A$ is an $S$-act. Then the following conditions are equivalent:*

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(i) *$A$ is Hopfian.*

(ii) *If $\{f_i\}_{i\in I}$ is a family of epimorphisms of $A$, then the sequence $ker(f_1) \subseteq ker(f_2) \subseteq \ldots$ is stationary.*

(iii) *For any epimorphism $f : A \longrightarrow A$, the sequence $ker(f) \subseteq ker(f^2) \subseteq \ldots$ is stationary.*

*Proof.* The implications (i) ⇒ (ii) ⇒ (iii) are clear. (iii) ⇒ (i) If $f : A \longrightarrow A$ is an epimorphism then we can easily see that there exists $n \in \mathbb{N}$ such that $\mathrm{Ker}f^n \cap \rho_{\mathrm{Im}f^n} = \Delta_A$. Thus $f^n$ is a monomorphism and so $f$ is a monomorphism as desired. □

**Proposition 2.2.** *Suppose $A$ is an $S$-act. Then the following conditions are equivalent:*

(i) *$A$ is co-Hopfian.*

(ii) *If $\{f_i\}_{i\in I}$ is a family of monomorphisms of $A$, then the sequence $Im(f_1) \supseteq Im(f_2) \supseteq \ldots$ is stationary.*

(iii) *For any monomorphism $f : A \longrightarrow A$, the sequence $Im(f) \supseteq Im(f^2) \supseteq \ldots$ is stationary.*

*Proof.* We only prove (iii) ⇒ (i). Suppose $f : A \longrightarrow A$ is a monomorphism. By assumption, there exists $n \in \mathbb{N}$ such that $\mathrm{Ker}f^n \vee \rho_{\mathrm{Im}f^n} = \nabla_A = A \times A$ and consequently $f^n$ is an epimorphism which implies that $f$ is an epimorphism and the result follows. □

Recall that a subact $B$ of an $S$-act $A$ is called $essential$ in $A$ denoted by $B \subseteq' A$, if any $S$-morphism $g : A \longrightarrow C$ such that $g|_B$ is a monomorphism is itself a monomorphism (see [3]). Also from [6], an $S$-act $A$ is called $torsion\ free$ if for any $s \in S$ and any elements $x, y \in A$, the equality $xs = ys$ implies that $x = y$. Additionally, from [6], an $S$-act $A$ is called *quasi-injective* if for any $S$-morphism $f : B \longrightarrow A$, there exists an $S$-morphism $h : A \longrightarrow A$ such that $h|_B = f$ where $B$ is a subact of $A$.
It can be easily checked that for an $S$-act $A$, if a monomorphism $f : A \longrightarrow A$ can be extended to an isomorphism $\bar{f} : E(A) \longrightarrow E(A)$, then $f(A) \subseteq' A$. Rgarding this observation we have the following result.

**Proposition 2.3.** *Suppose $A$ is a quasi-injective torsion free $S$-act. If $E(A)$ is co-Hopfian then $A$ is co-Hopfian.*

*Proof.* By assumption any monomorphism $f : A \longrightarrow A$ can be extended to an isomorphism $\bar{f} : E(A) \longrightarrow E(A)$. Since $A \subseteq' E(A)$ and $\bar{f}$ is an isomorphism, $\bar{f}(A) \subseteq' E(A)$. Moreover, $\bar{f}(A) = f(A) \subseteq A$. Thus $f(A) \subseteq' A$ and by Theorem 2.16 of [5] the proof is completed. □

From [2], an $S$-act $A$ is called $noetherian$ $(artinian)$ if the set of congruences of $A$ satisfies the ascending (descending) chain condition. For the sake of simplicity, we denote "ascending chain condition" and " descending chain condition" by "a.c.c" and "d.c.c ", respectively.

**Proposition 2.4.** *Suppose $\mathcal{P}$ is a property of $S$-acts which is preserved under isomorphism and $A$ is an $S$-act which has this property. If $A$ satisfies a.c.c on congruences $\lambda \in Con(A)$ such that $A/\lambda$ has the property $\mathcal{P}$, then $A$ is Hopfian.*

*Proof.* By way of contradiction, suppose $A$ is not Hopfian. Thus there exists a non-diagonal congruences $\sigma_1 \in Con(A)$ such that $A \cong A/\sigma_1$. Thus $A/\sigma_1$ has the property and is not Hopfian. By a similar way we can find a non-diagonal congruences $\sigma_2 \in Con(A)$ such that $\sigma_1 \subset \sigma_2$. Repeating this process yields an ascending chain of congruences of the form $\sigma_1 \subset \sigma_2 \subset \sigma_3 \subset \ldots$ such that for any $i = 1, 2, 3, \ldots, A/\sigma_i$ has the property $\mathcal{P}$, a contradiction. □

**Corollary 2.5.** *Suppose $A$ is an $S$-act which satisfies a.c.c on congruences $\lambda \in Con(A)$ such that $A/\lambda$ is not Hopfian, then $A$ is Hopfian.*

*Proof.* By Propositin 2.4 if $\mathcal{P}$ is the property of being not Hopfian, then the result follows. □

**Proposition 2.6.** *Suppose $\mathcal{P}$ is a property of $S$-acts which is preserved under isomorphism and $A$ is an $S$-act which has this property. If $A$ satisfies d.c.c on subacts with property $\mathcal{P}$, then $A$ is co-Hopfian.*

*Proof.* If $A$ is not co-Hopfian, then by a routine argument, we can obtain a strictly descending chain of proper subacts with the property $\mathcal{P}$ that cntradicts our hypothesis. □

**Corollary 2.7.** *Let $A$ be an $S$-act which has d.c.c on its non-co-Hopfian subacts, then $A$ is co-Hopfian.*

*Proof.* By Propositin 2.6 it is sufficient to let $\mathcal{P}$ the property of being non-co-Hopfian. □

From [5], an $S$-act $A$ is said to be $mono\text{-}uniform$ if for any monomorphism $f : A \longrightarrow A$, $f(A) \subseteq' A$.

**Proposition 2.8.** *Suppose $A$ is an $S$-act which satisfies d.c.c on its nonessential subacts, then $A$ is mono-unifrom.*

*Proof.* The proof is similar to the proof of Proposition 2.6. □

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# Some results on the soft hypergroupoids

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**Abstract**

Soft sets present an exceptionally effective approach to addressing issues of uncertainty. Among the various hyperstructures, the hypergroupoid stands out as a significant entity. In this paper, we introduce the notion of soft hypergroupoids, a concept that emerges from the foundational ideas of soft hypergroups and soft hypergroupoids. We meticulously define this concept and elucidate its properties, supported by illustrative examples that enhance understanding.

## 1. Introduction

To solve uncertainty problems, various methods have been stated and studied. Fuzzy set theory [7] and rough set theory [5], are among the famous theories that can be mentioned in this passage. For further reading, refer to [2, 3, 6]. These methods, with all the positive things they have, also have some problems. To improve the previous methods and solve problems in the study of uncertainty problems, in 1999, the concept of soft sets was presented by Molodtsov [4].
Let $U$ be an initial universe set and $E$ be a set of parameters. $\mathcal{P}(U)$ denotes the power set of $U$ and $A \subseteq E$.

**Definition 1.1.** A pair $(F, A)$ is called a soft set over $U$, where $F$ is a mapping given by $F : A \longrightarrow \mathcal{P}(U)$. In fact, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $\alpha \in A$, $F(\alpha)$ may be considered as the set of $\alpha$-approximate elements of the soft set $(F, A)$.

## 2. Soft hypergroupoids

We have highlighted the significant role of algebraic hyperstructures and acknowledged the significance of soft algebraic structures. One of the key outcomes of this section is the derivation of the concept of soft hypergroupoid from the definition of a soft groupoid. Subsequently, we will delve into exploring various results and properties stemming from this definition. An intriguing aspect to be explored is the relationship between soft sets and soft hypergroupoids, which promises to be engaging. Additionally, investigating the connection between soft hypergroups and soft hypergroupoids is deemed valuable and will provide useful insights.

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**Definition 2.1.** Suppose $H$ is a non-empty set and $* : H \times H \to \mathcal{P}^*(H)$ is a hyperoperation, such that $\mathcal{P}^*(H)$ is the family of non-empty subsets of $H$, then the couple $(H, *)$ is called a hypergroupoid.

**Remark 2.2.** For any two non-empty subsets $P$ and $Q$ of $H$, and $h \in H$, we have

$$P * Q = \bigcup_{\substack{p \in P \\ q \in Q}} p * q, \quad P * h = P * \{h\}, \quad h * Q = \{h\} * Q.$$

**Remark 2.3.** If $(H, *)$ is a hypergroupoid and $K$ is a non-empty subset of $H$, then $(K, *)$ is called a subhypergroupoid, when $(K, *)$ is an hypergroupoid. Also, if $\{(H_i, *_i) \mid i \in \Omega\}$ is a family of hypergroupoids, then $(\prod_{i\in\Omega} H_i, *)$ by the hyperoperation

$$(h_i) * (h_i') = \{(k_i) \mid k_i \in h_i *_i h_i',\ i \in \Omega\}$$

is a hypergroupoid. In addition, if $(H, *)$, $(H', *')$ are two hypergroupoids, then the map $f : H \to H'$ is called a good homomorphism, when, for all $h_1, h_2 \in H$ we have $f(h_1 * h_2) = f(h_1) *' f(h_2)$, and is called an inclusion homomorphism, if

$$f(h_1 * h_2) \subseteq f(h_1) *' f(h_2).$$

**Definition 2.4.** If $H$ is a hypergroupoid and $\mathcal{P}(H)$ is the set of all subhypergroupoids of $H$, and $S$ is a set of parameters, then a pair $(\mathcal{F}, S)$ is called a soft hypergroupoid over $H$, if $\mathcal{F}(v)$ is a subhypergroup of $H$ for all $v \in S$, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : S \to \mathcal{P}(H)$.

**Example 2.5.** If $H = \{r, s, t\}$ is a set, and "$*$" is a hyperoperation on $H$ as follows:

| * | r | s | t |
|---|---|---|---|
| r | {r,s,t} | {r,s} | {r,t} |
| s | {r,s} | {r,s,t} | {s,t} |
| t | {r,t} | {s,t} | {r,s,t} |

then $(H, *)$ is a hypergroupoid. If we defines $\mathcal{F} : \mathbb{Z} \to \mathcal{P}^*(H)$ by

$$\mathcal{F}(z) = \begin{cases} \{r, s\}, & 2 \mid z \\ \{r, s, t\}, & 2 \nmid z \end{cases}$$

then $(\mathcal{F}, S)$ is a soft hypergroupoid over $H$.

**Remark 2.6.** If $\mathcal{F}(v)$ is transitive as a hypergroupoid, for all $v \in S$, then the soft hypergroupoid $H$ is called transitive.

**Remark 2.7.** If $\mathcal{F}(v)$ is totally intransitive as a hypergroupoid, for all $v \in S$, then the soft hypergroupoid $H$ is called intransitive.

**Proposition 2.8.** *Every soft set can indeed be viewed as a soft hypergroupoid.*

*Proof.* Suppose $(\mathcal{F}, S)$ is a soft set over $X$. It is clear that $(\mathcal{F}, S)$ is a soft hypergroupoid. Since by definition $\mathrm{Ob}(H) = X$, hyperoperation as follows:

$$r * s = \begin{cases} \{\mathrm{Id}_r\}, & r = s \\ \emptyset, & r \neq s \end{cases}$$

for all $r, s \in X$. But $(\mathcal{F}, S)$ is a soft set over $X$, hence $\mathcal{F}(v)$ is a subset of $X$, for all $v \in S$. Therefore, $\mathcal{F}(v)$ with its identity morphism, is a subhypergroupoid of the hypergroupoid $H$. Hence, $(\mathcal{F}, S)$ is a soft hypergroupoid over $H$. □

This perspective offers a valuable insight into the relationship between soft sets and soft hypergroupoids, showcasing the versatility and applicability of soft hypergroupoid concepts in a variety of contexts.

**Proposition 2.9.** *Every soft hypergroup $(\mathcal{G}, \mathcal{F}, S)$, can be considered as a soft hypergroupoid.*

*Proof.* It is clear that, $\mathcal{F}(v)$ is a subhypergroup of $\mathcal{G}$ for all $v \in S$. On the other hand, from the hypergroupoid theory, every hypergroup is a hypergroupoid with only one object. In addition, the hypergroup $\mathcal{G}$ and $\mathcal{F}(v)$ are hypergroupoids for all $v \in S$. Hence $\mathcal{F}(v)$ is a subhypergroupoid of the hypergroupoid $H = \mathcal{G}$, for all $v \in S$. Thus, the $(H, \mathcal{F}, S)$ is a soft hypergroupoid. □

## 3. Soft action hypergroupoid

In this section, we establish the definition of soft action hypergroupoids and delve into exploring various results derived from them. Additionally, we aim to direct readers towards the category of soft hypergroupoids, which presents a promising avenue for research and exploration for those interested in this domain.

**Definition 3.1.** If $(H,\mathcal{F},S)$ is a soft hypergroup and $(X,\mathcal{F}',S)$ is a soft set, then, we can establish the following mappings:

$$\begin{array}{rcl} F:S & \longrightarrow & \mathcal{P}(H) \\ \nu & \longmapsto & \mathcal{F}(\nu) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathcal{F}':S & \longrightarrow & \mathcal{P}(X) \\ \nu & \longmapsto & \mathcal{F}'(\nu) \end{array}$$

Hence we can define:

$$\begin{array}{rcl} \mathcal{F}'':S & \longrightarrow & \mathcal{P}(H\times X) \\ \nu & \longmapsto & \mathcal{F}''(\nu)=\mathcal{F}(\nu)\times\mathcal{F}'(\nu) \end{array}$$

where is the hypergroup $H$ acts on the set $X$, the as follows:

$$\begin{array}{rcl} \circ: H\times X & \longrightarrow & \mathcal{P}^*(X) \\ (h,x) & \longmapsto & h\circ x, \end{array}$$

in addition, for all $h_i\in H$ and $x_i\in X$,

$$(h_i)(x_i)=\{(y_i)\ |\ y_i\in h_i\circ x_i\}.$$

If $(h,x)$ and $(h',x')\in H\times X$, then we can defines:

$$(h,x)*(h',x')=\{(r,s)\ |\ r\in hh',\ s=x\}$$

When combining the soft hypergroup $(H,\mathcal{F},S)$ and the soft set $(X,\mathcal{F}',S)$, the resulting structure $(H\times X,\mathcal{F}'',S)$ forms a soft hypergroupoid known as a soft action hypergroupoid. This concept of a soft action hypergroupoid demonstrates the interplay between soft hypergroup theory and soft set theory, showcasing their compatibility and utility in modeling mathematical structures.

**Definition 3.2.** Suppose $(\mathcal{F},S)$ and $(\mathcal{F}',S')$ are two non-empty soft hypergroupoids over $H_1$ and $H_2$, respectively. In addition $(f,g)$ is a soft function from $(\mathcal{F},S)$ to $(\mathcal{F}',S')$. Then $(f,g)$ is considered a soft inclusion (resp.good) homomorphism of hypergroupoids, when $f$ is an inclusion (resp.good) homomorphism from $H_1$ to $H_2$.

**Example 3.3.** Suppose that $P=\{e,r,s,t\}$ is a set, with a hyperoperation $*$ as follows:

| * | e | r | s | t |
|---|---|---|---|---|
| e | {e} | {r} | {s} | {t} |
| r | {r} | {e, r} | {t} | {s, t} |
| s | {s} | {t} | {e} | {r} |
| t | {t} | {s, t} | {r} | {e,r} |

and $Q=\{e\}\times P$ with a hyperoperation $\bullet$ as follows:

| • | (e,e) | (e,r) | (e,s) | (e,t) |
|---|---|---|---|---|
| (e,e) | {(e,e)} | {(e,r)} | {(e,s)} | {(e,t)} |
| (e,r) | {(e,r)} | {(e,e),(e,r)} | {(e,t)} | {(e,s),(e,t)} |
| (e,s) | {(e,s)} | {(e,t)} | {(e,e)} | {(e,r)} |
| (e,t) | {(e,t)} | {(e,s),(e,t)} | {(e,r)} | {(e,e),(e,r)} |

Consider sets $S$ and $T$ as follows:

$$S=\{e,r\},T=\{(e,e),(e,r)\}.$$

In addition, the the functions $\mathcal{F}, \mathcal{G}$ are as follows:

$$\begin{array}{rcl} \mathcal{F}: S & \longrightarrow & \mathcal{P}(P) \\ \nu & \longmapsto & \mathcal{F}(\nu) = \{z \in P \mid z \in \nu^2\} \end{array}$$

for $\nu \in S$, and

$$\begin{array}{rcl} \mathcal{G}: T & \longrightarrow & \mathcal{P}(Q) \\ (e,\beta) & \longmapsto & \mathcal{G}((e,\beta)) = \{(e,z) \in Q \mid (e,z) \in (e,\beta)^2\} \end{array}$$

for all $(e,\beta) \in T$. If we consider functions $f$ and $g$ as follows:

$$\begin{array}{rcl} f: P & \longrightarrow & Q \\ p & \longmapsto & \mathcal{F}(p) = (e,p) \end{array}$$

for $p \in P$, and

$$\begin{array}{rcl} g: S & \longrightarrow & T \\ \nu & \longmapsto & g(\nu) = (e,\nu) \end{array}$$

for $\nu \in S$, then, $f$ is a strong isomorphism, and $g$ is bijective. On the other hand, $f(\mathcal{F}(e)) = \{(e,e)\} = f(e)$, $f(\mathcal{F}(r)) = \{(e,e),(e,r)\} = f(\{e,r\})$, $\mathcal{G}(g(e)) = \{(e,e)\} = \mathcal{G}((e,e))$, $\mathcal{G}(g(r)) = \{(e,e),(e,r)\} = \mathcal{G}((e,r))$. Therefore, we have $f(\mathcal{F}(\nu)) = \mathcal{G}(g(\nu))$, for all $\nu \in S$. As a result, $(f,g)$ is a soft isomorphism, and hence $(\mathcal{F},S) \simeq (\mathcal{G},T)$.

**Remark 3.4.** From definition 3.2., we have a new category. The category whose objects are soft hypergroupoids and morphisms are soft hypergroupoid homomorphisms between them is formally known as the category of soft hypergroupoids, denoted by *SHGD. This category serves as a structured framework for studying and analyzing the properties and relationships within soft hypergroupoids, offering a systematic approach to exploring their characteristics and behaviors through the lens of morphisms and homomorphisms.*

**Proposition 3.5.** *If $(f,g)$ is a soft functor between the soft hypergroupoid $(H,\mathcal{F},S)$ and the soft category $(H',\mathcal{F}',S')$, then $(H',\mathcal{F}',S')$ is a soft hypergroupoid.*

*Proof.* $g$ functor is full. Hence $g$ is over the morphisms $(H',\mathcal{F}',S')$, and the soft category has a soft hypergroupoid structure. □

**Theorem 3.6.** *Suppose $(f,g)$ be a soft functor between the soft hypergroupoid $(H,\mathcal{F},S)$ and the soft category $(H',\mathcal{F}',S')$, then $(H',\mathcal{F}',S')$ is a soft hypergroupoid.*

*Proof.* If $(f,g)$ is a soft functor, then, the functor $f$ is full (surjection over morphisms), hence the soft category $(H',\mathcal{F}',S')$ is a soft hypergroupoid. □

**Theorem 3.7.** *If $(f,g)$ is a soft hypergroupoid homomorphism between soft hypergroupoids $(H,\mathcal{F},S)$ and $(H',\mathcal{F}',S')$, then, for all $h \in Mor(\mathcal{F}(\nu))$, $\nu \in S$, we have:*

1. $f(h^{-1}) = [f(h)]^{-1}$;
2. $f^{-1}(h) \cong f^{-1}(h^{-1})$.

*Proof.* 1. $f$ is a functor, and $\mathcal{F}(\nu)$ is a hypergroupoid for all $\nu \in S$. Since $(f,g)$ is a soft hypergroupoid homomorphism, hence, we have

$$f(h^{-1})f(h) = f(h^{-1}h) = f(1) = 1, \text{ and } f(h)f(h^{-1}) = f(hh^{-1}) = f(1) = 1,$$

for $h \in \mathrm{Mor}(\mathcal{F}(\nu))$. Thus, $f(h^{-1}) = [f(h)]^{-1}$.

2. For all $h' \in f^{-1}(h)$, we define:

$$\begin{array}{rcl} f_{h'} : f^{-1}(h) & \longrightarrow & f^{-1}(h^{-1}) \\ h' & \longmapsto & h'^{-1}. \end{array}$$

Mapping $f_{h'}$ is a bijection. Therefore $f_{h'}$ is an isomorphism. Hence $f^{-1}(h) \cong f^{-1}(h^{-1})$.

□

**Definition 3.8.** If $(H, \mathcal{F}, S)$ and $(H', \mathcal{F}', S')$ are two soft hypergroupoid, then $(H', \mathcal{F}', S')$ is called a soft subhypergroupoid of $(H, \mathcal{F}, S)$, when

(i) $S' \subset S$;

(ii) $\mathcal{F}'(v')$ is a subgroupoid of $\mathcal{F}'(v)$ for all $v' \in S'$.

**Example 3.9.** If $(H, \mathcal{F}, S)$ is a soft hypergroupoid, then $(H, \mathcal{F}, S)$ is a soft subhypergroupoid from $(H, \mathcal{F}, S)$.

**Definition 3.10.** If $(H', \mathcal{F}', S')$ is a soft subhypergroupoid of soft hypergroupoid $(H, \mathcal{F}, S)$, then

(i) $(H', \mathcal{F}', S')$ is called a full soft hypergroupoid, if $\mathcal{F}'(v')$ is a full subhypergroupoid of $\mathcal{F}'(v)$, for all $v' \in S'$;

(ii) $(H', \mathcal{F}', S')$ is called a wide soft subhypergroupoid of $\mathcal{F}(v')$, for all $v' \in S'$;

(iii) $(H', \mathcal{F}', S')$ is called a normal soft subhypergroupoid, if $\mathcal{F}'(v')$ is a normal subhypergroupoid of $\mathcal{F}(v')$, for all $v' \in S'$.

**Definition 3.11.** If the totally intransitive soft hypergroupoid $(H', \mathcal{F}', S')$ a normal soft subhypergroupoid of $(H, \mathcal{F}, S)$, then, the soft quotient hypergroupoid $(\frac{H}{H_1}, \mathcal{F}'', S')$ is as follows:

$$\begin{array}{rcl} \mathcal{F}'' : S'' & \longrightarrow & \mathcal{P}(\dfrac{H}{H'}) \\ v & \longmapsto & \mathcal{F}''(v) = \dfrac{\mathcal{F}(v)}{\mathcal{F}'(v)} \end{array}$$

**Definition 3.12.** If $(H, \mathcal{F}, S)$ is a soft hypergroupoid, and $\mathcal{F}(v)$ has an initial object as a hypergroupoid for all $v \in S$, then $(H, \mathcal{F}, S)$ is called a soft hypergroupoid with initial objects.

**Definition 3.13.** If $(\mathcal{F}, H, S)$ is a soft hypergroupoid, and $\mathcal{F}(v)$ has a terminal object as a hypergroupoid for all $v \in S$, then $(H, \mathcal{F}, S)$ is called a soft hypergroupoid with terminal objects.

**Theorem 3.14.** *If $(H, \mathcal{F}, S)$ is a soft hypergroupoid with initial (terminal) objects, then any two initial (terminal) objects in $(H, \mathcal{F}, S)$ are isomorphic.*

*Proof.* If soft hypergroupoid $(H, \mathcal{F}, S)$ has the initial objects $r$ and $s$, then $\mathcal{F}(v)$ has the initial objects $r$, $s$, such that can be as a hypergroupoid for all $v \in S$. We define morphisms $f : r \to t$ and $g : s \to t$, for all $t \in \mathcal{F}(v)$. But $\mathcal{F}(v)$ is a hypergroupoid, hence $f$ and $g$ are isomorphisms, that is to say $r \cong t$ and $s \cong t$, that means $r \cong s$. This completes the proof. □

**Remark 3.15.** In a soft hypergroupoid, the objects of the same type, are isomorphic.

## 4. Conclusion

In this paper, we delved into the realm of hypergroupoids as a generalization of hypergroups, focusing specifically on the study of soft hypergroupoids as an extension of soft hypergroups. Towards the conclusion, we guided readers towards exploring the category of soft hypergroupoids, which presents a promising avenue for further research and investigation for enthusiasts in this field. Additionally, akin to the developments in soft groupoid theory, we introduced the concept of trivial soft hypergroupoids.

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# A new family measures of noncompactness in the $L^p_{loc}(\mathbb{R}_+)$ space

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**Abstract**

The purpose of this paper is to introduce a new family measures of noncompactness in the space $L^p_{loc}(\mathbb{R}_+)$ $(1 \leq p < \infty)$ and study its properties. The technique of measure of noncompactness by applying fixed point theorem is the main tool in carrying out our proof.

## 1. Introduction

The notion of a measure of noncompactness (MNC) was introduced by Kuratowski [18] in 1930. In 1955, Darbo presented a fixed point theorem [11]associated with the measure of noncompactness. This result was applied, by many authers to study existence and behavior of solutions in considerable literatures to many classes of integral equations in the Banach space $BC(\mathbb{R}_+)$ (consisting of all real functions defined, bounded, and continuous on $\mathbb{R}_+$ ( see, for example, [2, 3, 5, 7, 9, 10, 12–14, 17, 20]). Later, Banas and Goebel [8] generalized and extended this notion axiomatically, which is more convenient in applications. Thereafter, Aghajani et al. [4], constructed a measure of non compactness on $L^p(\mathbb{R}^{\mathbb{N}})$ and Olszowy [19] presented a family of measure of noncompactness in $L^1_{loc}(\mathbb{R}_+)$ (consisting of all functions locally integrable on$(\mathbb{R}_+)$). In this paper we introduce a new family measures of noncompactness in the space $L^p_{loc}(\mathbb{R}_+)$. The structure of this article is as follows. In Section 2, some preliminaries and concepts are recalled. Section 3 is devoted to introduce a new family measure of noncompactness on the spaces $L^p_{loc}(\mathbb{R}_+)$.

## 2. Preliminaries

First, we introduce some notations and definitions which are used throughout this paper. Let $L^p(\mathbb{R}_+)$ denote the space of Lebesgue integrable functions on $\mathbb{R}_+$ with the standard norm

$$\|x\|_p = \Big(\int_0^\infty |x(t)|^p dt\Big)^{\frac{1}{p}}.$$

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We say that a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ belongs to $L^p_{loc}(\mathbb{R}_+)$ if $\chi_K f \in L^p(\mathbb{R}_+)$ for every compact set $K \subset \mathbb{R}_+$. Let us consider the set $L^p_{loc}(\mathbb{R}_+)$ equipped with the family of seminorms $\|\chi_{[0,T]}f\|_p$ for each $T > 0$. $L^p_{loc}(\mathbb{R}_+)$ becomes a Fréchet space furnished with the distance

$$d(x,y) = \sup\Big\{\frac{1}{2^n}\min\{1, \|\chi_{[0,n]}(x-y)\|_p\} : n \in \mathbb{N}\Big\} = \sup\Big\{\frac{1}{2^n}\min\{1, \Big(\int_0^n |x(t)-y(t)|^p dt\Big)^{\frac{1}{p}}\} : n \in \mathbb{N}\Big\}.$$

If X is a subset of $L^p_{loc}(\mathbb{R}_+)$, then $\overline{X}$ and $ConvX$ denote the closure and closed convex hull of $X$, respectively. A nonempty subset $X \subset L^p_{loc}(\mathbb{R}_+)$ is said to be bounded if

$$\sup\{\|\chi_{[0,T]}f\|_p = \Big(\int_0^T |f(t)|^p dt\Big)^{\frac{1}{p}} : f \in X\} < \infty$$

Further, let $\mathfrak{M}_{L^p_{loc}}$ denote the family of all nonempty and bounded subsets of $L^p_{loc}(\mathbb{R}_+)$ and $\mathfrak{N}_{L^p_{loc}}$ the family of all relatively compact subsets of $L^p_{loc}(\mathbb{R}_+)$.

Let us recall a fact, which we need in our considerations.
A sequence $(x_n)$ is convergent to $x$ in $L^p_{loc}(\mathbb{R}_+)$ if and only if for each $T > 0$, $(x_n)$ is convergent to $x$ in $L^p_{[0,T]}(\mathbb{R}_+)$.

**Definition 2.1.** ([8]) A family of functions $\{\mu_T\}$, where $\mu_T : \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)} \longrightarrow \mathbb{R}_+$, is said to be a family of measures of noncompactness in $L^p_{loc}(\mathbb{R}_+)$ if it satisfies the following conditions:

1° The family $\ker\{\mu_T\} = \{X \in \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)} : \mu_T(X) = 0 \; for \; T > 0\}$ is nonempty and $\ker\mu \subseteq \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$.

2° $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$.

3° $\mu_T(\overline{X}) = \mu_T(X)$ for $T \geq 0$.

4° $\mu_T(ConvX) = \mu_T(X)$ for $T \geq 0$.

5° $\mu_T(\lambda X + (1-\lambda)Y) \leq \lambda\mu_T(X) + (1-\lambda)\mu_T(Y)$, for $\lambda \in [0,1]$ and $T \geq 0$.

6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$, for $n = 1, 2, \cdots$ and if $\lim_{n\to\infty} \mu_T(X_n) = 0$ for each $T \geq 0$ then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$.

We say that a measure of noncompactness is regular [8 ], if it additionally satisfies the following conditions:

7° $\mu(X \cup Y) = max\{\mu(X), \mu(Y)\}$.

8° $\mu(X + Y) \leq \mu(X) + \mu(y)$.

9° $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}_+$.

10° $\ker\mu = \mathfrak{N}_E$.

The Hausdorff and Kuratowski measures of noncompactness satisfy all the above conditions (see [6, 8].

## 3. Main results

Before introducing a new measures of noncompactness on the spaces $L^p(\mathbb{R}_+)$, we need to characterize the compact subsets of $L^p(\mathbb{R}_+)$.

**Theorem 3.1.** ([16]) *Let $\mathcal{F}$ be a bounded subset in $L^p_{loc}(\mathbb{R}_+)$, $1 \leq p < \infty$. Then $\mathcal{F}$ is relatively compact if and only if for every $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\Big(\int_0^T |f(t) - f(t+h)|^p dt\Big)^{\frac{1}{p}} \leq \varepsilon$$

*for all $f \in \mathcal{F}$ and $|h| < \delta$.*

Now, we are ready to define a new measure of noncompactness on the spaces $L^p_{loc}(\mathbb{R}_+)$. Let $X$ be a bounded subset of the space $L^p_{loc}(\mathbb{R}_+)$, $1 \le p < \infty$. For $x \in X$, and $\varepsilon > 0$. Let us denote

$$\omega^T(x,\varepsilon) = \sup\{\Big(\int_0^T |x(t+h)-x(t)|^p dt\Big)^{\frac{1}{p}} : |h| < \varepsilon\},$$
$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\},$$

$$\omega^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon)$$

We have the following fact.

**Theorem 3.2.** *The family of mappings* $\{\omega^T\}$*, where* $\omega^T : \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)} \longrightarrow \mathbb{R}_+$ *is a family of measure of noncompactness on* $L^p_{loc}(\mathbb{R}_+)$ *and* $\ker\{\omega^T\} = \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$.

Proof. First we show that $\{\omega^T\}$ satisfies condition 1°. Let $X \in \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)}$ such that $\{\omega^T\} = 0$. Since $\{\omega^T\} = 0$, then $\lim_{\varepsilon\to 0} \omega^T(X,\varepsilon) = 0$. Thus, for all $T > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Big(\int_0^T |x(t)-x(t+h)|^p dt\Big)^{\frac{1}{p}} \le \varepsilon$$

for all $x \in X$ and $|h| < \delta$.
Therefore, from Theorem 3.1 we infer that the closure of $X$ in $L^p_{loc}(\mathbb{R}_+)$ is compact and $\ker\mu \subseteq \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$.
Satisfying $\{\omega^T\}$ in condition 2°, is obvious.
Now, we prove that condition 3° holds. Suppose that $X \in \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)}$ and $x \in \overline{X}$. Therefore, a sequence $x_n$ in $X$ exists such that $x_n \to x \in \overline{X}$ in $L^p_{loc}(\mathbb{R}_+)$. From definition of $\omega^T(X,\varepsilon)$, we have

$$\Big(\int_0^T |x_n(t+h)-x_n(t)|^p dt\Big)^{\frac{1}{p}} \le \omega^T(X,\varepsilon),$$

for any $n \in \mathbb{N}$, $T > 0$ and $|h| < \varepsilon$. By letting $n \to \infty$, we obtain

$$\Big(\int_0^T |x(t+h)-x(t)|^p dt\Big)^{\frac{1}{p}} \le \omega^T(X,\varepsilon)$$

Therefore,

$$\lim_{\varepsilon\to 0} \omega^T(\overline{X},\varepsilon) \le \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon),$$

Consequently,

$$\omega^T(\overline{X},\varepsilon) \le \omega^T(X,\varepsilon).$$

From 2°, we infer that $\omega^T(\overline{X}) = \omega^T(X)$ , and thus condition 3° holds.
For proof conditions 4° and 5°, in a similar manner , we use the inequality

$$\Big(\int_\Omega |\lambda x(t)+(1-\lambda)y(t)|^p dt\Big)^{\frac{1}{p}} \le \lambda\Big(\int_\Omega |x(t)|^p dt\Big)^{\frac{1}{p}} + (1-\lambda)\Big(\int_\Omega |y(t)|^p dt\Big)^{\frac{1}{p}},$$

where, $\Omega \in \mathbb{R}_+$ and $\lambda \in [0,1]$. Now, we prove that condition 6° satisfies in Definition 2.1. Suppose $\{X_n\}$ is a sequence of closed and nonempty sets from $L^p_{loc}(\mathbb{R}_+)$ such that $X_{n+1} \in X_n$, for $n = 1, 2, \cdots$, and $\lim_{n\to\infty} \omega^T(X_n) = 0$. For any $n \in \mathbb{N}$ take an $x_n \in X_n$. In the first step, we prove that $\mathcal{F} = \{\overline{x_n}\}$ is a compact set in $L^p_{loc}(\mathbb{R}_+)$. Equivalently, we need

to check that $\omega^T(\mathcal{F}) = 0$, where $T > 0$. Suppose $\varepsilon > 0$ be fixed and since $\lim_{n\to\infty} \omega^T(X_n) = 0$, $k \in \mathbb{N}$ exists such that $\omega^T(X_k) < \varepsilon$. Thus, we can find $\delta_1 > 0$ and $T > 0$ such that

$$\omega^T(X_k, \delta_1) < \varepsilon.$$

Thus, for $n \geq k$ and $|h| < \delta_1$, we can write

$$\Big(\int_0^T |x_n(t+h) - x_n(t)|^p dt\Big)^{\frac{1}{p}} \leq \omega^T(X_n, \varepsilon) < \varepsilon.$$

Since the set $\{x_1, x_2, \cdots, x_{k-1}\}$ is compact, then $\delta_2 > 0$ and $T > 0$ exist such that

$$\Big(\int_0^T |x_n(t+h) - x_n(t)|^p dt\Big)^{\frac{1}{p}} \leq \varepsilon,$$

for all $n = 1, 2, \cdots, k$ and $|h| < \delta_2$.
Therefore, we obtain $\omega^T(\mathcal{F}, \delta) < \varepsilon$, for $\delta < \min\{\delta_1, \delta_2\}$ and for all $T > 0$. We Know that

$$\omega^T(\mathcal{F}) = \lim_{\delta\to 0} \omega^T(\mathcal{F}, \delta)$$

Then, $\omega^T(\mathcal{F}) = 0$. Which means that $\mathcal{F}$ is compact. Therefore, a subsequence $\{X_{n_j}\}$ and $x_0 \in L^p_{loc}(\mathbb{R}_+)$ exist such that $X_{n_j} \to x_0$. Because $x_n \in X_n$, $X_{n+1} \subset X_n$ and $X_n$ is closed for all $n \in \mathbb{N}$, then we obtain

$$x_0 \in \cap_{n=1}^{\infty} X_n = X_\infty,$$

which completes the proof of condition 6°.
Finally, we show that $ker\mu = \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$. From condition 1°, we know that $ker\mu \subseteq \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$. We need to prove that $\mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)} \subseteq ker\mu$. For this aim, let $X \in \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}$. Therefore, the closure of $X$ in $L^p_{loc}(\mathbb{R}_+)$ is compact and hence from Theorem 3.1, for any $\varepsilon > 0$ and $T > 0$ there exists $\delta > 0$ such that

$$\Big(\int_0^T |x(t+h) - x(t)|^p dt\Big)^{\frac{1}{p}} \leq \varepsilon$$

for all $x \in X$ and $|h| < \delta$.
Thus,
for all $x \in X$, we have

$$\omega^T(x, \delta) = \sup\Big\{\Big(\int_0^T |x(t+h) - x(t)|^p dt\Big)^{\frac{1}{p}} : |h| < \delta\Big\} \leq \varepsilon.$$

Therefore,

$$\omega^T(X, \delta) = \sup\{\omega^T(x, \delta) : x \in X\} \leq \varepsilon.$$

This proves that

$$\lim_{\delta\to 0} \omega^T(X, \delta) = 0.$$

Since $\omega^T(X) = 0$, then we infer that $\mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)} \subseteq ker\mu$. Thus, we have

$$ker\mu = \mathfrak{N}_{L^p_{loc}(\mathbb{R}_+)}.$$

**Theorem 3.3.** *The familly measure of noncompactness* $\{\omega^T\}$ *defined in Theorem* 3.2 *is regular.*

*Proof.* Let $X, Y \in \mathfrak{M}_{L^p_{loc}(\mathbb{R}_+)}$. Since for every $\varepsilon > 0$, $\lambda > 0$ and $T > 0$ we have

- $\omega^T(X \cup Y, \varepsilon) \leq max\{\omega^T(X, \varepsilon), \omega^T(Y, \varepsilon)\}$,
- $\omega^T(X + Y, \varepsilon) \leq \omega^T(X, \varepsilon) + \omega^T(y, \varepsilon)$,
- $\omega^T(\lambda X, \varepsilon) \leq |\lambda|\omega^T(X, \varepsilon)$ for $\lambda \in \mathbb{R}_+$,

then the conditions 7°, 8° and 9° hold and the condition 10° in Theorem 3.2 was proved. □

Now we state some fixed point theorems which are used in the following.

**Theorem 3.4.** *Let C be a nonempty, bounded, closed, and convex subset of the space* $L^p_{loc}(\mathbb{R}_+)$ *and let* $F : C \to C$ *be continuous. Let* $C_0 = C$ *and* $C_{n+1} = Conv(FC_n)$ *for* $n = 0, 1, 2, \ldots$. *If* $\lim_{n\to\infty} \omega^T(C_n) = 0$, *for each* $T \geq 0$, *where* $\{\omega^T\}$ *is a family of measure of noncompactness on the spaces* $L^p_{loc}(\mathbb{R}_+)$ *then F has a fixed point.*

*Proof.* From properties 1° - 6° of the $\{\omega^T\}$ we have $C_\infty = \cap_{n=1}^{\infty} C_n$ is nonempty, closed, convex and compact in $L^p_{loc}(\mathbb{R}_+)$. Moreover $F$ is mapping $C_\infty \to C_\infty$. Applying Tichonov fixed point principle we infer that $F$ has at least one fixed point.

□

**Theorem 3.5.** *[11] Let Q be a nonempty, bounded, closed, and convex subset of the space* $L^p_{loc}(\mathbb{R}_+)$ *and let* $F : Q \to Q$ *be continuous mapping. Suppose that there exist numbers* $K_T \in [0, 1)$ *for* $T > 0$ *such that*

$$\omega^T(FX) \leq K_T \omega^T(X)$$

*for nonempty* $X \subset Q$ *and* $T \geq 0$. *Then F has at least one fixed point in the set* $Q$.

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# Nonparametric Bayesian optimal designs for Exponential regression model with Respect to Prior Processes(with Polya Urn Scheme as the base measure)

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**Abstract**

This study introduces optimal designs for the Exponential nonlinear model using nonparametric Bayesian approaches. Nonlinear regression models find extensive applications across various scientific disciplines. It is vital to accurately fit the optimal nonlinear model while considering the biases of the Bayesian optimal design. By utilizing the Dirichlet process as a prior, we present a Bayesian optimal design. In this research paper we employ a representation to approximate the D-optimality criterion considering the Dirichlet process as a functional tool. Through this approach, we aim to identify a Nonparametric Bayesian optimal design.

## 1. Introduction

Within the realm of experimental design, the concept of optimal design refers to a specific category of designs that are classified based on certain statistical criteria.
Optimal designs are sought using optimality criteria, typically based on the information matrix. In nonlinear models, the presence of unknown parameters introduced complexities in the design problem, as the optimality criteria depends on these unknown parameters [3, 6]. To address this challenge, researchers proposed various solutions, including local optimal designs [1, 8, 12, 19, 30], sequential optimal designs, minimax optimal designs, Bayesian optimal designs [28, 21-25], and pseudo-Bayesian designs [26]. Chernoff [8] introduced the concept of local optimality, which involves specifying fixed values for the unknown parameters and optimizing a function of the information matrix to determine the design for these specified parameter values. This approach aimed to overcome the difficulties associated with the dependence of the design problem on unknown parameters in nonlinear models. It's important to note that local designs for nonlinear models are derived subsequent to an initial linearization of the model, using the parameter set as a reference point.

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The selection of unknown parameters in local designs is typically obtained from previous studies or experiments specifically conducted for this purpose. The effectiveness of local designs heavily relies on the appropriate selection of these parameters. However, a significant challenge arises when the investigated problem lacks robustness in relation to weak parameter estimation. To address this, an alternative approach for local optimal designs involves utilizing a prior distribution for the unknown parameters instead of relying solely on initial guess. In the Bayesian method, the first step is to represent the available information in the form of a probability distribution for the model parameter, known as the prior distribution. A Bayesian optimal design aims to maximize the relevant optimality criterion over this prior distribution. Nevertheless, it is crucial to acknowledge that the selection of the prior distribution within the Bayesian framework can be problematic and may potentially lead to erroneous results. The choice of the prior distribution is subjective, relying on the researcher's beliefs, and it significantly influences the final outcome. Unfortunately, the Bayesian approach lacks a definitive method for selecting the prior distribution. Numerous researchers have investigated the effect of the prior distribution on determining design points in various types of optimal designs. For instance, Chaloner and Larentz [11], Chaloner and Duncan [9], Burghaus and Dette [5], Chaloner and Vardinelli [10], Pronzato and Walter [29], Mukhopadhyay and Haines [26], Dette and Ngobauer [13,14], Fedorov [15,16], and Firth and Hinde [18] have contributed extensively to this field. Chapter 18 of Atkinson et al.'s book [3] provides further reading on this topic. Moreover, in situations where there is insufficient evidence from previous studies on the topic of interest, specifying an appropriate prior distribution becomes challenging. In such cases, subjective or noninformative prior distributions are used, incorporating all available information regarding the uncertainty of the parameter values. For more information, refer to Burghaus and Dette [5].
This research paper presents the optimal design for nonlinear models in section 2. In Section3, the nonparametric Bayesian D-optimal design for exponential regresion model is presented. Finally, Section 4 concludes the paper with some closing remarks.

## 2. Introduction to optimal designs in nonlinear models

In the realm of nonlinear experimental design, a common scenario arises where in the relationship between the response variable $y$ and the independent variable $x$ is given by the equation $y = \eta(x,\theta)+\epsilon$ where $x \in \chi \subseteq \mathrm{R}$ and $y$ is a response variable and $\theta \in \Theta$ is the unknown parameter vector and $\epsilon$ is a normally distributed residual value with mean 0 and known variance $\sigma^2 > 0$. For simplicity, we assume $\sigma^2 = 1$ in this problem. If $\eta(x,\theta)$ is differentiable with respect to $\theta$ then, the information matrix $M(\xi,\theta)$ at a given point $x$ can be represented as follows:

$$I(\xi,\theta) = \frac{\partial}{\partial\theta}\eta(x,\theta)\frac{\partial}{\partial\theta^T}\eta(x,\theta). \tag{1}$$

There exist several optimality criteria used to obtain the optimal design, including D-optimality and A-optimality. These criteria are functions of the information matrix and can be expressed as follows:

$$\Psi_D(\xi,\theta) = -\log(\det(M(\xi,\theta))) \ , \Psi_A(\xi,\theta) = tr(M^{-1}(\xi;\theta)),$$

where $\xi$ denotes a design with two components; the first component represents specific values from the design space $\chi$ and the second component corresponds to the weights assigned to these values, so that design $\xi$ can be defined as follows:

$$\square = \begin{Bmatrix} x_1 & x_2 & \dots & x_\ell \\ w_1 & w_2 & \dots & w_\ell \end{Bmatrix} \in \Xi, \tag{2}$$

where $p$ represents the number of model parameters [25], and
$\Xi = \{\xi \mid 0 \le w_j \le 1; \sum_{j=1}^{\ell} w_j = 1, x \in \chi\}, p \le \ell \le \frac{p(p+1)}{2}$.
When considering a discrete probability measure $\xi$ with finite support, the information function of $\xi$ can be expressed as follows [3]:

$$M(\xi,\theta) = \sum_{j=1}^{\ell} w_j I(x_j,\theta). \tag{3}$$

Because of the dependence of the information matrix $M(\xi,\theta)$ to the unknown parameter $\theta$, one approach to address this issue is to employ the Bayesian method and incorporate a prior distribution of the parameter vector. The Bayesian D-optimality criterion can be formulated as follows:

$$\Psi_{\Pi}(\xi) = E(\psi(\xi;\theta)) = \int_{\Theta} \psi(\xi;\theta) d\Pi(\theta) = \int_{\Theta} -\log(\det(M(\xi,\theta))) d\Pi(\theta), \tag{4}$$

where Π represents the prior distribution for $\theta$ and the Bayesian D-optimal design is attained by minimizing (2.4). In certain situations, specifying a prior distribution on the parameter space Θ can be challenging for the experimenter. In such cases, an alternative approach is to consider an unknown prior distribution Π for the parameter $\theta$. In this condition, Π is treated as a parameter itself. Consequently, equation (2.4) becomes a random functional, and it becomes necessary to determine its distribution or approximation. From a Bayesian perspective, we construct a prior distribution on the space of all distribution functions to address this issue. To achieve this objective, Ferguson [17] introduced the concept of the DP that an overview of it will be provided in the following.

### *2.1. Nonparametric Bayesian D-optimal design*

Nonparametric models constitute an approach to model selection and fitting, where the size of the models is allowed to grow with the size of the data. It is unlike parametric models that use a fixed number of parameters. In this section, we introduce the nonparametric Bayesian optimal design. In the nonparametric Bayesian framework, it is assumed that $\theta \mid P \sim P$, where $P$ is a random probability distribution and $P \sim \Pi$. General method of construction a random measure is to start with the stochastic processes. Ferguson [17] formulated the requirements which must be imposed on a prior distribution and proposed a class of prior distributions, named DP. One of the main argument in using the Dirichlet distribution in practical applications is based on the fact that this distribution is a good approximation of many parametric probability distributions. Bondesson [6], Sethuraman [30], and Zarepour and Al Labadi [31] are among those who have contributed to this area. A method of producing samples from the Dirichlet process is to use the Polya urn process that in the upcoming section, we will discuss about it. Then the nonparametric Bayesian D-optimal design for the Exponential regresion model is discussed.

### *2.2. Polya Urn Scheme*

Polya Urn Scheme was used by Blackwell and McQueen (1973) to demonstrate the existence of the Dirichlet process. The method of producing a sample of the Dirichlet process is to use a Polya Urn Scheme [19]. Consider a Polya urn with $a(\chi)$ balls of which $a(i)$ are of color $i$ ; $i = 1, 2, ..., k$.[For the moment assume that $a(i)$ are whole numbers or 0]. Draw balls at random from the urn, replacing each ball drawn by two balls of the same color. Let $X_i = j$ if the $i$ th ball is of color $j$. Then:

$$P(X_1 = j) = \frac{a(j)}{a(\chi)}, \tag{5}$$

$$P(X_2 = j \mid X_1) = \frac{a(j) + \delta_{X_1}(j)}{a(\chi) + 1}, \tag{6}$$

and in general

$$P(X_{n+1} = j \mid X_1, X_2, ..., X_n) = \frac{a(j) + \sum_{1}^{n} \delta_{X_i}(j)}{a(\chi) + n}, \tag{7}$$

That $n$ is the number of extracted balls and $\delta_{X_i}(j)$ is equal to one if $X_i = j$, otherwise it is equal to zero.

## 3. Nonparametric Bayesian D-optimal design for Exponential regresion model with Respect to Prior Processes(with Polya Urn Scheme as the base measure)

Suppose we have the following regression model:

$$E(y|x) = \eta(x,\theta) = \exp(-\theta x), x > 0, \theta > 0. \tag{8}$$

therefore, the Bayesian D-optimality criterion, denoted as $\Psi_{\Pi}(\xi)$ can be expressed as follows:

$$\Psi_{\Pi}(\xi) = E(\psi(\xi;\theta)) = \int_{\Theta} \psi(\xi;\theta) d\Pi(\theta) = \int_{\Theta} -\log \sum_{j=1}^{\ell} w_j x_j^2 [\exp(-2\theta x_j)]) d\Pi(\theta) \quad (9)$$

where Π is the prior distribution for $\theta$. The Bayesian D-optimal design is attained by minimizing equation (3.2). In the nonparametric Bayesian framework, we consider $P \sim \mathrm{DP}(\alpha, P_0)$ and its collective representation as $P(.) = \sum_{i=1}^{\infty} p_i \, \delta_{\theta_i}(.)$. In this context, the optimality criterion can be expressed as follows:

$$\Psi_{\Pi}(\xi) = \sum_{i=1}^{\infty} p_i (-\log \sum_{j=1}^{\ell} w_j x_j^2 [\exp(-2\theta_i x_j)]). \quad (10)$$

Chernoff [8] demonstrated that when searching for a local optimal design, there exists an optimal design where all the mass is concentrated at a single point within the design supports. Caratheodory's theorem also confirms the existence of a one-point optimal design. However, when employing the Bayesian optimality criterion, a more complex situation arises. Dette [13] showed that with a uniform prior distribution, as the support of the prior distribution increases, the number of optimal design points for the single-parameter model also increases. Chaloner and Verdinelli [10] suggested that if the researcher aims to obtain a one-point optimal design, it is advisable to consider a small support for the uniform prior distribution. The same principle applies to nonparametric Bayesian designs. In this case, assuming a uniform distribution over the interval [0, B] as the basic distribution, the one-point optimal design can be achieved.

Equation (3.2) is a stochastic function of the DP. According to Ferguson's definition of the DP, the calculation (3.2) is not easily possible, so to address this challenge and obtain an approximation of the optimal nonparametric Bayesian criterion, methods such as the stick-breaking process is employed. Sethuraman (1994) introduced this method as a significant approach for generating realizations of the DP. Another method has been presented by Zarepour and Ellabadi [33] whose simulation speed and accuracy is much higher than the stick breaking process. We used this method in this paper.

Now, in this section we consider Polya Urn Scheme as the base measure in DP. We get the results by using a nonlinear optimization programing with R package Rsolnp. To better understanding of the effect of the $\alpha$ parameter, we tabulate the results for four different values of $\alpha$=1, 5, 10, 50, in Tables 1-4. Without loss of generality, we consider a bounded design space $\chi$=[0, 1]. Tables 1-4 represent the results when the concentration parameter and uncertainty in the base measure increase.

Table 1. Nonparametric Bayesian D-optimal design when $\alpha$=1.

| *Prior distribution* | *Design* | *Two − point* | | *Three − point* | | |
|---|---|---|---|---|---|---|
| $U[0,50]$ | $x$ | 0.054440 | 0.260513 | 0.030685 | 0.209087 | 0.387580 |
| | $w$ | 0.876842 | 0.123158 | 0.908720 | 0.002113 | 0.089167 |
| $U[0,100]$ | $x$ | 0.023030 | 0.219612 | 0.017941 | 0.195164 | 0.320225 |
| | $w$ | 0.905494 | 0.094506 | 0.977778 | 0.010774 | 0.009128 |
| $U[0,300]$ | $x$ | 0.008303 | 0.197470 | 0.008225 | 0.193860 | 0.306556 |
| | $w$ | 0.948465 | 0.051535 | 0.986071 | 0.010698 | 0.003231 |
| $U[0,1000]$ | $x$ | 0.003043 | 0.195013 | 0.002832 | 0.195338 | 0.299104 |
| | $w$ | 0.975145 | 0.024855 | 0.992532 | 0.007418 | 0.000050 |

According to the results, as the value of $\alpha$ increases, the support points in two-point design do not change significantly. The weight of the minimum point increases rapidly and the smallest point will have the most weight that this weight almost increases or remains fixed by increasing the concentration parameter.

## 4. Concluding Remarks And Future Works

Nonlinear regression models are widely used in various scientific fields, and the Bayesian method is commonly employed to obtain optimal designs in such models. However, one of the challenges in the Bayesian framework is the

Table 2. Nonparametric Bayesian D-optimal design when $\alpha$=5.

| *Prior distribution* | *Design* | *Two − point* | | *Three − point* | | |
|---|---|---|---|---|---|---|
| $U[0,50]$ | $x$ | 0.034587 | 0.388024 | 0.032450 | 0.236545 | 0.498389 |
| | $w$ | 0.811363 | 0.188637 | 0.894985 | 0.014646 | 0.090369 |
| $U[0,100]$ | $x$ | 0.018340 | 0.268389 | 0.018546 | 0.196677 | 0.374871 |
| | $w$ | 0.867598 | 0.132402 | 0.948225 | 0.017276 | 0.034499 |
| $U[0,300]$ | $x$ | 0.006756 | 0.190899 | 0.006666 | 0.172643 | 0.318524 |
| | $w$ | 0.922931 | 0.077069 | 0.948950 | 0.040386 | 0.010664 |
| $U[0,1000]$ | $x$ | 0.002193 | 0.194242 | 0.002133 | 0.188004 | 0.313534 |
| | $w$ | 0.966823 | 0.033177 | 0.977844 | 0.018185 | 0.003971 |

Table 3. Nonparametric Bayesian D-optimal design when $\alpha$=10.

| *Prior distribution* | *Design* | *Two − point* | | *Three − point* | | |
|---|---|---|---|---|---|---|
| $U[0,50]$ | $x$ | 0.036680 | 0.421636 | 0.034412 | 0.213537 | 0.559561 |
| | $w$ | 0.821632 | 0.178368 | 0.868929 | 0.029632 | 0.101439 |
| $U[0,100]$ | $x$ | 0.018490 | 0.271066 | 0.017731 | 0.194647 | 0.448192 |
| | $w$ | 0.867274 | 0.132726 | 0.932293 | 0.022384 | 0.045323 |
| $U[0,300]$ | $x$ | 0.006569 | 0.193873 | 0.006490 | 0.175472 | 0.333920 |
| | $w$ | 0.919503 | 0.080497 | 0.952076 | 0.035164 | 0.012760 |
| $U[0,1000]$ | $x$ | 0.002029 | 0.186047 | 0.002075 | 0.177670 | 0.307947 |
| | $w$ | 0.969009 | 0.030991 | 0.970351 | 0.025391 | 0.004258 |

Table 4. Nonparametric Bayesian D-optimal design when $\alpha$=50.

| *Prior distribution* | *Design* | *Two points* | | *Three points* | | |
|---|---|---|---|---|---|---|
| $U[0,50]$ | $x$ | 0.037439 | 0.511531 | 0.037144 | 0.211627 | 0.671521 |
| | $w$ | 0.898396 | 0.101604 | 0.895857 | 0.028447 | 0.075696 |
| $U[0,100]$ | $x$ | 0.018731 | 0.347769 | 0.018396 | 0.184127 | 0.565521 |
| | $w$ | 0.912504 | 0.087496 | 0.933661 | 0.020097 | 0.046242 |
| $U[0,300]$ | $x$ | 0.006397 | 0.200206 | 0.006329 | 0.141265 | 0.384839 |
| | $w$ | 0.934723 | 0.065277 | 0.951399 | 0.035986 | 0.012615 |
| $U[0,1000]$ | $x$ | 0.001972 | 0.170471 | 0.001942 | 0.153639 | 0.315340 |
| | $w$ | 0.962347 | 0.037653 | 0.961299 | 0.033890 | 0.004811 |

subjective selection of the prior distribution, which can potentially lead to incorrect results. By adopting a nonparametric Bayesian approach and utilizing the Dirichlet process, we aim to address the challenges associated with selecting the prior distribution in Bayesian optimal design construction. This allows us to account for uncertainty and mitigate the impact of restrictive parametric assumptions, providing more flexible and robust designs for nonlinear regression models.

In this study, we focus on utilizing the Polya Urn Scheme as the base distribution in the Dirichlet process. To better understand the influence of the concentration parameter $\alpha$, we present the results in tables for four different values of $\alpha$=1, 5, 10, 50.

In the investigated range, the results reveal interesting findings. As the uncertainty in the base measure and the concentration parameter in the Dirichlet process increase, the support points in the two-point designs do not undergo significant changes. The weight of the smallest point increases rapidly, and it becomes the point with the highest weight. This weight tends to either increase or remain relatively stable with an increase in the concentration parameter.

It is important to note that this approach can be applied to other optimality criteria and various models with two or more parameters. For example, nonparametric Bayesian optimal designs using the A- or E-optimality criterion for the nonlinear model discussed in this paper, along with a Dirichlet process prior, hold potential for further research. We hope to report new results in this area in the near future.

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# On the whiskered hypergroupoids and crossed hypermodules

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**Abstract**

In this study, we delve into the intricate relationship between crossed hypermodule frameworks and hypergroups by constructing a novel crossed hypermodule structure derived from a whiskered hypergroupoid, where the underlying object set is defined as a hypergroup. This innovative approach employs a standard functor that serves to bridge the categorical realms of crossed hypermodules and cat1-hypergroups. Additionally, we establish a corresponding whiskered hypergroupoid structure, wherein the object set is represented as a hypergroup, originating from a crossed hypermodule of hypergroups. The paper further elucidates the criteria necessary for bimorphisms within the context of a whiskered hypergroupoid, while also examining the interconnections between this structure and internal hypergroupoids situated within the category of whiskered hypergroupoids, specifically where the object set is a hypergroup. Moreover, we conduct a thorough investigation into the relationships that exist between the category of crossed hypermodules associated with hypergroups and the category of whiskered hypergroupoids. This exploration aims to uncover deeper insights into the theoretical underpinnings and applications of these mathematical constructs, fostering a greater understanding of their interplay and significance in the broader framework of hypergroup theory.

## 1. Introduction

The notion of whiskering applied to a groupoid emerges from the foundational principles of the tensor product of crossed complexes within the context of groupoids. In the realm of group theory, the pioneering work of Brown and Higgins [3] introduced a formal definition for the tensor product applicable to the category of crossed complexes over groupoids. Consider a crossed complex denoted as $H$, which is composed of hypergroupoids and equipped with a self-tensor product $w : H \otimes H \longrightarrow H$. By implementing a 1-truncation of $H$ that incorporates the biactions of the object set on the morphism sets, we can derive a whiskered groupoid. This construction not only enriches the structure of $H$ but also facilitates a deeper understanding of the interactions between its components, thereby enhancing the theoretical framework surrounding whiskered groupoids and their applications in higher algebraic contexts. Therefore, we have $w_{0,1} : H_0 \times H_1 \longrightarrow H_1$, $w_{1,0} : H_1 \times H_0 \longrightarrow H_1$ and $w_{0,0} : H_0 \times H_0 \longrightarrow H_0$ called whiskerings where $H_0$ is the set of objects and $H_1$ is the set of morphisms between objects. The operations $w_{0,1}$ and $w_{1,0}$ give the left and right actions

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of $H_0$ on $H_1$, respectively. Furthermore, the operation $w_{0,0}$ gives a monoid structure over $H_0$. A crossed complex $H$ over groupoids, along with the tensor product $\otimes$ over $H$, can be seen as a crossed differential graded algebra as defined by Baues in [4] and further investigated by Baues-Tonks in [5]. Therefore, the first component of a crossed differential graded algebra can also represent a whiskered groupoid. Essentially, the purpose of defining whiskering operations is to examine the conditions under whiHh the composition of morphisms exhibits commutativity for any given category. In the case of a group $G$, if the commutators within $G$ are trivial, then $G$ is considered an Abelian group. To define the notion of commutativity for any category $H$, considering the whiskering operations in $H$, the left and right multiplications have been introduced by Brown in [6]. In the case $H := (H_1, H_0)$ is a groupoid together with the whiskering $w_{1,0} : H_1 \times H_0 \longrightarrow H_1$ and $w_{0,1} : H_0 \times H_1 \longrightarrow H_1$, the commutator of $a : x \longrightarrow y$ and $b : u \longrightarrow v$ in $H$ Han be defined by $[a, b] = w_{10}(a, u)^{-1} \circ w_{1,0}(y, b)^{-1} \circ w_{1,0}(a, v) \circ w_{0,1}(x, b)$. In this equality, the left and right multiplications are given by $l(a, b) = w_{0,1}(y, b) \circ w_{1,0}(a, u)$ and $r(a, b) = w_{1,0}(a, v) \circ w_{0,1}(x, b)$. Thus, the commutator of the morphisms $a, b \in H$ is $[a, b] = l(a, b)^{-1} r(a, b)$. In the case $l(a, b) = r(a, b)$, the groupoid $H$ is called a commutative groupoid [6], and then $H$ is a strict monoidal category. On the other hand, if $H$ is a groupoid, then the automorphism structure $Aut(H)$ is equivalent to a crossed module introduced by Whitehead in [12]; $\partial : Sc(H) \longrightarrow Aut(H)$ where $Sc(H)$ is the set of sections of the source map $s$ and the target map is a bijection on $H_0$. Then the set $Sc(H)$ has a group structure with the Ehresmannian composition. By utilizing this composition, we establish the connection between crossed modules and whiskered groupoids with the object set in a group. In a 2-dimensional context, if $H$ is a crossed module, then $Aut(H)$ possesses a braided regular crossed module structure as defined by Brown and Gilbert [7]; refer to [8] for more information on this structure. The reduced cases of this structure in other contexts can be found in [2, 9]. This structure Han be viewed as a whiskered 2-groupoid with the object set as a group. Further exploration of this concept will be conducted in a separate paper. Brown also introduced the notion of whiskering for any R-category in [6]. Considering an R-algebroid as a small R-category, by employing the results from [11], the R-algebroid version of the results mentioned herein can be investigated.

## 2. Whiskered hypergroupoids and crossed hypermodules

According [10], a polygroup is a multi-valued system $\mathcal{M} =< P, \circ, e, {}^{-1} >$, with $e \in P$, ${}^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold, for all $r, s, t \in P$:

1. $(r \circ s) \circ t = r \circ (s \circ t)$
2. $e \circ r = r \circ e = r$
3. $r \in s \circ t$ implies $s \in r \circ t^{-1}$ and $t \in s^{-1} \circ r$.

$\mathcal{P}^*(P)$ is the set of all the non-empty subsets of $P$, and also if $x \in P$ and $R$, $T$ are non-empty subsets of $P$, then we have $R \circ T = \bigcup_{\substack{b \in T \\ a \in R}} a \circ b$, $x \circ T = \{x\} \circ T$ and $R \circ x = R \circ \{x\}$.
The following, are the facts that are clearly concluded from the principles of the polygroups: $e \in r \circ r^{-1} \cap r^{-1} \circ r$, $e^{-1} = e$ and $(r^{-1})^{-1} = r$.

**Example 2.1.** If we consider the set $P$ as $P = \{e, r, s\}$, then $P =< P, \circ, e, {}^{-1} >$ along with polyaction which have shown in the Table 2.1. is a polygroup.

| $\circ$ | $e$ | $r$ | $s$ |
|---|---|---|---|
| $e$ | $e$ | $r$ | s |
| $r$ | $r$ | $\{e, s\}$ | $\{r, s\}$ |
| $s$ | $s$ | $\{r, s\}$ | $\{e, r\}$ |

Table 2.1.

**Definition 2.2.** A crossed hypermodule $\chi = (C, P, \partial, \kappa)$ is consists of hypergroups $< C, *, e, {}^{-1} >$ and $< P, \circ, e, {}^{-1} >$ together with a strong homomorphism $\partial : C \longrightarrow P$ and a (left) action $\kappa : P \times C \longrightarrow \mathcal{P}^*(C)$ on $C$, satisfying the following conditions:

1. $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$, for all $c \in C$ and $p \in P$,

2. ${}^{\partial(c)}c' = c * c' * c^{-1}$, for all $c, c' \in C$.

**Example 2.3.** 1. In every hyperygroup, the set containing only the identity member is always a subhypergroup, and this subhypergroup is normal in the hypergroup. Therefore, we have crossed hypermodule $(1, P) = (1, P, c_1, id\ |_{c_1})$.
2. Every hypergroup $P$ contains the whole hypergroup $P$ as a normal subhypergroup. So, we always have crossed hypermodule $(P, P) = (P, P, c, id_P)$.
3. Consider the following hypergroup morphisms of an abelian hypergroup $P$, written multiplicatively,

$$l : 1 \to Aut(P) \qquad i \to id_P \qquad k : P \to 1 \qquad p \to 1$$

So, we have a crossed hypermodule $(P, 1) = (P, 1, l, k)$.

**Example 2.4.** 1. [1] A conjugation crossed hypermodule is an inclusion of a normal subhypergroup $N$ of $P$, with action given by conjugation. In fact, for any hypergroup $P$, the identity map $\mathrm{id}_P : P \longrightarrow P$ is a crossed hypermodule with the action of $P$ on itself by conjugation. Indeed, there are two canonical ways a hypergroup $P$ may be regarded as a crossed hypermodule: via the identity map or the inclusion of the trivial subhypergroup.
2. If $C$ is a $P$-hypermodule, then there is a well defined action $\kappa$ of $P$ on $C$. This, together with the zero homomorphisms, creates a crossed hypermodule $(C, P, \partial, \kappa)$.

**Definition 2.5.** Consider the crossed hypermodules $\chi = (C, P, \partial, \kappa)$ and $\chi' = (C', P', \partial', \kappa')$. A crossed hypermodule morphism $f = (\lambda, \Gamma) : \chi \to \chi'$ is a tuple of strong homomorphism, such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\lambda} & C' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ P & \xrightarrow[\Gamma]{} & P' \end{array}$$

commutes, and $\lambda(p\kappa c) = \Gamma(p)\kappa'\lambda(c)$, for all $p \in P, c \in C$.

Suppose that $H$ is a (small) category with set of morphisms (or 1-Hells) written by $H_1$ and the set of objects (or 0-cells) written by $H_0$. In $H_1$, particularly, the set of morphisms $a : x \longrightarrow y$ from $x$ to $y$ is denoted by $H_1(x, y)$, and $x, y$ are called the source and target of the morphism a, respectively. The source and target maps are written $s, t : H_1 \longrightarrow H_0$. Then, for $a \in H_1(x, y)$, we have $s(a) = x$ and $t(a) = y$. The category composition in $H$ of morphisms $a : x \longrightarrow y$ and $b : y \longrightarrow z$ can be defined by $b \circ a : x \longrightarrow z$. In this case, clearly, $s(b \circ a) = s(a)$ and $t(b \circ a) = t(b)$. We will write $H_1(x, x)$ as $H_1(x)$.

**Definition 2.6.** A whiskering on a category $H := (H_1, H_0)$ consists of operations $w_{i,j} : H_i \times H_j \longrightarrow H_{i+j}, i, j = 0, 1, i + j \leq 1$ satisfying the following axioms:

- Whisk 1: $w_{0,0}$ gives a monoid structure on $H_0$;
- Whisk 2: $w_{0,1} : H_0 \times H_1 \longrightarrow H_1$ is a left action of the monoid $H_0$ on the category $H$ in the sense that, if $x \in H_0$ and $a : u \longrightarrow v$ in $H_1$, then $w_{0,1}(x, a) : w_{0,0}(x, u) \longrightarrow w_{0,0}(x, v)$, in $H$, so that: $w_{0,1}(1, a) = a, w_{0,1}(w_{0,0}(x, y), a) = w_{0,1}(x, w_{0,1}(y, a))$,
and
$w_{0,1}(x, a \circ b) = w_{0,1}(x, a) \circ w_{0,1}(x, b), w_{0,1}(x, 1_y) = 1_{xy}$,
- Whisk 3: $w_{1,0} : H_1 \times H_0 \longrightarrow H_1$ is a right action of the monoid $H_0$ on $H_1$ with analogous rules,
- Whisk 4: $w_{0,1}(x, w_{1,0}(a, y)) = w_{1,0}(w_{0,1}(x, a), y)$, for all $x, y, u, v \in H_0, a, b \in H_1$.

A category $H$ together with a whiskering is called a whiskered category.

In a whiskered category, for $a : x \longrightarrow y$, $b : u \longrightarrow v$, there are two multiplications given by $l(a,b) := m_{0,1}(y,b) \circ m_{1,0}(a,u)$ and $r(a,b) := m_{1,0}(a,v) \circ m_{0,1}(x,b)$.
It is well-known that a hypergroupoid is a small category in which every arrow (or morphisms or 1-cells) is an isomorphism. That is, for any morphism a, there is a (necessarily unique) morphism $a^{-1}$ such that $a \circ a^{-1} = e_{s(a)}$ and $a^{-1} \circ a = e_{t(a)}$ where $e : H_0 \longrightarrow H_1$ gives the identity morphism at any object. We denote a hypergroupoid as $H := (H_1, H_0)$, where $H_0$ is the set of objects and $H_1$ is the set of morphisms. For any hypergroupoid $H$, if $H_1(x,y)$ is empty whenever $x, y$ are distinct (that is, if $s = t$) then $H$ is called totally disconnected hypergroupoid. A hypergroupoid $H := (H_1, H_0)$ together with the whiskering operations $w_{i,j} : H_i \times H_j \longrightarrow H_{i+j}$ for $i + j \leq 1$ satisfying the above conditions is called a whiskered hypergroupoid. We will denote a whiskered hypergroupoid by $(H, w)$. In a whiskered hypergroupoid, if the object set $H_0$ is a group with the multiplication given by $w_{0,0}$, we say that $(H_1, H_0)$ is a regular hypergroupoid. We will use the notation $WG$ to denote the category of whiskered hypergroupoids whose set of objects is a polygroup with the operation $w_{0,0}$.

## 3. Bimorphisms within Whiskered (Regular) Groupoids

In this section, we aim to clarify the profound relationship between the category of crossed hypermodules associated with polygroups and the category of whiskered hypergroupoids, where the object set is defined as a polygroup. The concept of crossed hypermodules was first introduced by Alp and Davvaz [1] in their seminal work, which laid the groundwork for understanding these structures in a broader mathematical context. Crossed hypermodules provide a robust algebraic framework that serves as a model for homotopy connected 2-types of topological spaces. This connection is pivotal, as it allows for the exploration of how algebraic properties can reflect and inform the topological characteristics of spaces. By examining the interplay between crossed hypermodules and whiskered hypergroupoids, we can uncover insights into the underlying structures that govern these mathematical entities. The implications of this relationship extend beyond mere theoretical curiosity; they offer a pathway to understanding complex interactions within algebraic topology and provide tools for analyzing the homotopical aspects of polygroups.
Let $\partial : M \longrightarrow N$ be a crossed polymodule. We will obtain a whiskered hypergroupoid $H := (H_1, H_0)$ together with the operations $w_{1,0}$ and $w_{0,1}$. Let $H_0 = N$. By using the action of $N$ on $M$, we can consider the semi-direct product polygroup $M \rtimes N$ with the polygroup operation given by $(m,n)(m',n') = (m(nm'), nn')$ for $m, m' \in M$ and $n, n' \in N$. Then, by taking $H_0 = N$ and $H_1 = M \rtimes N$, we can create a whiskered hypergroupoid as follows: The source and target maps from $H_1$ to $H_0$ are given by $s(m,n) = n$ and $t(m,n) = \partial(m)n$ for all $(m,n) \in H_1$. The hypergroupoid composition is given by $(m',n') \circ (m,n) = (m'm, n)$ if $n' = \partial(m)n$. Finally, the whiskering operations $w_{0,1}$ and $w_{1,0}$ are given by respectively $w_{0,1}(p,(m,n)) = (pm, pn)$ and $w_{1,0}((m,n),p) = (m, np)$ for all $m \in M$, $n, p \in N$. For these operations, we have $s(w_{0,1}(p,(m,n))) = s(pm,pn) = pn = ps(m,n)$ and $t(w_{0,1}(p,(m,n))) = t(pm,pn) = \partial(pm)pn = p\partial(m)p^{-1}pn$ (Since $\partial$ is crossed module) $= p\partial(m)n = pt(m,n)$ Similarly, we obtain easily that $s(w_{1,0}((m,n),p)) = s(m,np) = np = s(m,n)p$ and $t(w_{1,0}((m,n),p)) = t(m,np) = \partial(m)np = t(m,n)p$ for all $(m,n) \in H_1$ and $n, p \in H_0$. Consequently, we obtain a whiskered hypergroupoid. In this structure, the operation $w_{0,0}$ be taken as the group operation of $H_0 = N$. Hence, a functor can be defined from the category of crossed polymodules of polygroups to the category of whiskered hypergroupoids with objects set in a polygroup. We denote it by $S : XM \longrightarrow WG$.
Let $H := (H_1, H_0, w_{i,j})$ be a whiskered hypergroupoid with the set of objects $H_0$ is a polygroup according to the multiplication given by the operation $w_{00}$. In this case, we can say, using the Ehresmannian composition, that the set $K = \{a \in H_1 : s(a) = 1_{H_0}\}$ is a polygroup with the polygroup operation given by $a \odot b = w_{10}(a, t(b)) \circ b$ for any $a : 1_{H_0} \longrightarrow y$ and $b : 1_{H_0} \longrightarrow v$ in $K$, $y, v \in H_0$, and the target map $t$ from $K$ to $H_0$ is a homomorphism of polygroups. For $1_{H_0} \in H_0$, we have $e(1_{H_0}) : 1_{H_0} \longrightarrow 1_{H_0}$ is the identity element of $K$. Indeed for any $a : 1_{H_0} \longrightarrow y \in K$, we obtain $a \odot e(1_{H_0}) = w_{10}(a, 1_{H_0}) \circ 1_{H_0} = a = e(1_{H_0}) \odot a$ The inverse of $a : 1_{H_0} \longrightarrow y$ is $a^{-1} : 1_{H_0} \longrightarrow y^{-1}$ where $y^{-1}$ is the inverse of $y$ in the ploygroup $H_0$. Thus, we have $a \odot a^{-1} = w_{1,0}(a, y^{-1}) \circ a^{-1} = e(1_{H_0})$. We show that the target map $t$ is a homomorphism of polygroups from $K$ to $H_0$. For $a : 1_{H_0} \longrightarrow y$ and $b : 1_{H_0} \longrightarrow v$ in $K$, and $y, v \in H_0$, we obtain $t(a \odot b) = t(w_{1,0}(a, tb)) \circ b = yv = t(a)t(b)$. The polygroup action of $p \in H_0$ on $a : 1_{H_0} \longrightarrow y \in K$ is given by $pa = w_{1,0}(e(p), vp^{-1}) \circ w_{1,0}(a, p^{-1}) \circ e(p)$. The polygroup $H_0$ is acting on itself by conjugation. Thus, we obtain that the homomorphism t is $H_0$-equivariant relative to the action of $H_0$ on $K$ given above. Indeed, we have $t(pa) = pvp^{-1} = pt(a)p^{-1}$ for $p \in H_0$ and $a \in K$, and so $t$ is a pre-crossed polymodule of polygroups. Furthermore, for any $a : 1_{H_0} \longrightarrow y$, $b : 1_{H_0} \longrightarrow v \in K$, we have $a \odot b \odot a^{-1} = w_{1,0}(a, vy^{-1}) \circ w_{10}(b, y^{-1}) \circ a^{-1}$. Therefore,

we obtain $t(a)b = w_{1,0}(e(t(a)), vy^{-1}) \circ w_{1,0}(b, y^{-1}) \circ e(t(a))^{-1} = w_{1,0}(a, vy^{-1}) \circ w_{1,0}(b, y^{-1}) \circ e(t(a))^{-1}1 = a \odot b \odot a^{-1}$ and this is the second crossed-ploymodule axiom. So, we can say that t is a crossed ploymodule of polygroups. Thus, we have a crossed polymodule $t : K \longrightarrow H_0$ from the whiskered groupoid $(H, w) := (H_1, H_0, w_{i,j})$. We can define a functor from the category of whiskered hypergroupoids with objects set a polygroup to the category of crossed polymodules as $F : WG \longrightarrow XM$.
Now, we by employing the axioms of the crossed polymodule, we will delineate the bimorphism conditions in the whiskered hypergroupoid derived from a crossed polymodule.

**Definition 3.1.** Let $H$ be a category. A bimorphism $m : (H, H) \longrightarrow \Diamond H$ (where $\Diamond H$ is the double category) assigns to each pair of morphisms $a, b \in H$ a square $m(a, b) \in \Diamond H$ such that if $ad, bc$ are defined in $H$ then $m(ad, c) = m(a, c) \circ_1 m(d, c)$ and $m(a, bc) = m(a, b) \circ_2 m(a, c)$.

**Theorem 3.2.** *For the regular hypergroupoid*

$$(H_1, H_0) := \left( H_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} H_0 = N, \circ, w_{i,j} \right)$$

*which is obtained from the crossed polymodule $\partial : M \longrightarrow N$, the multiplication $a \star b$ given by:*

$$m(a, b) = a * b = \begin{pmatrix} w_{0,1}((m, n), n') & w_{0,1}(n, (m', n')) & w_{0,1}((m, n), \partial(m')n') \\ & w_{0,1}(\partial(m)n, (m', n')) & \end{pmatrix}$$

*is a bimorphism for $a = (m, n)$, $b = (m', n') \in M \rtimes N$.*

**Theorem 3.3.** *In the whiskered hypergroupoid*

$$(H_1, H_0) := \left( H_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} H_0 = N, \circ, w_{i,j} \right)$$

*associated to the crossed polymodule $\partial : M \longrightarrow N$, we have $l(a, b) = r(a, b)$ so this category is a strict monoidal category.*

**Theorem 3.4.** *In the whiskered hypergroupoid*

$$(H_1, H_0) := \left( H_1 = M \rtimes N \underset{e}{\overset{s,t}{\rightrightarrows}} H_0 = N, \circ, m_{ij} \right)$$

*associated to the crossed polymodule $\partial : M \longrightarrow N$, the interchange law is hold:*

$(a \circ c) \star (b \circ d) = (a \star b) \circ (c \star d)$. *Thus, $(H_1, H_0, )$ is an internal category in the category of whiskered hypergroupoids.*

## 4. Conclusion

In this paper, we established a significant connection between crossed polymodules and whiskered hypergroupoids with the object set as a group. Consequently, we observe that a whiskered hypergroupoid can be viewed as a crossed polymodule of polygroups. If H is a crossed polymodule, then the automorphism structure Aut(H), possesses a braided regular crossed polymodule structure. This structure can therefore be interpreted as a whiskered 2-hypergroupoid with the object set in a polygroup. In a forthcoming study, as a two-dimensional analogue of the findings presented here, the concept of a whiskered 2-hypergroupoid could be introduced by utilizing the properties of the braiding map on a crossed module. An R-algebroid can be seen as a small R-category. It is also plausible to investigate the R-algebroid version of the results.

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# Some results on the representations of crossed polymodules

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**Abstract**

Suppose $G$ be a group, also $V$ a K-vector space. We know that a K-linear representation of $G$ with representstion space $V$ is a homomorphism $\Phi : G \to GL(V)$. Also if $\Phi$ is a representation of $G$, then the character $\Phi$ is defined for $g \in G$ as $\psi_g(\phi) = Tr(\phi(g))$. Concept of symmetric crossed polymodule studied by Dehghanizadeh. In this paper we study the representstion of crossed polymodules. Our results extend the classical results of representation of crossed module to crossed polymodules.

## 1. Introduction

The Yang-Baxter equation plays a fundamental role in various areas of mathematics. In fact, this equation plays a fundamental role in such apparently distant fields as statistical mechanics, particle physics, quantum field theory and quantum group. Its solutions, called braidings, are built, among others,
1) from Yetter-Drinfel'd modules over a Hopf algebra,
2) from self-distributive structures,
3) from crossed modules of groups.
Also, Crossed modules and its applications play very important roles in category theory, homology and cohomology of groups, homotopy theory, algebra, k-theory etc. Therefore, study crossed modules and its all kinds automorphisms at least through this is very important. This is in fact one of the motivations of recent half-century studies in this field. We recall some definitions and properties of the crossed module category. A crossed module $(T, G, \partial)$ consist of a group homomorphism $\partial : T \longrightarrow G$ together with an action $(g, t) \longrightarrow {}^{g}t$ of $G$ on $T$ satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(s)}t = sts^{-1}$, for all $g \in G$ and $s, t \in T$ [1–4, 21]. Nilpotent, Solvable, n-Complete and Representations of crossed modules was studied by Dehghanizadeh and Davvaz[16–20]. Polygroups were studied by Comer[12], also see [13]. Specially, Comer and Davvaz developed the algebraic theory for polygroups. Alp and Davvaz in [4], introduced the notion of crossed polymodule of polygroups and they given some of its properties. Also they introduce new important classes by the fundamental relations. Alp and Davvaz, introduce the concept of pullback and pushout crossed polymodules and describe the construction of pullback and pushout crossed polymodules. Arvasi, Porter and

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Onarh in [5–8] , introduce the notion of an (co)-induced 2-crossed module, which extends the notion of an (co)-induced crossed module which were defined by Brown, Gilbert, Loday and Mosa [9–11]. In [14, 15] Dehghanizadeh, Davvaz and Alp, introduce the notion of crossed polysquare.
In addition to the inner automorphism map $\tau : N \to Aut(N)$; other standard examples of crossed modules are:
- The inclusion of a normal subgroup $N \to G$;
- A $G$-module $M$ with the zero homomorphism $M \to G$
- And any epimorphism $E \to G$ with central kernel.
In [18] Dehghanizadeh and Davvaz studied the reperesentations and characters of CaT$^1-$groups and crossed modules. In this paper we study the representation of crossed polymoduls.

## 2. polygroups and representation of polygroups

A polygroup is a completely regular, reversible in itself multigroup. According [12], a *polygroup* is a multi-valued system $\mathcal{M} =< P, \circ, e, {}^{-1} >$, with $e \in P$, ${}^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all $x, y, z$ in $P$:

1. $(x \circ y) \circ z = x \circ (y \circ z)$
2. $e \circ x = x \circ e = x$
3. $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

In this definition, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of $P$, and if $x \in P$ and $A$, $B$ are non-empty subsets of $P$, then $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.
The following elementary facts about polygroups follow easily from the axioms: $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $e^{-1} = e$ and $(x^{-1})^{-1} = x$.
If $K$ is a non-empty subset of $P$, then $K$ is called a *subpolygroup* of $P$ if $e \in K$ and $< K, \circ, e, {}^{-1} >$ is a polygroup. The subpolygroup $N$ of $P$ is said to be *normal* in $P$ if $a^{-1} \circ N \circ a \subseteq N$, for every $a \in P$. There are several kinds of homomorphisms between polygroups[13]. In this paper, we apply only the notion of strong homomorphisms. Let $< P, \circ, e, {}^{-1} >$ and $< P', *, e, {}^{-1} >$ be two polygroups. A mapping $\phi$ from $P$ into $P'$ is said to be a *strong homomorphism* if $\phi(e) = e$ and for all $a, b \in P$, $\phi(a \circ b) = \phi(a) * \phi(b)$ for all $a, b \in P$. A strong homomorphism $\phi$ is said to be an *isomorphism* if $\phi$ is one to one and onto.

**Definition 2.1.** Let $\mathcal{P} =< P, \circ, e, {}^{-1} >$ be a polygroup and $\Omega$ be a non-empty set. A map $\alpha : P \times \Omega \longrightarrow \mathcal{P}^*(\Omega)$, where $\alpha(p, \omega) := {}^{p}\omega$ is called a *(left) polygroupaction* on $\Omega$ if the following axioms hold:

1. ${}^{e}\omega = \omega$,
2. ${}^{h}({}^{p}\omega) = {}^{h \circ p}\omega$, where ${}^{p}A = \bigcup_{a \in A} {}^{p}a$ and ${}^{B}\omega = \bigcup_{b \in B} {}^{b}\omega$ for all $A \subseteq \Omega$ and $B \subseteq P$,
3. $\bigcup_{\omega \in \Omega} {}^{p}\omega = \Omega$,
4. for all $p \in P$, $a \in {}^{p}b \Rightarrow b \in {}^{p^{-1}}a$.

**Example 2.2.** Every polygroup P has its trivial subpolygroup 1 consisting of just the identity element of P.
This subpolygroup is always a normal subpolygroups. Therefore we have crossed polymodule $(1, P) = (1, P, c_1, id_c \mid_1)$.

**Example 2.3.** Every polygroup P contains the whole polygroup P as a normal subpolygroup. Therefore, we have crossed polymodule $(G, G) = (G, G, c, id_G)$.

**Definition 2.4.** A hyperring in the general sense is the largest class of multivalued systems that satisfies the ring-like axioms.
$(R, +, .)$ is a hyperring if $(R, +)$ is a hypergroup, is associative hyper operation and the distributive laws

$$x.(y + z) = x.y + x.z$$

$$(x+y).z = x.z + y.z$$

are satisfied for every $x, y, z \in R$.
$(R, +, .)$ is called a semihyperring if $(+, .)$ are associative hyperoperations, where . is distributive with respect to $+$.

**Definition 2.5.** [13] A hypermatrix is a matrix with entries from a semihyperring. The hyperproduct of two hypermatrices $(a_{ij})$ , $(b_{ij})$ which are of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner

$$(a_{ij})(b_{ij}) = \left\{ (c_{ij}) \mid c_{ij} \in \sum_{k=1}^{n} a_{ik} b_{kj} \right\}$$

**Definition 2.6.** [13] If $P =< P, \circ, e, ^{-1} >$ be a polygroup, and $R$ a semihyperring, with identity and $M_R = \{(a_{ij}) \mid a_{ij} \in R\}$, then a map $T : P \to M_R$ is called a representation if
(1) $T(x_1 \circ x_2) = \{T(x) \mid x \in x_1 \circ x_2\} = T(x_1)T(x_2)$ for all $x_1, x_2 \in P$.
(2) T(e)=I, where I is the identity matrix. If instead of the first condition, we have the condition $T(x_1 \circ x_2) \subseteq T(x_1)T(x_2)$ for all $x_1, x_2 \in P$, then $T$ is called an inclusion representation.

**Example 2.7.** Suppose that $P = \{e, a, b\}$ is a set, and $P =< P, \circ, e, ^{-1} >$ by the multiplication table

| $\circ$ | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | $e$ | $a$ | b |
| $a$ | $a$ | $\{e, b\}$ | $\{a, b\}$ |
| $b$ | $b$ | $\{a, b\}$ | $\{e, a\}$ |

is a polygroup. Then in $Z_3$, we define a hyperoperation $\oplus$ as follows.
$1 \oplus 1 = \{0, 2\}$, $2 \oplus 2 = \{0, 1\}$, $1 \oplus 2 = 2 \oplus 1 = \{1, 2\}$ and $\oplus = +$ be the usual sum for the other cases, and let $\odot$ be the usual product in $Z_3$.
Now $(\Omega_3, \oplus, \odot)$ is a semihyperring and the map $T : P \to M_R$ with

$$T(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T(a) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T(b) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a representation of the polygroup $P$.
Generally, if we choose $i_0, j_0, i_0 \neq j_0, 0 \leq i_0, j_0 \leq n$ and then put $T(e) = I_n$, $T(a) = A_n$ and $T(b) = B_n$, where

$$A_n = (a_{ij}) \quad with \quad \begin{cases} a_{ii} = 1 & for\ i = 1, 2, \ldots, n \\ a_{i_0 j_0} = 1 & \\ a_{ij} = 0 & otherwise \end{cases}$$

$$B_n = (b_{ij}) \quad with \quad \begin{cases} b_{ij} = a_{ij} & if \quad i \neq i_0\ ,\ j \neq j_0 \\ b_{i_0 j_0} = 2 & \end{cases}$$

then, $T$ is a representation of $P$.

## 3. Representation of crossed polymodules

In section we introduced the concept of representation of crossed polymodules. One should note that we have indeed to a definition of the symmetric crossed polymodule. Concept of symmetric crossed polymodule studied by Dehghanizadeh and according, we have:

**Theorem 3.1.** *Suppose $\chi = (Mor(\chi), Ob(\chi), (s,i,t), \bullet)$ be a category of polygroups. Also, consider the set*

$$Q_\chi = Aut(\chi) = \{F \mid F : \chi \longrightarrow \chi, F \text{ is an } autoploy functor\}$$

*together with the composition of functors (*), and $P_{Q_\chi} = Q_\chi \cup \{H\}$ such that $H \notin Q_\chi$. Then $P_{Q_\chi}$ is polygroup by a the appropriate hyperoperations.*

**Theorem 3.2.** *Suppose $\chi = (Mor(\chi), Ob(\chi), (s,i,t), \bullet)$ be a category of polygroups. Consider the set*

$$P_\chi = \{(id_\chi \xrightarrow{a} F) : F \in Aut(\chi) \text{ and } a \text{ is an } isotransformation\}$$

*and on $P_\chi$, we define a multiplication by*

$$(id_\chi \underset{\sim}{\xrightarrow{a}} F) * (id_\chi \underset{\sim}{\xrightarrow{b}} G) := (id_\chi \underset{\sim}{\xrightarrow{a \neq b}} FG) = a \bullet (Fb) = b \bullet (aG),$$

*then $P_{P_\chi} = P_\chi \cup \{H\}$ such that $H \notin P_\chi$, by a the appropriate hyperoperation is polygroup.*

**Theorem 3.3.** *Suppose $\chi = (Mor(\chi), Ob(\chi), (s,i,t), \bullet)$ be a category of polygroups, and suppose given functors $F, G : \chi \longrightarrow \chi$. Let given transformations $(id_\chi \xrightarrow{a} F)$ and $(id_\chi \xrightarrow{b} F)$ such that $a * b = b * a = id_{id_\chi}$ holds, then*
*(1) We have $F, G \in Aut(\chi)$, i.e. the functors $F$ and $G$ are autofunctors, and we have $G = F^{-1}$.*
*(2) The transformations a and b are isotransformations.*

**Theorem 3.4.** *Let $\chi = (Mor(\chi), Ob(\chi), (s,i,t), \bullet)$ be a category of polygroups, and $V = (P, Q, \alpha, f)$ be a crossed polymodule. Also $Q_\chi = Aut(\chi)$*

$$P_\chi = \{(id_\chi \xrightarrow{a} F) \mid F \in Aut(\chi) \text{ and } a \text{ is an } isotransformation\}.$$

*Then we have a polyaction of $Q_\chi$ on $P_\chi$, given by the polygroup morphism*

$$\alpha_\chi : Q_\chi \longrightarrow Aut(P_\chi)$$

$$Q \to (id_\chi \underset{\sim}{\xrightarrow{a}} F) \to (id_\chi \underset{\sim}{\xrightarrow{Q^{-1}aQ}} Q^{-1}FQ)$$

*and a polygroup morphism*

$$f_\chi : P_\chi \longrightarrow Q_\chi$$
$$(id_\chi \underset{\sim}{\xrightarrow{a}} F) \to F.$$

*Then $(P_\chi, Q_\chi, \alpha_\chi, f_\chi)$ is a crossed polymodule, (Symmetric Crossed Polymodule on $\chi$,) and we write*

$$S_\chi := (P_\chi, Q_\chi, \alpha_\chi, f_\chi).$$

*Also we write, for $Q \in Q_\chi$ and $(id_\chi \underset{\sim}{\xrightarrow{a}} F) \in P_\chi$,*

$$a^Q = (a)(Q\alpha_\chi) = Q^{-1}aQ : id_\chi \to F^Q = Q^{-1}FQ$$

*for the polyaction of $Q$ on $a$.*

**Definition 3.5.** $(P_\chi, Q_\chi, \alpha_\chi, f_\chi)$ is a crossed polymodule, and called the Symmetric Crossed Polymodule on $\chi$, and we write

$$S_\chi := (P_\chi, Q_\chi, \alpha_\chi, f_\chi)$$

**Theorem 3.6.** *If $S_\chi = (P_\chi, Q_\chi, \alpha_\chi, f_\chi)$ be the symmetric crossed polymodule on $\chi$, then*

$$P_\chi^R = \{\chi \xrightarrow{F} \chi \mid F \text{ is an } R-\text{linear autofunctor}\} = \{F \in P_\chi \mid F \text{ is } R-\text{linear}\} \leq P_\chi$$

*and*

$$Q_\chi^R = \{id_\chi \xrightarrow{a} \chi \mid F \in P_\chi^R \text{ and } a \text{ is an isotransformation}\} \leq Q_\chi$$

**Theorem 3.7.** *If $S_\chi = (P_\chi, Q_\chi, \alpha_\chi, f_\chi)$ be the symmetric crossed polymodule on $\chi$, and*

$$\alpha_\chi^R : Q_\chi^R \longrightarrow P_\chi^R$$
$$(id_\chi \underset{\sim}{\xrightarrow{a}} F) \to F$$
$$f_\chi^R : P_\chi^R \longrightarrow Aut(Q_\chi^R)$$
$$P \longmapsto \left((id_\chi \underset{\sim}{\xrightarrow{a}} F) \longrightarrow (id_\chi \underset{\sim}{\xrightarrow{P^{-1}aP}} P^{-1}FP)\right)$$

*be maps, then we have a crossed subpolymodule*

$$Aut_R^{CPM}(\chi) = Aut_R(\chi) = \left(Q_\chi^R, P_\chi^R, f_\chi^R, \alpha_\chi^R\right) \leq S_\chi.$$

*The upper index CPM in $Aut_R^{CPM}(\chi)$ should merely indicate the $Aut_R^{CPM}(\chi)$ is a crossed polymodule, and we call $Aut_R^{CPM}(\chi)$ the automorphism crossed polymodule of $\chi$.*

**Remark 3.8.** Consider the $\chi-$polycrossed category $\chi Pcat = (Q \ltimes P, Q, (s, i, t), \bullet)$; the composition in the category $\chi Pcat$ is given by

$$(q, p) \bullet (qp\alpha, p') = \{(q, y) \mid y \in pp'\}$$

**Definition 3.9.** A crossed polymodule $P = (\lambda, \Gamma) : \chi \to Aut_R^{CPM}(\chi)$ is called a representation of $\chi$ on $M_R$, where $M_R$ be an R-linear category.

## 4. Conclusion

In this paper, we introduced the representstion of crossed polymodules. Our results extended the classical results of representation of crossed module to crossed polymodules.

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# Stability of $p-$harmonic maps from Finsler manifolds

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**Abstract**

The harmonic (resp. p-harmonic) maps played a substantial role also in the recent studies in non-linear analysis, differential geometry, and geometric variational problems. The study of harmonic maps, which are critical points of a particular energy functional, has a long tradition on Riemannian manifolds. They can be generalized to p-harmonic maps, which have a parameter $p$ to finesse harmonic maps where $2 < p < \infty$, allowing for this framework to be adapted to more intricate environments. Such maps have been central to many incidence - type problems, stability properties, and energy minimization in geometric systems.

In this paper, we investigate p-harmonic maps in the context of Finsler geometry, a general framework to study non-quadratic metrics. The Finsler manifolds have the richer geometric structure and are well-suited for the study of higher-order variational problems. Stability and Liouville-type theorems for p-harmonic maps in this more general setting. In this context, stability is studied as the behavior of the energy functional under small perturbations of the map. In Riemannian geometry, classical results have shown that there are no (say, non-constant) stable harmonic maps between certain compact manifolds or from Euclidean $n$-spheres $n \geq 3$ into compact Riemannian manifold satisfying proper curvature conditions.

In this paper, based on this main results, we prove the stability of p-harmonic maps $\psi : \mathbb{S}^n \longrightarrow (N, F)$, where $(N, F)$ is a Finsler manifold with positive constant flag curvature. Moreover, if $F$ is a locally Minkowski manifold (a special class of Finsler manifolds with local flatness), the stability of p-harmonic maps from the standard $n$ -dimensional unit sphere $\mathbb{S}^n$ into $(N, F)$ is retained. On the other hand, thinking about hypersurfaces $M^{n-1}$ in Riemannian manifolds $N^n$, if $\psi : (M^{n-1}, g) \to (N^n, h)$ is a totally geodesic isometric immersion and $Ric_N \geq 0$, then it is shown that $\psi$ is p-unstable.

In this paper, we establish the existence and uniqueness of generalized solutions of p-harmonic maps between Finsler manifolds underaction through explicit regularization and limits of surjective maps results through long-wave approximation in explicit curvature conditions. The implications of this may govern energy distributions in Finsler manifolds, calling into question the ways in which we use energy densities for stability and minimization results.

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# Existence of $p-$harmonic maps between Riemannian manifolds

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**Abstract**

P-harmonic maps are a generalization of harmonic maps that have been studied in differential geometry and analysis and are based on the notion of the p-energy functional which is the p-th power of the norm of the differential of a map. It allows the investigation of the maps in various geometric and analytic settings, depending on the value of p, and has important applications to geometric analysis, mathematical physics and nonlinear partial differential equations. For instance, in physics, p-harmonic maps can describe minimal energy configurations and equilibrium states, showing features of elastic materials, fluid dynamic flows and general relativity. They are also useful in applied sciences, including image processing, where they are involved in denoising and image restoration.

We prove several main results concerning p-harmonic maps. To this end, we first show that any p-harmonic map into a codomain which admits a conformal vector field with a positive potential function is constant. Second, given a harmonic map $\psi$ from a compact Riemannian manifold $(M^m, g)$ to a Riemannian manifold $(N^n, h)$ with non-positive sectional curvature, we prove that there exists a smooth metric $\bar{g}$ conformally equivalent to $g$, $\bar{g} = e^{2f}(g)$, such that $\psi$ becomes p-harmonic. Third, we show that $\psi$ is p-harmonic if and only if $M$ is a minimal submanifold in $N$ (a generalization of this definition when $p = 2$ is an isometric embedding). Finally, the p-harmonic maps from a Riemannian manifold of negative sectional curvature into any Riemannian manifold admitting a nontrivial conformal vector field is shown to be constant. In addition to these results, there are many more outstanding results in the field of p-harmonic maps. In particular, there has been a large amount of regularity theory developed regarding p-harmonic maps, which has produced important information regarding the smoothness and singularity structures of solutions. Moreover, existence outcomes are obtained with respect to different boundary conditions and typologies, which broaden their applicability. Substantial work has also been done studying the interactions of p-harmonic maps with geometric flows including the heat flow of p-energy functionals, leading to dynamic techniques for constructing and studying such maps. Additionally, relations among p-harmonic maps and other variational problems, such as quasiconformal mappings and elastic deformations, have also paved the way for the interdisciplinary study. P-harmonic maps thus reflect a wide applicability and deep elegance as a mathematical tool addressing intricate problems spanning both mathematics and the applied sciences at large.

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# Liouville type theorem for exponentional harmonic maps

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**Abstract**

Theorems of Liouville-type are main tools in differential geometry and analysis of partial differential equations as they reveal important rigidity phenomena or classify solutions subject to certain geometric and analytic conditions. The present work is concerned with an important subclass of harmonic maps called exponential harmonic maps, which arise from the minimization of an energy functional involving an exponential term and that can describe a variety of topics from elasticity theory and liquid crystal modeling to generalized minimal surface theory.
We prove three main results which point out the subtle interplay between geometry and analysis in this context. We begin with some rigidity results: We first prov that any exponential harmonic map $\psi : (M, g) \longrightarrow (N, h)$ where $(M, g)$ is a complete Riemannian manifold and $(N, h)$ is a target manifold of non-positive sectional curvature is in fact constant. Second, we establish that exponentially harmonic maps, other non-harmonic maps and weakly conformal maps $\psi : \mathbb{S}^2 \longrightarrow N^n$ are the maps with non-minimal immersions, which provide structural information about the energy aspects of these maps. Finally, we obtain a conservation law for the exponential energy-momentum tensor of exponentially harmonic maps, which bears an intrinsic geometric quality and relates to their definition.
To obtain these results, we make use of sophisticated apparatus like the Bochner formula, integral estimates and various tools from geometric analysis. All together, this paper highlights the great importance of Liouville-type theorems in understanding the qualitative behavior of exponential harmonic maps, as well as their geometric and analytic foundations.

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# Uniform Topology on Rings of Countable Pointfree Functions

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Tpological ring

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**Abstract**

Let $\mathcal{R}L$ be the ring of continuous real-valued functions on a frame $L$, and let $\mathcal{R}_cL$ be its subring of functions with countable images. We investigate $\mathcal{R}_cL$ as a topological ring under $u_c$-topology. A frame $L$ is called countably pseudocompact if every element of $\mathcal{R}_cL$ is bounded. We show that a frame $L$ is countably pseudocompact if and only if the set of units of $\mathcal{R}_cL$ is open in the $u_c$-topology, which is equivalent to $\mathcal{R}_cL$ begin a topological ring under the $u_c$-topology, and also to it begin a topological vector space over $\mathbb{Q}$ in the $u_c$-topology.

## 1. Introduction

Let $C(X)$ be the ring of all real-valued continuous functions on a Tychonoff (completely regular and Hausdorff) topological space $X$ (see [6] for more details). We denote by $C_c(X)$ the subring of $C(X)$ consisting of those functions with countable images (see [5] for more details). As noted in [7], for a function $f \in C_c(X)$ and each positive real number $\epsilon$, we define the set $U_c(f,\epsilon)$ as follows:

$$U_c(f,\epsilon) = \{g \in C_c(X) : |g(x) - f(x)| \leq \epsilon, \text{ for all } x \in X\}.$$

The family $\{U_c(f,\epsilon) : f \in C_c(X), \epsilon \in \mathbb{R}^+\}$ forms a base for the neighborhood system of $f$, leading to the $u_c$-topology on $C_c(X)$. The topological ring structure of $C_c(X)$ with respect to the $u_c$-topology is studied in [7].
A frame is a complete lattice $L$ in which the distributive law

$$a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s.$$

holds for all $a \in L$ and $S \subseteq L$. We write 0 and 1 for the bottom and the top element of $L$, respectiveliy. A *frame homomorphism* is a lattice homomorphism $h : L \to M$ that preserves the bottom element, top element, binary meets, and arbitrary Joins.

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A *lattice-ordered ring* is a ring $A$ with a lattice structure satisfying

$$a + (b \vee c) = (a + b) \vee (a + c) \quad \text{or, equivalently,} \quad a + (b \wedge c) = (a + b) \wedge (a + c)$$

for all $a, b, c \in A$. Additionally, it holds that $ab \geq 0$ whenever $a \geq 0, b \geq 0$. Immediate consequences include:

$$-(b \vee c) = (-a) \wedge (-b), -(b \wedge c) = (-a) \vee (-b), \text{ and } a \leq b \Longrightarrow -b \leq -a.$$

Defining $|a| = a \vee (-a)$, we get:

$$|a| \geq 0, \qquad |a + b| \leq |a| + |b|, \quad \text{and} \quad |ab| \leq |a||b|.$$

Now, an *$f$-ring* is a lattice-ordered ring $A$ that satisfies $|ab| = |a||b|$ for all $a, b \in A$.
Recall [4] that the frame $\mathcal{L}(\mathbb{R})$ of reals is constructed using ordered pairs $(p, q)$ of rational numbers as generators and the following relations:
(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$
(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$
(R4) $1 = \bigvee\{(p, q) : p, q \in \mathbb{Q}\}$.
For any frame $L$, the ring $\mathcal{R}L$ consists of frame homomorphisms $\mathcal{L}(\mathbb{R}) \to L$. It is a reduced $f$-ring, meaning that it has no nonzero nilpotent element. Refer to [3] and [4] for properties of the ring $\mathcal{R}L$.
For each continuous operation $\diamond : \mathbb{Q}^2 \to \mathbb{Q}$ (including $+, ., \wedge, \vee$), there is a corresponding operation on $\mathcal{R}L$, also denoted by $\diamond$, defined as follows:

$$\alpha \diamond \beta(p, q) = \bigvee\{\alpha(r, s) \wedge \beta(u, w) : \langle r, s\rangle \diamond \langle u, w\rangle \leq \langle p, q\rangle\},$$

where

$$\langle p, q\rangle = \{t \in \mathbb{Q} : p < t < q\} \qquad \text{and} \qquad \langle r, s\rangle \diamond \langle u, w\rangle = \{x \diamond y : x \in \langle r, s\rangle, y \in \langle u, w\rangle\}.$$

For any $\alpha \in \mathcal{R}L$, The map $-\alpha$ is define by $(-\alpha)(p, q) = \alpha(-q, -p)$, where $p, q \in \mathbb{Q}$. The constant frame map $\mathbf{r} \in \mathcal{R}L$, where $r \in \mathbb{R}$, is given by

$$\mathbf{r}(p, q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

The identity element and the zero element of $\mathcal{R}L$ are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.
The cozero map $\mathrm{coz} : \mathcal{R}L \to L$ is given by $\mathrm{coz}(\alpha) = \alpha((-, 0) \vee (0, -))$, where

$$(0, -) = \bigvee\{(0, q)) : 0 < q \in \mathbb{Q}\}, \quad \text{and} \quad (-, 0) = \bigvee\{(p, 0)) : 0 > p \in \mathbb{Q}\}.$$

For properties of the cozero map, refer to [3] and [4].
For any $\alpha \in \mathcal{R}L$, let $R_\alpha := \{r \in \mathbb{R} : \mathrm{coz}(\alpha - \mathbf{r}) = \alpha(-, r) \vee \alpha(r, -) \neq 1\}$ as defined [9], where

$$(-, r) := \bigvee_{\substack{q \in \mathbb{Q} \\ q < r}} (-, q) \qquad \text{and} \qquad (r, -) := \bigvee_{\substack{p \in \mathbb{Q} \\ r < p}} (p.-).$$

Then $R_\alpha$ is the extension to arbitrary $\mathcal{R}L$ of the familiar correspondence between functions on spaces and their images (see [1, Irtroduction]). An element $\alpha$ in the ring $\mathcal{R}L$ has a *countable pointfree image* if $R_\alpha$ is countable. We denote the subring consisting of those functions with countable pointfree images by $\mathcal{R}_c L$. This ring, viewed as the pointfree topology counterpart of $C_c(X)$, is introduced and analyzed in [9], where the authors prove that $\mathcal{R}_c(\mathfrak{O}X) \cong C_c(X)$ for any space $X$. The frame of open subsets of $X$ is denoted by $\mathfrak{O}X$.

## 2. The uniform topology on $\mathcal{R}_cL$

Let us remind the reader that for a set $X$, if each $x \in X$ has an associated collection $\mathcal{A}_x$ of subsets satisfying the following conditions:

(i) If $U \in \mathcal{A}_x$, then $x \in U$.
(ii) If $U_1, U_2 \in \mathcal{A}_x$, then there exists $U_3 \in \mathcal{A}_x$ such that $U_3 \subseteq U_1 \cap U_2$.
(iii) If $U \in \mathcal{A}_x$, there is some $V \in \mathcal{A}_x$ such that $y \in V$, then there is some $W \in \mathcal{A}_y$ with $W \subseteq U$.

Then a topology can be defined on $X$ such that $\mathcal{A}_x$ is a base neighborhood at each point $x \in X$. Additionally, a subset $G \subseteq X$ is open if it contains a basic neighborhood for every point in $G$.
The uniform topology on $\mathcal{R}L$ (briefly, $u$-topology) is defined in [4]. Similarly, we define the uniform topology on $\mathcal{R}_cL$ by assigning basic neighborhoods to points in $\mathcal{R}_cL$, using its structure as an $f$-ring.
Let $\alpha \in \mathcal{R}_cL$. For each $r \in \mathbb{Q}^+$, let

$$B_c(\alpha, r) = \{\varphi \in \mathcal{R}_cL : |\varphi - \alpha| \leq \mathbf{r}\}$$

We verify that this is a valid assignment of basic neighborhoods. Let us first observe that for any $\alpha, \beta \in \mathcal{R}_cL$, we have $|\alpha + \beta| \leq |\alpha| + |\beta|$ (see [4]).

(i) Clearly, $\alpha \in B_c(\alpha, r)$ for any $\alpha \in \mathcal{R}_cL$) and $r \in \mathbb{Q}^+$.
(ii) For any $r, s \in \mathbb{Q}^+$, we have $B_c(\alpha, \frac{r+s}{2}) \subseteq B_c(\alpha, r) \cap B_c(\alpha, s)$.
(iii) Consider $B_c(\alpha, r)$, and let $G = B_c(\alpha, \frac{r}{2})$. Now if $\delta \in G$ and $\varphi \in B_c(\delta, \frac{r}{2})$, then we have

$$|\varphi - \alpha| = |(\varphi - \delta) + (\delta - \alpha)| \leq |\varphi - \delta| + |\delta - \alpha| \leq \frac{\mathbf{r}}{\mathbf{2}} + \frac{\mathbf{r}}{\mathbf{2}} = \mathbf{r}.$$

Thus, $B_c(\delta, \frac{r}{2}) \subseteq G$.

The following proposition is a direct consequence of the preceding discussion:

**Proposition 2.1.** *Let $L$ be a frame. Then the family $\{B_c(\alpha, r) : r \in \mathbb{Q}^+\}$ is a base neighborhood at $\alpha$, for each $\alpha \in \mathcal{R}_cL$.*

Now, similar to the classical case for $C_c(X)$, we present the following definition.

**Definition 2.2.** Let $L$ be a frame. The topology defined in the previous proposition is called the $u_c$-topology on $\mathcal{R}_cL$.

An element $\alpha$ of $\mathcal{R}L$ is bounded if there exist $p \in \mathbb{Q}^+$ such that $\alpha(-p, p) = 1$. The subring of bounded elements in $\mathcal{R}L$ is denoted by $\mathcal{R}^*L$. Similarly, the subring of bounded elements in $\mathcal{R}_cL$ is denoted by $\mathcal{R}_c^*L$. We note that $\mathcal{R}_c^*L = \mathcal{R}_cL \cap \mathcal{R}^*L$.
Following the approach from the discussion before proposition 2.1, We can similarly define the uniform topology on $\mathcal{R}_c^*L$. Let $\psi \in \mathcal{R}_c^*L$. For each $r \in \mathbb{Q}^+$, let

$$B_c^*(\psi, r) = \{\varphi \in \mathcal{R}_c^*L : |\varphi - \psi| \leq \mathbf{r}\}.$$

As in the earlier discussion before Proposition 2.1, it is straightforward to confirm that this assignment of basic neighborhoods is valid. Therefore, there is a unique topology on $\mathcal{R}_c^*L$ for which for any $\psi \in \mathcal{R}_c^*L$, the family $\{B_c^*(\psi, r) : r \in \mathbb{Q}^+\}$ forms a base for the neighborhood system of $\psi$. This topology is called the $u_c^*$-topology on $\mathcal{R}_c^*L$.
Proposition 3.3.1 in [3] states that an element $\alpha$ has inverse in $\mathcal{R}L$ if and only if $\mathrm{coz}(\alpha) = 1$. Additionally, Lemma 8.1 in [9] indicates that if $\alpha$ has inverse in $\mathcal{R}L$ and $\alpha \in \mathcal{R}_cL$, then $\alpha^{-1} \in \mathcal{R}_cL$. The following lemma directly follows from this Proposition and this Lemma.

**Lemma 2.3.** *Let $L$ be a frame. Then an element $\alpha$ has inverse in $\mathcal{R}_cL$ if and only if* $\mathrm{coz}(\alpha) = 1$.

The set of all units $\mathcal{R}_cL$ is denoted by $\mathrm{Inv}(\mathcal{R}_cL)$. Therefore, we have

$$\mathrm{Inv}(\mathcal{R}_cL) = \{\alpha \in \mathcal{R}_cL : \mathrm{coz}(\alpha) = 1\}.$$

We present a lemma analogous to Lemma 3.3 in [2]. Based on the proof of Proposition 3.3.1 in [3], it is easy to verify the equivalence of the first two parts. Additionally, it is straightforward to demonstrate that the second and third parts are also equivalent using the facts below.

1. $\alpha \leq \beta$ if and only if $\alpha(p,-) \leq \beta(p,-)$ for all $p \in \mathbb{Q}$ (see [4, Lemma 4]), and
2. $(\alpha \vee \beta)(p,-) = \alpha(p,-) \vee \beta(r,-)$ for all $p \in \mathbb{Q}$ (see [8, Ch. XIV, 5.3.3]).

**Lemma 2.4.** *Let $L$ be a frame. The following statements are equivalent for an element $\alpha \in \mathcal{R}_c^*L$.*

1. *The element $\alpha$ has inverse in $\mathcal{R}_c^*L$.*
2. *There exists $r \in \mathbb{Q}^+$ such that $\alpha(-,-r) \vee \alpha(r,-) = 1$.*
3. *There exists $r \in \mathbb{Q}^+$ such that $\mathbf{r} \leq |\alpha|$.*

The set of all units $\mathcal{R}_c^*L$ is denoted by $\mathrm{Inv}(\mathcal{R}_c^*L)$. Therefore, we have

$$\mathrm{Inv}(\mathcal{R}_c^*L) = \{\alpha \in \mathcal{R}_c^*L : \alpha(-,-r) \vee \alpha(r,-) = 1 \text{ for some } r \in \mathbb{Q}\}.$$

In the proof of the next theorem, we use the fact that $||x| - |y|| \leq |x - y|$ holds in a totally ordered ring, and, thus in every $f$-ring.

**Theorem 2.5.** *Let $L$ be a frame. Then $\mathrm{Inv}(\mathcal{R}_c^*L)$ is an open subset of $\mathcal{R}_c^*L$ in the $u_c^*$-topology.*

*Proof.* Let $\alpha \in \mathrm{Inv}(\mathcal{R}_c^*L)$. By Lemma 2.4, there exists $r \in \mathbb{Q}^+$ such that $\mathbf{r} \leq |\alpha|$. We will show that $B_c^*(\alpha, \frac{r}{2}) \subseteq \mathrm{Inv}(\mathcal{R}_c^*L)$. For $\beta \in B_c^*(\alpha, \frac{r}{2})$, by the discussion before theorem, it is easy to show that $|\beta| \geq \frac{\mathbf{r}}{\mathbf{2}}$. Applying Lemma 2.4 again, we conclude that $\beta \in \mathrm{Inv}(\mathcal{R}_c^*L)$. Therefor, we have $B_c^*(\alpha, \frac{r}{2}) \subseteq \mathrm{Inv}(\mathcal{R}_c^*L)$, indicating that $\mathrm{Inv}(\mathcal{R}_c^*L)$ is an open subset of $\mathcal{R}_c^*L$ . □

The ring $\mathcal{R}L$ is called a *topological ring* if it is endowed with a topology and the operations $(\alpha,\beta) \mapsto \alpha + \beta$ and $(\alpha,\beta) \mapsto \alpha\beta$ are continuous. Let us remind the reader that an element $\varphi$ of $\mathcal{R}L$ is bounded if and only if there exist $n \in \mathbb{N}$ such that $|\varphi| \leq \mathbf{n}$ (see [4]).

**Theorem 2.6.** *Let $L$ be a frame. Then $\mathcal{R}_c^*L$ is a topological ring under the $u_c^*$-topology.*

*Proof.* We need to demonstrate the continuity of the operations $(\alpha,\beta) \mapsto \alpha + \beta$ and $(\alpha,\beta) \mapsto \alpha\beta$ on $\mathcal{R}_c^*L$. To prove that addition is continuous, let $\alpha$ and $\beta$ be arbitrary elements of $\mathcal{R}_c^*L$, and let $r$ be an arbitrary positive rational number. Consider the neighborhoods $B_c^*(\alpha, \frac{r}{2})$ and $B_c^*(\beta, \frac{r}{2})$. The product $B_c^*(\alpha, \frac{r}{2}) \times B_c^*(\beta, \frac{r}{2})$ forms a neighborhood of $(\alpha,\beta)$ in $\mathcal{R}_c^*L \times \mathcal{R}_c^*L$. For any $(\alpha_1,\beta_1) \in B_c^*(\alpha, \frac{r}{2}) \times B_c^*(\beta, \frac{r}{2})$, it is easy to show that $\alpha_1 + \beta_1 \in B_c^*(\alpha + \beta, r)$, proving that addition is continuous at $(\alpha,\beta)$ and, consequently, throughout $\mathcal{R}_c^*L \times \mathcal{R}_c^*L$.
Next, To show that multiplication is continuous, let $\alpha$, $\beta$, and $r$ be as defined above. Since $\alpha, \beta \in \mathcal{R}_c^*L$, there exist natural numbers $m, n$ such that $|\alpha| \leq \mathbf{n}$ and $|\beta| \leq \mathbf{m}$. Define $p = \frac{r}{2(1+m+n+r)}$, which is positive and satisfies $p \leq r$. Consider the neighborhoods $B_c^*(\alpha, p)$ and $B^*(\beta, p)$. For any $\alpha_1 \in B_c^*(\alpha, p)$ and $\beta_1 \in B_c^*(\beta, p)$, it is straightforward to check that $|\alpha_1\beta_1 - \alpha\beta| \leq \mathbf{r}$. This shows that $\alpha_1\beta_1 \in B_c^*(\alpha\beta, r)$, establishing the continuity of multiplication at $(\alpha,\beta)$ and throughout $\mathcal{R}_c^*L \times \mathcal{R}_c^*L$. Therefore, $\mathcal{R}_c^*L$ is a topological ring. □

The following corollary directly follows from Theorem 2.5 Theorem 2.6.

**Corollary 2.7.** *The following statements hold for a frame $L$.*

1. *The closure of any ideal of $\mathcal{R}_c^*L$ with $u_c^*$-topology is a ideal.*
2. *The closure of any proper ideal of $\mathcal{R}_c^*L$ with $u_c^*$-topology is a proper ideal. Thus, maximal ideals in $\mathcal{R}_c^*L$ are closed.*

If $\mathcal{R}L$ is considered a vector space (with scalar multiplication given as $(r,\alpha) \mapsto \mathbf{r}\alpha$ ), it qualifies as *topological vector space* if equipped with a topology where the operations $(\alpha,\beta) \mapsto \alpha+\beta$ and $(r,\alpha) \mapsto \mathbf{r}\alpha$ are continuous. The following corollary directly follows from the preceding theorem.

**Corollary 2.8.** *For any frame $L$, the ring $\mathcal{R}_c^*L$ with the $u_c^*$-topology is a topological vector space over $\mathbb{Q}$.*

The subring of bounded elements in $C_c(X)$ is denoted by $C_c(X)^*$. A space $X$ is called *countably pseudocompact* (briefly, *$c$-pseudocompact*) if $C_c(X) = C_c^*(X)$. Now, we present the next definition.

**Definition 2.9.** A frame $L$ is called countably pseudocompact (briefly, $c$-pseudocompact) if $\mathcal{R}_cL = \mathcal{R}_c^*L$.

For any space $X$, $C_c(X) \cong \mathcal{R}_c(\mathfrak{O}(X))$ indicates that $X$ is a $c$-pseudocompact space if and only $\mathfrak{O}(X)$ is a $c$-pseudocompact frame.
A frame $L$ is called *pseudocompact* if $\mathcal{R}L = \mathcal{R}L^*$. In [2], the authors characterize pseudocompact frames $L$ through the $u$-topology on the rings $\mathcal{R}L$. Similarly, we will characterize $c$-pseudocompact frames $L$ using the $u_c$-topology on $\mathcal{R}_cL$. We begin by presenting the next lemma. First, note that for any $\alpha$, $\beta$, and $\delta$ of $\mathcal{R}_cL$, we have $-(\alpha\vee\beta) = (-\alpha)\wedge(-\beta)$ and $\alpha + (\beta\wedge\delta) = (\alpha+\beta)\wedge(\alpha+\delta)$ (see [4]).

**Lemma 2.10.** *For any $\alpha \in \mathcal{R}_cL$ and $r \in \mathbb{Q}$, the map $(\alpha - \mathbf{r})\vee\mathbf{0}$ belong to $B_c(\alpha,r)$.*

*Proof.* Since

$$|\alpha - ((\alpha-\mathbf{r})\vee\mathbf{0})| = |\alpha + ((\mathbf{r}-\alpha)\wedge\mathbf{0})| = |(\square + \mathbf{r} - \square)\wedge(\square+\mathbf{0})| = |\mathbf{r}\wedge\square| \le \mathbf{r},$$

we have $((\alpha-\mathbf{r})\vee\mathbf{0}) \in B_c(\alpha,r)$. □

Proposition 11 in [4] state any $\alpha \ge \mathbf{1}$ in $\mathcal{R}L$ has an inverse. Next, if $\alpha$ is an unbounded element in $\mathcal{R}L$, then $|\alpha|$ is also unbounded since $\alpha \le |\alpha|$. Consequently, $\mathbf{1} + |\alpha|$ is an unbounded positive unit in $\mathcal{R}L$.

**Lemma 2.11.** *When $L$ is not a pseudocompact frame, the following statements are true.*

1. $\mathrm{Inv}(\mathcal{R}_cL)$ *is not an open subset of $\mathcal{R}_cL$ with respect to the $u_c$-topology.*
2. *$\mathcal{R}_cL$ is not a topological ring under the $u_c$-topology.*
3. *$\mathcal{R}_cL$ with the $u_c$-topology is not a topological vector space over $\mathbb{Q}$.*

*Proof.* (1). Since $L$ is not $c$-pseudocompact, there exists a positive unit $\alpha$ of $\mathcal{R}_cL$ that is not a unit in $\mathcal{R}_c^*L$. By Lemmas 2.3 and 2.10, for any positive rational $r$, we have $\alpha(-,-r)\vee\alpha(r,-) \neq 1$ and $((\alpha-\mathbf{r})\vee\mathbf{0}) \in B_c(\alpha,r)$. It is apparent that $(\alpha-\mathbf{r})\vee\mathbf{0} \notin \mathrm{Inv}(\mathcal{R}_cL)$. Consequently, for $\alpha \in \mathrm{Inv}(\mathcal{R}_cL)$, the neighborhood $B_c(\alpha,r)$ ) cannot be a subset of $\mathrm{Inv}(\mathcal{R}_cL)$, indicating that $\mathrm{Inv}(\mathcal{R}_cL)$ is not an open subset of $\mathcal{R}_cL$.
(2) Since $L$ is not $c$-pseudocompact, there exists an unbounded element $\alpha$ in $\mathcal{R}_cL$. We will demonstrate that the multiplication in $\mathcal{R}_cL$ is not continuous at the point $(\mathbf{0},\alpha)$. Consider the neighborhood $B_c(\mathbf{0},1) = \{\beta \in \mathcal{R}_cL : |\beta| = |\beta - | \le \mathbf{1}\}$. We need to show that for any neighborhood $B_c(\mathbf{0},r) \times B_c(\alpha,s)$ of $(\mathbf{0},\alpha)$ in $\mathcal{R}_cL \times \mathcal{R}_cL$, it holds that $B_c(\mathbf{0},r).B_c(\alpha,s) \nsubseteq B_c(\mathbf{0},1)$. Take $(\frac{\mathbf{r}}{2},\alpha) \in B_c(\mathbf{0},r) \times B_c(\alpha,s)$. It is easy to prove that $\frac{\mathbf{r}}{2}\alpha \notin B_c(\mathbf{0},1)$. □

The following theorem is derived by combining Theorems 2.5 and 2.6 with the preceding Lemma.

**Theorem 2.12.** *The following statements are equivalent for a frame $L$.*

1. *$L$ is $c$-pseudocompact.*
2. *The set $\mathrm{Inv}(\mathcal{R}_cL)$ is open subset of $\mathcal{R}_cL$ equipped with the $u_c$-topology.*
3. *The ring $\mathcal{R}_cL$ equipped with the $u_c$-topology is a topological ring.*
4. *the ring $\mathcal{R}_cL$ equipped with the $u_c$-topology is a topological vector space over $\mathbb{Q}$.*

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# Modules whose submodules are comparable in coverage

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**Abstract**

A module is said to be *uniserial* if its submodules are linearly ordered by inclusion. Also an $R$-module $M$ is called *weakly uniserial* if for any two submodules $N$, $K$ of $M$, $\mathrm{Hom}_R(N,K)$ or $\mathrm{Hom}_R(K,N)$ contains an injective element. Here, we are interested in modules whose submodules are comparable in coverage. In fact, a module is called *epic-submodules* if for any two its submodules there is an epimorphism from one to the other. In this note, we first give some properties of these modules, then we show that if $R$ is a local right hereditary ring or is a principal right ideal domain, then every projective right $R$-module is epic-submodules. Then we prove that a ring $R$ is an Artinian simple ring if and only if every right $R$-module is epic-submodules.

## 1. Definitions, Examples and some properties of epic-submodules

In this note we use the following concepts;
A ring $R$ is local if $R/\mathrm{rad}(R)$ is a division ring, where $\mathrm{rad}(R)$ denotes the Jacobson radical of $R$. This is equivalent to this ring has a unique maximal left (right) ideal. A ring $R$ is called right hereditary, if any right ideal of $R$ is projective as a right $R$-module. Also a module is called hereditary if any its submodule is projective. A nonzero ring $R$ is said to be simple ring if $(0)$ and $R$ are the only ideals in $R$. This requires that for any nonzero element $a$ in $R$, the ideal generated by $a$ is $R$. A ring $R$ is said to be right (left) Artinian if it satisfies ascending chain condition on right (left) ideals. A semisimple module is called homogeneous semisimple if any two its simple submodules are isomorphic. For any two $R$-modules $M$ and $N$ if there exists an $R$-epimorphism from $M$ to $N$, we write $M \twoheadrightarrow N$, otherwise we write $M \not\twoheadrightarrow N$.

**Example 1.1.** (i) If $M$ is epic-submodules, then every submodule of $M$ is also epic-submodules.
(ii) $\mathbb{Z}$ and $\mathbb{Z}_{p^n}$ are $\mathbb{Z}$-epic-submodules, where $p$ is a prime number.
(iii) Any homogeneous semisimple module is epic-submodules; in particular any vector space is epic-submodules.
(iv) If $R$ is a commutative principal ideal domain, then for any index set $I$, it is well-known that any submodule of $\bigoplus_I R$ is free. So $\bigoplus_I R$ is epic-submodules as an $R$-module.
(v) $\mathbb{Q}$ is not epic-submodules as a $\mathbb{Z}$-module. For see this if we consider two submodules $H_p = \{\frac{a}{p^k} \mid a \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\}\}$ and $H_q = \{\frac{a}{q^k} \mid a \in \mathbb{Z}, k \in \mathbb{N} \cup \{0\}\}$, where $p$ and $q$ are two distinct prime numbers, then we can show that $\mathrm{Hom}_{\mathbb{Z}}(H_p, H_q) = \mathrm{Hom}_{\mathbb{Z}}(H_p, H_q) = 0$.

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**Proposition 1.2.** *A semisimple right $R$-module $M$ is epic-submodules if and only if $M$ is homogeneous semisimple.*

*Proof.* Suppose that $M$ is homogeneous semisimple. Then we may set $M = \bigoplus_I S$, where $S$ is a simple right $R$-module. Now if $N$ and $K$ are two submodules of $M$, then $N \cong \bigoplus_{I_1} S$ and $K \cong \bigoplus_{I_2} S$, for some two index subsets $I_1$ and $I_2$ of $I$. Without loss of generality we assume that $|I_1| \geq |I_2|$. Then there exists an onto function $I_1 \twoheadrightarrow I_2$ and this implies that $N \twoheadrightarrow K$. Therefore $M$ is epic-submodules. Conversely, suppose that $M = \bigoplus_{i\in I} S_i$ is a semisimple epic-submodules. Then for any $i, j \in I$; $S_i \twoheadrightarrow S_j$ or $S_j \twoheadrightarrow S_i$. In any case $S_i \cong S_j$ and hence $M$ is homogeneous semisimple. □

**Proposition 1.3.** (i) *A projective module over a principal right ideal domain is epic-submodules.*
(ii) *A projective module over a local right hereditary ring is epic-submodules.*

An $R$-module $M$ is called *weakly prime* (*semiprime*) if for any nonzero submodule $N$ of $M$, $\mathrm{Ann}_R(N)$ is a prime (semiprime) ideal of $R$.

**Proposition 1.4.** *Let $M$ be a semiprime right $R$-module. If $M$ is epic-submodules, then $M$ is weakly prime.*

*Proof.* Suppose that $N$ is a nonzero submodule of $M$. We prove that $\mathrm{Ann}_R(N)$ is prime. Let $IJ \subseteq \mathrm{Ann}_R(N)$ such that $I \nsubseteq \mathrm{Ann}_R(N)$ and $J \nsubseteq \mathrm{Ann}_R(N)$, where $I$ and $J$ are ideals of $R$. Then;

$$\begin{aligned} N(JI)^2 = NJIJI \subseteq NIJI = 0 &\Rightarrow (JI)^2 \subseteq \mathrm{Ann}_R(N) \\ &\Rightarrow JI \subseteq \mathrm{Ann}_R(N) \\ &\Rightarrow NJI = 0. \end{aligned}$$

Now since $M$ is epic-submodules, $NI \twoheadrightarrow NJ$ or $NJ \twoheadrightarrow NI$. If $NI \overset{f}{\twoheadrightarrow} NJ$, then;

$$\begin{aligned} NJ = f(NI) &\Rightarrow NJ^2 = f(NI)J = f(NIJ) = f(0) = 0 \\ &\Rightarrow J^2 \subseteq \mathrm{Ann}_R(N). \end{aligned}$$

Since $\mathrm{Ann}_R(N)$ is semiprime, $J \subseteq \mathrm{Ann}_R(N)$. Similarly if $NJ \twoheadrightarrow NI$, we have $I \subseteq \mathrm{Ann}(N)$. □

**Proposition 1.5.** *Let $R$ be a ring such that all 2-generated right $R$-modules are epic-submodules. Then*
(i) *All simple right $R$-modules are isomorphic;*
(ii) *All ideals of $R$ are comparable.*

If $X$ is a submodule of a right $R$-module $M$, then the ideal $\{r \in R \mid Xr = 0\}$ is denoted by $\mathrm{Ann}_R(X)$. An $R$-module $M$ is called *coprime* if $M \neq 0$ and for any proper submodule $N$ of $M$, $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(\frac{M}{N})$. Also an ideal $P$ of $R$ is called an *attached prime* of $M$ if there exists a submodule $N$ of $M$ such that $N$ is coprime and $P = \mathrm{Ann}_R(N)$. The set of all attached primes of $M$ is denoted by $\mathrm{Att}(M)$ (see [3]).

**Lemma 1.6.** *Let $M$ be a right $R$-module and $N \subseteq M$ be a coprime as an $R$-module. If $M$ is epic-submodules, then* $\mathrm{Ann_R(N)}$ *is a maximal element of the set* $\{\mathrm{Ann}(K) \mid 0 \neq K \subseteq M\}$.

*Proof.* Suppose that $\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(K)$, where $0 \neq K \subseteq M$. Since $M$ is epic-submodules, $N \twoheadrightarrow K$ or $K \twoheadrightarrow N$. If $N \overset{f}{\twoheadrightarrow} K$, then $\mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(K)$. On the other hand $\frac{N}{\ker f} \cong K$ and since $N$ is coprime, $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(K)$. If $K \twoheadrightarrow N$, then $\mathrm{Ann}_R(K) \subseteq \mathrm{Ann}_R(N)$ and by hypothesis, $\mathrm{Ann}_R(K) = \mathrm{Ann}_R(N)$. □

**Proposition 1.7.** *If a right $R$-module $M$ is epic-submodules, then* $|\mathrm{Att}(M)| \leq 1$. *In addition if $R$ satisfies in the ascending chain condition on two-sided ideals, then* $|\mathrm{Att}(M)| = 1$.

The converse of Proposition 1.7 is not true in general. For example $\mathbb{Q}$ is not epic-submodules, but $\mathrm{Att}(\mathbb{Q}) = \{(0)\}$.

Recall that a ring $R$ is *normal* if any idempotent of $R$ is central. Also a module is called *hereditary* if any its submodule is projective.

**Proposition 1.8.** (i) *Any hereditary module over a local ring is epic-submodules.*
(ii) *Any normal von Neumann regular ring that is epic-submodules as a right (left) module over itself is a division ring.*

**Proposition 1.9.** *Let $R$ be a commutative epic-submodules ring and $I_n = \{a \in R \mid a^n = 0\}$, for any $n \in \mathbb{N}$. Then*
(i) *$I_n$ is an ideal of $R$ and $(I_n)^n = 0$, for any $n \in \mathbb{N}$.*
(ii) *$\frac{R}{I}$ is a commutative epic-submodules ring where $I \in \{\mathrm{Nil}(R), I_1, I_2, ...\}$.*

**Theorem 1.10.** *Let $R$ be a simple ring. The following statements are equivalent:*
$(a)$ *$R$ is left Artinian.*
$(b)$ *$R$ is (left) semisimple.*
$(c)$ *$R$ has a minimal left ideal.*
$(d)$ *$R \cong M_n(D)$, for some natural number $n$ and some division ring $D$.*

*Proof.* [4, Theorem 3.10]. □

**Theorem 1.11.** *A ring $R$ is an Artinian simple ring if and only if every right $R$-module is epic-submodules.*

In this note we have used [1, 2, 5, 6].

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# Intersection large exact sequences in the category of $S$-acts

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**Abstract**

In this paper, we investigate the concept of $\cap$-exact sequences within $S$-acts and $S$-homomorphisms, focusing on their structural properties and applications. By introducing the notions of intersection large subacts and $\cap$-epic homomorphisms, we establish fundamental definitions and provide illustrative examples. Additionally, we examine the preservation of $\cap$-exactness under localization and study the interplay between homomorphisms and intersection large subacts through various propositions and theorems.

## 1. Introduction

The study of $S$-acts, which are algebraic structures defined over monoids, has garnered significant attention in recent decades due to their wide-ranging applications in category theory, semigroup theory, and universal algebra. An $S$-act is essentially a set equipped with a right action of a monoid $S$, providing a versatile framework for studying morphisms and substructures. See [4].
In particular, the investigation of exact sequences in $S$-acts has been a central theme in understanding the relationships between various subacts and homomorphisms. See, [1], [2], and [3].
Building upon earlier works that introduced notions like kernels, images, and exactness in modules and monoids, researchers have developed specialized concepts, such as intersection large subacts and $\cap$-exact sequences, to extend these ideas to the realm of $S$-acts. This paper aims to contribute to this growing body of research by analyzing $\cap$-exact sequences in depth, presenting new results that generalize previous findings, and examining their properties under localization and homomorphism mappings. The following section delves into the main definitions and results, offering a clearer understanding of these algebraic constructs. Let $S$ be a monoid with zero. A non-empty set $A$ is called right $S$-act if there exists an action $A \times S \to A$, $(a, s) \mapsto as$ such that $a(st) = (as)t$ for any $a \in A$ and $s, t \in S$, and $a.1 = a$ for all $a \in A$. A subact $B$ of $S$-act $A$ is a subset $B$ of $A$ such that for any $b \in B$ and $s \in S$, $bs \in B$. A subset $I$ of monoid $S$ is right ideal of $S$ if it is subact of $S$, as right $S$-act. An element $\theta \in A$ is called fixed element of $A$ if for all $s \in S$, $\theta s = \theta$. In this study, all $S$-acts are right $S$-act and have a unique fixed element. Recall that a subset $B$ of a right $S$-act $A$ is called intersection large if $B \cap C \neq \theta_A$, for each non-zero subact $C$ of $A$.
Clearly, if $B$ is intersection large subact of $A$, for any $\theta_A \neq a \in A$, there exists $0_S \neq s \in S$ such that $\theta_A \neq as \in B$.

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Let $A$ and $B$ be two right $S$-acts. A function $f : A \to B$ is called $S$-homomorphism if for any $a \in A$ and $s \in S$, $f(as) = f(a)s$.
Let $T$ be a multiplicatively closed subset of a monoid $S$, i.e $1 \in T$ and $ab \in T$ for any $a, b \in S$. Define $T^{-1}S$ to be a monoid with elements $\frac{s}{t}$, $s \in S$, $t \in T$, where $\frac{s}{t} = \frac{s'}{t'}$ if there is an element $u \in T$ such that $u(st') = u(s't)$. The multiplication in $T^{-1}S$ is induced by $s$, $(\frac{s}{t})(\frac{s'}{t'}) = \frac{ss'}{tt'}$. The monoid $T^{-1}S$ is called the localization of $S$ at $T$.
Let $A$ be an $S$-act. Then $T^{-1}A$, is the localization of $A$ at $T$ is the localization of $A$ at $T$ is the $T^{-1}S$-act with elements $\frac{a}{t}$, $a \in A$, $t \in T$, where $\frac{a}{t} = \frac{a'}{t'}$ when $u(t'a) = u(ta')$ for some $u \in T$. The action of $T^{-1}S$ on $T^{-1}A$ is $(\frac{a}{t})(\frac{s}{t'}) = \frac{as}{tt'}$.

## 2. Main Result

In this section, the study provides a comprehensive analysis of ∩-exact sequences in the context of $S$-acts. Through detailed definitions and propositions, we establish the foundational properties of intersection large subacts and ∩-epic homomorphisms. These results demonstrate how ∩-exactness is preserved under localization and how it relates to homomorphism mappings.

**Definition 2.1.** A sequence of $S$-acts and $S$-homomorphisms

$$\cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots,$$

is said to be intersection large exact, briefly ∩-exact at $A_i$, if $Im\ f_{i-1}$ is intersection large subact of $\ker f_i$. It is called ∩-exact if it is ∩-exact at each $A_i$.

In particular, a sequence of $S$-acts and $S$-homomorphisms $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$ is a short ∩-exact sequence if and only if $\ker f_1 = 0$, $Im\ f_1$ is intersection large subact of $\ker f_2$, and $Im\ f_2$ is intersection large subact of $A_3$.

**Definition 2.2.** An $S$-homomorphism $f : A \to B$ is called ∩-epic if $Im\ f$ is ∩-large subact of $B$.

**Example 2.3.** The following sequence is ∩-exact sequence

$$0 \to 4\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

We recall that the set $E(S)$ is the set of idempotent elements of monoid $S$.

**Lemma 2.4.** *Let $S$ be a commutative monoid and $e \in E(S)$. If $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$ is ∩-exact sequence, then $0 \to A_1e \xrightarrow{\bar{f}} A_2e \xrightarrow{\bar{g}} A_3e \to 0$ such that $\bar{f} = f|_{A_1e}$, $\bar{g} = g|_{A_2e}$, is ∩-exact sequence.*

*Proof.* We show that the sequence $0 \to A_1e \xrightarrow{\bar{f}} A_2e \xrightarrow{\bar{g}} A_3e \to 0$ is ∩-exact. Clearly, $\ker \bar{f} = 0$. We show that $Im\bar{f}$ is an intersection large subact of $\ker \bar{g}$. Let $\theta \neq a_2e \in \ker \bar{g}$, for arbitrary element $a_2 \in A_2$. Obviously $\theta \neq a_2e \in \ker \bar{g}$. Now, since $Imf$ is an intersection large subact of $\ker g$, we have $Im\ f \cap a_2eS \neq \theta$. So, there exists, $a_2' \in A_2$ and $t \in S$ such that $f(a_2') = a_2et$.
Thus $f(a_2'e) = a_2ete = a_2eet = a_2et$. Therefore, $Im\bar{f} \cap a_2eS \neq \theta$. Now, we show that $Im\bar{g}$ is an intersection large subact of $A_3e$. For this, consider non-zero element $a_3e \in A_3e$. Since $Im\ g \cap a_3$, there exists $a_2 \in A_2$ and $t \in S$ such that $g(a_2) = a_3et$. So, $\bar{g}(a_2e) = a_3ete = a_3et$ and the proof is complete. □

**Proposition 2.5.** *Consider the following commutative digram of S-acts and S-homomorphisms*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{f} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow[f']{} & B_2 & \xrightarrow[g']{} & B_3 & \longrightarrow & 0 \end{array}$$

*where $\alpha$, $\beta$ and $\gamma$ are isomorphism.*
*Then the above sequence is $\cap$-exact if and only if the below sequence is $\cap$-exact.*

**Proposition 2.6.** *Consider the following commutative digram of S-acts and S-homomorphisms with $\cap$-exact sequence in the rows.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & A_2 & \xrightarrow{f} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow[f']{} & B_2 & \xrightarrow[g']{} & B_3 & \longrightarrow & 0 \end{array}$$

*Then the sequence*

$$0 \to \ker\alpha \xrightarrow{k} \ker\beta \xrightarrow{h} \ker\gamma,$$

*which $k(x) = f(x)$, for any $x \in \ker\alpha$, and $h(y) = g(y)$, for any $y \in \ker\beta$ is also $\cap$-exact sequence.*

*Proof.* We show that $\ker k = 0$ . For this, let $x \in \ker k$. So, we have $f(x) = k(x) = 0$. Then $x = 0$, since $\ker f = 0$. Now, we show that $Im\ k$ is an intersection large subact of $\ker h$. Consider non-zero element $y \in \ker h$. So, we have $h(y) = g(y) = 0$. Thus, $y \in \ker g$ and so there exists element $x \in A_1$ and $s \in S$, such that $f(x) = ys$. Therefore $k(x) = f(x) = ys$ and we have $Imk \cap \langle y\rangle \neq \theta$. It is sufficient to show that $x \in \ker\alpha$. We have $f'\alpha(x) = \beta f(x) = \beta(ys) = \theta$. So, $\alpha(x) \in \ker f' = \theta$ and we can conclude $x \in \ker\alpha$. □

**Proposition 2.7.** *Consider the $\cap$-exact sequence $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to 0$.*
*Then the sequence*

$$0 \to T^{-1}A_1 \xrightarrow{T^{-1}f_1} T^{-1}A_2 \xrightarrow{T^{-1}f_2} T^{-1}A_3 \to 0,$$

*is also $\cap$-exact sequence.*

**Theorem 2.8.** *The sequence of S-acts and S-homomorphisms*

$$0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

*is $\cap$-exact if and only if for any S-act B, the sequence*

$$0 \to Hom(B, A_1) \xrightarrow{\bar{f_1}} Hom(B, A_2) \xrightarrow{\bar{f_2}} Hom(B, A_3)$$

*is $\cap$-exact for any S-act B.*

**Theorem 2.9.** *Let S be cancellable monoid. Consider the $\cap$-exact sequence $0 \to A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$.*
*Then for any torsion free S-act B, the sequence*

$$0 \to Hom(A_3, B) \xrightarrow{\bar{f_2}} Hom(A_2, B) \xrightarrow{\bar{f_1}} Hom(A_1, B)$$

*is $\cap$-exact.*

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# The dual of weakly uniserial modules

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**Abstract**

An $R$-module $M$ is said to be weakly uniserial if its submodules are comparable regarding embedding, i.e., for every pair $N, K$ of submodules of $M$, $N \hookrightarrow K$ or $K \hookrightarrow N$, where $N \hookrightarrow K$ means that $N$ is embedded in $K$. Here we introduce and study the dual of this concept. In fact an $R$-module $M$ is said to be co-weakly uniserial if for every pair $N, K$ of submodules of $M$, $M/N \twoheadrightarrow M/K$ or $M/K \twoheadrightarrow M/N$, where $M/N \twoheadrightarrow M/K$ means that an epimorphism from $M/N$ to $M/K$. In this note, we first give some properties of these modules, then we show that we show that a ring $R$ is homogeneous semisimple if and only if every (projective) right $R$-module is co-weakly uniserial. Also if $R$ is a semi-Artinian ring $R$, then $R$ is simple Artinian if and only if every 2-generated right $R$-module is co-weakly uniserial.

## 1. Examples and properties of co-weakly uniserial modules

In this note $R$ is a ring with nonzero identity element and $M$ is a unitary right $R$-module. We use the notation $N \leq M$, when $N$ is a submodule of $M$. Also $N \leq M$ is called essential, denoted by $N \leq_e M$, if for any nonzero submodule $K$ of $M$, $K \cap N \neq 0$. The right singular ideal and the Jacobson radical of $R$ are denoted by $Z(R_R)$ and $J(R)$, respectively. We usually write $E(M), Z(M)$ and $Soc(M)$ for the injective hull, the singular submodule and the socle of $M$ respectively. Also, if $X$ is a subset of $M$, the annihilator of $X$ in $R$ is the right ideal $Ann_r(X) = \{r \in R | Xr = 0\}$. In this note we have used [1–7].

**Example 1.1.** (1) Any homogeneous semisimple module is co-weakly uniserial. In particular, any vector space is a co-weakly uniserial.
(2) It is easy to see that any uniserial module is co-weakly uniserial. In particular, $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^\infty}$ are co-weakly uniserial $\mathbb{Z}$-modules where $p$ is a prime number and $n \geq 0$ is an integer number.
(3) In a right co-weakly uniserial ring, every two ideals are comparable. Because if $A$ and $B$ are two ideals in a right co-weakly uniserial ring $R$, then we may assume $R / A \twoheadrightarrow R / B$ and so $A = ann_r(R / A) \subseteq ann_r(R / B) = B$.
(4) A commutative ring is co-weakly uniserial if and only if it is uniserial.
(5) $\mathbb{Z}_n$ is a co-weakly uniserial $\mathbb{Z}$-module if and only if $n$ is a power of a prime.
(6) $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is a co-weakly uniserial $\mathbb{Z}$-module if and only if either $m = n$ is a prime number or $m = n = 1$.
(7) $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ is a co-weakly uniserial $\mathbb{Z}$-module if and only if either $n_1 = \cdots = n_k$ is a prime number or

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$n_1 = \cdots = n_k = 1$.
(8) If $D$ is a division ring and $n \geq 2$, then the ring $M_n(D)$ is co-weakly uniserial while neither left nor right uniserial.

A ring $R$ is called simple Artinian if $R \cong M_n(D)$, for some division ring $D$. Also the direct sum of all minimal right ideals of $R$ is denoted by $Soc(R_R)$

**Proposition 1.2.** *Let $R$ be a right co-weakly uniserial ring. Then $J(R)$ is a maximal ideal and $R$ is simple Artinian or $Soc^2(R_R) = 0$.*

*Proof.* Since $R$ is a right co-weakly uniserial ring, for any two maximal right ideals $m_1$ and $m_2$ of $R$, we have $ann_r(R / m_1) = ann_r(R / m_2)$. Thus $J(R) = ann_r(R / m)$, where $m$ is a maximal right ideal of $R$. Now if $I$ is a proper ideal of $R$, then $I \subseteq m$ for some maximal right ideal $m$ of $R$. Thus $I = ann_r(R / I) \subseteq ann_r(R / m)$ and hence $I \subseteq J(R)$. Also, if $R \neq Soc(R_R)$, then $Soc(R_R) \subseteq J(R)$ and so $Soc^2(R_R) = 0$. □

**Proposition 1.3.** *Let $R$ be a commutative principal ideal domain and $Q$ be its quotient field. Then $R$ is local if and only if $Q$ is co-weakly uniserial as an $R$-module.*

**Corollary 1.4.** *The field of rational numbers is not co-weakly uniserial as a $\mathbb{Z}$-module.*

**Proposition 1.5.** *Being co-weakly uniserial is a Morita invariant property.*

Let $M$ be a nonzero right $R$-module. Recall that $M$ is said to be coprime (or second) if $ann_r(M) = ann_r(M / N)$, for any proper submodule $N$ of $M$. A submodule $N$ of $M$ is called coprime if $M / N$ is a coprime $R$-module. Also $M$ is called weakly coprime (or weakly second) if for any $N \subsetneq M$, $ann_r(M / N)$ is a prime ideal of $R$. Moreover, $M$ is called semicoprime if for any $N \subsetneq M$, $ann_r(M / N)$ is a semiprime ideal of $R$. An ideal $P$ of $R$ is an attached prime of $M$ if $P = ann_r(M / N)$, for some coprime submodule $N$ of $M$. The set of attached primes of $M$ is denoted by $Att(M)$ (for more details see [1,5,8]).

**Proposition 1.6.** *(1) A co-weakly uniserial right $R$-module is semicoprime if and only if it is weakly coprime.*
*(2) A right co-weakly uniserial ring $R$ is semiprime if and only if it is prime.*

*Proof.* (1). Assume that $M$ is a semicoprime co-weakly uniserial right $R$-module. Let $N \subseteq M$ and $IJ \subseteq ann_r(M/N))$, where $I$ and $J$ are ideals of $R$. Then $MIJ \subseteq N$ and so $(JI)^2 = JIJI \subseteq IJ \subseteq ann_r(M / N)$. Since $M$ is semicoprime, $JI \subseteq ann_r(M / N)$. On the other hand since $M$ is co-weakly uniserial, $M / MI \twoheadrightarrow M / MJ$ or $M / MJ \twoheadrightarrow M / MI$. If $M / MI \twoheadrightarrow M / MJ$, then $ann_r(M / M) \subseteq ann_r(M / MJ)$ and hence $MI \subseteq MJ$. Thus $MI^2 \subseteq MJI \subseteq N$ and so $I \subseteq ann_r(M / N)$, because $M$ is semicoprime. Similarly, if $M / MJ \twoheadrightarrow M / MI$, then $J \subseteq ann_r(M / N)$. The converse is clear.
(2). Suppose that $R$ is a semiprime right co-weakly uniserial ring and $IJ = 0$, where $I$ and $J$ are two ideals of $R$. By Example 1.1(3), $I \subseteq J$ or $J \subseteq I$. Thus $I^2 \subseteq IJ = 0$ or $J^2 \subseteq IJ = 0$. Since $R$ is semiprime, $I = 0$ or $J = 0$, as desired. The converse is clear. □

**Corollary 1.7.** *For any semiprime right co-weakly uniserial ring $R$, $Soc(R_R) = 0$ or $J(R) = Z(R_R) = 0$.*

**Lemma 1.8.** *Let $R$ be a ring such that every 2-generated right $R$-module is co-weakly uniserial. Then:*
(1) *all simple right $R$-modules are isomorphic.*
(2) *every two ideals of $R$ are comparable.*
(3) *$R$ has only one maximal ideal.*

*Proof.* Proof. (1). It is clear because $S_1 \oplus S_2$ is co-weakly uniserial for any two simple right $R$-modules $S_1$ and $S_2$.
(2). Since $R / I \oplus R / J$ is co-weakly uniserial, for any two ideals $I$ and $J$, we may assume that $R / I \twoheadrightarrow R / J$ and hence $I \subseteq J$.
(3). Follows from (2). □

**Theorem 1.9.** *For any ring R, the following statements are equivalent:*
*(1)* $R \cong M_n(D)$*, where D is a division ring.*
*(2) Every right R-module is co-weakly uniserial.*
*(3) Every projective right R-module is co-weakly uniserial.*
*(4) The left-right symmetric of (b) and (c).*

A ring $R$ is right semi-Artinian if every right $R$-module contains a simple submodule. Also $R$ is called a right V-ring, if every simple right $R$-module is injective.

**Theorem 1.10.** *Let R be a right semi-Artinian ring. If every 2-generated right R-module is co-weakly uniserial, then* $R \cong M_n(D)$ *for some division ring D.*

*Proof.* Let $m_1, m_2$ be two right maximal ideals in $R$. Then $R/m_1 \oplus R/m_2$ is co-weakly uniserial and so $R/m_1 \twoheadrightarrow R/m_2$ or $R/m_2 \twoheadrightarrow R/m_1$. In any case we conclude that $R/m_1 \cong R/m_2$. Thus $R$ has only one simple right $R$-module (up to isomorphism), say $S$. Now we show that $S$ in injective. Suppose that $S \subsetneq E(S)$, where $E(S)$ is the injective hull of $S$ and $x \in E(S) - S$. Since $S \oplus xR$ is 2-generated right $R$-module, $S \twoheadrightarrow xR$ or $xR \twoheadrightarrow S$. If $S \twoheadrightarrow xR$, we conclude that $S \cong xR$ and so $S = xR$, a contradiction. Thus $xR \twoheadrightarrow S$. Also if $m$ is a maximal submodule of $xR$, then $S \oplus xR/m \twoheadrightarrow xR$ or $xR \twoheadrightarrow S \oplus xR/m$. But $xr/m \cong S$ and since $xR$ is uniform we have $xR \twoheadrightarrow S \oplus S$. By continuing this way, we get $\underbrace{S \oplus S \oplus \ldots \oplus S}_{n-times}$ is cyclic, for any $n$, a contradiction. Thus $S = E(S)$ and so $S$ is injective. Therefore $R$ is a V-ring and so $Soc(R_R)^2 = Soc(R_R)$. Since $R$ is semi-Artinian, $Soc(R_R) \neq 0$ and hence by Proposition 1.2 $R \cong M_n(D)$ for some division ring $D$. □

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# Stanley's conjectuere on the independence complexes of graphs

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**Article Info**

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**Abstract**

Let $G$ be a simple graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G)$. In this paper, we prove that the graph $G$ is vertex decomposable if it does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph. As a consequence, we show that Stanley's conjecture holds for the Stanley-Reisner ring of independence complex of graph $G$.

## 1. Introduction

Let $R = K[x_1, \dots, x_n]$, where $K$ is a field. Let $G$ be a simple graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G)$. We associate to $G$ a quadratic squarefree monomial

$$I(G) = (x_i x_j : ij \in E(G)) \subset R,$$

which is called the edge ideal of $G$. An independent vertex set of a graph $G$ is a set of vertices of the graph in which no two vertices are adjacent. The independence complex of $G$, denoted by $\Delta(G)$, is the set of independent sets of $G$. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let $R$ be a $\mathbb{N}^n$- graded ring and $M$ a $\mathbb{Z}^n$- graded $R$- module. Then Stanley [8] conjectured that

$$(M) \leq (M)$$

He also conjectured in [9] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [4] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. This paper is organized as follows. In next Section we recall several definitions and terminology which we need later. In Section 3, we show that graph $G$ is vertex decomposable if it does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph. Also, as a consequence, it is shown that Stanley's conjecture holds for $K[\Delta(G)]$.

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## 2. Preliminaries

In this section we recall some definitions and results which will be needed later.

**Definition 2.1.** A *simplicial complex* $\Delta$ over a set of vertices $V = \{x_1, \dots, x_n\}$, is a collection of subsets of $V$, with the property that:

(a) $\{x_i\} \in \Delta$, for all $i$;

(b) if $F \in \Delta$, then all subsets of $F$ are also in $\Delta$ (including the empty set).

An element of $\Delta$ is called a *face* of $\Delta$ and complement of a face $F$ is $V \setminus F$ and it is denoted by $F^c$. Also, the complement of the simplicial complex $\Delta = \langle F_1, \dots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \dots, F_r^c \rangle$. The *dimension* of a face $F$ of $\Delta$, $\dim F$, is $|F| - 1$ where, $|F|$ is the number of elements of $F$ and $\dim \emptyset = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of $\Delta$ is a subset $F$ of $V$ with $F \notin \Delta$. we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of $\Delta$. The maximal faces of $\Delta$ under inclusion are called *facets* of $\Delta$. The *dimension* of the simplicial complex $\Delta$, $\dim \Delta$, is the maximum of dimensions of its facets. If all facets of $\Delta$ have the same dimension, then $\Delta$ is called *pure*.
Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ be the facet set of $\Delta$. It is clear that $\mathcal{F}(\Delta)$ determines $\Delta$ completely and we write $\Delta = \langle F_1, \dots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex $\Gamma$ is called a *subcomplex* of $\Delta$, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.
For $v \in V$, the subcomplex of $\Delta$ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \langle F \in \Delta : \quad v \notin F \rangle.$$

The *link* of a face $F \in \Delta$, denoted by $\mathrm{link}_\Delta(F)$, is a simplicial complex on $V$ with the faces, $G \in \Delta$ such that, $G \cap F = \emptyset$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\mathrm{link}_\Delta(v)$.

$$\mathrm{link}_\Delta(v) = \{F \in \Delta : \quad v \notin F, \quad F \cup \{v\} \in \Delta\}.$$

**Definition 2.2.** Let $\Delta$ be a simplicial complex over $n$ vertices $\{x_1, \dots, x_n\}$. For $F \subset \{x_1, \dots, x_n\}$, we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of $\Delta$, denoted by $I(\Delta)$, to be the ideal of $S$ generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of $\Delta$, denoted by $I_\Delta$, is the ideal of $S$ generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of $\Delta$.

**Definition 2.3.** A simplicial complex $\Delta$ is recursively defined to be *vertex decomposable*, if it is either a simplex, or else has some vertex $v$ so that,

(a) Both $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable, and

(b) No face of $\mathrm{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.

A vertex $v$ which satisfies in condition (b) is called a *shedding vertex*.

**Definition 2.4.** A simplicial complex $\Delta$ is *shellable*, if the facets of $\Delta$ can be ordered $F_1, \dots, F_s$ such that, for all $1 \le i < j \le s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \dots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

**Definition 2.5.** A graph $G$ is called vertex decomposable, if the independence complex $\Delta(G)$ is vertex decomposable.

**Definition 2.6.** A graded $S$-module $M$ is called *sequentially Cohen-Macaulay* (over $K$), if there exists a finite filtration of graded $S$-modules,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each $M_i/M_{i-1}$ is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A simplicial complex $\Delta$ is called (sequentially) Cohen-Macaulay over $K$, if the ring $K[\Delta] = S/I_\Delta$ is (sequentially) Cohen-Macaulay. A simplicial complex $\Delta$ is called disconnected, if the vertex set $V$ of $\Delta$ is a disjoint union $V = V_1 \cup V_2$ such that no face of $\Delta$ has vertices in both $V_1$ and $V_2$. Otherwise $\Delta$ is connected.

**Definition 2.7.** A graph $G$ is chordal if every cycle of length strictly greater than three has a chord.

A chord of a cycle is an edge joining two nonconsecutive vertices of the cycle.

**Definition 2.8.** Suppose that $V_1$ be a subset of the vertex set $V$ of a graph $G$. Then the subgraph of $G$ whose vertex set is $V_1$ and whose edge set is the set of edges of $G$ that have both end vertices in $V_1$ is denoted by $G[V]$ called a vertex induced subgraph (induced subgraph) of $G$.

## 3. Vertex decomposability of the independence complexes of graphs

As the main result of this section, we show that graph $G$ is vertex decomposable if it does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph. Also, as a consequence, it is shown that Stanley's conjecture holds for $K[\Delta(G)]$. Dochtermann and Engrstrom in[2] showed that if $G$ is a chordal graph then the independence complex $\Delta(G)$ is vertex decomposable. Now, we are ready that prove one of the main result of this paper.

**Theorem 3.1.** *Let $G$ be a simple graph. If the graph $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph then $G$ is vertex decomposable.*

*Proof.* We first prove that $G$ is chordal. Suppose $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph. Because any $C_n$, $n > 5$ contains a $2K_2$ as an induced subgraph, $G$ must not contain any induced cycles larger than $C_3$. Therefore, $G$ is chordal. So by [2] independence complex $\Delta(G)$ is vertex decomposable. This implies that $G$ is vertex decomposable □

Let $R$ be any standard graded $K$- algebra over an infinite field $K$, *i.e*, $R$ is a finitely generated graded algebra $R = \oplus_{i\geq 0} R_i$ such that $R_0 = K$ and $R$ is generated by $R_1$. There are several characterizations of the depth of such an algebra. We use the one that $(R)$ is the maximal length of a regular $R$- sequence consisting of linear forms. Let $x_F = \sqcap_{i\in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \dots, x_n\}$. The $K$- subspace $x_F K[Z]$ of $S = K[x_1, \dots, x_n]$ is the subspace generated by monomials $x_F u$, where $u$ is a monomial in the polynomial ring $K[Z]$. It is called a squarefree Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let $\Delta$ be a simplicial complex on $\{x_1, \dots, x_n\}$. A squarefree Stanley decomposition $\mathcal{D}$ of $K[\Delta]$ is a finite direct sum $\oplus_i u_i K[Z_i]$ of squarefree Stanley spaces which is isomorphic as a $\mathbb{Z}^n$- graded $K$- vector space to $K[\Delta]$, *i.e*.

$$K[\Delta] \cong \bigoplus_i u_i K[Z_i].$$

We denote by $(\mathcal{D})$ the minimal dimension of a Stanley space in $\mathcal{D}$ and we define $(K[\Delta]) = \max\{(\mathcal{D})\}$, where $\mathcal{D}$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in [8] the upper bound for the depth of $K[\Delta]$ as the following:

$$(K[\Delta]) \leq (K[\Delta]).$$

Also we recall another conjecture of Stanley. Let $\Delta$ be again a simplicial complex on $\{x_1, \dots, x_n\}$ with facets $G_1, \dots, G_t$. The complex $\Delta$ is called partitionable if there exists a partition $\Delta = \cup_{i=1}^{t} [F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of $\Delta$. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [9] and [10] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [4] proved that for Cohen-Macaulay simplicial complex $\Delta$ on $\{x_1, \dots, x_n\}$ we have that $(K[\Delta]) \leq (K[\Delta])$ if and only if $\Delta$ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results we obtain :

**Corollary 3.2.** *Let $G$ be a simple graph. If the graph $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph then $\Delta(G)$ is partitionable and Stanley's conjecture holds for $K[\Delta(G)]$.*

*Proof.* Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. By theorem 3.1 proof is completed. □

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# A class of vertex decomposable flag simplicial complexes

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**Abstract**

Let $G$ be a simple graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G)$. In this paper, we prove that the clique complex $CL(G)$ is vertex decomposable if the graph $G$ does not contain a $2K_2$, $C_4$, or $C_5$ as an induced subgraph. As a consequence, we show that Stanley's conjecture holds for $K[CL(G)]$. Finally, we introduce a class of vertex decomposable flag simplicial complexes.

## 1. Introduction

Let $R = K[x_1, \dots, x_n]$, where $K$ is a field. Let $G$ be a simple graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G)$. We associate to $G$ a quadratic squarefree monomial

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which is called the edge ideal of $G$. An independent vertex set of a graph $G$ is a set of vertices of the graph in which no two vertices are adjacent. The independence complex of $G$, denoted by $\Delta(G)$, is the set of independent sets of $G$. The clique complex $CL(G)$ is the simplicial complex whose faces are complete subgraphs of $G$. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let $R$ be a $\mathbb{N}^n$- graded ring and $M$ a $\mathbb{Z}^n$- graded $R$- module. Then Stanley [8] conjectured that

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He also conjectured in [9] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [4] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. This paper is organized as follows. In next Section we recall several definitions and terminology which we need later. In Section 3, we show that the clique complex $CL(G)$ is vertex decomposable if the graph $G$ does not contain a $2K_2$, $C_4$, or $C_5$ as an induced subgraph. Also, as a consequence, it is shown that Stanley's conjecture holds for $K[CL(G)]$. Finally, we show that there exist a class of vertex decomposable flag simplicial complexes.

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## 2. Preliminaries

In this section we recall some definitions and results which will be needed later.

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(b) if $F \in \Delta$, then all subsets of $F$ are also in $\Delta$ (including the empty set).

An element of $\Delta$ is called a *face* of $\Delta$ and complement of a face $F$ is $V \setminus F$ and it is denoted by $F^c$. Also, the complement of the simplicial complex $\Delta = \langle F_1, \dots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \dots, F_r^c \rangle$. The *dimension* of a face $F$ of $\Delta$, $\dim F$, is $|F| - 1$ where, $|F|$ is the number of elements of $F$ and $\dim \emptyset = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of $\Delta$ is a subset $F$ of $V$ with $F \notin \Delta$. we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of $\Delta$. The maximal faces of $\Delta$ under inclusion are called *facets* of $\Delta$. The *dimension* of the simplicial complex $\Delta$, $\dim \Delta$, is the maximum of dimensions of its facets. If all facets of $\Delta$ have the same dimension, then $\Delta$ is called *pure*.
Let $\mathcal{F}(\Delta) = \{F_1, \dots, F_q\}$ be the facet set of $\Delta$. It is clear that $\mathcal{F}(\Delta)$ determines $\Delta$ completely and we write $\Delta = \langle F_1, \dots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex $\Gamma$ is called a *subcomplex* of $\Delta$, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.
For $v \in V$, the subcomplex of $\Delta$ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \langle F \in \Delta : \quad v \notin F \rangle.$$

The *link* of a face $F \in \Delta$, denoted by $\mathrm{link}_\Delta(F)$, is a simplicial complex on $V$ with the faces, $G \in \Delta$ such that, $G \cap F = \emptyset$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\mathrm{link}_\Delta(v)$.

$$\mathrm{link}_\Delta(v) = \{F \in \Delta : \quad v \notin F, \quad F \cup \{v\} \in \Delta\}.$$

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$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of $\Delta$, denoted by $I(\Delta)$, to be the ideal of $S$ generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of $\Delta$, denoted by $I_\Delta$, is the ideal of $S$ generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of $\Delta$.

**Definition 2.3.** A simplicial complex $\Delta$ is recursively defined to be *vertex decomposable*, if it is either a simplex, or else has some vertex $v$ so that,

(a) Both $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable, and

(b) No face of $\mathrm{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.

A vertex $v$ which satisfies in condition (b) is called a *shedding vertex*.

**Definition 2.4.** A simplicial complex $\Delta$ is *shellable*, if the facets of $\Delta$ can be ordered $F_1, \dots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \dots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

**Definition 2.5.** A simplicial complex is called a flag complex if all minimal non-faces are two element sets.

**Definition 2.6.** A graded $S$-module $M$ is called *sequentially Cohen-Macaulay* (over $K$), if there exists a finite filtration of graded $S$-modules,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each $M_i/M_{i-1}$ is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A simplicial complex $\Delta$ is called (sequentially) Cohen-Macaulay over $K$, if the ring $K[\Delta] = S/I_\Delta$ is (sequentially) Cohen-Macaulay. A simplicial complex $\Delta$ is called disconnected, if the vertex set $V$ of $\Delta$ is a disjoint union $V = V_1 \cup V_2$ such that no face of $\Delta$ has vertices in both $V_1$ and $V_2$. Otherwise $\Delta$ is connected.

**Definition 2.7.** A graph $G$ is chordal if every cycle of length strictly greater than three has a chord.

A chord of a cycle is an edge joining two nonconsecutive vertices of the cycle.

**Definition 2.8.** (Induced subgraph) Suppose that $V_1$ be a subset of the vertex set $V$ of a graph $G$. Then the subgraph of $G$ whose vertex set is $V_1$ and whose edge set is the set of edges of $G$ that have both end vertices in $V_1$ is denoted by $G[V]$ called a vertex induced subgraph (induced subgraph) of $G$.

## 3. Vertex decomposability on the clique complexes

As the main result of this section, we show that the clique complex $CL(G)$ is vertex decomposable if the graph $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph. Also, as a consequence, it is shown that Stanley's conjecture holds for $K[CL(G)]$. Dochtermann and Engrstrom in[2] showed that if $G$ is a chordal graph then the independence complex $\Delta(G)$ is vertex decomposable. Now, we are ready that prove one of the main result of this paper.

**Theorem 3.1.** *Let $G$ be a simple graph. If the graph $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph then the clique complex $CL(G)$ is vertex decomposable.*

*Proof.* We first prove that $G$ is chordal. Suppose $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph. Because any $C_n, n > 5$ contains a $2K_2$ as an induced subgraph, $G$ must not contain any induced cycles larger than $C_3$. Therefore, $G$ is chordal. Similarly, we prove that the complement of graph $G(G')$ is chordal. Since $C_4' = 2K_2$ and $C_5' = C_5$, $G'$ also does not contain $2K_2, C_4$, or $C_5$ as induced subgraphs. By the same argument, $G'$ is chordal. So by [2] independence complexes $\Delta(G)$ and $\Delta(G')$ are vertex decomposable. It is well-known that $\Delta(G') = CL(G)$. Hence $CL(G)$ is vertex decomposable. □

Let $R$ be any standard graded $K$- algebra over an infinite field $K$, *i.e*, $R$ is a finitely generated graded algebra $R = \bigoplus_{i\geq 0} R_i$ such that $R_0 = K$ and $R$ is generated by $R_1$. There are several characterizations of the depth of such an algebra. We use the one that $(R)$ is the maximal length of a regular $R$- sequence consisting of linear forms. Let $x_F = \sqcap_{i\in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \dots, x_n\}$. The $K$- subspace $x_F K[Z]$ of $S = K[x_1, \dots, x_n]$ is the subspace generated by monomials $x_F u$, where $u$ is a monomial in the polynomial ring $K[Z]$. It is called a squarefree Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let $\Delta$ be a simplicial complex on $\{x_1, \dots, x_n\}$. A squarefree Stanley decomposition $\mathcal{D}$ of $K[\Delta]$ is a finite direct sum $\bigoplus_i u_i K[Z_i]$ of squarefree Stanley spaces which is isomorphic as a $\mathbb{Z}^n$- graded $K$- vector space to $K[\Delta]$, *i.e*.

$$K[\Delta] \cong \bigoplus_i u_i K[Z_i].$$

We denote by $(\mathcal{D})$ the minimal dimension of a Stanley space in $\mathcal{D}$ and we define $(K[\Delta]) = \max\{(\mathcal{D})\}$, where $\mathcal{D}$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in [8] the upper bound for the depth of $K[\Delta]$ as the following:

$$(K[\Delta]) \leq (K[\Delta]).$$

Also we recall another conjecture of Stanley. Let $\Delta$ be again a simplicial complex on $\{x_1, \dots, x_n\}$ with facets $G_1, \dots, G_t$. The complex $\Delta$ is called partitionable if there exists a partition $\Delta = \cup_{i=1}^t [F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of $\Delta$. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [9] and [10] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [4] proved that for Cohen-Macaulay simplicial complex $\Delta$ on $\{x_1, \dots, x_n\}$ we have that $(K[\Delta]) \leq (K[\Delta])$ if and only if $\Delta$ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results we obtain :

**Corollary 3.2.** *Let $G$ be a simple graph. If the graph $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph then the clique complex $CL(G)$ is partitionable and Stanley's conjecture holds for $K[CL(G)]$.*

*Proof.* Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. By theorem 3.1 proof is completed. □

We know that the clique complex of any graph $G$ is flag complex. Therefore, we introduce a class of vertex decomposable flag simplicial complexes as the following:

**Corollary 3.3.** *Let $G$ be a simple graph. If the graph $G$ does not contain a $2K_2, C_4$, or $C_5$ as an induced subgraph then the clique complex $CL(G)$ is a vertex decomposable flag simplicial complex.*

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# Conditions that the polynomial ring is nil reversible

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**Abstract**

The notion of nil reversible ring is a generalization of reversible rings. The aim of this paper, is to provide conditions under which the polynomial ring over a nil reversible ring is nil reversible. We see that a semiprime ring $R$ is nil reversible if and only if the polynomial ring $R[x]$ is a reduced ring. Also, we show that a semiprime ring $R$ is nil reversible if and only if $R[x]$ is nil reversible. Further, if $R$ is a von Neumann ring and $X$ is a multiplicatively closed subset of $R$ which consists of central regular elements, then nil reversibility of $R$ is equivalent to nil reversibility of $R[x]$ and is equivalent to nil reversibility of $RX^{-1}$.

## 1. Introduction

Throughout this work, all rings are associative with identity. A ring $R$ is called *reversible* if for any $s,t \in R$, $st = 0$ implies $ts = o$. *Semiprime ring* is a ring $R$ in which $I^2 = 0$ implies $I = 0$, for all ideal $I$ of $R$. If a ring $R$ has no nonzero nilpotent element, then $R$ is called *reduced*. Also, a ring in which for each $r \in R$ there is $x \in R$ so that $r = rxr$, then $R$ is named *von Neumann ring.*
In this article, we study a generalization of reversible ring named nil reversible ring. The concept of nil reversible ring was first introduced in [5]. A ring $R$ in which for all $r \in R$ and for each nilpotent element $a$, $ra = 0$ iff $ar = 0$, is called *nil reversible ring*. We first study some properties of nil reversible rings. We note that the polynomial ring over a ring $R$ is written by $R[x]$. We inquire conditions under which the polynomial ring of a nil reversible ring is nil reversible. A ring in which every nilpotent element is central, is said to be *central reduced*. Also, if for all $x,y,z$ of a ring $R$, $xyz = 0$ yields $yxz = 0$, $R$ is called *symmetric ring* . It is obvious that central reduced and symmetric rings are nil reversible. We show that in a semiprime ring $R$ the concepts central reduced, symmetric and nil reversible are equivalent. Also, we show that a semiprime ring $R$ is nil reversible if and only if the polynomial ring $R[x]$ is reduced. Finally, we prove that a semiprime ring $R$ is a nil reversible ring if and only if $R[x]$ is nil reversible. This result tells us that nil reversibility of a von Neumann ring $R$ is equivalent to reversibility of $R[x]$ and is equivalent to nil reversibility of $RX^{-1}$ when $X$ is a multiplicatively closed subset of $R$ which consists of central regular elements.

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### *1.1. Some properties of nil reversible rings*

In here, we begin by reminding the definition of nil reversible rings. In the following, we state some properties of nil reversible rings.

**Definition 1.1.** A ring $R$ is called *nil reversible*, if for any its nilpotent element $a$ and $r \in R$, we have $ra = 0$ if and only if $ar = 0$.

**Example 1.2.** 1. Clearly, commutative rings and reversible rings are nil reversible ring.
2. Any subring of a nil reversible ring is nil reversible.
3. Let $R = \mathbb{Z}_2[x, y]$ such that $xy \neq yx$ and $L =< xy, yx^2, y^2x >$. Then $T = \frac{R}{L}$ is a nil reversible ring. Because, the set of all nilpotent elements of $T$ is $\{0 + L, yx + L\}$. Hence, we can see easily that $T$ is nil reversible.
4. The set of all $2 \times 2$ matrices over a nil reversible ring is not nil reversible. For example, if $R = M_2(\mathbb{Z})$ then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is a nilpotent element of $R$ and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

But

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0.$$

Thus $R$ is not nil reversible.

In Example 1.2, was stated that reversible rings are nil reversible. But, a nil reversible ring is not necessarily reversible. The ring $T$ in the Example 1.2 is a nil reversible ring but, it is not reversible. The next Example shows that the quotient ring of a nil reversible ring is not necessarily nil reversible.

**Example 1.3.** Let $F$ be a division ring and $R = F[x, y]$ such that $yx \neq xy$. Since $R$ is a domin, $R$ is nil reversible. Assume $L$ is the ideal generated by $xy$ and $T = \frac{R}{L}$. It is clear that $yx + L$ is a nilpotent element of $T$. On the other hand,

$$(x + L)(yx + L) = xyx + L = 0.$$

But,

$$(yx + L)(x + L) = (yx^2 + L) \neq 0.$$

This tells us $T$ is not a nil reversible ring.

**Lemma 1.4.** *Any finite direct product of nil reversible rings is a nil reversible ring.*

## 2. Main results

In this section, we study polynomial ring over nil reversible rings. We provide conditions that polynomial ring over a nil reversible ring is nil reversible.First, we study [1, P.P 361] in the following Lemma.

**Lemma 2.1.** *For a reduced ring R, the following hold:*

1. *R is a symmetric ring.*
2. *R is a reversible ring.*

We know that each reduced ring is nil reversible. In [3, Proposition 2.2] we saw that in semiprime rings the notions of nil reversible ring and reduced ring coincide. To refer to this result, we state the following Lemma.

**Lemma 2.2.** *Let R be a semiprime ring. Then R is nil reversible ring if and only if R is reduced.*

*Proof.* If $R$ is a reduced ring, then Lemma 2.1 tells us that $R$ is reversible and so $R$ is nil reversible. Conversely, assume $R$ is nil reversible and for $a \in R$, $a^2 = 0$. Then $ra = ar$, by nil reversibility of $R$. So $0 = ra^2 = raa = ara$, for all $r \in R$. From this $a = 0$, since $R$ is semiprime. Therefore $R$ is reduced. □

**Lemma 2.3.** *Let $R$ be a semiprime ring. Then the following sets coincide:*

1. *$R$ is reduced.*
2. *$R$ is symmetric.*
3. *$R$ is reversible.*
4. *$R$ is nil reversible.*

*Proof.* $1 \Rightarrow 2$. Every reduced ring is symmetric, by Lemma 2.1.
$2 \Rightarrow 3$. It is easily seen that every symmetric ring is reversible.
$3 \Rightarrow 4$. It is clear.
$4 \Rightarrow 1$. We obtain this result by Lemma 2.2.

□

Note that a ring $R$ is called *central reduced* if every nilpotent element of $R$ is central. It is clear that symmetric ring and central reduced ring are nil reversible. In Theorem 2.4, we see that in a semiprime ring the notions symmetric, central reduced and nil reversible are equivalent.

**Theorem 2.4.** *Let $R$ be a semiprime ring. Then the following conditions are equivalent:*

1. *$R$ is a symmetric ring.*
2. *$R$ is a reversible ring.*
3. *$R$ is a reduced ring.*
4. *$R$ is a nil reversible ring.*
5. *$R$ is a central reduced ring.*

*Proof.* We conclude this result from Lemma 2.3 and [2, Proposition 2.1]. □

In the following, we study Theorem 2.9 from [2] which is needed here.

**Theorem 2.5.** *Let $R$ be a semiprime ring. Then the following sets are equivalent:*

1. *$R$ is a symmetric ring.*
2. *$R$ is a reduced ring.*
3. *$R$ is a central reduced ring.*
4. *$R[x]$ is a reduced ring.*

The next Theorem follows from Theorems 2.5 and 2.4.

**Theorem 2.6.** *For a semiprime ring $R$, the following conditions are equivalent:*

1. *$R$ is a symmetric ring.*
2. *$R$ is a reduced ring.*
3. *$R$ is a nil reversible ring.*
4. *$R$ is a central reduced ring.*
5. *$R[x]$ is a reduced ring.*

**Corollary 2.7.** *In any semiprime ring $R$, the following sets coincide:*

1. *$R$ is a nil reversible ring.*
2. *$R[x]$ is a nil reversible ring.*

*Proof.* If $R$ is nil reversible, then Theorem 2.6 yields $R[x]$ is reduced. From this $R[x]$ is nil reversible ring. Conversely, since $R$ is a subring of $R[x]$, if $R[x]$ is nil reversible ring then $R$ is nil reversible. □

Remind that a ring $R$ is named *von Neumann*, if for any $r \in R$ there exists $x \in R$ that $r = rxr$. It is easy to see that von Neumann rings are semprime. From this we have the below result.

**Corollary 2.8.** *A von Neumann ring $R$ is nil reversible if and only if $R[x]$ is nil reversible.*

Let $R$ be a ring and $X$ a multiplicative set of regular elements in $R$. We recall $RX^{-1}$ is the right ring of fractions for $R$ with respect to $X$. In here, we study [4, Corollary 2.3] which we use in Theorem 2.10.

**Lemma 2.9.** *Let $R$ be a von Neumann ring and $X$ be a multiplicatively closed subset of $R$ which consists of central regular elements. Then the following conditions coincide:*

1. *$R$ is reversible.*
2. *$R$ is nil reversible.*
3. *$RX^{-1}$ is reversible.*
4. *$RX^{-1}$ is nil reversible.*

In the following, we study conditions that nil reversibility of $R[x]$ is equivalent to nil reversibility of $RX^{-1}$.

**Theorem 2.10.** *Let $R$ be a von Neumann ring and $X$ a multiplicatively closed subset of $R$ which consists of central regular elements. If there exists the right ring of fractions $RX^{-1}$, then the following sets are equivalent:*

1. *$R$ is reversible*
2. *$R$ is nil reversible.*
3. *$R[x]$ is nil reversible.*
4. *$RX^{-1}$ is reversible.*
5. *$RX^{-1}$ is nil reversible.*

*Proof.* Lemma 2.3 yields statemens $1, 2$ are equivalent. Also, Corollary 2.8 leads to 2 and 3 coincide. Further, we conclude from Lemma 2.9, the statemens $1, 2, 4, 5$ are equivalent. Therefore all of the above statemens are equivalent. □

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# Bilinear Operations of Modules on Quasi-Ideals

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**Abstract**

In this paper, some properties and characterizations of bilinear module actions over quasi-ideals of a Banach algebra has been taken into account. We apply our results to the quasi-ideals of the group algebra $M(G)$ and prove that $M(G)$ is isomorphic to a subspace of $\mathcal{B}_{mod}(L_1(G)^*)$ of all bilinear and separately continuous right module actions over $L_1(G)^*$.

## 1. Introduction

In [21, 22], Rieffel thoroughly investigated the Banach module $\hom_A(A,X)$, which represents the space of continuous homomorphisms between $A$ and $X$.
Consider $E$ and $X$ as topological left $A-$modules, where $E$ and $X$ are topological vector spaces (TVS), and $A$ is a topological algebra. Let $\hom_A(E,X)$ represent the vector space of continuous linear left $A-$module homomorphisms from $E$ to $X$. When $E$ is an $A-$bimodule, the operation $(a*T)(x)=T(x\cdot a)$ defines a left $A-$module structure on $\hom_A(E,X)$. In particular, for any $b\in A$ and $x\in E$, this satisfies the relation:

$$(a*T)(b\cdot x)=T(b\cdot x)\cdot a=T(b\cdot(x\cdot a))=b\cdot T(x\cdot a)=b\cdot(a*T)(x).$$

Notably, $\hom_A(A,X)$ also acquires a left $A-$module structure under this definition. Furthermore, if $A$ is commutative, the operation $(T*a)(x)=T(a\cdot x)$ gives $\hom_A(E,X)$ a right $A-$module structure. If $E=X=A$, then $\hom_A(A,A)$ is the usual multiplier algebra of $A$, and is denoted by $M(A)$. Indeed, a strong correlation exists between the concepts of module homomorphisms, multipliers and quasi-multipliers.
For any Banach space $X$, by $X^*$ we denote the first dual space of $X$ and for each $x\in X$ and $\zeta\in X^*$, by $\langle x,\zeta\rangle$ (and also $\langle\zeta,x\rangle$) we denote the natural duality between $X$ and $X^*$. We always consider $X$ as naturally embedded into $X^{**}$ through the mapping $\pi$, which is given by $\langle\pi(x),\zeta\rangle=\langle\zeta,x\rangle\quad(x\in X,\ \zeta\in X^*)$.
Let $A$ be a Banach algebra. On $A^{**}$ there exist two natural multiplications called the first and second Arens products. In this paper, we equip $A^{**}$ with the first Arens product whose definition are recalled below. Let $a\in A,\zeta\in A^*$, and

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$F,\ G \in A^{**}$ be arbitrary. Then one defines $\zeta \cdot a$ and $G \cdot \zeta$ as $\langle \zeta \cdot a, b\rangle = \langle \zeta, ab\rangle$ and $\langle G \cdot \zeta, b\rangle = \langle G, \zeta \cdot b\rangle$, where $b \in A$ is arbitrary. Now, the first Arens product of $F$ and $G$ is an element $F \circ G$ in $A^{**}$ which is given by $\langle F \circ G, \zeta\rangle = \langle F, G \cdot \zeta\rangle$, where $\zeta \in A^*$ is arbitrary. The second Arens product, which we denote by $\circ'$, is defined in a similar way. Equipped with these multiplications, $A^{**}$ is a Banach algebra and $A$ is a subalgebra of it.
A Banach algebra $A$ is called factorable if $AA = A$. By $A^*A$ we denote the subspace $\{\zeta \cdot a;\ \zeta \in A^*,\ a \in A\}$ of $A^*$. Similarly, $AA^* = \{a \cdot \zeta;\ a \in A,\ \zeta \in A^*\}$. If $A^*A = A^*$, then we say that $A^*$ factors on the left. Similarly, $A^*$ factors on the right if $AA^* = A^*$.
In [1], [2], [3], [4] and [5], we investigated the extent to which the study of quasi-multipliers could be made beyond Banach algebras. We considered the notion of quasi-multipliers on various frameworks of $f-$algebra, $\ell-$algebra, dual of a Banach algebra, general topological algebra setting and gave an account, how far one could get beyond Banach algebras, using combination of standard methods.
In this paper, we define the quasi-multipliers on quasi-ideals of a Banach algebra and extend the notion of strict topology on the space $\mathcal{B}_{mod}(A^*)$ of all bilinear module actions over $A^*$.

## 2. A generalization of bilinear module actions

**Definition 2.1.** A bilinear mapping $m : A^* \times A \to A^*$ is called a right bilinear module action over $A^*$ if

$$m(a \cdot \zeta, b) = a \cdot m(\zeta, b) \qquad \text{and} \qquad m(\zeta, ba) = m(\zeta, b) \cdot a \tag{1}$$

hold for arbitrary $\zeta \in A^*$ and $a,\ b \in A$.
Similarly, a bilinear mapping $m' : A \times A^* \to A^*$ is called a left bilinear module action over $A^*$ if

$$m'(ab, \zeta) = a \cdot m'(b, \zeta) \qquad \text{and} \qquad m'(b, \zeta \cdot a) = m'(b, \zeta) \cdot a$$

hold for arbitrary $\zeta \in A^*$ and $a,\ b \in A$.

Let $\mathcal{B}_{mod}(A^*)$ be the set of all bilinear and separately continuous right bilinear module actions over $A^*$. It is obvious that $\mathcal{B}_{mod}(A^*)$ is a linear space. Moreover, it is a Banach space with respect to the norm

$$||m|| = \sup\{||m(\zeta, a)||;\quad \zeta \in A^*,\ a \in A,\ ||\zeta|| \le 1,\ ||a|| \le 1\}.$$

Let $A$ be a general Banach algebra. Then a map $T : A^* \to A^*$ is called a right module action over $A^*$ if

$$T(a \cdot \zeta) = a \cdot T(\zeta),$$

for all $\zeta \in A^*, a \in A$. With $\mathcal{M}_{mod}(A^*)$ we denote the space of all bounded linear right module actions over $A^*$.

**Definition 2.2.** A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra $A$ is said to be ultra$^*-$approximate identity if, for all $m \in \mathcal{B}_{mod}(A^*)$ and $\zeta \in A^*$, the net $\{m(\zeta, e_\lambda) : \lambda \in I\}$ is Cauchy in $A^*$.

**Theorem 2.3.** *Let $A$ be factorable with an ultra$^*-$approximate identity $\{e_\lambda\}$. Then the mapping $\rho : \mathcal{M}_{mod}(A^*) \to \mathcal{B}_{mod}(A^*)$, defined by*

$$\rho_T(\zeta, a) = (T\zeta) \cdot a \qquad (T \in \mathcal{M}_{mod}(A^*),\ \zeta \in A^*,\ a \in A),$$

*is a bijective with norm $\|\rho\| \le 1$. If $\{e_\lambda\}$ is of norm one, then $\rho$ is an isometry.*

*Proof.* Let $T \in \mathcal{M}_{mod}(A^*)$ be arbitrary. It is obvious that $\rho_T$ is a bilinear map from $A^* \times A$ to $A^*$ and that it is bounded with $\|T\|$. For $\zeta \in A^*$, and $a, b \in A$, we have

$$\rho_T(a \cdot \zeta, b) = T(a \cdot \zeta) \cdot b = (a \cdot T\zeta) \cdot b = a \cdot (T\zeta \cdot b) = a \cdot \rho_T(\zeta, b)$$

and

$$\rho_T(\zeta, ba) = (T\zeta) \cdot (ba) = (T\zeta \cdot b) \cdot a = \rho_T(\zeta, b) \cdot a.$$

Thus, $\rho_T \in \mathcal{B}_{mod}(A^*)$. It follows from the definition that $\rho : \mathcal{M}_{mod}(A^*) \to \mathcal{B}_{mod}(A^*)$ is linear. Obviously, $\|\rho_T\| \le \|T\|$, which gives $\|\rho\| \le 1$. If $\rho_T = 0$, then we have $(T\zeta) \cdot a = 0$ for every $\zeta \in A^*$ and $a \in A$. So for each $b \in A$,

$\langle T\zeta \cdot a, b\rangle = \langle T\zeta, ab\rangle = 0$. Since $A$ is factorable, consequently $T = 0$. Now, let $m \in \mathcal{B}_{mod}(A^*)$ be arbitrary. The mapping $T\zeta = \lim_\lambda m(\zeta, e_\lambda)$ belong to $\mathcal{M}_{mod}(A^*)$ and $\rho_T(\zeta, a) = (T\zeta)\cdot a = \lim_\lambda m(\zeta, e_\lambda)\cdot a = m(\zeta, a)$ hold for all $\zeta \in A^*$ and $a \in A$. Which means that $\rho$ is onto.
At the end assume that $\{e_\lambda\}$ is a approximate identity for $A$ of norm one. Let $T \in \mathcal{M}_{mod}(A^*)$ and $\varepsilon > 0$ be arbitrary. Let $\zeta \in A^*$ is such that $\|\zeta\| \leq 1$ and $\|T\| - \varepsilon < \|T\zeta\|$. Since for arbitrary $a \in A$, $\lim_\lambda \langle T\zeta\cdot e_\lambda, a\rangle = \lim_\lambda \langle T\zeta, e_\lambda.a\rangle = \langle T\zeta, a\rangle$, we have

$$\|\rho_T\| \geq \lim_\lambda \|\rho_T(\zeta, e_\lambda)\| = \lim_\lambda \|T\zeta \cdot e_\lambda\| > \|T\| - \varepsilon,$$

and so, $\rho$ is an isometry. □

Let $A$ be factorable with an ultra$^*$−approximate identity. We may therefore use the Theorem 2.3 to define a multiplication in $\mathcal{B}_{mod}(A^*)$ making it a Banach algebra. We outline the details as follows.
Let $m_1,\ m_2 \in \mathcal{B}_{mod}(A^*)$. By virtue of the above Theorem, there exist $T_1,\ T_2 \in \mathcal{M}_{mod}(A^*)$ such that $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. Then

$$m_1 \circ_\rho m_2 = \rho_{T_1} \circ_\rho \rho_{T_2} := \rho_{T_2 T_1}$$

gives a well defined multiplication.
Note that $\mathcal{B}_{mod}(A^*)$ can be turned into an A-bimodule in the following way: For each $m \in \mathcal{B}_{mod}(A^*)$ and $a \in A$, the products $a * m$ and $m * a$ can be defined as mappings from $A^* \times A$ into $A^*$ by

$$\begin{aligned} (a * m)(\zeta, b) &= m(\zeta \cdot a, b), \\ (m * a)(\zeta, b) &= m(\zeta, ab), \quad \zeta \in A^*,\ b \in A. \end{aligned}$$

It is easy to see that $a * m,\ m * a \in \mathcal{B}_{mod}(A^*)$, so that $\mathcal{B}_{mod}(A^*)$ is an $A$-bimodule.

**Definition 2.4.** The strict topology $\beta$ on $\mathcal{B}_{mod}(A^*)$ is defined by the seminorm

$$m \to \|m * a\| \qquad (a \in A,\ m \in \mathcal{B}_{mod}(A^*)).$$

Let $A$ and $B$ be two factorable Banach algebras with an ultra$^*$−approximate identity, and let $\varphi : B \to A$ be a homomorphism such that $\varphi^* : A^* \to B^*$ be onto. We define $\widetilde{\varphi} : \mathcal{B}_{mod}(A^*) \to \mathcal{B}_{mod}(B^*)$ by $[\widetilde{\varphi}(m)](\varphi^*(\zeta), b) = \varphi^*(m(\zeta, \varphi(b)))$ for each $\zeta \in A^*$ and $b \in B$.

**Theorem 2.5.** *Let $A, B, \varphi, \varphi^*$ and $\widetilde{\varphi}$ be as above and $\beta_A$ and $\beta_B$ denote the strict topology $\beta$ on $\mathcal{B}_{mod}(A^*)$ and $\mathcal{B}_{mod}(B^*)$, respectively. If $\varphi : B \to A$ is continuous, then so is $\widetilde{\varphi} : (\mathcal{B}_{mod}(A^*), \beta_A) \to (\mathcal{B}_{mod}(B^*), \beta_B)$ a continuous homomorphism.*

**Definition 2.6.** A subalgebra $A$ of an algebra $B$ is called a quasi-ideal of $B$ if $ABA \subset A$.

**Definition 2.7.** Let $A$ be a quasi-ideal of the algebra $B$, then there is a linear mapping $\Psi_B : B \to \mathcal{B}_{mod}(A^*)$ defined by, for any $b \in B$

$$\Psi_B(b)(\zeta, a) = \zeta \cdot ba, \quad \textit{for all} \quad \zeta \in A^*,\ a \in A.$$

Note that for arbitrary $a' \in A$, we may define $\langle \zeta \cdot ba, a'\rangle = \langle \zeta, a'ba\rangle$.

**Definition 2.8.** The topology $\{\Psi_B^{-1}(\vartheta) : \vartheta \in \beta_A\}$ on $B$, will be denoted by $u_{(B,A)}$. A net $b_\alpha$ in $B$ is $u_{(B,A)}$−converges to some $b \in B$ precisely when $\lim_\alpha \|\zeta \cdot b_\alpha a - \zeta \cdot ba\| = 0$ for all $a \in A,\ \zeta \in A^*$

**Theorem 2.9.** *Let $A$ be factorable with an ultra$^*$−approximate identity $\{e_\alpha\}$. If $A$ is a quasi-ideal of an algebra $B$, then the mapping $\Psi_B$ is a $(u_{(B,A)}, \beta_A)$-continuous homomorphism of $B$ to $\mathcal{B}_{mod}(A^*)$.*

**Corollary 2.10.** *If the mapping $\phi_B$ is one to one. Then the algebra $B$ may be regarded as a subset of $\mathcal{B}_{mod}(A^*)$.*

Now, if we note that every ideal is a quasi-ideal and $L_1(G)$ and $L_\infty(G)$ are quasi-ideals of $M(G)$, we have the following example.
Let $M(G)$ denote the convolution algebra of all bounded regular measures on $G$. Recall that the convolution product of $f \in L_1(G)$ and $\mu \in M(G)$ is defined by

$$f * \mu(x) = \int_G f(xy^{-1})\, d\mu(y).$$

**Example 2.11.** Let $G$ be a compact group and $A = L_1(G)$. Then the equation

$$(\Psi_\mu(\zeta, f) := (\zeta * \mu) * f \qquad (\mu \in M(G), \zeta \in L_\infty(G),\ f \in L_1(G)).$$

defines a linear isomorphism between $M(G)$ and a subspace of $\mathcal{B}_{mod}(A^*)$.

*Proof.* By the definition of module action, we have $(\zeta * \mu) * f = \zeta * (\mu * f)$ which shows that $\Psi_\mu \in \mathcal{B}_{mod}(L_1(G)^*)$. Clearly, $\Psi : M(G) \to \mathcal{B}_{mod}(L_1(G)^*)$ is a bounded linear map. We claim that $\Psi$ is injective.To see this, accume $\Psi_\mu = 0$. This implies that $(\zeta * \mu) * f = 0$ for all $\zeta \in L_\infty(G)$ and $f \in L_1(G)$. Since $L_1(G)$ has a bounded approximate identity, it follows that $\zeta \circ \mu = 0$. Given that the measure algebra $M(G)$ is the dual space of $C_0(G)$ and has a bounded approximate identity, it shows that $\mu = 0$, as required.

□

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# Topologies and Generalized Actions of Modules on Banach Algebras

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**Abstract**

In this paper, we generalize the concept of module actions over a general Banach algebra $A$. We investigate the strict topology and quasi-strict topology on the space $\mathcal{B}_{mod}(A^*)$, which encompasses all bilinear and separately continuous module actions over $A^*$. Additionally, we explore the various properties of these topologies.

## 1. Introduction

In [21, 22], Rieffel thoroughly investigated the Banach module $\hom_A(A, W)$, which represents the space of continuous homomorphisms between $A$ and $W$.
Consider $V$ and $W$ as topological left $A-$modules, where $V$ and $W$ are topological vector spaces (TVS), and $A$ is a topological algebra. Let $\hom_A(V, W)$ represent the vector space of continuous linear left $A-$module homomorphisms from $V$ to $W$. When $V$ is an $A-$bimodule, the operation $(a * T)(v) = T(v \cdot a)$ defines a left $A-$module structure on $\hom_A(V, W)$. In particular, for any $b \in A$ and $v \in V$, this satisfies the relation:

$$(a * T)(b \cdot v) = T(b \cdot v) \cdot a = T(b \cdot (v \cdot a)) = b \cdot T(v \cdot a) = b \cdot (a * T)(v).$$

Notably, $\hom_A(A, W)$ also acquires a left $A-$module structure under this definition. Furthermore, if $A$ is commutative, the operation $(T * a)(v) = T(a \cdot v)$ gives $\hom_A(V, W)$ a right $A-$module structure. If $V = W = A$, then $\hom_A(A, A)$ is the usual multiplier algebra of $A$, and is denoted by $M(A)$. Indeed, a strong correlation exists between the concepts of module homomorphisms, multipliers and quasi-multipliers.
For any Banach space $A$, by $A^*$ we denote the first dual space of $A$ and for each $a \in A$ and $\zeta \in A^*$, by $\langle a, \zeta \rangle$ (and also $\langle \zeta, a \rangle$) we denote the natural duality between $A$ and $A^*$. We always consider $A$ as naturally embedded into $A^{**}$ through the mapping $\pi$, which is given by $\langle \pi(a), \zeta \rangle = \langle \zeta, a \rangle \quad (a \in A,\ \zeta \in A^*)$.

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Consider $A$ to be a Banach algebra. On the bidual space $A^{**}$, two natural types of multiplication are defined, referred to as the first and second Arens products. In this paper, we focus on the first Arens product, whose definition is recalled below. Let $a \in A, \zeta \in A^*$, and $F,\ G \in A^{**}$ be arbitrary. The elements $\zeta \cdot a$ and $G \cdot \zeta$ are defined as follows:

$$\langle \zeta \cdot a, b\rangle = \langle \zeta, ab\rangle \ \text{ and } \ \langle G \cdot \zeta, b\rangle = \langle G, \zeta \cdot b\rangle,$$

where $b \in A$ is arbitrary. The first Arens product of $F$ and $G$, denoted $F \circ G$, is an element of $A^{**}$ defined by $\langle F \circ G, \zeta\rangle = \langle F, G \cdot \zeta\rangle$, where $\zeta \in A^*$ is arbitrary. Similarly, the second Arens product, denoted by $\circ'$, is defined in an analogous manner. With these two multiplications, $A^{**}$ becomes a Banach algebra, and $A$ forms a subalgebra of $A^{**}$.
A Banach algebra $A$ is called factorable if $AA = A$. Let $A^*A$ denotes the subspace $\{\zeta \cdot a;\ \zeta \in A^*,\ a \in A\}$ of $A^*$. Similarly, $AA^*$ is defined as $\{a \cdot \zeta;\ a \in A,\ \zeta \in A^*\}$. We say that $A^*$ factors on the left if $A^*A = A^*$, and we say that $A^*$ factors on the right if $AA^* = A^*$.
In [1], [2], [3], [4] and [5], we investigated the extent to which the study of quasi-multipliers could be made beyond Banach algebras. We examined the notion of quasi-multipliers within the framework of $f-$algebra, $\ell-$algebra, dual of a Banach algebra, general topological algebra and provided an analysis of the extent to which the theory can be extended beyond Banach algebras through the application of standard methods.
In this paper, we extend the notion of module actions and introduce several notions of strict topology and quasi-strict topology on the space $\mathcal{B}_{mod}(A^*)$ of all bilinear module actions over $A^*$.

## 2. Generalized module actions

**Definition 2.1.** Let $A$ be a general Banach algebra. A bilinear mapping $m : A^* \times A \to A^*$ is called a right bilinear module action over $A^*$ if the following condition hold for all $\zeta \in A^*$ and $a,\ b \in A$ :

$$m(a \cdot \zeta, b) = a \cdot m(\zeta, b) \qquad \text{and} \qquad m(\zeta, ba) = m(\zeta, b) \cdot a. \tag{1}$$

Similarly, a bilinear map $m' : A \times A^* \to A^*$ is called a left bilinear module action over $A^*$ if

$$m'(ab, \zeta) = a \cdot m'(b, \zeta) \qquad \text{and} \qquad m'(b, \zeta \cdot a) = m'(b, \zeta) \cdot a$$

hold for arbitrary $\zeta \in A^*$ and $a,\ b \in A$.

Let $\mathcal{B}_{mod}(A^*)$ denotes the set of all bilinear and separately continuous right bilinear module actions over $A^*$. It is clear that $\mathcal{B}_{mod}(A^*)$ forms a linear space. Furthermore, it is a Banach space when equipped with the norm

$$||m|| = \sup\{||m(\zeta, a)||;\quad \zeta \in A^*,\ a \in A,\ ||\zeta|| \le 1,\ ||a|| \le 1\}.$$

Let $A$ be a general Banach algebra. A map $T : A^* \to A^*$ is called a right module action over $A^*$ if

$$T(a \cdot \zeta) = a \cdot T(\zeta),$$

for all $\zeta \in A^*, a \in A$. We denote by $\mathcal{M}_{mod}(A^*)$, the space of all bounded linear right module actions over $A^*$. It is clear that for each $a \in A$, the right multiplication operator $R_a$ defined by $R_a\zeta = \zeta \cdot a$ is a right module action over $A^*$.

**Definition 2.2.** A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a Banach algebra $A$ is called ultra$^*-$approximate identity if, for every $m \in \mathcal{B}_{mod}(A^*)$ and $\zeta \in A^*$, the net $\{m(\zeta, e_\lambda) : \lambda \in I\}$ is Cauchy in $A^*$.

**Theorem 2.3.** *Let A be a factorable Banach algebra with an ultra$^*-$approximate identity $\{e_\lambda\}$. Then the map $\rho : \mathcal{M}_{mod}(A^*) \to \mathcal{B}_{mod}(A^*)$, defined by*

$$\rho_T(\zeta, a) = (T\zeta) \cdot a \qquad (T \in \mathcal{M}_{mod}(A^*),\ \zeta \in A^*,\ a \in A),$$

*is a bijective with $\|\rho\| \le 1$. Furthermore, if $\{e_\lambda\}$ is a bounded approximate identity with norm one, then $\rho$ is an isometry.*

Let $A$ be a factorable Banach algebra with an ultra$^*-$approximate identity. Using Theorem 2.3, we can define a multiplication operation in $\mathcal{B}_{mod}(A^*)$ that turns it into a Banach algebra. The procedure is outlined below:
Let $m_1,\ m_2 \in \mathcal{B}_{mod}(A^*)$. By the results of the previous theorem, there exist $T_1,\ T_2 \in \mathcal{M}_{mod}(A^*)$ such that $m_1 = \rho_{T_1}$ and $m_2 = \rho_{T_2}$. The multiplication of $m_1$ and $m_2$ is then given by

$$m_1 \circ_\rho m_2 = \rho_{T_1} \circ_\rho \rho_{T_2} := \rho_{T_2T_1},$$

which defines a well-defined multiplication in $\mathcal{B}_{mod}(A^*)$.

## 3. Strict topology and quasi-strict topology on $\mathcal{B}_{mod}(A^*)$

This section is dedicated to defining the strict topology and quasi-strict topology on $\mathcal{B}_{mod}(A^*)$, and to extending various related results from [6, 8, 10] to bilinear module actions.
Note that $\mathcal{B}_{mod}(A^*)$ can be turned into an A-bimodule in the following way. Let $m \in \mathcal{B}_{mod}(A^*)$ and $a \in A$. The operations $a * m$ and $m * a$ can be defined as functions that map from $A^* \times A$ to $A^*$, defined by

$$\begin{aligned}(a * m)(\zeta, b) &= m(\zeta \cdot a, b),\\ (m * a)(\zeta, b) &= m(\zeta, ab), \quad \zeta \in A^*,\ b \in A.\end{aligned}$$

It is easy to see that $a * m, m * a \in \mathcal{B}_{mod}(A^*)$, thus $\mathcal{B}_{mod}(A^*)$ forms an $A$-bimodule.
In the following, $\tau$ represents the topology on $\mathcal{B}_{mod}(A^*)$ generated by the norm.

**Definition 3.1.** The strict topology $\beta$ on $\mathcal{B}_{mod}(A^*)$ is characterized by the seminorm

$$m \to \|m * a\| \qquad (a \in A,\ m \in \mathcal{B}_{mod}(A^*)).$$

**Definition 3.2.** The quasi-strict topology $\gamma$ on $\mathcal{B}_{mod}(A^*)$ is characterized by the seminorm

$$m \to \|m(\zeta, a)\| \qquad (\zeta \in A^*,\ a \in A,\ m \in \mathcal{B}_{mod}(A^*)).$$

**Lemma 3.3.** *If $A$ is factorable, then $\gamma \subseteq \beta \subseteq \tau$.*

*Proof.* Consider the net $\{m_\alpha\}_{\alpha \in I}$ in $\mathcal{B}_{mod}(A^*)$, which converges in the $\beta$-topology to $m \in \mathcal{B}_{mod}(A^*)$. Let $\zeta \in A^*$ be arbitrary. Since $A$ is factorable, for any $a \in A$, there exist $b, c \in A$ such that $a = bc$. By the definition of the $\beta$-topology, we have $||m_\alpha * b - m * b|| \to 0$. Therefore,

$$\begin{aligned}||m_\alpha(\zeta, a) - m(\zeta, a)|| &= ||m_\alpha(\zeta, bc) - m(\zeta, bc)||\\ &= ||(m_\alpha * b)(\zeta, c) - (m * b)(\zeta, c)|| \to 0.\end{aligned}$$

This implies that $\{m_\alpha\}_{\alpha \in I}$ is $\gamma$-convergent to $m$. Clearly, $\beta \subseteq \tau$. □

**Proposition 3.4.** *Let $A$ be a factorable Banach algebra with an ultra$^*$−approximate identity. If $A^*$ factors on the right then the map*

$$\phi_A : (A, \tau) \to (\mathcal{B}_{mod}(A^*), \beta)$$

*defined by*

$$(\phi_A(a))(\zeta, b) = \zeta \cdot ab$$

*is a continuous monomorphism.*

*Proof.* Consider a net $\{a_\alpha\}$ in $A$, which converges to $a \in A$ in the topology $\tau$. For each $\zeta \in A^*$ and $b, c \in A$, we have

$$\begin{aligned}\|(\phi_A(a_\alpha) * b)(\zeta, c) - (\phi_A(a) * b)(\zeta, c)\| &= \|(\phi_A(a_\alpha))(\zeta, bc) - (\phi_A(a))(\zeta, bc)\|\\ &= \|\zeta \cdot a_\alpha bc - \zeta \cdot abc\| \to 0.\end{aligned}$$

Hence $\phi_A(a_\alpha) \to^\beta \phi_A(a)$. Now, we show that $\phi_A$ is a multiplicative. Let $a_1, a_2 \in A$. According to Theorem 2.3, there exist $T_1, T_2 \in \mathcal{M}_{mod}(A^*)$ such that $\phi_A(a_1) = \rho_{T_1}$ and $\phi_A(a_2) = \rho_{T_2}$. Therefore, for any $\zeta \in A^*, b \in A$, we obtain

$$T_1(\zeta) \cdot b = \zeta \cdot a_1 b \qquad \text{and} \qquad T_2(\zeta) \cdot b = \zeta \cdot a_2 b.$$

It follows

$$\begin{aligned}(\phi_A(a_1) \circ_\rho \phi_A(a_2))(\zeta, b) &= \rho_{T_2 T_1}(\zeta, b) = T_2(T_1(\zeta)) \cdot b = T_1\zeta \cdot (a_2 b)\\ &= \zeta \cdot a_1 a_2 b = \phi_A(a_1 a_2)(\zeta, b).\end{aligned}$$

Assume that $\phi_A(a) = 0$ for $a \in A$. So for arbitrary $\zeta \in A^*$ and $b \in A$, $\langle \zeta, ab \rangle = 0$ which show that $\langle \pi(a), b \cdot \zeta \rangle = \langle b \cdot \zeta, a \rangle = \langle \zeta, ab \rangle = 0$. From the assumption that $A^*$ factors on the right, we deduce that $a = 0$. Thus, $\phi_A$ is injective. □

**Proposition 3.5.** *Let $A$ be factorable with an ultra$^*-$approximate identity. If $A^*$ factors on the right then $\phi_A(A)$ is closed in the $\beta$-topology.*

*Proof.* Let $m \in \mathcal{B}_{mod}(A^*)$ with $m \in \beta - cl(\phi_A(A))$. There exist a net $\{a_\alpha\}$ such that $\phi_A(a_\alpha) \to^\beta m$. By the definition of the $\beta$-topology, the net $\{a_\alpha\}$ is $\tau-$Cauchy. Since $A$ is complete, there exist $a \in A$ such that $a_\alpha \to^\tau a$. By Proposition 3.4, $\phi_A$ is one to one and $(\tau, \beta)-$continuous, it is clear that $a_\alpha \to^\tau a$ iff $\phi_A(a_\alpha) \to^\beta \phi_A(a)$. So for each $\zeta \in A^*$ and $y \in A$,

$$m(\zeta, y) = \lim_\alpha \varphi_A(a_\alpha)(\zeta, y) = \phi_A(a)(\zeta, y).$$

This shows that $m = \phi_A(a)$, so $\phi_A(A)$ is $\beta-$closed.

□

**Theorem 3.6.** *Assume that $A$ is a factorable algebra with an ultra$^*-$approximate identity. Then $(\mathcal{B}_{mod}(A^*), \gamma)$, $(\mathcal{B}_{mod}(A^*), \tau)$ and $(\mathcal{B}_{mod}(A^*), \beta)$ have identical bounded sets.*

**Theorem 3.7.** *Suppose that $A$ is factorable and has an ultra$^*$-approximate identity.*
*(i) The space $(\mathcal{B}_{mod}(A^*), \gamma)$ is complete.*
*(ii) In the case where $A$ has an approximate identity of norm one, the space $(\mathcal{B}_{mod}(A^*), \beta)$ is complete.*

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# Study of a class of intuitionistic fuzzy modules associated with small submodules

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**Abstract**

In this research, we first introduce intuitionistic fuzzy modules, which are a generalization of fuzzy modules, and investigate their basic properties. Then, by introducing small intuitionistic fuzzy submodules of a module, we study their relationship with small classical submodules. For example, we will examine the relationship between a small intuitionistic fuzzy submodule and its level sets in the classical case.

## 1. Introduction and Preliminaries

A map $\mu : X \longrightarrow [0,1]$ is called a fuzzy subset of the nonempty set $X$. The complement of $\mu$, denoted by $\mu^c$, is a fuzzy subset of $X$ defined by $\mu^c(x) = 1 - \mu(x)$ for every $x \in X$

**Definition 1.1.** ([1]) An *intuitionistic fuzzy set* (briefly an $IFS$) $A$ of a non-void set $X$ is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \longrightarrow [0,1]$ and $\nu_A : X \longrightarrow [0,1]$, are fuzzy subsets of $X$, denote respectively the degree of membership namely $\mu_A(x)$ and the degree of non-membership namely $\nu_A(x)$ for each element $x \in X$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.
For the sake of simplicity, we denote an $IFS$, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set $X$ by $A = (\mu_A, \nu_A)$ or briefly $A$, and the set of all $IFS$ of $X$ by $IFS(X)$.

([1]) Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i \in I}$ be a family of $IFS$ of $X$. Then
$\bigcap_{i \in I} A_i = (\mu_{(\cap_{i \in I} A_i)}, \nu_{(\cap_{i \in I} A_i)}) = \{(x, \bigwedge_{i \in I} \mu_{A_i}(x), \bigvee_{i \in I} \nu_{A_i}(x)); x \in X\}$ and
$\bigcup_{i \in I} A_i = (\mu_{(\cup_{i \in I} A_i)}, \nu_{(\cup_{i \in I} A_i)}) = \{(x, \bigvee_{i \in I} \mu_{A_i}(x), \bigwedge_{i \in I} \nu_{A_i}(x)); x \in X\}$

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**Definition 1.2.** ([3]) Let $M$ be an $R-$module and $A = (\mu_A, \nu_A)$ an $IFS$ of $M$. Then $A$ is called an *intuitionistic fuzzy submodule of* $M$ if $A$ satisfies the following

1. $\mu_A(0) = 1, \nu_A(0) = 0$
2. $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$, for all $x, y \in M$
   $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$, for all $x, y \in M$
3. $\mu_A(rx) \geq \mu_A(x)$, for all $x \in M$ and $r \in R$
   $\nu_A(rx) \leq \nu_A(x)$, for all $x \in M$ and $r \in R$

**Definition 1.3.** ([1]) Let $A = (\mu_A, \nu_A)$ and $B = \mu_B, \nu_B)$ be two $IFS$'s of $M$. Then the $IFS$, $A + B$ of $M$ is $A + B = \{(x, \mu_{A+B}(x), \nu_{A+B}(x); x \in M\}$ defined as

$$\mu_{A+B}(x) = \bigvee \{\mu_A(y) \wedge \mu_B(z) \mid x = y + z;\ y, z \in M\}$$

$$\nu_{A+B}(x) = \bigwedge \{\nu_A(y) \vee \nu_B(z) \mid x = y + z;\ y, z \in M\}$$

**Definition 1.4.** ([1]) Let $M$ be an $R$-module, $N \subseteq M$ and $\alpha \in [0, 1]$. Define the $IFS$ $\alpha_N = (\mu_{\alpha_N}, \nu_{\alpha_N})$ of $M$ as follows

$$\mu_{\alpha_N}(x) = \begin{cases} \alpha & x \in N \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \square_{\square_N}(x) = \begin{cases} 1-\alpha & x \in N \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in M$.

If $\alpha = 1$, then $\mu_{\alpha_N} = \chi_N$ and $\nu_{\alpha_N} = \chi_N^c$, where $\chi_N$ denotes the characteristic function of $N$. In this case we write $\alpha_N = \chi_N^{IF} = (\chi_N, \chi_N^c)$.

Let $A \subseteq B$ be two $IFM$'s of the module $M$. Define the quotient intuitionistic fuzzy module $\frac{B}{A}$ as an $IFM$ of the module $\frac{M}{A^*}$ by $\frac{B}{A} = (\mu_{\frac{B}{A}}, \nu_{\frac{B}{A}})$ such that for every $x \in B^*$

$$(\mu_{\frac{B}{A}})([x]) = \bigvee \{\mu_B(z) \mid z \in [x]\} \quad \text{and} \quad (\nu_{\frac{B}{A}})([x]) = \bigwedge \{\nu_B(z) \mid z \in [x]\}$$

$$\text{where} \quad [x] = x + A^* \in \frac{M}{A^*}.$$

**Definition 1.5.** ([3]) Let $M, N$ be two $R$-modules and $f : M \longrightarrow N$ an $R$-homomorphism. Let $A = (\mu_A, \nu_A) \leq_{IF} M$ and $B = (\mu_B, \nu_B) \leq_{IF} N$. Then $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ and $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ are $IFM$'s of $N$ and $M$ respectively, such that for all $y \in N$

$$(\mu_{f(A)})(y) = \begin{cases} \vee\{\mu_A(x) \mid\ y = f(x)\} & y \in Im(f) \\ 0 & y \notin Im(f) \end{cases}$$

and

$$(\nu_{f(A)})(y) = \begin{cases} \wedge\{\nu_A(x) \mid\ y = f(x)\} & y \in Im(f) \\ 1 & y \notin Im(f) \end{cases}$$

and for every $x \in M$

$$(\mu_{f^{-1}(B)})(x) = \mu_B(f(x)) \quad \text{and} \quad (\nu_{f^{-1}(B)})(x) = \nu_B(f(x))$$

## 2. Intuitionistic fuzzy small submodules

**Definition 2.1.** ([4]) Let $M$ be a module and $K \leq M$. $K$ is called a *small submodule of* $M$ (denoted by $K \ll M$) if, $K + L \neq M$ for every proper submodule $L$ of $M$.

In any module $M$, trivially zero is a small submodule of $M$ called trivial small submodule. Also $M$ is not small in $M$.

**Proposition 2.2.** *Let $M$ be a module and $K \leq N \leq M$ and $L \leq M$. Then*

1. *$L + K \ll M$ if and only if $L \ll M$ and $K \ll M$.*
2. *$N \ll M$ if and only if $K \ll M$ and $\frac{N}{K} \ll \frac{M}{K}$.*
3. *If $K \ll N$, then $K \ll M$.*
4. *If $M = M_1 \oplus M_2$ and $K_i \leq M_i$ for $i = 1, 2$; then $K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$.*

Now we define the concept of intuitionistic fuzzy small submodule.

**Definition 2.3.** Let $M$ be an $R$-module. An $IFM$, $A = (\mu_A, \nu_A)$ is called an *intuitionistic fuzzy small submodule of $M$* (denoted by $A \ll_{IF} M$) if, $A + B \neq \chi_M^{IF}$ for ever $IFM$, $B \neq \chi_M^{IF}$. Equivalently if $A + B = \chi_M^{IF}$, then $B = \chi_M^{IF}$.

**Proposition 2.4.** *Let $M$ be a module and $N \leq M$. Then $N \ll M$ if and only if $\chi_N^{IF} \ll_{IF} M$.*

Since 0 is a small submodule in any module $M$, so $\chi_0^{IF} \ll_{IF} M$ by above proposition.

**Example 2.5.**
1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. For every $0 \neq n \in \mathbb{Z}$ the submodule $n\mathbb{Z}$ is not small in $M$. Hence $\chi_{n\mathbb{Z}}^{IF} \not\ll_{IF} M$ for every $0 \neq n \in \mathbb{Z}$, by above proposition.
2. Consider the submodules $K = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}\}$ and $L = \{\bar{0}, \bar{6}\}$ of $M = \mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{56}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$ as $\mathbb{Z}$-module. It is easy to check that $K \not\ll M$ and $L \ll M$. So $\chi_K^{IF} \not\ll_{IF} M$ and $\chi_L^{IF} \ll_{IF} M$, by above proposition.

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two $IFM$'s of the module $M$ such that $A \subseteq B$. We say $A$ is an *small intuitionistic fuzzy submodule of $B$* (denoted by $A \ll_{IF} B$) if, $A \ll_{IF} B^*$; that is, for ever $IFM$, $C$ of $M$ satisfying $C|_{B^*} \neq \chi_{B^*}^{IF}$, we have $A|_{B^*} + C|_{B^*} \neq \chi_{B^*}^{IF}$, where $A|_{B^*} = (\mu_A|_{B^*}, \nu_A|_{B^*})$ such that $\mu_A|_{B^*}$ and $\nu_A|_{B^*}$ are the restriction mapping of $\mu_A$ and $\nu_A$ on $B^*$ respectively.

**Definition 2.6.** Let $M, N$ be any two modules over a ring $R$. An epimorphism $f : M \longrightarrow N$ is called an *intuitionistic fuzzy small epimorphism* if $f^{-1}(\chi_0^{IF}) \ll_{IF} M$ so that $f^{-1}(\chi_0^{IF}) = \chi_{Ker(f)}^{IF}$ clearly.

**Proposition 2.7.** *Let $M$ be a module and $A = (\mu_A, \nu_A) \leq_{IF} M$. Then $A \ll_{IF} M$ if and only if $A_* \ll M$.*

**Example 2.8.**
1. Let $M = \mathbb{Z}_6$ and $S = \{\bar{0}, \bar{2}, \bar{4}\}$. Define the $IFM$, $A = (\mu_A, \nu_A)$ of $\mathbb{Z}_6$ by

$$\mu_A = \begin{cases} 1 & x \in S \\ \frac{1}{2} & \text{otherwise} \end{cases} \quad \text{and} \quad \square_A = \begin{cases} 0 & x \in S \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

   Then $A_* = S$ is not a small submodule of $M$ ($S + \{\bar{0}, \bar{3}\} = M$), so by above proposition $A \not\ll_{IF} M$.
2. Let $M = Z_8$ and $S = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ which is a small submodule of $M$. Define the $IFM$, $A = (\mu_A, \nu_A)$of $M$ by

$$\mu_A = \begin{cases} 1 & x \in S \\ \alpha & \text{otherwise} \end{cases} \quad \text{and} \quad \square_A = \begin{cases} 0 & x \in S \\ \beta & \text{otherwise} \end{cases}$$

   where $0 < \alpha + \beta < 1$. Then $A_* = S$ is small in $M$, so $A \ll_{IF} M$ by above proposition.

**Corollary 2.9.** *Let $A, B$ be two $IFM$'s of an $R$-module $M$ such that $A \subseteq B$. Then $A \ll_{IF} B$ if and only if $A_* \ll B^*$.*

**Example 2.10.** Consider $M = \frac{\mathbb{Z}}{24\mathbb{Z}}$ and $N = \frac{12\mathbb{Z}}{24\mathbb{Z}}$ as $\mathbb{Z}$-modules. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two $IFM$'s of $M$ such that for every $x \in M$:

$$\mu_A(x) = \begin{cases} 1 & x \in N \\ \frac{3}{4} & \text{otherwise} \end{cases}$$

$$\nu_A(x) = \begin{cases} 0 & x \in N \\ \frac{1}{6} & \text{otherwise} \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 1 & x = 0 \\ \frac{3}{4} & x \in N\ \{0\} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

$$\nu_B(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{8} & x \in N\ \{0\} \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

It is not difficult to check that $A_* = N, B^* = M$ and $N \ll M$. So by Corollary 3.9, $A \ll_{IF} B$.

**Lemma 2.11.** *Let $M$ be a module and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ and $C = (\mu_C, \nu_C)$ be $IFM$'s of $M$. Then $A \cap (B+C) \supseteq (A \cap B) + (A \cap C)$. Moreover if $B \subseteq A$, then $A \cap (B + C) = B + (A \cap C)$.*

**Definition 2.12.** Let $M$ be a module and $A, B, C$ be $IFM$'s of $M$ such that $B \subseteq A$ and $c \subseteq A$. Then $A$ is called an $IF$ *direct sum* of $B$ and $C$ if,

$$A = B + C \text{ and } B \cap C = \chi_0^{IF}$$

In this case we write $A = B \oplus_{IF} C$ and say $B, C$ are $IF$ *direct summands* of $A$.

If whenever $A \cap B = \chi_0^{IF}$ for two $IFM$'s of any module $M$, then we denote $A \oplus_{IF} B$ instead of $A + B$.

**Proposition 2.13.** *Let $M$ be a module and $A, B$ two $IFM$'s of $M$ such that $A \subseteq B$. Then $A \ll_{IF} B$ if and only if $A_* \ll B_*$.*

**Lemma 2.14.** *Let $M$ be a module and $A, B$ be two $IFM$'s of $M$ such that $\chi_M^{IF} = A \oplus_{IF} B$. Then $M = A^* \oplus B^* = A_* \oplus B_*$.*

**Proposition 2.15.** *Let $A, B$ be two $IFM$'s of any module $M$ such that $A \subseteq B$ and $B$ is an $IF$ direct summand of $M$. Then $A \ll_{IF} M$ if and only if $A \ll_{IF} B$.*

Let $M = N \oplus K$ be modules, then it is easy to see that $\chi_M^{IF} = \chi_N^{IF} \oplus \chi_K^{IF}$.

**Example 2.16.** Let $M = \frac{\mathbb{Z}}{12\mathbb{Z}}$ and $M_1 = \frac{3\mathbb{Z}}{12\mathbb{Z}}, M_2 = \frac{4\mathbb{Z}}{12\mathbb{Z}}$. Put $A = \chi_M^{IF}, B = \chi_{M_1}^{IF}$ and $C = \chi_{M_2}^{IF}$. It is easy to see that $N = \frac{6\mathbb{Z}}{12\mathbb{Z}} \ll M$, so $\chi_N^{IF} \ll_{IF} M$ by Proposition 3.4. We have $M = M_1 \oplus M_2$ and hence $\chi_M^{IF} = A \oplus_{IF} B$. Since $\chi_N^{IF} \subseteq B$, so $\chi_N^{IF} \ll_{IF} B$ by Proposition 3.17.

**Lemma 2.17.** *Let $A, B$ be two $IFM$'s of the module $M$. Then $A \ll_{IF} M$ and $B \ll_{IF} M$ if and only if $A + B \ll_{IF} M$.*

**Proposition 2.18.** *Let $A, B$ be two $IFM$'s of $M$ such that $A \oplus_{IF} B = \chi_M^{IF}$. Moreover let $C, D$ be two $IFM$'s of $M$ such that $C \subseteq A$ and $D \subseteq B$. Then $C \oplus_{IF} D \ll_{IF} A \oplus_{IF} B$ if and only if $C \ll_{IF} A$ and $D \ll_{IF} B$.*

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# A generalization of Ky Fan's Minimax inequality

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**Abstract**

In this paper, we derive generalized minimax inequality results by using an intersection theorem related to set-valued maps. These theorems are stated under less restrictive closedness and coercivity conditions than are typically needed, enabling us to solve problems that cannot be addressed using standard closedness and compactness assumptions.

## 1. Introduction and Preliminaries

Finding a point of intersection associated with set-valued maps is of particular importance. Let $X$ and $Y$ be two sets and $S, T : X \rightrightarrows Y$ be set-valued maps. One kind of intersection problems for the pair $(S, T)$, especially related to some minimax inequalities in mathematical economics is to find a point $y_0 \in X$ such that

$$T(y_0) \cap S(x) \neq \emptyset, \quad \forall x \in X.$$

Liu [6] studied this problem and its applications to minimax inequalities, offering a new approach to KKM theory. Later, Balaj [3] introduced weak KKM mappings and established intersection results. Due to the applications of this problem, numerous studies have been done on this subject; see for example [1, 2].
Let $X$ be a convex subset of a vector space, $Y$ be a nonempty set and $S, T : X \rightrightarrows Y$ be set-valued mappings. Then $S$ is called weak KKM with respect to $T$ if for any finite set $A \subseteq X$ and $y \in \mathrm{conv}A$,

$$T(y) \cap \bigcup_{t \in A} S(t) \neq \emptyset$$

where conv$A$ denotes the convex hull of $A$.
This concept is a weaker version of generalized KKM mappings which is defined for the pair $(S, T)$. Clearly, if $S$ is a weak KKM mapping with respect to the identity mapping, then this definition reduces to the standard definition of KKM mappings. Recall that a set-valued mapping $S : X \rightrightarrows X$ is KKM if conv$A \subseteq \bigcup_{t \in A} S(t)$, for any finite set $A \subseteq X$. In [3], the following result was obtained.

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**Theorem 1.1.** *Suppose that $X$ is a compact and convex subset of a topological vector space, $Y$ is a nonempty set and $S, T : X \rightrightarrows Y$ are set-valued mappings with nonempty values such that*

*(i) $S$ is a weak KKM mapping with respect to $T$;*

*(ii) for any $x \in X$, the set $\{y \in X : T(y) \cap S(x) \neq \emptyset\}$ is closed;*

*Then, there exists $y_0 \in X$ such that $T(y_0) \cap S(x) \neq \emptyset$, for each $x \in X$.*

In this paper, inspired by Theorem 1.1, we first present an intersection theorem for $(S, T)$ that relaxes the standard compactness and closedness conditions. Then, by applying this result, we study the minimax inequalities under less restrictive hypotheses.
Throughout this paper, the family of all nonempty finite subsets of a set $X$ is denoted by $\langle X \rangle$.
Let $X$ and $Y$ be topological spaces. A set-valued mapping $F : X \rightrightarrows Y$ is said to be upper semicontinuous on $X$, if for any open set $V \subseteq Y$, the set $\{x \in X : F(x) \subseteq V\}$ is open in $X$. In other words, for any closed subset $B$ of $Y$ the set $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed.
The set-valued mapping $F$ is called to be intersectionally closed on $A \subseteq X$, if

$$\bigcap_{x \in A} \mathrm{cl}(F(x)) = \mathrm{cl}(\bigcap_{x \in A} F(x))$$

## 2. An intersection result

In the following, we assume that $X$ is a nonempty convex subset of a Hausdorff topological vector space, $Y$ is a nonempty set and $S, T : X \rightrightarrows Y$ are set-valued mappings. A nonempty set $A \subseteq X$ is called to be $c$-compact if $\mathrm{conv}(K \cup A)$ is compact for each $K \in \langle X \rangle$.
Here, we introduce a new concept of topological pseudomonotonicity for a pair of set-valued mappings $(S, T)$.

**Definition 2.1.** A pair of set-valued mappings $(S, T)$ is called generalized topological pseudomonotone provided that for all $x, x_0 \in X$, and every net $(x_\alpha)$ in $X$ converging to $x_0$ with

$$T(x_\alpha) \cap S(tx + (1-t)x_0) \neq \emptyset, \ \forall t \in [0,1],$$

we have $T(x_0) \cap S(x) \neq \emptyset$.

In the following definition, the concept of intersectionally closedness is generalized to the pair $(S, T)$.

**Definition 2.2.** Let $S, T : X \rightrightarrows Y$ be set-valued mappings and $A \subseteq X$. Then we call $S$ to be intersectionally closed on $A$ with respect to $T$ if for any $y_0 \in A$, the implication presented below is satisfied.:
if there is a neighborhood $U$ of $y_0$ in $A$ such that for all $y \in U$ we have $S(x_y) \subseteq Y \setminus T(y)$, for some $x_y \in A$, then we can find $x_0 \in A$ and a neighborhood $U'$ of $y_0$ in $A$ with $S(x_0) \subseteq Y \setminus T(x)$ for each $x \in U'$.

In [4], KKM-type theorems for generalized KKM mappings were derived and their applications were investigated. Here, inspired by this work, an intersection theorem for the pair of set-valued mappings under condition of weak KKM is presented.

**Theorem 2.3.** *Let $S, T : X \rightrightarrows Y$ be set-valued mappings such that*

*(I) $S$ is weak KKM mapping with respect to $T$;*

*(II) $S$ is intersectionally closed on* conv$A$ *with respect to $T$, for each $A \in \langle X \rangle$;*

*(III) $(S, T)$ is generalized topological pseudomonotone;*

*(IV) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, there is $x \in \mathrm{conv}(A \cup \{t\})$ such that $T(t) \cap S(x) = \emptyset$.*

*Then, we can find a point $y_0 \in X$ such that for each $x \in X$,*

$$T(y_0) \cap S(x) \neq \emptyset.$$

**Example 2.4.** Let $Y = \mathbb{R}, X = (0, 2]$ and the mappings $S, T : X \rightrightarrows Y$ are defined as follows:

$$S(x) = \begin{cases} \{1, 2\}, & 0 < x < 1, \\ (1, 2], & 1 \leq x \leq 2. \end{cases}$$

and

$$T(x) = \begin{cases} \{1\}, & 0 < x < 1, \\ \{2\}, & 1 \leq x \leq 2. \end{cases}$$

Then, one can easily verify that the coercivity condition (IV) in Theorem 2.3 holds. Moreover, other conditions of this theorem are also satisfied but $X$ is not compact.

## 3. Minimax inequalities

Inspired by [4] we present the following definition.

**Definition 3.1.** Suppose that $f : X \times Y \to \mathbb{R}$ is a bifunction and $T : X \rightrightarrows Y$ is a set-valued mapping, $A \subseteq X$ and $\alpha \in \mathbb{R}$, then we say that $f$ is $\alpha$-intersectionally upper continuous in the second argument with respect to $T$ on $A$ if for each $y_0 \in K$, the following implication holds:
if there is a neighborhood $U$ of $y_0$ in $A$ such that for any $y \in U$ there exists $x_y \in A$ such that $f(x_y, z) < \alpha$, for all $z \in T(y)$, then we can find an $x_0 \in A$ and a neighborhood $U'$ of $y_0$ in $A$ such that

$$f(x_0, z) < \alpha, \quad \forall z \in T(U').$$

**Lemma 3.2.** *Let $f : X \times Y \to \mathbb{R}, T : X \rightrightarrows Y$ be a set-valued mapping, $A \subseteq X$ and $\alpha \in \mathbb{R}$. Then $f$ is $\alpha$-intersectionally upper continuous in the second argument with respect to $T$ on $A$, if and only if $S : X \rightrightarrows Y$ is intersectionally closed on $Y$ with respect to $T$, where $S(x) = \{y \in Y : f(x, y) \geq \alpha\}$.*

Let $f, g : X \times Y \to \mathbb{R}$ be two bifunctions and $W \subseteq Y$. Then, $f$ is $g$-quasiconvex in the first variable on $X \times W$, if for each $y \in W$ and any $K \in \langle X \rangle$, $f(x, y) \leq \min_{t \in K} g(t, y)$ for all $x \in \mathrm{conv}K$.

Here, we present a generalization of Ky Fan's Minimax inequality [5].

**Theorem 3.3.** *Suppose that $T : X \rightrightarrows Y$ is a set-valued mapping and $f, g : X \times Y \to \mathbb{R}$ are bifunctions with $\alpha_0 = \inf_{x \in X} \sup_{y \in T(x)} f(x, y) > -\infty$. Suppose that the following conditions hold*

*(I) $g$ is $\alpha$-intersectionally upper continuous in the second argument with respect to $T$ on $\mathrm{conv}A$, for each $\alpha < \alpha_0$ and $A \in \langle X \rangle$;*

*(II) $f$ is $g$-quasiconvex in the first variable on $X \times T(X)$;*

*(III) for each $\beta < \alpha_0$, $(S, T)$ is generalized topological pseudomonotone, where $S : X \rightrightarrows Y$ is defined by $S(x) := \{y \in Y : g(x, y) \geq \beta\}$;*

*(IV) there exists a c-compact set $A$, there is $\lambda < \alpha_0$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, and $z \in T(t)$, we have $\inf\{g(x, z) : x \in \mathrm{conv}(A \cup \{t\})\} < \lambda$.*

*Then,*

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \sup_{x \in X} \inf_{z \in X} \sup_{y \in T(x)} g(z, y).$$

**Corollary 3.4.** *Suppose that $Y$ be a topological space, $T : X \rightrightarrows Y$ be an upper semicontinuous set-valued mapping and $f, g : X \times Y \to \mathbb{R}$ are bifunctions with $\alpha_0 = \inf_{x \in X} \sup_{y \in T(x)} f(x, y) > -\infty$. Let the following conditions hold*

*(I) for any $x \in X, g(x,.)$ is upper semicontinuous;*

*(II) $f$ is $g$-quasiconvex in the first variable on $X \times T(X)$;*

*(III) there exists a c-compact set $A$ and there is $\lambda < \alpha_0$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, and $z \in T(t)$, we have* $\inf\{g(x,z) : x \in \mathrm{conv}(A \cup \{t\})\} < \lambda$.

*Then,*

$$\inf_{x \in X} \sup_{y \in T(x)} f(x,y) \leq \sup_{x \in X} \inf_{z \in X} \sup_{y \in T(x)} g(z,y).$$

**Remark 3.5.** Theorem 3.3 generalizes most of such Fan type minimax results such as [1, 6]. Indeed, the compactness assumption of $X$ in similar results has been replaced by the coercivity condition (IV) which is weaker condition. Moreover, in most of such minimax theorems, $T$ is upper semicontinuous and $f$ or $g$ is upper semicontinuous in the second argument which have been replaced by weaker condition. For more information about applications of this kind of minimax inequalities in game theory and mathematical economics, see [6].

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# Hypercyclic $C_0$-semigroups on the algebra of the Hilbert-Schmidt operators

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**Abstract**

We show in this paper that a $C_0$-semigroup $(T_t)_{t\geq 0}$ on a Hilbert space $H$ satisfies the hypercyclicity criterion if and only if the corresponding multiplication semigroup satisfies the hypercyclicity criterion on the algebra of the Hilbert-Schmidt operators. We state that if $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion, then $T_t$ fulfills the hypercyclicity criterion for any $t > 0$. Furthermore, we state some conditions for hypercyclicity of a $C_0$-semigroup and its corresponding multiplication semigroup.

## 1. Introduction and Preliminaries

Assume $H$ is a separable Hilbert space. The set of bounded and linear operators on $H$ is denoted by $B(H)$. Suppose $T \in B(H)$. If there exists $h \in H$ so that

$$\overline{orb(T,h)} = \overline{\{T^n h : n \geq 0\}} = H,$$

then $T$ is called a hyercyclic operator [7]. Hypercyclicity of various types of operators have been investigated extensively by mathematicians. One can see a good history of hypercyclicity in [8]. For some newer results one can see [2].
Assume $\{e_i\}$ is a basis for $H$. For $T \in B(H)$, $\|T\|_2$ is defined by,

$$\|T\|_2 = (\Sigma_{i=1}^{\infty}\|Te_i\|^2)^{\frac{1}{2}}, \tag{1}$$

is well defined and does not depend on the basis $\{e_i\}$ [6]. $B_2(H)$ is described with

$$B_2(H) = \{T \in B(H) : \|T\|_2 < \infty\},$$

and it is named the algebra of Hilbert-Schmidt operators. Any $T \in B_2(H)$ is named a Hilbert-Schmidt operator [6]. $B_2(H)$ is a two sided ideal of $B(H)$, and it is a Hilbert space with $\|.\|_2$. Moreover, any operator in $B_2(H)$ is compact [6,

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p. 267]. By [7, Theorem 5. 11], compact operators are not hypercyclic. So, $B_2(H)$ contains no hypercyclic operator. In this paper, we consider $B_2(H)$ with $\|.\|_2$-topology that is defined by (1).
One of the notable operators that are defined on the $B_2(H)$ is a (left) multiplication operator that is denoted by $L_T$. $L_T$ on $B_2(H)$ is defined by $L_T A = TA$, for any $A \in B(H)$. If $\overline{orb(L_T, A)} = \overline{\{T^n A : n \geq 0\}} = B_2(H)$ for some $A \in B_2(H)$, then $L_T$ is hypercyclic by definition of hyercyclicity.
One can see some newer results about them in [10]. Hypercyclicity of $L_T$ and satisfying $T$ in hypercyclicity criterion are related to each other as follows.

**Theorem 1.1.** *([12]) $T \in B(H)$ fulfills the hypercyclicity criterion if and only if $L_T$ is hypercyclic on $B_2(H)$ with $\|.\|_2$-topology.*

Remember $T \in B(H)$ satisfies the hypercyclicity criterion if two subsets $X$ and $Y$ of $H$ exist so that $\overline{X} = H$ and $\overline{Y} = H$, and a sequence $(n_k)$ of positive integers and maps $S_{n_k} : Y \to H$ exist so that $T^{n_k}x \to 0$ and $S_{n_k}y \to 0$, where $x \in X$ and $y \in Y$ are arbitrary, and $T^{n_k}S_{n_k}y \to y$ for any $y \in Y$ [7]. If $T$ satisfies the hypercyclicity criterion, then $T \oplus T$ is hypercyclic [7, Theorem 3.12]. Recall that $T \oplus T$ is named the direct sum of $T$ with itself and for any $(h_1, h_2) \in H \oplus H$ is defined by $(T \oplus T)(h_1, h_2) = (Th_1, Th_2)$. Clearly, hypercyclicity of $T \oplus T$ implies the hypercyclicity of $T$.
Now, this question arises that can we extend Theorem1.1 for the notable structure of operators that is named a $C_0$-semigroup? Recall that a $C_0$-semigroup $(T_t)_{t\geq 0}$ on $H$ is a family of operators on $H$ so that $T_0 = I$ and for any $s, t \geq 0$ and any $h \in H$,

$$T_{t+s}h = T_t T_s h \quad \text{and} \quad lim_{s\to t} T_s h = T_t h.$$

As mentioned in [7], $(T_t)_{t\geq 0}$ is hypercyclic on $H$ if for some $h \in H$, $\overline{orb((T_t)_{t\geq 0}, h)} = \overline{\{T_t h : t \geq 0\}} = H$. Hypercyclicity of $T_t$ for some $t > 0$ indicates hypercyclicity of $(T_t)_{t\geq 0}$ and vice versa [5, Theorem 2.3]. One can also see [1] and [11] for more information. A criterion for $C_0$-semigroups named hypercyclicity criterion can be found in [7, Theorem 7.27]. Consider $X$ and $Y$ are dense subsets of $H$. Assume that $(t_n)$ is an increasing sequence consisting of positive integers. Then $(T_t)_{t\geq 0}$ satisfies the hypercyclicity criterion if for any $n \in \mathbb{N}$, $S_{t_n} : Y \to H$ exist where for any $x \in X$ and $y \in Y$,

$$\lim_{n\to\infty} T_{t_n} x = 0, \quad \lim_{n\to\infty} S_{t_n} y = 0 \quad \text{and} \quad \lim_{n\to\infty} T_{t_n} S_{t_n} y = y.$$

It is proved in [7, Theorem 7.28] that $(T_t)_{t\geq 0}$ satisfies the hypercyclicity criterion if and only if $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic.
In this paper, we state a theorem similar to Theorem 1.1 for $C_0$-semigroups. Furthermore, we state some equivalent conditions for hypercyclicity of a $C_0$-semigroup by using the multiplication semigroup.

## 2. Main Results

First, we assert three equivalent conditions as follows.

**Theorem 2.1.** *For a $C_0$-semigroup $(T_t)_{t\geq 0}$ on $H$, the following conditions are equivalent:*

*(i) The hypercyclicity criterion is fulfilled by $T_t$ for any $t > 0$.*

*(ii) The hypercyclicity criterion is fulfilled by $T_t$ for some $t > 0$.*

*(iii) The hypercyclicity criterion for $C_0$-semigroups is satisfied by $(T_t)_{t\geq 0}$.*

*Proof.* $(i) \to (ii)$. is clear.
$(ii) \to (iii)$. By hypothesis, the hypercyclicity criterion is fulfilled by $T_t$ for some $t > 0$. Thus, $T_t \oplus T_t$ is hypercyclic for some $t > 0$ [7, Theorem 3.15]. Hence, $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic. Now, [7, Theorem 7.28] asserts that the hypercyclicity criterion is fulfilled by $(T_t)_{t\geq 0}$.
$(iii) \to (i)$. By hypothesis, $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. So, $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic [7, Theorem 7.28]. Therefore, by [7, Theorem 2.3], $T_t \oplus T_t$ is hypercyclic for any $t > 0$. Hence, the hypercyclicity criterion is fulfilled by $T_t$ [7, Theorem 3.15], for every $t > 0$.

□

Applying Theorem 2.1, we can demonstrate the other main conclusion of this paper. First, we define multiplication $C_0$-semigroup.
Consider $C_0$-semigroup $(T_t)_{t\geq 0}$ on $H$. Consider $(L_{T_t})_{t\geq 0}$, where $L_{T_t} : B_2(H) \to B_2(H)$ defines with $L_{T_t}A = T_tA$ for any $A \in B_2(H)$, and any $t \geq 0$. Then $(L_{T_t})_{t\geq 0}$ is a $C_0$-semigroup on $B_2(H)$ since, if $I$ indicates the identity operator on $H$, and if $A$ is an arbitrary member of $B_2H$ and $t, s \geq 0$ are arbitrary, then:

(i) $L_{T_0}A = T_0A = IA = A$.

(ii) $L_{T_{t+s}}A = T_{t+s}A = T_tT_sA = L_{T_t}L_{T_s}A$,

(iii) $\lim_{t\to s} L_{T_t}A = \lim_{t\to s} T_tA = T_sA$.

We call $(L_{T_t})_{t\geq 0}$ a (left)multiplication $C_0$-semigroup.
In the next theorem, we describe the relation between satisfying $(T_t)_{t\geq 0}$ in the hypercyclicity criterion and hypercyclicity of $(L_{T_t})_{t\geq 0}$ on $B_2(H)$.

**Theorem 2.2.** *Assume $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on $H$. Then, $(T_t)_{t\geq 0}$ satisfies the hypercyclicity criterion if and only if $(L_{T_t})_{t\geq 0}$ is hyercyclic on $B_2(H)$ with $\|.\|_2$-topology.*

*Proof.* Presume that $(L_{T_t})_{t\geq 0}$ is a hypercyclic $C_0$-semigroup on $B_2(H)$. So, $L_{T_t}$ is hypercyclic for any $t > 0$ [7, Theorem 7.27]. Hence, Theorem 2.1 asserts that for any $t > 0$, $T_t$ fulfills the hypercyclicity criterion. Hence, $(T_t)_{t\geq 0}$ is hypercyclic because $T_t$ is hypercyclic.
Now, consider $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. Theorem 2.1 asserts that there exists $t_0 > 0$ so that $T_{t_0}$ fulfills the hypercyclicity criterion. By Theorem 1.1, $L_{T_{t_0}}$ is hypercyclic on $B_2(H)$, and so $(L_{T_t})_{t\geq 0}$ is hypercyclic. □

The subsequent corollary is obtained by exerting Theorem 2.1 and Theorem 2.2.

**Corollary 2.3.** *Presume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on $H$. If the hypercyclicity criterion is satisfied by $T_{t_0}$ for some $t_0 > 0$, then $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$ with $\|.\|_2$-topology.*

*Proof.* Assume that $T_{t_0}$ fulfills the hypercyclicity criterion for some $t_0 > 0$. Theorem 2.1 asserts that $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. Applying Theorem 2.2 finishes the proof. □

**Theorem 2.4.** *Presume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on $H$, and consider $(L_{T_t})_{t\geq 0}$ be the corresponding (left)multiplication $C_0$-semigroup. If one of the following conditions occurs, then $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic.*
*Especially, $(T_t)_{t\geq 0}$ is a hypercyclic $C_0$-semigroup.*

*(i) $T_t$ fulfills the hypercyclicity criterion for some $t > 0$.*

*(ii) $T_t \oplus T_t$ is hypercyclic on $H \oplus H$ for some $t > 0$.*

*(iii) there is some $t > 0$ such that $\overline{\bigcup_{n=1}^{\infty} ker(T_t^n)} = H$, and $T_t$ is hypercyclic.*

*(iv) $L_{T_t}$ is hypercyclic on $B_2(H)$ for some $t > 0$.*

*(v) $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$.*

*Proof.* $(i)$ Suppose that $T_t$ fulfills the hypercyclicity criterion for some $t > 0$. Applying Theorem 2.1 indicates that $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. By [7, Theorem 7.28], $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic.
$(ii)$. The proof is done by considering the fact that hypercyclicity of $T_t \oplus T_t$ implies hypercyclicity of $(T_t \oplus T_t)_{t\geq 0}$.
$(iii)$. In this case, $T_t$ satisfies the hypercyclicity criterion by [3, Remark 2.6]. So, similar to part $(i)$, $(T_t \oplus T_t)_{t\geq 0}$ is hypercyclic.
$(iv)$. Hypercyclicity of $L_{T_t}$ on $B_2(H)$ for some $t > 0$ points the hypercyclicity of $(L_{T_t})_{t\geq 0}$. Applying Theorem 2.2, $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. The remain of the proof is similar to the proof of the part $(i)$.
$(v)$. If $(L_{T_t})_{t\geq 0}$ is hypercyclic, then applying Theorem 2.2, $(T_t)_{t\geq 0}$ fulfills the conditions of the hypercyclicity criterion for $C_0$-semigroups. The remain of the proof is the same as part $(i)$.

□

**Remark 2.5.** If any of the conditions in Theorem 2.4 occurs for a $C_0$-semigroup $(T_t)_{t\geq 0}$, then by [5, Theorem 2.3], $T_t$ is hypercyclic for any $t > 0$.

A hypercyclic operator $T$ is chaotic if it has a dense set of periodic points [7]. This operator fulfills the hypercyclicity criterion [7, Theorem 3.1]. Now, the subsequent theorem can be resented.

**Theorem 2.6.** *Assume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on $H$. Consider that $T_t$ is chaotic for some $t > 0$. Then $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$.*

*Proof.* Suppose that there is some $t > 0$ such that $T_t$ is a chaotic operator on $H$. So, the hypercyclicity criterion is satisfied by $T_t$[7, Theorem 3.1]. By Theorem 2.1, the hypercyclicity criterion is satisfied by $(T_t)_{t\geq 0}$. Thus, Theorem 2.2 asserts that $(L_{T_t})_{t\geq 0}$ is hypercyclic. □

A hypercyclic $C_0$-semigroup is called a chaotic $C_0$-semigroup if it has a dense set of periodic points [7, Definition 7.9]. Remember that $h \in H$ is a periodic point for $(T_t)_{t\geq 0}$ if $T_t h = h$ for some $t > 0$. It is not hard to see that if $T_t$ is chaotic for some $t > 0$, then $(T_t)_{t\geq 0}$ is chaotic. We assert that the chaoticity of $(T_t)_{t\geq 0}$ indicates that it fulfills the hypercyclicity criterion.

**Lemma 2.7.** *Consider a a $C_0$-semigroup $(T_t)_{t\geq 0}$ on $H$. If $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion, then it is chaotic.*

*Proof.* $(T_t)_{t\geq 0}$ is chaotic by hypothesis. By [7, Theorem 7.23] and proof of [7, Theorem 7.25], the hypercyclicity criterion is satisfied by $T_t$ for some $t > 0$. Thus, by Theorem 2.1, $(T_t)_{t\geq 0}$ satisfies the conditions of the hypercyclicity criterion for $C_0$-semigroups. Now, Theorem 2.2 completes the proof. □

For an operator $T$ on a Banach space, chaoticity of $T$ and $L_T$ are equivalent [9, Theorem 2.3]. Now, this question arises that can we establish an statement same to Theorem 2.2 for chaotic semigroups?
Employing Lemma 2.7, we can say the next corollary.

**Corollary 2.8.** *Assume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on $H$. Then $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$, when $(T_t)_{t\geq 0}$ is chaotic.*

*Proof.* Presume that $(T_t)_{t\geq 0}$ is chaotic. Thus, Lemma 2.7 asserts that $(T_t)_{t\geq 0}$ fulfills the hypercyclicity criterion. Now, applying Theorem 2.2 asserts that $(L_{T_t})_{t\geq 0}$ is hypercyclic on $B_2(H)$. □

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# Bayesian Inference on the Parameters of the Lomax-Lindley Distribution

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**Abstract**

This paper is devoted to the problem of Bayesian estimation for the parameters of a new flexible three-parameters distribution called the Lomax-Lindley distribution. The estimation of parameters are obtained by using the MCMC methods under squared error, linear-exponential, and Stein's loss functions. An application to real data set is provided for illustrative purposes. Finally, a Monte Carlo simulation study is conducted to investigate and compare the performance of different types of Bayes estimators presented in this paper.

## 1. Introduction

The Lomax and Lindley distributions are two well-known distributions that have been extensively used over the past decades for modeling data in many fields such as business, economics, actuarial modeling, queuing problems, biological sciences, reliability and life testing problems. Although these distributions are very useful for modeling lifetime data, they have monotone hazard rate functions. The hazard rate function (hrf) of Lomax distribution is decreasing and that of Lindley distribution is increasing. Thus, these distributions may not provide a reasonable parametric fit for modeling phenomena with non-monotone hazard rates such as the bathtub-shaped and unimodal hazard rates which are often encountered in practice. As a solution, several authors have considered modified or generalized forms and extensions of these distributions to give them more flexibility in describing various types of data. For instance, McDonald Lomax distribution by Lemonte and Cordeiro [4], Burr X exponentiated Lomax distribution by Aboraya [1], new extended generalized lindley distribution by Maya and Irshad [5] and odd log-logistic Marshal-Olkin Lindley distribution by Alizadeh et al. [2].

Recently, Tarvirdizade [6] has introduced a new flexible three-parameter distribution using the combination of the Lomax and the Lindley distributions in a serial system which is called the Lomax-Lindley (L-L) distribution. The cumulative distribution function (cdf) and the probability density function (pdf) of the L-L distribution with parameters $\alpha$, $\beta$ and $\theta$ are given by

$$F(x) = 1 - (1+\beta x)^{-\alpha}\left(1 + \frac{\theta x}{\theta+1}\right)e^{-\theta x}, \quad x > 0,\ \alpha > 0,\ \beta, \theta \geq 0, \tag{1}$$

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and

$$f(x)=\left(\frac{\theta^2(1+x)}{1+\theta}+\frac{\alpha\beta}{1+\beta x}\left(1+\frac{\theta x}{\theta+1}\right)\right)(1+\beta x)^{-\alpha}e^{-\theta x},\quad x>0,\ \alpha>0,\ \beta,\theta\geq 0, \tag{2}$$

respectively. The L-L distribution could be applied effectively for analyzing different types of lifetime data since it has a simple form of hrf which can accommodate decreasing, increasing, bathtub-shaped and unimodal hazard rates. Some statistical properties and estimation of the parameters of the L-L distribution using maximum likelihood method were studied by Tarvirdizade [6]. Also, it was shown that the L-L distribution provides a better fit than other three-parameter distributions for data with a bathtub-shaped hazard rate while its hrf is very simple in comparison with those of the competitor distributions. In this paper, we apply a Bayesian approach to estimate the parameters of the L-L distribution based on a complete sample. First the likelihood function of the parameters are presented and then, the Bayes estimates of the parameters based on symmetric and asymmetric loss functions are obtained.
The rest of the paper is organized as follows. In Section 2, the Bayes estimates of unknown parameters under the squared error, linear-exponential (LINEX) and Stein loss functions via the Metropolis-Hastings method are obtained. In Section 3, a real data set is analyzed to illustrate the estimation procedures discussed in the previous section. In Section 4, to investigate and compare the performance of the different methods of estimation presented in this paper, outcomes of a Monte Carlo simulation study are presented. Finally, some conclusions are provided in Section 5.

## 2. Bayesian Estimation

In this section, we discuss Bayesian estimation of the parameters of the L-L distribution under different loss functions. One of the most commonly used loss function is squared error loss (SEL) function which is given by $L(\Theta,\delta(X))=(\delta(X)-\Theta)^2$, where $\delta$ is a decision rule based on the data and $\Theta$ is the unknown parameter. The symmetric nature of SEL function gives equal weight to overestimation and underestimation of the parameters under consideration. However, in life testing, overestimation may be more serious than underestimation or vice versa. In these cases, the use of an asymmetric loss function which assigns greater importance to overestimation or underestimation may be more appropriate. For this purpose, Varian [8] proposed a convex but asymmetric loss function which is known as LINEX loss function and is defined as

$$L(\Theta,\delta)=e^{d(\delta-\Theta)}-d(\delta-\Theta)-1,\quad d\neq 0 \tag{3}$$

The shape parameter $d$ is known and gives the degree of asymmetry. If $d>0$, the overestimation is more serious than underestimation and if $d<0$, underestimation is more serious than overestimation. If $d$ close to zero, the LINEX loss is approximately SEL and therefore almost symmetric. Under the LINEX loss function (3), the Bayes estimator of $\Theta$ that minimizes the posterior risk $E\left[L(\Theta,\delta(X))|X\right]$ is given by

$$\delta_{BL}(X)=-\frac{1}{d}\log E\left(e^{-d\Theta}|X\right), \tag{4}$$

provided that the expectation exists and is finite. Another useful asymmetric loss function is the Stein loss function which is also known as entropy loss function. This loss function has the form

$$L(\Theta,\delta)=\frac{\delta}{\Theta}-\log\frac{\delta}{\Theta}-1, \tag{5}$$

This loss is a convex function of $\delta$ and more penalized underestimation than overestimation. Under this loss function the Bayes estimator $\delta_{BST}(X)$ that minimizes the posterior risk $E\left[L(\Theta,\delta(X))|X\right]$ is given by

$$\delta_{BST}(X)=\left\{E\left(\frac{1}{\Theta}|X\right)\right\}^{-1}, \tag{6}$$

provided that the expectation exists.

In Bayesian inference, we need to determine the likelihood function and prior distributions for unknown parameters. Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be $n$ observations of a random sample from the L-L$(\alpha, \beta, \theta)$. The likelihood function of this sample using (2) can be written as

$$L(\alpha, \beta, \theta|x) = \prod_{i=1}^{n} \left( \frac{\theta^2(1+x_i)(1+\beta x_i) + \alpha\beta(1+\theta+\theta x_i)}{1+\theta} \right) (1+\beta x_i)^{-(\alpha+1)} e^{-\theta x_i}, \tag{7}$$

The MLEs of $\alpha$, $\beta$ and $\theta$, say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$, can be obtained through the solution of the three nonlinear equations which are obtained by setting the first partial derivatives of the log-likelihood function with respect to $\alpha$, $\beta$ and $\theta$ equal to zero. These equations cannot be solved analytically and therefore, we have to solve the equations numerically. We can use iterative techniques such as a Newton-Raphson type algorithm to obtain the MLEs of the parameters $\alpha$, $\beta$ and $\theta$. To find the standard error of the MLEs of the parameters $\alpha$, $\beta$ and $\theta$, we can obtain the asymptotic variance-covariance matrix of the MLEs, which need to calculate the observed information matrix. The variance-covariance matrix $\mathbf{V}$ can be approximated by the reciprocal of the observed information matrix, i.e., $\mathbf{V} = I^{-1}$. Since $\mathbf{V}$ involves the parameters $\alpha$, $\beta$ and $\theta$, we replace the parameters by the corresponding MLEs in order to obtain an estimate of $\mathbf{V}$, which is denoted by

$$\hat{V} = \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} = \begin{pmatrix} \hat{I}_{11} & \hat{I}_{12} & \hat{I}_{13} \\ \hat{I}_{21} & \hat{I}_{22} & I_{23} \\ \hat{I}_{31} & \hat{I}_{32} & \hat{I}_{33} \end{pmatrix}^{-1}, \tag{8}$$

where $\hat{I}_{ij}$ is the $(i,j)$th element of the observed information matrix $I$ with $\alpha$, $\beta$ and $\theta$ replaced by $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$, respectively. The likelihood equations and the elements of the observed information matrix $I$ are given by Tarvirdizade [6].

For the prior distributions, we assumed that $\alpha$, $\beta$ and $\theta$ have independent gamma priors as $Gamma(\gamma_1, \lambda_1)$, $Gamma(\gamma_2, \lambda_2)$ and $Gamma(\gamma_3, \lambda_3)$, respectively, with the pdf's given by

$$\pi(\Theta|\gamma_i, \lambda_i) = \frac{\lambda_i^{\gamma_i} \Theta^{\gamma_i - 1} e^{-\lambda_i \Theta}}{\Gamma(\gamma_i)}, \quad \Theta > 0, \quad \gamma_i, \lambda_i > 0, \tag{9}$$

where $\Theta$ can be each of the parameters $\alpha$, $\beta$ and $\theta$ and the hyperparameters $(\gamma_i, \lambda_i)$, $i = 1, 2, 3$, are assumed to be known. Now, using (7) and (9), the joint posterior density function of $\alpha$, $\beta$ and $\theta$ can be written as

$$\begin{aligned} \pi(\alpha, \beta, \theta|\mathbf{x}) &= \frac{\pi(\alpha)\pi(\beta)\pi(\theta)L(\alpha, \beta, \theta|\mathbf{x})}{\int_\theta \int_\beta \int_\alpha \pi(\alpha)\pi(\beta)\pi(\theta)L(\alpha, \beta, \theta|\mathbf{x}) d\alpha d\beta d\theta} \\ &\propto \prod_{i=1}^{n} \left( \frac{\theta^2(1+x_i)(1+\beta x_i) + \alpha\beta(1+\theta+\theta x_i)}{(1+\theta)(1+\beta x_i)^{\alpha+1}} \right) \alpha^{\gamma_1 - 1} \beta^{\gamma_2 - 1} \theta^{\gamma_3 - 1} e^{-\lambda_1 \alpha - \lambda_2 \beta - (\lambda_3 + \sum_{i=1}^{n} x_i)\theta}. \end{aligned} \tag{10}$$

Since the above multiple integrals can not be solved analytically, the expression for $\pi(\alpha, \beta, \theta|\mathbf{x})$ in (10) can not be written in a closed form. Therefore, we need a simulation technique to compute the Bayes estimate of the parameters. We adopt the Gibbs sampling technique which requires a decomposition of the joint posterior distribution into full conditional distributions for each parameter and then sampling from them. The full conditional distributions of $\alpha$, $\beta$ and $\theta$ can be obtained as follows

$$\pi(\alpha|\beta, \theta, \mathbf{x}) \propto \prod_{i=1}^{n} \left( \frac{\theta^2(1+x_i)(1+\beta x_i) + \alpha\beta(1+\theta+\theta x_i)}{(1+\theta)(1+\beta x_i)^{\alpha+1}} \right) \alpha^{\gamma_1 - 1} e^{-\lambda_1 \alpha}, \tag{11}$$

$$\pi(\beta|\alpha, \theta, \mathbf{x}) \propto \prod_{i=1}^{n} \left( \frac{\theta^2(1+x_i)(1+\beta x_i) + \alpha\beta(1+\theta+\theta x_i)}{(1+\theta)(1+\beta x_i)^{\alpha+1}} \right) \beta^{\gamma_2 - 1} e^{-\lambda_2 \beta}, \tag{12}$$

$$\pi(\theta|\alpha,\beta,\mathbf{x}) \propto \prod_{i=1}^{n}\left(\frac{\theta^2(1+x_i)(1+\beta x_i)+\alpha\beta(1+\theta+\theta x_i)}{(1+\theta)(1+\beta x_i)^{\alpha+1}}\right)\theta^{\gamma_3-1}e^{-(\lambda_3+\sum_{i=1}^{n}x_i)\theta}. \tag{13}$$

The full conditional distributions for $\alpha$, $\beta$ and $\theta$ cannot be reduced analytically to well-known distributions and therefore it is not possible to sample directly by standard methods. In order to generate values of $\alpha$, $\beta$ and $\theta$ from (11)–(13), we use the Metropolis-Hastings algorithm into the Gibbs sampling algorithm as explained by Tierney [7]. Therefore, we can use the following algorithm for Gibbs sampling:

**Step 1.** Start with an initial guess $(\alpha^{(0)},\beta^{(0)},\theta^{(0)})$ and set t = 1.
**Step 2.** Using Metropolis-Hastings method, generate $\alpha^{(t)}$ from $\pi(\alpha|\beta^{(t-1)},\theta^{(t-1)},\mathbf{x})$ with the proposal distribution

$$q(\alpha) \propto N(\alpha^{(t-1)}, C_\alpha \hat{V}_{11})I(\alpha>0),$$

where $C_\alpha$ is a scaling factor, $\hat{V}_{11}$ is given in (8) and $N(\mu,\sigma^2)\ I(\alpha>0)$ denote the normal distribution $N(\mu,\sigma^2)$ truncated on $(0,\infty)$.
**Step 3.** Using Metropolis-Hastings method, generate $\beta^{(t)}$ from $\pi(\beta|\alpha^{(t)},\theta^{(t-1)},\mathbf{x})$ with the proposal distribution

$$q(\beta) \propto N(\beta^{(t-1)}, C_\beta \hat{V}_{22})I(\beta>0),$$

where $C_\beta$ is a scaling factor and $\hat{V}_{22}$ is given in (8).
**Step 4.** Using Metropolis-Hastings method, generate $\theta^{(t)}$ from $\pi(\theta|\alpha^{(t)},\beta^{(t)},\mathbf{x})$ with the proposal distribution

$$q(\theta) \propto N(\theta^{(t-1)}, C_\theta \hat{V}_{33})I(\theta>0),$$

where $C_\theta$ is a scaling factor and $\hat{V}_{33}$ is given in (8).
**Step 5.** Set t = t + 1.
**Step 6.** Repeat Steps 2–5, N times, and obtain the posterior sample $(\alpha^{(t)},\beta^{(t)},\theta^{(t)})$, $t=1,\ldots,N$.

After generating a sample from the above algorithm, we can compute the Bayes estimate of the parameters $\alpha$, $\beta$ and $\theta$ under the SEL, LINEX and Stein loss functions as follow

$$\tilde{\Theta}_{BS} = \hat{E}(\Theta|X) = \frac{1}{N-M}\sum_{i=M+1}^{N}\Theta^{(i)}, \tag{14}$$

$$\tilde{\Theta}_{BL} = -\frac{1}{d}\log\hat{E}(e^{-d\Theta}|X) = -\frac{1}{d}\log\left(\frac{1}{N-M}\sum_{i=M+1}^{N}e^{-d\Theta^{(i)}}\right), \tag{15}$$

$$\tilde{\Theta}_{BST} = \left\{\hat{E}\left(\frac{1}{\Theta}|X\right)\right\}^{-1} = \left\{\frac{1}{N-M}\sum_{i=M+1}^{N}\frac{1}{\Theta^{(i)}}\right\}^{-1}, \tag{16}$$

respectively, where $M$ is the burn-in period and $\Theta$ can be each of the parameters $\alpha$, $\beta$ and $\theta$.

## 3. Real Data Analysis

In this section, we analyze the real data to illustrate the use of our proposed estimation methods. We consider a real data set consist of the times between failures (in hours) of load-haul-dump (LHD) machine used to pick up rock or waste. The data has been used by Kumar et al. [3] and the TTT-plot presented by them for this data set exhibits a bathtub-shaped hrf. Tarvirdizade [6] has fitted the L-L distribution to this data with the corresponding MLEs as follows:

$$\hat{\alpha} = 0.4658, \quad \hat{\beta} = 0.0911, \quad \hat{\theta} = 0.0207.$$

To obtain Bayes estimates of the parameters $\alpha$, $\beta$ and $\theta$, we used small values of the hyperparameters as $(\gamma_1, \lambda_1) = (0.5, 0.25)$, $(\gamma_2, \lambda_2) = (0.75, 0.5)$ and $(\gamma_3, \lambda_3) = (1, 0.5)$ to reflect little prior information. We computed the Bayes estimates of the parameters based on $N = 10000$ MCMC samples and discard the first $M = 2000$ values as burn-in period. The simulated values and Histogram of the parameters $\alpha$, $\beta$ and $\theta$ generated by the algorithm of Gibbs sampling are plotted in Figure 1. Based on these simulated values, the Bayes estimate of the parameters $\alpha$, $\beta$ and $\theta$ under the SEL function using (14) are computed as

$$\tilde{\alpha}_{BS} = 0.23594, \quad \tilde{\beta}_{BS} = 0.52337, \quad \tilde{\theta}_{BS} = 0.02631.$$

From (15), the Bayes estimate of the parameters $\alpha$, $\beta$ and $\theta$ under the LINEX loss function for $d = 2$ are given as

$$\tilde{\alpha}_{BL} = 0.22338, \quad \tilde{\beta}_{BL} = 0.40158, \quad \tilde{\theta}_{BL} = 0.02628,$$

and for $d = -1$ are given as

$$\tilde{\alpha}_{BL} = 0.24318, \quad \tilde{\beta}_{BL} = 0.67735, \quad \tilde{\theta}_{BL} = 0.02632.$$

Also under the Stein loss function, the Bayes estimate of the parameters $\alpha$, $\beta$ and $\theta$ using (16) are given as

$$\tilde{\alpha}_{BST} = 0.18617, \quad \tilde{\beta}_{BST} = 0.28926, \quad \tilde{\theta}_{BST} = 0.02527.$$

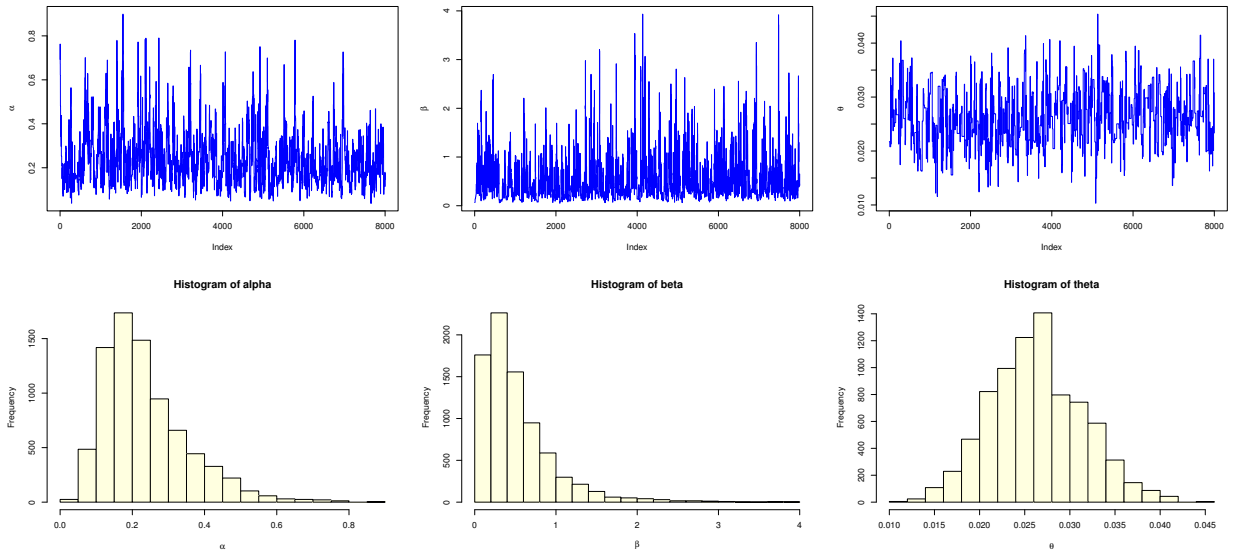


Fig. 1. Simulated values and Histogram of the parameters $\alpha$, $\beta$ and $\theta$

## 4. A Simulation Study

In this section, a Monte Carlo simulation study is conducted to investigate and compare the performance of the Bayes estimators presented in Section 2. In this simulation study we generate the samples of size $n = 20, 50, 100$ from the L-L distribution with different parameter combinations $(\alpha, \beta, \theta)$, namely (0.25, 1, 0.75), (1, 0.75, 0.5) and (0.5, 0.25, 1), respectively. To compute different Bayes estimates, we use the small hyperparameter values as $(\gamma_1, \lambda_1) = (0.5, 0.75)$, $(\gamma_2, \lambda_2) = (1, 0.1)$ and $(\gamma_3, \lambda_3) = (0.25, 1)$. We generate $N = 10000$ MCMC samples and discard the first $M = 2000$ values as burn-in period as described in Section 2. The performance of the Bayes estimators is

compared in terms of their estimated risk (ER). When $\theta$ is estimated by $\hat{\theta}$, the ER of $\theta$ under the SEL function is given by

$$ER_{BS}(\theta) = \frac{1}{T}\sum_{i=1}^{T}(\hat{\theta}_i - \theta)^2,$$

where $T$ is the number of replications and $\hat{\theta}_i$ is the estimate of $\theta$ in $i$th replication. Moreover, the ER of $\theta$ under the LINEX and Stein loss functions are given by

$$ER_{BL}(\theta) = \frac{1}{T}\sum_{i=1}^{T}\left(e^{d(\hat{\theta}_i-\theta)} - d(\hat{\theta}_i - \theta) - 1\right),$$

$$ER_{BST}(\theta) = \frac{1}{T}\sum_{i=1}^{T}\left(\frac{\hat{\theta}_i}{\theta} - \ln\frac{\hat{\theta}_i}{\theta} - 1\right),$$

respectively. We report the average estimates and ER of the parameters in Tables 1–3. All the results are reported based on 1000 replications.

Table 1. Average estimates and estimated risk (in parentheses) of the parameters with ($\alpha = 0.25$, $\beta = 1$, $\theta = 0.75$).

| $n$ | $\tilde{\alpha}_{BS}$ | $\tilde{\alpha}_{BL}(d=2)$ | $\tilde{\alpha}_{BL}(d=-1)$ | $\tilde{\alpha}_{BST}$ |
|---|---|---|---|---|
| 20 | 0.2934 (0.0321) | 0.1906 (0.0335) | 0.4882 (0.0879) | 0.0308 (1.6352) |
| 50 | 0.2748 (0.0158) | 0.2126 (0.0134) | 0.3632 (0.0183) | 0.0513 (1.2645) |
| 100 | 0.2674 (0.0098) | 0.2414 (0.0089) | 0.3247 (0.0116) | 0.0983 (0.9903) |
| $n$ | $\tilde{\beta}_{BS}$ | $\tilde{\beta}_{BL}(d=2)$ | $\tilde{\beta}_{BL}(d=-1)$ | $\tilde{\beta}_{BST}$ |
| 20 | 1.4165 (0.4626) | 1.0236 (0.1817) | 1.7563 (0.3308) | 0.8469 (0.1305) |
| 50 | 1.2207 (0.1368) | 0.9849 (0.1381) | 1.3597 (0.1329) | 0.8661 (0.1049) |
| 100 | 1.1084 (0.1143) | 0.9773 (0.1004) | 1.2068 (0.0580) | 0.8732 (0.0771) |
| $n$ | $\tilde{\theta}_{BS}$ | $\tilde{\theta}_{BL}(d=2)$ | $\tilde{\theta}_{BL}(d=-1)$ | $\tilde{\theta}_{BST}$ |
| 20 | 0.9378 (0.2132) | 0.6534 (0.1824) | 1.2182 (0.2180) | 0.4178 (0.3305) |
| 50 | 0.8301 (0.1357) | 0.6481 (0.1499) | 0.9913 (0.0979) | 0.4268 (0.2883) |
| 100 | 0.7798 (0.0833) | 0.6609 (0.1152) | 0.8781 (0.0608) | 0.4347 (0.2007) |

Table 2. Average estimates and estimated risk (in parentheses) of the parameters with ($\alpha = 1$, $\beta = 0.75$, $\theta = 0.5$).

| $n$ | $\tilde{\alpha}_{BS}$ | $\tilde{\alpha}_{BL}(d=2)$ | $\tilde{\alpha}_{BL}(d=-1)$ | $\tilde{\alpha}_{BST}$ |
|---|---|---|---|---|
| 20 | 0.8234 (0.6809) | 0.3237 (0.7125) | 1.4203 (0.4035) | 0.2116 (1.3978) |
| 50 | 0.6770 (0.3110) | 0.3188 (0.4529) | 0.9065 (0.1803) | 0.2906 (1.0027) |
| 100 | 0.5836 (0.2655) | 0.3119 (0.3672) | 0.7811 (0.1144) | 0.3883 (0.8403) |
| $n$ | $\tilde{\beta}_{BS}$ | $\tilde{\beta}_{BL}(d=2)$ | $\tilde{\beta}_{BL}(d=-1)$ | $\tilde{\beta}_{BST}$ |
| 20 | 1.4824 (0.6326) | 0.9346 (0.2514) | 1.8847 (0.6891) | 0.8619 (0.1406) |
| 50 | 1.3913 (0.5805) | 0.9801 (0.2032) | 1.7036 (0.6329) | 0.8932 (0.1185) |
| 100 | 1.2703 (0.5132) | 1.0033 (0.1856) | 1.6330 (0.5905) | 0.9707 (0.0902) |
| $n$ | $\tilde{\theta}_{BS}$ | $\tilde{\theta}_{BL}(d=2)$ | $\tilde{\theta}_{BL}(d=-1)$ | $\tilde{\theta}_{BST}$ |
| 20 | 0.9350 (0.3302) | 0.6339 (0.1494) | 1.2567 (0.4037) | 0.3662 (0.4103) |
| 50 | 0.6037 (0.0991) | 0.5816 (0.0883) | 0.8704 (0.1103) | 0.3392 (0.3292) |
| 100 | 0.5379 (0.0680) | 0.4843 (0.0534) | 0.5889 (0.0809) | 0.2955 (0.1782) |

Table 3. Average estimates and estimated risk (in parentheses) of the parameters with ($\alpha = 0.5, \beta = 0.25, \theta = 1$).

| $n$ | $\tilde{\alpha}_{BS}$ | $\tilde{\alpha}_{BL}(d=2)$ | $\tilde{\alpha}_{BL}(d=-1)$ | $\tilde{\alpha}_{BST}$ |
|---|---|---|---|---|
| 20 | 0.2256 (0.1268) | 0.1436 (0.2330) | 0.3239 (0.0835) | 0.0089 (1.5204) |
| 50 | 0.2404 (0.0953) | 0.1561 (0.2108) | 0.2889 (0.0661) | 0.0137 (1.1046) |
| 100 | 0.2697 (0.0824) | 0.1845 (0.1784) | 0.2807 (0.0457) | 0.0188 (0.9283) |
| $n$ | $\tilde{\beta}_{BS}$ | $\tilde{\beta}_{BL}(d=2)$ | $\tilde{\beta}_{BL}(d=-1)$ | $\tilde{\beta}_{BST}$ |
| 20 | 0.4417 (0.0596) | 0.3932 (0.0903) | 0.4725 (0.0326) | 0.2813 (0.1308) |
| 50 | 0.3789 (0.0321) | 0.3510 (0.0473) | 0.3883 (0.0158) | 0.2846 (0.0976) |
| 100 | 0.3514 (0.0156) | 0.3255 (0.0274) | 0.3606 (0.0079) | 0.2868 (0.0858) |
| $n$ | $\tilde{\theta}_{BS}$ | $\tilde{\theta}_{BL}(d=2)$ | $\tilde{\theta}_{BL}(d=-1)$ | $\tilde{\theta}_{BST}$ |
| 20 | 1.0283 (0.1234) | 0.8553 (0.1573) | 1.1325 (0.0814) | 0.6728 (0.1732) |
| 50 | 1.0147 (0.0704) | 0.8945 (0.0956) | 1.0982 (0.0456) | 0.7743 (0.0909) |
| 100 | 0.9904 (0.0438) | 0.9134 (0.0686) | 1.0443 (0.0259) | 0.7951 (0.0737) |

From Tables 1–3, it can be seen that the ERs of all estimates decrease as the sample sizes increase in all cases, as expected. The results shows that the performances of the Bayes estimators under various loss functions are different. Also, it is clear that the Bayes estimates under the LINEX loss function are sensitive to the values of the shape parameter $d$. It can be observed that the performances of Bayes estimates under the LINEX loss function for small values of $d$ become close to the performances of Bayes estimates under the SEL function, as expected. In most of the cases, the performances of Bayes estimates under Stein loss function are not suitable especially in the estimation of parameter $\alpha$.

## 5. Conclusions

In this paper, the problem of Bayesian inference on the parameters of a three-parameters distribution, called the Lomax-Lindley distribution, is considered. Bayesian estimation of the parameters under SEL, LINEX and Stein loss functions are obtained by using the MCMC methods. A real lifetime data set is analyzed for illustrative purposes. Finally, to investigate and compare the performance of different types of estimators presented in this paper a Monte Carlo simulation study is conducted. Based on simulation results, we observed that the performance of the Bayes estimators are different under various loss functions and the Bayes estimates under the LINEX loss function are sensitive to the values of the shape parameter $d$. It is clear that the performance of the Bayes estimators under SEL and LINEX loss functions are better than the Bayes estimators under Stein loss function in most of the cases.

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# Supercyclicity of a $C_0$-semigroup and Its Corresponding Left Multiplication $C_0$-semigroup

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**Abstract**

Corresponding to each $C_0$-semigroup $(T_t)_{t\geq 0}$ on a Hilbert space $H$, we have the left multiplication $C_0$-semigroup $(L_{T_t})_{t\geq 0}$ on $B_2(H)$, the algebra of Hilbert-Schmidt operators. In this paper, we investigate the supercyclicity of $C_0$-semigroups and some relations between the supercyclicity of a $C_0$-semigroup and its left multiplication $C_0$-semigroup. We show that for a $C_0$-semigroup $(T_t)_{t\geq 0}$ on a Hilbert space $H$ satisfying the supercyclicity criterion is equivalent to that $(L_{T_t})_{t\geq 0}$, satisfies the supercyclicity criterion on $B_2(H)$, with $\|.\|_2$-topology.
We state some conditions for supercyclicity of the direct sum of a $C_0$-semigroup with itself. We prove that if $L_{T_t}$ is supercyclic on $B_2(H)$ for some $t > 0$ or $(L_{T_t})_{t\geq 0}$ is supercyclic on $B_2(H)$, then $(T_t \oplus T_t)_{t\geq 0}$ is supercyclic. We state that the supercyclicity of $(T_t)_{t\geq 0}$ is concluded from the supercyclicity of $(T_t \oplus T_t)_{t\geq 0}$. Also, we establish that if for some $t_0 > 0$, we have $\overline{\cup_{n=1}^{\infty} ker(T_{t_0}^n)} = H$ and $T_{t_0}$ has a dense range, then $(T_t)_{t\geq 0}$ is supercyclic.

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# Imaging Time Series using Recurrence Plots

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**Abstract**

The present paper aims to familiarize readers with recurrence plot, a tool for analyzing dynamical systems by visualizing the recurrences of states in time series data. The recurrence plot transforms an observed time series into an image and it provides new approaches to estimate various nonlinear dynamical characteristics such as the Lyapunov exponent, information dimension, and correlation dimension. In order to help better understand, the recurrence plots of some synthetic and a real-world time series data are presented using the Python package `pyts`.

## 1. Introduction

The states of natural or technical systems frequently undergo changes over time, sometimes in a rather intricate manner. The study of such complex dynamics is an important task within numerous scientific disciplines and their applications. It is important for our daily life to understand, describe, and predict these changes [9]. Forecasting the weather, oil price or mortality rate are just three examples among many.

Recurrence plots have been initially introduced by Eckmann et al. [2] as a visualization tool for the analysis of complex systems. More recently, these plots have proven to be a powerful technique to quantify and reveal important dynamical features in time series data. The recurrence plot is a simple visual and easily estimable tool to characterize the dynamics of a system. It is based on the observed time series and it provides new approaches to estimate various nonlinear dynamical characteristics such as the Lyapunov exponent, information dimension, and correlation dimension [5]. The approaches based on recurrence plots are called *recurrence quantification analysis*. An important advantage of these approaches over other nonlinear techniques is that they perform reasonably well for short and non-stationary time series [5].

The recurrence plot-based approaches have been widely used in a variety of fields such as finance, physiology, cognitive science, information technology, neuroscience, music, geophysics, astrophysics and engineering. One of interesting applications of recurrence plots is the classification of time series. For this purpose, first, the time series is transformed into an image using the recurrence plot. Then, deep learning methods, especially convolutional neural

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networks, are used to classify this imaged time series [6]. The review of all applications of recurrence plot-based approaches is beyond the scope of this paper, thus we refer interested readers to [5, 9].
The aim of this article is to familiarize readers with recurrence plot. For this purpose, first, the theory of this technique is briefly reviewed and then, the recurrence plots of some synthetic and a real-world data are presented. There are many software and programs available for creation and application of recurrence plots and their quantitative analysis, which some of them have been listed in [8]. In this paper, all calculations related to recurrence plot have been performed using the free available Python package `pyts`. More details on this package can be found in [4].
The rest of the paper is organized as follows. A brief introduction to dynamical system is presented in Section 2. The method of reconstruction of state space is explained in Section 3. Section 4 is dedicated towards introducing recurrence plots and in Section 5, the recurrence plots of some synthetic and a real-world time series are presented.

## 2. What Is a Dynamical System?

The set of all possible states of a system is called ***state*** (or ***phase***) space and denoted by $\mathcal{X}$. A dynamical system is given by following items:

1) a state space,
2) a continuous or discrete time,
3) a time evolution law.

Assume $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_d(t))^T$ is the state of a system at a fixed time $t$ that is specified by $d$ components in the $d$-dimensional state space of the system. The vectors $\vec{x}(t)$ define a trajectory in state space and the elements of them could be, e.g., temperature, wind speed, air pressure, vapor pressure, relative humidity. The development of the systems is described by these vectors.
In a broad context, the time-evolution law is a rule that allows determining the state of the system at each moment of time $t$ from its states at all previous times. However, often these rules are restricted to time-evolution laws that enable calculating all future states given a state at any particular moment [9].
Formally, the time-evolution law of a dynamical system can be formulated in terms of either (i) a system of differential equations that result in ***flows***, that is, continuous time-evolving trajectories, or (ii) a system of discrete iterative rules often called ***maps*** [5]. For continuous-time systems, the time-evolution law (or flow) is given by a set of differential equations [9]:

$$\dot{\vec{x}}(t) = \vec{F}\left(\vec{x}(t); \theta\right), \tag{1}$$

where $\vec{F} : \mathcal{X} \to \mathcal{X}$ is a $d$-dimensional function, which maps every point in $\mathcal{X} \subseteq \mathbb{R}^d$ to another point in $\mathcal{X}$ itself, $\theta$ is the set of parameters of the system, and $\dot{\vec{x}}(t)$ indicates the first derivative of vector $\vec{x}(t)$ with respect to time, that is,

$$\dot{\vec{x}}(t) = \frac{d}{dt}\vec{x}(t) = \left(\frac{d}{dt}x_1(t), \frac{d}{dt}x_2(t), \dots, \frac{d}{dt}x_d(t)\right)^T.$$

Without loss of generality, Eq.(1) can be used to denote the dynamic of any flow. For example, the Rössler model is described by the following differential equations system:

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c), \end{aligned} \tag{2}$$

where $\theta = \{a, b, c\}$ is the set of parameters of the system. This model can be written in the form of Eq.(1) by setting $\vec{x}(t) = (x(t), y(t), z(t))^T \in \mathcal{X} \subseteq \mathbb{R}^3$ and $\vec{F}\left(\vec{x}(t); \theta\right) = (-y - z, x + ay, b + z(x - c))^T \in \mathcal{X} \subseteq \mathbb{R}^3$.
For discrete-time systems, the generic form of the time-evolution law (or map) is given by discrete iterative rules as follows [5]:

$$\vec{x}_{t+1} = \vec{F}\left(\vec{x}_t; \theta\right). \tag{3}$$

For example, the Hénon map is described by following set of discrete-time functions:

$$\begin{aligned} x_{t+1} &= 1 - ax_t^2 + y_t, \\ y_{t+1} &= bx_t, \end{aligned} \tag{4}$$

where $x_t$ denotes the value of $x$ at time $t$ and $\theta = \{a, b\}$ is the set of parameters of the system. This map can be written in the form of Eq.(3) by setting $\vec{x}_t = (x_t, y_t)^T \in \mathcal{X} \subseteq \mathbb{R}^2$ and $\vec{F}(\vec{x}_t; \theta) = \left(1 - ax_t^2 + y_t, bx_t\right)^T \in \mathcal{X} \subseteq \mathbb{R}^2$.
The function $\vec{F}$, which is called ***dynamic***, indicates how to move from the state at a time $t$ to a next time $t + \delta t$ (continuous time) or $t + 1$ (discrete time). The state space $\mathcal{X}$ and the dynamic $\vec{F}$ together form a dynamical system and denoted by $\left(\mathcal{X}, \vec{F}\right)$.

## 3. Reconstruction of State Space

In experimental settings, it is often difficult to construct the state vector ($\vec{x}(t)$ or $\vec{x}_t$) because not all relevant components are known or cannot be measured. Frequently, we are confronted with a discrete-time measurement of only one observable. This produces a univariate discrete time series $\{u_t\}_{t=1}^N$ [9]. In such a case, the state space has to be reconstructed [10]. However, in what approaches can high dimensional dynamics be analyzed utilizing only univariate time series?
A commonly used method for the reconstruction of state space is the time-delay embedding approach that first proposed by Takens in 1981 [13]. In this approach, a delay coordinates map $\vec{G} : \mathcal{X} \to \mathbb{R}^m$ is defined, such that $m < d$, $\vec{G}(\vec{x}_t) = \vec{y}_t \in \mathbb{R}^m$, where

$$\vec{y}_t = \left(u_t, u_{t+\tau}, u_{t+2\tau}, \dots, u_{t+(m-1)\tau}\right)^T, \tag{5}$$

and $m$ is the ***embedding dimension*** and $\tau$ is the ***time delay***. Indeed, the embedding $\vec{G}$ maps a point $\vec{x}_t$ in $d$-dimensional state space $\mathcal{X}$ to a point $\vec{y}_t$ of the reconstructed state space in $\mathbb{R}^m$ with lower $m$ dimension. It has been shown that it is possible to investigate and conceptualize the equilibrium dynamics of a high-dimensional system $\left(\mathcal{X}, \vec{F}\right)$ through the utilization of a significantly reduced-dimensional embedding $\vec{G}$ [12, 13].

## 4. Recurrence Plots

In 1987, the *recurrence plots* were introduced by Eckmann et al. [2] to visualize the recurrences of dynamical systems. Let $\{\vec{x}_i\}_{i=1}^N$ be a trajectory of the dynamical system $\left(\mathcal{X}, \vec{F}\right)$. It is said that the system recurs when a state $\vec{x}_i$ at time $t = i$ is *approximately close* to a different state $\vec{x}_j$ at time $t = j$, that is, $\vec{x}_i \approx \vec{x}_j$ [5]. It is noteworthy that the notion *'approximately close'* is not clear. In order to eliminate ambiguity, a norm (e.g. the Euclidean norm, $L_1$ norm or the maximum norm) and a distance threshold should be determined.
The recurrence plot of the system is based on the recurrence matrix **R** that its elements are defined as follows:

$$R_{ij} = \begin{cases} 1 & , \vec{x}_i \approx \vec{x}_j \\ 0 & , \vec{x}_i \not\approx \vec{x}_j \end{cases} \qquad i, j = 1, 2, \dots, N. \tag{6}$$

Formally, the entries of recurrence matrix in (6) can be expressed as follows:

$$R_{ij} = \Theta\left(\varepsilon - \left\|\vec{x}_i - \vec{x}_j\right\|\right), \qquad i, j = 1, 2, \dots, N, \tag{7}$$

where $\|\cdot\|$ is a proper norm, $\varepsilon$ is distance threshold and $\Theta$ is the Heaviside function defined as follows:

$$\Theta(x) = \begin{cases} 1 & , x \geq 0 \\ 0 & , x < 0 \end{cases}. \tag{8}$$

It is important to note that the distance threshold $\varepsilon$ is necessary as systems often do not recur exactly to a previously visited state but just approximately [9]. The recurrence matrix **R** consists of 1s and 0s and compares the states of a

system at times $i$ and $j$. In this matrix, the 1s indicate pairs of time points where the states are similar and 0s denote pairs of time points where the states are rather different. Therefore, the matrix **R** reveals that when similar states of the underlying system occur [9]. This matrix is symmetric and all elements of its main diagonal are equal to one ($\vec{x}_i = \vec{x}_i, i = 1, 2, \dots, N$). It has been proved that, given a recurrence matrix **R**, the time evolution of the dynamical system can be reconstructed up to a change in the coordinate system [11].
As mentioned in Section 3, in practice, often the state vector $\vec{x}_t$ can not be measured. Instead of that, only a univariate time series $\{u_t\}_{t=1}^{N}$ is measured. In this case, first the time series is embedded using time-delay embedding approach (Section 3) to obtain an embedded state-space trajectory $\{\vec{y}_t\}_{t=1}^{K}$, where $K = N-(m-1)\tau$. Then, the recurrence matrix **R** is estimated from the embedded trajectory $\vec{y}_t$, which is defined in (5). In other words, the elements of recurrence matrix **R** are estimated as follows:

$$\widehat{R}_{ij} = \Theta\left(\varepsilon - \left\|\vec{y}_i - \vec{y}_j\right\|\right), \qquad i, j = 1, 2, \dots, K, \tag{9}$$

where $\widehat{R}_{ij}$ indicates the estimation of $R_{ij}$ defined in (7). We denote the estimated recurrence matrix by $\widehat{\mathbf{R}}$.
A recurrence plot is a visualization of the square $K \times K$ recurrence matrix $\widehat{\mathbf{R}}$. In this plot, black and white dots correspond to 1 and 0, respectively. The recurrence plot enables us to investigate the $d$-dimensional state space trajectory using a two-dimensional representation of its recurrences[8]. Reconstruction of time series is another interesting application of recurrence plots [7].

## 5. Recurrence Plots of some Synthetic and Real-World Data

In this section, the recurrence plots of some synthetic and a real-world data are presented, based on the Euclidean norm. For the first example, consider $N = 200$ random samples drawn from standard normal (Gaussian) distribution. The time series plot of these samples is shown on the top of Figure 1. Also, the recurrence plots are depicted on the middle and bottom of this figure for different values of embedding dimension ($m$) and time delay ($\tau$), where the distance threshold equals to 0.5 ($\varepsilon = 0.5$). It can be easily seen from Figure 1 that there is not a recognizable pattern in the recurrence plots for $m = 2$. In addition, the black anti-diagonal dots of recurrence plots disappear as $m$ increases.

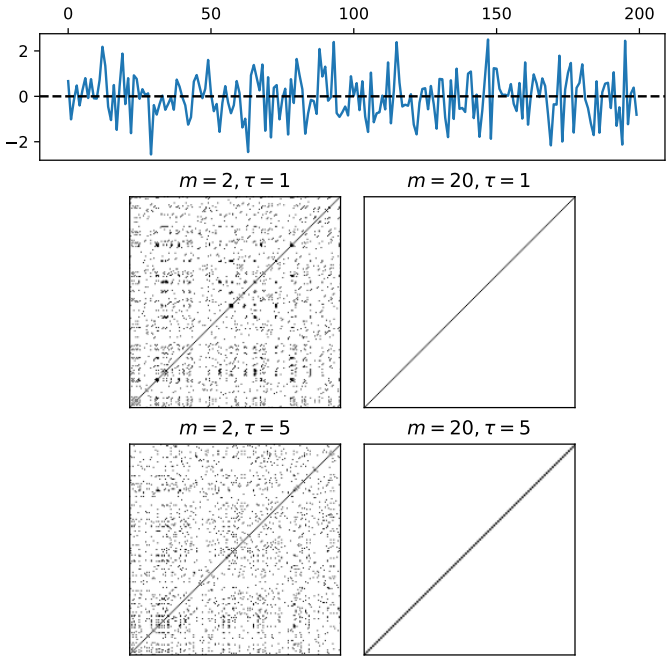


Fig. 1. The time series and recurrence plots of 200 random samples from standard normal distribution.

For the second example, consider $N = 1000$ data points of deterministic sine function, $u_t = \sin(t), \quad 0 \le t \le 4\pi$, where $\varepsilon = \frac{\pi}{18}$. The time series plot and recurrence plots of deterministic sine series are shown in Figure 2. As can be seen in this figure, the recurrence plots have distinguishable patterns, especially for $m = 2$.

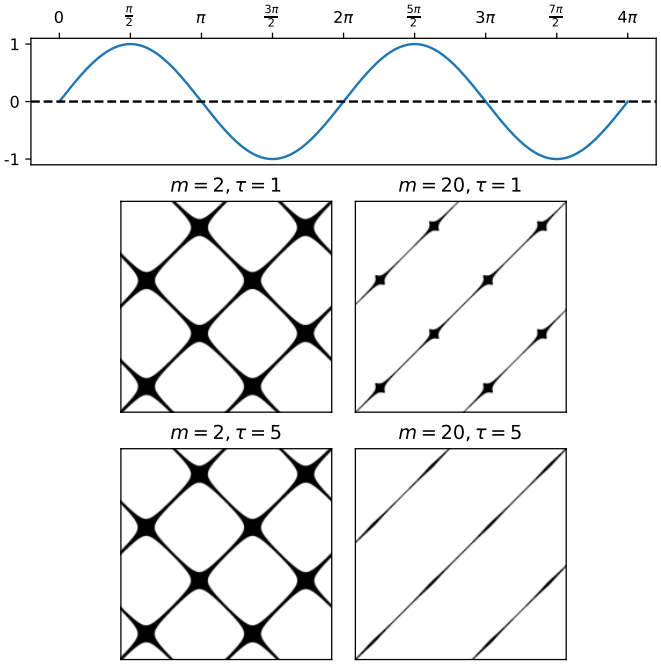


Fig. 2. The time series and recurrence plots of deterministic sine series.

Now, let us add some noise that has a normal distribution with a mean of zero and a standard deviation of 0.2 to the sine series. In other words, $N = 1000$ samples are generated from the following stochastic sine series:

$$u_t = \sin(t) + n_t, \quad 0 \le t \le 4\pi, \tag{10}$$

where $\varepsilon = \frac{\pi}{18}$ and $n_t \sim N(0, 0.2)$. Figure 3 shows the time series plot and recurrence plots of this series. This figure is slightly similar to Figure 2, with the difference that due to the presence of noise in the data, the patterns are somewhat distorted.

For the forth example, the sum of two sinusoidal functions with different frequencies are considered. In other words, the $N = 1000$ data points of following model are obtained:

$$u_t = \sin(t) + \sin(3t), \quad 0 \le t \le 4\pi, \tag{11}$$

where $\varepsilon = \frac{\pi}{18}$. The time series plot and recurrence plots of this series are depicted in Figure 4. Again, the recurrence plots have obvious patterns, especially for $m = 2$.

What would happen if some noise were added to the time series defined in (11)? In order to answer this question, we add Gaussian noise with a mean of zero and a standard deviation of 0.2 to the sum of two sine series defined in (11). In other words, $N = 1000$ samples are generated from the following stochastic sum of two sine series:

$$u_t = \sin(t) + \sin(3t) + n_t, \quad 0 \le t \le 4\pi, \tag{12}$$

where $\varepsilon = \frac{\pi}{18}$ and $n_t \sim N(0, 0.2)$. Figure 5 depicts the time series plot and recurrence plots of this series. As expected, this figure is somewhat similar to Figure 4, with the difference that due to the presence of noise in the data, the patterns have become slightly distorted.

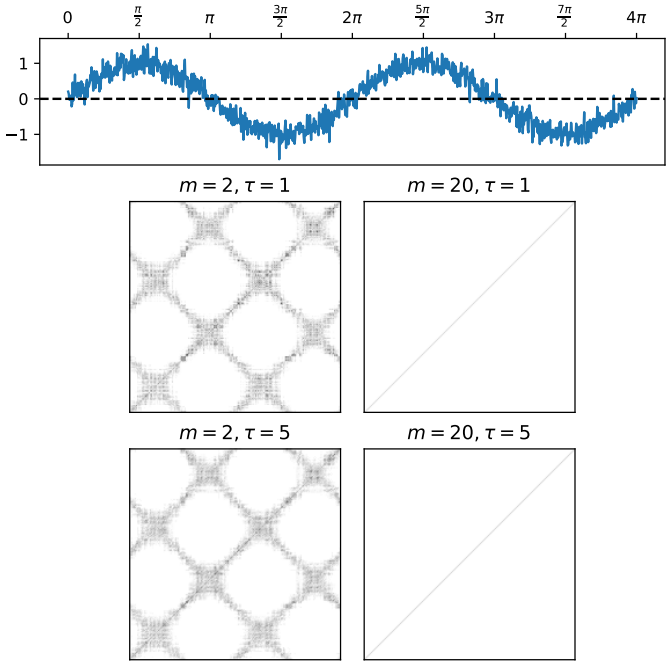


Fig. 3. The time series and recurrence plots of stochastic sine series.

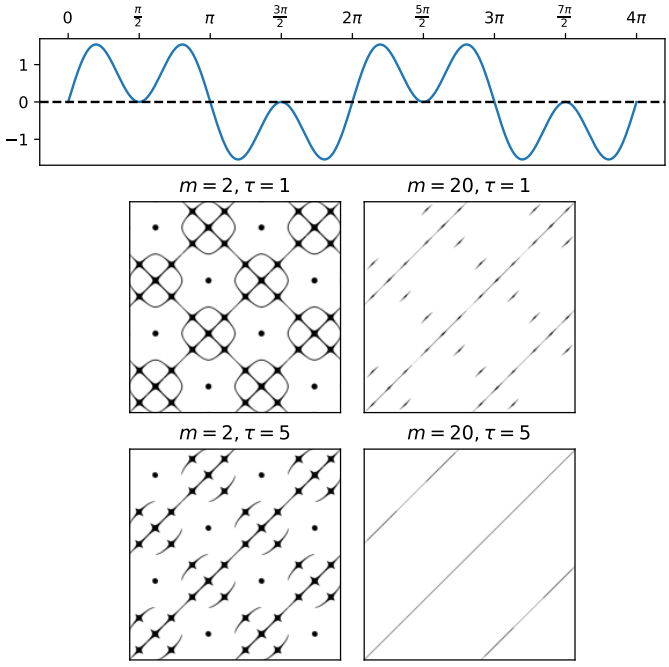


Fig. 4. The time series and recurrence plots for the sum of two sine series with different frequencies.

Now, let us consider a weather time series dataset recorded at the weather station at the Max Planck Institute for Biogeochemistry in Jena, Germany [3]. This dataset includes fourteen different variables (such as temperature, airtight, pressure, humidity, wind direction, and so on) that were recorded every 10 minutes over several years [1]. The original

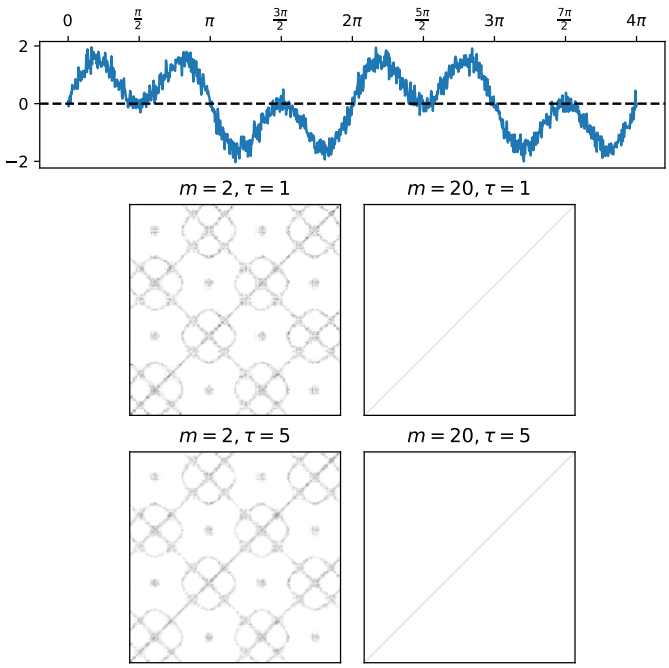


Fig. 5. The time series and recurrence plots for the stochastic sum of two sine series with different frequencies.

data goes back to 2003, but we use the subset of the data limited to 2009–2016. In particular, we utilize only monthly mean temperature (in degrees Celsius) in this paper from 2009 to 2016. Figure 6 shows the time series plot and recurrence plots of this time series, where $\varepsilon = 7$. Notice how the periodic pattern in the data is also reflected in this figure. Also, the pattern of recurrence plots disappears as $m$ increases.

As can be seen in Figures 1-6, the pattern appeared in the recurrence plot highly depends on the value of three parameters: embedding dimension ($m$), time delay ($\tau$), and distance threshold ($\varepsilon$). Therefore, choosing these parameters correctly is very important. Until now, there are not unique rigorous mathematical methods to determine $m$, $\tau$, and $\varepsilon$. Consequently, various heuristic methods are used to determine these parameters. For more information on these methods see [5, 9].

As previously mentioned, the recurrence plot can visualize trajectories in state space and it yields important insights into the time evolution of these trajectories. It is noteworthy that typical patterns in recurrence plots are linked to a specific behavior of the dynamical system. There are very diverse patterns in recurrence plots that reviewing all of them is beyond the purpose of this paper. More details on this topic can be found in [9].

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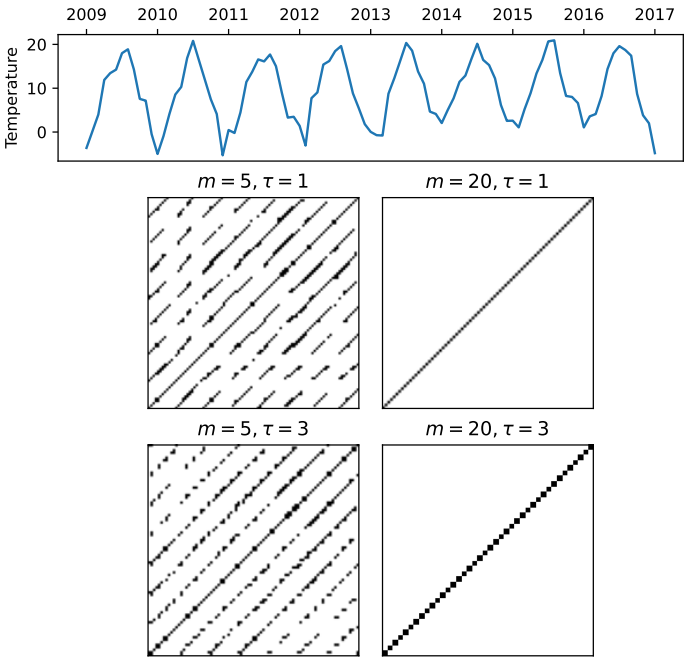


Fig. 6. The time series and recurrence plots of monthly mean temperature.

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The 5th National Congress on Mathematics and Statistics, AS–180, pp. 111–115

# Empirical assessment of two time series forecasting methods: A case study on road traffic accident

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Time series

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**Abstract**

Singular Spectrum Analysis (SSA) and Holt Winters (HW) methods are frequently utilized for time series forecasting, which typically involves a comparative assessment of their accuracy and appropriateness for the specific dataset being analysed. In this study, we applied both methods to forecast road traffic accidents in Golestan province for the years 1400 and 1401 as a case study. The objective is to assess and distinguish between these techniques by exploring their individual strengths and limitations through the calculation of the root mean squared error (RMSE) and the mean absolute percentage error (MAPE).

## 1. Introduction

Singular Spectrum Analysis (SSA) and Holt Winters (HW) are both effective methodologies for the analysis and forecasting of time series data, yet they differ significantly in their approaches ([1], [2]). SSA is a non-parametric technique that breaks down a time series into its fundamental components: trend, seasonal variations, and noise. This method is especially advantageous for dealing with non-linear and non-stationary time series, and there has been a growing interest in its application due to its promising predictive capabilities across various domains. Moreover, SSA is adept at managing missing data, exhibits robustness against noise, and possesses the flexibility to capture intricate patterns while effectively isolating different components of the time series. Holt Winters (HW) is widely recognized as one of the leading methods for forecasting in time series analysis, in contrast to the SSA approach ([1], [3]). This technique is extensively utilized across various time series applications, including monitoring tasks such as anomaly detection and capacity planning. HW enhances the analysis of time series data by incorporating level, trend, and seasonality through an updating equation, which facilitates multi-step forecasting via a straightforward linear combination of these components. Additionally, HW method is particularly effective for seasonal data, enabling the modelling of both trend and seasonality. Moreover, its advantages include simplicity and ease of implementation, making it a well-established methodology with clear interpretations, especially effective for data exhibiting strong seasonal patterns. Nevertheless, regarding to the SSA method, its limitations include the assumption of a constant

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seasonal pattern,which may not be applicable to all datasets, and its sensitivity to outliers and data anomalies ([4], [5]). A notable application of both methods lies in forecasting daily road traffic accidents, a responsibility that is becoming increasingly vital in contemporary society. In this study, we employed both methodologies to forecast road traffic accidents in Golestan province for the years 1400 and 1401 as a case study. The objective is to conduct an empirical evaluation of these time series forecasting methodologies by analysing their individual strengths and weaknesses through the computation of the root mean squared error (RMSE) and the mean absolute percentage error (MAPE) [6].

## 2. Forecasting Methods

### *2.1. Singular Spectrum Analysis (SSA)*

SSA is a model-independent method employed for the purposes of separating sources in the analysis of time series data. It consists of four key stages: embedding, Singular Value Decomposition (SVD), eigentriple grouping, and reconstruction. The SSA algorithm can be articulated in the following manner:
Consider a time series denoted as $X_t$ where $t = 1, 2, ..., N$. An integer number L is selected such that $1 < L < N$ referred to as the window length. Let $K = N - -L + 1$ and define, $W_j = [X_j, ..., X_{j+L--1}]^T$ for $j = 1, ..., K$ ([1], [2]). The trajectory matrix $W$ is formed as $W = [X_1, ..., X_k]$.
SVD of matrix $R = WW^T$ with dimension $L \times L$ is computed. Let $\lambda_1, ..., \lambda_L$ represent the eigenvalues arranged in descending order, such that $\lambda_1 \geq \cdots \geq \lambda_L$, and $U_1, ..., U_L$ being the corresponding eigenvectors. At the final step the forecasting values of $X_t$ can be found based on the diagonal averaging algorithm transforms $W$ into the reconstructed time series $\hat{X_1}, ..., \hat{X_N}$.

### *2.2. Holt Winters (HW)*

The HW method, often known as triple exponential smoothing, is utilized for forecasting time series data that exhibit both trends and seasonal fluctuations. This technique consists of three fundamental components: level, which denotes the mean value of the series; trend, reflecting the directional movement over time; and seasonality, which identifies the repetitive patterns occurring at consistent intervals. The method employs smoothing equations to refine these components as time advances. For a given time series $X_t$, the updating equations for the HW method are outlined below [3].

$$a_t = \theta(X_t - s_{t-p}) + (1 - \theta)(a_{t-1} + b_{t-1}) \tag{1}$$

$$b_t = \beta(a_t - a_{t-1}) + (1 - \beta)b_{t-1} \tag{2}$$

$$s_t = \gamma(X_t - a_t) + (1 - \gamma)s_{t-p} \tag{3}$$

where $a_t$, $b_t$ and $s_t$ are the smoothed estimates of level, trend, and seasonality at time $t$, respectively. The parameters $\theta$, $\beta$ and $\gamma$ serve as smoothing coefficients and $p$ represents a chosen duration.. These coefficients differentiate the significance of recent observations from those that are older, with the influence of older observations diminishing exponentially. The k-step forecasting equation for $X_{m+k}$ at time $m$ and for $k \leq p$ is:

$$\hat{X}_{m+k|m} = a_m + kb_m + s_{m+k-p}k \tag{4}$$

## 3. Real time series data

The retrospective analysis encompasses all road traffic incidents documented by the Emergency Medical Services (EMS 115) from 1400 to (1401 in Golestan province, situated in northeastern Iran. During this two-year time frame, EMS 115 recorded a total of 37,409 accidents. An examination of daily time-series data can yield significant insights into the trends, patterns, and possible influences on traffic accidents within the study region. The year variable indicates the specific year during which each accident occurred, ranging from Farvardin, the first month of the year in Persian, 1400 to Esfand, the final month of the year in Persian, 1401. Table 1 presents the cumulative count of traffic accidents occurring each month in Golestan province from Farvardin 1400 to Esfand 1401, with the month names referred to in Persian.

Table 1. The cumulative count of road traffic accidents occurring each month in Golestan province from Farvardin 1400 to Esfand 1401, with the month names referred to in Persian.

| Month<br>Year | Far. | Ord. | Kor. | Tir | Mor. | Sha. | Meh. | Aba. | Aza. | Day | Beh. | Esf. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1400 | 1193 | 1202 | 1667 | 1487 | 1231 | 1347 | 1490 | 1390 | 1418 | 1266 | 1130 | 1196 |
| 1401 | 1614 | 1711 | 1914 | 2221 | 1889 | 2009 | 2079 | 1743 | 1523 | 1496 | 1564 | 1630 |

### *3.1. Road traffic accidents analysis*

We utilize SSA and HW time series forecasting methods on our dataset. Their performance is evaluated based on various error statistics, including Root Mean Square Error (RMSE) and Mean Absolute Percentage Error (MAPE). The calculations for these two error statistics are conducted as follows.

$$RMSE = \sqrt{\frac{\sum_{t=k+1}^{t}(X_t - \hat{X}_t)^2}{(n-k)}} \tag{5}$$

$$MAPE = (\frac{1}{k}\sum_{t=1}^{k}\frac{X_t - \hat{X}_t}{X_t}) \times 100 \tag{6}$$

where $X_t$ and $\hat{X}_t$ are the actual observed and the predicted value respectively for all time $t$, $n$ is the total number of observations and a samples size of $k < n$. RMSE effectively quantifies the square root of the mean of the squared discrepancies between predicted and actual values. A lower RMSE signifies superior model performance. Similarly, MAPE offers a percentage error metric, facilitating interpretation across various scales. Once more, lower values reflect enhanced accuracy.

### *3.2. Empirical assessment*

SSA offers greater flexibility for analysing complex time series that exhibit irregular patterns, making it particularly suitable for exploratory data analysis and cases where seasonality is not consistently periodic. In contrast, the HW method is favoured for time series characterized by pronounced seasonal trends, as it aims to generate forecasts based on recognized patterns. When a time series displays distinct trends and seasonal behaviours, HW may be the more appropriate option. However, for those dealing with intricate time series that may include non-linearities or irregular seasonal variations, SSA should be considered. The efficacy of the HW approach in capturing stable seasonal patterns is often observed in contexts with reliable production cycles. Conversely, SSA may demonstrate superior performance in situations where production is subject to unpredictable external influences, offering enhanced adaptability and precision. Ultimately, the selection between SSA and Holt Winters should be guided by the specific characteristics of the time series, the existence of seasonal elements, and the computational resources at hand. Further, Table 2 shows that the SSA method outperforms the HW method with respect to both RMSE and MAPE. A MAPE value of less than 10 signifies highly accurate predictions, while values between 10 and 20 indicate good forecasting. Values ranging from 20 to 50 reflect reasonable forecasting, and values exceeding 50 suggest poor forecasting [6]. In order to further understand the underlying trend and seasonality within the data, decomposition was used to divide the data into its constituent parts. Figure 1 shows the decomposition plots for the road traffic accidents volume in Golestan province from Farvardin 1400 to Esfand 1401. An upward trend in road traffic accidents has been noted throughout the examined period. Figure 1 further illustrates a significant increase in road accidents during the spring and summer months over the years examined. Figure 2 illustrates the accident count for the year 1402, utilizing SSA and HW methods. The data presented in this graph indicates that the SSA method foretasted a higher number of accidents compared to the HW method. The SSA method might have detected emerging trends or changes in behaviour that the HW method did not capture. Variability in data quality, seasonal effects, or recent changes in regulations or enforcement might have influenced the SSA's predictions more than the HW's.

## 4. Discussion

This study examined the daily incidence of traffic accidents, emphasizing the significance of forecasting accident numbers during specific times of the year for effective traffic management in a particular region. The Singular Spectrum Analysis (SSA) method may have identified emerging trends or behavioural changes that the Holt Winters (HW) method failed to recognize. Factors such as data quality variability, seasonal influences, or recent modifications in regulations or enforcement may have had a greater impact on the predictions made by SSA compared to those made by HW. Our findings suggest that SSA is not only a suitable method for analysing the characteristics of specific time series data, such as road traffic accidents, but it may also prove to be more effective for future forecasting. The benefits of SSA likely arise from its capacity to provide a clearer understanding of underlying trends, seasonality, and noise, thereby improving predictive accuracy.

Table 2. Comparison of SSA and HW time series forecasting methods through RMSE and MAPE (%).

| Methods | RMSE | MAPE (%) |
|---|---|---|
| SSA | 14.93 | 10.25 |
| HW | 25.24 | 18.52 |

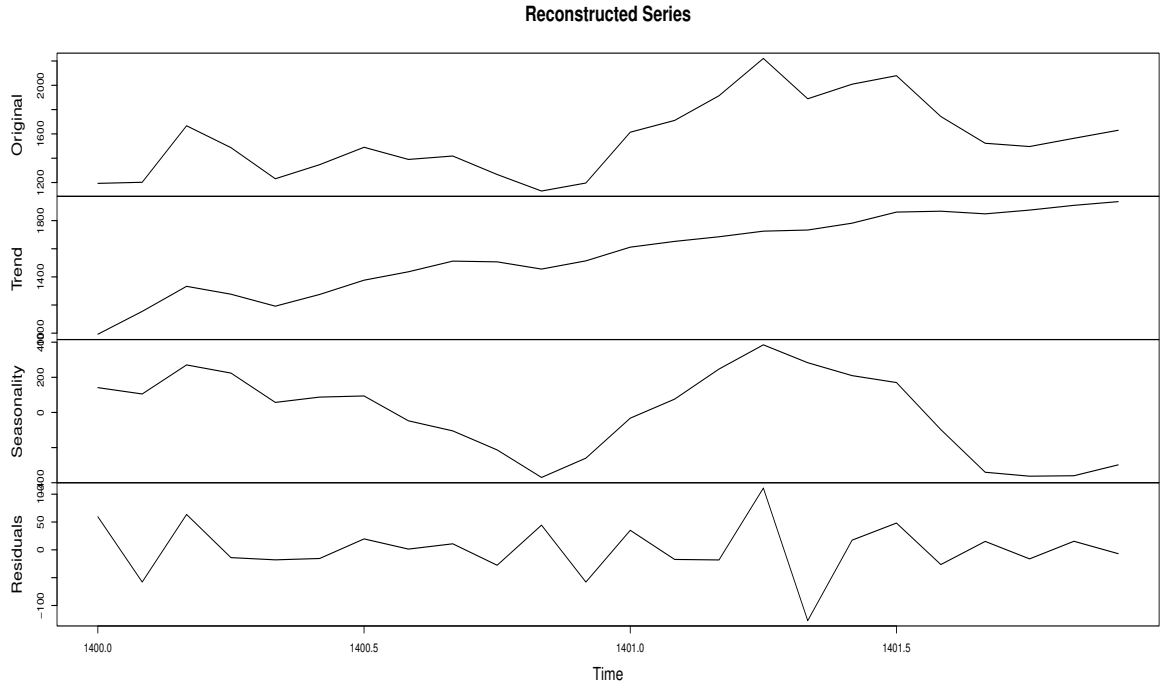


Fig. 1. Decomposition plot for the cumulative count of road traffic accidents in Golestan province from Farvardin 1400 to Esfand 1401.

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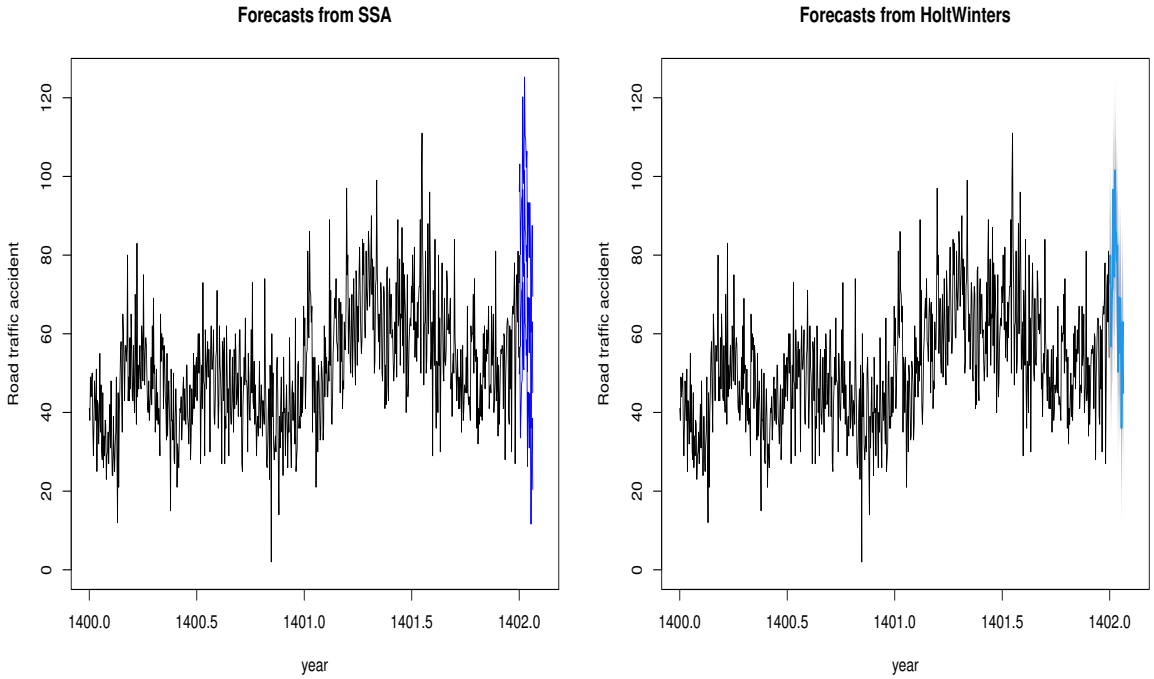


Fig. 2. Forecasting daily road traffic accidents in Golestan province for the year 1402 are illustrated by blue lines. The recorded data from Farvardin 1400 to Esfand 1401 is presented by the black lines.

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# A Remark on the Arithmetic Mean of Prime Numbers

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**Abstract**

Let $A_n$ be the arithmetic mean of the first $n$ prime numbers. It is known that the sequence $(\sqrt[n]{A_n})_{n\geqslant 1}$ is strictly decreasing, or equivalently, the inequality $A_{n+1} < A_n^{1+1/n}$ holds for any $n \geqslant 1$. In this note we refine this fact by showing that

$$A_n^{1+\beta_0/n} \leqslant A_{n+1} \leqslant A_n^{1+\alpha_0/n}$$

holds for any $n \geqslant 1$ with the best possible constants $\beta_0 = 0$ and $\alpha_0 = Q(4)$, where $Q(n) = n((\log A_{n+1})/(\log A_n) - 1)$.

## 1. Introduction and main result

Let $p_n$ be the $n$-th prime number, $S_n$ be the sum of the first $n$ prime numbers, and $A_n = S_n/n$ be the arithmetic mean of the first $n$ prime numbers. Motivated by the unsolved Firoozbakht conjecture (see [4, p. 185]) asserting that the sequence $(\sqrt[n]{p_n})_{n\geqslant 1}$ is strictly decreasing, Sun in [6, Thm. 2.1] proved that the sequences $(\sqrt[n]{S_n})_{n\geqslant 2}$ and $(\sqrt[n]{A_n})_{n\geqslant 1}$ are strictly decreasing, hence confirming two averaged forms of Firoozbakht conjecture. In this paper, we read Sun's result on $A_n$ as the following inequality

$$A_{n+1} < A_n^{1+\frac{1}{n}} \qquad (n \geqslant 1). \tag{1}$$

Our main motivation of writing this paper is to improve on the exponent $1/n$ by reducing it as may as possible. Accordingly, first we consider the inequality

$$A_{n+1} \leqslant A_n^{1+\frac{\alpha}{n}} \qquad (n \geqslant 1). \tag{2}$$

Following an argument like in [6], we observe that the inequality (2) is equivalent with

$$\frac{S_{n+1}}{n+1} \leqslant \left(\frac{S_n}{n}\right)^{1+\frac{\alpha}{n}}.$$

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Since $S_{n+1} = S_n + p_{n+1}$, we have

$$p_{n+1} + S_n \leqslant (n+1)\left(\frac{S_n^{1+\frac{\alpha}{n}}}{n^{1+\frac{\alpha}{n}}}\right) = \frac{n+1}{n^{1+\frac{\alpha}{n}}} S_n S_n^{\frac{\alpha}{n}}.$$

Thus, the inequality (2) is equivalent with

$$p_{n+1} \leqslant S_n\left(\frac{n+1}{n^{1+\frac{\alpha}{n}}} S_n^{\frac{\alpha}{n}} - 1\right).$$

Also, it is possible to obtain a similar equivalent to the inequality $A_n^{1+\beta/n} \leqslant A_{n+1}$. Hence, the double sided inequality

$$A_n^{1+\frac{\beta}{n}} \leqslant A_{n+1} \leqslant A_n^{1+\frac{\alpha}{n}}, \tag{3}$$

is equivalent with the following double sided inequality

$$S_n\left(\frac{n+1}{n^{1+\frac{\beta}{n}}} S_n^{\frac{\beta}{n}} - 1\right) \leqslant p_{n+1} \leqslant S_n\left(\frac{n+1}{n^{1+\frac{\alpha}{n}}} S_n^{\frac{\alpha}{n}} - 1\right). \tag{4}$$

To examine possibility of the validity of (4), we analyse asymptotic behaviour of sides of it as $n \to \infty$. Such an analysis shed light to the inequalities (4) and then (3), and ends in the following result.

**Proposition 1.1.** *For any positive fixed $\alpha_0 > 0$ there exists integer $n_0 \geqslant 1$ such that the double sided inequality (3) holds for $n \geqslant n_0$ with the best possible constants $\alpha = \alpha_0$ and $\beta = 0$. Moreover, the left hand side inequality in (3) is strict.*

Sun's result (1) asserts that the truth of the above proposition is true with $\alpha_0 = 1$ and $n_0 = 1$, that is, with the value $\alpha_0 = 1$ the inequality (2) holds for the global range $n \geqslant 1$. Computations show that this value of $\alpha_0$ is not the best possible for global validity of (2). Regarding to this issue, we prove the following explicit result.

**Theorem 1.2.** *The truth of the above proposition is true with*

$$\alpha_0 = 4\left(\frac{\log\frac{28}{5}}{\log\frac{17}{4}} - 1\right) \approxeq 0.76258,$$

*and $n_0 = 1$.*

**Remark 1.3.** Letting

$$Q(n) = n\left(\frac{\log A_{n+1}}{\log A_n} - 1\right),$$

the inequality (2) is equivalent with $Q(n) \leqslant \alpha$. Moreover, we mention that the optimal value $\alpha_0$ in the above theorem is actually $Q(4)$.

**Remark 1.4.** Very recently, Alzer and the author of this note [1] improved on the above double sided inequality by proving validity of

$$A_n^{1+\widehat{\beta}_0/(n\log n)} \leqslant A_{n+1} \leqslant A_n^{1+\widehat{\alpha}_0/(n\log n)} \tag{5}$$

for any $n \geqslant 2$ with the best possible constants $\widehat{\beta}_0 = \widehat{Q}(2)$ and $\widehat{\alpha}_0 = \widehat{Q}(9)$, where $\widehat{Q}(n) = (\log n)Q(n)$. Our result here, however is weaker than (5), but its proof is simple and contains asymptotic analysis of the inequalities under study.

## 2. Proofs

*Proof of Proposition 1.1.* Let $n \to \infty$. First we observe that

$$\frac{n+1}{n^{1+\frac{\alpha}{n}}} = 1 + \frac{1}{n} - \frac{\alpha \log n}{n} + O\left(\frac{\log^2 n}{n^2}\right). \tag{6}$$

Axler in [2, Thm. 1.4] obtained a full asymptotic for $S_n$, from which we get

$$S_n = \frac{n^2}{2}(\log n + O(\log\log n)) = \frac{n^2 \log n}{2}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right). \tag{7}$$

This implies that

$$\begin{aligned}\log S_n &= 2\log n + \log\log n - \log 2 + \log\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)\\ &= 2\log n + \log\log n - \log 2 + O\left(\frac{\log\log n}{\log n}\right).\end{aligned}$$

From the above last asymptotic, the following truncated form works for our purpose as well

$$\log S_n = 2\log n + O(\log\log n). \tag{8}$$

We deduce from (8) that

$$\begin{aligned}S_n^{\frac{\alpha}{n}} &= \mathrm{e}^{\frac{\alpha}{n}\log S_n} = \mathrm{e}^{\frac{\alpha}{n}(2\log n + O(\log\log n))} = \mathrm{e}^{\frac{2\alpha\log n}{n}}\mathrm{e}^{\frac{O(\log\log n)}{n}}\\ &= \left(1 + \frac{2\alpha\log n}{n} + O\left(\frac{\log^2 n}{n^2}\right)\right)\left(1 + O\left(\frac{\log\log n}{n}\right)\right).\end{aligned}$$

Thus,

$$S_n^{\frac{\alpha}{n}} = 1 + \frac{2\alpha\log n}{n} + O\left(\frac{\log\log n}{n}\right). \tag{9}$$

Multiplying asymptomatic relations (6) and (9) gives

$$\frac{n+1}{n^{1+\frac{\alpha}{n}}} S_n^{\frac{\alpha}{n}} = 1 + \frac{\alpha\log n}{n} + O\left(\frac{\log\log n}{n}\right).$$

Thus,

$$\frac{n+1}{n^{1+\frac{\alpha}{n}}} S_n^{\frac{\alpha}{n}} - 1 = \frac{\alpha\log n}{n}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right).$$

Multiplying this last asymptotic and (7), eventually we obtain

$$S_n\left(\frac{n+1}{n^{1+\frac{\alpha}{n}}} S_n^{\frac{\alpha}{n}} - 1\right) = \frac{\alpha}{2}\left(n\log^2 n\right)\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right). \tag{10}$$

Thus, the exact order of the sides of (4) are respectively $\frac{\beta}{2} n\log^2 n$ and $\frac{\alpha}{2} n\log^2 n$. Since $p_{n+1} \sim n\log n$ as $n \to \infty$, we deduce the desired result. □

*Proof of Theorem 1.2.* In order to prove explicit bounds in Theorem 1.2 we need some explicit bounds for $p_n$ and $S_n$. Regarding to $p_n$, Rosser and Schoenfeld in [5, Thm. 3] proved that

$$\mathcal{B}\left(n; \frac{3}{2}\right) < p_n < \mathcal{B}\left(n; \frac{1}{2}\right), \tag{11}$$

holds receptively for $n \geqslant 2$ and $n \geqslant 20$, with

$$\mathcal{B}(n;\eta) := n\left(\log n + \log\log n - \eta\right).$$

Regarding to $S_n$, Dusart in [3, Lemme 1.7] proved that

$$S_n \geqslant \ell(n), \tag{12}$$

holds for $n \geqslant 305494$ with

$$\ell(n) = \frac{n^2}{2}\left(\log n + \log\log n - \frac{3}{2}\right).$$

Also, Axler in [2, Cor. 9.1] proved that

$$S_n < u(n), \tag{13}$$

holds for $n \geqslant 115149$ with

$$u(n) = \ell(n) + \frac{n^2}{2}\left(\frac{\log\log n - \frac{5}{2}}{\log n}\right).$$

Now, we have enough materials to prove our explicit results. Considering equivalency of the inequalities (3) and (4), we need to find a lower bound for the function

$$R(n) := S_n\left(\frac{n+1}{n^{1+\frac{\alpha_0}{n}}}\, S_n^{\frac{\alpha_0}{n}} - 1\right),$$

and compare it with an upper bound for $p_{n+1}$. By using (12) we deduce that $R(n) \geqslant R_\ell(n)$ for $n \geqslant 305494$, where

$$R_\ell(n) = \ell(n)\left(\frac{n+1}{n^{1+\frac{\alpha_0}{n}}}\, \ell(n)^{\frac{\alpha_0}{n}} - 1\right).$$

By computation, we observe that $R_\ell(n) > 0.38n\log^2 n$ holds for $n \geqslant 8$. Thus, $R(n) > 0.38n\log^2 n$, for $n \geqslant 305494$. On the other hand, considering the upper bound in (11) we observe that $\mathcal{B}(n+1;1/2) < 0.38n\log^2 n$ holds for $n \geqslant 29$. Thus, we get validity of $p_{n+1} < R(n)$ for $n \geqslant 305494$, and then by computation for $5 \leqslant n \leqslant 305494$ and $n = 1, 2, 3$, too. Note that $R(4) = 11 = p_5$. Thus, the right hand side inequality in (3) with $\alpha = \alpha_0$ is strict for any positive integer $n \neq 4$, and the equality holds only for $n = 4$. □

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# A Remark on the Linear Congruences

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**Abstract**

For given integers $a, b, n$ with $a, n > 0$, we provide an explicit method to exchange a solution of the equation $ny \equiv -b \pmod{a}$ in $y$ to a solution of the equation $ax \equiv b \pmod{n}$ in $x$. This simple result allows us to reduce solving linear congruences with large moduli to some linear congruences with smaller moduli.

## 1. Introduction

For odd distinct primes $p$ and $q$, the law of quadratic reciprocity (for example see [1]) connects, by means of the Legendre symbol, the solubility of the following truncated quadratic congruences

$$x^2 \equiv p \pmod{q}, \qquad \text{and} \qquad x^2 \equiv q \pmod{p}.$$

The significance of the above connection is when $q$ is larger than $p$. Hence, the left congruence is harder than the right one. Indeed, the main point in the practical use of the law of quadratic reciprocity is to reduce the modulus of the congruence to a smaller one. This point will be useful also for higher degree congruences, motivating the formulation of several reciprocity laws such as cubic and quartic reciprocities [3].
Motivated by the usefulness of reducing modulo in congruences, in [2] we provide a "linear reciprocity law" as follows.

**Theorem 1.1.** *Let $a, b, n$ be integers and $a, n > 0$. If $y_0$ is a solution of the equation $ny \equiv -b \pmod{a}$, then*

$$x_0 = \frac{ny_0 + b}{a}$$

*is a solution of the equation $ax \equiv b \pmod{n}$.*

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## 2. Proof and Examples

Our short proof of the above result is independent of the theory of linear congruences, and only relates the solutions of the congruences $ny \equiv -b \pmod{a}$ and $ax \equiv b \pmod{n}$ in the explicit manner, providing a way of reducing moduli.

*Proof of Theorem 1.1.* Since $ny_0 \equiv -b \pmod{a}$, we see that $a|ny_0 + b$. Hence, the ratio $(ny_0 + b)/a$ is an integer. Let us denote it by $x_0$. We have

$$ax_0 = ny_0 + b \equiv b \pmod{n}.$$

This is the desired conclusion. □

**Example 2.1.** As an example, we solve the congruence $5x \equiv 1 \pmod{97}$. According to the above theorem, we consider the congruence $97y \equiv -1 \pmod{5}$, which is equivalent to $2y \equiv 4 \pmod{5}$. Since $\gcd(2, 5) = 1$, the last congruence is equivalent to $y \equiv 2 \pmod{5}$. Therefore $y_0 = 2$ and consequently,

$$x_0 = \frac{97 \times 2 + 1}{5} = 39.$$

This implies that the multiplicative inverse of 5 modulo 97 is 39.

**Example 2.2.** Let us give another example, providing the multiplicative inverse of $n$ modulo $n! + 1$, for given integer $n \geqslant 1$. We note that $\gcd(n! + 1, n) = 1$. Thus, the equation

$$nx \equiv 1 \pmod{n! + 1}$$

has a unique solution. According to Theorem 1.1, we consider the congruence $(n! + 1)y \equiv -1 \pmod{n}$. Since $n|n!$, we have $y \equiv -1 \pmod{n}$, and hence, $y_0 = n - 1$. We deduce that

$$x_0 = n! - (n - 1)! + 1$$

is the unique solution of the above congruence.

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# An interesting property of a class of integral graph on dihedral group

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**Abstract**

In this paper, we consider the Cayley graph $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$, where $\mathbb{D}_{2n} =< a, b \mid a^n = b^2 = 1, ba = a^{n-1}b >$ is the dihedral group of order $2n$ ($n \geq 4$), and $\Omega_1 = \{b, a^{n-1}b\}$, $\Omega_2 = \Omega_1 \cup \{ab, a^{n-2}b\}$, ..., $\Omega_k = \Omega_{k-1} \cup \{a^{k-1}b, a^{n-k}b\}$ and $S_1 = \{a, a^{n-1}\}$, $S_2 = S_1 \cup \{a^2, a^{n-2}\}$, ..., $S_m = S_{m-1} \cup \{a^m, a^{n-m}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$ for any $k, m \in \mathbb{N}$, $1 \leq k, m \leq [\frac{n}{2}]$. Also, we show that if $n$ is an odd integer, and ($k = m = [\frac{n}{2}]$), then $\Pi$ is an integral graph. In addition, if $n$ is an even integer, and $k = \frac{n}{2}$, $m = \frac{n}{2} - 1$, then $\Pi$ is an integral graph. Moreover, we determine the automorphism group of graph $\Pi$.

## 1. Introduction

Let $\Gamma = (V, E)$ be a simple connected graph with $n$ vertices, where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. We assume $V(\Gamma) = \{v_1, v_2, ..., v_n\}$, where $v_1, v_2, ..., v_n$ are indexed in the non increasing order of degrees. The adjacency matrix $A = A(\Gamma) = [a_{ij}]$ of a graph $\Gamma$ is the $n \times n$ symmetric matrix with entries 0 and 1 whose rows and columns are indexed by the vertices of $\Gamma$, and where $A_{xy} = 1$ if and only if $x$ is adjacent to $y$, we write $x \sim y$. The characteristic polynomial of $\Gamma$ with respect to the adjacency matrix $A$ is the polynomial $P(\Gamma) = P(\Gamma, \lambda) = det(\lambda I_n - A)$, where $I_n$ denotes the $n \times n$ identity matrix. The eigenvalues of a graph $\Gamma$ are the eigenvalues of the adjacency matrix of $\Gamma$. The spectrum of $\Gamma$ is the list of the eigenvalues of the adjacency matrix of $\Gamma$ together with their multiplicities, and it is denoted by $Spec(\Gamma)$. If all the eigenvalues of the adjacency matrix of graph $\Gamma$ are integers, then we say that $\Gamma$ is an integral graph. The notion of integral graphs was first introduced by F. Harary and A. J. Schwenk in 1974 [10]. In general, the problem of characterizing integral graphs seems to be very difficult. Known characterizations of integral graphs are restricted to certain graph classes, see [1–3, 11–13].
Let $T = \{t_1, ..., t_n\}$ be a set and $K$ be a group then we write $Fun(T, K)$ to denote the set of all functions from $T$ into $K$, we can turn $Fun(T, K)$ into a group by defining a product:

$$(fg)(t) = f(t)g(t) \quad for\ all \quad f, g \in Fun(T, K) \quad and \quad t \in T,$$

where the product on the right is in $K$. Since $T$ is finite, the group $Fun(T, K)$ is isomorphic to $K^n$ (a direct product of $n$ copies of $K$) via the isomorphism $f \to (f(t_1), ..., f(t_n))$. Let $H$ and $K$ be groups and suppose $H$ acts on the

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nonempty set $T$. Then the wreath product of $K$ by $H$ with respect to this action is defined to be the semidirect product $Fun(T,K) \rtimes H$ where $H$ acts on the group $Fun(T,K)$ via

$$f^x(t) = f(t^{x^{-1}}) \qquad for\ all \quad f \in Fun(T,K), t \in T \quad and \quad x \in H.$$

We denote this group by $Kwr_TH$. Consider the wreath product $G = Kwr_TH$. If $K$ acts on a set $\Delta$ then we can define an action of $G$ on $\Delta \times T$ by

$$(\delta, t)^{(f,h)} = (\delta^{f(t)}, \gamma^h) \qquad for\ all \quad (\delta, t) \in \Delta \times T,$$

where $(f,h) \in Fun(T,K) \rtimes H = Kwr_TH$ [7].
Suppose $\Gamma$ is a graph with automorphism group $Aut(\Gamma)$, we say that $\Gamma$ is vertex transitive if for any $x, y \in V(\Gamma)$, there is some $\pi$ in $Aut(\Gamma)$, the automorphism group of $\Gamma$, such that $\pi(x) = y$. Also, we say that $\Gamma$ is symmetric if, for all vertices $u, v, x, y$ of $\Gamma$ such that $u$ and $v$ are adjacent, also, $x$ and $y$ are adjacent, then there is an automorphism $\pi$ such that $\pi(u) = x$ and $\pi(v) = y$. We say that $\Gamma$ is distance transitive if, for all vertices $u, v, x, y$ of $\Gamma$ such that $\partial(u,v) = \partial(x,y)$, there is an automorphism $\pi$ such that $\pi(u) = x$ and $\pi(v) = y$, where $\partial(u,v)$ denotes the distance in $\Gamma$ between the vertices $u$ and $v$. It is clear that we have a hierarchy of conditions: see [5].
**distance transitive ⇒ symmetric ⇒ vertex transitive**

Let $G$ be a finite group and $\Omega$ a subset of $G$ that is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = Cay(G, \Omega)$ is a graph whose vertex set and edge set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} \mid x^{-1}y \in \Omega\}.$$

It is well known that every Cayley graph is vertex transitive [5].
Let $\mathbb{D}_{2n} =< a, b \mid a^n = b^2 = 1, ba = a^{n-1}b >$ be the dihedral group of order $2n$ ($n \geq 4$), such that $\Omega_1 = \{b, a^{n-1}b\}$, $\Omega_2 = \Omega_1 \cup \{ab, a^{n-2}b\}$, ..., $\Omega_k = \Omega_{k-1} \cup \{a^{k-1}b, a^{n-k}b\}$ and $S_1 = \{a, a^{n-1}\}$, $S_2 = S_1 \cup \{a^2, a^{n-2}\}$, ..., $S_m = S_{m-1} \cup \{a^m, a^{n-m}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$ for any $k, m \in \mathbb{N}$, $1 \leq k, m \leq [\frac{n}{2}]$. In this paper, we construct a class of integral graph on the dihedral group $\mathbb{D}_{2n}$ as follows:
Suppose, $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is a Cayley graph on the dihedral group $\mathbb{D}_{2n}$ ($n \geq 4$), where $\Omega_k$ and $S_m$ which are defined already. We show that if $n$ is an odd integer, and ($k = m = [\frac{n}{2}]$), then $\Pi$ is an integral graph. In addition, if $n$ is an even integer, and $k = \frac{n}{2}$, $m = \frac{n}{2} - 1$, then $\Pi$ is an integral graph. Moreover, we determine the automorphism group of graph $\Pi$.

## 2. Definitions And Preliminaries

**Definition 2.1.** [5, 6] For any vertex $v$ of a connected graph $\Gamma$, we define the $r$-distance graph as

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid \partial(u,v) = r\},$$

where $r$ is a non-negative integer not exceeding $d$, the diameter of $\Gamma$. It is clear that $\Gamma_0(v) = \{v\}$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(v), ..., \Gamma_d(v)$, for each $v$ in $V(\Gamma)$. The graph $\Gamma$ is called distance regular with diameter $d$ and intersection array $\{b_0, ..., b_{d-1}; c_1, ..., c_d\}$ if it is regular of valency $k$ and, for any two vertices $u$ and $v$ in $\Gamma$ at distance $r$, we have $|\Gamma_{r+1}(v) \cap \Gamma_1(u)| = b_r$, and $|\Gamma_{r-1}(v) \cap \Gamma_1(u)| = c_r$ $(0 \leq r \leq d)$. The intersection numbers $c_r, b_r$ and $a_r$ satisfy

$$a_r = k - b_r - c_r \quad (0 \leq r \leq d),$$

where $a_r$ is the number of neighbours of $u$ in $\Gamma_r(v)$ for $\partial(u,v) = r$.

**Remark 2.2.** [5] It is clear that if $\Gamma$ is distance transitive graph, then $\Gamma$ is distance regular.

**Proposition 2.3.** *[5] Let $\Gamma$ be a distance regular graph with diameter $d$. Then $\Gamma$ has exactly $d+1$ distinct eigenvalues.*

**Theorem 2.4.** *[6, 12] Let* $\Gamma$ *be a distance regular graph with valency* $k$*, diameter* $d$*, adjacency matrix* $A$*, and intersection array*

$$\{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}.$$

*Then, the tridiagonal* $(d+1)\times(d+1)$ *matrix*

$$B = \begin{bmatrix} a_0 & b_0 & 0 & 0 & ... & & & \\ c_1 & a_1 & b_1 & 0 & ... & & & \\ 0 & c_2 & a_2 & b_2 & 0 & ... & & \\ & & & ... & & & & \\ 0 & ... & & 0 & c_{d-2} & a_{d-2} & b_{d-2} & 0 \\ 0 & ... & & 0 & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & ... & & 0 & 0 & 0 & c_d & a_d \end{bmatrix},$$

*determines all the eigenvalues of* $\Gamma$.

**Theorem 2.5.** *[4] Let* $\Gamma$ *be a graph such that contains* $n$ *components* $\Gamma_1, ..., \Gamma_n$*. If, for any* $i \in I = \{1, ..., n\}$, $\Gamma_i \cong \Gamma_1$, *then* $Aut(\Gamma) \cong Aut(\Gamma_1) wr_I Sym(n)$.

**Definition 2.6.** [9] Let $k$ be a positive integer, a $k$-colouring of a graph $\Gamma$ is a mapping $f: V(\Gamma) \longrightarrow \{1, ..., k\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices $x$ and $y$ in $\Gamma$, and if such a mapping exists we say that $\Gamma$ is $k$-colorable. The chromatic number $\chi(\Gamma)$ of $\Gamma$ is the minimum number $k$ such that $\Gamma$ is $k$-colorable.

**Definition 2.7.** [9] Let $\Gamma$ be a graph and $I(\Gamma)$ denote the set of all independent sets of the graph $\Gamma$. A fractional colouring of a graph $\Gamma$ is a weight function $\mu: I(\Gamma) \longrightarrow [0, 1]$ such that for any vertex $x$ of $\Gamma$, $\sum_{x \in I \in I(\Gamma)} \mu(I) \geq 1$, and if such a weight function exists we say that $\Gamma$ is fractional colouring. The fractional chromatic number of a graph $\Gamma$ is denoted by $\chi_f(\Gamma)$ and defined in [ [9], page 134]. Also a fractional clique of a graph $\Gamma$ is denoted by $\omega_f(\Gamma)$ and defined in [ [9], page 134].

**Proposition 2.8.** *[8] For any graph* $\Gamma$ *we have*

$$\omega(\Gamma) \leq \omega_f(\Gamma) \leq \chi_f(\Gamma) \leq \chi(\Gamma).$$

**Proposition 2.9.** *[8] If* $\Gamma$ *is vertex transitive graph, then we have*

$$\omega_f(\Gamma) = \frac{|V(\Gamma)|}{\alpha(\Gamma)}.$$

## 3. Main results

**Proposition 3.1.** *Let* $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ *be the Cayley graph on the dihedral group* $\mathbb{D}_{2n} (n \geq 4)$*, where* $\Omega_k$ *and* $S_m$ *which are defined as before. If* $n$ *is an odd integer and* $(k = m = [\frac{n}{2}])$ *then* $\Pi$ *is an integral graph.*

*Proof.* We know that if $n$ is an odd integer and $k = m = [\frac{n}{2}]$ then $\Omega_k = \{b, ab, a^2b, ..., a^{n-1}b\} - \{a^k b\}$, and $S_m = \{a, a^2, ..., a^{n-1}\}$. It is not hard to see that $\Pi$ is a distance transitive graph. Also, let $V(\Pi) = \{b, ab, ..., a^{n-1}b, a, a^2, ..., a^n\}$ be the vertex set of $\Pi$. Consider the vertex $v = a^n$ in the $V(\Pi)$, then $\Gamma_0(v) = \{a^n\}$, $\Gamma_1(v) = \mathbb{D}_{2n} - \{a^n, a^k b\}$ and $\Gamma_2(v) = \{a^k b\}$. Let be $u$ in the $V(\Pi)$ such that $\partial(u, v) = 0$ then $u = v = a^n$ and $|\Gamma_1(v) \cap \Gamma_1(u)| = 2n - 2$, hence $b_0 = 2n - 2$ and by Definition (2.1), $a_0 = 2n - 2 - b_0 = 0$. Also, if $u$ in the $V(\Pi)$ and $\partial(u, v) = 1$ then two vertices $u, v$ are adjacent in $\Pi$, so $|\Gamma_0(v) \cap \Gamma_1(u)| = 1$ and $|\Gamma_2(v) \cap \Gamma_1(u)| = 1$, hence $c_1 = 1$, $b_1 = 1$ and $a_1 = 2n - 2 - b_1 - c_1 = 2n - 4$. Finally, if $u$ in the $V(\Pi)$ and $\partial(u, v) = 2$, then two vertices $u, v$ are not adjacent in $\Pi$, so $|\Gamma_1(v) \cap \Gamma_1(u)| = 2n - 2$, hence $c_2 = 2n - 2$ and $a_2 = 2n - 2 - c_2 = 0$. So the intersection array of $\Pi$ is $\{2n - 2, 1; 1, 2n - 2\}$. Therefore by Theorem (2.4), the tridiagonal $(3) \times (3)$ matrix,

$$\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 2n-2 & 0 \\ 1 & 2n-4 & 1 \\ 0 & 2n-2 & 0 \end{bmatrix},$$

determines all the eigenvalues of Π, because by Remark (2.2), it is clear that Π is a distance regular graph. Also, it is easy to show that all the eigenvalues of Π are $2n-2, 0, -2$ and their multiplicities are $1, n, n-1$, respectively. So Π is an integral graph. □

**Proposition 3.2.** *Let* $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ *be the Cayley graph on the dihedral group* $\mathbb{D}_{2n}(n \geq 4)$*, where* $\Omega_k$ *and* $S_m$ *which are defined as before. If* $n$ *is an even integer and* ($k = \frac{n}{2}$, $m = \frac{n}{2} - 1$) *then* Π *is an integral graph.*

*Proof.* We know that, if $n$ is an even integer and $k = \frac{n}{2}$, $m = \frac{n}{2} - 1$, then $\Omega_k = \{b, ab, a^2b, ..., a^{n-1}b\}$, and $S_m = \{a, a^2, ..., a^n\} - \{a^{\frac{n}{2}}, a^n\}$. It is easy to prove that Π is a distance transitive graph. Also, let $V(\Pi) = \{b, ab, ..., a^{n-1}b, a, a^2, ..., a^n\}$ be the vertex set of Π. Consider the vertex $v = a^n$ in the $V(\Pi)$, then $\Gamma_0(v) = \{a^n\}$, $\Gamma_1(v) = \mathbb{D}_{2n} - \{a^n, a^{\frac{n}{2}}\}$ and $\Gamma_2(v) = \{a^{\frac{n}{2}}\}$. Similarly, by proof of Proposition (3.1), we can show that the intersection array of Π is $\{2n-2, 1; 1, 2n-2\}$. Therefore by Theorem (2.4), the tridiagonal $(3) \times (3)$ matrix,

$$\begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 2n-2 & 0 \\ 1 & 2n-4 & 1 \\ 0 & 2n-2 & 0 \end{bmatrix},$$

determines all the eigenvalues of Π, because by Remark (2.2), it is clear that Π is a distance regular graph. Also, it is easy to show that all the eigenvalues of Π are $2n-2, 0, -2$ and their multiplicities are $1, n, n-1$, respectively. So Π is an integral graph. □

**Theorem 3.3.** *Let* $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ *be a Cayley graph on the dihedral group* $\mathbb{D}_{2n}(n \geq 4)$*, where* $\Omega_k$ *and* $S_m$ *which are defined as before. If* $n$ *is an odd integer and* ($k = m = [\frac{n}{2}]$) *then* $Aut(\Pi) = \mathbb{Z}_2 wr_I Sym(n)$*, where* $I = \{1, 2, ..., n\}$.

*Proof.* By assumptions and definitions of $\Omega_k$ and $S_m$, we can show that $\chi(\Pi) = \omega(\Pi) = n$. So by Proposition (2.8), $\chi_f(\Pi) = \omega_f(\Pi) = n$. Also we know that Π is a vertex transitive graph, so by Proposition (2.9), $n = \omega_f(\Pi) = \frac{|V(\Pi)|}{\alpha(\Pi)}$. Thus, the size of the every independent set of vertices in the Π is 2. Therefore for any $x \in V(\Pi)$, there is exactly one $y \in V(\Pi)$ such that $x^{-1}y \notin \Omega_k \cup S_m$. Hence, if $x^{-1}y \notin \Omega_k \cup S_m$ then two vertices $x$ and $y$ are adjacent in the complement $\overline{\Pi}$ of Π, so $\overline{\Pi}$ contains $n$ components $\Pi_1, ..., \Pi_n$ such that for any $i \in I = \{1, ..., n\}$, $\Pi_i \cong K_2$, where $K_2$ is the complete graph on 2 vertices. Therefore $\overline{\Pi} \cong nK_2$, hence by Theorem (2.5), $Aut(\overline{\Pi}) \cong Aut(K_2) wr_I Sym(n) = \mathbb{Z}_2 wr_I Sym(n)$, so $Aut(\Pi) \cong \mathbb{Z}_2 wr_I Sym(n)$. In fact, it is well known that for any graph Π, $Aut(\Pi) = Aut(\overline{\Pi})$; see [8]. □

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# A strong convergence algorithm for mixed equilibrium problems

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**Abstract**

In this paper, we introduce a new iterative algorithm for finding a common element of the set of solutions of the set of fixed points for a $\phi$-nonexpansive mapping and the mixed equilibrium problems in Banach spaces by using sunny generalized nonexpansive retraction in Banach spaces.

## 1. Introduction

Let $C$ be a nonempty closed convex subset of a real Banach space $X$ with norm $\|.\|$ and $X^*$ be the dual of $X$. Let $J$ be the duality mapping from $X$ into $X^*$ such that $J(C)$ is a cloded and convex subset of $X^*$. Let $F : J(C) \times J(C) \to \mathbb{R}$ be a bifunction and $\varphi : J(C) \to \mathbb{R}$ be a real valued function.
The mixed equilibrium problem is to find $X^* \in C$ such that

$$F(Jx^*, Jy) + \varphi(Jy) - \varphi(Jx^*) \geq 0, \tag{1}$$

for all $y \in C$. The solution set of (1) is denoted by $MEP(F, \varphi)$.

$$J_p(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for every $x \in X$. If $p = 2$, then $J_2 = J$ is the normalized duality mapping. Let $X$ be a smooth Banach space and let $J_X$ be the duality mapping on $X$. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J_X y\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{2}$$

Clearly, from (2), we can conclude that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{3}$$

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**Definition 1.1.** [1] Let $C$ be a nonempty closed subset of $X$. A mapping $R : X \to C$ is called:

(i) a retraction if $R^2 = R$,
(ii) sunny if $R(Rx + \lambda(x - Rx)) = Rx$ for all $x \in X$ and $\lambda > 0$.

**Definition 1.2.** [1] A nonempty closed subset $C$ of a smooth Banach space $X$ is called a sunny generalized nonexpansive retract of $X$ if there exists a sunny generalized nonexpansive retraction $R$ from $X$ onto $C$.

For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to R$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,
(A2) $F$ is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$,
(A3) $\limsup_{\lambda \to \infty} F(x + t(z - x), y) \leq F(x, y)$ for all $x, y, z \in C$,
(A3) the function $y \mapsto F(x, y)$ is convex and lower semicontinuous.

**Lemma 1.3.** *[2] Let $X$ be a smooth and uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $h : [0, 2r] \to \mathbb{R}$ such that $h(0) = 0$ and*

$$h(\|x - y\|) \leq \phi(x, y), \ \forall x, y \in B_r.$$

**Lemma 1.4.** *[3] Let $E$ be a strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $T : C \to C$ be a closed $\phi$-nonexpansive mapping. Then $F(T)$ is a closed convex subset of $C$.*

**Lemma 1.5.** *[1] Let $C$ be a nonempty closed subset of a uniformly smooth, strictly convex and reflexive Banach space $X$ such that $J(C)$ is closed and convex. Let $\varphi : J(C) \to \mathbb{R}$ be convex and lower semicontinuous and mapping $F : J(C) \times J(C) \to \mathbb{R}$ be a bifunction mapping satisfying the conditions (A1)–(A4). For any $r > 0$ and $x \in X$, define a mapping $T_r : X \to C$ by*

$$T_r(x) = \{z \in C : F(Jz, Jy) + \varphi(Jy) - \varphi(Jz) + \frac{1}{r}\langle z - x, Jy - Jz\rangle \geq 0, \forall y \in C\}$$

*for all $x \in X$. Then the following statements hold:*

*(1) $T_r$ is single-valued,*
*(2) For all $x, y \in X$,*

$$\langle T_r(x) - T_r(y), J(T_r(x)) - J(T_r(y))\rangle \leq \langle x - y, J(T_r(x)) - J(T_r(y))\rangle$$

*(3) $F(T_r) = MEP(F, \varphi)$ and $J(MEP(F, \varphi))$ is closed and convex,*
*(4) $\phi(x, T_r(x)) + \phi(T_r(x), p) \leq \phi(x, p)$ for all $x \in X$ and $p \in F(T_r)$.*

**Lemma 1.6.** *[1] Let $C$ be a nonempty closed subset of a smooth and strictly convex Banach space $X$ such that there exists a sunny generalized nonexpansive retraction $R$ from $X$ onto $C$. Then, for any $x \in X$ and $z \in C$, the following statements hold:*

*(1) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$,*
*(2) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.*

**Lemma 1.7.** *[4] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \forall n \geq 1.$$

*If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.*

## 2. Main results

**Theorem 2.1.** *Let $X$ be a uniformly smooth and uniformly convex Banach space and let $C$ be a nonempty closed and convex subset of $X$ such that $J(C)$ is closed and convex of $X^*$. Suppose that $F : J(C) \times J(C) \to \mathbb{R}$ is a bifunction satisfying the conditions $(A1) - (A4)$. let $T : C \to C$ be a closed and $\phi$-nonexpansive mapping such that $\mathcal{F} := F(T) \cap MEP(F, \phi) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by $x_0 \in C$ and*

$$\begin{cases} u_n \in C \text{ such that} \\ F(Ju_n, Jy) + \varphi(Jy) - \varphi(Ju_n) + \frac{1}{r_n}\langle u_n - x_n, Jy - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tu_n, \end{cases} \tag{4}$$

*where $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ satisfy the following conditions:*

*(i)* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n < \infty$,
*(ii)* $\{\beta_n\} \subset [a,b] \subset (0,1)$ *and* $\sum_{n=0}^{\infty} \beta_n < \infty$,
*(iii)* $\{r_n\} \subset [d,\infty)$ *for some* $d > 0$ *and* $\liminf_{n\to\infty} r_n > 0$.

*Then the sequence $\{R_{\mathcal{F}} x_n\}$ converges strongly to a point $q \in \mathcal{F}$, where $R_{\mathcal{F}}$ is the sunny generalized nonexpansive retraction of $X$ onto $\mathcal{F}$.*

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# Zero Sets of Pointfree Functions with Countable Images

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**Abstract**

In 2015, the pointfree version of the zero set of a real-valued continuous function $f$ on a topological $X$ was defined by viewing prime elements of a frame $L$ as pointfree points, replacing them for points $x$ of $X$. Moreover, the zero set of a frame homomorphisms $\alpha : \mathfrak{O}(\mathbb{R}) \to L$ is characterized by its cozero element. Using this characterization, we define the zero set of a frame homomorphisms $\alpha : \mathfrak{O}(\mathbb{R}) \to L$ with a countable pointfree image and provide basic relations arising from these sets. We show that for a zero-dimensional frame $L$, the family $Z_cL$ of all zero sets in $L$ forms a base for the closed sets of all primes in $L$. We also introduce the concept of $z_c$-filters on a frame $L$ and examine their connections to ideals, particularly maximal ideals.

## 1. Introduction

Let $C(X)$ be the ring of all real-valued continuous functions on a topological space $X$ (see [11] for more details), and let $C_c(X)$ be the subring of $C(X)$ consisting of those functions with countable images (see [9, 10] for more details). For each $f \in C(X)$, the *zero set* of $f$, denoted by $Z(f)$, is the set $\{x \in X \mid f(x) = 0\}$, and $Coz(f) = X \setminus Z(f)$ is the *cozero-set* of $f$. To study the rings $C(X)$ and $C_c(X)$, $X$ is a topological space, zero sets have an important role.
The pointfree topology version of $C(X)$ was introduced by Banaschewski and Gilmour in [3]. This structure is denote by $\mathcal{R}L$, where $L$ is a frame. For any frame $L$, the elements of $\mathcal{R}L$ are frame homomorphisms $\alpha : \mathcal{L}(\mathbb{R}) \to L$, where $\mathcal{L}(\mathbb{R})$ is the frame of reals, which is isomorphic to $\mathfrak{O}(\mathbb{R})$ the frame of open subsets of $\mathbb{R}$. It is a reduced $f$-ring; for further details on its properties, see [2] and [4]. For each $\alpha \in \mathcal{R}L$, let $\mathrm{coz}(\alpha) = \alpha(-, 0) \vee \alpha(0, -)$, where

$$(0, -) = \bigvee\{(0, q)) \mid 0 < q \in \mathbb{Q}\}, \quad \text{and} \quad (-, 0) = \bigvee\{(p, 0)) \mid 0 > p \in \mathbb{Q}\}.$$

A *cozero element* of a frame $L$ is of the form $\mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}L$. The set $\mathrm{Coz}L = \{\mathrm{coz}(\alpha) \mid \alpha \in \mathcal{R}L\}$ is a sub-$\sigma$-frame of $L$. For detailed properties of cozero elements, refer to [2] and [4].

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Let $r \in \mathbb{R}$. The constant frame homomorphism $\mathbf{r} \in \mathcal{R}L$ is defined as follows:

$$\mathbf{r}(p,q) = \begin{cases} \top & \text{if } p < r < q \\ \bot & \text{otherwise.} \end{cases}$$

For any $\alpha \in \mathcal{R}L$, define $R_\alpha := \{r \in \mathbb{R} \mid \operatorname{coz}(\alpha - \mathbf{r}) = \alpha(-,r) \vee \alpha(r,-) \neq \top\}$, as stated in [12], where

$$(-,r) := \bigvee_{\substack{q \in \mathbb{Q} \\ q<r}} (-,q) \qquad \text{and} \qquad (r,-) := \bigvee_{\substack{p \in \mathbb{Q} \\ r<p}} (p.-).$$

Thus, $R_\alpha$ generalizes the familiar relationship between functions on spaces and their images (see [1, Irtroduction] and [12]).
The pointfree topology version of $C_c(X)$ was introduced in [12], where it is denoted by $C_c(L)$. We refer to it as $\mathcal{R}_c(L)$. The elements of the ring $\mathcal{R}_c L$ are frame homomorphisms $\alpha : \mathcal{L}(\mathbb{R}) \to L$, where $\alpha$ has a countable pointfree image. A real-valued continuous function $\alpha$ on a frame $L$ is said to have a *countable pointfree image* if $R_\alpha$ is a countable set. For any frame $L$, [8] establishes the following:

1. For any $\alpha \in \mathcal{R}_c L$, $\operatorname{coz}(\alpha)$ is a countable join of complemented elements of $L$.
2. The set $\operatorname{Coz}_c L = \{\operatorname{coz}(\alpha) \mid \alpha \in \mathcal{R}_c L\}$ is a sublattice of $L$ that includes $\bot$ and $\top$.
3. $L$ is a zero-dimensional frame if and only if it is generated by $\operatorname{Coz}_c L$.
4. $\operatorname{Coz}_c L$ is closed under countable joints when $L$ is a completely regular frame.

A *frame* is a complete lattice $L$ in which the distributive law

$$a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s$$

holds for all $a \in L$ and $S \subseteq L$. Throughout this context $L$ will denote a frame. We denote the top element and the bottom element of $L$ by $\top$ and $\bot$ respectively. An element $p \in L$ is called *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We denoted the set of all points of $L$ by $Pt(L)$.The *pseudocomplement* of an element $a \in L$ is the element $a^* = \bigvee\{x \in L \mid a \wedge x = \bot\}$. When $a \vee a^* = 1$, then $a$ is called a *complemented* element in $L$.The frame of open subsets of a topological space $X$ is denoted by $\mathfrak{O}(X)$.
Let $a \in L$, and $\alpha \in \mathcal{R}L$. We define the sets $L(a,\alpha) = \{r \in \mathbb{Q} \mid \alpha(-,r) \leq a\}$ and $U(a,\alpha) = \{s \in \mathbb{Q} \mid \alpha(s,-) \leq a\}$. For $a \neq \top$, it is evident that for each $r \in L(a,\alpha)$ and $s \in U(a,\alpha)$, $r \leq s$. Indeed, as shown in [5], if $p$ is a prime element in $L$, then the pair $(L(p,\alpha), U(p,\alpha))$ forms a Dedekind cut for a real number denoted by $\alpha[p]$.
In paper [6], the authors present a pointfree version of zero sets by treating prime elements of a frame $L$ as pointfree points. They define the trace of an element $\alpha$ of $\mathcal{R}L$ on a point $p$ in $L$ by $\alpha[p]$. The *zero set* of an element $\alpha \in \mathcal{R}L$ is define by $Z(\alpha) = \{p \in Pt(L) : \alpha[p] = 0\}$. They show that for any prime element $p \in L$ and $\alpha \in \mathcal{R}L$, $\alpha[p] = 0$ if and only if $\operatorname{coz}(\alpha) \leq p$. Consequently, we have $Z(\alpha) = \{p \in Pt(L) \mid \operatorname{coz}(\alpha) \leq p\}$.
We define a countable zero set in pointfree topology based on thecharacterization of zero sets mentioned earlier. For any $\alpha \in \mathcal{R}_c L$, we define $Z(\alpha) = \{p \in Pt(L) \mid \operatorname{coz}(\alpha) \leq p\}$. This note explore the relation between countable zero sets and cozero elements in $\operatorname{Coz}_c L$. We demonstrate that for any $\alpha \in \mathcal{R}_c L$, the pointfree zero set $Z(\alpha)$ corresponds to a topological zero set $Z(f)$ for some continuous function $f : Pt(L) \to \mathbb{R}$. Additionally, we show that in a zero-dimensional frame $L$, the family $Z_c L$ forms a base for the closed sets of $Pt(L)$. We also introduce the concept of $z_c$-filters on a frame $L$ and give their connections to ideals of $\mathcal{R}_c L$, especially maximal ideals.

## 2. Countably Zero Sets

Here, we introduce the pointfree version of the zero set of $f \in C_c(X)$, where prime elements $p \in L$ substitute for points $x \in X$.

**Definition 2.1.** Let $\alpha \in \mathcal{R}_c L$. We define

$$Z(\alpha) = \{p \in Pt(L) \mid \mathrm{coz}\alpha \leq p\}.$$

This set is known as a countable zero set (or zero set) in $L$. For a subset $D \subseteq \mathcal{R}_c L$, we denote the family of zero sets by $Z[D]$. The collection of all zero sets in $L$ is simplified to $Z_c L$.

Recall that the contravariant functor $\Sigma$ which assigns to each frame $L$ its *spectrum* $Pt(L)$ with

$$\Sigma_a = \{p \in Pt(L) \mid a \not\leq p\}$$

for any $a \in L$. The set $\{\Sigma_a \mid a \in L\}$ forms a topology on $Pt(L)$, satisfying the following properties:

$$\Sigma_\bot = \emptyset \quad , \quad \Sigma_\top = Pt(L)$$
$$\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b} \quad , \quad \bigcup_{i \in I} \Sigma_{a_i} = \Sigma_{\bigvee_{i \in I} a_i}.$$

Thus, definition 2.1 directly lead to the following lemma.

**Lemma 2.2.** *For $\alpha \in \mathcal{R}_c L$, we have $Z(\alpha) = Pt(L) - \Sigma_{coz(\alpha)}$.*

The above lemma is crucial for describing zero sets in pointfree topology. We will now use it to derive the fundamental relations we anticipate from these sets. Before we continue, we will review the basic properties of cozero elements as outlined in [2] and [4]. Let $\alpha, \beta \in \mathcal{R}L$ (or $\mathcal{R}_c L$). Then we have

1. $\mathrm{coz}(\alpha) = \mathrm{coz}(|\alpha|) = \mathrm{coz}(\alpha^n)$ for all $n \in \mathbb{N}$.
2. $\mathrm{coz}(\alpha^2 + \beta^2) = \mathrm{coz}(|\alpha| + |\beta|) = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$
3. $\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta)$.
4. $\mathrm{coz}(\alpha + \beta) \leq \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$.
5. $\mathrm{coz}(\alpha) = \bot$ if and only if $\alpha = \mathbf{0}$.
6. $\mathrm{coz}(\alpha) = \top$ if and only if $\alpha$ is invertible.

**Proposition 2.3.** *Let $L$ be a frame. Then for every $\alpha, \beta \in \mathcal{R}_c L$, we have*

1. *For all $n \in \mathbb{N}$, $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.*
2. *$Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$. Note that $Z(\alpha) \cap Z(\beta) \subseteq Z(\alpha + \beta)$.*
3. *$Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.*
4. *For every $\alpha \in \mathcal{R}_c L$, $Z(\alpha)$ is a closed set in $Pt(L)$ and $Pt(L) - Z(\alpha) = \Sigma_{coz(\alpha)}$.*
5. *$Z_c(L)$ is closed under countable intersection, when $L$ is completely regular.*
6. *If $\alpha = \mathbf{0}$, then $Z(\alpha) = Pt(L)$.*
7. *If $\alpha$ is a unit of $\mathcal{R}_c L$, then $Z(\alpha) = \emptyset$.*

*Proof.* (1): For any $n \in \mathbb{N}$, since $coz(\alpha) = coz(|\alpha|) = coz(\alpha^n)$, it follows from Lemma 2.2 that $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.
(2): For any $p \in Pt(L)$, by Definition 2.1, we have:

$$\begin{aligned} p \in Z(\alpha) \cap Z(\beta) &\Leftrightarrow \mathrm{coz}(|\alpha|) = \mathrm{coz}(\alpha) \leq p \quad \text{and} \quad \mathrm{coz}(|\beta|) = \mathrm{coz}(\beta) \leq p \\ &\Leftrightarrow \mathrm{coz}(|\alpha|) \vee \mathrm{coz}(|\beta|) \leq p \\ &\Leftrightarrow \mathrm{coz}(|\alpha| + |\beta|) \leq p \\ &\Leftrightarrow p \in Z(|\alpha| + |\beta|). \end{aligned}$$

This demonstrates that $Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|)$. A similar argument shows that

$$Z(\alpha) \cap Z(\beta) = Z(\alpha^2 + \beta^2) \qquad \text{and} \qquad Z(\alpha) \cap Z(\beta) \subseteq Z(\alpha + \beta).$$

(3): For any $p \in Pt(L)$, by Definition 2.1, we have:

$$\begin{aligned} p \in Z(\alpha) \cup Z(\beta) &\Leftrightarrow \operatorname{coz}(\alpha) \leq p \quad \text{or} \quad \operatorname{coz}(\beta) \leq p \\ &\Leftrightarrow \operatorname{coz}(\alpha\beta) = \operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta) \leq p \\ &\Leftrightarrow p \in Z(\alpha\beta). \end{aligned}$$

This implies that $Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.
(4) By Definition 2.1, we have $Z(\alpha) = \{p \in Pt(L) \mid \operatorname{coz}(\alpha) \leq p\}$. Thus,

$$q \in Pt(L) - Z(\alpha) \Leftrightarrow q \notin Z(\alpha) \Leftrightarrow \operatorname{coz}(\alpha) \not\leq q \Leftrightarrow q \in \Sigma_{\operatorname{coz}(\alpha)}.$$

Consequently, $Pt(L) - Z(\alpha) = \Sigma_{\operatorname{coz}(\alpha)}$ is open set in $Pt(L)$, implying that $Z(\alpha)$ is closed.
(5) Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a subset of $\mathcal{R}_cL$. There exists $\varphi \in \mathcal{R}_cL$ such that $\bigvee_{n\in\mathbb{N}} \operatorname{coz}(\alpha_n) = \operatorname{coz}(\varphi)$. We have

$$Pt(L) - \bigcap_{n\in\mathbb{N}} Z(\alpha_n) = \bigcup_{n\in\mathbb{N}} (Pt(L) - Z(\alpha_n)).$$

From this, it follows that

$$Pt(L) - \textstyle\bigcap_{n\in\mathbb{N}} Z(\alpha_n) = \bigcup_{n\in\mathbb{N}} \Sigma_{\operatorname{coz}(\alpha_n)} = \Sigma_{\bigvee_{n\in\mathbb{N}} \operatorname{coz}(\alpha_n)} = \Sigma_{\operatorname{coz}(\varphi)} = Pt(L) - Z(\varphi).$$

Thus, we conclude that $\bigcap_{n\in\mathbb{N}} Z(\alpha_n) = Z(\varphi) \in Z_c(L)$.
(6): If $\alpha = \mathbf{0}$, then $\operatorname{coz}(\alpha) = \bot$. This implies that $\Sigma_{\operatorname{coz}(\alpha)} = \Sigma_\bot = \emptyset$, and by Lemma 2.2, we have $Z(\alpha) = Pt(L)$.
(7): If $\alpha$ is a unit of $\mathcal{R}_cL$, then $\operatorname{coz}(\alpha) = \top$. Consequently, for any $p \in Pt(L)$, $\operatorname{coz}(\alpha) \not\leq p$. This implies that $\Sigma_{\operatorname{coz}(\alpha)} = Pt(L)$, and by Lemma 2.2, we have $Z(\alpha) = \emptyset$. □

**Remark 2.4.** Part (5) of the previous proposition establishes zero sets are closed. Let $X$ be a topological space, and let $f \in C_c(X)$. Consider the frame map $\mathfrak{O}f : \mathfrak{O}(\mathbb{R}) \to \mathfrak{O}(X)$, defied by $\mathfrak{O}f(p,q) = \{x \in X \mid p < f(x) < q\}$. Notice that $\mathfrak{O}f \in \mathcal{R}L$ corresponds to $f$(see [12]), and we have

$$\operatorname{coz}(\mathfrak{O}f) = \mathfrak{O}f((-,0) \vee (0,-)) = \{x \in X \mid f(x) \neq 0\} = X - Z(f).$$

If $X$ is a sober space, the prime elements of $\mathfrak{O}(X)$ are of the form $X - \overline{\{x\}}$ for $x \in X$. Thus, we deduce:

$$\begin{aligned} X - \overline{\{x\}} \in Z(\mathfrak{O}f) &\Leftrightarrow \operatorname{coz}(\mathfrak{O}f) \leq X - \overline{\{x\}} \\ &\Leftrightarrow (X - Z(f)) \cap \overline{\{x\}} = \emptyset \\ &\Leftrightarrow x \in Z(f). \end{aligned}$$

Therefore, $Z(\mathfrak{O}f) = \{X - \overline{\{x\}} \mid x \in Z(f)\}$ relates $Z(f)$ and $Z(\mathfrak{O}f)$.

Let us remind the reader that a frame $L$ is called *weakly spatial* if for any $a \in L$, the condition $\Sigma_a = \Sigma_\top$ implies $a = \top$. We will show that the converse of the last part of the previous proposition holds true when $L$ is a weakly spatial frame.

**Corollary 2.5.** *Let $L$ be a weakly spatial frame and $\alpha \in \mathcal{R}_cL$. If $Z(\alpha) = \emptyset$, then $\alpha$ is a unit of $\mathcal{R}_cL$.*

*Proof.* By Lemma 2.2, $\Sigma_{\operatorname{coz}(\alpha)} = Pt(L) - Z(\alpha) = Pt(L) = \Sigma_\top$. Therefore, weakly spatiality implies $\operatorname{coz}(\alpha) = \top$, showing that $\alpha$ is a unit of $\mathcal{R}_cL$. □

We will now outline the conditions for the reverse of part (6) of the previous proposition to hold, starting with the following definition.

**Definition 2.6.** A frame $L$ is called $c$-coz-dense if $\alpha \in \mathcal{R}_cL$ and $\Sigma_{\operatorname{coz}(\alpha)} = \emptyset$ imply $\alpha = \mathbf{0}$.

**Corollary 2.7.** *Let $L$ be a $c$-coz-dense frame and $\alpha \in \mathcal{R}_cL$. If $Z(\alpha) = Pt(L)$, then $\alpha = \mathbf{0}$.*

*Proof.* By Lemma 2.2, $\Sigma_{\operatorname{coz}(\alpha)} = Pt(L) - Z(\alpha) = Pt(L) - Pt(L) = \emptyset = \Sigma_\bot$. Therefore, $c$-coz-density implies $\operatorname{coz}(\alpha) = \bot$, showing that $\alpha = \mathbf{0}$. □

Recall from [7, Lemma 3.2] a frame $L$ is weakly spatial if and only if there exists $p \in Pt(L)$ such that $a \leq p < \top$, for any $a < \top$

**Proposition 2.8.** *Let $L$ be weakly spatial and $\alpha \in \mathcal{R}_c L$. If $\Sigma_{coz(\alpha)} = \emptyset$, then $coz(\alpha) = \bot$.*

*Proof.* Let $r, s \in \mathbb{Q}$ such that $r < 0 < s$ and $p \in Pt(L)$. Then, $p \notin \Sigma_{coz(\alpha)}$, which implies $coz(\alpha) \leq p$. We claim that $\alpha(r,s) \not\leq p$. If $\alpha(r,s) \leq p$, then $\top = coz(\alpha) \vee \alpha(r,s) \leq p$, leading to contradiction. Therefore $\Sigma_{\alpha(r,s)} = Pt(L)$ implies $\alpha(r,s) = \top$ since $L$ is weakly spatial. We also have

$$\bot = (\alpha(-,r) \vee \alpha(s,-)) \wedge \alpha(r,s) = (\alpha(-,r) \vee \alpha(s,-)) \wedge \top = \alpha(-,r) \vee \alpha(s,-).$$

Therefore, $coz(\alpha) = \bigvee\{\alpha(-,r) \vee \alpha(s,-) : r < 0 < s\} = \bot$. □

The next theorem demonstrate that for any $\alpha \in \mathcal{R}_c L$, the pointfree zero set $Z(\alpha)$ corresponds to a topological zero set $Z(f)$ for some function $f : Pt(L) \to \mathbb{R}$. First, we need some tools. There exists a homeomorphism $\tau : Pt(\mathcal{L}(\mathbb{R})) \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r,s) \not\leq p$ for all $p \in Pt(\mathcal{L}(\mathbb{R}))$ and for all $r, s \in \mathbb{Q}$ (see [4, Proposition 1]). Each prime element of $\mathcal{L}(\mathbb{R}))$ has the form $p_x = \bigvee\{(-,r) \vee (s,-) \mid r,s \in \mathbb{Q}, r \leq x \leq s\}$ for some $x \in \mathbb{R}$ with $\tau(p_x) = x$. In particular, for each $r \in \mathbb{Q}$, $p_r = (-,r) \vee (r,-)$, and $\tau((-,r) \vee (r,-)) = r$ (see [6, Lemma 3.6]). A frame homomorphism $f : L \to M$ is associated with its right adjoint $f_* : M \to L$, defined by $f_*(a) = \bigvee\{x \in L \mid f(x) \leq a\}$. Note that $f_*$ preserves primes and arbitrary meets. Moreover, for a frame map homomorphism $f : L \to M$, the map $\Sigma h : Pt(M) \to Pt(L)$ sends $p \in Pt(M)$ to $h_*(p) \in Pt(L)$.

**Theorem 2.9.** *Let $L$ be a frame and $\alpha \in \mathcal{R}_c L$. Then $Z(\alpha) = Z(\tau \circ \Sigma\alpha)$.*

*Proof.* Let $p \in Pt(L)$. We have $\tau \circ \Sigma\alpha(p) = 0$ if and only if $\tau \circ \alpha_*(p) = 0$, where $\alpha_*$ is a right adjoint of $\alpha$. As stated above, $\tau \circ \alpha_*(p) = 0$ if and only if $\alpha_*(p) = (-,0) \vee (0,-)$ if and only if $\alpha((-,0) \vee (0,-)) \leq p$, since $\alpha\alpha_* \leq id$. Thus, $p \in Z(\tau \circ \Sigma\alpha)$ if and only if $\mathrm{coz}(\alpha) \leq p$, completing the proof as per Definition 2.1 the proof is complete. □

We now identify when the family $Z_c L$ constitutes a base for the closed sets of $Pt(L)$.

**Theorem 2.10.** *For any frame $L$, the following statements hold:*

1. *If $L$ is zero-dimensional, then the family $Z_c L$ of zero sets forms a base for the closed sets of $Pt(L)$.*
2. *If $L$ is a spatial frame and $Z_c L$ forms a base for the closed sets of $Pt(L)$, then $L$ is zero-dimensional.*

*Proof.* (1): Let $F$ be a closed set of $Pt(L)$. Then, there exists an $a \in L$ such that $Pt(L) - F = \Sigma_a$. Since $L$ is zero-dimensional, we can find a family $\{\alpha_i\}_{i \in I} \subseteq \mathcal{R}_c L$ such that $a = \bigvee_{i \in I} \mathrm{coz}(\alpha_i)$. By Lemma 2.2, we have:

$$F = Pt(L) - \Sigma_a = Pt(L) - \Sigma_{\bigvee_{i \in I} \mathrm{coz}(\alpha_i)} = Pt(L) - \bigcup_{i \in I} \Sigma_{\mathrm{coz}(\alpha_i)} = \bigcap_{i \in I} (Pt(L) - \Sigma_{\mathrm{coz}(\alpha_i)}) = \bigcap_{i \in I} Z(\alpha_i).$$

Thus, the proof is complete.
(2): Let $a \in L$ and assume that $\{\alpha_i\}_{i \in I} \subseteq \mathcal{R}_c L$ satisfies $Pt(L) - \Sigma_a = \bigcap_{i \in I} Z(\alpha_i)$. By Lemma 2.2, we have:

$$\Sigma_a = Pt(L) - \bigcap_{i \in I} Z(\alpha_i) = \bigcup_{i \in I} (Pt(L) - Z(\alpha_i) = \bigcup_{i \in I} \Sigma_{\mathrm{coz}(\alpha_i)} = \Sigma_{\bigvee_{i \in I} \mathrm{coz}(\alpha_i)}.$$

Since $L$ is spatial frame, we conclude that $a = \bigvee_{i \in I} coz(\alpha_i)$, and the proof is complete. □

## 3. $z_c$-filters

Continuing our exploration of the connections between the algebraic properties of $\mathcal{R}_c L$ and lattice properties of $L$, we will now focus on the unique characteristics of the zero sets of an ideal of functions. This family exhibits properties similar to those of a filter, a fact that will be crucial for our discussion.

**Definition 3.1.** A nonempty subfamily $\mathcal{F}$ of $Z_c L$ is called a $z_c$-filter on $L$ if it satisfies the following conditions:

1. $\emptyset \notin \mathcal{F}$.
2. for any $Z, W \in \mathcal{F}$, the intersection $Z \cap W \in \mathcal{F}$.
3. If $Z \in \mathcal{F}$, $W \in Z_cL$, and $Z \subseteq W$, then $W \in \mathcal{F}$.

It is clear that, by (3), we conclude that $Pt(L)$ belongs to every $z_c$-filter. A nonempty family $\mathcal{A}$ of setshas the finite intersection property if the intersection of any finite number of its members is nonempty. Any family $\mathcal{A}$ of zero sets of a frame $L$ with the finite intersection property is contained in a $z_c$-filter. Moreover,

$$\mathcal{F} = \{Z \in Z_cL \mid \text{ there exists a finite subset } \mathcal{B} \text{ of } \mathcal{A} \text{ such that } \bigcap \mathcal{B} \subseteq Z\}$$

is the smallest $z_c$-filter containing $\mathcal{A}$.

**Lemma 3.2.** *The following statements hold for each frame L.*

1. *If L is a weakly spatial frame and I is a proper ideal in $\mathcal{R}_cL$, then the family $Z[I] = \{Z(\alpha) \mid \alpha \in I\}$ is a $z_c$-filter on L.*
2. *If $\mathcal{F}$ is a $z_c$-filter on L, then the family $Z^{\leftarrow}[\mathcal{F}] = \{\alpha \in \mathcal{R}L \mid Z(\alpha) \in \mathcal{F}\}$ is a proper ideal in $\mathcal{R}_cL$.*

*Proof.* (1): Since $I$ is a proper ideal in $\mathcal{R}_cL$, it contains no units, and by Corollary 2.5, $\emptyset \notin Z[I]$. For $\alpha, \beta \in I$, we have $\alpha^2 + \beta^2 \in I$, and by Proposition 2.3(2), $Z(\alpha) \cap Z(\beta) = Z(\alpha^2 + \beta^2) \in Z[I]$. If $Z \in Z[I]$ and $W \in Z_cL$ with $Z \subseteq W$, there exists $\alpha \in I$ and $\beta \in \mathcal{R}_cL$ such that $Z = Z(\alpha)$ and $W = Z(\beta)$. Since $\alpha\beta \in I$, we conclude from Proposition 2.3(3) that $W = Z(\alpha) \cup Z(\beta) = Z(\alpha\beta) \in Z[I]$.
(2): Let $J = Z^{\leftarrow}[\mathcal{F}]$. By Proposition 2.3(7), $J$ contains no units. For $\alpha, \beta \in J$ and $\gamma \in \mathcal{R}_cL$. Parts (2) and (3) of Proposition 2.3 imply that $Z(\alpha - \beta) \supseteq Z(\alpha) \cap Z(\beta) \in \mathcal{F}$, and $Z(\gamma\alpha) \supseteq Z(\alpha) \in \mathcal{F}$. Therefore $\gamma\alpha$ and $\alpha - \beta$ are both in $J$. □

A $z_c$-ultrafilter on $L$ refers to a maximal $z_c$-filter, meaning it is not contained in any other $z_c$-filter. For any frame $L$, the following statements are true:

1. A $z_c$-ultrafilter is a maximal subfamily of $Z_cL$ that has the finite intersection property.
2. Any subfamily of $Z_cL$ with the finite intersection property is contained in a $z_c$-ultrafilter.

**Proposition 3.3.** *For a weakly spatial frame L, the following statements are true:*

1. *If M is a maximal ideal in $\mathcal{R}_cL$, then $Z[M]$ is a $z_c$-ultrafilter on L.*
2. *If $\mathcal{F}$ is a $z_c$-ultrafilter on L, then $Z^{\leftarrow}[\mathcal{F}]$ is a maximal ideal in $\mathcal{R}_cL$.*

*Proof.* (1) By Lemma 3.2(1), $Z[M]$ is a $z_c$-filter on $L$. Let $\mathcal{F}$ be a $z_c$-filter on $L$ with $Z[M] \subseteq \mathcal{F}$. From Lemma 3.2(2), $Z^{\leftarrow}[\mathcal{F}]$ is a proper ideal in $\mathcal{R}_cL$ and since $M \subseteq Z^{\leftarrow}[\mathcal{F}]$, we conclude that $M = Z^{\leftarrow}[\mathcal{F}]$. Thus, $Z[M] = \mathcal{F}$, establishing that $Z[M]$ is a $z_c$-ultrafilter on $L$.
(2) By Lemma 3.2(2), $Z^{\leftarrow}[\mathcal{F}]$ is a proper ideal in $\mathcal{R}_cL$. If $I$ is a proper ideal of $\mathcal{R}_cL$ with $Z^{\leftarrow}[\mathcal{F}] \subseteq I$, then $\mathcal{F} \subseteq Z[I]$. Since $\mathcal{F}$ is a $z_c$-ultrafilter on $L$, Lemma 3.2(1) implies that $\mathcal{F} = Z[I]$. Therefore, $Z^{\leftarrow}[\mathcal{F}] = I$, making $Z^{\leftarrow}[\mathcal{F}]$ is a maximal ideal in $\mathcal{R}_cL$. □

The proof of the next corollary directly follows from the preceding proposition and paragraph.

**Corollary 3.4.** *For a weakly spatial frame L, the following statements hold:*

1. *If M be a maximal ideal in $\mathcal{R}_cL$ and $Z(\alpha)$ intersects every member of $Z[M]$, then $\alpha \in M$.*
2. *If $\mathcal{F}$ be a z-ultrafilter on L and a zero-set Z intersects every member of $\mathcal{F}$, then $Z \in \mathcal{F}$.*

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# On Po-noetherian $S$-posets over pomonoids

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**Abstract**

In this paper, we introduce the concept of po-noetherian $S$-posets based on ascending chain conditions on order congruences, and study their general properties.

## 1. Introduction

The properties of being noetherian is of fundamental importance in abstract algebra. Some fundamental properties of right noetherian semigroups can be found in [2]. The concepts of noetherian $S$-acts were investigated in [1]. In this paper, we investigate po-noetherian $S$-posets.

A monoid $S$ is referred to as a *pomonoid* if it is also a partially ordered set with the binary operation and the order relation are compatible. A *right S-poset* $A_S$ is a right $S$-act $A_S$ equipped with a partial order $\leq$, and furthermore,, for every $s, t \in S$ and $a, b \in A$, if $s \leq t$, then $as \leq at$, and if $a \leq b$, then $as \leq bs$. An *S-subposet* of a right $S$-poset $A$ is a subset of $A$ that is closed under the $S$-action. The definition of an ideal remains consistent in the act case.

Let $A$ be a right $S$-poset. An *$S$-poset congruence* $\theta$ on $A$ is a right $S$-act congruence with the property that the $S$-act $A/\theta$ can be made into an $S$-poset in such a manner that the natural map $A \longrightarrow A/\theta$ is an $S$-poset map. For an $S$-act congruence $\theta$ on $A$ we write $a \leq_\theta a'$ if the so-called $\theta$-chain

$$a \leq a_1\ \theta b_1 \leq a_2 \theta b_2 \ldots \leq a_n \theta b_n \leq a'$$

from $a$ to $a'$ exists in $A$, where $a_i, b_i \in A,\ 1 \leq i \leq n$. It can be shown that an $S$-act congruence $\theta$ on a right $S$-poset $A$ is an $S$-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$.

Let $H \subseteq A \times A$. Then $a \leq_{\alpha(H)} b$ if and only if $a \leq b$ or there exist $n \geq 1, (c_i, d_i) \in H, s_i \in S, 1 \leq i \leq n$ such that

$$a \leq c_1 s_1 \quad d_1 s_1 \leq c_2 s_2 \quad \ldots \quad d_n s_n \leq b.$$

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The relation $\nu(H)$ given by $a\ \nu(H)\ b$ if and only if $a \leq_{\alpha(H)} b \leq_{\alpha(H)} a$ is *the $S$-poset congruence induced by $H$*. We have $[a]_{\nu(H)} \leq [b]_{\nu(H)}$ if and only if $a \leq_{\alpha(H)} b$. Moreover, $\nu(H \cup H^{-1})$ is denoted $\theta(H)$ and is called *the $S$-poset congruence generated by $H$*. The set of all congruences on $A_S$ is denoted by $Con(A)$. For more information on $S$-posets, refer to [3].

## 2. Main results

In this section, we introduce a po-noetherian $S$-poset, and study some of its general properties. First, we provide a definition for it.

**Definition 2.1.** Let $A_S$ be an $S$-poset. We call $A_S$ *po-noetherian* if $Con(A)$ satisfies the ascending chain condition.

The notions of *right po-noetherian pomonoid* is applied for a pomonoid $S$ with this property as a right $S$-poset. The following theorem presents a characterization of po-noetherian $S$-posets.

**Theorem 2.2.** *For an $S$-poset $A_S$, the following statements are equivalent.*

(i) *The $S$-act $A_S$ is po-noetherian.*

(ii) *Every congruence of $A_S$ is finitely generated.*

(iii) *Every non-empty subset of $Con(A)$ contains a maximal element.*

**Lemma 2.3.** *For a pomonoid $S$, the following statements hold.*

(i) *Every $S$-subposet of a po-noetherian $S$-poset is po-noetherian.*

(ii) *Every factor $S$-poset of a po-noetherian $S$-poset is po-noetherian.*

**Corollary 2.4.** *Let $S$ be a pomonoid. Then, $S_S$ is po-noetherian if and only if every cyclic $S$-poset is po-noetherian.*

Let $f : A \longrightarrow B$ be an $S$-epimorphism, the *subkernel* of an $S$-poset morphism $f$ is defined by

$$\overrightarrow{kerf} := \{(a, a') \in A \times A : f(a) \leq f(a')\}.$$

Then $\nu(\overrightarrow{kerf}) = kerf := \{(a, a') \in A \times A : f(a) = f(a')\}$, and we denote the subkernel of $f$ briefly by $\mathcal{K}_f$. Obviously, $f$ is a regular monomorphism if and only if

$$\mathcal{K}_f = \xi_A = \{(a, a') \in A \times A |\ a \leq a'\}.$$

Recall from [4] that if $f : A \longrightarrow B$, $g : B \longrightarrow C$ are order preserving maps, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* at $B$ when $\mathcal{I}_f = \mathcal{K}_g$, (i.e., $\leq_{\mathcal{I}m_f} = \leq_{kerg}$). It is also called a *short exact sequence* if $g$ is surjective, $f$ is a regular monomorphism, and $\mathcal{I}_f = \mathcal{K}_g$. The following theorem discusses the behavior of the properties of being po-noetherian for Rees short exact sequences.

**Theorem 2.5.** *Let $A \longrightarrow B \longrightarrow C$ be a Rees short exact sequence of $S$-posets. Then, $B_S$ is po-noetherian if and only if both $A_S$ and $C_S$ are po-noetherian.*

**Lemma 2.6.** *Let $A_S$ be an $S$-poset, and $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n = A$. Then, $A$ is po-noetherian if and only if $A_1$ and the factor $S$-posets $A_{i+1}/A_i$ are po-noetherian for all $1 \leq i \leq n-1$.*

**Proposition 2.7.** *For a pomonoid $S$, the following statements are true.*

(i) *If $A = \prod_{i \in I} A_i$ ($A = \coprod_{i \in I} A_i$) is po-noetherian, then $I$ is finite and each $A_i$ is po-noetherian.*

(ii) *$A_S$ is po-noetherian if and only if $A \coprod \Theta$ is po-noetherian.*

(iii) *If $A_S$ is po-noetherian, then $\coprod_{i=1}^{i=n} A$ is po-noetherian for each $n \in \mathbb{N}$.*

**Proposition 2.8.** *Let $f : S \longrightarrow T$ be a pomonoid homomorphism, and let $A$ be a $T$-act. If $A$ is a right po-noetherian $S$-poset, then it is also a right po-noetherian $T$-poset. If $f$ is an epimorphism, the converse is also true.*

The following corollary shows that po-noetherian property is closed under quotients.

**Corollary 2.9.** *If $\rho$ is a congruence on a right po-noetherian pomonoid $S$, then $S/\rho$ is a right po-noetherian pomonoid.*

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# Study of a class of integral graphs determined by their spectrum

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**Abstract**

In the present paper, we consider the Cayley graph $\Gamma = Cay(\mathbb{Z}_n, S)$, where $n = p^m$, ($p$ is a prime integer, $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_n \mid (a, n) = 1\}$. Although the automorphism group of this graph has been calculated, see [10], but we obtain the automorphism group of this graph by another way. Moreover, we show that $K_v \nabla r\Gamma$ is determined by the adjacency spectrum as well as Laplacian spectrum.

## 1. Introduction

The graphs in this paper are simple, undirected and connected. We always assume that $\overline{\Gamma}$ denotes the complement graph of $\Gamma$. The eigenvalues of a graph $\Gamma$ are the eigenvalues of the adjacency matrix of $\Gamma$. The spectrum of $\Gamma$ is the list of the eigenvalues of the adjacency matrix of $\Gamma$ together with their multiplicities, and it is denoted by Spec($\Gamma$), see [6]. If all the eigenvalues of the adjacency matrix of the graph $\Gamma$ are integers, then we say that $\Gamma$ is an integral graph. The notion of integral graphs was first introduced by F. Harary and A. J. Schwenk in 1974, see [7]. In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area, see [1].
Let $G$ be a finite group and $\Omega$ a subset of $G$ such that it is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = Cay(G, \Omega)$ is the graph whose vertex set and edge set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} \mid x^{-1}y \in \Omega\}.$$

Two graphs with the same spectrum are called cospectral. It is not hard to see that the spectrum of a graph does not determine its isomorphism class [6]. The authors in [4] proposed the question: which graphs are determined by their spectrum?. It seems hard to prove a graph to be determined by its spectrum(DS). Up to now, only a few classes of graphs are proved to be determined by their spectrum, such as: the path $P_n$, the complete graph $K_n$ and the cycle $C_n$. For a graph $\Gamma$, let $A(\Gamma)$ and $L(\Gamma) = D(\Gamma) - A(\Gamma)$ be respectively the adjacency matrix and Laplacian matrix of $\Gamma$, where $D(\Gamma)$ is the diagonal matrix of vertex degrees with $\{d_1, d_2, ..., d_n\}$ as diagonal entries. Laplacian spectrum and their applications are involved indiverse theoretical problems on complex networks [8, 13]. Many results have been devoted to studying Laplacian spectrum for complex networks [9].

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Two graphs $\Gamma_1$ and $\Gamma_2$ are isomorphic if there is a bijection, $\varphi$ say, from $V(\Gamma_1)$ to $V(\Gamma_2)$ such that $x$ is adjacent to $y$ in $\Gamma_1$ if and only if $\varphi(x)$ is adjacent to $\varphi(y)$ in $\Gamma_2$. We say that $\varphi$ is an isomorphism from $\Gamma_1$ to $\Gamma_2$. An isomorphism from a graph $\Gamma$ to itself is called an automorphism of $\Gamma$. The set of automorphisms of $\Gamma$ with the operation of composition of functions is a group, called the automorphism group of $\Gamma$ and denoted by $Aut(\Gamma)$, and it is well known that $Aut(\bar{\Gamma}) = Aut(\Gamma)$, see [6]. Let $T = \{t_1, ..., t_{k+1}\}$ be a set and $K$ be a group then we write $Fun(T, K)$ to denote the set of all functions from $T$ into $K$, we can turn $Fun(T, K)$ into a group by defining a product:

$$(fg)(t) = f(t)g(t) \quad for\ all \quad f, g \in Fun(T, K) \quad and \quad t \in T,$$

where the product on the right is in $K$. Since $T$ is finite, the group $Fun(T, K)$ is isomorphic to $K^{k+1}$ (a direct product of $k+1$ copies of $K$) via the isomorphism $f \to (f(t_1), ..., f(t_{k+1}))$. Let $H$ and $K$ be groups and suppose $H$ acts on the nonempty set $T$. Then the wreath product of $K$ by $H$ with respect to this action is defined to be the semidirect product $Fun(T, K) \rtimes H$ where $H$ acts on the group $Fun(T, K)$ via

$$f^x(t) = f(t^{x^{-1}}) \qquad for\ all \quad f \in Fun(T, K), t \in T \quad and \quad x \in H.$$

We denote this group by $Kwr_T H$. Consider the wreath product $G = Kwr_T H$. If $K$ acts on a set $\Delta$ then we can define an action of $G$ on $\Delta \times T$ by

$$(\delta, t)^{(f,h)} = (\delta^{f(t)}, t^h) \qquad for\ all \quad (\delta, t) \in \Delta \times T,$$

where $(f, h) \in Fun(T, K) \rtimes H = Kwr_T H$, see [5].
In the present paper, we consider the Cayley graph $\Gamma = Cay(\mathbb{Z}_n, S)$, where $n = p^m$, ($p$ is a prime integer, $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_n \mid (a, n) = 1\}$. The authors in [10], by using the theory of distance regular graphs showed that the Cayley graph $\Gamma = Cay(\mathbb{Z}_n, S)$ is an integral graph. In fact, the adjacency spectrum of this graph is $\{n - p^{m-1}, 0^{(n-p)}, (-p^{m-1})^{(p-1)}\}$. They showed the Cayley graph $\Gamma = Cay(\mathbb{Z}_n, S)$ and the multicone graph $K_v \nabla \Gamma$ are determined by the adjacency spectrum as well as Laplacian spectrum, where $K_v$ is the complete graph on $v$ vertices. In this paper, we obtain the automorphism group of this graph by another way. Moreover, we show that $K_v \nabla r\Gamma$ is determined by the adjacency spectrum as well as Laplacian spectrum.

## 2. Definitions And Preliminaries

**Theorem 2.1.** *[2] Let $\Gamma$ be a graph such that contains $k$ components $\Gamma_1, ..., \Gamma_k$. If for any $i \in I = \{1, ..., k\}$, we have $\Gamma_i \cong \Gamma_1$ then $Aut(\Gamma) \cong Aut(\Gamma_1) wr_I Sym(k)$, where the wreath product is defined already.*

**Theorem 2.2.** *[3] Let $\Gamma_i$ be an $r_i$-regular graph of $n_i$ $(i = 1, 2)$ vertices. Then*

$$P(\Gamma_1 \nabla \Gamma_2, \lambda) = \frac{P(\Gamma_1, \lambda) P(\Gamma_2, \lambda)}{(\lambda - r_1)(\lambda - r_2)} ((\lambda - r_1)(\lambda - r_2) - n_1 n_2).$$

**Definition 2.3.** [12] Let $\Gamma_1 \cup \Gamma_2$ denote the disjoint union of graphs $\Gamma_1$ and $\Gamma_2$. The join $\Gamma_1 \nabla \Gamma_2$ is the graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining every vertex of $\Gamma_1$ with every vertex of $\Gamma_2$. A multicone graph is defined to be the join of a clique and a regular graph.

**Theorem 2.4.** *[11] Let $\Gamma_1$ and $\Gamma_2$ be two graphs with the Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$, respectively. Then, the Laplacian spectrum of $\Gamma_1 \nabla \Gamma_2$, is $n + m, m + \lambda_1, m + \lambda_2, ..., m + \lambda_{n-1}, n + \mu_1, n + \mu_2, ..., n + \mu_{m-1}, 0$.*

**Theorem 2.5.** *[6] Let $\Gamma$ be a graph on $n$ vertices. Then, $n$ is a Laplacian eigenvalue of $\Gamma$ if and only if $\Gamma$ is the join of two graphs.*

**Lemma 2.6.** *[6] A connected graph $\Gamma$ has exactly one positive eigenvalue if and only if it is a complete multipartite graph.*

**Corollary 2.7.** *[10] Let $\Gamma = Cay(\mathbb{Z}_n, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$, where $n = p^m$, ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_n \mid (a, n)=1\}$. Then the adjacency spectrum of $\Gamma$ is $\{n - p^{m-1}, 0^{(n-p)}, (-p^{m-1})^{(p-1)}\}$.*

**Proposition 2.8.** *[10] Let $\Lambda$ be a graph cospectral with the multicone graph $K_v \nabla \Gamma$ with respect to its adjacency matrix spectrum, where $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before, then*

$$Spec(\Lambda) = \{0^{(n-p)}, (-p^{m-1})^{(p-1)}, -1^{(v-1)}, (\frac{S+\sqrt{S^2+4T}}{2}), (\frac{S-\sqrt{S^2+4T}}{2})\},$$

*where $S = v - 1 + p^m - p^{m-1}$ and $T = p^m + p^{m-1}v - p^{m-1}$.*

**Theorem 2.9.** *[10] Consider the multicone graph $K_v \nabla \Gamma$, where $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before. Then $K_v \nabla \Gamma$ is DS with respect to its adjacency matrix spectrum.*

**Proposition 2.10.** *[10] Consider the multicone graph $K_v \nabla \Gamma$, where $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before. Then $K_v \nabla \Gamma$ is DS with respect to its Laplacian spectrum.*

## 3. Main results

**Theorem 3.1.** *Let $\Gamma = Cay(\mathbb{Z}_n, S)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$, where $n = p^m$, ($p$ is a prime integer and $m \in \mathbb{N}$) and $S = \{a \in \mathbb{Z}_n \mid (a, n)=1\}$. Then,*

$$Aut(\Gamma) \cong Sym(p^{m-1}) wr_I Sym(p),$$

*where $I = \{1, 2, ..., p\}$.*

*Proof.* Note that if $m = 1$, then the result immediately follows. Because in this case, $\Gamma \cong K_p$, where $K_p$ is the complete graph on $p$ vertices. Hence in the sequel, we assume that $m \geq 2$. Let $V(\Gamma) = \{1, ..., n\}$ be the vertex set of $\Gamma$. It can be checked that the maximum size of an independent subset of vertices in $\Gamma$ is $p^{m-1}$. Because $\Gamma$ is a vertex transitive graph, and it is a regular graph with valency $\phi(p^m) = p^m - p^{m-1}$, where $\phi(p^m)$ denotes the Euler phi-function. Therefore, for any vertex $x$ in $V(\Gamma)$, there are exactly $p^{m-1}$ vertices $y \in V(\Gamma)$ such that $x^{-1}y \notin S$. Hence, if $x^{-1}y \notin S$, then two vertices $x$ and $y$ are adjacent in the complement $\overline{\Gamma}$ of $\Gamma$. So, $\overline{\Gamma}$ contains $p$ components $\Gamma_1, \Gamma_2, ..., \Gamma_p$ such that $\Gamma_i \cong K_{p^{m-1}}$ $(1 \leq i \leq p)$, where $K_{p^{m-1}}$ is the complete graph on $p^{m-1}$ vertices. Thus, we conclude $\overline{\Gamma} \cong pK_{p^{m-1}}$. Hence, by Theorem 2.1, $Aut(\overline{\Gamma}) \cong Aut(K_{p^{m-1}}) wr_I Sym(p) = Sym(p^{m-1}) wr_I Sym(p)$. On the other hand, it is well known that for any graph $\Gamma$, $Aut(\Gamma) = Aut(\overline{\Gamma})$, see [6]. □

**Proposition 3.2.** *Let $\Lambda$ be a graph cospectral with the multicone graph $K_v \nabla r\Gamma$ with respect to its adjacency matrix spectrum, where $r\Gamma$ denoted union $r$ copies of $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before. Then*

$$Spec(\Lambda) = \{(n - p^{m-1})^{(r-1)}, 0^{(n-p)r}, (-p^{m-1})^{(p-1)r}, -1^{(v-1)}, (\frac{S+\sqrt{S^2+4T}}{2}), (\frac{S-\sqrt{S^2+4T}}{2})\},$$

*where $S = v - 1 + p^m - p^{m-1}$ and $T = p^m + p^{m-1}v - p^{m-1} + 2p^m rv - p^m v$.*

*Proof.* By Theorem 2.2, this claim holds. □

**Proposition 3.3.** *Let $\Lambda$ be a graph cospectral with the multicone graph $K_v \nabla r\Gamma$ with respect to its adjacency Laplacian spectrum, where $r\Gamma$ denoted union $r$ copies of $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before. Then*

$$Spec(\Lambda) = \{(2nr + v)^{(v)}, (n + v)^{(p-1)r}, (n - p^{m-1} + v)^{(n-p)r}, v^{(r-1)}, 0\}$$

*Proof.* By Theorem 2.4, this claim holds. □

**Proposition 3.4.** *Consider the multicone graph $K_v \nabla r\Gamma$, where $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before. Then $K_v \nabla r\Gamma$ is DS with respect to its adjacency matrix spectrum.*

*Proof.* By similar way is done in the proof of Theorem 7, in [10], and a few changes this claim holds. □

**Proposition 3.5.** *Consider the multicone graph $K_v \nabla r\Gamma$, where $\Gamma = Cay(\mathbb{Z}_n, S)$ is defined as before. Then $K_v \nabla r\Gamma$ is DS with respect to its Laplacian spectrum.*

*Proof.* By similar way is done in the proof of Proposition 7, in [10], and a few changes this claim holds. □

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# New formulas for calculating Sz, Mo, and PI indices based on the structure of the $\rho$-dimensional hammer and MATLAB code for calculations

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**Abstract**

In this article, novel formulas for the Sz, Mo, and PI indices are presented based on the graph structure of the $\rho$-dimensional hammer. These formulas depend only on the length of the linear polyacene and do not require the derivation of polynomials or the use of complex methods such as edge partition. In this way, the new method simplifies the calculations and provides the relationships between these indices with minimal complexity. The values of these indices are calculated using new formulas for $\rho$-dimensional hammers. Finally, for analysis and understanding of these formulas, the Sz, Mo, and PI indices are compared. In addition, Matlab code is provided to calculate these indices using new formulas and for different values of $\rho$, which is also capable of drawing comparative plots to analyze and compare these indices.

## 1. Introduction

Chemical graph theory is an interdisciplinary field that integrates the principles of graph theory with chemical analysis and focuses on the mathematical representation of chemical phenomena. In this field, mathematical representations of molecular structures are expressed as molecular graphs, where vertices represent atoms and edges represent chemical bonds [1–3]. Among the tools of chemical graph theory, topological indices (TIs) are used to examine the structural properties of molecules through graph analysis. TIs are numerical values derived from a graph, which encapsulate its structural properties. These TIs are constant quantities, meaning they remain unchanged under graph automorphisms and are used to represent chemical structures numerically [1, 2]. They predict the physical, chemical, and biological properties of new structures obtained from a molecule or molecular compound. The predictive capability of these indices is a measure of their quality, as they enable the estimation of various molecular properties, especially for

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new compounds [1]. TIs are derived based on factors such as vertex degree, distance between vertices, eigenvalues, eccentricity, and other structural properties of a graph. In degree-based indices, the number of bonds each atom has in the graph, which corresponds to the degree of the vertex, is considered. In distance-based indices, topological distances between atoms in the graph are calculated. These distances can provide information about the molecular structure and communication properties of molecules. Topological polynomials serve as another important tool in mathematical chemistry. The application of topological polynomials is to calculate TIs and analyze their overall behavior. Numerous topological polynomials have been introduced for graphs, and they are applicable in mathematical chemistry, including Hosoya polynomials, M-polynomials, neighborhood M-polynomials (NM-polynomials), and SMP-polynomials. By utilizing these polynomials, various TIs can be computed more efficiently and swiftly [4]. Graph polynomials are typically expressed as polynomials in one or two variables with integer coefficients. The SMP polynomial, introduced by Knor and Tratnik in 2023, is a unified method for computing three important TIs in graph theory and molecular graphs: the Szeged index (Sz), the Mostar index (Mo), and the PI index. This bivariate polynomial allows computing all three indices with a polynomial instead of three separate polynomials. To obtain these indices, one only needs to compute the first derivative of the SMP polynomial at $x = 1$, which makes the process more efficient, effective, and faster [5, 6]. The SMP-polynomial of $G$ is defined as [7]:

$$SMP(G,x,y) = \sum_{e=uv\in E(G), n_u(e|G)\geq n_v(e|G)} x^{n_u(e|G)} y^{n_v(e|G)}.$$

where $n_u(e|G) = |N_u(e|G)|$ and $N_u(e|G) = \{x \in V(G) | d_G(x,u) < d_G(x,v)\}$ are number and set of vertices of $G$ lying closer to $u$, respectively, see also [8]. The Sz index is the oldest TI based on distance [8, 9] and is as:

$$Sz(G) = \sum_{e=uv\in E(G)} n_u(e|G) n_v(e|G).$$

The Mo used as a measure of peripherality in chemical graphs, is defined as [10]:

$$Mo(G) = \sum_{e=uv\in E(G)} |n_u(e|G) - n_v(e|G)|.$$

Khadikar introduced the vertex-PI index [8] as:

$$PI_v(G) = \sum_{e=uv\in E(G)} (n_u(e|G) + n_v(e|G)).$$

Researchers have made significant efforts in this field in recent years for example, Ghani et al. [11] calculated capacity-based indices for some drugs. Chaudhry et al. [12] computed the closed form of the M-polynomial for the tadpole graph. Moreover, they derived several degree-based TIs as well. Kamran et al. [13] explored TIs of oligothiophene dendrimer via neighborhood M-polynomials. Recently, formulas have been obtained for calculating the TIs of some hydrocarbons, which can be calculated using the apparent property of the structure of the desired hydrocarbon without polynomial calculation [6]. According to previous research, the indices can be obtained by counting edges and using their definition or as the derivative of the SMP polynomial according to Table 3. This article considers the chemical structure of the $\rho$-dimensional hammer ($H_\rho$), focusing on the hammer indices Sz, Mo, and PI, and presents new formulas for their calculation, independent of edge partitioning and without the need to write topological polynomials. The new formulas are based on the length of the linear polyacene connecting the two pyrene fragments in the hammer structure.

## 2. Main result

The $\rho$-dimensional hammer structure $H_\rho$ is obtained by terminating the ends of a linear polyacene of length $\rho$ with two pyrene fragments, see the $H_\rho$ graph in Figure 1. The structure $H_\rho$ consists of $4\rho + 30$ vertices, $5\rho + 37$ edges and $\rho + 8$ hexagons [14]. The edge partitioning of the graph $H_\rho$ based on the length of the linear polyacene is given in Tables 1 and 2. Table 3 shows the mathematical formula for the distance-based TIs used in this article and Table 3 includes some distance-based TIs computed via SMP polynomial.

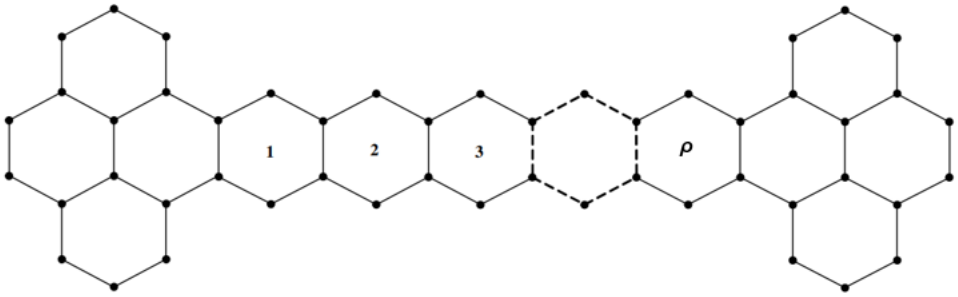


Fig. 1. Structure of $H_\rho$.

Table 1. Partition of the edges of $H_\rho$, when $\rho$ is even and $1 \le i \le \frac{\rho}{2}$.

| Edges types | $\{4\rho+27, 3\}$ | $\{4\rho+25, 5\}$ | $\{4\rho+19, 11\}$ | $\{2\rho+15, 2\rho+15\}$ | $\{4i+17, 13+4i\}$ |
|---|---|---|---|---|---|
| Number | 8 | 12 | 12 | $\rho+5$ | 8 |

**Theorem 2.1.** *Let $G$ be the base graph of $H_\rho$, where $\rho = 2i$ and $i = 0, 1, \dots$. Then the SMP polynomial of $H_\rho$ is as follows:*

$$SMP(H_\rho; x, y) = 8x^{4\rho+27}y^3 + 12x^{4\rho+25}y^5 + 12x^{4\rho+19}y^{11} + (\rho+5)x^{2\rho+15}y^{2\rho+15} + 8\sum_{i=1}^{\frac{\rho}{2}} x^{4i+17}y^{13+4i}.$$

*Proof.* By using Figure 1 and Table 1, the $SMP$-polynomial of $H_\rho$ is calculated as follows:

$$\begin{aligned} SMP(H_\rho; x, y) &= \sum_{e=uv\in E(H_\rho), n_u(e)\ge n_v(e)} x^{n_u(e|H_\rho)}y^{n_v(e|H_\rho)} \\ &= |E_{\{4\rho+27,3\}}|x^{4\rho+27}y^3 + |E_{\{4\rho+25,5\}}|x^{4\rho+25}y^5 + |E_{\{4\rho+19,11\}}|x^{4\rho+19}y^{11} \\ &+ |E_{\{2\rho+15,2\rho+15\}}|x^{2\rho+15}y^{2\rho+15} + |E_{\{4i+17,13+4i\}}|x^{4i+17}y^{13+4i} \\ &= 8x^{4\rho+27}y^3 + 12x^{4\rho+25}y^5 + 12x^{4\rho+19}y^{11} + (\rho+5)x^{2\rho+15}y^{2\rho+15} + 8x^{4i+17}y^{13+4i}. \end{aligned}$$

□

**Proposition 2.2.** *Let $G$ be the graph of $H_\rho$, where $\rho$ is even. Then TIs of $H_\rho$ are obtained as follows:*

- $Sz(H_\rho) = \frac{28}{3}\rho^3 + 216\rho^2 + (2513 + \frac{32}{3})\rho + 5781$,
- $Mo(H_\rho) = 8\rho^2 + 128\rho + 528$,
- $PI(H_\rho) = (4\rho+30)(5\rho+37) = 20\rho^2 + 298\rho + 1110$.

*Proof.* Let $SMP(H_\rho; x, y) = 8x^{4\rho+27}y^3 + 12x^{4\rho+25}y^5 + 12x^{4\rho+19}y^{11} + (\rho+5)x^{2\rho+15}y^{2\rho+15} + 8\sum_{i=1}^{\frac{\rho}{2}} x^{4i+17}y^{13+4i}$.

Table 2. Partition of the edges of $H_\rho$, when $\rho$ is odd and $1 \le i \le \frac{\rho-1}{2}$.

| Edges types | $\{4\rho+27,3\}$ | $\{4\rho+25,5\}$ | $\{4\rho+19,11\}$ | $\{2\rho+15,2\rho+15\}$ | $\{4i+21,13+4i\}$ |
|---|---|---|---|---|---|
| Number | 8 | 12 | 12 | $\rho+9$ | 8 |

Table 3. Topological indices.

| TI | Derivation of TIs from SMP-polynomial | Mathematical formula for the distance-based TIs |
|---|---|---|
| Sz index | $D_xD_ySMP(G;x,y)\|_{x=y=1}$ | $\sum_{e=uv\in E(G)} n_u(e\|G)n_v(e\|G)$ |
| Mo index | $D_xSMP(G;x,\frac{1}{x})\|_{x=1}$ | $\sum_{e=uv\in E(G)} \|n_u(e\|G)-n_v(e\|G)\|$ |
| PI index | $D_xSMP(G;x,x)\|_{x=1}$ | $\sum_{e=uv\in E(G)}(n_u(e\|G)+n_v(e\|G))$ |

The following results are obtained by applying the operators on the SMP-polynomial:

$$D_xD_ySMP(H_\rho;x,y) = 24(4\rho+27)x^{4\rho+27}y^3 + 60(4\rho+25)x^{4\rho+25}y^5 + 132(4\rho+19)x^{4\rho+19}y^{11} + (\rho+5)(2\rho+15)^2x^{2\rho+15}y^{2\rho+15} + 8\sum_{i=1}^{\frac{\rho}{2}}(13+4i)(4i+17)x^{4i+17}y^{13+4i},$$

$$D_xSMP(H_\rho;x,\frac{1}{x}) = 8(4\rho+24)x^{4\rho+24} + 12(4\rho+20)x^{4\rho+20} + 12(4\rho+8)x^{4\rho+8} + 8\sum_{i=1}^{\frac{\rho}{2}}4x^4,$$

$$D_xSMP(H_\rho;x,x) = 8(4\rho+30)x^{4\rho+30} + 12(4\rho+30)x^{4\rho+30} + 12(4\rho+30)x^{4\rho+30} + (\rho+5)(4\rho+30)x^{4\rho+30} + 8\sum_{i=1}^{\frac{\rho}{2}}(4\rho+30)x^{8i+30}.$$

Then, based on Table 3, it is concluded that:

- $Sz(H_\rho) = D_xD_ySMP(H_\rho;x,y)|_{x=y=1} = \frac{28}{3}\rho^3 + 216\rho^2 + (2513+\frac{32}{3})\rho + 5781,$
- $Mo(H_\rho) = D_xSMP(H_\rho;x,\frac{1}{x})|_{x=1} = 8\rho^2 + 128\rho + 528,$
- $PI(H_\rho) = D_xSMP(H_\rho;x,x)|_{x=1} = (4\rho+30)(5\rho+37) = 20\rho^2 + 298\rho + 1110.$

□

**Theorem 2.3.** *Let $G$ be the base graph of $H_\rho$, where $\rho = 2i+1$ and $i = 0,1,\dots$. Then the SMP polynomial of $H_\rho$ is as follows:*

$$SMP(H_\rho;x,y) = 8x^{4\rho+27}y^3 + 12x^{4\rho+25}y^5 + 12x^{4\rho+19}y^{11} + (\rho+9)x^{2\rho+15}y^{2\rho+15} + 8\sum_{i=1}^{\frac{\rho-1}{2}}x^{4i+21}y^{13+4i}.$$

*Proof.* By using Figure 1 and Table 2, the proof is similar to Theorem 2.1. □

**Proposition 2.4.** *Let $G$ be the graph of $H_\rho$, where $\rho$ is odd. Then TIs of $G$ are obtained as follows:*

- $Sz(H_\rho) = \frac{28}{3}\rho^3 + 232\rho^2 + (2721-\frac{16}{3})\rho + 5453,$
- $Mo(H_\rho) = 160\rho + 496,$

- $PI(H_\rho) = 12\rho^2 + 330\rho + 1086$.

*Proof.* The proof of this proposition is similar to the proof of Proposition 2.2. □

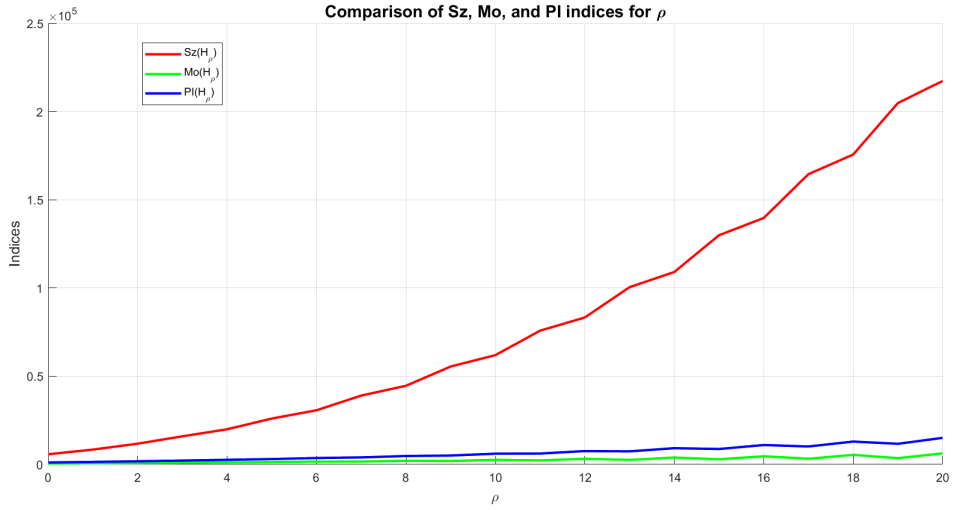


Fig. 2. Comparison of Sz, Mo, and PI indices of $H_\rho$.

Table 4. Calculation of Sz, Mo, and PI indices of $H_\rho$ (0 to 20).

| | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Sz($H_\rho$)** | 5781 | 8410 | 11767 | 15940 | 19929 | 25998 | 30715 | 39032 | 44573 | 55490 | 61951 |
| **Mo($H_\rho$)** | 528 | 656 | 816 | 976 | 1168 | 1296 | 1584 | 1616 | 2064 | 1936 | 2608 |
| **PI($H_\rho$)** | 1110 | 1428 | 1786 | 2184 | 2622 | 3036 | 3618 | 3984 | 4774 | 5028 | 6090 |

| | **11** | **12** | **13** | **14** | **15** | **16** | **17** | **18** | **19** | **20** |
|---|---|---|---|---|---|---|---|---|---|---|
| **Sz($H_\rho$)** | 75820 | 83297 | 100470 | 109060 | 129890 | 139680 | 164520 | 175620 | 204820 | 217320 |
| **Mo($H_\rho$)** | 2256 | 3216 | 2576 | 3888 | 2896 | 4624 | 3216 | 5424 | 3536 | 6288 |
| **PI($H_\rho$)** | 6168 | 7566 | 7404 | 9202 | 8736 | 10998 | 10164 | 12954 | 11688 | 15070 |

In the next section, we present a MATLAB function that calculates three graph indices (Sz, Mo, and PI) for values of $\rho$ within a specified range. The function first checks the validity of the input values and then calculates the indices for each value of $\rho$. The results are then plotted for comparison.

## 3. MATLAB Code

```
function calculate_and_plot_indices_for_range()
    start_rho = input('Please enter the starting value of rho (positive integer): ');
    end_rho = input('Please enter the ending value of rho (positive integer): ');
    if ~isnumeric(start_rho) || start_rho <= 0 || mod(start_rho, 1) ~= 0 || ...
       ~isnumeric(end_rho) || end_rho <= 0 || mod(end_rho, 1) ~= 0 || start_rho >
          end_rho
        error('Rho values must be positive integers, and the start value must be less
           than or equal to the end value.');
```

```
    end
    rho_values = start_rho:end_rho;
    Sz_values = zeros(1, length(rho_values));
    Mo_values = zeros(1, length(rho_values));
    PI_values = zeros(1, length(rho_values));
    for i = 1:length(rho_values)
        rho = rho_values(i);
        if mod(rho, 2) == 0
            Sz = (28/3) * rho^3 + 216 * rho^2 + (2513 + 32/3) * rho + 5781;
            Mo = 8 * rho^2 + 128 * rho + 528;
            PI = 20 * rho^2 + 298 * rho + 1110;
        else
            Sz = (28/3) * rho^3 + 232 * rho^2 + (2721 - 16/3) * rho + 5453;
            Mo = 160 * rho + 496;
            PI = 12 * rho^2 + 330 * rho + 1086;
        end
        Sz_values(i) = Sz;
        Mo_values(i) = Mo;
        PI_values(i) = PI;
    end
    figure;
    hold on;
    plot(rho_values, Sz_values, 'r', 'LineWidth', 2, 'DisplayName', 'Sz(H_{\rho})');
    plot(rho_values, Mo_values, 'g', 'LineWidth', 2, 'DisplayName', 'Mo(H_{\rho})');
    plot(rho_values, PI_values, 'b', 'LineWidth', 2, 'DisplayName', 'PI(H_{\rho})');
    hold off
    xlabel('\rho', 'FontSize', 12);
    ylabel('Indices', 'FontSize', 12);
    title('Comparison of Sz, Mo, and PI indices for \rho', 'FontSize', 14);
    legend('show');
    grid on;
end
```

Listing 1. Matlab code for calculating and comparing indices.

## 4. Conclusion

In this paper, new formulas were presented for Sz, Mo, and, PI indices according to $rho$, where $\rho$ is the length of the linear polyacene connecting two pyrene fragments in the hammer structure. The formulas obtained for these indices do not depend on the number of edges or degree of vertices. Instead, these formulas are more related to the structural properties of the graph. This approach allows us to calculate indices based on the structural properties of the graph for different values of $\rho$. The values of these indices were calculated to investigate the chemical properties and biological behavior of the structure of $H_\rho$ for $\rho = 1, \ldots, 20$, see Tables 4. Topological indices of the hammer graph are compared numerically and graphically. According to Tables 4 and Figure 2 with the increase of $\rho$, the indices increase but their growth rates are different. The sz index shows the highest growth, especially for larger values of $\rho$, indicating higher sensitivity to changes in $\rho$. In contrast, the Mo and PI indices increase at a lower rate, although the PI index shows a significant increase at higher values of $\rho$. Additionally, the formulas used to calculate indices differ for even and odd values of $\rho$, highlighting that the model exhibits distinct behaviors depending on whether $\rho$ is even or odd.

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# On Non-Cyclically Split Division Algebras

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**Abstract**

This paper explores the properties of non-cyclically split division algebras over Henselian fields. Our main theorems establish the relationship between the algebraic structure of a division algebra and its relative value group.

## 1. Introduction

Let $F$ be a field and $A$ be an $F$-central simple algebra of degree $n$, i.e., $[A : F] = n^2$. $A$ is called *cyclic* if it contains a maximal subfield $K/F$ with $\mathrm{Gal}(K/F) = \langle\sigma\rangle$. By Skolem-Noether Theorem [3, p. 39], $A = \bigoplus_{i=0}^{n-1} Kz^i$ for some $z \in A^*$ with $z^n = a \in F^*$ and $zk = \sigma(k)z$ for $k \in K$. We write $A = (K/F, \sigma, a)$. More generally, $A$ is said to be *cyclically split* if there exists a cyclic extension $K/F$ such that $A$ is Brauer equivalent to the cyclic algebra $(K/F, \sigma, b)$ for some $b \in F$, equivalently, $A \otimes_F K \cong M_n(K)$ where $n$ is the degree of $A$. While cyclic algebras have a simple structure, determining cyclicity is difficult. Albert gave the first example of a non-cyclic division algebra [1]. Amitsur and Saltman showed the existence of non-cyclic generic abelian crossed product $p$-algebras of degree $p^n$ $(n \geq 2)$ [2]. As another notable result, Tignol and Wadsworth proved that a tame, totally ramified $F$-central division algebra $D$ with $\mathrm{rank}(\Gamma_D/\Gamma_F) \geq 3$ is not split by any cyclic extension of $F$ [4, Theorem 4.7]. The main goal of this article is to generalize [4, Theorem 4.7] to the case of inertially split division algebras.

## 2. Preliminaries

Let $\Gamma$ be a totally ordered abelian group. A *valuation* $v$ on a division ring $D$ (possibly commutative) with values in $\Gamma$ is a function $v : D^* \to \Gamma$ satisfying the following properties:

1. $v(xy) = v(x) + v(y)$ for all $x, y \in D^*$.

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2. $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in D$.

A *valued division algebra* $D$ is a division algebra equipped with a valuation $v : D^* \to \Gamma$. The value group $\Gamma_D$ of $D$ is defined as $\Gamma_D = v(D^*)$, which captures the set of values assigned by the valuation. We define the valuation ring $V_D$ as:

$$V_D = \{d \in D^* \mid v(d) \geq 0\} \cup \{0\}.$$

It can be shown that $V_D$ is a local ring and its unique maximal ideal is

$$M_D = \{d \in D^* \mid v(d) > 0\} \cup \{0\}.$$

The *residue class division ring* $\overline{D}$ is defined as:

$$\overline{D} = V_D / M_D.$$

Restricting $v$ to $F$, the center of $D$, is a valuation on $F$. $D$ is said to be *unramified* over its center $F$ if the value group of $D$ coincides with the value group of $F$. To be more precise, $\Gamma_D = \Gamma_F$.
We say that $D$ is *tame* if the characteristic of $F$ does not divide the degree of $D$, i.e., $\text{char}(F) \nmid \deg(D)$. In this setting, $D$ is called *inertially split* if there exists an unramified extension $K$ of $F$ such that $D \otimes_F K \cong M_n(K)$, where $n = \deg(D)$.
A valued field $F$ is called *Henselian* if its valuation has a unique extension to each algebraic extension of $F$.

## 3. Results

Our first theorem establishes a relationship between the structure of the relative value group of a division algebra and the Galois group of its splitting field.

**Theorem 3.1.** *Let $D$ be a tame and inertially split $F$-central division algebra. If $D$ is split by an abelian extension $K$ of $F$, then*

$$\text{rank}(\Gamma_D/\Gamma_F) \leq 2 \cdot \text{rank}(\text{Gal}(K/F)).$$

The following corollary follows from Theorem 3.1.

**Corollary 3.2.** *If $D$ is a tame and inertially split $F$-central division algebra and the rank of $\Gamma_D/\Gamma_F$ is at least 3, then $D$ is not cyclically split.*

The next theorem provides a specific condition under which non-cyclicality can be guaranteed, emphasizing the role of roots of unity in the center of a division algebra.

**Theorem 3.3.** *For a tame and inertially split division algebra $D$ over a Henselian field $F$, if* $\text{rank}\,(\Gamma_D/\Gamma_F) = 2$ *and $F$ does not contain any $n$-th root of unity, then $D$ is not cyclically split.*

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# Rings with specific Lie ideals

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**Abstract**

We show that if $I$ is a non-central Lie ideal of a ring $R$ such that all non-zero elements of $I$ are invertible, then $R$ is a division ring. Also, we prove that if $R$ is an $F$-central algebra and $I$ is a Lie ideal not contained in the set of zero divisors such that cardinality of the set of multiplicative cosets $\{aF \mid a \in I\}$ is finite, then either $R$ is a field or $I$ is central. We show the only non-central Lie ideal without zero divisor of a non-commutative central $F$-algebra $R$ with $Char(R) \neq 2$ and radical over the center is $[R,R]$, the additive commutator subgroup of $R$ and in this case $R$ is a generalized quaternion algebra. Finally we prove that if $I$ is a Lie ideal without zero divisor in a central $F$-algebra with characteristic not 2 and if $(\frac{I+F}{F}, +)$ is a finite residual group, then $I$ is central.

## 1. Introduction and Preliminaries

A division ring is a nontrivial ring in which every nonzero element has a multiplicative inverse. Throughout this paper $R$ is a unitary ring with center $Z(R)$ and $F$ is a field. For a pair of elements $a, b$ of $R$ we denote by $[a,b] = ab - ba$ the Lie product of $a$ and $b$. For two subsets $A$ and $B$ let $A \setminus B = \{a \in A \mid a \notin B\}$. An additive subgroup $I$ of $R$ is said to be a Lie ideal if $[r,a] \in I$ for every $r \in R$ and $a \in I$. Also, for subsets $A, B$ of $R$ we denote by $[A,B]$ the additive subgroup of $R$ generated by all $[a,b]$ with $a \in A$ and $b \in B$. An element $a \in R$ is said to be radical over $Z(R)$ if there exists an integer $n = n(a)$ such that $a^n \in Z(R)$. A subset $S \subseteq R$ is said to be radical over $Z(R)$, if each element of $S$ is radical over $Z(R)$. We denote the characteristic of $R$ by $Char(R)$. For a subset $S \subseteq R$, the centralizer of $S$ in $R$ is defined by $C_R(S) = \{r \in R \mid rs = sr \; for \; all \; s \in S\}$. An element $a$ of $R$ is called a zero divisor if there exists a non-zero $b \in R$ such that $ab = 0$ or $ba = 0$. We say an $F$-algebra $R$ is central if $F = Z(R)$. A derivation on $R$ is an additive group homomorphism $d : R \longrightarrow R$ satisfying $d(r_1r_2) = (d(r_1))r_2 + r_1(d(r_2))$.
J. Bergen, I.N. Herstein and C. Lanski studied the structure of a ring $R$ with a derivation $d$ when the non-zero images of $d$ are invertible [3]. In this note, by an analogous approach we are interested in to know the structure of a ring $R$ when the non-zero elements of a Lie ideal $I$ of $R$ are invertible. We show that when $R$ is a ring with $Char(R) \neq 2$ and $I$ is a non-central Lie ideal of $R$ such that all non-zero elements of $I$ are invertible, then $R$ is a division ring. As a consequence we also present a commutativity condition over a ring. In particular, we show that if $I$ is a Lie ideal of an $F$-central algebra $R$ without zero divisor such that cardinality of the multiplicative cosets $\{aF \mid a \in I\}$ is finite, then either $R$ is a field or $I$ is central. Also we prove that any division ring $D$ with characteristic not 2 which contains

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a non-central Lie ideal $I$ radical over the center is isomorphic to the generalized quaternion algebra and $I = [D, D]$. At last we prove that when $R$ is an $F$-central algebra and $Char(R) \neq 2$ and $I$ is a Lie ideal without zero divisor, if the residual additive group $(\frac{I+F}{F}, +)$ is of finite cardinality, then $I$ is central. First we recall the following two theorems.

**Theorem 1.1.** [6] *Let $D$ be a division ring with center $F$, such that $(xy - yx)^{n(x,y)} \in F$, $n(x, y) \geq 1$ for all $x, y \in D$, then $dim_F(D) \leq 4$.*

**Theorem 1.2.** [5, p. 5] *Let $D$ be a division algebra with center $F$ and $Char(D) \neq 2$ and let $I$ be a Lie ideal of $D$. Then either $I \subseteq F$ or $[D, D] \subseteq I$.*

## 2. Main Results

We show that the invertibility condition on some special subsets or substructures of a ring implies the invertibility of all elements of the ring. In particular, we show that a ring, whose all non-zero additive commutators are invertible, is a division ring. This is the content of the following Theorem.

**Theorem 2.1.** [1] *Let $R$ be a ring with center $Z(R)$, whose all non-zero additive commutators are invertible. Then $R$ is a division ring.*

Clearly $[R, R]$ is a Lie ideal in any ring $R$, containing all additive commutators. When $R$ is a division ring by Theorem 1.2 all Lie ideals contain $[R, R]$ as a substructure, but in general there is not a clear relation between a Lie ideal and $[R, R]$. So one may ask what would be the case when the same condition, as above theorem, is on a Lie ideal of a ring. In the following theorem we show that only a division ring may contains a Lie ideal such that all of its non-zero elements are invertible.

**Theorem 2.2.** [1] *Let $R$ be a ring and $Char(R) \neq 2$. If $I$ is a non-central Lie ideal of $R$, whose all non-zero elements are invertible, then $R$ is a division ring.*

*Proof.* First, we show that all elements of $R \setminus C_R(I)$ are invertible. Let $x \in R \setminus C_R(I)$. There exists an element $a \in I$ such that $[a, x] \neq 0$. The equation $[a, x^2] + 2x[x, a] = [[a, x], x]$, implies that $2x[a, x] \in I$. Therefore $x(2[a, x])$ is invertible and so $x$ is invertible.
Now, we show that all elements of $C_R(I)$ are invertible. Let $x \in C_R(I) \setminus \{0\}$. Since $I$ is non-central we may find elements $u \in I$ and $y \in R$ such that $[y, u] \neq 0$. Thus $y \notin C_R(I)$. Using above argument we find that $y$ is invertible. We claim that $xy \notin C_R(I)$. Otherwise

$$(xy)u - u(xy) = (xy)u - (ux)y = (xy)u - (xu)y = x[y, u] = 0.$$

Since $[y, u] \in I$ implies that $x = 0$, a contradiction. Hence $xy \notin C_R(I)$ which is invertible by above. Now clearly $x$ is invertible. This completes the proof. □

We use the above theorem to present a commutativity condition in terms of Lie ideals not contained in the set of zero divisors.

**Theorem 2.3.** [1] *Let $F$ be a field and $R$ be an $F$-central algebra with a Lie ideal $I$ without zero divisor. If the set of multiplicative cosets $\{aF \mid a \in I\}$ has a finite cardinality, then either $R$ is a field or $I$ is central.*

To present our next result, we need to remind the following concepts. Let $F$ be a field with $Char(F) \neq 2$. By [2] when $R$ is a finite dimensional $F$-algebra, then $[R, R]$ is a hyperplane in $R$. The generalized quaternion algebra $D$ is defined of the form

$$D = \left(\frac{a, b}{F}\right) = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in F\},$$

where $i^2 = a, j^2 = b, ij = k$ and $a, b \in F$ [4, p. 136]. The classical instance where $F = \mathbb{R}$ is Hamilton's quaternions $(a = b = -1)$. Then one can easily show that $[D, D] = \{\alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_1, \alpha_2, \alpha_3 \in F\}$. In the following we show that only generalized quaternion algebras $D$ may contain non-central radical Lie ideal $I$ and in this case $I = [D, D]$.

**Theorem 2.4.** [1] *Let $R$ be a non-commutative $F$-central algebra and $Char(R) \neq 2$. If $I$ is a non-central Lie ideal of $R$ without zero divisor and radical over $F$, then $I = [R, R]$ and $R$ is a generalized quaternion algebra.*

*Proof.* By Theorem 2.2, $R$ is a division ring. By Theorem 1.1, $dim_F(R) \leq 4$. We know that every 4-dimensional central simple algebra is isomorphic to $\left(\frac{a,b}{F}\right)$, for some $a, b \in F \setminus \{0\}$[4, p. 136]. Since $R$ is non-commutative, $dim_F(R) = 4$ and the unique possibility is that $R = \left(\frac{a,b}{F}\right)$. By Theorem 1.2, we have $[R, R] \subseteq I$. Thus it suffices to prove $I \subseteq [R, R]$. Otherwise, suppose that $I$ contains an element $\alpha_0+\alpha_1 i+\alpha_2 j+\alpha_3 k \notin [R, R]$. Since $\alpha_1 i+\alpha_2 j+\alpha_3 k \in [R, R]$, we find that $\alpha_0 \in I \setminus \{0\}$. Therefore, for every $x \in [R, R]$ one can see $\alpha_0 + x \in I$. Hence $\alpha_0(1 + \alpha_0^{-1}x)$ is radical over $F$. Thus we conclude that $1 + \alpha_0^{-1}x$ is radical over $F$. It is easy to verify that $c(1 + \alpha_0^{-1}x)$ is radical over $F$, for every $c \in F$. Now, if we consider $x = c^{-1}\alpha_0\beta_1 i + c^{-1}\alpha_0\beta_2 j + c^{-1}\alpha_0\beta_3 k$, where $\beta_1, \beta_2, \beta_3 \in F$, then we obtain that $c + \beta_1 i + \beta_2 j + \beta_3 k$ is radical over $F$, for all $c, \beta_1, \beta_2, \beta_3 \in F$. This implies that $R$ is radical over the center which by Kaplansky's Theorem [7, p. 246] is a field and a contradiction. □

We need the following technical Lemma to give our next result.

**Lemma 2.5.** [1] *Let $R$ be a ring and $a, y \in R$ such that $a$ is not zero divisor. If $ay \in Z(R)$, then $ay = ya$.*

**Theorem 2.6.** [1] *Let $R$ be a central $F$-algebra with $Char(R) \neq 2$ and let $I$ be a Lie ideal of $R$ without zero divisor. If the residual additive group $(\frac{I+F}{F}, +)$ is of finite cardinality, then $I$ is central.*

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# Some studies on Lie ideals in division rings

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**Abstract**

In this note we study the structure of Lie ideals in associative algebras. In particular we prove that given any finite dimensional division algebra $A$ with the center $F$ such that $charF \neq 2$, any finitely generated $\mathbb{Z}$-module Lie ideal of $A$ is central. As a consequence we also prove that if $A$ is a division algebra of finite dimension over its center $F$ and $charF \neq 2$ then the additive commutator subgroup of $A$ or $[A, A]$ is not finitely generated $\mathbb{Z}$-module. Also some results on maximal Lie subring and maximal Lie ideal of $M_n(D)$ are studied.

## 1. Introduction

Let $A$ be an associative algebra. If we replace the usual multiplication $ab$ of two elements $a$ and $b$ of $A$ by their Lie product $[a, b] = ab - ba$, then simultaneously, we have a non associative Lie ring [7]. The question is that is there any kind of duality between these two structures over the set $A$. In what follows we try to test a series of properties from the associative structure to the another Lie structure one to give some kind of correspondence [3].
An additive subgroup $L$ of $A$ is called a Lie ideal of $A$ if for any $l \in L$ and any $x \in A$, their Lie product or $[l, x]$ is again in $L$. An additive subgroup $T$ of $A$ is called a Lie subring of $A$ if $[t, t^{'}] \in T$, where $t, t^{'} \in T$. A derivation of a ring $A$ is an additive group homomorphism $d : A \longrightarrow A$ satisfying $d(a_1a_2) = (d(a_1))a_2 + a_1(d(a_2))$. Inner derivation of a ring $A$ is denoted by $Inn(A)$ and is defined $Inn(A) = \{d_a;\ for\ all\ a \in A,\ where\ d_a(x) = ax - xa,\ for\ x \in A\}$. In a multiplicative group the normal subgroups usually is defined as the subgroups which are invariant under inner automorphisms. Equivalently in a Lie algebra, Lie ideals can be defined as the submodules which are invariant under all inner derivations.
Two main theorems which guide us for identifying a connection between the concept of normal subgroups and Lie ideals are as follows. The Skolem-Noether theorem states that if $A$ is finite dimensional central simple $F$-algebra then every $F$-automorphism of $A$ is inner [5]. The additive version of this theorem states that if $A$ is a finite dimensional central simple $F$-algebra then every $F$-linear derivation of $A$ is inner [5].
The other theorem which is a clue in identifying a sort of duality is the Cartan-Brauer-Hua theorem which states that if $A$ is a division ring and $B$ is a subdivision ring of $A$ such that $B^*$ is a normal subgroup of $A^*$ then either $B = A$ or

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$B \subseteq Z(A)$ [6]. The additive version of this theorem states that if $A$ is a division ring and $B$ is a subdivision ring of $A$ such that $B$ is a Lie ideal in $A$ and $charA \neq 2$ then either $B = A$ or $B \subseteq Z(A)$ [6].
S. Akbari [4] proved that if $D$ be a finite dimensional division algebra with center $F$ then any finitely generated normal subgroup of $D^*$ is central. Also he proved that if $D$ be an infinite division ring with center $F$ such that $[D : F] < \infty$ then $D^*$ contains no finitely generated maximal subgroups. In this note analogous to some related researches [3, 4], we show that if $A$ is a finite dimensional division algebra with center $F$ such that $charF \neq 2$ then any finitely generated $\mathbb{Z}$-module Lie ideal of $A$ is central. We also show that if $A$ is an infinite division ring with center $F$ such that $[A : F] < \infty$ then $A$ contains no finitely generated maximal Lie ideals as a $\mathbb{Z}$-module.

## 2. Main results

We present two theorems just as a dual Lie version of the same theorem about normal subgroups of a division ring [3].

**Theorem 2.1.** [1] *Let $A$ be a division ring which is finite dimension over its center $F$ and $charF \neq 2$. Let $L$ be a noncentral Lie ideal in $A$ such that as a $\mathbb{Z}$-module is finitely generated. Then there exists a finite subset $\Gamma$ of $F$ such that $F = P(\Gamma)$.*

**Theorem 2.2.** *Let $F$ be an algebraic extension of $\mathbb{Q}$ and $A$ be a division ring which is finite dimension over its center $F$. Then noncentral Lie ideals of $A$ are not finitely generated $\mathbb{Z}$-module.*

The following gives a more general version of the above theorem and just as an analogous result about division ring and normal subgroup given in [4].

**Theorem 2.3.** [1] *Let $A$ be a finite dimensional division algebra with center $F$ such that $charF \neq 2$. Then any finitely generated $\mathbb{Z}$-module Lie ideal of $A$ is central.*

Now we can say a main result as a corollary.

**Corollary 2.4.** [1] *Let $A$ be a noncommutative division algebra of finite dimension over its center $F$ and $charF \neq 2$. Then the additive commutator subgroup of $A$ or $[A, A]$ is not finitely generated $\mathbb{Z}$-module.*

**Example 2.5.** Let $F$ be a field with $charF \neq 2$. By [2] when $R$ is a finite dimensional $F$-algebra, then $[R, R]$ is a hyperplane in $R$. Consider the generalized quaternion algebra

$$D = \left(\frac{a, b}{F}\right) = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in F\},$$

where $i^2 = a$, $j^2 = b$, $ij = k$ and $a, b \in F$ [5, p. 136]. Then one can easily show that $[D, D] = \{\alpha_1 i + \alpha_2 j + \alpha_3 k \mid \alpha_1, \alpha_2, \alpha_3 \in F\}$ which is not finitely generated as a $\mathbb{Z}$-module.

Another consequences of the above theorem, about maximal Lie ideals are as follows.

**Corollary 2.6.** *Let $A$ be an infinite division ring with center $F$ such that $[A : F] < \infty$. Then $A$ contains no finitely generated maximal Lie ideals as a $\mathbb{Z}$-module.*

**Corollary 2.7.** [1] *Let $A$ be a division ring with center $F$ with $charF \neq 2$ and assume that $L$ is a maximal Lie ideal of $A$ containing $F$. If the additive group index of $L$ over $F$ is finite or $[L : F] < \infty$, then $A = F$.*

we continue our study with the following two lemmas about maximal Lie subrings.

**Lemma 2.8.** [1] *Let $A$ be an $F$-algebra and $L$ be a maximal Lie subring of $A$. Then we have*

*(i) $L$ contains either $F$ or $[A, A]$.*

*(ii) If $A$ is a division ring, then either $A = F(L)$ or $L\backslash\{0\}$ is a multiplicative group, where $F(L)$ is the division ring generated by $L$ and $F$.*

**Lemma 2.9.** [1] *Let $D$ be a division ring with center $F$ and assume that $L$ is a maximal Lie subring of $D$. Then either the multiplicative center of $L$ is equal to $F \cap L$ or $L$ is a maximal division subring of $D$.*

Now we can give the following results.

**Theorem 2.10.** [1] *Let $D$ be a non-commutative division ring with center $F$. Then $D$ contains no finitely generated $\mathbb{Z}$-module maximal Lie ideal.*

Now, we show that the converse of the above theorem is true. Let $D$ be a commutative division ring. Thus any additive subgroup of $D$ is a Lie ideal of $D$. This means that any finitely generated maximal additive subgroup of $D$ is a finitely generated $\mathbb{Z}$-module maximal Lie ideal of $D$. This implies that if $D$ is a commutative division ring then $D$ contains a finitely generated $\mathbb{Z}$-module maximal Lie ideal. Therefore the converse of the above theorem is true.

**Theorem 2.11.** [1] *Let $D$ be a division algebra algebraic over its center $F$ with $charF \neq 2$ and $n$ is a natural number. Assume that $L$ is a maximal Lie ideal of $M_n(D)$. If $L$ is finite, then $D = F$.*

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# Solving a Nonlinear Fractional Differential Equation Using Picard Iteration Method Based on Fractional Legendre Functions

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**Abstract**

This paper presents a novel approach for solving nonlinear fractional differential equations (FDEs) using the Picard iteration method combined with fractional Legendre function. The proposed method uses the orthogonal properties of fractional Legendre functions to approximate solutions iteratively, ensuring both accuracy and computational efficiency. The non-smooth numerical example is provided to demonstrate its effectiveness in handling nonlinear FDEs.

## 1. Introduction

Systems of fractional differential equations (SFDEs) have garnered significant attention in recent years due to their ability to model complex phenomena across various scientific and engineering disciplines, including fluid dynamics, viscoelasticity, and control theory. Unlike classical integer-order differential equations, SFDEs incorporate non-local operators, which provide a more accurate representation of systems exhibiting memory and hereditary properties. However, the inherent nonlinearity and complexity of SFDEs often present substantial challenges in deriving analytical solutions, necessitating the development of efficient numerical methods [5]. Recent advancements have led to the development of various approximate methods for numerically solving systems of fractional differential equations [2, 3]. Among the available numerical techniques, iterative methods have demonstrated considerable promise in addressing nonlinear SFDEs due to their simplicity and adaptability. The Picard iteration method, in particular, is a well-established approach for solving nonlinear equations. It generates a sequence of approximations that converge to the exact solution. When combined with suitable basis functions, such as fractional Legendre functions, the Picard method can be enhanced to achieve higher accuracy and faster convergence rates. In this study, we propose a hybrid approach that integrates the Picard iteration method with fractional Legendre function bases to solve nonlinear SFDEs. Fractional Legendre functions, renowned for their orthogonality and flexibility, serve as an effective tool for approximating solutions in the fractional domain. The proposed method's performance is validated through a series of numerical experiments. The results demonstrate the method's efficiency in handling a wide range of nonlinear SFDEs, offering a reliable and effective alternative to existing techniques.

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## 2. Preliminaries

In the initial part of this section, we revisit some fundamental properties of fractional calculus theory. Subsequently, we introduce several properties of the Legendre function, which will be utilized in the following sections of this paper.

**Definition 2.1.** [4] The Riemann–Liouville fractional integral operator of order $q > 0$, applied to a function $u(x)$, is defined as:

$$\mathcal{J}_0^q u(x) = \frac{1}{\Gamma(q)} \int_0^x (x-s)^{q-1} u(s) ds, \quad 0 < x < L, \tag{1}$$

$$\mathcal{J}_0^0 u(x) = u(x), \tag{2}$$

The operator $\mathcal{J}^q$ satisfies the following propertie

$$\mathcal{J}_0^q x^k = \frac{\Gamma(q+1)}{\Gamma(q+1+k)} x^{q+k}, \quad q \geq 0, \qquad k \geq -1. \tag{3}$$

**Definition 2.2.** [4] The Caputo fractional derivative of order $q$ is defined by the following expression:

$$^c\mathcal{D}^q u(x) = \mathcal{J}_0^{n-q} u^{(n)}(x) = \frac{1}{\Gamma(n-x)} \int_0^x (x-s)^{n-q-1} u^{(n)}(s) ds, \quad n-1 < q \leq n, \quad x > 0.$$

The operator $^c\mathcal{D}^q$, satisfies the following properties:

$$\mathcal{J}_0^q {}^c\mathcal{D}^q u(x) = u(x) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} x^k, \quad x > 0,$$

$$^c\mathcal{D}^q \mathcal{J}_0^q u(x) = u(x), \quad n-1 \leq q \leq n.$$

### *2.1. Shifted Legendre function*

The analytic form of the shifted Legendre polynomial on the interval $\Lambda = [0, L]$ is defined as [6]:

$$\mathcal{L}_n^*(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k} \Gamma(n+k+1)}{\Gamma(k+1)(n-k)!k!L^k} x^k. \tag{4}$$

Hence, the orthogonality condition is

$$\int_\Lambda \mathcal{L}_n^*(x) \mathcal{L}_m^*(x) dx = \left(\frac{L}{2m+1}\right) \delta_{nm},$$

where, $\delta_{nm}$ is Kronecker delta function.

The fractional Legendre functions $\overset{*}{\mathcal{L}}_n^{\lambda}(t)$ with $\lambda > 0$ and $t \in [0, L]$ are defined from the shifted Legendre polynomials through the coordinate transform $t = x^\lambda$ as follows [1]:

$$\overset{*}{\mathcal{L}}_n^{\lambda}(t) = \overset{*}{\mathcal{L}}_n(x^\lambda). \tag{5}$$

These functions have the following explicit formula

$$\overset{*}{\mathcal{L}}_n(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k} \Gamma(n+k+1)}{\Gamma(k+1)(n-k)!k!L^{\lambda k}} t^{\lambda k}.$$

It can easily be checked that these functions are mutually orthogonal concerning the weight function

$$\overset{*}{\omega}^{\lambda}(t) = 1,$$

i.e.,

$$\int_{\Lambda} \overset{*}{\mathcal{L}}_m^{\lambda}(t)\, \overset{*}{\mathcal{L}}_n^{\lambda}(t)\, \overset{*}{\omega}^{\lambda}(t)dt = \left(\frac{L^{\lambda}}{2m+1}\right)\delta_{nm}.$$

Thus, for any $u \in L^2_{\overset{*}{\omega}^{\lambda}}(\Lambda)$ we have

$$u(t) = \sum_{i=0}^{\infty} u_i \, \overset{*}{\mathcal{L}}_i^{\lambda}(t),$$

where the cofficients $u_i$ are given by

$$u_i = \frac{2i+1}{L^{\lambda}} \int_0^L u(t)\, \overset{*}{\mathcal{L}}_i^{\lambda}(t) \overset{*}{\omega}^{\lambda}(t)dt, \quad i = 0, 1, 2, \cdots. \tag{6}$$

In practical applications, only the first $(N+1)$ terms of the shifted Legendre functions are typically considered. As a result, the function $u(t)$ can be conveniently expressed in the following form:

$$u_N(t) \simeq \sum_{i=0}^{N} u_i \, \overset{*}{\mathcal{L}}_i^{\lambda}(t)$$

The integral in (6) can be approximated using the shifted Legendre-Gauss quadrature rule as

$$u_i(t) \simeq \frac{2i+1}{2} \sum_{j=0}^{N} u(t_j)\, \overset{*}{\mathcal{L}}_i^{\lambda}(t_j)\omega_j, \quad i = 0, 1, \cdots, N. \tag{7}$$

where

$$t_j = L\left(\frac{x_j+1}{2}\right)^{\frac{1}{\lambda}}, \qquad j = 0, 1, \cdots, N, \tag{8}$$

and $\{x_j\}_{j=0}^N$ are the roots of $\mathcal{L}_{N+1}(x)$ and $\{\omega_j\}_{j=0}^N$ are corresponding weights as

$$\omega_j = \frac{2}{(1-x_j^2)(\mathcal{L}'_{N+1}(x_j))^2} \tag{9}$$

$$= \frac{(2N+2)}{(N+1)^2 \mathcal{L}_N(x_j)\mathcal{L}'_{N+1}(x_j)}, \quad j = 0, 1, \cdots, N. \tag{10}$$

The article continues by addressing the calculation of the fractional integral of shifted Legendre functions. Since expressing the fractional integral of these functions in terms of the functions themselves is pivotal to the method discussed in this study, the following theorem establishes this essential relationship.

**Theorem 2.3.** *Let* $\overset{*}{\mathcal{L}}_n^{\lambda}(t)$ *be the shifted Legendre functions of degree* $n$*, Then we have*

$$\mathcal{J}_0^q \, \overset{*}{\mathcal{L}}_i^{\lambda}(t) = \sum_{j=0}^{N} \Theta(i,j)\, \overset{*}{\mathcal{L}}_j^{\lambda}(t), \quad i = 0, 1, \cdots, N, \tag{11}$$

*where*

$$\Theta(i,j) = \sum_{k=0}^{i} \zeta_{ijk}, \tag{12}$$

*and*

$$\zeta_{ijk} = \frac{(-1)^{i-k}\Gamma(i+k+1)\Gamma(k\lambda+1)}{\Gamma(k+1)\Gamma(k\lambda+q+1)(i-k)!k!} \times \sum_{h=0}^{j} \frac{(-1)^{j-h}\Gamma(j+h+1)\Gamma(h+k+\frac{q}{\lambda}+1)(2j+1)}{\Gamma(h+1)\Gamma(h+\frac{q}{\lambda}+k+2)(j-h)!h!} L^q. \tag{13}$$

**Lemma 2.4.** *According to Theorem 2.3 , in vector form, as indicated by [3], we have*

$$\mathcal{J}_0^q \overset{*}{\mathcal{L}}_i^{\lambda}(t) \simeq [\Theta(i,0), \Theta(i,1), \cdots, \Theta(i,N)]\psi(t), \quad i = 0,1,\cdots,N. \tag{14}$$

## 3. Fractional Legendre-Picard iteration method

In this section, our proposed method named as the FLPIM provides an iterative algorithm for solving a SFDEs of the form

$${}^{c}\mathcal{D}^{q_j}U_j(t) = G_j(t, U_j(t)), \quad 0 \le t \le L, \quad m_j - 1 \le q_j \le m_j, \quad m \in \mathbb{N}, \tag{15}$$

$$U_j^{(l)}(0) = U_{j0}^{(l)}, \qquad l = 0,1,\dots m_j - 1, \quad 1 \le j \le n. \tag{16}$$

By applying the fractional integral to equation (16), we have

$$U_j(t) = \sum_{l=0}^{m_j-1} \frac{t^l U_{j0}^{(l)}}{l!} + \frac{1}{\Gamma(q_j)} \int_0^t (t-s)^{q_j-1} G_j(s, U_j(s)) ds. \tag{17}$$

According to (17), the iteration sequence is generated in the following way

$$U_j^i(t) = \sum_{l=0}^{m_j-1} \frac{t^l U_{j0}^{(l)}}{l!} + \frac{1}{\Gamma(q_j)} \int_0^t (t-s)^{q_j-1} G_j(s, U_j^{i-1}(s)) ds, \tag{18}$$

where $U_j^0(t)$ is an appropriate initial function that corresponds to the initial conditions of the problem. The first step of the FLPIM is to approximate the function $f(t, U_j^{i-1}(t))$ using $\{\overset{*}{\mathcal{L}}_n^{\lambda}(t)\}_{n=0}^{N}$. Therefore

$$G_j(t, U_j^{i-1}(t)) \simeq \sum_{k=0}^{N} \tilde{G}_{jk}^{i-1} \overset{*}{\mathcal{L}}_k^{\lambda}(t), \tag{19}$$

where $\{\tilde{G}_{jk}^{i-1}\}_{j=0}^{N}$ obtained as $\tilde{G}_{jk}^{i-1} = \frac{2k+1}{2} \sum_{r=0}^{N} G_j(t_r, \mathcal{U}_j^{i-1}(t_r)) \overset{*}{\mathcal{L}}_k^{\lambda}(t_r)\omega_r$. With the placement of equation (19) at (18), and according to (11) we have

$$\begin{aligned} U_d^i(t) &= \sum_{l=0}^{m_j-1} \frac{t^l U_{d0}^{(l)}}{l!} + \mathcal{J}^{q_j}(G_d(t, U_d^{i-1}(t))) \\ &= \sum_{l=0}^{m_j-1} \frac{U_{d0}^{(l)}}{l!} \sum_{j=0}^{N} C_{lj} \overset{*}{\mathcal{L}}_j^{\lambda}(t) + \sum_{k=0}^{N} \tilde{G}_{dk}^{i-1} \sum_{j=0}^{N} \sum_{k=0}^{i} \Theta_{ijk} \overset{*}{\mathcal{L}}_j^{\lambda}(t) \\ &= \sum_{j=0}^{N} \overset{*}{\mathcal{L}}_j^{\lambda}(t) \Big( \sum_{l=0}^{m_j-1} \frac{U_{d0}^{(l)}}{l!} C_{lj} + \sum_{k=0}^{N} \tilde{G}_{dk}^{i-1} \sum_{k=0}^{i} \Theta_{ijk} \Big), \quad 0 \le d \le n. \end{aligned} \tag{20}$$

where $C_{lj} = (2j+1)L^l \sum_{f=0}^{j} \frac{(-1)^{j-f}\Gamma(j+f+1)\Gamma(f+l+1)}{\Gamma(f+1)\Gamma(f+l+2)(j-f)!f!}$. We assume that the approximate solution in the $i$-th step is shown as

$$U_d^i(t) \simeq \sum_{j=0}^{N} a_j^i \overset{*}{\mathcal{L}}_j^{\lambda}(t) \simeq \overset{*}{\mathcal{L}}_0^{\lambda}(t)\Big(\sum_{l=0}^{m_j-1} \frac{U_{d0}^{(l)}}{l!} C_{l0} + \sum_{k=0}^{N} \tilde{G}_{dk}^{i-1} \sum_{k=0}^{i} \Theta_{i0k}\Big)$$
$$+ \cdots + \overset{*}{\mathcal{L}}_N^{\lambda}(t)\Big(\sum_{l=0}^{m_j-1} \frac{U_{d0}^{(l)}}{l!} C_{lN} + \sum_{k=0}^{N} \tilde{G}_{dk}^{i-1} \sum_{k=0}^{i} \Theta_{iNk}\Big). \tag{21}$$

We obtain the coefficients $\{a_j^i\}_{j=0}^N$ directly from (21), as follows:

$$a_j^i = \sum_{l=0}^{m_j-1} \frac{U_{d0}^{(l)}}{l!} C_{lj} + \sum_{k=0}^{N} \tilde{G}_{dk}^{i-1} \sum_{k=0}^{i} \Theta_{ijk}, \quad j = 0, 1, \cdots N, \quad 0 \le d \le n.$$

After updating the cofficients, the new estimated solution is generated. The proposed algorithm continues to iterate until the stopping criterion is met, which is defined as $\|U_d^i(t) - U_d^{i-1}(t)\|_\infty < \epsilon$.

## 4. Numerical example

To assess the precision and effectiveness of our method, we introduce the following system of FDE with non-smooth solutions. First, we establish the absolute error as:

$$Absolute\ error = |U(t) - U_{App}(t)|.$$

where $U(t)$ and $U_{App}(t)$ represent the exact solution and approximate solution, respectively.

**Example 4.1.** Consider a system of non-smooth fractional differential equations:

$$^c\mathcal{D}^{q_1} U_1(t) = \frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{7}{2}-q_1)} t^{\frac{5}{2}-q_1} - t^{\frac{3}{2}} + U_2(t), \qquad 0 < q_1 \le 1,$$
$$^c\mathcal{D}^{q_2} U_2(t) = \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-q_2)} t^{\frac{3}{2}-q_2} - t^{\frac{5}{2}} + U_1(t), \qquad 0 < q_2 \le 1,$$
$$U_1(0) = 0, \ U_2(0) = 0, \qquad 0 < t \le 1.$$

The exact solution of this system is $U_1(t) = t^{\frac{5}{2}}$, $U_2(t) = t^{\frac{3}{2}}$. We find that the exact and approximation solutions closely line up. Table 1 shows the maximum absolute error for various values of $\lambda$. The results indicate that $\lambda = 0.5$ is the optimal value, providing the highest accuracy in solving the nonlinear fractional differential equation. Table 2 compares the proposed method with MFJPIM [3]. The proposed method demonstrates superior performance, achieving lower errors and faster convergence, which highlights its efficiency and effectiveness in solving nonlinear fractional differential equations.
These results confirm the advantages of combining the Picard iteration method with fractional Legendre function bases, offering a robust and accurate approach for such problems.

## 5. Conclusion

In this study, we propose a hybrid numerical approach that combines the Picard iteration method with fractional Legendre function bases to solve nonlinear systems of fractional differential equations (SFDEs). Using the orthogonality and

Table 1. The maximum absolute error of $U_1(t)$ and $U_2(t)$ is examined for $q_1 = q_2 = 0.5$ and $N = 10$ with different values of $\lambda$ in Example 4.1.

| $\lambda$ | $E_{U_1(t)}$ | $E_{U_2(t)}$ |
|---|---|---|
| 1 | $2.4957e-05$ | $2.8690e-04$ |
| 0.5 | $9.8323e-12$ | $5.7171e-11$ |

Table 2. Comparison of the proposed method with MFJPIM [3] with stopping criterion $\epsilon = 10^{-30}$.

| | FLPIM | | MFJPIM [3] | |
|---|---|---|---|---|
| | $q_1 = q_2 = \lambda = 0.5$ | | $q_1 = q_2 = 0.5$ | |
| | $E_{U_1(t)}$ | $E_{U_2(t)}$ | $E_{U_1(t)}$ | $E_{U_2(t)}$ |
| $N = 5$ | $9.2012e-11$ | $5.5783e-11$ | $3.6357e-04$ | $1.8609e-03$ |
| $N = 10$ | $9.8323e-12$ | $5.7171e-11$ | $2.4957e-05$ | $2.8690e-04$ |

flexibility of fractional Legendre functions, this method provides an effective framework for approximating solutions in the fractional domain. The proposed approach is rigorously validated through a series of numerical experiments, which demonstrate its effectiveness and reliability. The results highlight the method's ability to efficiently address a broad spectrum of nonlinear SFDEs, establishing it as a robust and competitive alternative to existing numerical techniques.

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# Cancellation Property for Acts Over Monoids

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**Abstract**

The cancellation property in the theory of actions over a monoid is introduced and examined in this paper. We will find some significant classes of acts which are cancellable. In addition, we give a characterization of cancellable acts.

## 1. INTRODUCTION

The cancellation property typically occurs in the study of any algebraic structure where the concept of a direct sum exists. The cancellation property for the given algebraic structure asks whether $M \oplus X \cong M \oplus Y$ implies $X \cong Y$. This property has been investigated by many authors in different mathematical subjects (e.g., see [5, 6] in categories of algebraic structures and abelian groups, [1, 3, 4, 7, 9, 10] in categories of modules, [11, 12] in categories of topological spaces). On the other hand, There are different ideas to study the notion of cancellation in modules over rings. For instance, for a module $M$, $M = X \oplus Y = X' \oplus Y'$ with $Y \cong Y'$, does it imply that $X \cong X'$? If the implication is true, $M$ is said to have *internal cancellation property* (or $M$ is internally cancellable, for short.)

In the following, we collect some basic notions that will be required in the sequel. The notation in this paper is conventional and was taken directly from [8].

In the following, $S$ will always stand for a monoid. A *right $S$-act* is a non-empty set $M$ on which $S$ acts unitarily form the right in the usual way, that is, to say $m(st) = (ms)t$, $m1 = m$, for all $m \in M$, and $s, t \in S$, where 1 denotes the identity of $S$. Left $S$-act are defined similarly. From now on, by an $S$-act we mean a right $S$-act unless othewise stated. The monoid $S$ with its operation as the action is clearly an $S$-act. A non-empty subset $N$ of an $S$-act $M$ is called a *subact* of $M$ whenever $ns \in N$ for all $s \in S$ and $n \in N$. If $M$ and $N$ are right $S$-acts and $f : M \to N$ is a mapping then $f$ is called an $S$-homomorphism if $f(ms) = f(m)s$, for $s \in S$, $m \in M$. The set of all $S$-homomorphisms from $M$ to $N$ is denoted by $\mathrm{Hom}(M, N)$. By **Act**-$S$ or $S$-**Act** we denote the categories of left or right $S$-acts.

Recall that the coproduct of any non-empty family of $S$-acts in **Act**-$S$ exists. In fact, if $\{M_i \mid i \in I\}$ is a none-empty family of $S$-acts, then the coproduct of $\{M_i : i \in I\}$ is their disjoint union denoted by $\dot{\bigcup}_{i\in I} M_i$, according to [8, Proposition 2.1.8]. An $S$-act $M$ is *decomposable* whenever there are two subacts $M_1$ and $M_2$ of $M$ such that

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$M = M_1 \cup M_2$ and $M_1 \cap M_2 = \emptyset$. In this case, $M = M_1 \dot{\cup} M_2$ is called a *decomposition* of $M$. Otherwise, $M$ is *indecomposable*. By [8, Theorem 1.5.10], every $S$-act $M$ has a unique decomposition into indecomposable subacts.
In section 2, we define and study the cancellation property in **Act**-$S$. In fact, first we show that every indecomposable $S$-act is cancellable (see Proposition 2.3 below). Then, by Theorem 2.7, we describe a large class of $S$-acts which are cancellable and finally we prove that cancellable $S$-acts coincide with internally cancellable.

## 2. Second Section

We begin this section with a basic definition.

**Definition 2.1.** An $S$-act $M$ is said to have *cancellation property*, if $M \dot{\cup} X \cong M \dot{\cup} Y$ implies that $X \cong Y$, for any two $S$-act $X$ and $Y$. In this case, $M$ is called *cancellable*.

Here, we give some examples to show that cancellation in **Act**-$S$ does not hold, in general.

**Lemma 2.2.** *Assume that $M$ and $N$ are two arbitrary $S$-acts. Then, the $S$-act $M \dot{\cup} N$ is cancellable if and only if both of $M$ and $N$ are cancellable.*

Now, we will find some classes of cancellable acts. The first class is the class of indecomposable $S$-acts.

**Proposition 2.3.** *Indecomposable $S$-acts are cancellable.*

**Definition 2.4.** Let $M = \dot{\bigcup}_{i \in I} M_i$ be the unique decomposition of an $S$-act $M$ into indecomposable subacts $M_i, i \in I$. Then $M$ is called *finitely decomposable* whenever $1 \leq |I| < \infty$. Otherwise, $M$ is called *infinitely decomposable*.

In the next proposition we present other classes of cancellable $S$-acts.

**Proposition 2.5.** *For a monoid $S$, all finitely decomposable $S$-acts are cancellable. In particular, all finitely generated $S$-acts are cancellable.*

We demonstrated earlier that every finitely decomposable $S$-act is cancellable. In general, the converse is not true, as demonstrated by the following result.

**Theorem 2.6.** *Suppose that $M$ is an infinitely decomposable $S$-act and $M = \dot{\bigcup}_{i \in I} M_i$ the unique decomposition of $M$ into indecomposable subacts, such that $M_i \ncong M_j$ for all pair of distinct elements $i, j \in I$. Then $M$ is cancellable.*

Suppose that $M = \dot{\bigcup}_{i \in I} M_i$ is the unique decomposition of an $S$-act $M$ into indecomposable subacts and we define an equivalence relation $\sim$ on $I$ by $i \sim j$ if and only if $M_i \cong M_j$, for $i, j \in I$. For each $i \in I$ the equivalence class of $i$ is given by $[i] = \{j \in I \mid M_i \cong M_j\}$. In the sequel, the set of all equivalence classes of $I$ is denoted by $I/\sim$.
These observations leads to the following result.

**Theorem 2.7.** *Suppose that $M = \dot{\bigcup}_{i \in I} M_i$ is the unique decomposition of an $S$-act $M$ into indecomposable subacts $M_i, i \in I$ such that the set*

$$P = \{\text{Card}[i] \mid i \in I\}$$

*is finite. Then $M$ is cancellable if and only if the equivalence class $[i]$ is finite for every $i \in I$.*

Next, we intend to define the notion of internal cancellation in **Act**-$S$. As stated in the introduction, we will demonstrate how this idea correlates with cancellation.

**Definition 2.8.** An $S$-act $A$ is said to have the *internal cancellation property* if for any subacts $A_1, A_2, B_1$, and $B_2$ of $A$ whenever $A = A_1 \dot{\cup} B_1 = A_2 \dot{\cup} B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$. In this case, $A$ is also called an *internally cancellable* $S$-act.

Now we prove our claim in the following result.

**Theorem 2.9.** *An $S$-act $A$ has the cancellation property if and only if $A$ has the internal cancellation property.*

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# Cancellation Problem and Refinement Monoids

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**Abstract**

In this talk, we will make use of cancellation properties in refinement monoids and consider some cancellation properties for $S$-acts with a unique zero element. We will show that any Dedekind-finite $S$-act is cancellable and has multiplicative cancellation in the category of Dedekind-finite $S$-acts with a unique zero

## 1. INTRODUCTION

In the following, $S$ will always stand for a monoid. A *right $S$-act* is a non-empty set $M$ on which $S$ acts unitarily form the right in the usual way, that is, to say $m(st) = (ms)t$, $m1 = m$, for all $m \in M$, and $s, t \in S$, where 1 denotes the identity of $S$. Left $S$-act are defined similarly. From now on, by an $S$-act we mean a right $S$-act unless othewise stated. The monoid $S$ with its operation as the action is clearly an $S$-act. A non-empty subset $N$ of an $S$-act $M$ is called a *subact* of $M$ whenever $ns \in N$ for all $s \in S$ and $n \in N$. If $M$ and $N$ are right $S$-acts and $f : M \to N$ is a mapping then $f$ is called an $S$-homomorphism if $f(ms) = f(m)s$, for $s \in S$, $m \in M$. The set of all $S$-homomorphisms from $M$ to $N$ is denoted by $\mathrm{Hom}(M, N)$. By **Act**-$S$ or $S$-**Act** we denote the categories of left or right $S$-acts [6].
An element $\theta$ of an $S$-act $M$ is called a zero of $M$ if $\theta s = \theta$ for any $s \in S$, that is, $\Theta = \{\theta\}$ is a one-element subact. In this paper we will assume that all $S$-acts have a unique zero element $\theta$ and any subact of an $S$-act contains the zero $\theta$. The category of all $S$-acts with a unique zero $\theta$ and $S$-homomorphisms preserving the zero (i.e., $S$-homomorphisms $f : M \to N$ for which $f(\theta) = \theta$), is denoted by $\mathbf{Act}_0$-$S$. The coproduct of any non-empty family of acts in $\mathbf{Act}_0$-$S$ exists. More precisely, if $I$ is a non-empty set and $M_i \in \mathbf{Act}_0$-$S, i \in I$ and $\theta_i$ be the zero of $M_i$, then by [6, Proposition 2.1.15] the coproduct of $\{M_i \mid i \in I\}$ is $(\dot{\bigcup}_{i\in I}(M_i \backslash \{\theta_i\}))\dot{\cup}\{\theta\}$ with $m_i s = \theta$, if $m_i s = \theta_i$ in $M_i$, $\theta s = \theta$ for $s \in S$. Likewise, an $S$-act $M$ is *decomposable* if there are subacts $M_1, M_2 \subseteq M$ such that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \{\theta\}$. In this case, $M = M_1 \cup M_2$ is a *decomposition* of $M$. Otherwise, $M$ is called *indecomposable*. By Theorem 1.5.10 of [6], every $S$-act $M$ has a unique decomposition into indecomposable subacts.
The cancellation property for the given algebraic structure asks whether $M \oplus X \cong M \oplus Y$ implies $X \cong Y$. This property has been investigated by many authors in different mathematical subjects (e.g., see [1–5].

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Our goal in this talk is to consider the Krull-Schmidt commutative monoid $M_S(\mathcal{C})$ for a suitable class $\mathcal{C}$ of $S$-acts, for a monoid $S$, and then to establish a submonoid consisting of all equivalence classes of Dedekind-finite $S$-acts. We shall show that this monoid has good properties related to cancellation, such as, it is both cancellative and multiplicative cancellative. certain set.

## 2. Second Section

We start with the following definition which is taken from notation 2.1 and definition 2.6 of [2].

**Definition 2.1.** Let $(T,+)$ be a commutative monoid with the identity element 0.

1. for $u,v \in T$, $u \leq v \Leftrightarrow \exists w \in T$ such that $u+w=v$.
2. for $u,v \in T$, $u \ll v \Leftrightarrow u+v \leq v$.
3. for $u,v \in T$, $u \equiv v \Leftrightarrow u \leq v$ and $v \leq u$.
4. $p \in T$ is *prime* if for all $u,v \in T$, $p \leq u+v$ implies $p \leq u$ or $p \leq v$.
5. $p \in T$ is *primely generated* if it is the sum of prime elements of $T$. The monoid $T$ itself is *primely generated* if all its elements are primely generated.
6. $u \in T$ is *proper* if $u \nleq 0$.
7. $T$ is *cancellative* if for all $u,v,w \in T$, $u+w=v+w$ implies $u=v$.
8. $T$ is *stably finite* if for all $u,x \in T$, $u+x=u$ implies $x=0$.
9. $T$ has *multiplicative cancellation* if for all $u,v \in T$ and $n \in \mathbb{N}$, $nu=nv$ implies $u=v$.
10. $T$ has *refinement* if for all $x_1,x_2,y_1,y_2 \in T$ with $x_1+x_2=y_1+y_2$, there exist $z_{11},z_{12},z_{21},z_{22} \in T$ such that

$$x_1 = z_{11}+z_{12}, \qquad x_2 = z_{21}+z_{22},$$
$$y_1 = z_{11}+z_{21}, \qquad y_2 = z_{12}+z_{22}.$$

It is convenient to record refinements using matrices. The refinement of $x_1,x_2,y_1,y_2$ from the definition would be written

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \end{array}.$$

This means that the entry labeling each row (column) matches the sum of the entries in each row (column).
Let $S$ be a monoid. Define an equivalence relation $\sim$ on the class of all $S$-acts with a unique zero element $\theta$ by

$$M \sim N \Leftrightarrow M \cong N \tag{1}$$

for $M,N \in \mathbf{Act}_0\text{-}S$ (where $\cong$ means "isomorphic to"). The equivalence class of any $M \in \mathbf{Act}_0\text{-}S$ is denoted by $[M]$. Let $\mathcal{C}$ be a class of right $S$-acts with a unique zero element. Then $\mathcal{C}$ is said to be small if the class $M_S(\mathcal{C}) = \{[M] \mid M \in \mathcal{C}\}$ is a set. Assume $\mathcal{C}$ is a small class of right S-acts with a unique zero element that is closed under isomorphism ($[M] \subseteq \mathcal{C}$, for every $M \in \mathcal{C}$) and finite direct sums (equivalently, zero act $\Theta = \{\theta\} \in \mathcal{C}$ and $M \coprod N \in \mathcal{C}$ in case $M,N \in \mathcal{C}$). The set $M_S(\mathcal{C})$ equipped with the addition defined by $[M]+[N] := [M \coprod N]$ is the Krull-Schmidt monoid of $\mathcal{C}$. clearly, the Krull-Schmidt monoid of $\mathcal{C}$, $(M_S(\mathcal{C}),+)$, is a commutative monoid and the class $[\Theta]$ is the identity element of $M_S(\mathcal{C})$.

**Lemma 2.2.** *Let $\mathcal{C}$ be a small class of right $S$-acts with a unique zero element closed under isomorphism and finite direct sums. Then the monoid $(M_S(\mathcal{C}),+)$ has refinement.*

**Definition 2.3.** By a *Dedekind-finite* $S$-act we mean an $S$-act $M$ which is not isomorphic to any proper direct summand of itself. Equivalently, $M$ is Dedekind-finite if and only if $N=\{\theta\}$ is the only $S$-act for which $M \coprod N \cong M$.

**Lemma 2.4.** *Assume that $M$ and $N$ are two arbitrary $S$-acts. Then the $S$-act $M \coprod N$ is Dedekind-finite if and only if both of $M$ and $N$ are Dedekind-finite.*

**Lemma 2.5.** *Let $S$ be a monoid and $\mathcal{C}$ be a small class of right $S$-acts with a unique zero element closed under isomorphism and finite direct sums. Then,*
(i) *If $M \in \mathcal{C}$ is an indecomposable $S$-act, then $[M] \in M_S(\mathcal{C})$ is prime.*
(ii) *If $M \in \mathcal{C}$ is a Dedekind-finite $S$-act and $[M] \in M_S(\mathcal{C})$ is prime, then $M$ is indecomposable.*
(iii) *$M_S(\mathcal{C})$ is primely generated.*

**Definition 2.6.** (See [2, Definition 2.8].) Let $(T, +)$ be a commutative monoid with the identity element 0. For $u \in T$, now define a congruence $\sim_u$ on $T$ by

$$x \sim_u y \Leftrightarrow x + u = y + u$$

for $x, y \in T$. We will write $\bar{t}_u$ for the $\sim_u$-congruence class containing $t \in T$. Now, define $G_u = \{\bar{t}_u \mid t \ll u\}$. One can easily show that $G_u$ is the set of all units (invertible elements) of the quotient monoid $T/\sim_u$ and so is an abelian group.

For an abelian group $G$, let

$$\tau(G) = \{a \in G \mid na = 0 \text{ for some } n \in \mathbb{N}\},$$

be the torsion subgroup of $G$.

**Theorem 2.7.** *Let $S$ be a monoid and $\mathcal{C}$ be a small class of Dedekind-finite right $S$-acts with a unique zero element closed under isomorphism and finite direct sums. Then, the commutative monoid $M_S(\mathcal{C})$ has multiplicative cancellation property.*

*Proof.* First of all, note that by Lemma 2.4, $M_S(\mathcal{C})$ is a monoid. It is enough to show that $M_S(\mathcal{C})$ satisfies the conditions of [2, Theorem 5.13]. Since any direct summand of a Dedekind-finite $S$-act is Dedekind-finite, therefore $M_S(\mathcal{C})$ has refinement, by Lemma 2.2. Also, $M_S(\mathcal{C})$ is primely generated by Lemma 2.5. Now, we show that $\tau(G_{[A]}) = 0$ for any $[M] \in M_S(\mathcal{C})$. We have

$$G_{[M]} = \{\overline{[N]}_{[M]} \mid [N] \ll [M]\}$$

where $\overline{[N]}_{[M]} = \{[L] \mid [N] \sim_{[M]} [L]\}$. First, we prove that $\overline{[\Theta]}_{[M]} = \{[\Theta]\}$. Suppose that $[D] \in \overline{[\Theta]}_{[M]}$, then $[D] \sim_{[M]} [\Theta]$ and so $D \dot{\cup} M \cong \Theta \dot{\cup} M \cong M$. Therefore $D \cong \Theta$, because $M$ is a Dedekind-finite $S$-act. So $[D] = [\Theta]$ and then $\overline{[\Theta]}_{[M]} = \{[\Theta]\}$. Now assume that $n\overline{[N]}_{[M]} = \overline{[\Theta]}_{[M]}$, where $\overline{[N]}_{[M]} \in \tau(G_{[M]})$. Then, $\dot{\bigcup}_{i=1}^{n} N_i \cong \Theta$, where $N_i = N$, for $1 \leq i \leq n$. Therefore, we obtain $N \cong \Theta$ and so $\overline{[B]} = \overline{[\Theta]}$. This implies that $\tau(G_{[M]}) = 0$. □

**Definition 2.8.** (See [2, Notation 2.1].) Let $(T, +)$ be a commutative monoid with the identity element 0. Consider the congruence $\equiv$ on $T$ defined by

$$u \equiv v \Leftrightarrow u \leq v \text{ and } v \leq u$$

for any $u, v \in T$. Now, let $\overline{T} := T/\equiv$ and $\mathbb{P}$ denotes the set of all proper prime elements of $\overline{T}$.

**Theorem 2.9.** *Let $S$ be a monoid and $\mathcal{C}$ be a small class of Dedekind-finite right $S$-acts with a unique zero element closed under isomorphism and finite direct sums. Then, the commutative monoid $M_S(\mathcal{C})$ is cancellative.*

*Proof.* In the proof of Theorem 2.7 we have shown that $M_S(\mathcal{C})$ is a primely generated refinement monoid. It is easy to check that $M_S(\mathcal{C})$ is stably finite. Since $\overline{M_S(\mathcal{C})} \cong M_S(\mathcal{C})$ and $G_{[\Theta]} = \{\overline{[\Theta]}_{[\Theta]}\}$ the result follows by [2, Theorem 5.14]. □

**Corollary 2.10.** *Let $M, P$, and $Q$ be Dedekind-finite $S$-acts. If $M \coprod P \cong M \coprod Q$, then $P \cong Q$.*

*Proof.* The proof is straightforward by Theorem 2.9. □

We conclude the paper by a result which is related to a special type of multiplicative cancellation.

**Proposition 2.11.** *Let $M$ and $N$ be two $S$-acts. If $M \coprod M \cong M \coprod N \cong N \coprod N$, then $M \cong N$.*

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# Some differences between Lie and Leibniz algebras

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**Abstract**

Lie and Leibniz algebras are closely related but have key differences. In a Lie algebra, the bracket $[x, y]$ is anti-symmetric, meaning $[x, y] = -[y, x]$. In a Leibniz algebra, this condition is relaxed, so the bracket may not be anti-symmetric. Leibniz algebras introduced by J. L. Loday (1993) are non-antisymmetric generalizations of Lie algebras. Investigations of nilpotent Leibniz algebras , show that many nilpotent properties of Lie algebras can be extended to the case of nilpotent Leibniz algebras.
Of course there are some theorems and facts in Lie case that they are not true for Leibniz algebras, also there are some theorems that they are correct in Leibniz algebras but we can not apply them for Lie case. Thus, every Lie algebra is a Leibniz algebra, but not every Leibniz algebra is a Lie algebra, if the bracket in a Leibniz algebra happens to be anti-symmetric it reduces to be Lie algebra.

## 1. Introduction

Leibniz algebras was first introduced by Loday in [9] and [10] as a non-antisymmetric versions of Lie algebras. The classification problem of complex nilpotent Leibniz algebras was first studied by Loday. In [10] he gave a complete classification of complex nilpotent Leibniz algebras of dimension $n \leq 2$. Later Ayupov and Omirov classified $3-$dimensional complex nilpotent Leibniz algebras in [7]. Recently, Albeverio, Omirov and Rakhimov have obtained a classification of 4-dimensional complex nilpotent Leibniz algebras in [6].
One of the techniques to classify nilpotent Lie algebras was introduced by Skjelbred and Sund in [13]. Rakhimov and Langari used Skjelbred-Sund method in Leibniz algebras [8]. They also applied in [12] and [8] this technique to obtain the classification of complex nilpotent Leibniz algebras of dimension $n \leq 4$. Comparing the results of [8] and [11] with classification in [6] we realized that the Skjelbred-Sund method could be used to check the validity of the main result of [6]. In this part we give the basic definitions and properties of Leibniz algebras.

**Definition 1.1.** A Lie algebra $L$ is a vector space over a field $F$ equipped with a bilinear map, $[\cdot, \cdot] : L \times L \to L$ which has the following properties :
1. $[x, x] = 0 \quad \forall x \in L$
2. $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad \forall x, y, z \in L.$

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Condition (2) is known as the Jacobi identity. As the Lie bracket $[\cdot,\cdot]$ is bilinear, we have

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x].$$

Hence condition (1) implies

$$(1') \; [x,y] = -[y,x] \text{ for all } x, y \in L \quad (anti - symmetry).$$

If the field $F$ does not have characteristic 2, then putting $x = y$ in $(1')$ shows that $(1')$ implies condition (1).

**Example 1.2.** Any vector space $V$ has a Lie bracket defined by $[x,y] = 0$ for all $x, y \in V$. This is the abelian Lie algebra structure on $V$. In particular, the field $F$ may be regarded as a $1-$dimensional abelian Lie algebra.

**Definition 1.3.** A Leibniz algebra $L$ is a vector space over a field $F$ equipped with a bilinear map

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad (x,y,z \in L).$$

The first pure algebraic motivation of J.-L. Loday to introduce this class of algebras was the search for an "obstruction" to the periodicity in algebraic $K$-theory. Besides this purely algebraic motivation, some relationships with classical geometry have recently been discovered, which could lead to an investigation of the (co)homological theory of Leibniz algebras in view of concrete applications in non-commutative geometry and its physical interpretations. Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if

$$[x,x] = 0, \quad (x \in L).$$

Let $n$ be the dimension of Leibniz algebra $L$. Let $\{e_1, e_2, ..., e_n\}$ be a basis in $L$. The structural constants of $L$ are the numbers $C_{ij}^k$ given by

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k \quad (i,j = 1, ..., n).$$

We can identify the Leibniz identity with its structural constants. These constants satisfy:

$$\sum_{l=1}^{n} (C_{jk}^l C_{il}^s - C_{ij}^l C_{lk}^s + C_{ik}^l C_{lj}^s) = 0 \quad (i,j,k,s = 1, ..., n).$$

**Definition 1.4.** Let $L$ is a Leibniz algebra. We define

$$L^1 = L, L^k = \left[L^{k-1}, L\right] \quad (k > 1).$$

The series

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq ...$$

is called the descending central series of $L$. If the series terminates for some positive integer $s$, then the Leibniz algebra $L$ is said to be nilpotent.

## 2. Some differences between Lie and Leibniz algebras

Investigations of nilpotent Leibniz algebras in [1], [2],[3] etc., show that many nilpotent properties of Lie algebras can be extended to the case of nilpotent Leibniz algebras.
Of course there are some theorems and facts in Lie case that they are not true for Leibniz algebras, also there are some theorems that they are correct in Leibniz algebras but we can not apply them for Lie case. For example, we remark that the next Theorem gives an explicit method for finding a basis for central extensions of abelian Lie algebras, that it is not true in Leibniz case [4].

**Theorem 2.1.** *Let $\theta : L \times L \longrightarrow F$ be a skew-symmetric bilinear form. Then there is a basis of L with respect to which*

$$\theta = \Delta_{12} + \Delta_{34} + \ldots + \Delta_{(2r-1)(2r)}.$$

Here, we give an application of Theorem 2.1.

**Example 2.2.** Find the 1-dimensional central extensions of dimension 5 and 7 of the abelian Lie algebras of dimensions 4 and 6, respectively.

By Theorem 2.1, there is a basis of 4-dimensional abelian Lie algebra with respect to which $\theta = \Delta_{12} + \Delta_{34}$. We get the Lie algebra

$$L : [e_1, e_2] = e_5, [e_3, e_4] = e_5.$$

There is a basis of $6-$dimensional abelian Lie algebra such that $\theta = \Delta_{12} + \Delta_{34} + \Delta_{56}$. In this case, we get the following Lie algebra

$$L : [e_1, e_2] = e_7, [e_3, e_4] = e_7, [e_5, e_6] = e_7.$$

By this Theorem there is not $1-$dimensional central extensions of even dimensions of abelian Lie algebras.
As we will see in sections **??** and **??** this Theorem is not true for central extensions of abelian Leibniz algebras.
Let $L$ be a Lie algebra, and $\theta : L \times L \to V$ a bilinear map. If $\theta(x, y) = 0$ for all $x$, $y$, then $\theta$ is said to be skew symmetric.

**Definition 2.3.** Let $L$ be a Lie algebra over an arbitrary field $F$. A skew-symmetric bilinear form $\theta : L \times L \to V$ is said to be cocycle if it satisfies the Jacobi identity $\theta([x, y], z) + \theta([y, z], x) + \theta([z, x], y) = 0$ for all $x, y, z \in L$.

Let $V$ be a vector space, a Leibniz $2-$cocycle on $L$ is a bilinear map $\theta : L \times L \to V$ satisfying the Leibniz identity:

$$\theta(x, [y, z]) = \theta([x, y], z) - \theta([x, z], y),$$

for all $x, y, z \in L$.

## 3. Central extensions of abelian Leibniz algebra in dimension 2 ($L_{2,1}$)

The center of $L_{2,1}$ is generated by $e_1, e_2$, and $[L_{2,1}, L_{2,1}] = (0)$. We need to determine all the 2-cocycles $\theta = \sum_{i,j=1}^{2} C_{ij}\Delta_{ij}$, but there is no constraints on $C_{i,j}$. Therefore we can get a basis for $HL^2\left(L_{2,1}, F\right)$, and write "Basis: $\Delta_{11}$, $\Delta_{12}$, $\Delta_{21}$, $\Delta_{22}$ ". In this case, we are considering the 1-dimensional central extensions of $L_{2,1}$. We need to find a set of representatives of the orbits of $1-$dimensional subspaces of $HL^2\left(L_{2,1}, F\right)$ under the action of the automorphism group Aut($L_{2,1}$). Here Aut($L_{2,1}$) consists of all matrices

$$\varphi = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{where } a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

With the chosen basis, we may represent an arbitrary element in $HL^2\left(L_{2,1}, F\right)$ by the following

$$\theta := [a, b, c, d] = a\Delta_{11} + b\Delta_{12} + c\Delta_{21} + d\Delta_{22}.$$

When a generic element $\varphi$ in Aut($L_{2,1}$) acts on $\theta$, we get

$$\varphi\theta = a'\Delta_{11} + b'\Delta_{12} + c'\Delta_{21} + d'\Delta_{22},$$

we simply write $a \longrightarrow a'$, $b \longrightarrow b'$, $c \longrightarrow c'$, $d \longrightarrow d'$. Now we have

$$\begin{cases} a \longmapsto a_{11}(aa_{11} + ba_{21}) + a_{21}(ca_{11} + da_{21}), \\ b \longmapsto a_{11}(aa_{12} + ba_{22}) + a_{21}(ca_{12} + da_{22}), \\ c \longmapsto a_{12}(aa_{11} + ba_{21}) + a_{22}(ca_{11} + da_{21}), \\ d \longmapsto a_{12}(aa_{12} + ba_{22}) + a_{22}(ca_{12} + da_{22}). \end{cases}$$

Case 1: $a \neq 0$. By taking $a_{21} = 0$ (and ensuring at the same time that the matrix of $\varphi$ is nonsingular). We set $a_{11} = \frac{1}{\sqrt{a}}$, then $a \longmapsto 1$. Now to fix $a$, we require $a_{11} = 1$. we get $a \longmapsto 1$, $b \longmapsto a_{12} + ba_{22}$, $c \longmapsto a_{12} + ca_{22}$, $d \longmapsto a_{12}^2 + ba_{12}a_{22} + ca_{12}a_{22} + da_{22}^2$.
By taking $a_{12} = -ca_{22}$, we get $c \longmapsto 0$, and to fix $c$, we require $a_{12} = 0$. In this case we get
$a \longmapsto 1$, $b \longmapsto ba_{22}$, $c \longmapsto 0$, $d \longmapsto da_{22}^2$.
Subcase 1: $b = 0$. Depending $d = 0$ or not, we would have two representatives (1) $[1, 0, 0, 0]$ and (2) $[1, 0, 0, 1]$. But (1) is split algebra (because the corresponding Leibniz algebra of (1) is given by $[e_1, e_1] = e_3$, and this algebra is isomorphic to $[e_1, e_1] = e_2$ that already we obtained it).
Subcase 2: $b \neq 0$. By taking $a_{22} = \frac{1}{\sqrt{b}}$, we get $b \longmapsto 0$, and to fix $b$, we require $a_{22} = 1$. Therefore we get the representative $[1, 1, 0, \alpha]$. The corresponding Leibniz algebra is given by

$$L_{3,4} : [e_1, e_1] = e_3, [e_1, e_2] = e_3, [e_2, e_2] = \alpha e_3, \alpha \in \mathbb{C}.$$

Case 2: If $a = 0$, then we have $a \longmapsto 0$, $b \longmapsto ba_{11}a_{22}$, $c \longmapsto ca_{11}a_{22}$, $d \longmapsto ba_{12}a_{22} + ca_{12}a_{22} + da_{22}^2$.
At least one of $b$, $c$ and $d$ is nonzero (because, if $b = c = d = 0$ then $\theta = [0, 0, 0, 0]$, and there is no representative in this case). Suppose that $b \neq 0$. We choose $a_{11}a_{22} = \frac{1}{b}$ and get $b \longmapsto 1$. Now to fix $b$, we require $a_{11} = a_{22} = 1$, and get $a \longmapsto 0$, $b \longmapsto 1$, $c \longmapsto c$, $d \longmapsto a_{12}(1 + c) + d$.
If $c = -1$, depending on $d = 0$ or not, we have two representatives $[0, 1, -1, 0]$ and (3) $[0, 1, -1, \alpha]$. The corresponding Leibniz algebra with $[0, 1, -1, 0]$ is

$$L_{3,3} : [e_1, e_2] = e_3,\ [e_2, e_1] = -e_3.$$

Note that $L_{3,3}$ is a Lie algebra and all Lie algebras are Leibniz. We also need all Lie algebras in dimension 3, because Leibniz central extensions of Lie algebras give us Leibniz algebras.
But (3) $\cong L_{3,4}$ ($\alpha = \frac{1}{4}$) by

$$e_1 \longmapsto e_1' + \frac{1}{\alpha}e_2',\ e_2 \longmapsto \frac{1}{2\alpha}e_2',\ e_3 \longmapsto \frac{1}{\alpha}e_3'.$$

*3.1. Lie homology and Leibniz homology*

For any Lie algebra $L$ over a field $F$ its homology $H_*(L)$ is the homology of a certain complex $(\wedge L, d)$ where $\wedge L$ is the exterior module over $L$. The Leibniz homology $HL_*(L)$, is defined as the homology of a complex $(TL, d)$, where $TL$ is the tensor module over $L$. In the below, the cohomology of Lie and Leibniz algebras have been explained to let the readers recognize the difference between them.
The Chevalley-Eilenberg chain complex of a Lie algebra $L$ is the sequence of chain modules given by the exterior powers $\wedge^* L$ and boundary operators $d : \wedge^n L \longrightarrow \wedge^{n-1} L$ classically defined as

$$d(x_1 \wedge x_2 \ldots \wedge x_n) :=$$

$$\sum_{1 \leq i \lneq j \leq n} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x_i} \wedge \ldots \wedge \hat{x_j} \wedge \ldots \wedge x_n,$$

where $\hat{x_i}$ and $\hat{x_j}$ indicate that the terms $x_i$ and $x_j$ are omitted. The property $dod = 0$, which makes this sequence a chain complex, is proved by using the antisymmetry $x \wedge y = -y \wedge x$ of the exterior product, the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0,$$

and the antisymmetry $[x,y] = -[y,x]$ of the Lie bracket given on $L$. Because, from the definition of $d$ for $n = 2$, we have $d(x_1 \wedge x_2) = [x_1, x_2]$, and for $n = 3$

$$dod\,(x_1 \wedge x_2 \wedge x_3) = d\,(d\,(x_1 \wedge x_2 \wedge x_3)) =$$

$$d([x_1,x_2] \wedge x_3) + d([x_2,x_3] \wedge x_1) + d([x_3,x_1] \wedge x_2) =$$

$$[[x_1,x_2],x_3] + [[x_2,x_3],x_1] + [[x_3,x_1],x_2] = 0 \text{ (Jacobi identity).}$$

In 1989, Loday remarked that if $L$ be a Leibniz algebra and we consider the sequence of chain modules and boundary operations $d : L^{\otimes n} \longrightarrow L^{\otimes n-1}$, with

$$d(x_1 \otimes \ldots \otimes x_n) :=$$

$$\sum_{1 \le i \lneqq j \le n} (-1)^{j+1} x_1 \otimes \ldots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \ldots. \otimes \hat{x_j} \otimes \ldots \otimes x_n,$$

where again the symbol ^ means that the variable below is omitted. Then the property $dod = 0$ is proved without making use of the antisymmetry properties of both the exterior product and of the Lie bracket on $L$ , it suffices that the bracket satisfies the so called Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

because, from definition of $d$ for $n = 2$ we have, $d(x_1 \otimes x_2) = -[x_1, x_2]$, and for $n = 3$ :

$$dod(x_1 \otimes x_2 \otimes x_3) = d(d(x_1 \otimes x_2 \otimes x_3)) =$$

$$-d([x_1,x_2] \otimes x_3) + d([x_1,x_3] \otimes x_2) + d(x_1,[x_2,x_3]) =$$

$$[[x_1,x_2],x_3] - [[x_1,x_3],x_2] - [x_1,[x_2,x_3]] = 0 \text{ (Leibniz identity),}$$

which is equivalent to the Jacobi identity any time the bracket is also antisymmetric. Hence this sequence defines a chain complex whose homology is called Leibniz homology of $L$, and denoted by $HL_*(L)$.
Of course this homology theory can be dualized into a cohomology theory, given by the sequence of cochain modules and cohomology operators

$$\delta : Hom(L^{\otimes n}, F) \longrightarrow Hom(L^{\otimes n+1}, F),$$

with $\delta(f) = fod$. This sequence defines a cochain complex whose homology is called Leibniz cohomology of $L$, and denoted by $HL^*(L)$.
Let $L$ be a Leibniz algebra. Given a corepresentation $M$ of $L$, it is possible to extend the Leibniz boundary operator to sequence $d : M \otimes L^{\otimes n} \longrightarrow M \otimes L^{\otimes n-1}$, with

$$d(x_0 \otimes x_1 \otimes \ldots \otimes x_n) :=$$

$$\sum_{1 \le i \lneqq j \le n} (-1)^{j+1} x_0 \otimes \ldots \otimes [x_i, x_j] \otimes \ldots. \otimes \hat{x_j} \otimes \ldots \otimes x_n + [x_0, x_1] \otimes \ldots \otimes x_{n+1} +$$

$$\sum_{i=1}^{n+1} (-1)^i [x_i, x_0] \otimes x_1 \otimes \ldots \otimes \hat{x_i} \otimes \ldots \otimes x_{n+1}, \text{ for } x_0 \in M \text{ and } x_1, \ldots, x_n \in L,$$

where again the symbol ^ means that the variable below is omitted.

For any Leibniz algebra $L$ and any $L-$module $M$, the map

$$d : M \otimes L^{\otimes n} \longrightarrow M \otimes L^{\otimes n-1}$$

defined above satisfies $d^2 = 0$. Therefore the sequence

$$\ldots \longrightarrow \ M \otimes L^{\otimes n} \xrightarrow{d_n} M \otimes L^{\otimes n-1} \xrightarrow{d_{n-1}} \ldots \longrightarrow M \otimes L \longrightarrow M$$

is a well-defined complex $(M \otimes TL, d)$.
The homology of this new complex is called Leibniz homology of $L$ with coefficients in $M$, and denoted by $HL_*(L, M)$ . We note that

$$HL_*(L, F) \ = \ HL_*(L),$$

where $F$ is the ground field.

**Lemma 3.1.** *For a Leibniz algebra* $(L, [.,.])$

$$d : Hom(L, L) \longrightarrow Hom(L^{\otimes 2}, L)$$

*is given by:*

$$(df)(x_1, x_2) = [x_1, f(x_2)] + [f(x_1), x_2] - f([x_1, x_2]).$$

**Lemma 3.2.** *Let* $(L, [.,.])$ *be a Leibniz algebra, then*

$$d : Hom(L^{\otimes 2}, L) \longrightarrow Hom(L^{\otimes 3}, L),$$

*is given by:*

$$df(x, y, z) = [x, f(y, z)] + [f(x, z), y] - [f(x, y), z] - f([x, y], z) + f([x, z], y) +$$

$$f(x, [y, z]).$$

It is well known that Levi−Malcev theorem is valid for Lie algebras and Malcev algebras. This theorem states that any finite dimensional Lie algebra $L$ is (as a vector space) the direct sum of two significant structural parts; namely, a solvable subalgebra and a semisimple subalgebra, however this theorem still is open problem for Leibniz algebras.

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# Importance of Randomness in Cryptography

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**Abstract**

This paper provides a brief overview of the utilization of randomness in cryptography, partially based on previous research. Initially, it offers insights into the construction and application of randomness and pseudo-randomness within a cryptographic framework. Subsequently, it examines the formal definition of pseudo-random sequences, introducing the concepts of distinguishers and prediction algorithms, and comparing these two notions.

## 1. Introduction

Randomness is one of the primary tools in cryptography. Numerous cryptographic primitives or protocols incorporate a random component. This is evident in applications such as stream ciphers, key construction, key exchange in the Ephemeral Unified Model, and the generation of initial values. Typically, randomness is simulated using a pseudo-random generator or, in cases where only a small number of isolated values are needed, by a built-in physical generator. It is essential to differentiate between two typical modes of usage. On one hand, there is the random selection of a medium-sized number, such as a secret key. On the other hand, there is the random selection of a very large sequence of bits, as seen in the case of a stream cipher. In the former scenario, a general-purpose primitive, such as a hash function or a block cipher, can be employed. The latter case presents more challenges. A common application that leads to this situation is the stream cipher, where speed is of utmost importance. Consequently, it is necessary to develop a system that operates faster than conventional block ciphers without compromising security, which is challenging because each bit must be processed as it arrives. Therefore, it is not feasible to iterate a round $R$ as is done with block ciphers.
In the first part of this article, a practical study of the concept of randomness in cryptography is presented. The second part is theoretical, where the notion of a pseudo-random generator is precisely defined. The concepts of distinguishers and predictions are then defined, and the equivalence between the indistinguishability of a pseudo-random generator and the unpredictability of the next bit is established from an asymptotic perspective, as stated in Yao's theorem [6]. This paper also presents modified versions of Yao's theorem [1] that may be relevant for the study of practical cryptographic primitives, particularly focusing on non-asymptotic versions. The analysis includes a single pseudo-random generator, a family of pseudo-random generators with a fixed length, and finally, the asymptotic case. The

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cost of reduction (in terms of complexity theory) between the two algorithms is computed in each case. Additional literature on pseudo-random generators, probabilistic algorithms, and proofs can be found in [2–4].

## 2. Pseudo-Random Number Generator

*How to Generate a Seed and Construct a Pseudo-Random Generator*

Typically, operating systems provide a physical source of randomness based on the behavior of various components, such as the keyboard, mouse, clock, and processes. Since this source of randomness does not yield a large number of bits, it is primarily used to generate an occasional number as a seed.

Let $H$ be a hash function (for example, SHA-256). From a seed $s$ (at least 128 bits), we can construct the following pseudo-random sequence $S_n$ of bits:

$$s_0 = H(s); \quad s_1 = H(s\|s_0); \quad \dots; \quad s_n = H(s\|s_{n-1});$$

Thus, the pseudo-random sequence $S_n$ is given by:

$$S_n = s_1\|s_2\| \dots \|s_n.$$

It is also possible to use AES (as in counter mode) to create a pseudo-random generator. This type of pseudo-random generator is often referred to as a key derivation function, as its primary purpose is to generate cryptographic keys. For practical implementation details, one can refer to the standard ISO 18033-2 [5]. However, as previously mentioned, for stream ciphers, it is crucial to utilize a specific construction if a faster encryption device than AES is desired, while being cautious of potential attacks against stream ciphers.

Let $H$ denote a cryptographic hash function (e.g., SHA-256). Starting with a seed value $s$ (of at least 128 bits), a pseudo-random bit sequence $S_n$ can be generated through iterative hashing as follows:

$$s_0 = H(s); \quad s_1 = H(s\|s_0); \quad \dots; \quad s_n = H(s\|s_{n-1});$$

The resulting sequence $S_n$ is formed by concatenating the outputs:

$$S_n = s_1\|s_2\| \dots \|s_n.$$

Alternatively, AES in counter mode (CTR) can serve as a pseudo-random generator. Such constructions are commonly classified as *key derivation functions* (KDFs), as their primary role is to derive cryptographic keys. Detailed implementation guidelines can be found in the ISO 18033-2 standard [5].

For stream cipher applications, specialized designs may be necessary if a faster alternative to AES-based encryption is required. However, these designs must account for known vulnerabilities in stream ciphers, such as biases or state recovery attacks, necessitating rigorous security analysis.

## 3. Theoretical Point of View

### *3.1. Definition of a Pseudo-Random Number Generator*

**Definition 3.1.** Let $k$ and $n$ be two integers such that $n > k$. A pseudo-random generator (PRNG) is a function $f$ from a subset $U$ of $\{0,1\}^k$ into $\{0,1\}^n$:

$$f : U \to \{0,1\}^n.$$

This function maps a seed $X_0 \in U$ to a pseudo-random finite sequence $f(X_0) = (x_1, x_2, \dots, x_n)$. We denote this PRNG as $(f, U, k, n)$.

A typical case occurs when $u$ is a bijection from $U$ onto itself, $X_i$ is a secret internal state recursively built from $X_0$ by $X_i = u(X_{i-1})$, and the bit $x_i$ is extracted from $X_i$ by a function $v$:

$$x_i = v(X_i).$$

Thus, we have:

$$f(X_0) = (v \circ u(X_0), v \circ u^2(X_0), \dots, v \circ u^n(X_0)).$$

*3.2. Distinguisher*

Roughly speaking, a distinguisher is a probabilistic algorithm capable of distinguishing a true random sequence from a pseudo-random one. To specify this informal definition, let $(f, U, k, n)$ be a PRNG. Let $A$ be a probabilistic algorithm that takes as input a binary vector $Y = (Y_1, \dots, Y_n)$ and outputs a single bit.

**True randomness experiment:** Let $p_{f,0}$ denote the probability of the following event: we randomly draw an element $Y$ from $\{0,1\}^n$ and $A(Y) = 1$.

**Pseudo-randomness experiment:** Let $p_{f,n}$ denote the probability of the following event: we randomly draw an element from $U$, compute $Y = f(U)$, and $A(Y) = 1$.

We can define the Advantage of a distinguisher as follows:

$$\mathrm{Adv}_{\mathrm{dist}}(f, A) = |p_{f,0} - p_{f,n}|.$$

Next, we define a $(T, \epsilon)$-distinguisher:

**Definition 3.2.** Let $f$ be a pseudo-random generator. Let $T$ and $\epsilon$ be positive real numbers. A $(T, \epsilon)$-distinguisher for $f$ is a probabilistic algorithm $A$ such that:

1. The maximum running time of $A$ is $\leq T$.
2. The input of $A$ is an element of $\{0,1\}^n$.
3. The output of $A$ is a bit.
4. The algorithm $A$ can distinguish the pseudo-random generator from the uniform distribution, namely $\mathrm{Adv}_{\mathrm{dist}}(f, A) > \epsilon$.

*3.3. Prediction*

Let $f$ be a pseudo-random generator whose output is in $\{0,1\}^l$. A prediction algorithm is a probabilistic algorithm that can predict the next bit of a finite sequence.

To specify this informal definition, let $1 \leq s < l$. The following random experiment involves a probabilistic algorithm $B$ that takes as input a sequence of $s$ bits and outputs a bit.

**Experiment B:**

$\mathrm{Expt}_{\mathrm{pred}}(f, s, B)$
  $X_0 \leftarrow U$
  $X \leftarrow f(X_0)$ (where $X = (x_1, x_2, \dots, x_l)$)
  $Y \leftarrow (x_1, x_2, \dots, x_s)$
  $b \leftarrow B(Y)$
  if $b = x_{s+1}$
    then return 1
    else return0

Let $r_{f,s}$ be the probability that the experiment $\mathrm{Expt}_{\mathrm{pred}}(f, s, B)$ returns 1.

**Definition 3.3.** The advantage of the algorithm $B$ in predicting the bit at index $s+1$ computed by $f$ is defined as:

$$\mathrm{Adv}_{f,s}^{\mathrm{pred}}(B) = |r_{f,s} - \frac{1}{2}|.$$

**Definition 3.4.** Let $f$ be a pseudo-random generator. Let $T$ and $\epsilon$ be positive real numbers, and let $s$ be an integer such that $1 \leq s < l$. A $(T, s, \epsilon)$-prediction algorithm $B$ is a probabilistic algorithm such that:

1. The maximum running time of $B$ is $\leq T$.
2. The input of $B$ is an element of $\{0,1\}^s$.
3. The output of $B$ is a bit.
4. The algorithm $B$ can predict the next bit, namely, $\mathrm{Adv}_{f,s}^{\mathrm{pred}}(B) > \epsilon$.

We now define the notion of a $(T, s, \epsilon)$-unpredictable pseudo-random generator.

**Definition 3.5.** Let $f$ be a pseudo-random generator and $s$ be an integer such that $1 \leq s < l$. The generator $f$ is $(T, s, \epsilon)$-unpredictable if there does not exist any $(T, s, \epsilon)$-prediction algorithm.

*3.4. A Static Version of Yao's Theorem*

The following two theorems summarize the relationships between prediction algorithms and distinguishers in a static context.

**Theorem 3.6.** *Consider the following pseudo-random generator:*

$$f : U \subset \{0,1\}^k \to \{0,1\}^l.$$

*If there exists a $(T, s, \epsilon)$-prediction algorithm for $f$, then we can construct a $(T + c, \epsilon)$-distinguisher, where $c$ is the constant time required to compare two bits.*

**Theorem 3.7.** *Let $f$ be a pseudo-random generator:*

$$f : U \subset \{0,1\}^k \to \{0,1\}^l.$$

*If there exists a $(T, \epsilon)$-distinguisher for $f$, then there exists an $s$ such that $1 \leq s \leq l$ and a $(T + c_1 l + c_2, s, \frac{\epsilon}{l})$-prediction algorithm, where $c_1$ is the constant time needed to draw one bit at random, and $c_2$ is the constant time needed to test the value of a bit and return a bit based on the result of the test.*

*3.5. Family of Pseudo-Random Generators*

In realistic scenarios, it is necessary to draw the function $f$ from a family according to a probability distribution. For instance, consider the family of Blum, Blum, Shub generators. We select two Blum primes $p$ and $q$, where both $p$ and $q$ satisfy $p \equiv 3 \mod 4$ and $q \equiv 3 \mod 4$. From a secret seed $s_0$, we construct a sequence defined by:

$$s_k = s_{k-1}^2 \mod (pq).$$

Here, $s_k$ is an internal state that must remain secret. The bit $x_k$ of the pseudo-random sequence is the last bit of $s_k$. If we fix a size for the product $pq$, we can consider the family of pseudo-random generators constructed from all pairs $(p, q)$ of distinct primes such that $pq$ has the required size. By slightly modifying the definitions according to this new context, we can derive similar results.

**Theorem 3.8.** *Let $\mathcal{F}$ be a family of pseudo-random generators of the same size, where each $f \in \mathcal{F}$ is a function:*

$$f : U_f \subset \{0,1\}^k \to \{0,1\}^l.$$

*If there exists a $(T, s, \epsilon)$-prediction algorithm for $\mathcal{F}$, then we can construct a $(T + c, \epsilon)$-distinguisher for $\mathcal{F}$, where $c$ is the constant time needed to compare two bits.*

**Theorem 3.9.** *Let $\mathcal{F}$ be a family of pseudo-random generators of the same size, where each $f \in \mathcal{F}$ is a function:*

$$f : U_f \subset \{0,1\}^k \to \{0,1\}^l.$$

*If there exists a $(T, \epsilon)$-distinguisher algorithm for $\mathcal{F}$, then we can construct a $(T + c_1 l + c_2, s, \frac{\epsilon}{l})$-prediction algorithm for $\mathcal{F}$ for some value of $s$ (where $1 \leq s < l$), with $c_1$ being the constant time needed to draw one bit at random, and $c_2$ being the constant time needed to test the value of a bit and return a bit based on the result of the test.*

### *3.6. Asymptotic Behavior*

As a consequence of the previous results for fixed $k$ and $l$, we can derive results on the asymptotic theory of pseudo-random generators, specifically when $k$ approaches infinity and $l = l(k) > k$ is a polynomial function of $k$.
Let $k$ be a positive integer (the security parameter) and $l(k)$ a polynomial function of $k$ such that $l(k) > k$. For any $k$, we have a set $\mathcal{F}_k$ of deterministic functions such that:

1. If $f \in \mathcal{F}_k$, then $f$ is a function from a subset $U_f$ of $\{0,1\}^k$ into $\{0,1\}^{l(k)}$.
2. There exists a polynomial function $t(k)$ such that for any $k$, any $f \in \mathcal{F}_k$, and any $X \in U_f$, the computation time of $f(X)$ is upper-bounded by $t(k)$.
3. For any $k$, we provide a probability distribution $\delta_k$ on the set $\mathcal{F}_k$.

The asymptotic notions of indistinguishability and unpredictability are derived from the previous definitions. We define a distinguisher $A$ to be a probabilistic polynomial algorithm that takes as inputs the security parameter $k$, a function $f \in \mathcal{F}_k$, and a vector $Y \in \{0,1\}^{l(k)}$, and outputs a bit. For any integer $k$, we denote by $A_k$ the probabilistic algorithm obtained from $A$ by fixing the first entry to the value $k$.

**Definition 3.10.** The family $\mathcal{F} = \{\mathcal{F}_k\}_{k>0}$ of sets of pseudo-random generators is said to be asymptotically secure if for any polynomial $S(k)$, any positive integer $u$, and any distinguisher $A$ with running time $\leq S(k)$, the advantage of the algorithm $A_k$ is a negligible function of $\frac{1}{k^u}$, namely:

$$\lim_{k\to\infty} k^u \operatorname{Adv}^{\text{dist}}_{\mathcal{F}_k}(A_k) = 0.$$

Let $s = \{s_k\}_{k\geq 1}$ be an increasing sequence of positive integers such that $1 \leq s_k < l(k)$. We define an $s$-prediction algorithm to be a probabilistic polynomial algorithm $B$ that takes as inputs the security parameter $k$, a function $f \in \mathcal{F}_k$, and a vector $Z \in \{0,1\}^{s_k}$, and outputs a bit. For any integer $k$, we denote by $B_k$ the probabilistic algorithm obtained from $B$ by fixing the first entry to the value $k$.

**Definition 3.11.** The family $\mathcal{F} = \{\mathcal{F}_k\}_{k>0}$ of sets of pseudo-random generators is said to be asymptotically unpredictable if for any polynomial $S(k)$, any sequence $s$, and any $s$-prediction algorithm $B$ with running time $\leq S(k)$, the advantage of the $s_k$-prediction algorithm $B_k$ is a negligible function of $\frac{1}{k^u}$, namely:

$$\lim_{k\to\infty} k^u \operatorname{Adv}^{\text{pred}}_{\mathcal{F}_k, s_k}(B_k) = 0.$$

We can now state the asymptotic version of Yao's theorem:

**Theorem 3.12.** *Let $l(k)$ be a polynomial function of a single integer variable k such that $l(k) > k$. Let $\mathcal{F} = \{\mathcal{F}_k\}_{k>0}$ be a family of sets, where each set $\mathcal{F}_k$ is a probability set of random generators mapping a subset of $\{0,1\}^k$ into $\{0,1\}^{l(k)}$ (more precisely, each $f \in \mathcal{F}_k$ has its own definition of subset $U_f \subseteq \{0,1\}^k$). The family $\mathcal{F}$ is asymptotically secure if and only if it is asymptotically unpredictable.*

*Proof.* See [1]. □

## 4. Conclusion

Randomness is a fundamental component in cryptographic protocols and primitives, essential for ensuring security in various applications such as key generation, key exchange, and stream ciphers. This paper discusses the construction and theoretical underpinnings of pseudo-random generators (PRNGs), emphasizing their role in simulating randomness. It highlights the necessity of distinguishing between true randomness and pseudo-randomness, introducing concepts such as distinguishers and prediction algorithms. The paper presents a formal definition of PRNGs and explores the relationship between distinguishers and prediction algorithms through Yao's theorem. It provides insights into the complexity and efficiency of these algorithms, establishing a foundation for understanding their security. The paper concludes with a discussion on the asymptotic security and unpredictability of PRNGs, establishing that a family of PRNGs is asymptotically secure if and only if it is asymptotically unpredictable.

The findings suggest avenues for further research in improving the efficiency and security of randomness generation in cryptographic systems, particularly in the context of evolving computational capabilities and attack strategies. Additionally, the integration of physical randomness sources with pseudo-random generators to create hybrid systems that enhance the overall quality of randomness in cryptographic applications is explored. Investigating the application of machine learning techniques to improve the testing and validation of randomness in cryptographic systems is also proposed, potentially leading to more robust detection of weaknesses.

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# Upper bound of second teoplitz determinant for the subclasses of analytic functions involving of q-derivative operator

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**Abstract**

In this paper, we investigate an interesting class of analytic and bi-univalent functions in the open unit disk $\Delta$ which is defined using the q-derivative operator. We apply the subordination method to the functional of coefficients problem.
Furthermore, we obtain the bounds of the Toeplitz determinants $T_2(1) = 1 - a_2^2$ and $T_2(2) = a_2^2 - a_3^2$ defined for the coefficients of a functions $f$ which belongs to the classes $\mathcal{H}_{\Sigma}^{q,\alpha}$ and $\mathcal{H}_{\Sigma}^{q}(\beta)$.

## 1. Introduction

The class of all analytic functions was denoted by $\mathcal{A}$. Functions belonging to this class can be displayed in the form of the following power series

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n. \tag{1}$$

The class of univalent functions in $\mathcal{A}$ which normalized with the conditions $f(0) = f'(0) - 1 = 0$, was represented by $\mathcal{S}$.
Since each function which belongs to the class $\mathcal{S}$ has an inverse, hence, we can easily calculate,

$$\left(f^{-1}(w)\right)' = \frac{1}{f'(z)}.$$

Therefore, we conclude that $f^{-1}$ is analytic[6]. So $f^{-1}$ can also be displayed in the form of a power series as follows,

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots \tag{2}$$

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where

$$f^{-1}(f(z)) = z, \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w, \quad \left(w \in \Delta,\ |w| < r_0(f); r_0(f) > \frac{1}{4}\right).$$

A function is called bi-univalent in open unit disk Δ, if $f$ and $f^{-1}$ are univalent in open unit disk Δ.
Σ is considered a symbol of the class of bi-univalent functions Δ. For more information on the class Σ, readers can refer to [7].

Let Ω be the family functions $\omega(z)$ in the unit disc Δ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$. Note that $f(z) \prec g(z)$ if there is a function $\omega(z) \in \Omega$ such that $f(z) = g(\omega(z))$. [2].

In recent years, the subject of quantum calculus has been considered by many researchers. There are many applications of quantum calculus in various branches of mathematics and physics. q-derivatives and q-integrals by Jackson [3, 4] were first systematically way introduced and studied, also, he was the first to introduce applications of quantum calculus.

Let $0 < q \leq p \leq 1$. The (p,q)-bracket $[n]_{p,q}$ is defined by

$$[n]_{p,q} = \begin{cases} \dfrac{p^n - q^n}{p - q}, & p \neq q \\ np^{n-1}, & p = q. \end{cases}$$

Notice that $\lim_{q \to p} [n]_{p,q} = [n]_{p,p}$.
The (p,q)-derivative operator $D_{p,q}$ of a function $f \in \mathcal{A}$ is given by

$$D_{p,q} f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}, \quad p, q \in (0, 1],$$

For a function $f \in \mathcal{A}$, it can be easily seen that

$$D_{p,q} f(z) = \frac{f(pz) - f(qz)}{(p - q)z}, \quad z \in \Delta, z \neq 0, p, q \in (0, 1], p \neq q,$$

$D_{p,q} f(0) = 1$ and $D_{p,p} f(z) = f'(z)$.
The (1,q)-derivative operator is known as the q-derivative operator and is denoted by $D_q$; for a function $f \in \mathcal{A}$ and $z \neq 0$ is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1 - q)}, \quad z \in \Delta, z \neq 0, q \in (0, 1). \tag{3}$$

Using equations (1) and (3), for $n \in \mathbb{N}$ and $z \in \Delta$ we obtained

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad q \in (0, 1), \tag{4}$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + \sum_{k=1}^{n-1} q^k; \ [0]_q = 0. \tag{5}$$

It is clear that $[1]_q = 1$.
Also, the $n-$th order q-derivative of $f \in \mathcal{A}$ is defined by

$$D_q^{(n)} f(z) = (-1)^n (1-q)^{-n} z^{-n} q^{\frac{-n(n-1)}{2}} \times \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\frac{k(k-1)}{2}} f\left(q^{n-k} z\right).$$

For this results, we refer to [9].
One of the important subclass of analytic functions defined by q-derivative is $\mathcal{S}_q^*$, the class $\mathcal{S}_q^*$ of q-starlike functions consists of $f \in \mathcal{A}$ that satisfies

$$f(0) = f'(0) - 1 = 0$$

and

$$\left| \frac{z D_q f(z)}{f(z)} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}.$$

S. Bulut introduced the two subclass $\mathcal{H}_\Sigma^{q,\alpha}$ and $\mathcal{H}_\Sigma^q(\beta)$ of analytic functions in $\Delta$[8].
Let $\mathcal{H}_\Sigma^{q,\alpha}$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following conditions:

$$f \in \Sigma \quad and \quad |\arg(D_q f(z))| < \frac{\alpha\pi}{2} \quad (z \in \Delta) \tag{6}$$

and

$$|\arg(D_q g(\omega))| < \frac{\alpha\pi}{2} \quad (\omega \in \Delta) \tag{7}$$

where the function $g$ is given by

$$g(\omega) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - \cdots \tag{8}$$

And let $\mathcal{H}_\Sigma^q(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following conditions:

$$f \in \Sigma \quad and \quad Re(D_q f(z)) > \beta \quad (z \in \Delta), \tag{9}$$

and

$$Re(D_q g(\omega)) > \beta \quad (\omega \in \Delta), \tag{10}$$

where the function $g$ is defined by 8.

Toeplitz determinants are closely related to Hankel determinants [5]. Toeplitz matrices have constant entries along the diagonal. Toeplitz matrices have some applications in pure and applied mathematics[11].
Thomas and Halim in [10] introduced the symmetric determinant $T_q(n)$ for analytic functions $f$ of the form 1 defined by,

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix} \qquad q \in \mathbb{N} \setminus 1, \quad n \in \mathbb{N}.$$

The study of exact upper bound of $|T_q(n)|$ for different subclasses of analytic functions has attracted some authors. The Toeplitz determinant $T_q(n)$ for class $\mathcal{S}$ of univalent functions was studied and improved by Ali et al[1] and [6].

In the particular cases
$q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Toeplitz determinant simplifies respectively to

$$T_2(1) = 1 - a_2^2, \quad and \quad T_2(2) = a_2^2 - a_3^2.$$

In this paper,we find sharp upper bounds for the functional $|1 - a_2^2|$ and $|a_2^2 - a_3^2|$ for functions belonging to the two classes $\mathcal{H}_{\Sigma}^{q,\alpha}$ and $\mathcal{H}_{\Sigma}^{q}(\beta)$, where, $0 < q < 1$, $0 \leq \alpha, \beta < 1$.

## 2. Main Results

Suppose that $\mathcal{P}$ denote the class of analytic functions $p$ of the type

$$p(z) = 1 + \sum_{n=2}^{\infty} c_n z^n, \tag{11}$$

then

$$|c_n| \leq 2, \qquad (n = 1, 2, \dots)$$

where $\mathcal{P}$ be the class of functions with positive real part consisting of analytic functions $p : \Delta \to \mathcal{C}$ satisfying $p(0) = 1$ and $Re(p(z)) > 0$[2].
To prove our main results, we need following lemmas and theorems.

**Lemma 2.1.** *If $p \in \mathcal{P}$ and of the form (11), then*

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$

*for some $x$, with $|x| \leq 1$.*

**Theorem 2.2.** *If $f(z) \in \mathcal{H}_{\Sigma}^{q,\alpha}(0 < q < 1,\ 0 \leq \alpha < 1)$, then*

$$|T_2(1)| = |1 - a_2^2| \leq 1 + \frac{4\alpha^2}{2[3]_q\alpha + (1-\alpha)[2]_q^2}, \tag{12}$$

*and*

$$|T_2(2)| = |a_2^2 - a_3^2| \leq \frac{4\alpha^2}{2[3]_q\alpha + (1-\alpha)[2]_q^2} \tag{13}$$

$$+ \left[\frac{4\alpha^2}{[2]_q^2} + \frac{2\alpha}{[3]_q}\right]^2 \tag{14}$$

**Theorem 2.3.** *If $f(z) \in \mathcal{H}_{\Sigma}^{q}(\beta)(0 < q < 1,\ 0 \leq \beta < 1)$, then*

$$|T_2(1)| = |1 - a_2^2| \leq 1 + \left[\min\left\{\frac{2(1-\beta)}{[2]_q}, \sqrt{\frac{2(1-\beta)}{[3]_q}}\right\}\right]^2, \tag{15}$$

*and*

$$|T_2(2)| = |a_2^2 - a_3^2| \leq \left[\min\left\{\frac{2(1-\beta)}{[2]_q}, \sqrt{\frac{2(1-\beta)}{[3]_q}}\right\}\right]^2 \tag{16}$$

$$+ \frac{4(1-\beta)^2}{[3]_q^2}. \tag{17}$$

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# Prediction and Analysis of Molecular Energies of Anticancer Drugs Using Topological Indices and Machine Learning in Python

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**Abstract**

In this paper, topological indices such as the Sombor index, reduced Sombor index, and average Sombor index are calculated for the molecular graphs of several anticancer drugs. Then, using the adjacency matrix, the Sombor energy, reduced Sombor energy, and average Sombor energy of these compounds are extracted and analyzed. By optimizing parameters and utilizing advanced regression models within the framework of Quantitative Structure-Property Relationship (QSPR), high accuracy in predicting molecular properties is achieved, providing deeper insights into the relationship between structure, topological indices, and the functionality of the drugs. For modeling the relationship between the topological features and molecular properties, machine learning techniques, particularly Linear Regression, are applied. The dataset is preprocessed by removing outliers, selecting relevant features based on correlation with molecular properties, and standardizing the data. The model is trained using training data, and its performance is evaluated with metrics such as Mean Squared Error (MSE), Root Mean Squared Error (RMSE), Mean Absolute Error (MAE), and $R^2$ Score. This machine learning approach, implemented in Python, accurately predicts the physical and chemical properties of these drugs, providing valuable insights into their functionality.

## 1. Introduction

In the drug discovery process, determining the properties of molecular compounds in laboratories can be highly expensive due to the requirement for costly equipment and rare, valuable materials. Additionally, these experiments are time-consuming and require skilled professionals. Therefore, utilizing graphs and topological indices as a mathematical and theoretical approach to estimate the physicochemical properties of molecular compounds provides an efficient solution. These methods can predict various physicochemical properties of molecular structures and serve as essential tools for analyzing drug properties and structures. In recent years, researchers have increasingly focused on employing these indices in the analysis of pharmaceutical structures, particularly in fields such as Regular Domination in Vague Graphs, dendrimers, and drug compounds[1, 11, 14]. In graph theory, graph energy is defined as the sum of the absolute values of the eigenvalues of a graph's adjacency matrix, which serves as a useful parameter for evaluating

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molecular properties. In this study, along with the calculation of Sombor indices for cancer drugs, Sombor energy is also determined using the Sombor energy matrix. These approaches contribute to improving the design of cancer drugs and enhancing their therapeutic efficiency[3]. Jahanbani et al. (2023) investigated the relationships between the energy and topological indices of graphs, demonstrating a strong correlation between these properties[11]. In another study, Ülker et al. (2022) analyzed the energy and Sombor index of graphs, examining their impact on graph theory [16]. Subsequently, Reja and Nayeem (2024) explored the Sombor index and graph energy for certain chemically significant graphs, providing various examples and counterexamples [13]. These studies indicate that the Sombor index and graph energy serve as powerful tools for analyzing the mathematical and chemical properties of complex networks, with numerous applications across various scientific fields. Numerous studies have utilized topological indices to analyze the structure-property relationships of drugs and various compounds [5–7]. These indices represent complex relationships between molecular structures and physicochemical properties, making them particularly useful in developing Quantitative Structure-Property Relationship (QSPR) models for predicting molecular characteristics [8–10, 15]. The application of these models enables researchers to simulate intricate relationships between molecular structures and physicochemical properties, facilitating the design of more effective drugs with enhanced therapeutic potential.

## 2. Preliminaries

In this article, $G = (V, E)$ is considered a simple and connected graph, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. The degree of an arbitrary vertex $r$ is denoted by $d_r$.
The Sombor index was introduced by Gutman in late 2020 [12], and since then, many researchers have studied it.

**Definition 2.1.** The Sombor index for a graph $G$ is defined as:

$$SO(G) = \sum_{rs \in E(G)} \sqrt{d_r^2 + d_s^2}$$

We also have for the reduced Sombor index and the average Sombor index, respectively:

$$SO_{\text{red}}(G) = \sum_{rs \in E(G)} \sqrt{(d_r - 1)^2 + (d_s - 1)^2}$$

$$SO_{avr}(G) = \sum_{rs \in E(G)} \sqrt{(d_r - \bar{d}(G))^2 + (d_s - \bar{d}(G))^2},$$

where

$$\bar{d}(G) = \frac{2|E(G)|}{|V(G)|}$$

**Definition 2.2.** The adjacency matrix or connectivity matrix of a graph is a matrix with rows and columns labeled by the vertices of the graph and whose values are either 1 or 0. If the vertices of $x$ and $y$ are adjacent, the value of their similarity is 1, otherwise it is 0 [1, 12].

**Definition 2.3.** Let $G$ be a graph and $A = A(G)$ be its adjacency matrix. The energy of the graph $G$ is given by

$$\varepsilon(G) = \sum_{i=0}^{n} |\lambda_i|$$

where $\lambda_i$'s are the eigenvalues of the adjacency matrix of $G$ [4] .

### *2.1. Evaluation Metrics*

To assess the accuracy of predictions, three main metrics are used. These metrics evaluate how accurate and effective each model is.

## 3. Methodology

In this section, the Sombor index, reduced Sombor index, average Sombor index, Sombor energy, reduced Sombor energy, and average Sombor energy are calculated for the molecular graphs of the drugs Dacarbazine, Imiquimod, Vismodegib, Cobimetinib, Baicalein, Picato, Naringenin, Daurismo, Dabrafenib, Vemurafenib, and Myricetin based on the adjacency matrix, and the results are presented in Table 3. As an example, the calculations for the drug Naringenin are explained in detail, and the same process is followed for the other drugs. Figure 1 illustrates the chemical structure of Naringenin. Figure 2 presents the labeled vertices in the molecular graph of Naringenin and its Sombor matrix SO(N).

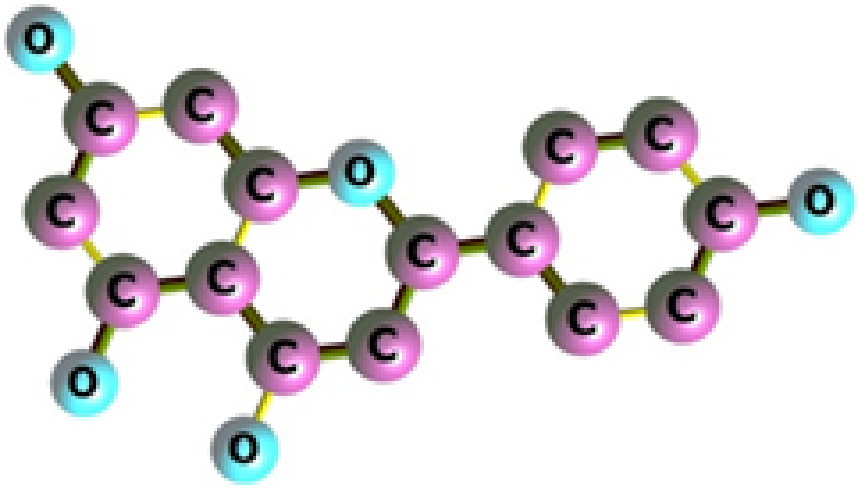

Fig. 1. Naringenin chemical structure of the drug

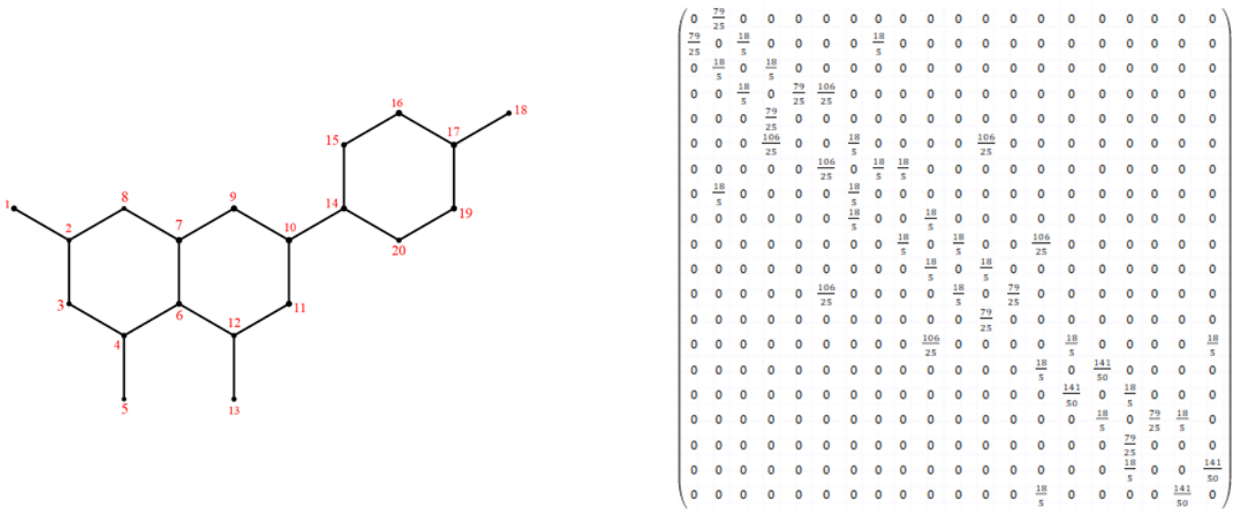

Fig. 2. The labelled vertices on the molecular graph of Naringenin and its Sombor matrix SO(N).

There are four types of edges in the molecular graph of Naringenin as follows:

$$E_{13} = \{(d_u, d_v) \mid uv \in E(N), d_u = 1, d_v = 3\},$$
$$E_{22} = \{(d_u, d_v) \mid uv \in E(N), d_u = 2, d_v = 2\},$$
$$E_{23} = \{(d_u, d_v) \mid uv \in E(N), d_u = 2, d_v = 3\},$$
$$E_{33} = \{(d_u, d_v) \mid uv \in E(N), d_u = 3, d_v = 3\}.$$

The types of edges in the molecular graph of Naringenin are shown in Table 1.

| Edge type | $E_{13}$ | $E_{22}$ | $E_{23}$ |
|---|---|---|---|
| Number of edges | 4 | 2 | 12 |
| $\bar{d}(G)$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{6}{5}$ |

Table 1. Edge types and values of $\bar{d}(G)$ for Naringenin.

According to the structure of the molecular graph of Naringenin, we obtain the Sombor matrix of graph SO(N) which is shown in Figure 2.

Theorem 3.1: Let N be the molecular graph of Naringenin. Then Sombor indices of N are computed as,

$$i.SO(N) = \sum_{uv \in E(N)} \sqrt{d_v^2 + d_u^2} = 4\sqrt{1+9} + 2\sqrt{4+4} + 12\sqrt{4+9} + 4\sqrt{9+9} = 78.96$$

$$ii.S_{O_{red}}(N) = \sum_{uv \in E(N)} \sqrt{(d_u - 1)^2 + (d_v - 1)^2} = 4\sqrt{4} + 2\sqrt{2} + 12\sqrt{5} + 4\sqrt{8} = 48.86$$

$$iii.S_{O_{avr}}(N) = \sum_{uv \in E(N)} \sqrt{(d_u - \bar{d})^2 + (d_v - \bar{d})^2} = 4\sqrt{1.44} + 2\sqrt{0.28} + 12\sqrt{0.82} + 4\sqrt{1.13} = 20.68$$

Theorem 3.2: Let N be the molecular graph of Naringenin. Then Sombor energy of N is computed as,

$$\text{E (SO (N))} = \sum_{i=1}^{20} |\lambda_i| \sim 88.144$$

Proof:

In order to calculate the Sombor energy of the molecular graph of Naringenin SO(N), we obtain the characteristic polynomial of the matrix SO(N). Characteristic polynomial of Sombor matrix of the molecular graph of Naringenin is as

$$P(N, X) = x^4\left(x^2 - \left(\frac{141}{50}\right)^2\right)\left(x^2 - (9.302)^2\right)\left(x^2 - (4.850)^2\right)\left(x^2 - (5.897)^2\right)$$
$$\times \left(x^2 - (2.829)^2\right)\left(x^2 - (4.049)^2\right)\left(x^2 - (5.897)^2\right)\left(x^2 - (7.860)^2\right)$$

The roots of polynomial P (N, X) are the eigenvalues of the molecular graph SO(N) which are shown in Table 2.

| Eigenvalues of the molecular graph SO(N) | | | | |
|---|---|---|---|---|
| 141/5 | 7.860 | 5.897 | 4.049 | 4.850 |
| -141/5 | -7.860 | -5.897 | -4.049 | -4.850 |

Table 2. Eigenvalues of the Sombor graph of Naringenin.

We get the edge energy as follows

$$E(SO(N)) = \sum_{i=1}^{20} |\lambda_i| \approx 88.144$$

## 4. Comparison of actual and predicted values of the model

In Table 3, the Sombor indices and the energy of each index for cancer drugs have been calculated using Python code in the IDLE Shell 3.12.7 environment, with links provided. The data related to physicochemical properties (Boiling point (BP), Enthalpy (EN), Flash Point (FP), Molar Refractivity (MR), Polar Surface Area (PSA), Polarizability (PO), Surface Tension (ST)) [2] and the prediction results obtained using Python code in the linear regression model, a machine learning approach, within the IDLE Shell 3.12.7 environment are presented in Table 4. Evaluation metrics for comparing the model are shown in Table 5. Additionally, Figure 3 provides a graphical representation of the regression method based on four evaluation criteria, while Figure 4 illustrates the analysis of the performance of actual and predicted property values using regression. The algorithm is also visible.

**Machine Learning Model Algorithm with Linear Regression**

1. **Data Preparation:**
• Load data from Excel and convert numeric values.
• Remove outliers using the IQR method for BP.
2. **Feature Selection:**
• Select features that have a correlation greater than 0.3 with BP.
3. **Data Preprocessing:**
• Apply the Box-Cox transformation on y if needed.
• Split the dataset into training and testing sets (80/20).
• Standardize the features (X).
4. **Modeling and Training:**
• Train the Linear Regression model on the training data.
5. **Prediction and Evaluation:**
• Predict y_test using the trained model.
• Evaluate the model using MAE, MSE, RMSE, and $R^2$ metrics.
6. **Results:**
• Save and display the model evaluation results.

| Drugs | SO) G) | SOR(G) | SOAV(G) | ESO(G) | ESOR(G) | ESOAV(G) |
|---|---|---|---|---|---|---|
| Dacarbazine | 126.7421 | 118.3849 | 2.934199 | 33.2518 | 0.368967 | 59.19246 |
| Imiquimod | 360.5874 | 347.0184 | 2.368258 | 45.89154 | 0.150337 | 173.5092 |
| Vismodegib | 550.5969 | 514.9066 | 4.377426 | 85.86864 | 0.32912 | 257.4533 |
| Cobimetinib | 693.8927 | 660.8518 | 3.39665 | 79.15253 | 0.191875 | 330.4259 |
| Baicalein | 78.16 | 48.4 | 20.22 | 89.856 | 55.22 | 30.348 |
| Picato | 901.9648 | 838.6402 | 2.9193 | 87.5087 | 0.141683 | 419.3201 |
| Naringenin | 78.96 | 48.86 | 20.68 | 88.144 | 39.478 | 20.68 |
| Daurismo | 623.1533 | 603.4705 | 3.034629 | 67.66663 | 0.163337 | 301.7353 |
| Dabrafenib | 843.2172 | 785.0541 | 3.762052 | 101.3705 | 0.210452 | 392.5271 |
| Vemurafenib | 762.9345 | 726.3321 | 3.685783 | 86.8169 | 0.207326 | 363.1661 |
| Myricetin | 92.04 | 58.86 | 28.07 | 101.002 | 56.422 | 32.71 |

Table 3. Prediction of physicochemical properties using linear regression model

| Drugs | Actual BP | Pred. BP | Actual EN | Pred. EN | Actual FP | Pred. FP | Actual MR | Pred. MR | Actual PO | Pred. PO | Actual PSA | Pred. PSA | Actual ST | Pred. ST |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Dacarbazine | 456.3 | 455.2 | 71.6 | 69.9 | 191.4 | 226.4 | 46.2 | 46.2 | 18.3 | 18.3 | 57 | 59.6 | 48.8 | 50.3 |
| Imiquimod | 456.7 | 457.7 | 71.7 | 75.3 | 224.7 | 180.2 | 69.9 | 69.8 | 27.7 | 27.7 | 65 | 102.6 | 54.7 | 52.2 |
| Vismodegib | 561.6 | 567.6 | 84.4 | 86.1 | 225.3 | 201.4 | 70.3 | 70.4 | 27.9 | 27.8 | 85 | 102.5 | 59 | 60.8 |
| Cobimetinib | 565.9 | 595.0 | 89.4 | 94.2 | 229.7 | 213.0 | 71 | 71.0 | 28.2 | 28.1 | 87 | 82.0 | 60.7 | 58.8 |
| Baicalein | 575.9 | 569.3 | 89.5 | 89.3 | 230 | 259.7 | 75.2 | 75.2 | 29.8 | 29.7 | 87 | 80.5 | 61 | 63.6 |
| Picato | 576.9 | 578.5 | 89.7 | 91.7 | 285.9 | 304.8 | 105.5 | 105.5 | 41.8 | 41.7 | 97 | 124.4 | 61.2 | 59.2 |
| Naringenin | 577 | 565.1 | 93.6 | 100.2 | 293.4 | 276.2 | 106.9 | 106.7 | 42.4 | 42.4 | 100 | 90.5 | 61.7 | 54.2 |
| Daurismo | 633.4 | 636.5 | 96.3 | 100.7 | 296.1 | 323.5 | 115.2 | 115.1 | 45.6 | 45.6 | 100 | 108.2 | 72.9 | 99.4 |
| Dabrafenib | 653.7 | 652.4 | 99.2 | 91.8 | 336.9 | 382.0 | 115.3 | 115.1 | 45.7 | 45.7 | 104 | 115.4 | 73.2 | 57.9 |
| Vemurafenib | 711.4 | 709.2 | 104 | 103.4 | 349.2 | 322.2 | 121.6 | 121.5 | 48.2 | 48.2 | 147 | 118.2 | 79.6 | 79.9 |
| Myricetin | 747.6 | 749.7 | 114.4 | 114.3 | 384 | 384.9 | 127.4 | 127.4 | 50.5 | 50.4 | 148 | 151.1 | 133 | 132.6 |

Table 4. The sombor indices and Sombor energy indices measurements for the drugs

| physicochemical | MAE | MSE | RMSE | $R^2$ |
|---|---|---|---|---|
| BP | 6.004234 | 99.41913 | 9.970914 | 0.986828 |
| EN | 3.023736 | 15.06636 | 3.88154 | 0.897498 |
| FP | 26.03287 | 828.2199 | 28.77881 | 0.761161 |
| MR | 0.053521 | 0.004564 | 0.067556 | 0.999993 |
| PO | 0.022983 | 0.000842 | 0.02902 | 0.999992 |
| PSA | 14.34632 | 334.5838 | 18.29163 | 0.547131 |
| ST | 5.65082 | 92.6196 | 9.623908 | 0.803944 |

Table 5. Evaluation of the linear regression model using MAE, MSE, RMSE, and $R^2$ to predict drug properties.

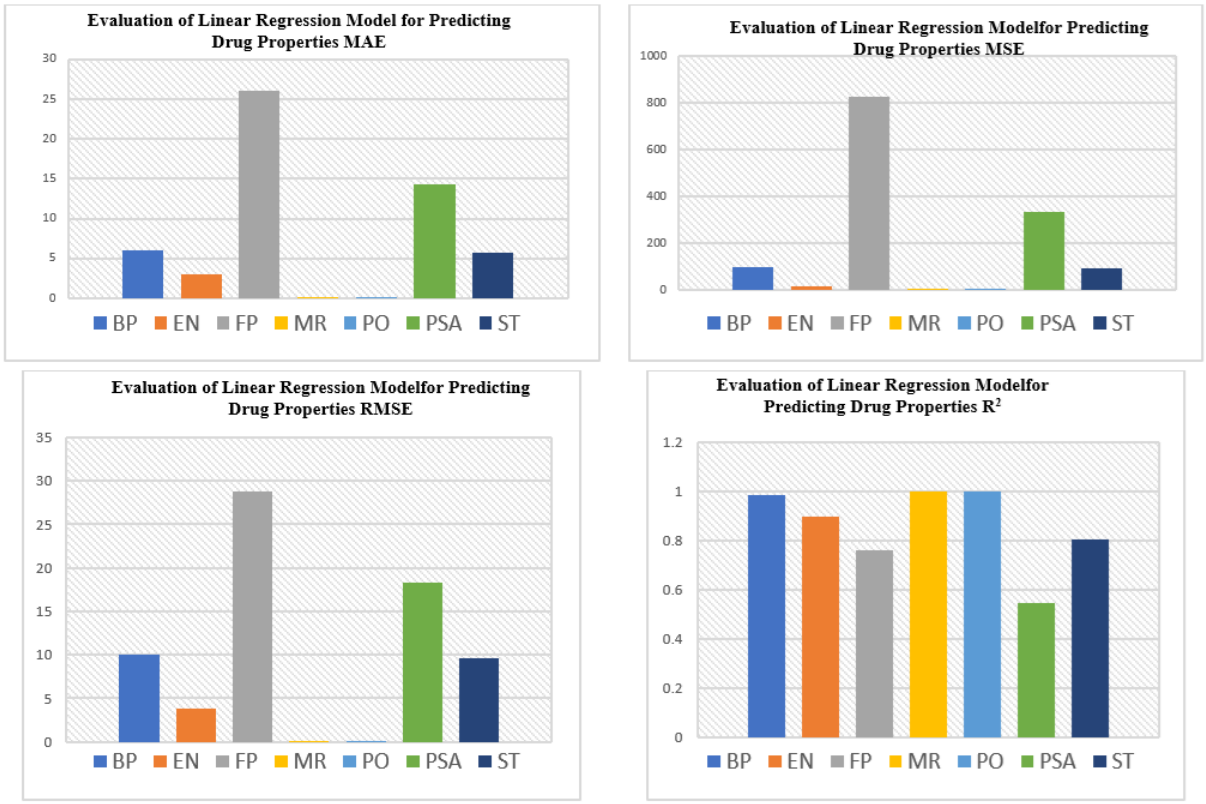


Fig. 3. Comparison of the linear regression model approach for predicting the physicochemical properties of cancer drugs using $R^2$, MSE, MAE, and RMSE.

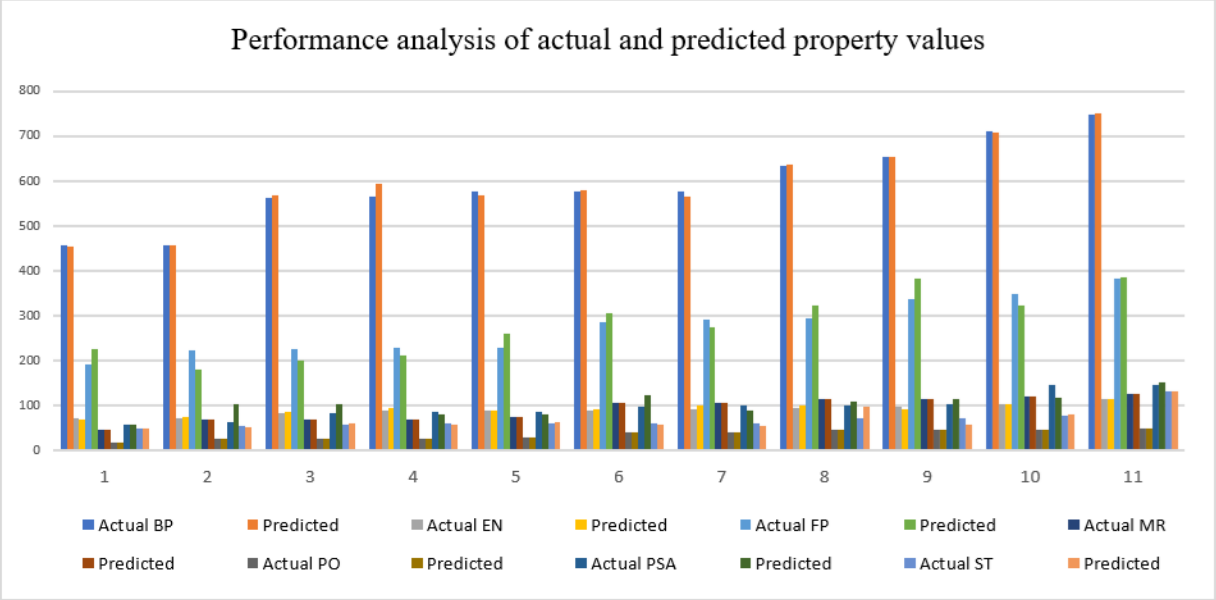


Fig. 4. Analysis of the performance of actual and predicted property values using regression

## 5. Conclusion

This study demonstrates the strong predictive power of Sombor indices and Sombor energy in estimating the physicochemical properties of drugs. Linear regression, which is considered an efficient and interpretable machine learning method for simple models, has been used. The boiling point (BP) with MAE = 6.00 and RMSE = 9.97 shows acceptable accuracy, and the value of $R^2 = 0.9868$ indicates a very precise prediction. Electronegativity (EN) also performs well with MAE = 3.02 and RMSE = 3.88, though its accuracy is slightly lower than BP ($R^2 = 0.8975$). In contrast, the flash point (FP) with MAE = 26.03 and RMSE = 28.77 exhibits high error, and $R^2 = 0.7611$ indicates that the model performs poorly in predicting this feature. The refractive index (MR) and oil solubility (PO) both demonstrate very accurate performance, as their $R^2 = 0.9999$ values indicate high accuracy and negligible model error (MAE = 0.053, RMSE = 0.067 for MR, and MAE = 0.022, RMSE = 0.029 for PO). On the other hand, the molecular polar surface area (PSA) with MAE = 14.34 and RMSE = 18.29 shows moderate error, and $R^2 = 0.5471$ indicates that the model performs poorly in predicting this feature. Surface tension (ST) with MAE = 5.65 and RMSE = 9.62 has moderate accuracy, with $R^2 = 0.8039$ showing lower accuracy than MR and PO. Linear regression, an efficient and interpretable machine learning method for simple models, has been utilized. This study confirms the effectiveness of the regression model in QSPR analysis for pharmaceutical applications and provides valuable insights for the future development of more effective drugs. Using the code for the predicted BP, the other properties are also predicted in the same way.
https://drive.google.com/file/d/15nY8CTuj11IPtEOZZJNYq5YHWSADaOrn/view?usp=sharing
• In the link below, the Sombor adjacency matrix and Sombor energy are calculated, and the other parameters are calculated in the same way.
https://drive.google.com/file/d/18tqhzoO0HgrTlzjZQr1-KobSXoehHDAr/view?usp=sharing

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# Metric dimension of a class of graphs on dihedral group

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**Abstract**

For an arranged subset $R = \{r_1, ..., r_k\}$ of vertices in a connected graph $\Gamma$ the metric representation of a vertex $v$ in $\Gamma$, is the $k$-vector $r(v|R) = (d(v, r_1), ..., d(v, r_k))$ relative to $R$. Also, the subset $R$ is considered as resolving set for $\Gamma$ if any pair of vertices of $\Gamma$ is distinguished by some vertices of $R$. If the set $R$ is as small as possible, then it is called a metric basis of graph $\Gamma$. We recall that the metric dimension of $\Gamma$, denoted by $dim(\Gamma)$ is defined as the minimum cardinality of a resolving set for $\Gamma$. In this paper, we consider the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$, where $\mathbb{D}_{2n} =< a, b \mid a^n = b^2 = 1, ba = a^{n-1}b >$ is the dihedral group of order $2n$ ($n \geq 4$), and $\Omega_1 = \{b, a^{n-1}b\}$, $\Omega_2 = \Omega_1 \cup \{ab, a^{n-2}b\}$, ..., $\Omega_k = \Omega_{k-1} \cup \{a^{k-1}b, a^{n-k}b\}$ and $S_1 = \{a, a^{n-1}\}$, $S_2 = S_1 \cup \{a^2, a^{n-2}\}$, ..., $S_m = S_{m-1} \cup \{a^m, a^{n-m}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$ for any $k, m \in \mathbb{N}$, $1 \leq k, m \leq [\frac{n}{2}]$, and we show that if $n$ is an even integer and ($k = \frac{n}{2}, m = \frac{n}{2} - 1$ ), then the minimal resolving set of the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is $n$. Moreover, we show that if $n$ is an odd integer and ($k = m = [\frac{n}{2}]$), then the metric dimension of the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is $n$.

## 1. Introduction

In this paper we consider finite, simple, and connected graphs. The distance $d_\Gamma(x, y)$ between two vertices $x$ and $y$ in a connected graph $\Gamma$ is the length of the shortest path from $x$ to $y$. A vertex $x \in V(\Gamma)$ is said to resolve a pair $u, v \in V(\Gamma)$ if $d_\Gamma(u, x) \neq d_\Gamma(v, x)$. For an arranged subset $R = \{r_1, ..., r_k\}$ of vertices in a connected graph $\Gamma$ the metric representation of a vertex $v$ in $\Gamma$, is the $k$-vector $r(v|R) = (d(v, r_1), ..., d(v, r_k))$ relative to $R$. Also, the subset $R$ is considered as resolving set for $\Gamma$ if any pair of vertices of $\Gamma$ is distinguished by some vertices of $R$. A resolving set with least number of vertices is referred as metric basis for $\Gamma$ and the cardinality of such resolving set is know as metric dimension denoted by $dim(\Gamma)$. The metric dimension of a graph $\Gamma$ is the least number of vertices in a set with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex. The concept of resolving set and that of metric dimension date back to the 1950s. They were defined by Bluementhal [4] in the context of metric space. Also, the concept of the metric dimension in algebraic graph theory date back to the 1970s. It was defined independently by Harary and Melter [6] and by Slater [10], and it is known that the problem of computing this invariant is NP-hard. In recent years, a considerable literature has developed [1]. These concepts have different applications in the areas of network discovery and verification [2], combinatorical optimization [9], chemistry [5],

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and robot navigation [7].

If $G$ is a finite group and a subset $\Omega$ of $G$ is closed under taking inverses and does not contain the identity, then the Cayley graph $Cay(G,\Omega)$ is the graph with vertex set $G$ and the edge set $E(Cay(G,\Omega)) = \{\{x,y\} \mid x^{-1}y \in \Omega\}$. For any vertex $v$ of a connected graph $\Gamma$, we define the $r$-distance graph as

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid \partial(u,v) = r\},$$

where $r$ is a non-negative integer not exceeding $d$, the diameter of $\Gamma$. It is clear that $\Gamma_0(v) = \{v\}$, and $V(\Gamma)$ is partitioned into the disjoint subsets $\Gamma_0(v), ..., \Gamma_d(v)$, for each $v$ in $V(\Gamma)$. The graph $\Gamma$ with automorphism group $Aut(\Gamma)$, is called vertex transitive if for any $x, y \in V(\Gamma)$, there is some $\pi$ in $Aut(\Gamma)$, the automorphism group of $\Gamma$, such that $\pi(x) = y$, [3].

Let $\mathbb{D}_{2n} =< a,b \mid a^n = b^2 = 1, ba = a^{n-1}b >$ be the dihedral group of order $2n$ ($n \geq 4$). The metric dimension, of the Cayley graph $Cay(\mathbb{D}_{2n}, \Psi)$, where $\mathbb{D}_{2n}$ is the dihedral group of order $2n$, and $\Psi = \{ab, a^2b, ..., a^{n-1}b, b\} \cup \{a^{\frac{n}{2}}\}$ is an inverse closed subset of $\mathbb{D}_{2n} - \{1\}$, is computed, see [8]. Now, let $\Omega_1 = \{b, a^{n-1}b\}$, $\Omega_2 = \Omega_1 \cup \{ab, a^{n-2}b\}$, ..., $\Omega_k = \Omega_{k-1} \cup \{a^{k-1}b, a^{n-k}b\}$ and $S_1 = \{a, a^{n-1}\}$, $S_2 = S_1 \cup \{a^2, a^{n-2}\}$, ..., $S_m = S_{m-1} \cup \{a^m, a^{n-m}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$ for any $k, m \in \mathbb{N}$, $1 \leq k, m \leq [\frac{n}{2}]$. In this paper, we show that if $n$ is an even integer and ($k = \frac{n}{2}, m = \frac{n}{2} - 1$ ), then the minimal resolving set of the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is $n$. Moreover, we show that if $n$ is an odd integer and ($k = m = [\frac{n}{2}]$), then the metric dimension of the Cayley graph $Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ is $n$.

## 2. Main results

**Proposition 2.1.** *Let $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n}$($n \geq 4$), where $\Omega_k$ and $S_m$ which are defined as before. If $n$ is an even integer and ($k = \frac{n}{2}, m = \frac{n}{2} - 1$ ), then the metric dimension of $\Pi$ is $n$.*

*Proof.* Let $V(\Pi) = \{b, ab, ..., a^{n-1}b, a, a^2, ..., a^n\}$ be the vertex set of $\Pi$. We know that, if $n$ is an even integer and $k = \frac{n}{2}, m = \frac{n}{2} - 1$, then $\Omega_k = \{b, ab, a^2b, ..., a^{n-1}b\}$, and $S_m = \{a, a^2, ..., a^n\} - \{a^{\frac{n}{2}}, a^n\}$. Also, it is well known that every Cayley graph is a vertex transitive graph. Now, if we consider the vertex $a^n$ in $V(\Pi)$, then $\Gamma_0(v) = \{a^n\}$, $\Gamma_1(v) = \mathbb{D}_{2n} - \{a^n, a^{\frac{n}{2}}\}$ and $\Gamma_2(v) = \{a^{\frac{n}{2}}\}$. Hence, for any subset $R_1$ of $V(\Pi)$ so that $a^n, a^{\frac{n}{2}} \notin R_1$ the metric representation of the vertices $a^n$ and $a^{\frac{n}{2}}$ is identical relative to $R_1$, and hence, such a set cannot be a resolving set of $\Pi$. Especially, the set $\Gamma_1(v) = \mathbb{D}_{2n} - \{a^n, a^{\frac{n}{2}}\}$, cannot be a resolving set of $\Pi$. Also, we can verify that, any clique in the graph $\Pi$ of the size $n-1$ cannot be a resolving set of $\Pi$. Now, let $R_2$ be an ordered subset of vertices in the graph $\Pi$ such that $R_2$ is a clique in graph $\Pi$ of the size $n-1$ and let $a^n \notin R_2$, hence, if we consider $R_3 = R_2 \cup \{a^n\}$ then by the following cases we have:

Case 1. If there is an element of $R_2$ so that is not adjacent to $a^n$, then the set $R_3 = R_2 \cup \{a^n\}$ cannot be a resolving set of $\Pi$.

Case 2. If every element of an ordered subset $R_2$ of vertices in the graph $\Pi$ is adjacent to $a^n$, then the set $R_3 = R_2 \cup \{a^n\}$ is a resolving set of $\Pi$, because, in this case the metric representations of all the vertices $V(\Pi) - R_3$ are not identical relative to $R_3$, and hence $R_3 = R_2 \cup \{a^n\}$ is a resolving set of $\Pi$.
Therefore by above Cases any clique in the graph $\Pi$ of the size $n$ is a minimal resolving set of $\Pi$. □

**Theorem 2.2.** *Let $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n}$($n \geq 4$), where $\Omega_k$ and $S_m$ which are defined as before. If $n$ is an even integer and ($k = \frac{n}{2}, m = \frac{n}{2} - 1$ ), then the cardinality of every minimal resolving set of $\Pi$ is $n$.*

**Proposition 2.3.** *Let $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n}$($n \geq 4$), where $\Omega_k$ and $S_m$ which are defined as before. If $n$ is an odd integer and ($k = m = [\frac{n}{2}]$), then the metric dimension of $\Pi$ is $n$.*

*Proof.* Let $V(\Pi) = \{b, ab, ..., a^{n-1}b, a, a^2, ..., a^n\}$ be the vertex set of $\Pi$. We know that, if $n$ is an odd integer and $k = m = [\frac{n}{2}]$ then $\Omega_k = \{b, ab, a^2b, ..., a^{n-1}b\} - \{a^k b\}$, and $S_m = \{a, a^2, ..., a^{n-1}\}$. Now, if we consider the vertex $a^n$ in $V(\Pi)$, then $\Gamma_0(v) = \{a^n\}$, $\Gamma_1(v) = \mathbb{D}_{2n} - \{a^n, a^k b\}$ and $\Gamma_2(v) = \{a^k b\}$. Hence, by similar way is done in the Proposition 2.1, we can show that the metric dimension of $\Pi$ is $n$. □

**Theorem 2.4.** *Let* $\Pi = Cay(\mathbb{D}_{2n}, \Omega_k \cup S_m)$ *be the Cayley graph on the dihedral group* $\mathbb{D}_{2n}$*(*$n \geq 4$*), where* $\Omega_k$ *and* $S_m$ *which are defined as before. If* $n$ *is an odd integer and (*$k = m = [\frac{n}{2}]$*), then the cardinality of every minimal resolving set of* $\Pi$ *is* $n$*.*

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# Hodge Theory for a Class of Singular Complex Manifolds

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**Abstract**

We introduce a new class of singular complex manifolds and we develop a degenerate Weak -Hodge theory for this class of singular manifolds.

## 1. Introduction

Recently we have proved that the solutions to degenerate complex Monge-Ampère equations (DCMA) have a regular behavior along the locus of degeneration which is supposed to be a smooth divisor in a Kähler manifold [1]. More precisely given a Kähler manifold $(X, \omega)$ of dimension $n$ and a smooth divisor $[D]$ in $X$ then the solution to the following degenerate complex Monge Ampère equation

$$(\omega + \partial\bar{\partial}\phi)^n = e^F |S|^2 \omega^n$$

is smooth $C^\infty$ on $D$ if the function $F$ belongs to $C^\infty(X)$. Here $S \in H^0(X, [D])$ is a holomorphic section of $[D]$ vanishing along $D$.
If $D$ is assumed to be a canonical divisor in $X$, then one can apply the solutions to the above Monge-Ampère equation to equip $X \setminus D$, with a hyperkähler structure with very special degeneration along $D$. The twistor sphere $\mathbb{S}$ on $X \setminus D$ has itself a smooth extension to a the whole space $X$.
We develop a Hodge Kodaira theory for the degenerate complex manifold obtained by a degenerate complex structure $J'$ in $\mathbb{S}$, which is orthogonal to the initial complex structure $J$ on $X$. This type of structures are called involutive structures following Treves [3], where $D$ constitute the corresponding characteristic set.

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### 1.1. Definition of Degenerate Complex Manifolds

We consider the following definition of singular complex manifolds

**Definition 1.1.** Let $X$ be a smooth real manifold of dimension 2n and let $D \subset X$ be a real smooth submanifold of real dimension k. Assume that $X \setminus D$ is endowed with a Kähler structure $(w', g', J')$ with the following assymptotic behavior near $D$:

1) The Kähler metric itself and its Ricci curvature tensor both are bounded with respect to the sup norm $|.|_{sup}$ induced by the degenerate metric $g'$ in $X \setminus D$

2) Around each point $p \in D$ there exists an open set $U$ and a local coordinates system $\{(x_1, \ldots x_{2n}) | x_i \in \mathbb{R}, 1 \leq i \leq 2n\}$ defined on $U$ such that $D \cap U = \pi^{-1}(0)$ and $det(g') = |r|^s + o(r^s)$. Where $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-k}$ denotes projection on the last $2n - k$ coordinates, $r = \sqrt{\sum_{i=2n-k+1}^{2n} x_i^2}$, and $g'$ is the singular Kähler metric.

3) If $\mathcal{S}$ denotes the sphere bundle of the normal bundle $N$ of $D$ in $X$ then there exists an $\epsilon > 0$ and a smooth application $T : \mathcal{S} \times [0, \epsilon) \to U_\epsilon$ onto an open neighborhood $U_\epsilon$ of $D$ in $X$ such that $T|_{\mathcal{S}\times,(0,\epsilon)}$ defines a diffeomorphism onto $U_\epsilon \setminus D$, $T|_{D\times\{0\}} : \mathcal{S} \times \{0\} \to D$ sends $(x, 0)$ to $p(x)$ where $p : N \to D$ is the bundle projection. And such that $T_*(\frac{\partial}{\partial r}) \perp D_r$ where for $0 < r < \epsilon$, $D_r = T(\mathcal{S} \times \{r\})$ and $\frac{\partial}{\partial r}$ is the vector field on $S \times (0, \epsilon)$ tangent to the direction of the interval $(0, \epsilon)$ at each point.

We also assume that if a neighborhood of a point $p \in D$ is endowed with a coordinates system such that $r$ is one of its coordinates then $\|dr\|_{g'} = r^t + o(r^t)$ for some $t > 0$.

4) The parameters $s$ and $t$, defined respectively in parts 2 and 4, satisfy

$$s > 2t - 2n + k + 1.$$

### 1.2. Main Examples

Let $D$ be a smooth canonical divisor in the complex surface $X$ and if $\Omega$ denotes a holomorphic $(2, 0)$-form vanishing along $D$, then by solving the degenerate Monge-Ampère equation

$$\omega' \wedge \omega' = \lambda \Omega \wedge \bar{\Omega} \tag{1}$$

for appropriate constant $\lambda$, we obtain a Ricci-flat metric $g'$ on $X \setminus D$ degenerating transvarsally along $D$.

Let

$$\Omega = \alpha + i\beta$$

be the decomposition of $\Omega$ into real and imaginary parts. Consider the map

$$p : S^2 \to \Omega^2(X),$$
$$(a, b, c) \to a\omega' + b\alpha + c\beta,$$

where $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ is the unit sphere. Then, for any positively oriented orthonormal frame $(v_1, v_2, v_3)$ of unit vectors $v_1, v_2, v_3 \in S^2$ the subbundle

$$\mathcal{V} := Ker\,(p\,(v_1) + ip\,(v_2)) \subset T_{\mathbb{C}}X, \tag{2}$$

defines an involutive structure on $X$ ([3]). This means that the complex subbundle of $T_{\mathbb{C}}X$ defined by $\mathcal{V}$ satisfies Frobenious integrabiluty condition $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$. Moreover we have $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$ and $\mathcal{V} \oplus \bar{\mathcal{V}} = T_{\mathbb{C}}$ on $X \setminus D$. In other words we have a sphere $\mathbb{S}$ of involutive structures on $X$ inducing a hyperKähler structure on $X \setminus D$.

The degenerate complex structure compatible with the degenerate Kähler form $\Re\Omega$ and the degenerate Ricci Flat metric $g'$ is denoted by $J'$. The manifold $(X, g', J')$ constitute the main class of examples we study in this paper.

## 2. Hodge Theory on Degenerate Complex Manifolds

### 2.1. $L^2$ Hodge Theory on manifolds with non-isolated conical singularities following Cheeger-Dai

The study of $L^2$ cohomology of spaces with non-isolated conical singularities has been carried out by Cheeger and Dai in [6].

### 2.2. Definition of Degenerate Complex Manifolds

The following definition is a generalization of the degenerate complex manifolds associated with the structures $J'$ in the twistor sphere $\mathbb{S}$ described in section 2. All the theorems in this section, including $L^2$ Hodge decomposition, fundamental inequality and Sobolev embedding theorems are proved for the class of singular complex manifolds introduced in the definition (2.1). Although the weak and strong Hodge theory discussed in section 3 and afterwards are restricted to the above mentioned case.

**Definition 2.1.** Let $X$ be a smooth real manifold of dimension 2n and let $D \subset X$ be a real smooth submanifold of real dimension k. Assume that $X \setminus D$ is endowed with a Kähler structure $(w', g', J')$ with the following assymptotic behavior near $D$:
1) The Kähler metric itself and its Ricci curvature tensor both are bounded with respect to the sup norm $|.|_{sup}$ induced by the degenerate metric $g'$ in $X \setminus D$

2) Around each point $p \in D$ there exists an open set $U$ and a local coordinates system $\{(x_1, \dots x_{2n}) | x_i \in \mathbb{R}, 1 \leq i \leq 2n\}$ defined on $U$ such that $D \cap U = \pi^{-1}(0)$ and $det(g') = |r|^s + o(r^s)$. Where $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-k}$ denotes projection on the last $2n-k$ coordinates, $r = \sqrt{\sum_{i=2n-k+1}^{2n} x_i^2}$, and $g'$ is the singular Kähler metric.

3) If $\mathcal{S}$ denotes the sphere bundle of the normal bundle $N$ of $D$ in $X$ then there exists an $\epsilon > 0$ and a smooth application $T : \mathcal{S} \times [0, \epsilon) \to U_\epsilon$ onto an open neighborhood $U_\epsilon$ of $D$ in $X$ such that $T|_{\mathcal{S}\times,(0,\epsilon)}$ defines a diffeomorphism onto $U_\epsilon \setminus D$, $T|_{D\times\{0\}} : \mathcal{S} \times \{0\} \to D$ sends $(x, 0)$ to $p(x)$ where $p : N \to D$ is the bundle projection. And such that $T_*(\frac{\partial}{\partial r}) \perp D_r$ where for $0 < r < \epsilon$, $D_r = T(\mathcal{S} \times \{r\})$ and $\frac{\partial}{\partial r}$ is the vector field on $S \times (0, \epsilon)$ tangent to the direction of the interval $(0, \epsilon)$ at each point.
We also assume that if a neighborhood of a point $p \in D$ is endowed with a coordinates system such that $r$ is one of its coordinates then $\|dr\|_{g'} = r^t + o(r^t)$ for some $t > 0$.

4) The parameters $s$ and $t$, defined respectively in parts 2 and 4, satisfy

$$s > 2t - 2n + k + 1.$$

The assumption (3) about $\|dr\|_{g'}$ implies that the total volume of the submanifolds $D_r$ is vanishing at least of order $O(r^{2n-k+t})$ around $D$ since $\|J'dr\|_{g'} = \|dr\|_{g'}$ and this covector lives in the cotangent bundle of $D_r$.
It is clear that for the degenerate complex manifolds obtained through degenerate Monge Ampère equation we have $s = 2$ and $t = 1$ and it satisfies the conditions of the above definition.

### 2.3. Weighted $L^2$ Spaces and $L^2$ Hodge Decomposition

Now for a singular Kähler manifold $X$ as in definition (2.1) we denote by $W_l'^{p,q}(X)$ the Sobolev space of $(p, q)$-forms on $X \setminus D$ with respect to the degenerate metric $g'$

$$W_l'^{p,q}(X) := \mathcal{H}_l(X \setminus D, \Lambda^{p,q}(X \setminus D)),$$

where the sobolev $l$-norm $\|.\|'_{2,l}$ is defined by

$$\|\phi\|'_{2,l} = \|\phi\|'_2 + \|\nabla_{g'}\phi\|'_2 + \dots + \|\nabla^l_{g'}\phi\|'_2.$$

Here $\nabla_{g'}$ is the Levi-Civita connection associated to $g'$ on $X \setminus D$, and $\mathcal{H}_l(M,E)$ denotes the Sobolev space of order $l$ of sections of a vector bundle $E$ over a manifold $M$.

We represent by $L_2'(X)$, the space of functions in $X \setminus D$ which are $L^2$ integrable with respect to the volume form $d\Omega_{g'}$ induced by the metric $g'$ and let $L_2'^{p,q}$ (resp. $L_2'^{m}$) be the space of $(p,q)$-forms (resp. $m$-forms) $\alpha$ on $X \setminus D$ s.t. $\int \alpha \wedge *\alpha < \infty$ .

Also $W_1'(X)$ denotes the space of functions $f \in L_2'$ s.t. $df \in L_2'^1$.

**Lemma 2.2.** *If $A_0^{p,q}(X \setminus D)$ denotes the space of smooth $(p,q)$-forms with compact suppport in $X \setminus D$ then $A_0^{p,q}(X \setminus D)$ is dense in $W_l'^{p,q}(X)$.*

**Lemma 2.3.** *If $f \in L'^2(X)$ is such that $lim_{r\to 0}\int_{D_r} f d\Omega'(r)$ exists, then this limit must be zero:*

$$lim_{r\to 0}\int_{D_r} f d\Omega'(r) = 0$$

*Here $d\Omega'(r)$ denotes the volume form induced by the metric $g'$ on $D_r$ .*

**Corollary 2.4.** *i) If $\eta \in W_1'^{2n-1}$, then $\int_X d\eta = 0$.*
*ii )If $\eta \in A^{2n-1}(X \setminus D) \cap L_2'^{2n-1}$ is such that $d\eta \in L'^2$, then $\int_X d\eta = 0$*
*iii) If $\alpha \in W_1'^{p}$, $\beta \in W_1'^{q}$ and $p+q = 2n-1$, then*

$$\int_X d\alpha \wedge \beta = (-1)^p \int_X \alpha \wedge d\beta$$

Clearly $W_1'^{p,q} \subset L_2'^{p,q} = W_0'^{p,q}$, and the following $L^2$-decomposition holds:

**Theorem 2.5.**

$$L_2'^{p,q} = \mathcal{H}_2^{p,q} \oplus \overline{\bar\partial_{J'}(A^{p,q}(X \setminus D) \cap L_2'^{p,q})} \oplus \overline{\bar\partial_{J'}^{*}(A^{p,q}(X \setminus D) \cap L_2'^{p,q})} = V_0 \oplus V_1 \oplus V_2,$$

*where the space $\mathcal{H}_2^{p,q}$ is defined by $\mathcal{H}_2^{p,q} := \{\alpha \in L_2'^{p,q} \mid \bar\partial_{J'}\alpha = \bar\partial_{J'}^{*}\alpha = 0\}$.*

According to lemma (2.2), we also have

**Lemma 2.6.**

$$V_1 = \overline{\bar\partial_{J'} A_0^{p,q}(X \setminus D)}, \qquad V_2 = \overline{\bar\partial_{J'}^{*} A_0^{p,q}(X \setminus D)},$$

*where $A_0^{p,q}(X \setminus D)$ denotes the space of smooth $(p,q)$ forms with compact support on $X \setminus D$.*

## 3. Weak Hodge Theorem

We set

$$C^{p,q}(X) := A^{p,q}(X \setminus D) \cap L_2'^{p,q}(X), \tag{3}$$

and we introduce the following $\bar\partial_{J'}$ -complex of differential forms:

$$\ldots \to C^{p,q-1} \xrightarrow{\bar\partial_{J'}} C^{p,q} \xrightarrow{\bar\partial_{J'}} C^{p,q+1} \to \ldots \tag{4}$$

Obviously the domain of definition of $\bar\partial_{J'}$ in the above complex is strictly smaller than $C^{*,*}$.
Then we have the following theorem

**Theorem 3.1.** *For every $\bar\partial_{J'}$-closed $(p,q)$-form $\alpha \in C^{p,q}(X)$, there exists a unique harmonique form $\alpha_0 \in \mathcal{H}_2^{p,q}$ and $\beta \in C^{p,q-1}(X)$ such that, $\alpha = \alpha_0 + \bar\partial_{J'}\beta$. Moreover if $\alpha \in W_1'^{p,q}$, then $\beta$ can be chosen to belong in $W_1'^{p,q-1}$ as well.*

*Proof:* If $\alpha \in C^{p,q}(X)$ belongs also to $W_1'^{p,q}$ the proof is complete by our previous discussion. Otherwise by the solvability of $\bar{\partial}_{J'}$-equation in a neighborhood of $D$ we can find $\gamma \in C^{p,q-1}$ such that $\alpha - \bar{\partial}_{J'}\gamma \in W_1'^{p,q}$ and we can repeat our previous argument.

The cohomology group associated with the above complex is denoted by $H_{(2)}^{p,q}(X)$. From the above theorem it follows that $H_{(2)}^{p,q}(X) \simeq \mathcal{H}_2^{p,q}$.
As in ([6]) we can also define $H_{(2),\#}^{p,q}$ the cohomology associated with the closure $\bar{\partial}_{J'}^{c}$ of the operator $\bar{\partial}_{J'}$ . From theorem (2.5) and theorem (3.1) it follows that

**Corollary 3.2.** *We have the isomorphism*

$$H_{(2)}^{p,q}(X) \simeq H_{(2),\#}^{p,q}(X)$$

*where $H_{(2),\#}^{p,q}$ denotes the cohomology associated with the closure $\bar{\partial}_{J'}^{c}$ of the operator $\bar{\partial}_{J'}$*

**Corollary 3.3.** *If $\alpha \in W_1'^{p,q}$ then in the Hodge decomposition $\alpha = \alpha^h + \alpha^1 + \alpha^2$ where $\alpha^h \in \mathcal{H}_2^{p,q}$ and $\alpha^i \in V_i$, for $i = 1, 2$, we have in fact $\alpha^1 \in W_1'^{p,q-1}$ and $\alpha^1 \in W_1'^{p,q+1}$*

*Proof.* As in the prof in section 4.4.1, we find a sequence $\alpha_0 + \bar{\partial}_{J'}\theta_n + \bar{\partial}_{J'}^{*}\eta_n$ such that $\bar{\partial}_{J'}^{*}\theta_n = \bar{\partial}_{J'}^{*}\eta_n = 0$ ans which converges in $L_2'$ to $\alpha$. Then $\Delta_{g'}\theta_n \to \bar{\partial}_{J'}^{*}\alpha$ and $\Delta_{g'}\eta_n \to \bar{\partial}_{J'}\alpha$ in the weak sense in $L_2'^{p,q}$. The rest of argument goes as before. □

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# QSPR analysis of drugs for anti-diabetes using degree based Entropy topological indices

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**Abstract**

Topological entropy is a quantitative measure used to characterize the complexity and disorder of a molecular graph. It is derived from the degree of vertices within the graph, which represents the connectivity of atoms in a molecular structure. The index reflects the number of distinct configurations or arrangements that can be formed by the molecular graph, indicating its potential for structural diversity. In this study, we calculated the entropy indices associated with six diabetes medications. Subsequently, we employed SPSS software to analyze the correlation between these indices and the physicochemical properties of the drugs. Additionally, a linear regression analysis was conducted to assess the relationships identified. The findings indicate that the identification of optimal regression models based on these indices enhances the predictive capability for the physicochemical properties of diabetes medications, including Boiling Point (BP), Enthalpy of Vaporization (EV), Flash Point (FP), Molar Refractivity (MR), Polar Surface Area (PSA), Polarizability (PO), Molar Volume (MV) and Index of Refraction (IR) for diadetes medications.

## 1. Introduction

Diabetes is a metabolic disorder characterized by the pancreas's inability to produce adequate insulin or by the body's ineffective utilization of insulin. The history of diabetes spans millennia, with early references dating back to ancient Egyptian papyri and classical Greek texts. The term "diabetes" is derived from the Greek word "siphon", which describes the frequent urination observed in affected individuals. In the early 19th century, the full term "Diabetes Mellitus" was adopted, with "mellitus"-Latin for "honey-sweet"-reflecting the sweet-smelling urine characteristic of the disease[1, 2].
This chronic condition leads to elevated blood glucose levels, commonly referred to as "blood sugar." Glucose is derived from the food we consume and serves as an energy source in our bodies. The pancreas produces However, effective management strategies can greatly improve the standard of living for individuals impacted by the illness. Pharmacological treatments are employed to control this serious condition, and numerous clinical trials are being conducted to combat its complications. Timely diagnosis, screening, and appropriate medication are essential for

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helping patients manage this potentially life-threatening disease in the long term. Among the essential medications for diabetes management are Sitagliptin, Tolazamide, Metformin, Miglitol, Chlorpropamide, Alogliptin[3, 4]. Figure 1 depicts the chemical structure of the drugs.

Graph theory in chemistry represents a crucial domain within computational chemistry and drug design, where degree based Topological descriptors hold great importance[5–7]. While Topological Indices (TIs) find applications across mathematics, bioinformatics, informatics, biology, and other fields, their most significant application to date remains in nonempirical QSPR[8, 9].

In this study, we construct the molecular graph of diabetes medications based on the molecular structure sourced from ChemSpider, ensuring that it is free from loops and redundant edges. Let V(H) and W(H) represent the collections of vertices and edges of a graph H, correspondingly. The degree of a node $r \in V(H)$, represented as $\eta(r)$ (we know that the degree of each vertex represents a Count of incident edges to that vertex).

Subsequently, eight TIs were calculated, which include: First Zagreb Index, Second Zagreb Index, Modified Second Zagreb, Symmetric Division Index, Harmonic Index, Inverse Sum Index, Inverse Randić Index, Ordinary Geometric Arithmetic Index. Subsequently, the entropy related to each of the indices has been calculated[10].

The characteristics estimated through this approach demonstrate a robust relationship with the properties of diabetes therapeutics, as established through linear regression analysis. It has been observed that a significant association exists between the properties of these compounds and their corresponding Entropy Topological Indices (ENT_TI).

## 2. Material and Method

In the molecular configurations of medications, atoms are modeled as vertices, while the bonds that connect these atoms are represented as edges. The graph H = (V, W) is defined as a simple, finite, and connected graph, where V and W denote the collections of nodes and edges, correspondingly, within the chemical graph. The formula for calculating the TIs is given in Table 1. Due to its extensive applicability is given in various articles. For further reading, refer to references [11–15].

Table 1. Exploration of TIs and Their Formulation [16–20]

| TI | $FormulaH(\eta(r),\eta(s))$ |
|---|---|
| First Zagreb Index: $Z_1$ | $\sum_{rs\in W(H)}(\eta(r)+\eta(s))$ |
| Second Zagreb Index: $Z_2$ | $\sum_{rs\in W(H)}\eta(r)+\eta(s)$ |
| Modified Second Zagreb Index: mZ | $\sum_{rs\in W(H)}\frac{1}{\eta(r)\eta(s)}$ |
| Symmetric Division Index: SDD | $\sum_{rs\in W(H)}\frac{(\eta^2(r)+\eta^2(s))}{\eta(r)\eta(s)}$ |
| Harmonic Index: H | $\sum_{rs\in W(H)}\frac{2}{\eta(r)+\eta(s)}$ |
| Inverse Sum Index: ISI | $\sum_{rs\in W(H)}\frac{\eta(r)\eta(s)}{\eta(r)+\eta(s)}$ |
| Inverse Randić Index: IR | $\sum_{rs\in W(H)}\frac{1}{\sqrt{\eta(r)\eta(s)}}$ |
| Ordinary Geometric Arithmetic Index: OGA | $\sum_{rs\in W(H)}\frac{2\sqrt{\eta(r)\eta(s)}}{(\eta(r)+\eta(s))}$ |

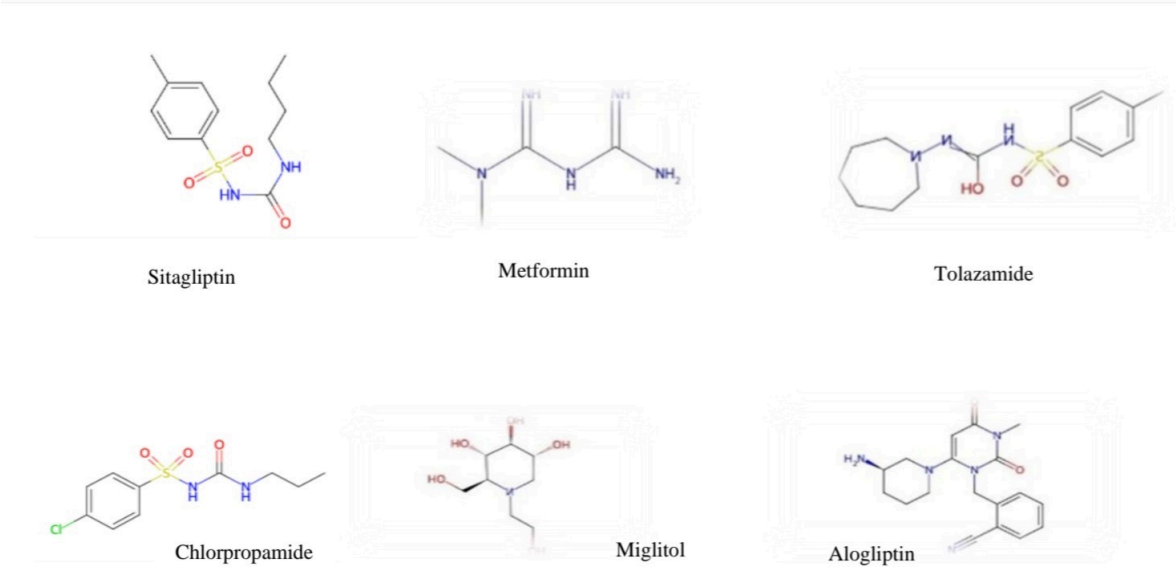


Fig. 1. Molecular structure of Sitagliptin, Tolazamide, Metformin, Miglitol, Alogliptin, Chlorpropamide

The concept of graph entropy involves assigning a probability function to the edges of a chemical graph based on the topological descriptor. In this context, let $\Upsilon(H)$ represent the topological descriptor of a chemical graph H, and let $\psi$ denote a functional characterization of this topological descriptor.

$$\Upsilon(H) = \sum_{rs \in w} \psi(rs)$$

The graph entropy is denoted by ENT $\Upsilon(H)$ and is defined as,

$$ENT_{\Upsilon}(H) = log(\Upsilon(H)) - \frac{1}{\Upsilon(H)} log\left[\prod_{rs \in W(H)} \psi(rs)^{\psi(rs)}\right] \quad (1)$$

If we consider the First Zagreb index, then the functional $\psi(rs) = \eta(r) + \eta(s)$ and $\Upsilon(H) = Z_1(H)$. Thus Equation (1) reduces to the following expression, which is called as the First Zagreb Entropy.

$$ENT_{Z_1}(H) = log(Z_1(H)) - \frac{1}{Z_1(H)} log\left[\prod_{rs \in W(H)} (\eta(r) + \eta(s))^{(\eta(r)+\eta(s))}\right] \quad (2)$$

In a similar manner, we can express the formulas for the remaining degree-based graph entropies.

## 3. Entropy Topological Indices and QSPR

Based on the aforementioned information, eight Entropy TIs ($ENT\_TI$) were analyzed for sex diabetic drugs, with the results presented in Tables 3. The modeling of eight physical features, including BP, EV, FP, MR, PSA, PO, MV, and IR, was conducted for the drugs Sitagliptin, Chlorpropamide, Tolazamide, Metformin, Alogliptin, and Miglitol, extracted from the ChemSpider database and presented in Table 2. Furthermore, the normality of these data was evaluated, and the correlations among them are detailed in Table 4.

Table 2. Physical features of diabetes drugs

| NAME | BP | EV | FP | MR | PSA | PO | MV | IR |
|---|---|---|---|---|---|---|---|---|
| Sitagliptin | 529.9 | 80.5 | 274.3 | 85.2 | 77 | 33.8 | 252.4 | 1.59 |
| Metformin | 172.5 | 40.9 | 58.1 | 33.4 | 89 | 13.2 | 100.8 | 1.576 |
| Miglitol | 453.7 | 82.3 | 284.3 | 48.5 | 104 | 19.2 | 142.1 | 1.598 |
| Chlorpropamide | - | - | - | 66.3 | 84 | 26.3 | 207.4 | 1.553 |
| Alogliptin | 519.2 | 79.2 | 267.8 | 93.3 | 94 | 37 | 252.9 | 1.66 |
| Tolazamide | 484.5 | 79 | 246.8 | 82.5 | 90 | 32.7 | 237.9 | 1.61 |

Subsequently, a linear regression model was employed to evaluate and analyze the data.
The physical characteristics of pharmaceutical agents employed in the treatment of diabetes are detailed in Table 2.

The calculation results of section 2 are shown in Table 3.

Table 3. The computed values of Entropy TIs for the pharmaceutical agents

| NAME | ENT_Z1 | ENT_Z2 | ENT_mZ | ENT_SDD | ENT_H | ENT_ISI | ENT_IR | ENT_OGA |
|---|---|---|---|---|---|---|---|---|
| Sitagliptin | 1.4725556 | 1.4474796 | 1.4464818 | 1.4606408 | 1.4726557 | 1.4647546 | 1.4693708 | 1.476007 |
| Metformin | 0.8979745 | 0.8595576 | 0.874275 | 0.8933885 | 0.8988284 | 0.885537 | 0.8947195 | 0.902201 |
| Miglitol | 1.1354422 | 1.095129 | 1.0833051 | 1.1369332 | 1.1336931 | 1.1271989 | 1.1300981 | 1.145498 |
| Chlorpropamide | 1.2227875 | 1.1933171 | 1.1954085 | 1.2139172 | 1.222681 | 1.2179737 | 1.2216041 | 1.229332 |
| Alogliptin | 1.4095424 | 1.3851582 | 1.3831798 | 1.4057595 | 1.4096793 | 1.4040256 | 1.4068878 | 1.414381 |
| Tolazamide | 1.3372054 | 1.3167688 | 1.3230059 | 1.3272875 | 1.3378794 | 1.3341404 | 1.3372287 | 1.341476 |

### *3.1. Regression models*

In this section, the analysis is performed utilizing the Entropy TIs calculated in Section 2. The linear regression model is articulated as follows:

$$P = B + A(ENT\_TI) \tag{3}$$

Where P represents the features of the diabetes medication, B is a constant, A is the regression coefficient. This analysis was conducted using SPSS software, incorporating eight specific characteristics and eight $ENT_{TI}$ for a total of sex diabetes medications.
Utilizing equation (3), various linear regression models for Entropy TIs are presented as follows:
1. $ENT_{Z1}$
$BP = -303.354 + 587.995[ENT\_Z1], MR = -65.252 + 107.111[ENT\_Z1],$
$BP = -303.354 + 587.995[ENT\_Z1], MR = -65.252 + 107.111[ENT\_Z1].$
2. $ENT_{Z2}$
$BP = -260.027 + 566.822[ENT\_Z2], MR = -58.598 + 104.254[ENT\_Z2],$
$PO = -23.361 + 41.435[ENT\_Z2], MV = -148.997 + 286.058[ENT\_Z2].$
3. $ENT_{mZ}$
$BP = -260.985 + 567.035[ENT\_mZ], MR = -61.373 + 106.416[ENT\_mZ],$
$PO = -24.464 + 42.294[ENT\_mZ], MV = -158.846 + 292.181[ENT\_mZ].$
4. $ENT_{SDD}$
$BP = -313.138 + 598.568[ENT\_SDD], MR = -66.017 + 108.270[ENT\_SDD],$
$PO = -26.309 + 43.030[ENT\_SDD], MV = -168.537 + 296.416[ENT\_SDD].$
5. $ENT_{H}$
$BP = -302.171 + 587.048[ENT\_H], MR = -65.353 + 107.194[ENT\_H],$
$PO = -26.046 + 42.603[ENT\_H], MV = -167.498 + 294.096[ENT\_H].$
6. $ENT_{ISI}$
$BP = -291.915 + 582.300[ENT\_ISI], MR = -63.237 + 106.088[ENT\_ISI],$
$PO = -25.205 + 42.164[ENT\_ISI], MV = -161.800 + 291.150[ENT\_ISI].$

7. $ENT_{IR}$
$BP = -298.439 + 585.414[ENT\_IR], MR = -64.809 + 106.979[ENT\_IR],$
$PO = -25.830 + 42.518[ENT\_IR], MV = -166.149 + 293.622[ENT\_IR].$
8. $ENT_{OGA}$
$BP = -312.049 + 592.405[ENT\_OGA], MR = -66.106 + 107.318[ENT\_OGA],$
$PO = -26.346 + 42.653[ENT\_OGA], MV = -169.574 + 294.443[ENT\_OGA].$

Table 4. The computed values of correlation coefficients of physical features for the pharmaceutical agents

| Entropy TIs | BP | EV | FP | MR | PSA | PO | MV | IR |
|---|---|---|---|---|---|---|---|---|
| ENT_Z1 | 0.93 | 0.811 | 0.805 | 0.963 | 0.35 | 0.963 | 0.97 | 0.477 |
| ENT_Z2 | 0.925 | 0.804 | 0.796 | 0.967 | 0.357 | 0.967 | 0.974 | 0.477 |
| ENT_mZ | 0.911 | 0.783 | 0.773 | **0.972** | **0.382** | **0.972** | **0.98** | 0.473 |
| ENT_SDD | **0.936** | **0.819** | **0.814** | 0.962 | 0.332 | 0.962 | 0.967 | **0.489** |
| ENT_H | 0.929 | 0.809 | 0.802 | 0.964 | 0.352 | 0.964 | 0.971 | 0.477 |
| ENT_ISI | 0.932 | 0.813 | 0.806 | 0.964 | 0.347 | 0.964 | 0.971 | 0.476 |
| ENT_RI | 0.929 | 0.809 | 0.802 | 0.964 | 0.353 | 0.964 | 0.972 | 0.475 |
| ENT_OG | 0.934 | 0.817 | 0.811 | 0.961 | 0.342 | 0.962 | 0.968 | 0.477 |

Bold numbers indicate the strongest correlations

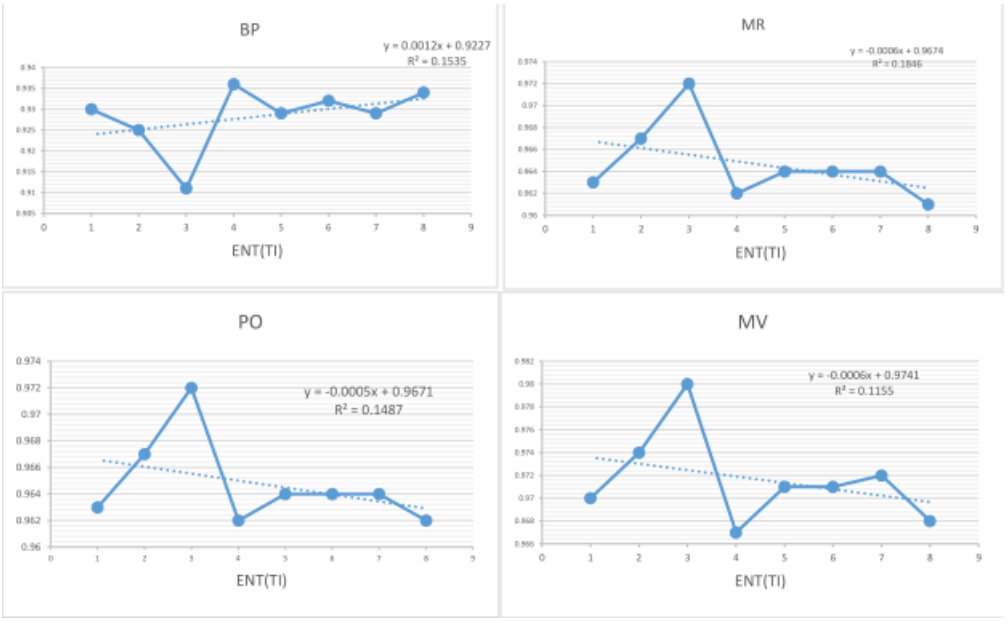


Fig. 2. The chart illustrates some of the strongest correlations between entropy indices and physicochemical properties

Table 5. Calculation of statistical metrics for the linear QSPR about ENT_mZ

| Physical features | N | A | B | R | $R^2$ | SE | F | sig | Indicant |
|---|---|---|---|---|---|---|---|---|---|
| BP | 6 | 567.035 | -260.985 | .911 | .831 | 70.3749 | 14.719 | .031 | significant |
| MR | 6 | 106.416 | -61.373 | .972 | .944 | 6.1654 | 67.719 | .001 | significant |
| PO | 6 | 42.294 | -24.464 | .972 | .945 | 2.4436 | 68.094 | .001 | significant |
| MV | 6 | 292.181 | -158.846 | **.980** | **.960** | 14.2792 | 95.175 | .001 | significant |

Bold numbers indicate the strongest correlations.

## 4. Conclusion

Table 4 and Figure 2 illustrate the correlation between the physical-chemical features of diabetes medications and the defined topological indices in this study, the majority of the indicators showed a strong and positive correlation. The $ENT_{mZ}$ showed a positive and excellent correlation with MR, MV and PO with "r=0.972", "r=0.980" and "r=0.972" values. The $ENT_{SDD}$ showed a positive and excellent correlation with BP with "r=0.936" value. The BP property provided a high positive correlation with the $ENT_{Z1}$, $ENT_{Z2}$, $ENT_{mZ}$, $ENT_{SDD}$, $ENT_{H}$, $ENT_{ISI}$, $ENT_{IR}$ and $ENT_{OGA}$ with "r=0.930", "r=0.925", "r=0.911", "r=0.936", "r=0.929", "r=0.932", "r=0.929" and "r=0.934" values. Additionally, the properties of MR, PO, and MV exhibit strong correlations with all eight indices. In contrast, the PSA and IR indices do not show good correlations with any of the indices. Table 5 presents the regression model related to the $ENT_{mZ}$ (we have omitted similar tables for other indices here) for various physical and chemical features. The results show that the regression values (r) exceed 0.5, while the p-values are less than 0.05, indicating reliable predictors for linear regression. Equations are derived based on criteria such as minimum standard error (Std. E), maximum coefficient of determination (R square), and maximum F-statistic (F). Therefore, it can be concluded that all physical and chemical features are highly significant. This underscores the potential significance of these $ENT_{TI}$ in the analysis of QSPR for diabetes drugs, as evidenced by the plotted regression lines (thirty two lines). The results of this study may be employed to inform the production, advancement, and enhancement of more effective diabetes therapeutics. Furthermore, the methodology employed in this study can be extended to explore the structural characteristics of other pharmaceutical compounds.

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# The Number of Spanning Trees of Some Graphs

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**Abstract**

For any connected graph, a spanning tree is a subgraph that includes all the vertices and is a single connected acyclic structure. By the Matrix-Tree Theorem, finding the number of spanning tree in a graph has been previously discussed. In this paper, The number of spanning tree of some graphs has been presented, then we show a exact formula for the number of spanning tree in a these graphs.

## 1. Introduction

Let $\Gamma$ be a simple graph on a finite vertex set $V$ and edge set $E$ and let $A$ be its adjacency matrix. The number of spanning trees in a graph is determined by a fundamental concept in graph theory. It is well known [3] that the number of labelled spanning trees of the complete bipartite graph on $m$ and $n$ vertices is equal to $m^{n-1}n^{m-1}$. In [2] author give a different proof of this fact and prove a formula for the number of labelled spanning trees of the complete graph on n vertices. Also, for computing the number of spanning tree in graphs, there is a technique using adjacency and Laplacian matrix of graph.

Spanning trees have a variety of practical applications in the real world. For examples: Network Design and Optimization, Circuit Design, Transportation and Logistics, Biology and Ecology, Data Structures and Algorithms, Clustering and Data Mining, Image Processing and Computer Vision. Spanning trees are used to design efficient communication networks to minimize the cost of laying cables while ensuring all nodes are connected. Also, Spanning trees help in ensuring there are no loops in the network, which can cause data packets to circulate indefinitely. Spanning trees play a crucial role in ensuring efficiency and connectivity in various systems and applications. For more information see [5], [4], [1].

For a given graph $G$ with $n$ vertices, the number of spanning trees $T(G)$ can be found using Matrix-Tree Theorem. This theorem involves the use of the Laplacian matrix of the graph. There are the steps to determine the number of spanning trees.

In next section, we see how to calculate the number of spanning trees in graphs. In section 3, we provide an exact formula for calculating the number of spanning trees in some graphs.

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## 2. Laplacian matrix and number of spanning tree

For each edge $e_j = \{v_i, v_k\}$, choose one of $v_i, v_k$ to be the positive end of $e_j$ and the other to be the negative end. Thus $G$ is given an orientation. The vertex-edge incidence matrix (or cross-linking matrix) afforded by an orientation of $G$ is the $n$-by-$m$ matrix $Q = Q(G) = (q_{ij})$, where $q_{ij} = +1$ if $v_i$ is the positive end of $e_j$, $-1$ if it is the negative end, and 0 otherwise. It turns out that the Laplacian matrix, $L(G) = QQ^t$, is independent of the orientation. In fact,$L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the $(0,1)$ adjacency matrix.

One may also describe $L(G)$ by means of its quadratic form $xL(G)x^t = \sum(x_i - x_j)^2$, where $x = (x_1, x_2, \dots, x_n)$, and the sum is over the pairs $i < j$ for which $\{v_i, v_j\} \in E$. So $L(G)$ is a symmetric, positive semidefinite. Also, we have $detL(G) = 0$. Let $L^-$ denote the matrix obtained from the Laplacian matrix by removing the last row and the last column. We have the Matrix -tree theorem: the number of spanning trees of a connected graph equals $det(L^-)$. Let us express the determinant of $L^-$ as the sum of products of its elements, representing each diagonal element as a sum of units or minus units, and expanding brackets. So, here we have steps to determine the number of spanning trees:

1. Construct the Laplacian Matrix $L$: The Laplacian matrix is defined as $L = D - A$, where $D$ is the degree matrix (a diagonal matrix where each diagonal entry is the degree of the corresponding vertex), and $A$ is the adjacency matrix (a matrix where each entry $A_{ij}$ is 1 if there is an edge between vertices $i$ and $j$), and 0 otherwise).
2. Delete any Row and the Corresponding Column from $L$: Let the resulting matrix be $L^-$.
3. Calculate the Determinant of $L^-$: The determinant of $L^-$ gives the number of spanning trees in the graph $G$.

Apply the Matrix-tree theorem and calculate the determinant of matrix $L^-$. For a complete graph it is an $(n-1)\times(n-1)$ matrix of the form:

$$det\begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{bmatrix} = det\begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{bmatrix} = det\begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & n \end{bmatrix} = n^{n-2} \tag{1}$$

Another example of apply the Matrix-tree theorem and calculate the determinant of matrix $L^-$ is for a complete bipartite graph$K_{k,l}$. The Laplacian matrix here looks as:

$$L = \begin{bmatrix} k & \dots & 0 & -1 & \dots & -1 \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & k & -1 & \dots & -1 \\ -1 & \dots & -1 & l & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ -1 & \dots & -1 & 0 & \dots & l \end{bmatrix} \tag{2}$$

We will find the determinant of matrix $M = L_{l,l+1}$:

$$M = \begin{bmatrix} k & \dots & 0 & 0 & -1 & \dots & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & k & 0 & -1 & \dots & -1 \\ -1 & \dots & -1 & -1 & 0 & \dots & 0 \\ -1 & \dots & -1 & -1 & l & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & \dots & -1 & -1 & 0 & \dots & l \end{bmatrix} \tag{3}$$

We have $det(L^-) = -detM$. For calculations of the determinant of matrix M. It is not difficult to check that the sum of weights of paths from $i$-th inlet to $j$-th outlet equals $m_{ij}$. Therefore, $detM$ equals the sum of weights of the sets consisting of $n$ non-intersecting paths. It is easy to understand that there exists exactly one such path, and its weight equals $-k^{l-1}l^{k-1}$, which finishes the calculation.

## 3. Number of spanning tree in another graph

In this section, we compute number of spanning tree in some graph. Consider cycle graph $C_n$. Matrix $L^-$ of order $(n-1)\times(n-1)$ of this graph is:

$$L=\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \ddots & 0 \\ \vdots & 0 & -1 & 2 & -1 & 0 \\ \vdots & \ddots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{bmatrix} \tag{4}$$

Because the $L^-$ matrix is tri-diagonal, therefore determinant of this matrix is $n$. This shows that the number of spanning tree of $C_n$ is $n$.

## 4. Conclusion

From this discussion, By the Matrix- tree theorem and structure of $L^-$, we can find the number of spanning tree in some graphs. Also, We can test upper bounds on the number of spanning trees in a graph. Open problem in this category, is finding exact formula for the number of spanning tree in terms of the number of vertices, the number of edges and the vertex degrees.

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**Note:** We have made corrections to the results presented in papers [2] and [10].

# Investigation of Degree-Based Topological Indices in Nano Star and Zinc Porphyrin-Cored Dendrimers

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**Abstract**

Graph theory functions as a robust mathematical framework for modeling complex systems, encompassing the chemical structures of pharmaceutical compounds. This article presents a comprehensive analysis of three molecular graphs: Nano Star Dendrimers $NSC5C6[n]$, $NSD[n]$, and Zinc Porphyrin Dendrimer $DPZ_n$. Furthermore, we investigate various topological indices, including Sombor, a reduced version of Sombor, a multiplicative version of Sombor, Revan, and Gourava indices. The results are derived through combinatorial computations utilizing vertex degree counting and edge partitioning methods. The primary objective of this study is to compare the topological indices of the selected dendrimers. Among the indices evaluated—namely $SO$, $SO_{red}$, $SO_{rev}$, $GO_2$, $R_1(G)$, $R_2(G)$, and $HR_1(G)$—the Nano Star Dendrimer $NSD[n]$ exhibits the highest values, particularly from the third generation onwards, whereas $NSC5C6[n]$ demonstrates the lowest values.

## 1. Introduction

Graph theory, a robust mathematical framework, finds applications in various disciplines, including operations research, chemistry, and genetics. It has developed into a distinct branch of mathematics. The use of graphs that incorporate vertices and edges to depict real-world situations aids in the simplification of intricate concepts.
Dendrimers are synthesized by adding successive layers of branching groups. The structure of dendrimers consists of a core molecule at the center, with branching groups to which other branching molecules are added in layers. Each layer is referred to as a generation and is denoted by $G_n$. Dendrimers are highly branched macromolecules with nanoscale dimensions, making them advantageous as carriers for drugs and genes. Research has demonstrated that some dendrimers exhibit medicinal properties, including antibacterial, antifungal, and cytotoxic effects. Topological indices, also known as molecular descriptors, are numerical quantities associated with a graph that can be used to predict the characteristics of chemical compounds without the need for laboratory experiments. These indices help specify

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features such as melting points, boiling points, and flash points, among others. Hasani et al. (2024) examined the quantitative structure-property relationship (QSPR) of specific drugs used for treating heart diseases [3, 5]. Furthermore, they utilized topological indices and MATLAB programming to predict the physical and chemical characteristics of drugs for Parkinson's disease in the same year [4]. This researchers performed quantitative structure-property relationship (QSPR) analysis on medications for pyelonephritis, employing entropy graphs that are weighted by topological indices, facilitated through MATLAB programming [6].

In this study, we computed the Sombor index, the reduced version of the Sombor index, the multiplicative version of the Sombor index, the Revan index, and the Gourava index of the molecular graphs Nano Star Dendrimer $NSC5C6[n]$, Nano Star Dendrimer $NSD[n]$, and Zinc Porphyrin Dendrimer $DPZ_n$. The results were visualized using Maple software to demonstrate the dependence of some topological indices on the involved parameters. Throughout the study, all graphs were simple and connected. Notations used include the degree of vertex $v$ denoted as $d_v$, the vertex set as $V(G)$, the edge set as $E(G)$, the number of edges in $G$ as $e(G)$, the largest of all degrees of $G$ as $\Delta(G)$, and the smallest of all degrees of $G$ as $\delta(G)$.

The degree-based topological indices are of significant importance. In 2021, Gutman defined the Sombor indices, which consist of five types: the Sombor index, the reduced Sombor index [1], the multiplicative Sombor [8], the reverse Sombor index [11], and the Revan-Sombor index [7].

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

$$SO_{red}(G) = \sum_{uv\in E(G)} \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2}$$

$$\prod_{so}(G) = \prod_{uv\in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$$

$$SO_{rev} = SO_{rev}(G) = \sum_{uv\in E(G)} \sqrt{[\Delta - d_G(u) + 1]^2 + [\Delta - d_G(v) + 1]^2}$$

$$RSO(G) = \sum_{uv\in E(G)} \sqrt{r_G(u)^2 + r_G(v)^2} = \sum_{uv\in E(G)} \sqrt{[\Delta + \delta - d_G(u)]^2 + [\Delta + \delta - d_G(v)]^2}$$

If $\delta = 1$, then the reverse Sombor index coincides with the Revan-Sombor index. In 2017, Kulli [9] defined the first Gourava index based on the definitions of the Zagreb indices and their applications.

$$GO_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v) + d_G(u)d_G(v)].$$

The redefined third Zagreb index was defined as:

$$GO_2(G) = ReZG_3(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)][d_G(u)d_G(v)].$$

In 2017, Kulli [9] presented the equation for the Revan vertex degree u in a graph $G$ as:

$$r_G(u) = \Delta(G) + \delta(G) - d_G(u).$$

The first and second Revan indices were defined as follows:

$$R_1(G) = \sum_{uv \in E(G)} [r_G(u) + r_G(v)]$$

$$R_2(G) = \sum_{uv \in E(G)} r_G(u) r_G(v)$$

In 2018, Kulli [9] introduced the first and second Hyper Revan indices.

$$HR_1(G) = \sum_{uv \in E(G)} [r_G(u) + r_G(v)]^2$$

$$HR_2(G) = \sum_{uv \in E(G)} [r_G(u) r_G(v)]^2$$

In this study, we analyze the Sombor index and its variations, including the reduced version, multiplicative version, reverse version, Revan-Sombor index, Gourava index,and Revan index for the Nano star dendrimers $NSC5C6[n]$ and $NSD[n]$, as well as the Zinc porphyrin dendrimer $DPZ_n$. Additionally, we consider $E_{ij} = \{uv \in E(P) \mid d_u = i, d_v = j\}$.

## Main results

### 2. Nano star Dendrimer NSC5C6[n]

The first molecular graph is the Nano star dendrimer $NSC5C6[n]$ of generation $G_n$ with $n \geqslant 1$ growth stage (see Figure 1). The central core of $NSC5C6[n]$ consists of 10 vertices and 10 edges, and the first generation ($G_1$) consists of 28 vertices and 30 edges. Therefore, in $NSC5C6[n]$, there are a total of $9 \times 2^{n+2} - 44$ vertices and $10 \times 2^{n+2} - 50$ edges [10]. The edges set of $NSC5C6[n]$ can be observed in Figure 1, which can be divided into seven classes, and their cardinalities are computed in Table 1.

(**Note:** The number of edges $E_{(2,3)}$ and $E_{(3,3)}$ in [10] was miscalculated. The correct numbers have been updated in Table 1.)

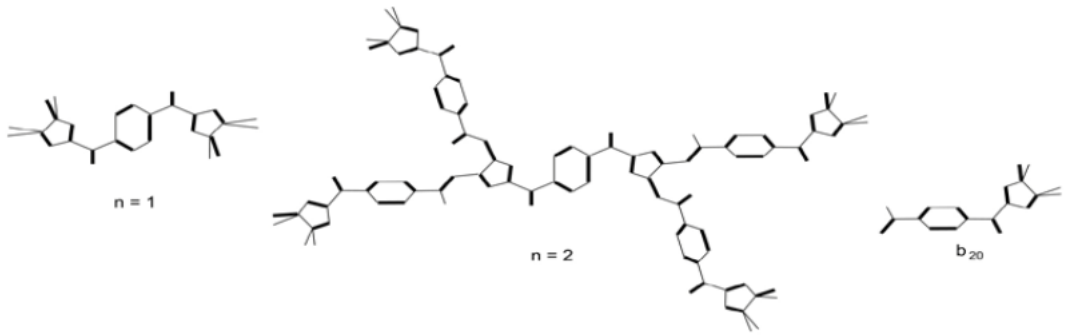


Fig. 1. The molecular graph of Nano star dendrimer $NSC5C6[n]$

**Theorem 2.1.** *The* $SO$, $SO_{red}$, $\prod_{so}$, $SO_{rev}$, *and,* $RSO$ *indices for* $NSC5C6[n]$ *are as follows :*
**(i)** $SO(G) = (33\sqrt{2} + 4\sqrt{10} + 4\sqrt{17} + 18\sqrt{13} + 4\sqrt{5})2^n - 42\sqrt{2} - 6\sqrt{10} - 28\sqrt{13}$.

Table 1. The edges partition of $NSC5C6[n]$.

| $\mathbf{E_{(u,v)}}$ | **Number of edges** $(\mathbf{d_u}, \mathbf{d_v})$ |
|---|---|
| $E_{(1,3)}$ | $2^{n+2}-6$ |
| $E_{(1,4)}$ | $2^{n+2}$ |
| $E_{(2,2)}$ | $2^{n+2}-6$ |
| $E_{(2,3)}$ | $9\times 2^{n+1}-28$ |
| $E_{(2,4)}$ | $2^{n+1}$ |
| $E_{(3,3)}$ | $7\times 2^{n}-10$ |
| $E_{(4,4)}$ | $2^{n}$ |

**(ii)** $SO_{red}(G)=(20+24\sqrt{2}+36\sqrt{5}+2\sqrt{10})2^n-26\sqrt{2}-28\sqrt{5}-12.$

**(iii)** $\prod_{so}(G)=15360\sqrt{221}(2^{7n+10}-377\times 2^{6n+4}+3383\times 2^{5n+2}$
$-843\times 2^{4n+4}+1040\times 2^{3n}).$

**(iv)** $SO_{rev}(G)=(2\sqrt{5}+\sqrt{17}+3\sqrt{2})2^{n+2}+(9\sqrt{13}+\sqrt{10})2^{n+1}$
$+(15\sqrt{2})2^n-(12\sqrt{5}+28\sqrt{13}+38\sqrt{2}).$

**(v)** $RSO(G)=(2\sqrt{5}+\sqrt{17}+3\sqrt{2})2^{n+2}+(9\sqrt{13}+\sqrt{10})2^{n+1}$
$+(15\sqrt{2})2^n-(28\sqrt{13}+38\sqrt{2}+12\sqrt{5}).$

*Proof.* From the edge partition of $NSC5C6[n]$ provided in Table 1, we can observe...

**(i)** $SO(NSC5C6[n])=\sum_{uv}\sqrt{d_u^2+d_v^2}$
$=\sqrt{10}\,(2^{n+2}-6)+\sqrt{17}\,(2^{n+2})+2\sqrt{2}\,(2^{n+2}-6)+4(9\times 2^{n+1}-28)$
$+2\sqrt{5}\,(2^{n+1})+3\sqrt{2}(7\times 2^n-10)+4\sqrt{2}\,(2^n)$
$=(33\sqrt{2}+4\sqrt{10}+4\sqrt{17}+18\sqrt{13}+4\sqrt{5})\,2^n-42\sqrt{2}-6\sqrt{10}-28\sqrt{13}.$

**(ii)** $SO_{red}(NSC5C6[n])=\sum_{uv}\sqrt{(d_u-1)^2+(d_v-1)^2}$
$=2(2^{n+2}-6)+3(2^{n+2})+\sqrt{2}(2^{n+2}-6)+\sqrt{5}(9\times 2^{n+1}-28)+\sqrt{10}\,(2^{n+1})$
$+2\sqrt{2}\,(7\times 2^n-10)+3\sqrt{2}\,(2^n)$
$=(20+24\sqrt{2}+4\sqrt{10}+36\sqrt{5}+2\sqrt{10})\,2^n-26\sqrt{2}-28\sqrt{5}-12.$

**(iii)** $\prod_{so}(NSC5C6[n])=\prod_{uv}\sqrt{d_u^2+d_v^2}$
$=\sqrt{10}\,(2^{n+2}-6)\times\sqrt{17}\,(2^{n+2})\times 2\sqrt{2}\,(2^{n+2}-6)\times\sqrt{13}(9\times 2^{n+1}-28)$
$\times 2\sqrt{5}(2^{n+1})\times 3\sqrt{2}(7\times 2^n-10)\times 4\sqrt{2}(2^n)$
$=(2^{7n+10}-377\times 2^{6n+4}+3383\times 2^{5n+2}-843\times 2^{4n+4}+5040\times 2^{3n})$
$\times 15360\sqrt{221}$

**(iv)** $SO_{rev}(NSC5C6[n])=\sum_{uv}\sqrt{[\Delta-d_u+1]^2+[\Delta-d_v+1]^2}$
$=2\sqrt{5}(2^{n+2}-6)+\sqrt{17}(2^{n+2})+3\sqrt{2}(2^{n+2}-6)+\sqrt{13}(9\times 2^{n+1}-28)$
$+\sqrt{10}(2^{n+1})+2\sqrt{2}(7\times 2^n-10)+\sqrt{2}(2^n)$
$=\left(2\times\sqrt{5}+\sqrt{17}+3\sqrt{2}\right)2^{n+2}+(9\sqrt{13}+\sqrt{10})2^{n+1}+(15\sqrt{2})2^n)$
$-(12\sqrt{5}+28\sqrt{13}+38\sqrt{2}).$

**(v)** $RSO(NSC5C6[n]) = \sum_{uv} \sqrt{{r_u}^2 + {r_v}^2}$

$= \sum_{uv} \sqrt{[\Delta + \delta - d_u]^2 + [\Delta + \delta - d_v]^2}$

$= 2\sqrt{5}(2^{n+2} - 6) + \sqrt{17}(2^{n+2}) + 3\sqrt{2}(2^{n+2} - 6) + \sqrt{13}(9 \times 2^{n+1} - 28)$
$+\sqrt{10}(2^{n+1}) + 7\sqrt{2}(7 \times 2^n - 10) + \sqrt{2}(2^n)$
$= (2\sqrt{5} + \sqrt{17} + 3\sqrt{2})2^{n+2} + (9\sqrt{13} + \sqrt{10})2^{n+1} + (15\sqrt{2})2^n$
$-(28\sqrt{13} + 38\sqrt{2} + 12\sqrt{5}).$

□

**Theorem 2.2.** *The $GO_1$, $GO_2$, $R_1$, $R_2$, $HR_1$, $HR_2$ indices for $NSC5C6[n]$ are as follows :*

**(i)** $GO_1(G) = 24 \times 2^{n+2} + 113 \times 2^{n+1} + 129 \times 2^n - 548.$
**(ii)** $GO_2(G) = 48 \times 2^{n+2} + 3182 \times 2^{n+1} + 506 \times 2^n - 1548.$
**(iii)** $R_1(G) = 17 \times 2^{n+2} + 49 \times 2^{n+1} + 30 \times 2^n - 252.$
**(iv)** $R_2(G) = 21 \times 2^{n+2} + 57 \times 2^{n+1} + 29 \times 2^n - 310.$
**(v)** $HR_1(G) = 97 \times 2^{n+2} + 241 \times 2^{n+1} + 116 \times 2^n - 1292.$
**(vi)** $HR_2(G) = 161 \times 2^{n+2} + 333 \times 2^{n+1} + 113 \times 2^n - 2038.$

*Proof.* From the edge partition of $NSC5C6[n]$ as shown in Table 1, it is evident that...

**(i)** $GO_1(NSC5C6[n]) = 7\left(2^{n+2} - 6\right) + 9(2^{n+2}) + 8(2^{n+2} - 6) + 11(9 \times 2^{n+1} - 28)$
$+14(2^{n+1}) + 15(7 \times 2^n - 10) + 24(2^n) = 24 \times 2^{n+2} + 113 \times 2^{n+1}$
$+129 \times 2^n - 548.$

**(ii)** $GO_2(NSC5C6[n]) = 12\left(2^{n+2} - 6\right) + 20(2^{n+2}) + 16(2^{n+2} - 6)$
$+30(9 \times 2^{n+1} - 28) + 48(2^{n+1}) + 54(7 \times 2^n - 10) + 128(2^n)$
$= 48 \times 2^{n+2} + 3182 \times 2^{n+1} + 506 \times 2^n - 1548.$

**(iii)** $R_1(NSC5C6[n]) = 6\left(2^{n+2} - 6\right) + 5(2^{n+2}) + 6(2^{n+2} - 6) + 5(9 \times 2^{n+1} - 28)$
$+4(2^{n+1}) + 4(7 \times 2^n - 10) + 2(2^n) = 17 \times 2^{n+2} + 49 \times 2^{n+1} + 30 \times 2^n - 252.$
$= 17 \times 2^{n+2} + 49 \times 2^{n+1} + 30 \times 2^n - 252.$

**(iv)** $R_2(NSC5C6[n]) = 8\left(2^{n+2} - 6\right) + 4(2^{n+2}) + 9(2^{n+2} - 6) + 6(9 \times 2^{n+1} - 28)$
$+3(2^{n+1}) + 4(7 \times 2^n - 10) + 2^n = 21 \times 2^{n+2} + 57 \times 2^{n+1} + 29 \times 2^n - 310.$

**(v)** $HR_1(NSC5C6[n]) = 36\left(2^{n+2} - 6\right) + 25(2^{n+2}) + 36(2^{n+2} - 6)$
$+25(9 \times 2^{n+1} - 28) + 16(2^{n+1}) + 16(7 \times 2^n - 10) + 4(2^n)$
$= 97 \times 2^{n+2} + 241 \times 2^{n+1} + 116 \times 2^n - 1299.$

**(vi)** $HR_2(NSC5C6[n]) = 64\left(2^{n+2} - 6\right) + 16(2^{n+2}) + 81(2^{n+2} - 6)$
$+36(9 \times 2^{n+1} - 28) + 9(2^{n+1}) + 16(7 \times 2^n - 10) + 2^n$
$= 161 \times 2^{n+2} + 333 \times 2^{n+1} + 113 \times 2^n - 2038.$

□

## 3. Nano star Dendrimer NSD[n]

Our second molecular graph is known as Nano star Dendrimer $NSD[n]$. It is a molecule of generation $G_n$ with $n \geqslant 1$ growth stage, composed of four similar branches, with a central core at its center. The central core comprises six vertices of degree 2, six vertices of degree 3, and thirteen edges, as illustrated in Figure 2. In one branch of $NSD[n]$, there are $30+2\times30+2^2\times30+\cdots+2^{n-1}\times30 = 30(2^n-1)$ vertices, and $35+2\times35+2^2\times35+\cdots+2^{n-1}\times35 = 35(2^n-1)$ edges. Hence, in $NSD[n]$, there is a total of $120 \times 2^n - 108$ vertices and $140 \times 2^n - 127$ edges, as shown in [10]. Figure 2 depicts that the edge set of $NSD[n]$ can be divided into the following three categories. Table 2 presents the number of edges in each of these categories.

(**Note:** The correct number of edges $E_{(2,2)}$ in [10] was inaccurately calculated and has been corrected in Table 2.)

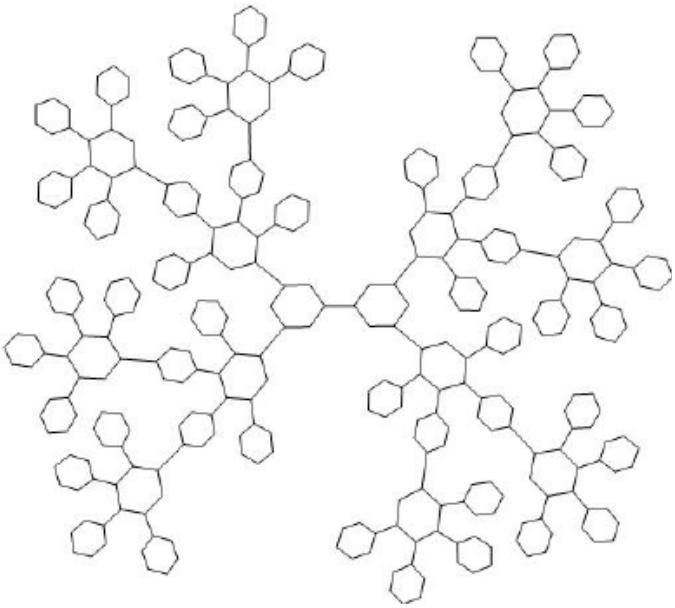

Fig. 2. The molecular graph of Nano star dendrimer NSD[n]

Table 2. The edges partition of $NSD[n]$

| $\mathbf{E_(u,v)}$ | **Number of edges** $(\mathbf{d_u}, \mathbf{d_v})$ |
|---|---|
| $E_{(2,2)}$ | $56 \times 2^n - 48$ |
| $E_{(2,3)}$ | $48 \times 2^n - 44$ |
| $E_{(3,3)}$ | $36 \times 2^n - 35$ |

**Theorem 3.1.** *Let $NSD[n]$ be a molecule graph of generation $G_n$ with $n \geq 1$ growth stage. Then the $SO$, $SO_{red}$, $\prod_{so}$, $SO_{rev}$, and $RSO$ indices for $NSD[n]$ are as follows:*

**(i)** $SO(G) = (220\sqrt{2} + 48\sqrt{13})\ 2^n - (201\sqrt{2} + 44\sqrt{13})$.
**(ii)** $SO_{red}(G) = (128\sqrt{2} + 48\sqrt{5})\ 2^n - (118\sqrt{2} + 44\sqrt{5})$.
**(iii)** $\prod_{so}(G) = 12\sqrt{13}\ (96768 \times 2^{3n} - 171648 \times 2^{2n} + 242912 \times 2^n - 168000)$.
**(iv)** $SO_{rev}(G) = (148\sqrt{2} + 48\sqrt{5})\ 2^n - (131\sqrt{2} + 44\sqrt{5})$.
**(v)** $RSO(G) = (240\sqrt{2} + 48\sqrt{13})\ 2^n - (214\sqrt{2} + 44\sqrt{13})$.

*Proof.* The results above are derived from the edge partition of $NSD[n]$ presented in table 2.

□

**Theorem 3.2.** *Suppose $NSD[n]$ be a molecule graph of generation $G_n$ with $n \geq 1$ growth stage. Then the $GO_1, GO_2, R_1, R_2, HR_1$, and $HR_2$ indices for $NSD[n]$ are as follows:*

**(i)** $GO_1(G) = 1516 \times 2^n - 1393.$
**(ii)** $GO_2(G) = 4280 \times 2^n - 3978.$
**(iii)** $R_1(G) = 720 \times 2^n - 648.$
**(iv)** $R_2(G) = 936 \times 2^n - 836$
**(v)** $HR_1(G) = 3792 \times 2^n - 3388$
**(vi)** $HR_2(G) = 6840 \times 2^n - 6032$

*Proof.* The results above are derived from the edge partition of $NSD[n]$ presented in table 2.

□

## 4. Zinc Porphyrin dendrimer $\mathbf{DPZ_n}$

The Zinc Porphyrin dendrimer $DPZ_n$ of generation $G_n$ with $n \geqslant 1$ growth stage is our third molecular graph. $DPZ_n$ is composed of four identical branches with a core at its center. The central core consists of 24 vertices of degree 2, 24 vertices of degree 3, 1 vertex of degree 4, and 60 edges, as shown in Figure 3. In one branch of $DPZ_n$, there are $14+2\times14+2^2\times14+\cdots+2^{n-1}\times14 = 14\times2^n-14$ vertices; among them, $9+2\times9+2^2\times9+\cdots+2^{n-2}\times9+2^{n-1}\times11 = 10 \times 2^n - 9$ vertices are of degree 2, and the remaining $14 \times 2^n - 14 - (10 \times 2^n - 9) = 4 \times 2^n - 5$ vertices are of degree 3. Therefore, there are a total of $56 \times 2^n - 7$ vertices in $DPZ_n$, among them, $40 \times 2^n - 12$ vertices are of degree 2, and $16 \times 2^n + 4$ vertices are of degree 3, and one vertex is of degree 4. According to Figure 3, the edges set of $DPZ_n$ can be divided into the following four classes. Their cardinalities are computed in Table 3.
(**Note:** The number of edges $E_{(2,2)}$ and $E_{(2,3)}$ in [2] was incorrectly calculated. The correct numbers have been updated in Table 3.)

Table 3. The edge partition of $DPZ_n$

| $\mathbf{E_{(u,v)}}$ | **Number of edges** $(\mathbf{d_u}, \mathbf{d_v})$ |
|---|---|
| $E_{2,2)}$ | $24 \times 2^n - 4$ |
| $E_{(2,3)}$ | $32 \times 2^n - 16$ |
| $E_{(3,3)}$ | $8 \times 2^n + 12$ |
| $E_{(3,4)}$ | 4 |

**Theorem 4.1.** *Let $DPZ_n$ be a molecule graph of generation $G_n$ with $n \geq 1$ growth stage. Then the $SO$, $SO_{red}$, $\prod_{so}$, $SO_{rev}$, and $RSO$ indices for $DPZ_n$ are as follows:*

**(i)** $SO(G) = (72\sqrt{2} + 32\sqrt{13})\ 2^n + 28\sqrt{2} - 16\sqrt{13}.$
**(ii)** $SO_{red}(G) = (40\sqrt{2} + 32\sqrt{5})\ 2^n + 20\sqrt{2} - 16\sqrt{5} + 4\sqrt{13}.$
**(iii)** $\prod_{so}(G) = 240\sqrt{13}\ (6144 \times 2^{3n} - 512 \times 2^n + 768).$
**(iv)** $SO_{rev}(G) = (88\sqrt{2} + 32\sqrt{13})\ 2^n + 20\sqrt{2} - 16\sqrt{13}.$
**(v)** $RSO(G) = (152\sqrt{2} + 32\sqrt{41})\ 2^n + 28\sqrt{2} - 16\sqrt{41} + 20.$

*Proof.* The results are derived based on the edge divisions of $DPZ_n$ presented in Table 3.

□

**Theorem 4.2.** *Suppose $DPZ_n$ be a molecule graph of generation $G_n$ with $n \geq 1$ growth stage. Then the $GO_1, GO_2, R_1, R_2, HR_1$, and $HR_2$ indices for $DPZ_n$ are as follows:*

**(i)** $GO_1(G) = 664 \times 2^n + 48.$

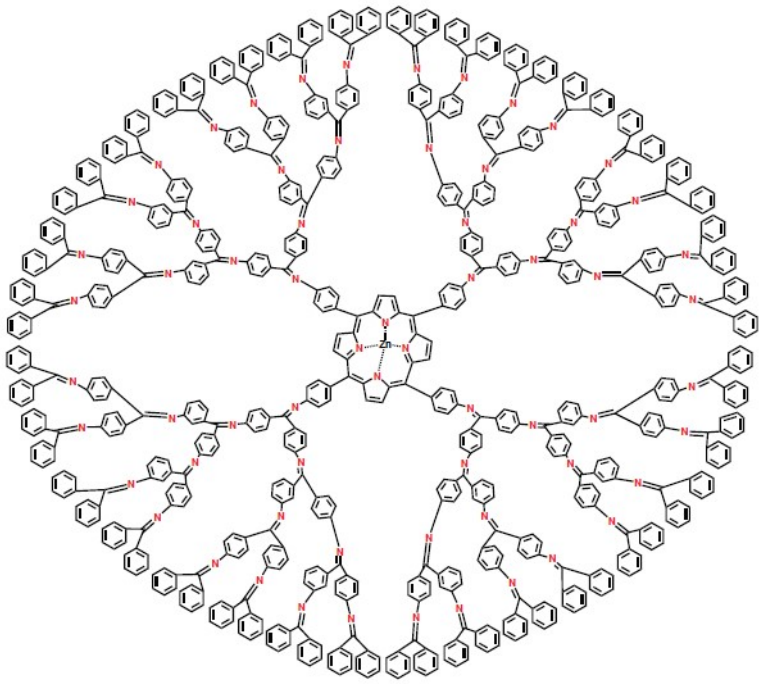

Fig. 3. The molecular graph of Zinc porphyrin dendrimer

**(ii)** $GO_2(G) = 1776 \times 2^n + 440.$
**(iii)** $R_1(G) = 464 \times 2^n - 52.$
**(iv)** $R_2(G) = 840 \times 2^n - 84.$
**(v)** $HR_1(G) = 3392 \times 2^n - 508.$
**(vi)** $HR_2(G) = 11400 \times 2^n - 2212.$

*Proof.* The results are derived based on the edge divisions of $DPZ_n$ presented in Table 3.

□

## 5. Conclusion

The primary objective was to compare the topological indices of the selected dendrimers. Our findings indicate that the Nano Star Dendrimer $NSD[n]$ exhibits the highest values among the evaluated indices—namely $SO$, $SO_{red}$, $SO_{rev}$, $GO_2$, $R_1(G)$, $R_2(G)$, and $HR_1(G)$—especially from the third generation onward. In contrast, the Nano Star Dendrimer $NSC5C6[n]$ demonstrates the lowest values. Figure 4 illustrates that all diagrams represent exponential functions with a base of 2, indicating a consistent increase in topological indices as the number of generations increases. These results highlight the significant differences in topological properties among the dendrimers, suggesting that $NSD[n]$ may be more advantageous for applications requiring higher topological indices.

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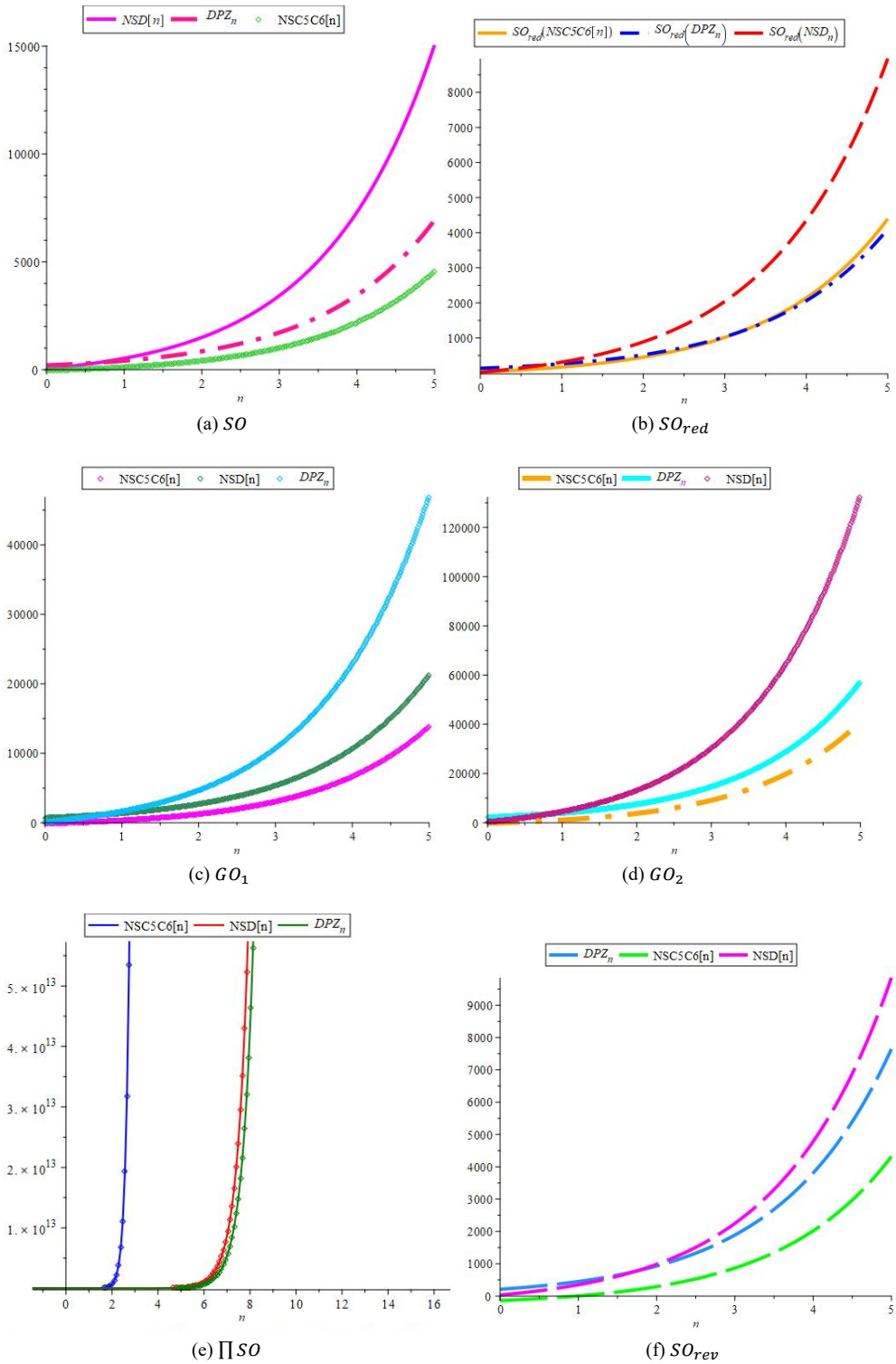


(a) $SO$ (b) $SO_{red}$

(c) $GO_1$ (d) $GO_2$

(e) $\prod SO$ (f) $SO_{rev}$

Fig. 4. Comparison of topological indices of $SO$, $SO_{red}$, $\prod SO$, $SO_{rev}$

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# Quotient Ideal Amenabiliy in Tensor Banach Algebras

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**Abstract**

A. Jabbari and O.T. Mewomo have studied the ideal amenability property in tensor Banach algebras and have proven several results. In this paper, we explore a different form of ideal amenability within tensor Banach algebras, specifically examining the concept of quotient ideal amenability. This concept applies to tensor Banach algebras constructed from the tensor product of two Banach algebras. Let $A$ be a Banach algebra, so $A\hat{\otimes}A$ is a tensor Banach algebra. We consider closed ideal generated in the projective tensor product $A\hat{\otimes}A$, then we generate closed ideal for $A$ and investigate the quotient ideal amenability of both $A$ and $A\hat{\otimes}A$. Additionally, we examine the hereditary properties of quotient ideal amenability for tensor Banach algebras and explore the relationship between the quotient ideal amenability of $A$ and its projective tensor product.

## 1. Introduction and preliminaries

The notion of amenability in Banach algebras was first introduced by B. E. Johnson [5] in 1972. He showed that for locally compact group $G$, the group algebra $L^1(G)$ is amenable if and only if $G$ is amenable in [5]. In [3], Gorgi and Yazdanpanah studied and introduced the concept of $I$-weak amenability and ideal amenability of Banach algebras and they showed that ideal amenability is different from amenability and weak amenability in Banach algebras, also they showed that every $C^*$-algebra is ideally amenable. The Banach algebra $A$ is $I$-weakly amenable if $\mathcal{H}^1(A, I^*) = \{0\}$ and is ideally amenable if it is $I$-weakly amenable for every closed two-sided ideal $I$ of $A$. Kazemipour and Fozouni in [7], defined a new amenability in Banach algebras named quotient amenability. The Banach algebra $A$ is known $I$-quotient amenable, if satisfy the condition $\mathcal{H}^1(\frac{A}{I}, X^*) = \{0\}$, for every Banach $\frac{A}{I}$-bimodule $X$, where $I$ is a closed two-sided ideal of $A$ and $X^*$ is the dual module of $X$, so $X^*$ is a Banach $\frac{A}{I}$-bimodule. They also showed that, $A$ is quotient amenable, if $A$ is $I$-quotient amenable for every closed two-sided ideal $I$ of $A$. The following module actions imply that the Banach $\frac{A}{I}$-bimodule $X$ becomes a Banach $A$-bimodule:

$$a \bullet x = (a + I) \cdot x, \quad x \bullet a = x \cdot (a + I) \qquad (a \in A, x \in X). \tag{1}$$

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Influenced by these concepts, Teymouri and et. al, in [12], studied some derivations into annihilators of the closed ideals of Banach algebras and introduce $\frac{A}{I}$-weak amenability and quotient ideal amenability of Banach algebra $A$, where $I$ is a closed two-sided ideal of $A$.

**Definition 1.1.** Let $A$ be a Banach algebra and $I$ be a closed two-sided ideal of $A$. $A$ is colled $\frac{A}{I}$-weakly amenable if $\mathcal{H}^1(A,(\frac{A}{I})^*) = \{0\}$. Also, $A$ is said to be quotient ideally amenable if it is $\frac{A}{I}$-weakly amenable for every closed two-sided ideal $I$ of $A$.

**Remark 1.2.** Let $A$ be a Banach algebra and $I$ be a closed two-sided ideal of $A$. The annihilator of $I$ is denoted by $I^{\perp} = \{f \in A^*; f(I) = 0\}$, so $(\frac{A}{I})^* \simeq I^{\perp}$.

Since $\frac{A}{I}$ is a Banach $\frac{A}{I}$-bimodule, by (1), $\frac{A}{I}$ is a Banach $A$-bimodule and with following actions, $(\frac{A}{I})^*$ becomes a Banach $A$-bimodule

$$\langle a' + I, a \bullet f\rangle = \langle a'a + I, f\rangle, \quad \langle a' + I, f \bullet a\rangle = \langle aa' + I, f\rangle \qquad (a \in A, a' + I \in \frac{A}{I}, f \in (\frac{A}{I})^*).$$

In this paper, we explore the concept of quotient ideal amenability and the conditions under which we can deduce the quotient ideal amenability of $A\hat{\otimes}A$ from that of $A$ and conversely. We begin by considering the unital Banach algebra $A$ and constructing the unital tensor Banach algebra $(A\hat{\otimes}A)$. After this, we investigate the relationship between the quotient ideal amenability of $A$ and its tensor product.

## 2. Quotient ideal amenability in tensor Banach algebra $(A\hat{\otimes}A)$

Let $A$ be an unital Banach algebra with unit element $e_A$, then $(A\hat{\otimes}A)$ is an unital Banach algebra with unit element $(e_A \otimes e_A)$. The concepts of weak amenability and ideal amenability in a unital Banach algebra $A$ are equivalent for every closed ideal $I$ of $A$. In this section, we examine the quotient ideal amenability of the tensor Banach algebra $(A\hat{\otimes}A)$ and its relationship with the quotient ideal amenability of the unital Banach algebra $A$. The projective tensor product of the Banach algebra $A$ with itself, defined by

$$A\hat{\otimes}A = \{\sum_{i=1}^{\infty} a_i \otimes a_i' : \quad \sum_{i=1}^{\infty} \|a_i\|\|a_i'\| < \infty, \quad a_i, a_i' \in A\},$$

the projective tensor product $(A\hat{\otimes}A)$ is an $A$-bimodule with following module actions

$$a \cdot (a_1 \otimes a_2) = aa_1 \otimes a_2 \quad , \quad (a_1 \otimes a_2) \cdot a = a_1 \otimes a_2 a \qquad (a, a_1, a_2 \in A),$$

also, $(A\hat{\otimes}A)$ is a Banach algebra with the product and norm defined by

$$(a_1 \otimes a_2) \cdot (a_4 \otimes a_3) = a_1a_3 \otimes a_2a_4 \quad , \qquad (a_1, a_3, a_2, a_4 \in A)$$

$$\|a\| = \inf\{\sum_{i=1}^{\infty} \|a_i\|\|a_i'\|, \quad a = \sum_{i=1}^{\infty} a_i \otimes a_i'\}.$$

According to the [12, Lemma 3.3], for Banach algebra $A$ and it's closed two-sided ideal $I$, if $I$ is complemented of $A$, then $(I\hat{\otimes}A)$ is a closed two-sided ideal of $(A\hat{\otimes}A)$ and is an $(A\hat{\otimes}A)$-bimodule. Also for complemented closed two-sided ideal of $A$, such as $I$, the following exact sequence of Banach $A$-bimodules splits

$$\{0\} \to I \xrightarrow{i} A \xrightarrow{q} \frac{A}{I} \to \{0\},$$

where $q : A \to \frac{A}{I}$ is canonical quotient map.

Let $A$ and $\mathfrak{A}$ are Banach algebras such that $A$ becomes $\mathfrak{A}$-bimodule. And let $J$ be a closed ideal generated in $(A\hat{\otimes}A)$ by the elements of the form $(a_1\cdot\alpha\otimes a_2-a_1\otimes\alpha\cdot a_2)$ for any $\alpha\in\mathfrak{A}$ and $a_1,a_2\in A$. So the module projective tensor product $(A\hat{\otimes}_{\mathfrak{A}}A)$ is the quotient $\frac{A\hat{\otimes}A}{J}$, [6]. Let $I$ be the closed ideal in $A$ generated by $\Delta(J)$, where $\Delta: A\hat{\otimes}A\to A$ is the multiplication map. By [11, Lemma 3], the definition of the module projective tensor product implies that $(A\hat{\otimes}_{\mathfrak{A}}A)\simeq\frac{A\hat{\otimes}A}{I_0}$, where $I_0$ is the closed linear span of the set $\{a_1\cdot\alpha\otimes a_2-a_1\otimes\alpha\cdot a_2;\quad a_1,a_2\in A,\alpha\in\mathfrak{A}\}$. Since $A$ is a Banach $A$-$\mathfrak{A}$-bimodule, by identity map $\mathbf{id}_A: A\to A$ and canonical quotient map $q: A\to\frac{A}{I}$, we have following exact sequence of Banach $A$-bimodules

$$\{0\}\longrightarrow I\hat{\otimes}A\xrightarrow{i\hat{\otimes}\mathbf{id}_A}A\hat{\otimes}A\xrightarrow{q\hat{\otimes}\mathbf{id}_A}\frac{A}{I}\hat{\otimes}A\longrightarrow\{0\},$$

Then the map

$$\Phi:=q\otimes\mathbf{id}_A: A\hat{\otimes}A\to\frac{A}{I}\hat{\otimes}A$$
$$a_1\otimes a_2\mapsto(a_1+I)\otimes a_2$$

is the quotient map for any $a_1,a_2\in A$. Therefore we have an isometric isomorphism

$$\frac{A\hat{\otimes}A}{\mathbf{ker}\Phi}\simeq\frac{A}{I}\hat{\otimes}A.$$

According to the [10, Theorem 2.20], $\mathbf{ker}\Phi=I\hat{\otimes}A$. Finally, we conclude that $I\hat{\otimes}A=I_0$

**Lemma 2.1.** *Let $A$ be a Banach algebra and a $\mathfrak{A}$-bimodule. For closed ideal of $A$ such as $I$ that is equal to the closed linear span of the set of elements of the form $((a_1\cdot\alpha)a_2-a_1(\alpha\cdot a_2))$ for any $\alpha\in\mathfrak{A}$ and any $a_1,a_2\in A$, we have*

$$\frac{A\hat{\otimes}A}{I\hat{\otimes}A}\simeq\frac{A}{I}\hat{\otimes}A$$

**Theorem 2.2.** *Let $A$ be a Banach algebra and a $\mathfrak{A}$-bimodule and a commutative and essential Banach algebra, and $I$ be a complemented closed two-sided ideal of $A$ as in Lemma (2.1). If $A\hat{\otimes}A$ is $(\frac{A\hat{\otimes}A}{I\hat{\otimes}A})$-weakly amenable then $A$ is $\frac{A}{I}$-weakly amenable.*

*Proof.* Let $I$ be a complemented closed two-sided ideal of $A$, as in the previous, then $(I\hat{\otimes}A)$ is a closed two-sided ideal of $(A\hat{\otimes}A)$. Since $(A\hat{\otimes}A)$ is $(\frac{A\hat{\otimes}A}{I\hat{\otimes}A})$-weakly amenable, we have $\mathcal{H}^1(A\hat{\otimes}A,(\frac{A\hat{\otimes}A}{I\hat{\otimes}A})^*)=\{0\}$.
Let $D: A\longrightarrow(\frac{A}{I})^*$ be a derivation. Since $A$ is essential, let $\lambda$ be a non-zero element in $A^*$ and there are $c,d\in A$ such that $\langle cd,\lambda\rangle=1$. According to isomorphism in Lemma (2.1), define $\tilde{D}: A\hat{\otimes}A\to(\frac{A\hat{\otimes}A}{I\hat{\otimes}A})^*\simeq((\frac{A}{I})\hat{\otimes}A)^*$ by

$$\langle(a''+I)\otimes a',\tilde{D}(a_1\otimes a_2)\rangle=\langle[a'']_I,D(a_1)\rangle\langle a',a_2\cdot\lambda\rangle,$$

$\tilde{D}$ is a bounded linear map and for each $a'', a', a_i \in A$, we have

$$\begin{aligned}
\langle [a'']_I \otimes a', \tilde{D}((a_1 \otimes a_2)(a_3 \otimes a_4))\rangle &= \langle [a'']_I \otimes a', \tilde{D}(a_1a_3 \otimes a_2a_4)\rangle \\
&= \langle [a'']_I, D(a_1a_3)\rangle\langle a', a_2a_4 \cdot \lambda\rangle \\
&= \langle [a'']_I, D(a_1)\cdot a_3\rangle\langle a', a_2a_4 \cdot \lambda\rangle \\
&\quad + \langle [a'']_I, a_1 \cdot D(a_3)\rangle\langle a', a_2a_4 \cdot \lambda\rangle \\
&= \langle a_3 \cdot [a'']_I, D(a_1)\rangle\langle a_4 \cdot a', a_2 \cdot \lambda\rangle \\
&\quad + \langle [a'']_I \cdot a_1, D(a_3)\rangle\langle a' \cdot a_2, a_4 \cdot \lambda\rangle \\
&= \langle (a_3[a'']_I) \otimes (a_4a'), \tilde{D}(a_1 \otimes a_2)\rangle \\
&\quad + \langle ([a'']a_1) \otimes (a'a_2), \tilde{D}(a_3 \otimes a_4)\rangle \\
&= \langle (a_3 \otimes a_4)([a'']_I \otimes a'), \tilde{D}(a_1 \otimes a_2)\rangle \\
&\quad + \langle ([a'']_I \otimes a')(a_1 \otimes a_2), \tilde{D}(a_3 \otimes a_4)\rangle \\
&= \langle [a'']_I \otimes a', \tilde{D}(a_1 \otimes a_2)\cdot(a_3 \otimes a_4)\rangle \\
&\quad + \langle [a'']_I \otimes a_1', (a_1 \otimes a_2)\cdot \tilde{D}(a_3 \otimes a_4)\rangle,
\end{aligned}$$

therefore $\tilde{D}$ is a derivation. According to the hypothesis, there is $\varphi \in (\frac{A\hat{\otimes}A}{I\hat{\otimes}A})^*$ such that $\tilde{D} = \mathbf{ad}_\varphi$. We define $\gamma \in (\frac{A}{I})^*$ by $\gamma([a'']_I) = \varphi([a'']_I \otimes cd)$, so $\gamma$ is a bounded linear functional and for any $c, d, a \in A$ we have

$$\begin{aligned}
\langle [a'']_I, D(a)\rangle &= \langle [a'']_I, D(a)\rangle\langle cd, \lambda\rangle \\
&= \langle [a'']_I, D(a)\rangle\langle c, d\lambda\rangle \\
&= \langle [a'']_I \otimes c, \tilde{D}(a \otimes d)\rangle \\
&= \langle [a'']_I \otimes c, \mathbf{ad}_\varphi(a \otimes d)\rangle \\
&= \langle [a'']_I \otimes c, (a \otimes d)\cdot\varphi\rangle - \langle [a'']_I \otimes c, \varphi\cdot(a \otimes d)\rangle \\
&= \langle ([a'']_I \otimes c)\cdot(a \otimes d), \varphi\rangle - \langle (a \otimes d)\cdot([a'']_I \otimes c), \varphi\rangle \\
&= \langle [a'']_I \cdot a \otimes cd, \varphi\rangle - \langle a \cdot [a'']_I \otimes dc, \varphi\rangle \\
&= \langle ([a'']_I \cdot a - a \cdot [a'']_I) \otimes cd, \varphi\rangle \\
&= \langle ([a'']_I \cdot a - a \cdot [a'']_I), \gamma\rangle \\
&= \langle [a'']_I, a\cdot\gamma - \gamma\cdot a\rangle \\
&= \langle [a'']_I, \mathbf{ad}_\gamma(a)\rangle,
\end{aligned}$$

therefore $D$ is an inner derivation, so $\mathcal{H}^1(A, (\frac{A}{I})^*) = \{0\}$, hence the Banach algebra $A$ is $\frac{A}{I}$-weakly amenable. □

**Theorem 2.3.** *Let $A$ be a essential Banach algebra. If $A$ is quotient ideally amenable, then $(A\hat{\otimes}A)$ is quotient ideally amenable for any complemented closed two-sided ideals of $A$.*

*Proof.* Let $I$ be a closed two-sided ideal of $A$. Since $A$ is quotient ideally amenable, then $\mathcal{H}^1(A, (\frac{A}{I})^*) = \{0\}$. According to the [9, Theorem 2.6], This implies that $\frac{A}{I}$ is weakly amenable, and indicates that $A$ is weakly amenable. On the other hand, based on [1, Theorem 2.8.71], the weak amenability of the Banach algebra $A$, ensures the weak amenability of Banach algebra $(A\hat{\otimes}A)$. Furthermore, applying the same theorem, the weak amenability of $(\frac{A}{I})$ and $A$, leads to the conclusion that $(\frac{A}{I}\hat{\otimes}A)$, is weakly amenable. Therefore, we have $\mathcal{H}^1(\frac{A}{I}\hat{\otimes}A, (\frac{A}{I}\hat{\otimes}A)^*) = \{0\}$. By the first isomorphism in Lemma (2.1), $\mathcal{H}^1((\frac{A\hat{\otimes}A}{I\hat{\otimes}A}), (\frac{A\hat{\otimes}A}{I\hat{\otimes}A})^*) = \{0\}$. For the complemented closed two-sided ideal $I$ of essential Banach algebra $A$, by [12, Lemma 3.3], $(I\hat{\otimes}A)$ is an essential closed ideal of $(A\hat{\otimes}A)$. Therefore, by the converse of [9, Theorem 2.6], we have $\mathcal{H}^1((A\hat{\otimes}A), (\frac{A\hat{\otimes}A}{I\hat{\otimes}A})^*) = \{0\}$. This reveals that $(A\hat{\otimes}A)$ is $(\frac{A\hat{\otimes}A}{I\hat{\otimes}A})$-weakly amenable, indicating that $(A\hat{\otimes}A)$ is quotient ideally amenable for any complemented ideal of $A$. □

## 3. Quotient ideal amenability in tensor Banach algebra $(A, \bullet_\varphi)\hat{\otimes}(B, \bullet_\psi)$

In this section, we consider a Banach algebra $A$ and its character space denoted by $\Phi_A$. For new product $\bullet_\varphi$ on $A$ as follows

$$a_1 \bullet_\varphi a_2 = \varphi(a_1)a_2 \quad (a_1, a_2 \in A, \varphi \in \Phi_A).$$

For $a_1, a_2, a_3 \in A$, the product operates as follows:

$$(a_1 \bullet_\varphi a_2) \bullet_\varphi a_3 = \varphi(a_1)a_2 \bullet_\varphi a_3 = \varphi(a_1)\varphi(a_2)a_3,$$

$$a_1 \bullet_\varphi (a_2 \bullet_\varphi a_3) = a_1 \bullet_\varphi (a_2)a_3 = \varphi(a_1)\varphi(a_2)a_3.$$

$A$ equipped with this product is an algebra, denoted by $(A, \bullet_\varphi)$. The norm on $A$ can be preserved in $(A, \bullet_\varphi)$ if we continue to use the original norm of Banach algebra $A$. The new multiplication satisfies the inequality

$$\|a_1 \bullet_\varphi a_2\| \leq \|\varphi\|\|a_1\|\|a_2\|.$$

Therefore, $(A, \bullet_\varphi)$ forms a Banach algebra. For $(A, \bullet_\varphi)$ to have an identity element, there must exist $e_A \in A$ such that

$$e_A \bullet_\varphi a = \varphi(e_A)a = a \quad \text{and} \quad a \bullet_\varphi e_A = \varphi(a)e_A = a.$$

The product $\bullet_\varphi$ is generally not commutative, because for any $a_1, a_2 \in A$

$$a_1 \bullet_\varphi a_2 = \varphi(a_1)a_2 \quad \text{and} \quad a_2 \bullet_\varphi a_1 = \varphi(a_2)a_1.$$

In [12], Teymouri et al. investigated the quotient ideal amenability of the algebraic structures $(A, \bullet_\varphi)$ and $(A, \bullet_\varphi)\hat{\otimes}(B, \bullet_\psi)$. Additionally, in [4], Jabbari researched the ideal amenability of $(A, \bullet_\varphi)\hat{\otimes}(B, \bullet_\psi)$. In this section of the paper, We explore the relationship between the quotient ideal amenability of $(A, \bullet_\varphi)$ and $(B, \bullet_\psi)$ with $(A, \bullet_\varphi)\hat{\otimes}(B, \bullet_\psi)$.

**Theorem 3.1.** *Let A and B be unital Banach algebras and let I and J be closed two-sided ideals of A and B, respectively. Suppose that* $\tilde{D} : A\hat{\otimes}B \to (\frac{A\hat{\otimes}B}{K})^*$ *is a map where K is a closed two-sided ideal of* $A\hat{\otimes}B$*. Then* $\tilde{D}$ *is a derivation if and only if there exist derivation maps* $D_A : A \to (\frac{A}{I})^*$ *and* $D_B : B \to (\frac{B}{J})^*$ *such that* $\tilde{D}(a \otimes b) := D_A(a) \otimes b + a \otimes D_B(b)$*. Moreover* $\tilde{D}$ *is an inner if and only if* $D_A$ *and* $D_B$ *are inner.*

*Proof.* Firstly, we show that $\tilde{D} \in \mathcal{Z}^1((A\hat{\otimes}B), (\frac{A\hat{\otimes}B}{K})^*)$ if and only if $D_A \in \mathcal{Z}^1(A, (\frac{A}{I})^*)$ and $D_B \in \mathcal{Z}^1(B, (\frac{B}{J})^*)$. Let $D_A$ and $D_B$ be derivation maps, for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$\begin{aligned}
\tilde{D}((a_1 \otimes b_1)(a_2 \otimes b_2)) &= \tilde{D}(a_1a_2 \otimes b_1b_2) \\
&= D_A(a_1a_2) \otimes (b_1b_2) + (a_1a_2) \otimes D_B(b_1b_2) \\
&= (D_A(a_1)a_2 + a_1D_A(a_2)) \otimes (b_1b_2) + (a_1a_2) \otimes (D_B(b_1)b_2 + b_1D_B(b_2)) \\
&= (D_A(a_1)a_2 \otimes b_1b_2) + (a_1D_A(a_2) \otimes b_1b_2) \\
&\quad + (a_1a_2 \otimes D_B(b_1)b_2) + (a_1a_2 \otimes b_1D_B(b_2)) \\
&= (D_A(a_1) \otimes b_1)(a_2 \otimes b_2) + (a_1 \otimes b_1)(D_A(a_2) \otimes b_2) \\
&\quad + (a_1 \otimes D_B(b_1))(a_2 \otimes b_2) + (a_1 \otimes b_1)(a_2 \otimes D_B(b_2)) \\
&= (D_A(a_1) \otimes b_1 + a_1 \otimes D_B(b_1))(a_2 \otimes b_2) \\
&\quad + (a_1 \otimes b_1)(D_A(a_2) \otimes b_2 + a_2 \otimes D_B(b_2)) \\
&= \tilde{D}(a_1 \otimes b_1)(a_2 \otimes b_2) + (a_1 \otimes b_1)\tilde{D}(a_2 \otimes b_2),
\end{aligned}$$

therefore, $\tilde{D}$ is a derivation.

Conversely, let $\tilde{D}$ be a derivation, for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$\begin{aligned}
D_A(a_1a_2) \otimes (b_1b_2) + (a_1a_2) \otimes D_B(b_1b_2) &= \tilde{D}(a_1a_2 \otimes b_1b_2) \\
&= \tilde{D}((a_1 \otimes b_1)(a_2 \otimes b_2)) \\
&= \tilde{D}(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) + (a_1 \otimes b_1) \cdot \tilde{D}(a_2 \otimes b_2) \\
&= (D_A(a_1) \otimes b_1 + a_1 \otimes D_B(b_1))(a_2 \otimes b_2) \\
&\quad + (a_1 \otimes b_1)(D_A(a_2) \otimes b_2 + a_2 \otimes D_B(b_2)) \\
&= (D_A(a_1) \otimes b_1)(a_2 \otimes b_2) + (a_1 \otimes D_B(b_1))(a_2 \otimes b_2) \\
&\quad + (a_1 \otimes b_1)(D_A(a_2) \otimes b_2) + (a_1 \otimes b_1)(a_2 \otimes D_B(b_2)) \\
&= (D_A(a_1)a_2 \otimes b_1b_2) + (a_1a_2 \otimes D_B(b_1)b_2) \\
&\quad + (a_1D_A(a_2) \otimes b_1b_2) + (a_1a_2 \otimes b_1D_B(b_2)) \\
&= (D_A(a_1)a_2 + a_1D_A(a_2)) \otimes (b_1b_2) \\
&\quad + (a_1a_2) \otimes (D_B(b_1)b_2 + b_1D_B(b_2)),
\end{aligned}$$

therefore $D_A$ and $D_B$ are derivation maps.

In the following, we show that $\tilde{D} \in \mathcal{B}^1((A\hat{\otimes}B), (\frac{A\hat{\otimes}B}{K})^*)$ if and only if $D_A \in \mathcal{B}^1(A, (\frac{A}{I})^*)$ and $D_B \in \mathcal{B}^1(B, (\frac{B}{J})^*)$. Let $D_A$ and $D_B$ be inner derivation, so there exist $\varphi \in (\frac{A}{I})^*$ and $\psi \in (\frac{B}{J})^*$, respectively such that

$$D_A(a) = \mathbf{ad}_\varphi(a) = a \cdot \varphi - \varphi \cdot a \quad , \qquad (a \in A)$$

$$D_B(b) = \mathbf{ad}_\psi(b) = b \cdot \psi - \psi \cdot b \quad , \qquad (b \in B).$$

Define $\phi \in ((A\hat{\otimes}B), (\frac{A\hat{\otimes}B}{K})^*)$ by $\phi := \varphi \otimes \mathbf{id}_B + \mathbf{id}_A \otimes \psi$. Obviously, $(\varphi \otimes \mathbf{id}_B)$ and $(\mathbf{id}_A \otimes \psi)$ are functional on $A\hat{\otimes}B$. We claim that $\phi$ generates $\tilde{D}$ to inner derivation. For $a \in A$ and $b \in B$, we have

$$\begin{aligned}
\tilde{D}(a \otimes b) &= D_A(a) \otimes b + a \otimes D_B(b) \\
&= \mathbf{ad}_\varphi(a) \otimes b + a \otimes \mathbf{ad}_\psi(b) \\
&= (a \cdot \varphi - \varphi \cdot a) \otimes b + a \otimes (b \cdot \psi - \psi \cdot b) \\
&= a \cdot \varphi \otimes b - \varphi \cdot a \otimes b + a \otimes b \cdot \psi - a \otimes \psi \cdot b \\
&= a \cdot \varphi \otimes b \cdot \mathbf{id}_B(e_B) - \varphi \cdot a \otimes \mathbf{id}_B(b) + a \cdot \mathbf{id}_A(e_A) \otimes b \cdot \psi - \mathbf{id}_A(a) \otimes \psi \cdot b \\
&= a \cdot \varphi \otimes b \cdot \mathbf{id}_B(e_B) + a \cdot \mathbf{id}_A(e_A) \otimes b \cdot \psi - \varphi \cdot a \otimes \mathbf{id}_B(b) - \mathbf{id}_A(a) \otimes \psi \cdot b \\
&= (a \otimes b)(\varphi \otimes \mathbf{id}_B) + (a \otimes b)(\mathbf{id}_A \otimes \psi) - ((\varphi \otimes \mathbf{id}_B)(a \otimes b) + (\mathbf{id}_A \otimes \psi)(a \otimes b)) \\
&= (a \otimes b)((\varphi \otimes \mathbf{id}_B) + (\mathbf{id}_A \otimes \psi)) - ((\varphi \otimes \mathbf{id}_B) + (\mathbf{id}_A \otimes \psi))(a \otimes b) \\
&= (a \otimes b)\phi - \phi(a \otimes b) \\
&= \mathbf{ad}_\phi(a \otimes b).
\end{aligned}$$

Therefore $\tilde{D}$ is an inner derivation, when $D_A$ and $D_B$ are inner derivations.

Conversely, let $\tilde{D}$ be an inner derivation. There exists $\phi \in (\frac{A\hat{\otimes}B}{K})^*$ such that $\tilde{D} = \mathbf{ad}_\phi$. This means that for $a \otimes b \in (A\hat{\otimes}B)$, we have

$$\tilde{D}(a \otimes b) = \mathbf{ad}_\phi(a \otimes b) = (a \otimes b)\phi - \phi(a \otimes b). \tag{2}$$

Assuming $b = e_B$, we can express $\tilde{D}(a \otimes e_B)$ as follows

$$\tilde{D}(a \otimes e_B) = D_A(a) \otimes e_B + a \otimes D_B(e_B) = D_A(a) \otimes e_B.$$

Using the equation derived from the relationship above (2), we find

$$D_A(a) \otimes e_B = (a \otimes e_B)\phi - \phi(a \otimes e_B).$$

Similarly, assuming $a = e_A$, we have

$$\tilde{D}(e_A \otimes b) = D_A(e_A) \otimes b + e_A \otimes D_B(b) = e_A \otimes D_B(b).$$

By applying the earlier formulation, it follows that

$$e_A \otimes D_B(b) = (e_A \otimes b)\phi - \phi(e_A \otimes b).$$

From the preceding statements, the action of $\tilde{D}$ on $(a \otimes e_B)$ reduces to $D_A(a)$, while on $(e_A \otimes b)$ it reduces to $D_B(b)$. Thus, the behaviors of $D_A$ and $D_B$ can be fully reconstructed from the restrictions of $\tilde{D}$. Consequently, $D_A$ and $D_B$ represent inner derivations, similar to the notation used for $\tilde{D}$. □

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# Homogenously linked ideals over a module

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**Abstract**

Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be a standard graded ring, $M$ be a finitely generated graded $R$-module and $R_+ := \bigoplus_{n\in\mathbb{N}} R_n$ denotes the irrelevant ideal of $R$. In this talk, we study some basic properties of homogeneously linked ideals over $M$. Due to the importance of irrelevant ideal in a standard graded ring, we show whether a homogeneous ideal could be homogeneously linked with $R_+$ over $M$.

## 1. Introduction

Throughout the paper, $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a standard graded Noetherian ring, i.e. $R_0$ is a commutative Noetherian ring and $R$ is generated, as an $R_0$-algebra, by finitely many elements of degree one, $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$ is the irrelevant ideal of $R$ and $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals of $R$. Also, $M$ denotes a finitely generated graded $R$-module.
In ([4]), the authors introduced the concept of linkage of ideals over a module, which is a generalization of its classical concept introduced by Peskine and Szpiro ([10]). In this paper, we consider the above concept in the graded case. More precisely, the homogeneous ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be homogeneously linked (or h-linked) by $I$ over $M$, denoted by $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, if $I$ is generated by a homogeneous $M$-regular sequence and $\mathfrak{a}$ and $\mathfrak{b}$ are linked by $I$ over $M$.
We study some basic properties of homogeneously linked ideals. To be more precise, we show by examples that if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, this doesn't imply that $\mathfrak{a}\cap R_0 \sim_{(I\cap R_0;M_n)} \mathfrak{b}\cap R_0$ for all $n \in \mathbb{Z}$ and vice versa. Although, in some cases it does.
Due to the importance of irrelevant ideal in a standard graded ring, it is natural to ask whether a homogeneous ideal could be h-linked with $R_+$. We answer this question in some cases.
We keep the notations introduced in the introduction, throughout the paper.

## 2. Results

We start by the basic concept of the paper.

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**Definition 2.1.** Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by a homogeneous $M$-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneously linked (or h-linked) by $I$ over $M$, denoted $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically h-linked by $I$ over $M$ if, in addition, $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is h-linked over $M$ if there exist homogeneous ideals $\mathfrak{b}$ and $I$ of $R$ such that $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$. $\mathfrak{a}$ is h-$M$-selflinked by $I$ if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{a}$.

**Remark 2.2.** Note that, this definition is a special case of linkage of ideals over a module, studied in [4]. Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ are h-linked by $I$ over $M$ and $\operatorname{grade}(\mathfrak{a}, M) = t$ then the following statements hold.

(i) If $\mathfrak{a}M \cap \mathfrak{b}M \neq IM$, then $IM :_M (\mathfrak{a} + \mathfrak{b}) \neq IM$. So, $(\mathfrak{a} + \mathfrak{b}) \subseteq Z(M/IM)$, the set of zero divisors of $M/IM$, that results $\operatorname{grade}(\mathfrak{a} + \mathfrak{b}, M) = t$.

(ii) If $\mathfrak{a}M \cap \mathfrak{b}M = IM$ (i.e. $\mathfrak{a}$ and $\mathfrak{b}$ are geometrically h-linked), then, by [5, 2.9], $\operatorname{grade}(\mathfrak{a} + \mathfrak{b}, M) = t + 1$.

In the next example, we show that there is no bilateral relation between h-linkedness of ideals $\mathfrak{a}$ and $\mathfrak{b}$ by $I$ over $M$ and linkage of $\mathfrak{a} \cap R_0$ and $\mathfrak{b} \cap R_0$ by $I \cap R_0$ over homogeneous components of $M$.

**Example 2.3.** Let $(R_0, \mathfrak{m}_0)$ be local with depth $R_0 > 0$, $\mathfrak{m} = \mathfrak{m}_0 + R_+$ be the homogeneous maximal ideal of $R$ and $x_1, x_2, \dots, x_s (s \geq 2)$ be a homogeneous $R$-regular sequence in $\mathfrak{m}$. Assume that $1 \leq l < s$ such that $\deg x_i = 0$ for all $1 \leq i \leq l$ and $\deg x_i \geq 1$ for all $l < i \leq s$.

- Set $\mathfrak{a} := (x_1, x_2, \dots, x_s)$, $I := (x_1, \dots, x_l, x_{l+1}^2, \dots, x_s)$, $\mathfrak{a}_0 := \mathfrak{a} \cap R_0 = (x_1, \dots, x_l)R_0$ and $I_0 := I \cap R_0 = (x_1, \dots, x_l)R_0$. So, by [4, 2.2], $\mathfrak{a}$ is h-$R$-selflinked by $I$. But, since $\mathfrak{a}_0 = I_0$, $\mathfrak{a}_0$ is not $R_n$-selflinked by $I_0$, for all n.
- Again, set $\mathfrak{a} := (x_1, x_2, \dots, x_s)$, $I := (x_1^2, x_2, \dots, x_{s-1})$, $\mathfrak{a}_0 := \mathfrak{a} \cap R_0 = (x_1, \dots, x_l)R_0$ and $I_0 := I \cap R_0 = (x_1^2, x_2, \dots, x_l)R_0$. Then, $\operatorname{grade}(\mathfrak{a}, R) = s \neq \operatorname{grade}(I, R)$, so, by [4, 2.6(i)], $\mathfrak{a}$ is not h-$R$-selflinked by $I$. But, $\mathfrak{a}_0$ is $R_n$-selflinked by $I_0$ for all $n$, using [4, 2.2] and the fact that $x_1^2, x_2, \dots, x_l$ is an $R_n$-regular sequence for all $n$.

**Lemma 2.4.** Assume that $(R_0, \mathfrak{m}_0)$ is local and $\mathfrak{a}$ and $\mathfrak{b}$ are generated by elements of degree zero. Then, $\mathfrak{a} \overset{h}{\sim}_{(0;M)} \mathfrak{b}$ if and only if $(\mathfrak{a} \cap R_0) \sim_{(0;M_n)} (\mathfrak{b} \cap R_0)$ for all $n \in \mathbb{Z}$.

In view of the importance of the irrelevant ideal in a standard graded ring, it is natural to study homogeneous ideals which are h-linked with $R_+$.
If $R = R_0[x_1, \dots, x_n]$ is a polynomial ring graded in the usual way, then $R_+ = (x_1, \dots, x_n)$ is h-$R$-selflinked by $(x_1^2, x_2, \dots, x_n)$, using [4, 2.2]. In the next example, we find some homogeneous ideals that are h-linked with $R_+$. It will be used in the next section, too.

**Example 2.5.**

1. Let $R = R_0[x]$ and $x \notin Z(M)$, then $R_+ \overset{h}{\sim}_{((r_0x^t);M)} (r_0x^{t-1})$ for all non-unit elements $r_0 \in R_0 \backslash Z(M)$ and all $t \geq 1$.
2. Let $R = R_0[x, y]$ and $r_1x, r_2y$ be an $M$-regular sequence where $r_1, r_2 \in R_0$. Then, it is straight forward to see that $R_+ \overset{h}{\sim}_{((r_1x^t, r_2y^{t'});M)} (r_1x^t, r_2y^{t'}, r_1r_2x^{t-1}y^{t'-1})$ for all $t, t' \geq 1$.

It's natural to ask whether a homogeneous ideal which is linked could be an h-linked ideal? In the following, we answer it in a special case.

**Remark 2.6.** Let $R_0$ be reduced and $R_+$ be a linked ideal by $I$ over $R$. As $R_0$ is reduced, $R_+$ is radical. Also, in view of [1, 16.1.2], there exists an ideal, say $I'$, generated by a homogeneous $R$-regular sequence of length $t$ in $R_+$, where $t := \operatorname{grade} R_+$ and $I' \neq R_+$. So, using [4, 2.8], [3, Theorem 1] and [12, 1.4],

$$\begin{aligned} \operatorname{Ass} R/R_+ \subseteq \operatorname{Ass} R/I \cap V(R_+) &= \operatorname{Ass} Hom_R(R/R_+, R/I) \\ &= \operatorname{Ass} Ext_R^t(R/R_+, R) = \operatorname{Ass} R/I' \cap V(R_+), \end{aligned}$$

where $V(R_+)$ denotes the set of prime ideals of $R$ containing $R_+$. This implies that $R_+$ is an h-linked ideal by $I'$, by [6, 2.8].

**Definition 2.7.** Following [9, 2.1], a sequence $x_1, x_2, \dots, x_t$ of homogeneous elements of $\mathfrak{b}$ is said to be a homogeneous $\mathfrak{b}$-filter regular sequence on $M$ if $x_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(\frac{M}{(x_1,\dots,x_{i-1})M})\backslash V(\mathfrak{b})$ and all $i = 1, \dots, t$.

Assume that $\mathfrak{a}$ is generated by elements of positive degrees and $\mathfrak{a} \subseteq \mathfrak{b}$. By [7, 1.5], if $\mathrm{Supp}(\frac{M}{\mathfrak{a}M}) \nsubseteq V(\mathfrak{b})$, then all maximal homogeneous $\mathfrak{b}$-filter regular sequences in $\mathfrak{a}$ on $M$ have the same finite length, that is denoted by $\text{f-grade}(\mathfrak{b}, \mathfrak{a}, M)$. Also $\text{f-grade}(\mathfrak{b}, \mathfrak{a}, M) := \infty$ whenever $\mathrm{Supp}(\frac{M}{\mathfrak{a}M}) \subseteq V(\mathfrak{b})$. Note that $\mathrm{grade}(\mathfrak{a}, M) \leq \text{f-grade}(\mathfrak{b}, \mathfrak{a}, M)$.
Moreover, Chu and Gu in [**?** , 2.4] in the case where $\mathfrak{b} = R_+$, show that if $\mathrm{Supp}(\frac{M}{\mathfrak{a}M}) \nsubseteq V(R_+)$ then

$$\text{f-grade}(R_+, \mathfrak{a}, M) = \max\{i \mid H^j_{\mathfrak{a}}(M)_n = 0, \text{ for all } n \gg 0 \text{ and all } j < i\}.$$

In the following theorem, we consider a polynomial ring and see whether a homogeneous ideal could be h-linked with $R_+$.

**Theorem 2.8.** Let $(R_0, \mathfrak{m}_0)$ be a regular local ring containing a field of characteristic zero and $R = R_0[x_1, \dots, x_t]$ be the polynomial ring graded in the usual way, that is $\deg(x_i) = 1$ for all $i = 1, \dots, t$. Then $R_+$ can't be h-linked with any ideal $\mathfrak{a} \supsetneq R_+$. Moreover, if $\mathfrak{a} \overset{h}{\sim}_{(I;R)} R_+$ and $\mathfrak{a} \nsubseteq R_+$, then $\mathfrak{a}$ and $R_+$ are geometrically h-linked by $I$ over $R$.

*Proof.* Let $\mathfrak{a} \overset{h}{\sim}_{(I;R)} R_+$ and suppose to the contrary that $R_+ \subsetneq \mathfrak{a}$. Since $t := \mathrm{grade}(R_+) \leqslant f - \mathrm{grade}(\mathfrak{a}, R_+, R)$, so $H^t_{\mathfrak{a}}(R)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$, by [7, 1.7]. Thus $H^t_{\mathfrak{a}}(R) = 0$, using [11, 8.1], that is a contradiction, in view of [4, 2.6(i)].
Now, assume that $\mathfrak{a} \overset{h}{\sim}_{(I;R)} R_+$ and $\mathfrak{a} \nsubseteq R_+$. By [7, 1.7] and [11, 8.1], $H^i_{\mathfrak{a}+R_+}(R) = 0$ for all $i \leq f-\mathrm{grade}(\mathfrak{a}+R_+, R_+, R)$. So, $f - \mathrm{grade}(\mathfrak{a} + R_+, R_+, R) \lneq \mathrm{grade}(\mathfrak{a} + R_+)$. Thus $\mathrm{grade}(R_+) \lneq \mathrm{grade}(\mathfrak{a} + R_+)$ and, by 2.2(i), $\mathfrak{a} \cap R_+ = I$. □

In the following theorem, we study the set $\mathrm{Ass}_{R_0}(M/\mathfrak{a}M)$ where $\mathfrak{a}$ is an h-linked ideal over $M$.

**Theorem 2.9.** Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are geometrically h-linked by $I$ over $M$ and $\mathfrak{b} \supseteq R_+$. Then

(i) $\mathrm{Ass}_{R_0}(M/\mathfrak{a}M) = \mathrm{Ass}_{R_0}(M/IM) \cap V(\mathfrak{a}_0)$;

(ii) $\mathrm{Ass}_{R_0}(M/\mathfrak{a}M) \cap \mathrm{Ass}_{R_0}(M/\mathfrak{b}M) = \emptyset$;

(iii) $\mathrm{Ass}_{R_0}(M/\mathfrak{b}M) \cap V(\mathfrak{a}_0) = \emptyset$.

The first case also holds if $\mathfrak{a}$ and $\mathfrak{b}$ are just h-linked and $\mathrm{Ass}_R(M/IM) = \mathrm{minAss}_R(M/IM)$ (e.g. $M$ is a Cohen-Macaulay module).

*Proof.* (i) By [4, 2.9] and [8, Exercise 6.7], $\mathrm{Ass}_{R_0}(M/\mathfrak{a}M) = \{\mathfrak{p} \cap R_0 \mid \mathfrak{p} \in \mathrm{Ass}_R(M/IM) \cap V(\mathfrak{a})\}$ and that $\mathrm{Ass}_{R_0}(M/IM) = \{\mathfrak{p} \cap R_0 \mid \mathfrak{p} \in \mathrm{Ass}_R(M/IM)\}$. This implies that $\mathrm{Ass}_{R_0}(M/\mathfrak{a}M) \subseteq \mathrm{Ass}_{R_0}(M/IM) \cap V(\mathfrak{a}_0)$. Now, let $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(M/IM) \cap V(\mathfrak{a}_0)$. Then, there exists $\mathfrak{p} \in \mathrm{Ass}_R(M/IM)$ such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$. $\sqrt{0 : M + I} = \sqrt{0 : M/IM} \subseteq \mathfrak{p}$. Thus, by [5, 2.2] and the assumption, $\sqrt{0 : M + \mathfrak{a} \cap R_+} \subseteq \mathfrak{p}$. So, $\mathfrak{p} \supseteq \mathfrak{a}$ and, again by [4, 2.9], $\mathfrak{p} \in \mathrm{Ass}_R(M/\mathfrak{a}M)$.

(ii) Let $\mathfrak{p}_0 \in \mathrm{Ass}_{R_0}(M/\mathfrak{b}M)$ then, by [4, 2.9], there exists $\mathfrak{p} \in \mathrm{Ass}_R(M/IM) \cap V(\mathfrak{b})$ such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$ and $\mathfrak{p} \notin \mathrm{Ass}_R(M/\mathfrak{a}M)$. So, $\mathfrak{p} \notin V(\mathfrak{a})$. On the other hand, $\mathfrak{p} \supseteq \mathfrak{b} \supseteq R_+$ thus $\mathfrak{p}_0 \not\supseteq \mathfrak{a}_0$ and by (i), $\mathfrak{p}_0 \notin \mathrm{Ass}_{R_0}(M/\mathfrak{a}M)$.

(iii) Follows from (i) and (ii).

□

If we remove the condition $\mathfrak{b} \supseteq R_+$, then the above proposition does not hold any more, as the following example shows.

**Example 2.10.** Let $\mathfrak{b}$ and $R_+$ be geometrically h-linked by $I$ over $R$, then $\mathrm{Ass}_{R_0}(R/R_+) \neq \mathrm{Ass}_{R_0}(R/I)$.

*Proof.* Since $I = \mathfrak{b} \cap R_+$, so $\mathfrak{b} \not\supseteq R_+$. Assume that $\mathrm{Ass}_{R_0}(R/R_+) = \mathrm{Ass}_{R_0}(R/I)$, then by 2.9(i), $\mathrm{Ass}_{R_0}(R/\mathfrak{b}) \subseteq \mathrm{Ass}_{R_0}(R/R_+)$ that is a contradiction, by 2.9(ii). □

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# Graded components of local cohomology modules with respect to h-linked ideals

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**Abstract**

Let $R = \oplus_{n\in\mathbb{N}_0} R_n$ be a standard graded ring, $M$ be a finitely generated graded $R$-module and $R_+ := \oplus_{n\in\mathbb{N}} R_n$ denotes the irrelevant ideal of $R$. In this paper, we study the graded components $H^i_{\mathfrak{a}}(M)_n$ where $\mathfrak{a}$ is a homogeneously linked ideal over $M$. More precisely, we consider tameness of $H^i_{\mathfrak{a}}(M)$ in some special cases.

## 1. Introduction

Throughout the paper, $R = \oplus_{n\in\mathbb{N}_0} R_n$ is a standard graded Noetherian ring, i.e. $R_0$ is a commutative Noetherian ring and $R$ is generated, as an $R_0$-algebra, by finitely many elements of degree one, $R_+ = \oplus_{n\in\mathbb{N}} R_n$ is the irrelevant ideal of $R$ and $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals of $R$. Also, $M$ denotes a finitely generated graded $R$-module.
For $i \in \mathbb{N}_0$, the set of non-negative integers, and $n \in \mathbb{Z}$, the set of integers, let $H^i_{\mathfrak{a}}(M)_n$ denotes the $n$-th component of graded local cohomology module $H^i_{\mathfrak{a}}(M)$ of $M$ with respect to $\mathfrak{a}$ (our terminology on local cohomology comes from [1]). It is well-known that $H^i_{R_+}(M)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$ and $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$ ([1, 16.1.5]). The asymptotic behavior of the components $H^i_{R_+}(M)_n$ when $n \to -\infty$ has been studied by many authors, too. But, we know not much about the graded components $H^i_{\mathfrak{a}}(M)_n$ where $\mathfrak{a}$ is an arbitrary homogeneous ideal of $R$, although, there are some studies in this topic.
In ([3]), the authors introduced the concept of linkage of ideals over a module, which is a generalization of its classical concept introduced by Peskine and Szpiro ([10]).
In this talk, we consider this concept in the graded case and study some of their cohomological properties. More precisely, we show that $H^i_{\mathfrak{a}}(M)$ is tame for some $i \in \mathbb{N}_0$ in the case where $\mathfrak{a}$ is a homogeneous linked ideal with $R_+$ over $M$.
We keep the notations introduced in the introduction, throughout the paper.

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## 2. Results

We start by homogeneously linked ideals, which is the basic concept of the paper.

**Definition 2.1.** Assume that $\mathfrak{a}M \neq M \neq \mathfrak{b}M$ and $I \subseteq \mathfrak{a} \cap \mathfrak{b}$ be an ideal generated by a homogeneous $M$-regular sequence. Then we say that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneously linked (or h-linked) by $I$ over $M$, denoted $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, if $\mathfrak{b}M = IM :_M \mathfrak{a}$ and $\mathfrak{a}M = IM :_M \mathfrak{b}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be geometrically h-linked by $I$ over $M$ if, in addition, $\mathfrak{a}M \cap \mathfrak{b}M = IM$. Also, we say that the ideal $\mathfrak{a}$ is h-linked over $M$ if there exist homogeneous ideals $\mathfrak{b}$ and $I$ of $R$ such that $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$. $\mathfrak{a}$ is h-$M$-selflinked by $I$ if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{a}$.

For a graded $R$-module $N = \oplus_{n\in\mathbb{Z}} N_n$, set

$$end(N) := sup\{n \in \mathbb{Z} |\ N_n \neq 0\}.$$

Note that $end(N)$ could be $\infty$ and that the supremum of the empty set is to be taken as $-\infty$.
The following lemma considers a case where $end(H^i_{\mathfrak{a}}(M)) < \infty$.

**Lemma 2.2.** Let $t \in \mathbb{N}_0$ and assume that $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \neq t$. Then for all $n \gg 0$ and all $i \in \mathbb{N}_0$,

$$H^i_{\mathfrak{a}+\mathfrak{b}}(M)_n \cong \begin{cases} H^{i-t}_{\mathfrak{b}}(H^t_{\mathfrak{a}}(M))_n, & i \geq t \\ 0 & i \lneq t. \end{cases}$$

*Proof.* We have the following convergence of spectral sequences, by [12, 11.38],

$$(E_2^{i,j})_n = H^i_{\mathfrak{b}}(H^j_{\mathfrak{a}}(M))_n \overset{i}{\Rightarrow} H^{i+j}_{\mathfrak{a}+\mathfrak{b}}(M)_n.$$

Since $end(H^j_{\mathfrak{a}}(M)) < \infty$ for all $j \neq t$, $H^j_{\mathfrak{a}}(M)$ is $R_+$-torsion for all $j \neq t$. So, by [1, 2.1.9],

$$H^i_{\mathfrak{b}}(H^j_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{b}+R_+}(H^j_{\mathfrak{a}}(M)) \cong H^i_{\mathfrak{b}_0 R}(H^j_{\mathfrak{a}}(M)) \qquad \text{for all } j \neq t \text{ and all } i \geq 0,$$

where $\mathfrak{b}_0 := \mathfrak{b} \cap R_0$. Hence, by [1, 14.1.12] and the assumption, $(E_2^{i,j})_n = H^i_{\mathfrak{b}_0}(H^j_{\mathfrak{a}}(M)_n) = 0$ for all $j \neq t$ and all $n \gg 0$. As a result, $H^i_{\mathfrak{a}+\mathfrak{b}}(M)_n \cong (E_2^{i-t,t})_n$ for all $i \geq t$ and that $H^i_{\mathfrak{a}+\mathfrak{b}}(M)_n = 0$ for all $i \lneq t$, when $n \gg 0$. □

The following corollary, which is immediate by the above lemma, generalizes [6, 1.1].

**Corollary 2.3.** Let $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \in \mathbb{N}_0$. Then for any homogeneous ideal $\mathfrak{b} \supseteq \mathfrak{a}$, $end(H^i_{\mathfrak{b}}(M)) < \infty$ for all $i \in \mathbb{N}_0$.

**Lemma 2.4.** Let $\mathfrak{a}$ be linked by $I$ over $M$. Then $\operatorname{Supp} H^t_{\mathfrak{a}}(M) = \operatorname{Supp} M/\mathfrak{a}M$, where $t := \operatorname{grade}(\mathfrak{a}, M)$.

*Proof.* By [3, 2.8], [2, Theorem 1] and [13, 1.4],

$$\operatorname{Ass} M/\mathfrak{a}M \subseteq \operatorname{Ass} M/IM \cap V(\mathfrak{a}) = \operatorname{Ass} Hom_R(R/\mathfrak{a}, M/IM) = \operatorname{Ass} Ext^t_R(R/\mathfrak{a}, M) = \operatorname{Ass} H^t_{\mathfrak{a}}(M).$$

On the other hand, $\operatorname{Supp} H^t_{\mathfrak{a}}(M) \subseteq \operatorname{Supp} M/\mathfrak{a}M$, which proves the claim. □

In the following, we show some equivalent conditions for $end(H^i_{\mathfrak{a}}(M)) < \infty$, where $\mathfrak{a}$ is an h-linked ideal over $M$.

**Proposition 2.5.** Let $\mathfrak{a}$ be an h-linked ideal by $I$ over $M$ with $\operatorname{grade}(\mathfrak{a}, M) = t$. Then the following statements are equivalent.

(i) $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \in \mathbb{N}_0$,

(ii) $end(H^t_{\mathfrak{a}}(M)) < \infty$,

(iii) $\operatorname{Supp} M/\mathfrak{a}M \subseteq V(R_+)$.

Also, if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$ and one of the above conditions holds, then

$$H^i_{\mathfrak{b}}(M)_n \cong \begin{cases} 0 & i \neq t \\ H^t_I(M)_n & i = t, \end{cases}$$

for all $i$ and all $n \gg 0$.

*Proof.* "$(ii) \Rightarrow (iii)$" Since $end(H^t_{\mathfrak{a}}(M)) < \infty$, $H^t_{\mathfrak{a}}(M)$ is $R_+$-torsion and Ass $H^t_{\mathfrak{a}}(M) \subseteq V(R_+)$. So, the result follows from 2.4.
"$(iii) \Rightarrow (i)$" Since $\sqrt{\mathfrak{a} + 0 : M} \supseteq \sqrt{R_+}$, using [1, 2.1.9], [1, 16.1.5(ii)] and 2.3, the statement holds.
The last statement follows from [4, 2.2(i)], 2.3 and the following homogeneous Mayer-Vietoris sequence

$$\ldots \longrightarrow H^i_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{b}}(M) \longrightarrow H^i_I(M) \longrightarrow H^{i+1}_{\mathfrak{a}+\mathfrak{b}}(M) \longrightarrow \ldots .$$

□

Note that if $\mathfrak{b} \subseteq R_+$ and one of the above conditions holds, then $\mathfrak{a}$ can't be geometrically h-linked with $\mathfrak{b}$. Otherwise, by 2.5(iii), $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}+R_+}(M)$ for all $i$, so grade$(\mathfrak{a}, M) =$ grade$(\mathfrak{a} + \mathfrak{b}, M)$, that is a contradiction.

**Definition 2.6.**

- We say that the ideal $I$ is generated by an $M$-regular sequence under radical if there exists an $M$-regular sequence $\underline{x} = x_1, \ldots, x_t$ such that $\sqrt{I + 0 : M} = \sqrt{\underline{x} + 0 : M}$.
- $M$ is called relative Cohen-Macaulay with respect to $\mathfrak{a}$ of degree $n$ if $H^i_{\mathfrak{a}}(M) = 0$ for all $i \neq n$.

**Corollary 2.7.** If $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$ and $t := \text{grade}(\mathfrak{a}, M)$, then the following statements are equivalent.

(i) $\max\{end(H^t_{\mathfrak{a}}(M)), end(H^t_{\mathfrak{b}}(M))\} < \infty$,

(ii) Supp $M/IM \subseteq V(R_+)$.

In particular, if $\mathfrak{b} = R_+$ and $end(H^t_{\mathfrak{a}}(M)) < \infty$, then $\mathfrak{a}$ is radically h-$M$-licci with $R_+$ of length 1.

*Proof.* The results follow from [3, 2.6(iii)], 2.5 and the fact that $end(H^i_{R_+}(M)) < \infty$ for all $i \in \mathbb{N}_0$. □

The above corollary shows, if $\mathfrak{a} \overset{h}{\sim}_{(I;M)} \mathfrak{b}$, then $end(H^i_{\mathfrak{a}}(M))$ or $end(H^i_{\mathfrak{b}}(M))$ is infinite for some $i \in \mathbb{N}_0$ if and only if Supp $M/IM \nsubseteq V(R_+)$.

**Theorem 2.8.** If $\mathfrak{a} \overset{h}{\sim}_{(I;M)} R_+$ and $t := \text{grade}(R_+, M)$, then $end(H^i_{\mathfrak{a}}(M)) < \infty$ for all $i \neq t$ and $end(H^t_{\mathfrak{a}}(M)) < \infty$ or $H^t_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.
In a special case, $H^t_{\mathfrak{a}}(M)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$.

*Proof.* The case $i \neq t$ follows from 2.5 and [1, 16.1.5(ii)]. Also, we have $H^t_{\mathfrak{a}}(M)_n \cong H^t_I(M)_n$ for all $n \gg 0$. We consider two cases:

case 1: Let Supp$(M/IM) \nsubseteq V(R_+)$. By 2.7 and [1, 16.1.5(ii)], $end(H^t_{\mathfrak{a}}(M)) = \infty$. Now, we prove, by induction on $t$, that $H^t_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.
If $t = 0$, then $\Gamma_{\mathfrak{a}}(M)_n = \Gamma_{\underline{0}}(M)_n = M_n$ for all $n \gg 0$. On the other hand, by [7, Theorem 1], $R_1 M_n = M_{n+1}$ for all $n \gg 0$, thus $\Gamma_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.
Let $t > 0$ and assume, inductively, that the claim holds for $t - 1$. Let $I = (x_1, x_2, \ldots, x_t)$ and $deg(x_1) = l$. Now, the homogeneous exact sequence

$$0 \longrightarrow M \xrightarrow{.x_1} M(l) \longrightarrow (M/x_1M)(l) \longrightarrow 0$$

and [3, 2.6(i)] yield the following exact sequence of $R_0$-modules for all $n \in \mathbb{Z}$,

$$0 \longrightarrow H^{t-1}_{\mathfrak{a}}(M/x_1M)_{n+l} \longrightarrow H^t_{\mathfrak{a}}(M)_n \xrightarrow{.x_1} H^t_{\mathfrak{a}}(M)_{n+l} \longrightarrow \dots .$$

Since $x_1 \in I$, $\mathfrak{a}/(x_1) \overset{h}{\sim}_{(I/(x_1);M/x_1M)} R_+/(x_1)$ and, by the inductive hypothesis, $H^{t-1}_{\frac{\mathfrak{a}}{(x_1)}}(M/x_1M)_n \neq 0$ for all $n \gg 0$. Hence, $H^t_{\mathfrak{a}}(M)_n \neq 0$ for all $n \gg 0$.

case 2: Now, assume that $\mathrm{Supp}(M/IM) \subseteq V(R_+)$. Then $\sqrt{0 : M + I} = \sqrt{0 : M + R_+}$ and $H^i_I(M) \cong H^i_{R_+}(M)$ for all $i$. So, by [1, 16.1.5], we have

1. $end(H^t_I(M)) < \infty$,
2. $H^t_I(M)_n$ is a finitely generated $R_0$-module for all $n$,
3. $M$ is relative Cohen-Macaulay with respect to $R_+$ of degree $t$.

Therefore, in view of 2.5, $end(H^t_{\mathfrak{a}}(M)) < \infty$. Also, using [9, 3.4], there are homogeneous isomorphisms

$$H^i_{\mathfrak{a}}(M) \cong H^{i-t}_{\mathfrak{a}}(H^t_I(M)) \cong H^{i-t}_{\mathfrak{a}+R_+}(H^t_I(M)) \cong H^i_{\mathfrak{a}+R_+}(M) \qquad \text{for all } i \geq t. \tag{3.1}$$

Using [3, 2.6], $t \leq$ f-grade$(\mathfrak{a}_0 + R_+, R_+, M)$. Therefore, $H^t_{\mathfrak{a}_0+R_+}(M)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$, by [6, 1.7]. As a result, by (3.1), $H^t_{\mathfrak{a}}(M)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$.

□

As wee have seen in the proof of 2.8, if $R_+$ is h-linked by $I$ over $M$ and $\mathrm{Supp}(M/IM) \subseteq V(R_+)$ then $M$ is relative Cohen-Macaulay with respect to $R_+$. However, the converse does not hold any more, as the following example shows. Although, it does in some special cases, see Proposition 2.10.

**Example 2.9.** Assume that $(R_0, \mathfrak{m}_0)$ is a domain and $\dim R_0 = 2$. Set $R = R_0[x]$. So, there exists a non zero prime ideal $\mathfrak{p}_0$ of $R_0$ such that $\mathfrak{p}_0 \subsetneq \mathfrak{m}_0$. By **??**, for any $0 \neq r_0 \in \mathfrak{p}_0$, $(r_0) \overset{h}{\sim}_{((r_0x);R)} (x)$ and $\mathrm{Supp}(R/(r_0x)) = V(r_0) \cup V(x) \nsubseteq V(x)$, while $R$ is relative Cohen-Macaulay with respect to $(x)$ of degree 1.

The following proposition considers a case where the irrelevant ideal can be generated by an $M$-regular sequence under radical.

**Proposition 2.10.** Let $(R_0, \mathfrak{m}_0)$ be local and $M$ be relative Cohen-Macaulay with respect to $R_+$ of degree $t$. Then there exists a maximal homogeneous $M$-regular sequence $I$ in $R_+$ such that $\mathrm{Supp}(M/IM) \subseteq V(R_+)$. In other words, $R_+$ can be generated by a homogeneous $M$-regular sequence under radical.

*Proof.* Assume that $t = 0$, so $\dim M/\mathfrak{m}_0M = 0$. Therefore, $M/\mathfrak{m}_0M$ is Artinian and $end(M/\mathfrak{m}_0M) < \infty$, using [7, Theorem 1]. Hence, by Nakayama Lemma, $end(M) < \infty$. This implies that $M$ is $R_+$-torsion and that $\mathrm{Supp}\, M \subseteq V(R_+)$.
Now, let $t > 0$ and assume inductively that the statement holds for $t - 1$. As $t > 0$, $R_+ \nsubseteq (\bigcup_{\mathfrak{p} \in MinAss(M/\mathfrak{m}_0M)} \mathfrak{p}) \cup Z(M)$, where $Z(M)$ denotes the set of zero divisors on $M$. So, by [1, 16.1.2], there exists a homogeneous element

$$x \in R_+ \backslash \Big( \bigcup_{\mathfrak{p} \in MinAss(M/\mathfrak{m}_0M)} \mathfrak{p} \Big) \cup Z(M).$$

Therefore, $\dim \frac{M/xM}{\mathfrak{m}_0(M/xM)} = t - 1 = \mathrm{grade}(R_+, M/xM)$. In other words, $M/xM$ is relative Cohen-Macaulay with respect to $R_+$ of degree $t - 1$ and, by the induction hypothesis, there is a maximal homogeneous $M/xM$-regular sequence $I'$ in $R_+$ such that $\mathrm{Supp}\, \frac{M}{(I' + <x>)M} = \mathrm{Supp}\, \frac{M/xM}{I'(M/xM)} \subseteq V(R_+)$. Now, the result follows by induction. □

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# Prediction and simulation of stock returns using Heston model, GRU neural network and their combination using PSO

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**Abstract**

Stock return prediction is of fundamental significance to risk management, investment decision, and trading strategy optimization, for improved accuracy of prediction can lead to reduced losses and increased gains. In the Article, apple's stock price return using the Heston model and a GRU neural network was predicted in this paper, and to further improve the accuracy, the PSO algorithm was applied to optimally combine these two models. The performance showed that the PSO-based hybrid model was better with an MSE of 0.000148, while the Heston model was 0.000428 and the GRU model was 0.000170. The RMSE and MAE of the PSO model were 0.0122 and 0.0082, respectively, showing improvement over the individual models. These findings demonstrate that the use of financial models and deep learning combined with optimization techniques can be utilized to enhance stock return prediction accuracy.

## 1. Introduction

Stock return prediction is one of the fundamental challenges in the field of finance and investment, with a direct impact on investor decision-making, risk management, and portfolio optimization [1]. Higher accuracy in predicting returns can lead to reduced losses, increased profits, and improved trading strategies [2].
One of the conventional methods for predicting stock returns is the use of stochastic models that simulate the dynamic behavior of stock prices based on mathematical processes. The Heston model is one of these approaches, which considers price volatility as a time-dependent random variable and is particularly suitable for modeling assets with high volatility changes [3]. Alongside these stochastic process-based methods, recurrent neural networks like GRU (Gated Recurrent Unit) are also highly effective for forecasting financial time series [4].
These models can identify complex patterns in historical data and predict future trends based on past information. While each of these models can independently predict stock returns, their combination can lead to higher accuracy in financial predictions.With the advancement of mathematical methods and deep learning algorithms, hybrid models have gained attention for more accurate prediction of financial market behavior [5].

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Two methodologies were used in this research to forecast Apple (AAPL) stock returns. Firstly, the Heston model, which has the ability to model the volatility of stock prices, was employed. The model is driven by stochastic processes and considers volatility as a dynamic variable that changes over time. Secondly, the GRU recurrent neural network, which is adept at capturing temporal patterns, was employed to handle historical stock data. Both these models produce adequate results independently, but combined they have the capacity to bring in higher precision. For blending these two models and raising prediction accuracy, a Particle Swarm Optimization (PSO) algorithm has been utilized. The algorithm adjusts the predictive models' coefficients and provides a balanced composition of Heston and GRU outcomes. The results indicate that the PSO-based hybrid model performs better with lower prediction errors (MSE, RMSE, and MAE) compared to individual models.
There are four major sections in the paper's structure. Section 2, the Methodology, presents the stochastic modeling techniques and deep learning networks with a focus on the usage of the Heston model for predicting the stock's return and GRU (Gated Recurrent Unit) network. Section 3 presents the quantitative findings, i.e., comparing the individual models (Heston and GRU) and the hybrid model optimized through Particle Swarm Optimization (PSO), on the basis of performance measures MSE, RMSE, and MAE. Finally, Section 4 gives the conclusion, summarizing the findings and the improvement in predictive accuracy achieved through the PSO-based hybrid model.

## 2. Research methodology

In this study, we employ a hybrid approach combining stochastic modeling and deep learning for stock return prediction. The Heston model is used to capture market volatility dynamics, while the GRU neural network learns complex temporal patterns in stock price data.

### *2.1. Heston model*

The Heston model is a stochastic volatility model that describes the evolution of stock prices by incorporating a mean-reverting stochastic process for volatility. This model allows for the volatility of the asset to change over time, capturing the observed market behavior of volatility clustering and sudden shifts. It is defined by the following system of stochastic differential equations (SDEs) [7]

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v$$

In this model, $S_t$ represents the stock price, and $v_t$ denotes the variance of the asset price. The Wiener processes $W_t^S$ and $W_t^v$ drive the stochastic dynamics with correlation coefficient $\rho$.
To estimate the parameters $\kappa, \theta, \sigma, \rho$, and the initial variance $v_0$, we employ the **Maximum Likelihood Estimation (MLE)** method. This approach maximizes the likelihood function, which is derived from the probability distribution of the data given the parameters. By finding the parameter values that best fit the observed data, the MLE method provides the most efficient and unbiased estimates for the model parameters [6].
We use the following final log-likelihood function to estimate the parameters:

$$\mathcal{L}(\mu, \kappa, \theta, \sigma, \rho, v_0) = \sum_{t=1}^{n} \left[ -\frac{1}{2} \ln(2\pi |\Sigma_{t+1}|) - \frac{1}{2} z_{t+1}^{\top} \Sigma_{t+1}^{-1} z_{t+1} \right].$$

The covariance matrix $\Sigma_{t+1}$ at time $t+1$ captures the variability and correlations of the process, while $z_{t+1}$ represents the difference between the observed and predicted values for $x_{t+1}$ and $v_{t+1}$.

### *2.2. Gated recurrent unit*

Artificial Neural Networks (ANNs) have revolutionized time series forecasting by providing powerful tools for capturing complex, nonlinear patterns in financial data [8].
Unlike traditional statistical models, which often rely on linear assumptions, neural networks can automatically learn intricate relationships from historical data without requiring explicit feature engineering. Recurrent architectures, such as Long Short-Term Memory (LSTM) and Gated Recurrent Unit (GRU), are particularly effective for sequential data, as they retain past information and adapt dynamically to changing market conditions [9]. These models are widely used in financial applications, including stock price prediction, risk management, and algorithmic trading, due to their ability to analyze vast datasets and uncover hidden patterns that influence asset prices.
Recurrent Neural Networks (RNNs) are widely used for time series forecasting due to their ability to capture temporal dependencies in sequential data. However, traditional RNNs suffer from vanishing gradient problems, making them inefficient for long-term dependencies. To address this issue, **Gated Recurrent Unit (GRU)** networks have been introduced as a more efficient alternative to standard RNNs.
GRU is a type of recurrent neural network that incorporates gating mechanisms to regulate the flow of information through the network. It consists of two main gates [10]

1. **Reset Gate** $(r_t)$ – Determines how much past information to discard.
2. **Update Gate** $(z_t)$ – Controls the amount of past information to carry forward to the future.

The mathematical formulation of GRU is as follows:

$$r_t = \sigma(W_r \cdot [h_{t-1}, x_t] + b_r) \tag{1}$$
$$z_t = \sigma(W_z \cdot [h_{t-1}, x_t] + b_z) \tag{2}$$
$$\tilde{h}_t = \tanh(W_h \cdot [r_t \odot h_{t-1}, x_t] + b_h) \tag{3}$$
$$h_t = (1 - z_t) \odot h_{t-1} + z_t \odot \tilde{h}_t \tag{4}$$

where:

- $x_t$ is the input at time $t$.
- $h_t$ is the hidden state at time $t$.
- $W_r, W_z, W_h$ are weight matrices, and $b_r, b_z, b_h$ are bias terms.
- $\sigma$ is the sigmoid activation function, and tanh is the hyperbolic tangent activation function.
- $\odot$ represents element-wise multiplication.

### *2.3. Particle Swarm Optimization*

Particle Swarm Optimization (PSO) is a population-based metaheuristic algorithm inspired by the coordinated movement of bird flocks and fish schools [11]. It is widely employed for solving complex optimization problems, including parameter tuning in machine learning models. The algorithm begins by initializing a swarm of particles, where each particle represents a potential solution within the search space. Throughout the iterations, particles update their positions based on their own best-known positions and the best position identified by the entire swarm. This movement follows these update equations:

$$v_i^{t+1} = \omega v_i^t + c_1 r_1 (pbest_i - x_i^t) + c_2 r_2 (gbest - x_i^t)$$

$$x_i^{t+1} = x_i^t + v_i^{t+1}$$

Here, $x_i^t$ denotes the position of particle $i$ at iteration $t$, while $v_i^t$ represents its velocity, dictating movement within the search space. The inertia weight $\omega$ balances global exploration and local exploitation, influencing convergence. The

acceleration coefficients $c_1$ and $c_2$ determine the influence of a particle's own experience and the swarm's collective knowledge, respectively. Random values $r_1$ and $r_2$, drawn from a uniform distribution between 0 and 1, introduce stochasticity into the update process. Each particle tracks its best position, $pbest_i$, representing its most optimal solution so far, while $gbest$ corresponds to the best position found by the swarm, steering all particles toward improved solutions.
In this study, PSO is utilized to optimize the integration of the Heston model and GRU-based predictions by fine-tuning the weight distribution between the two models. The goal is to determine the optimal combination that minimizes prediction error. Each particle represents a set of parameters for model fusion, updating their positions and velocities according to the given equations.

## 3. Numerical results

This section presents numerical results obtained by using the Heston model, the GRU network, and their optimized PSO combination to forecast Apple (AAPL) stock returns. The performance of each model is evaluated using error metrics such as MSE, RMSE, and MAE to compare their forecasting accuracy. The results demonstrate the effectiveness of the combined approach in reducing forecasting errors and improving the overall forecasting performance.
In this study, we analyze stock return data for Apple Inc. (AAPL) to evaluate the performance of different predictive models. The dataset consists of historical price information, from which log returns are computed to model the stock's volatility dynamics. Given the inherent stochastic nature of financial markets, accurately predicting AAPL's returns is essential for risk management and investment decision-making 1.

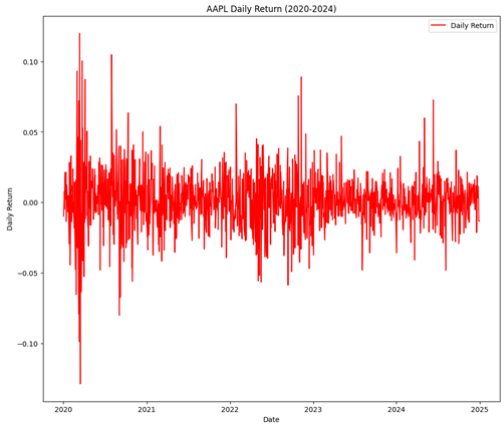


Fig. 1. Apple Daily Return (2020-2024).

### *3.1. Heston model results*

The optimized parameters for the Heston model, obtained using the Maximum Likelihood Estimation (MLE) method, ensure an optimal fit to the stock return data . These parameters help capture the stochastic volatility dynamics of the stock, making the model suitable for financial market analysis.

Table 1. optimized parameters for the Heston model

| $\kappa$ | $\theta$ | $\sigma$ | $v_0$ | $\rho$ |
|---|---|---|---|---|
| 0.09909 | 0.020020 | 0.10188 | 0.0200 | 0.506 |

The model's performance was evaluated using standard error metrics, including Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and Mean Absolute Error (MAE) 2. The results indicate that while the Heston model effectively captures volatility patterns, there is still room for improvement in predictive accuracy2.

Table 2. optimized parameters for the Heston model

| MSE | RMSE | MSE |
|---|---|---|
| 0.0004 | 0.02 | 0.016 |

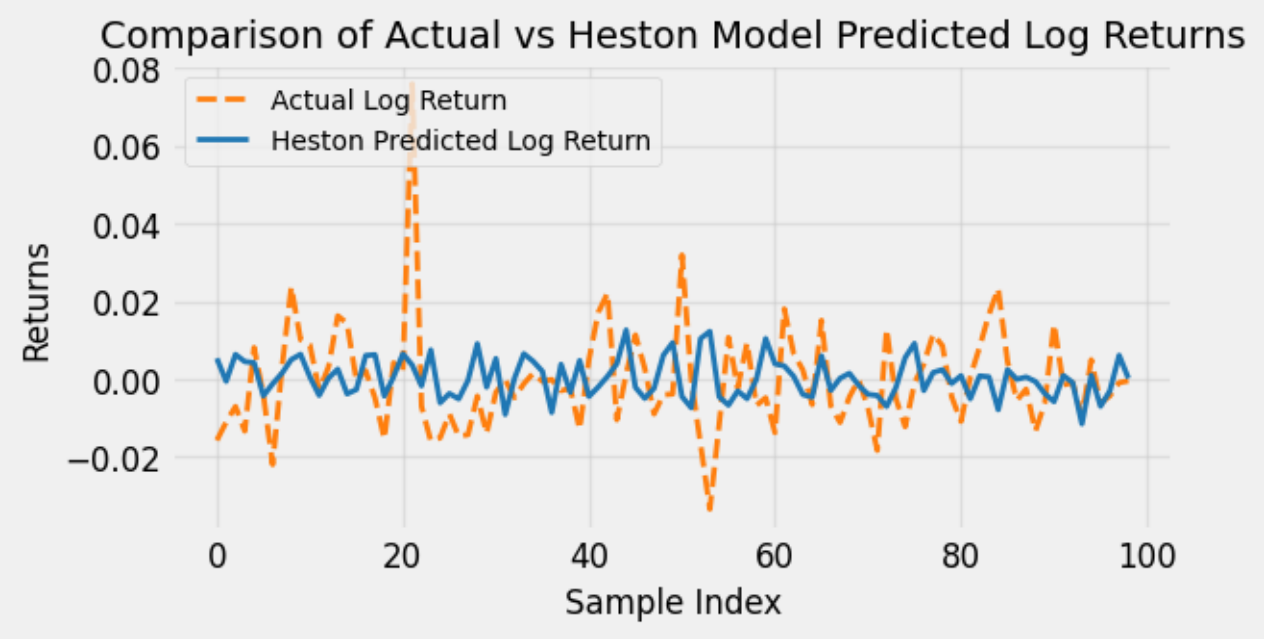


Fig. 2. Predicted by Heston vs Actual AAPL Daily Return (Test Data)

### *3.2. Gated recurrent unit results*

The GRU-based model in this study is designed to predict time-series data with high accuracy. The dataset is split into training (90%) and testing (10%) sets to ensure proper evaluation of the model's performance. Before training, the input data is reshaped into a three-dimensional format to meet the requirements of recurrent neural networks, where the dimensions correspond to the number of samples, time steps, and features (channels). The model architecture begins with a GRU layer containing 64 units and utilizes the ReLU activation function, enabling it to capture temporal dependencies within the sequence data effectively. This is followed by a Dense layer with a number of neurons equal to the prediction step and employing the SELU activation function to generate the final outputs. The model is compiled with mean squared error (MSE) as the loss function and the Adam optimizer, ensuring efficient training. After 100 epochs with a batch size of 100, the model achieved promising performance metrics:

Table 3. optimized parameters for the Heston model

| MSE | RMSE | MSE |
|---|---|---|
| 0.00017 | 0.013 | 0.0089 |

### *3.3. PSO results*

The PSO algorithm in this study is structured to optimize the integration of the Heston model with GRU-based predictions by fine-tuning the weight coefficients assigned to each model. The process begins by initializing a swarm of candidate solutions, where each particle represents a potential set of weights. During each iteration, particles adjust their positions in the search space based on their own best-known positions and the best-known position of the entire

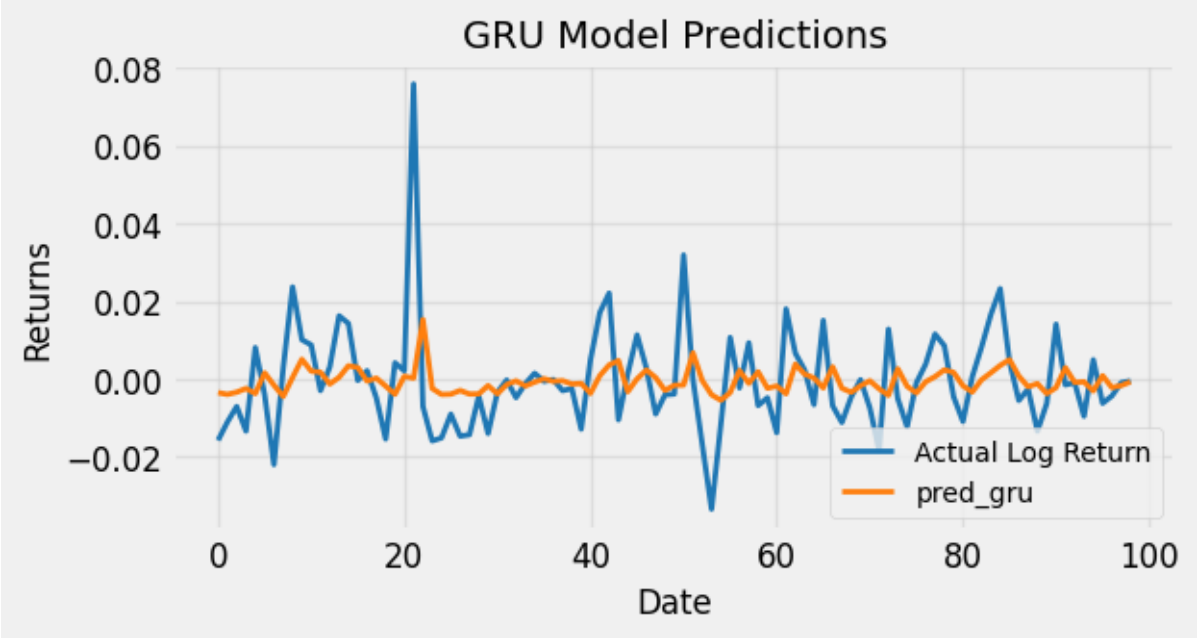


Fig. 3. Predicted by gru vs Actual AAPL Daily Return (Test Data)

swarm, following the established update rules. This iterative, collaborative search strategy allows the algorithm to explore and exploit the solution space effectively, converging on an optimal set of parameters that minimizes prediction error. The final optimized results achieved by PSO :

Table 4. optimized parameters for the Heston model

| MSE | RMSE | MSE |
|---|---|---|
| 0.00014 | 0.012 | 0.0082 |

## 4. Conclusion

The Particle Swarm Optimization (PSO) algorithm effectively optimized the integration of the Heston model with GRU-based predictions, significantly improving the model's capability to minimize prediction errors. By adjusting the weight coefficients assigned to each component, PSO facilitated a more precise and reliable forecasting process. The results highlighted a substantial reduction in error metrics, achieving a final MSE of 0.00014, an RMSE of 0.0122, and an MAE of 0.0082, underscoring the success of the optimization approach.
From an economic perspective, these enhanced prediction outcomes are of considerable importance. Accurate forecasting plays a crucial role in mitigating risks and uncertainties, thereby fostering more informed decision-making within economic and financial frameworks. By combining advanced machine learning techniques, such as GRU, with traditional financial models and optimizing them through PSO, this study offers a robust methodology for assessing market conditions. The improved predictive accuracy has the potential to support superior investment strategies, more effective risk management, and informed economic policy development.

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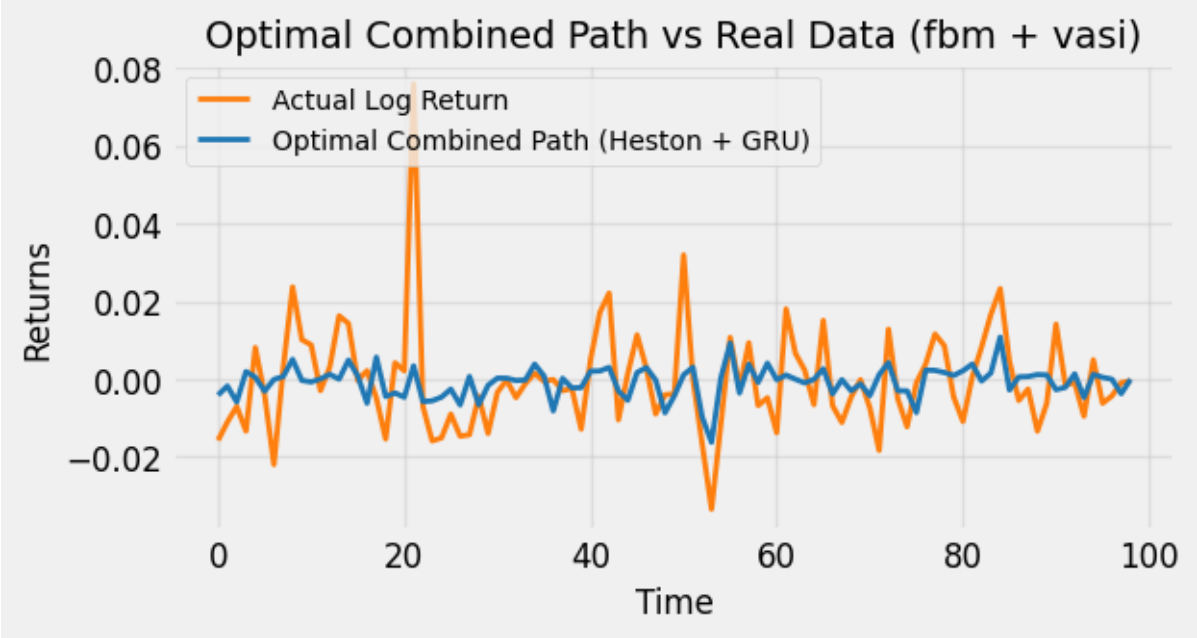


Fig. 4. Predicted by pso vs Actual AAPL Daily Return (Test Data)

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# Enhancing Stock Price Forecasting by Combining LightGBM and AdaBoost Models with a Genetic Algorithm

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**Abstract**

In this research, we propose a novel hybrid ensemble model that integrates LightGBM (LGBM) and AdaBoost, optimized using a genetic algorithm, to enhance stock price prediction accuracy. While both LGBM and AdaBoost exhibit strong individual performance, their combination through evolutionary optimization significantly improves predictive efficiency. We apply this model to Tesla stock price data and evaluate its effectiveness using standard error metrics. The results demonstrate a remarkable improvement, with the genetic algorithm reducing MAE from 9.91 (LGBM) and 8.40 (AdaBoost) to 1.23 and MAPE from over 3

## 1. Introduction

Stock price movement is affected by the broad set of macroeconomic reports, market sentiment, geopolitical strife, and company-specific news[1]. Financial time series are extremely noisy and volatile, making the problem of constructing dependable forecasting models arduous. Classical approaches suffer from enforcing linearity of relations and time series stationarity, which makes them less adaptive with respect to rapidly changing conditions on the market[2]. However, machine learning models are capable of learning from huge data sets, detecting hidden trends, and adapting to new market conditions, and thus emerge as a strong choice for stock price prediction. [3] Stock price prediction is a critical problem in the financial markets, and accurate prediction can assist investors, analysts, and financial organizations in making sound decisions and mitigating risks[7]. Conventional statistical approaches, including autoregressive integrated moving average (ARIMA) and generalized autoregressive conditional heteroskedasticity (GARCH), have been extensively employed for time series prediction. These approaches are likely to fail to capture the complicated, nonlinear relationships in the stock market data[8]. Hence, machine learning algorithms have received growing interest as they are able to capture intricate relationships and enhance predictive performance.
Recent studies have explored various hybrid and ensemble models to enhance stock price forecasting accuracy. Tian et al. proposed an LSTM-based hybrid model optimized with Bayesian techniques and combined with LightGBM,

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demonstrating superior predictive performance across multiple stock datasets[5]. Yang et al. investigated the integration of XGBoost and LightGBM, highlighting the effectiveness of feature engineering in refining predictions on large datasets[4]. Zhang et al. introduced a novel status box method combined with AdaBoost, a genetic algorithm, and a probabilistic support vector machine to classify stock trends effectively. These approaches underscore the growing role of ensemble learning and hybrid models in improving financial time series forecasting[6].
Ensemble learning techniques like LightGBM (LGBM) and AdaBoost have been effective in the prediction of stock prices. The models utilize decision trees and boosting to effectively learn from the historical data[9]. Nevertheless, the optimal model combination and hyperparameter tuning are still of utmost importance in best performance. Evolutionary algorithms like genetic algorithms (GA) have offered an effective optimization technique through emulating natural selection to optimize the model parameters. By incorporating ensemble models with a genetic algorithm, it is possible to make the predictions more accurate and the prediction system more adaptive.
This research suggests a hybrid ensemble method of LGBM and AdaBoost, optimized using a genetic algorithm, for predicting stock prices. The process entails training the models individually, merging their predictions using an optimization technique, and optimizing the ensemble using GA for better performance. By exploiting the strengths of each element, the suggested methodology seeks to create a strong and efficient framework for financial time series prediction. The remainder of this paper is organized as follows: Section 2 describes the methodology of this study, covering data preprocessing, models architecture, and the application of the genetic algorithm. Section 3 shows the numerical results and the discussion and conclusions are presented in the last section.

## 2. Research methodology

In this study, we propose a hybrid ensemble learning approach for stock price prediction by integrating LightGBM and AdaBoost, optimized using a genetic algorithm. LightGBM, a gradient boosting framework, efficiently handles large datasets and captures complex patterns, while AdaBoost enhances weak learners to improve predictive accuracy. To further refine the model's performance, a genetic algorithm is employed for optimal hyperparameter selection and model combination. This evolutionary optimization ensures a robust and adaptive framework, improving the overall predictive capability in financial time series forecasting.

### *2.1. lightgbm model*

The core formula of LightGBM builds upon the general structure of gradient boosting algorithms but introduces key differences in how trees are constructed and how splits are handled. The objective function in LightGBM consists of two main components: the **loss function** and the **regularization term** [11].
The objective function is expressed as:

$$\mathcal{L} = \sum_{i=1}^{N} \mathcal{L}_i(f(x_i)) + \sum_{k=1}^{K} \Omega(f_k)$$

In LightGBM, the objective function is composed of two main terms: the loss function and the regularization term. The term $N$ represents the total number of data points in the training dataset, with each data point indexed as $i$. The feature vector for the $i$-th sample is denoted as $x_i$, while $y_i$ represents the actual target value corresponding to that sample. The predicted output for the $i$-th data point is given by $f(x_i)$, and the loss function $\mathcal{L}_i(f(x_i))$ quantifies the difference between the predicted value and the actual target value for that data point. The model learns functions $f_k$ for each of the $k$-th trees, and the term $\Omega(f_k)$ represents the regularization term associated with the complexity of each tree, which helps to prevent overfitting by penalizing excessively complex models.
**Loss Function:** The loss function $\mathcal{L}_i(f(x_i))$ is problem-dependent. In the case of regression tasks, **mean squared error (MSE)** is often used, defined as:

$$\mathcal{L}_i(f(x_i)) = (y_i - f(x_i))^2$$

For classification tasks, a **logistic loss** function is typically employed.
**Regularization Term:** The regularization term $\Omega(f_k)$ serves to prevent overfitting by discouraging excessive complexity in the individual trees. It is typically expressed as:

$$\Omega(f_k) = \gamma T + \frac{\lambda}{2}\|w_k\|^2$$

In LightGBM, the regularization term $\Omega(f_k)$ plays a crucial role in controlling the complexity of the decision trees. The parameter $T$ refers to the number of leaves in the $k$-th tree, which is an important factor in determining the tree's complexity. Additionally, $\|w_k\|$ represents the weight vector of the $k$-th tree, capturing the significance of the tree's features. To prevent overfitting, regularization parameters $\gamma$ and $\lambda$ are introduced, where $\gamma$ penalizes the number of leaves in the tree, and $\lambda$ helps to control the magnitude of the weights. Together, these components ensure that the model does not become too complex and is better generalized to unseen data.
After $K$ boosting rounds, the final model prediction $f(x_i)$ is computed as the sum of the predictions from each tree:

$$f(x_i) = \sum_{k=1}^{K} f_k(x_i)$$

Where each tree $f_k(x_i)$ contributes to the final prediction, with its influence modulated by the learning rate and the overall optimization process.
Through minimizing the loss function, applying regularization to control complexity, and using boosting to iteratively enhance the model, LightGBM effectively captures complex relationships in the data, making it a powerful method for predictive modeling in various tasks.

### *2.2. Adaboost Algorithm*

AdaBoost, short for Adaptive Boosting, is an ensemble learning method that combines multiple weak learners to form a strong learner. It works by iteratively training a series of weak models, typically decision trees, and combining them to improve prediction accuracy. AdaBoost assigns higher weights to misclassified instances, emphasizing the importance of correctly classifying difficult-to-predict samples in subsequent iterations[12].
The core idea behind AdaBoost is to update the model's predictions by considering the errors made by previous models. The algorithm aims to minimize the weighted error through an additive model, adjusting the model weights after each iteration. The final model is a weighted combination of all the weak models, where the contribution of each model depends on its performance during training.
Mathematically, AdaBoost can be formulated as follows:

$$F(x) = \sum_{t=1}^{T} \alpha_t h_t(x)$$

In AdaBoost, the final prediction of the model, $F(x)$, is obtained after $T$ iterations by aggregating the outputs of individual weak learners. Specifically, at each iteration $t$, a weak model produces a prediction $h_t(x)$, and its contribution to the final prediction is weighted by $\alpha_t$, a parameter determined by the model's accuracy on the training data. More accurate weak models receive higher weights, ensuring that the final prediction $F(x)$ is a robust, weighted sum of all the weak learners' predictions.
The weight $\alpha_t$ for each weak model is computed based on the model's performance, with more accurate models receiving a higher weight. This iterative adjustment ensures that AdaBoost focuses on the most challenging instances for prediction, thereby enhancing the overall model performance. Through this process, AdaBoost can achieve high accuracy even when individual weak models perform poorly.

### *2.3. Genetic Algorithm for Optimizing the Ensemble Model*

The Genetic Algorithm (GA) is an evolutionary optimization technique inspired by natural selection. It is particularly effective in complex optimization problems where the search space is vast, such as hyperparameter tuning in machine learning models. In this study, we employ GA to optimize the ensemble model combining LightGBM and AdaBoost, ensuring that the model achieves high predictive accuracy while maintaining robustness in stock price forecasting.
The GA starts by generating an initial population of 30 individuals, where each individual represents a unique combination of hyperparameters for LightGBM and AdaBoost as well as their ensemble weights. The hyperparameters

include the learning rate, maximum depth of decision trees, number of estimators, and boosting parameters. The ensemble weights determine the contribution of each base model to the final prediction. These individuals are randomly initialized within predefined ranges to ensure diversity in the search space.
Each individual in the population is evaluated using a fitness function, which measures the performance of the model based on specific error metrics. In this study, the Mean Absolute Error (MAE) and Mean Squared Error (MSE) are used as fitness criteria. The lower the error, the better the individual is at predicting stock prices. The fitness function is computed as follows:

$$Fitness = \frac{1}{MAE + \epsilon}$$

where $\epsilon$ is a small constant to prevent division by zero. This ensures that models with lower MAE receive higher fitness scores.
After evaluating the fitness of each individual, the best-performing candidates are selected using roulette wheel selection, ensuring that models with lower prediction errors have a higher probability of passing their traits to the next generation. The crossover operation then combines the hyperparameters of two parent models using uniform crossover, creating new offspring with mixed traits from both parents. To maintain genetic diversity and avoid premature convergence, a mutation operation is applied to 5
The genetic algorithm iterates through 50 generations, continuously refining the population through selection, crossover, and mutation. Over time, the average fitness score of the population improves, leading to an optimal configuration of LightGBM and AdaBoost with fine-tuned parameters. The process stops when either 50 generations are reached or further improvements become negligible. The best-performing model from the final generation is then selected for stock price prediction, ensuring a robust and adaptive forecasting system.
The integration of Genetic Algorithm (GA) optimization enhances our ensemble model by efficiently exploring the hyperparameter space, reducing computational costs, and avoiding exhaustive searches. GA navigates this space through selection, crossover, and mutation, iteratively improving model performance.
Additionally, GA improves model generalization by optimizing the balance between LightGBM and AdaBoost, preventing overfitting while capturing complex stock price patterns. It also enables automatic feature selection and weighting, leading to higher predictive accuracy. By leveraging GA, our hybrid model becomes more adaptive and robust in forecasting stock prices, making it a powerful tool for financial time series prediction.

## 3. Numerical results

This section presents the numerical results obtained from our proposed hybrid ensemble model, which combines LightGBM and AdaBoost, optimized using a Genetic Algorithm (GA), to forecast Tesla (TSLA) stock prices. The performance of each individual model, as well as the optimized ensemble, is evaluated using standard error metrics such as Mean Absolute Error (MAE), Root Mean Squared Error (RMSE), and Mean Absolute Percentage Error (MAPE). The results highlight the effectiveness of our approach in improving predictive accuracy and reducing forecasting errors, demonstrating the advantages of GA-driven optimization in financial time series prediction.
In this study, we analyze stock price data for Tesla Inc. (TSLA) 1 to evaluate the performance of our proposed hybrid ensemble model. The dataset consists of historical price information, which is used to train and test the predictive models. By leveraging machine learning techniques and evolutionary optimization, our approach aims to enhance predictive accuracy and provide a robust framework for financial time series forecasting.

### *3.1. Ensemble Model Results*

This section presents the numerical results obtained from our proposed ensemble model, which integrates LightGBM and AdaBoost with Genetic Algorithm (GA) optimization for stock price forecasting. The performance of each model is evaluated using standard error metrics, including Mean Absolute Error (MAE), Mean Squared Error (MSE), Root Mean Squared Error (RMSE), and Mean Absolute Percentage Error (MAPE). The comparison highlights the improvement achieved by the GA-optimized ensemble over individual models.


Fig. 1. Tesla (2017-2024).

### *3.2. LightGBM and AdaBoost Performance*

The LightGBM model demonstrates strong predictive capabilities but exhibits relatively higher error values due to its sensitivity to market fluctuations. Conversely, AdaBoost achieves better performance by reducing MAE and MSE, indicating enhanced generalization. However, both models still struggle to fully capture nonlinear dependencies in stock price data.

Table 1. Performance Metrics for LightGBM and AdaBoost Models

| Model | MAE | MSE | RMSE | MAPE |
|---|---|---|---|---|
| LightGBM | 9.9062 | 293.8897 | 17.1432 | 3.7012% |
| AdaBoost | 8.3992 | 181.4914 | 13.4719 | 3.2720% |

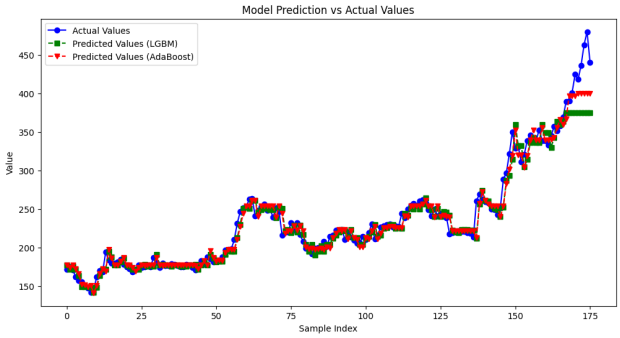


Fig. 2. Predicted vs Actual Stock Prices Using GA-Optimized Ensemble Model

### *3.3. Genetic Algorithm Optimization*

To enhance predictive accuracy, we optimize the ensemble model using a Genetic Algorithm (GA). The GA effectively fine-tunes the weighting of LightGBM and AdaBoost contributions, ensuring better generalization and improved forecasting performance. The optimized model significantly outperforms the individual models, achieving substantial reductions in error metrics.

Table 2. Performance Metrics for GA-Optimized Ensemble Model

| MAE | MSE | RMSE | MAPE |
|---|---|---|---|
| 1.1944 | 2.3784 | 1.5422 | 0.53% |

As illustrated in Table 2, the GA-optimized ensemble model significantly reduces MAE, MSE, and RMSE, highlighting its ability to capture complex stock price movements more effectively. The drastic improvement in MAPE (from 3.2720% in AdaBoost to 0.53%) further emphasizes the robustness of our approach.

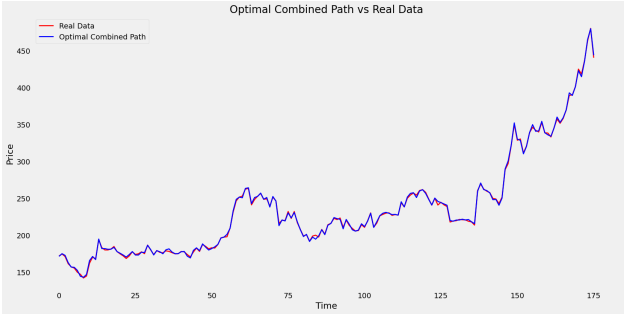


Fig. 3. Predicted vs Actual Stock Prices Using GA-Optimized Ensemble Model

The results confirm that GA-based optimization enhances stock price prediction accuracy, making it a valuable technique for financial time series forecasting. The integration of boosting models with evolutionary algorithms allows for more adaptive and precise stock price predictions in highly volatile market conditions.

## 4. Conclusion

This study demonstrated the effectiveness of Genetic Algorithm (GA) optimization in enhancing the predictive performance of an ensemble model that integrates LightGBM and AdaBoost. By intelligently optimizing hyperparameters and weighting contributions from each model, GA significantly improved forecasting accuracy while reducing computational complexity. The results revealed a substantial reduction in error metrics, with the GA-optimized model achieving a MAE of 1.1944, an MSE of 2.3784, an RMSE of 1.5422, and a MAPE of just 0.53%. These improvements highlight the superiority of our approach in capturing complex stock price dynamics compared to individual boosting models.

From a financial perspective, the enhanced forecasting accuracy achieved through GA optimization holds significant practical implications. More precise stock price predictions contribute to better risk assessment, improved portfolio management, and more informed investment decisions. By integrating advanced ensemble learning techniques with evolutionary optimization, this study presents a scalable and adaptive methodology for financial time series forecasting. The findings suggest that GA-based optimization can serve as a powerful tool for traders, financial analysts, and policymakers seeking to navigate volatile market conditions with greater confidence and precision. Future research could explore hybrid optimization techniques or apply this framework to broader financial markets to further refine predictive performance.

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# Application of Fuzzy Ant Community Optimization Algorithm for Disease Model Interpretation

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**Abstract**

This paper addresses the problem of automatic disease diagnosis and uses fuzzy inference systems for automatic disease diagnosis. The proposed method uses rule-based fuzzy systems, which are a set of fuzzy if-then rules, to retrieve the required knowledge and perform the classification operation. Over the past few decades, computer-aided diagnosis tools have been introduced with the intention of helping doctors to somehow remove irregularities and inconsistencies from data. A main goal for such computer tools is related to the scope of diagnosing cancer, heart diseases, hepatitis, etc. The results obtained from running on the medical dataset available in the UCI data repository of the University of California prove that the proposed method ,which is based on two methodologies of fuzzy systems and ant colony algorithm, can classify and diagnose diseases with high accuracy so that the generated rules can be easily interpreted by an expert.

## 1. Introduction

An important category of problems in medical science is related to the diagnosis of diseases, which are usually based on various tests that can be performed on the patient. The number of these tests may make it difficult to diagnose the disease even for an expert doctor.
In references [1–4], various algorithms and computer methods have been mentioned in relation to disease diagnosis, which clearly shows the important position of data mining and artificial intelligence methods in medical science. So far, several methods have been proposed for the automatic diagnosis of complex and different diseases. In all of these methods, the desired output is a binary output. That is, it displays two states of benign or malignant in the output. In addition, the accuracy and interpretability of the presented methods are also low. The proposed method presented in this article can improve all the parameters mentioned compared to the methods that have been done so far.

## 2. Introducing the Ant Colony Optimization Algorithm (ACO)

Today, due to the large volume of data and their complexity, a suitable tool is needed to analyze this existing data and access the knowledge hidden in them. Data mining has made a tremendous impact in academic and industrial

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environments in recent years and has found many applications in various fields. For example, commercial applications of medical and sports fraud management and detection, text mining, and web mining can be mentioned. The term data mining refers to all aspects of an automated or semi-automated process for extracting unknown and useful knowledge and patterns from large databases. This process consists of two main stages; the first stage is data preprocessing, which includes cleaning, integrating, selecting attributes, and converting data into a format used by data mining. In the second stage, the data obtained from the first stage are used for pattern recognition, which is done with the help of algorithms such as classification and clustering. Then, the obtained patterns are evaluated based on a series of criteria such as accuracy and knowledge interpretation ability.

Systems based on if-then rules have been proposed in many different application areas to automatically generate and modify these rules. One such method is the ant colony optimization algorithm. The ant colony algorithm has been proven both theoretically and experimentally to be able to search for features in complex search spaces and to provide a valid method for problems that require efficient and effective search.

Ant Colony Optimization (ACO) is a heuristic method that is used to simulate the natural behavior of ants. The initial idea of this method was proposed by Dorigo [5]. In this algorithm, the main task of each artificial ant, like its natural counterpart, is to find the shortest path between a pair of nodes in a graph on which the problem is appropriately written. Therefore, the problem is transformed into subproblems in which the artificial ants are tasked with choosing the next node based on the pheromone emitted and the distance to the next node. The decision rule for ant $k$ located at node i that wants to choose one of the nodes from the set of unvisited nodes $N_i$ is obtained from Formula (1). Here $\tau_{ij}$ represents the amount of pheromone on edge $(i,j)$, while $\eta_{i,j}$ represents the inverse of the distance between two nodes $i,j$.

Of course, each of these two has a power that can be changed to change the importance of each relative to the other.

$$P_{ij}^{k} = \begin{cases} \frac{\tau_{ij}^{\alpha}\eta_{i,j}^{\beta}}{\sum_{j\in N_i}\tau_{ij}^{\alpha}\eta_{i,j}^{\beta}} & \text{if } j \in N_i \\ 0 & \text{if } j \notin N_i \end{cases} \tag{1}$$

Also, while moving from one node $i$ to another node $j$, ants usually drop pheromone information $\delta\tau_{ij}$ on the corresponding edge in the opposite amount of edge $(i,j)$. This operation is done using formula (2).

$$\tau_{ij}(t) \leftarrow \tau_{ij}(t) + \delta\tau_{ij} \tag{2}$$

In addition, the algorithm, like its natural version, uses the pheromone evaporation mechanism to avoid rapid convergence of all ants to a suboptimal path. That is, the pheromone concentration is automatically reduced by the value ρ in each iteration. In other words, if $\tau$ is the pheromone matrix present on the edges of the corresponding graph, then this matrix is updated in each iteration by formula (3).

$$\tau \leftarrow (1-\phi)\tau \quad \rho \in (0,1] \tag{3}$$

Therefore, ACO is an optimization search idea to escape from local optimal solutions and converge towards the global optimal solution.

The corresponding process of the ant community and the fuzzy if-then rule classification problem is as follows:

2-1. Number of ants in the ant community: Number of rules used for classification.
2-2. Ant characteristics: Traits and parameters related to the disease domain in question.
2-3. Amount of pheromone secreted by each ant: Amount of evaluation function related to that ant.
2-4. Update the pheromone related to the selected ant: Optimize the fuzzy if-then rule set of the classifier.
2-5. Current ant community: Current rule set.
2-6. Disrupt the current ant community: Apply changes to the current rule set.
2-7. New ant community: New rule set.
2-8. Calculate the fitness of the new ant community Calculate the value of the evaluation function for the new rule set.
2-9. Accept the new ant community with a specific probability and provided that its fitness value is greater than the current fitness value: Accept the new rule set provided that the value of its evaluation function is greater than the value of the evaluation function of the current rule set.

## 3. Proposed Fuzzy Inference System

A fuzzy inference system is a system based on if-then rules that uses fuzzy logic to reason about data.The decision-making process is carried out by the inference engine using the rules in the rule base in the knowledge base. In a fuzzy inference system, the knowledge base is called the heart of the system. The stronger the knowledge base obtained, the more powerful the system's inference and decision-making power will be. To build the rule base in the knowledge base of a fuzzy inference system, a good algorithm must be used to generate the best rules.
The proposed fuzzy inference system uses the ant colony optimization algorithm.The overall design of the proposed algorithm based on ACO for building the existing rule set in the rule base includes the following steps:

3-1. Data preprocessing, which itself includes two steps:
Data normalization.
Data fuzzification.
3-2. Creating an initial set of fuzzy if-then rules and determining the result category and degree of certainty of each rule.
3-3. Evaluating the cost of the current rule set using the evaluation function.
3-4. Changing the current rule set using a random change of one of the rules and creating a new rule set.
3-5. Calculate the evaluation value of the new rule set using the evaluation function.
3-6. Insert the new rule set into the current rule set provided that the evaluation function value of the new rule set is greater than the evaluation function value of the current rule set, store the best evaluation function value and the best rule set and accept the new rule set with a probability.
3-7. Repeat steps 4 to 6 a specified number of times.
3-8. Return the best rule set.

After the best rule set is returned, a fuzzy inference system is created that uses it to automatically determine the type of disease and its degree of certainty.

## 4. Evaluation of the Proposed Method

The proposed algorithm is evaluated on three medical datasets from the UCI[1] data repository. The number of breast cancer cases is 785, hepatitis is 225, and heart disease is 346.
The efficiency of the proposed algorithm compared to three well-known algorithms in the field of rule inference is shown in Table 1. The first algorithm is the C4.5 algorithm [6]. This algorithm is based on decision trees and uses pruning techniques to remove redundant branches. The second algorithm is the nearest neighbor algorithm (K-NN). The procedure of this algorithm is that for each new sample, the result category is determined by comparing it with k closest training samples. Therefore, it is necessary to specify a criterion for determining the distance between samples. Euclidean distance is used to determine the distance between two samples. Finally, the XCS algorithm is used for comparison [7]. The results of comparing the proposed method with the three methods mentioned are shown in Table 1. The results show that the classification accuracy of the test sets for the proposed method is higher than other well-known methods in this field.
Of course, it should be noted that in the experiment, both the results of the training set and the results of the test set are obtained, which is due to the importance of the test set and the fact that the final accuracy of each system is obtained from its test set, so only the accuracy of the test set is given in the table below.

## 5. Conclusion

In general, the proposed method, which is based on two methodologies of fuzzy systems and ant colony algorithm, provides efficient results for classifying different disease samples and, as seen in Table 1, detects different disease samples with high accuracy. In addition to acceptable accuracy in disease diagnosis, the proposed method has another

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Table 1. Results obtained from implementing the proposed algorithm.

| method | breast cancer | hepatitis | heart disease |
|---|---|---|---|
| C4.5 | 94.5 | 91.3 | 83.5 |
| k-NN | 93.2 | 93.6 | 89.6 |
| XCS | 94.31 | 95.8 | 89.4 |
| Proposed Methods | 96.5 | 97.2 | 89.9 |

advantage over other methods, which is related to the criterion of simplicity and interpretability of the disease diagnosis model because the average length of the rules in it is shorter than other systems and a specialist doctor can easily understand this system.

According to the results obtained, it can be concluded that the proposed method has both characteristics of a fuzzy inference system, namely high reliability and appropriate interpretability, and can help a specialist doctor in the field of disease diagnosis as an expert system.

This system can be easily expanded to detect more complex diseases in addition to diagnosing these three diseases; of course, this requires having a standard data set that must be collected with high accuracy.

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# An efficient algorithm based on descent direction for nonsmooth optimization and its application in image restoration

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**Abstract**

In this paper, we present a novel optimization algorithm for minimizing nonsmooth locally Lipschitz functions under box constraints. The proposed Algorithm combines an efficient descent line search strategy to determine a descent direction with the quasisecant method for approximating the subdifferential set. We prove global covergences of proposed Algorithm.This optimization framework is particularly useful for image restoration problems, where box constraints help preserve image fidelity and enhance solution accuracy. Numerical experiments demonstrate its effectiveness restoring images.

## 1. Introduction

In this paper, we consider the following box constrainted optimization problem:

$$\begin{aligned} &\min f(x), \\ &s.t.\ \ a \leq x \leq b. \end{aligned} \tag{1}$$

where, the objective function $f$ is nonosmooth and locally Lipschitz, and the vectors $a$ and $b$ represent the lower and upper bounds on the variables, respectively.
Box-constrained optimization problems arise in various fields, including machine learning, artificial intelligence, engineering, industrial design, economics, and management [1, 2].
These constraints are particularly useful in image processing applications, where they ensure that solutions remain within meaningful ranges[3, 4].
Image restoration is a fundamental task in image processing that aims to recover high-quality images from degraded observations affect by noise, blurring, or compression artifacts. The general mathematical model for image restoration involves reconstructing an unknown image from a given noisy observation using a linear transformation and an

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additive noise term. A widely used optimization-based formulation for image restoration involves minimizing an objective function that balances data fidelity and regularization. The total variation (TV) function is often employed as a regularization term to preserve sharp edges while reducing noise, with isotropic and anisotropic total variation that provide image gradients[5].
Several methods have been developed for solving box-constrained optimization problems, including Newton's method, the Lagrange method, and the quasisecant method. Newton's method, known for its quadratic convergence, is highly effective for smooth functions but computationally expensive due to the need for second-order derivatives[6].
The Lagrange method reformulates constrained optimization problems using multipliers, with the Karush-Kuhn-Tucker (KKT) conditions determining optimality. However, solving the resulting system can be complex, especially in large-scale problems[7].
The quasisecant method, designed for cases where derivative computations are challenging, approximates search directions using secant-like updates, making it a robust alternative when function evaluations are costly[8].
The goal of image denoising problem, given by $y = Ax + c$ is to approximate the clear image $x \in \mathbb{R}^{m\times n}$ from the noisy observed image $y \in \mathbb{R}^{m\times n}$. Here, $A$ is linear and bluring operator, and $c$ represents unknown noises. The general formulation of image restoration is

$$\min_{0\le x\le 1} \frac{1}{2}\|Ax - y\|_2^2 + \lambda TV(x), \tag{2}$$

where $TV(x)$, is the total variation function, $\frac{1}{2}\|Ax - y\|_2^2$ represents the exact axplicit data fidelity, and $\lambda$ is regulirization parametrs.
The total variation function computes the sum of absolute value differences between adjacent pixels in an image $x$, with both isotropic and anisotropic formulation available.
The structure of the paper is organized as the following:
In section 2, we introduce a novel global algorithm for solving nonsmooth box-constrained optimization problems. Section 3 presents numerical experiments demonstrating the algorithm's effectiveness. Section 4 is dedicated to the conclusion.

## 2. Global method for nonsmooth box constraints problem

we will solve problem (1) using the following approach, to determine the descent direction, we approximate the differential set. At a given point, the descent direction is obtained by solving the following problem:

$$\bar{v} = argmin\frac{1}{2}\|v\|^2, \qquad v \in \partial_c f(x^0). \tag{3}$$

The descent direction at $x^0$ is given by $d = -\frac{\bar{v}}{\|\bar{v}\|}$. if $\partial_c f(x^0) = \nabla f(x^0)$, then the descent direction simplifies, $d = -\frac{\nabla f(x^0)}{\|\nabla f(x^0)\|}$, which corresponding to the steepest descent direction.
Now, we consider a set of subgradients $u^1, \dots, u^m$ of function $f$ at the $x^0$, where $m > 0$. The following problem provides an apprpximation of problem (3)

$$\min \frac{1}{2}\|v\|^2, \quad v \in conv\{u^1, \dots, u^m\}. \tag{4}$$

Let the parameters $\varepsilon, \lambda_{min}, > 0$ and $\alpha, c_1 \in (0,1), c_2 \in (0, c_1]$ be given, along with the bundle size $m_b$.

$x^k$ is the output of Algorithm 1, which minimizes problem (2). The key difference between the local and global search algorithms is the parameter $\lambda_k$. In Algorithm 1, $\lambda_k$ decreases, while in Algorithm 2, it increases as $k$ grows, indicating local and global search methods, respectively. Additionally, Algorithm 2 does not include search a line search, which facilitiates global exploration.
Let the parameters $\varepsilon, K > 0$ and $m_b$ be given.

**Algorithm 1** Local search method

Step0: Choose random $x^0 \in [a,b]$; $k := 0$ counter is , calculate $\lambda_{max} = \max\limits_{i=1,\dots,n}\{\max\{|x_i^k - a_i|, |x_i^k - b_i|\}\}$,
Step1: Compute $\lambda_k = \alpha^k \lambda_{max}$. **If** $\lambda_k < \lambda_{min}$, **then** stop, $x^k$ is output.
Step2: Choose random $d^0 \in S_1$ and put $m := 0$.
Step3: **If** $f$ is differentiable on $x^k + \lambda_k d^1$, **then** set $v = \nabla f(x^k + \lambda_k d^1)$, **else** select arbitrary $v \in \partial f(x^k + \lambda_k d^1)$.
Step4: Run Algorithm 1 of [9] and put and update set

$$W(x^k) = \begin{cases} \{v^m\}, & m = 0, \\ W(x^k) \cup \{v_m\}, & m \geq 1. \end{cases} \tag{5}$$

Step5: (Compute a descent direction) Solve $\bar{v} = \min\limits_{v \in conv W_k} \|v\|$. **If** $\|\bar{v}\| \leq \varepsilon$ **then** stop, **else** put $d^{m+1} = -\frac{\bar{v}}{\|\bar{v}\|}$.
Step6: (Stopping condition) **If** $f(x^k + \lambda_k d^{m+1}) - f(x^k) \leq -c_1 \lambda_k \|\bar{v}\|\}$ **then**, go to step6, **else** set $m := m + 1$ **If** $m > m_b$ **then** go to step6 **else** go to step3.
Step7: (Compute a search direction) Compute $\bar{h} = argmax\{0 < h < 1 : f(x^k + h\lambda_k d^{m+1}) - f(x^k) \leq -c_2 h\|\bar{v}\|\}$, put $x^k := x^k + hd^{m+1}$ go to step3.
Step8: **If** $m \leq m_b$ **then**, go to step 2.
Step9: set $x^{k+1} := x^k, k := k + 1$ and go to step1

**Algorithm 2** Global search method

Step0: Start with minimizer $x \in [a,b]$ from Algorithm 1; calculate $\lambda_{max} = \max\limits_{i=1,\dots,n}\{\max\{|x_i^k - a_i|, |x_i^k - b_i|\}\}$, number $\lambda_{min} \in (0, \lambda_{max})$ and calculate $\beta = (\lambda_{min} - \lambda_{max})/K$ and $\lambda_k = \lambda_{min} + k\beta$.
Step1: **If** $\lambda_k > \lambda_{max}$, **then** stop, output $S, T$.
Step2: Choose random $d^0 \in S_1$ and put $m := 0$. Compute $y^m = x + \lambda_k d^m$ and $f(y^m)$.
Step3: Compute $y^m = x^k + \lambda_k d^m$ and subgradient $v^m \in \partial f(y^m)$ and update set (5)
Step4: Solve quadratic problem(4),
Step5: **If** $m > m_b$ or $\|\bar{v}\| \leq \varepsilon$ **then**, go to next step. **else** compute descent direction $d^{m+1} = -\|\bar{v}\|^{-1}\bar{v}$. put $m := m + 1$. go to step3.
Step6: put $S = S \cap \{y^m\} := x_k$ and $T = T \cap \{f(y^m)\} := x^k$, $k = k + 1$ and go to step2.

The sets $S = \{y^1, \dots, y^K\}$ and $T = \{f(y^1), \dots, f(y^K)\}$ are the outputs of Algorithm 2. The algorithm employs two counters: $k$ for the outer loop and $m$ for the inner loop. Initially, the parameter $\lambda_k$ is determined based on box constraints, and in each iteration of the outer loop, $x^k$ is updated. This process continues until falls below a predefined threshold $\lambda_k \leq \lambda_{min}$.
Within the inner loop, spanning Steps 2–5, the algorithm determines the search direction by solving a quadratic optimization problem. A new point is then obtained via line search. Throughout the $k$-th iteration of the inner loop, remains constant to ensure that remains $y_m$ within a unit sphere. The inner loop terminates if either the distance between the origin and the quasisecant polytope is reduced to less than $\varepsilon$, or if the number of computed quasisecants exceeds $m_b$.

**Algorithm 3** Global subdifferential method

Step0: Choose random $\tilde{x} \in \mathbb{R}^n$; counter $k := 0$.
Step1: Run Algorithm 1 with starting point $\tilde{x}$, $x^k$ will result, then put $x_{opt} = x^k$,$f_{opt} = f(x_{opt})$.
Step2: Run algorithm 2 with starting point $x^k$, sets $S$, $T$ will generate.
Step3: Suppose $\bar{y} = arg\min_{y \in S} f(y)$, **If** $f(\bar{y} < f_{opt})$, **then** set $\tilde{x} := \bar{y}, k := k + 1$ and go to Step1; **else**, go to step4.
Step4: put $y^0 := x^k$ and calculate $f(y^m)$. Start from $m = 0$, Repeatedly check value in set $T$, generate index set $I = \{i | f(y^m) \leq f(y^{i-1}, f(y^i) \leq f(y^{i+1}), i = 1, ..., K - 1\}$ and update $S$, $T$.
Step5: Each point $y \in S$ is a starting point for Algorithm 1. **If** $f(\bar{x}) < f_{opt}$ **then** set $x^{k+1} = x_{opt} = \bar{x}$ and $f_{opt} = f(x_{opt})$ and go to step3, **else** stop and output $x_{opt}$ and $f_{opt}$.

The outputs of Algorithm 3 are $x_{opt}$ and its corresponding function value $f_{opt} = f(x_{opt})$.
It uses two sub-algorithms, Algorithm 1 for local exploitation and Algorithm 2 for global exploration. Algorithm 1 is applied in Step1 when a better starting point is found and in Step5 when new promising basins are detected.
Algorithm 2 starts from a local minimizer $x_k$ and generates candidate starting points for Algorithm 1. The algorithm restart a new round of local search when a better point than the current optimal is found (Step3) or when new promising basins are heuristically identified (Steps 4–5). If neither condition results in further progress, the algorithm terminates, returning $x_{opt}$ and $f_{opt}$ as the final results.

## 3. Numerical result

In this section, performance of the proposed denoising method is evaluated. Each iteration of the algorithm requires the solution of problem (2), which total variation function as following form:

$$TV_{anisotropic}(x) = \|TV\|_1 = \sum_{i,j} (|x_{i+1,j} - x_{i,j}| + |x_{i,j+1} - x_{i,j}|).$$

where, $TV$ is considered anisotropic.
We have used several gray images (Cameraman, House, Lake, Lena, Mandrill, Peppers, Chart) from [10] as test images.
The Fig1. show the original clean images with a size of 256 × 256 and corresponding added noise are Gaussian noise and Salt and peper noise. Image restoration results, compared with two denoising methods such as, Adams Algorithm and Chambolle Algorithm. The PSNR and CPU TIME values for these images are listed in following table. It reveals that the proposed Algorithm have better performance than Adam Algorithm and Chambolle Algorithm in PSNR. Numerical results of CPU Time for proposed Algorithm is upper than Chambolle Algorithm and Adam Algorithm.

Table 1. Comparison of PSNR and CPU Time of Algorithm 3 and Adams Algorithm and Chambolle Algorithm.

| | | Images | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | House | Cameraman | Lena | Lake | Mandrill | Peppers | Chart |
| PSNR | Algorithm 3 | 70.90 | 69.88 | 71.00 | 71.00 | 70.21 | 70.39 | 71.18 |
| | Adams Algorithm | 70.77 | 70.30 | 70.32 | 70.28 | 69.51 | 70.31 | 70.07 |
| | Chambolle Algorithm | 69.70 | 70.15 | 69.23 | 69.80 | 69.10 | 70.28 | 66.12 |
| CPU Time | Algorithm 3 | 18.33 | 13.91 | 15.75 | 14.45 | 14.47 | 14.52 | 15.39 |
| | Adams Algorithm | 6.12 | 6.20 | 4.77 | 6.98 | 6.94 | 4.83 | 5.27 |
| | Chambolle Algorithm | 7.80 | 7.10 | 6.08 | 7.70 | 6.51 | 7.39 | 7.18 |

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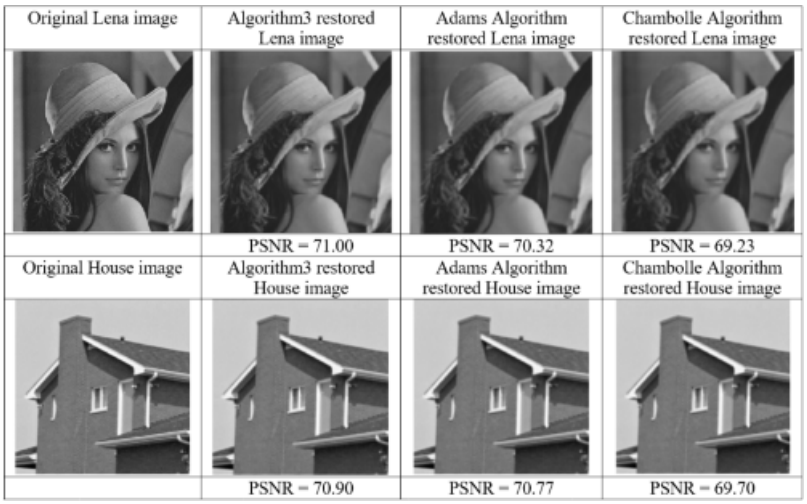


Fig. 1. PSNR results of Lena and House

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# Operation-Inverter-Based Graphs of $EQ$-algebras

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**Abstract**

In this research we introduce the new concept of operation inverter for each element of the $EQ$-algebras and we examine some properties of this new concept. Next, the concepts of inverter graph $\Gamma(X)$ is introduced based on inverter set of $EQ$-algebras and check their attributes.

## 1. Introduction

Algebraic logic as the oldest approach to formal logic was introduced by George Boole in 1847 that it has many applications in interdisciplinary sciences. From the combination of any regular set with the principles of logical algebra, lattices are created, which are very important in logical algebra. The theory of residual lattices is of double importance in logical algebra, and for this reason, a new concept called $EQ$-algebras was presented by Novak and De Baets in 2009 [11]. Many researchers have studied and written many articles about $EQ$-algebras and their application [1, 2, 8–10, 12]. Graph theory is a practical and valuable tool for modeling problems and finding the shortest solutions and paths. Also, directed graphs are widely used in today's world, for example, computer science, network science, excavate public communication, decompose electrical flow and network theory. In this research, for each member of the $EQ$-algebras, we present an inverter operation, , which plays a fundamental role in our article. In fact, we make the inverter graph $\Gamma(X)$ based on the inverter set.

### 1.1. Preliminaries

**Definition 1.1.** [11] A system $(X, \wedge, \odot, \backsim, 1)$ where $\wedge, \odot, \backsim$ " are binary operations, is a (commutative) $EQ$-*algebra*, if $\forall x, y, z, t \in X$:

$(EQ1)$ $(X, \wedge, 1)$ is a $\wedge$-semilattice which contains the element above 1. We place $x \leq y \Leftrightarrow \wedge(x, y) = x$,

$(EQ2)$ $(X, \odot, 1)$ is a (commutative) monoid and $\odot$ is isotone,

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$(EQ3)$ $x \backsim x = 1$,

$(EQ4)$ $((x \wedge y) \backsim z) \odot (t \backsim x) \leq_X (z \backsim (t \wedge y))$,

$(EQ5)$ $(x \backsim y) \odot (z \backsim t) \leq_X (x \backsim z) \backsim (y \backsim t)$,

$(EQ6)$ $(x \wedge y \wedge z) \backsim x \leq_X (x \wedge y) \backsim x$,

$(EQ7)$ $(x \wedge y) \backsim x \leq_X (x \wedge y \wedge z) \backsim (x \wedge z)$,

$(EQ8)$ $x \odot y \leq_X x \backsim y$.

The action " $\odot$ " is multiplication, and " $\backsim$ " is fuzzy equality. Based [2], put $x \hookrightarrow y = (x \wedge y) \backsim x, \neg x = x \backsim 0$ and $\overset{\smile}{x} = x \backsim 1, \forall x, y \in X$.
We say that $X$ is good if for all $x \in X$, $x \backsim 1 = x$.

**Theorem 1.2.** *[2, 11] Let $(X, \wedge, \odot, \backsim, 1)$ be an $EQ$-algebra. Then $\forall a, b, c \in X$:*

(1) $a \hookrightarrow (b \wedge c) \leq_X a \hookrightarrow b$,

(2) $a \hookrightarrow b \leq_X (a \wedge c) \hookrightarrow b$,

(3) $a \hookrightarrow d \leq_X (b \hookrightarrow a) \hookrightarrow (b \hookrightarrow d)$,

(4) $b \hookrightarrow a \leq_X (a \hookrightarrow d) \hookrightarrow (b \hookrightarrow d)$,

(5) $a \hookrightarrow b \leq_X (a \wedge c) \hookrightarrow (b \wedge c)$,

(6) $a \hookrightarrow b = a \hookrightarrow (a \wedge b)$,

(7) $b \leq_X \overset{\smile}{b} \leq_X a \hookrightarrow b$,

(8) $a \backsim b \leq_X a \hookrightarrow b$ *and* $a \hookrightarrow a = 1$,

(9) $a \hookrightarrow b \leq_X \neg b \hookrightarrow \neg a$,

(10) $a \hookrightarrow b = (a \vee b) \hookrightarrow b$,

(11) $a \hookrightarrow b = (a \vee c) \hookrightarrow (b \vee c)$,

(12) *Presume* $a \backsim b = 1$, *then* $(a \wedge c) \backsim (b \wedge c) = 1$ *and* $(a \backsim c) \backsim (b \backsim c) = 1$.

**Theorem 1.3.** *[2, 11] Let $(X, \wedge, \odot, \backsim, 1)$ be a good $EQ$-algebra. Then $\forall a, b, c \in X$:*

(1) $a \hookrightarrow (b \hookrightarrow c) = b \hookrightarrow (a \hookrightarrow c)$,

(2) $a \hookrightarrow (b \hookrightarrow c) \leq_X a \odot b \hookrightarrow c$.

## 2. Main results

In this section, we define the significance of inverter set in $EQ$-algebra and inverter graph $\Gamma(X)$ based on inverter set of $EQ$-algebras.

**Definition 2.1.** Let $(X, \wedge, \odot, \backsim, 1)$ be an $EQ$-algebra. Then for $z \in X$, define $Tr(\hookrightarrow, z) = \{x \in X \mid x \hookrightarrow z = 1\}$ as $(\hookrightarrow)$-inverter of $z$ in $X$.

It is clear that in any $EQ$-algebra $(X, \wedge, \odot, \backsim, 1)$, $|Tr(\hookrightarrow, z)| \geq 1$, because of $z \in Tr(\hookrightarrow, z)$.

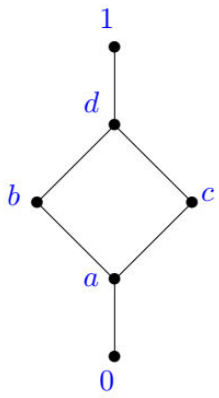


Fig. 1.

Table 1. $(X, \odot)$

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | $a$ | $a$ | $a$ | $b$ |
| $c$ | 0 | 0 | $a$ | 0 | $a$ | $c$ |
| $d$ | 0 | 0 | $a$ | $a$ | $a$ | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

Table 2. $(X, \backsim)$

| $\backsim$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 1 | $d$ | $d$ | $d$ | $d$ |
| $b$ | 0 | $d$ | 1 | $d$ | $d$ | $d$ |
| $c$ | 0 | $d$ | $d$ | 1 | $d$ | $d$ |
| $d$ | 0 | $d$ | $d$ | $d$ | 1 | 1 |
| 1 | 0 | $d$ | $d$ | $d$ | 1 | 1 |

Table 3. $(X, \hookrightarrow)$

| $\hookrightarrow$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | 0 | 1 | 1 | 1 | 1 | 1 |
| $b$ | 0 | $d$ | 1 | $d$ | 1 | 1 |
| $c$ | 0 | $d$ | $d$ | 1 | 1 | 1 |
| $d$ | 0 | $d$ | $d$ | $d$ | 1 | 1 |
| 1 | 0 | $d$ | $d$ | $d$ | 1 | 1 |

**Example 2.2.** Suppose $X = \{0, a, b, c, d, 1\}$ and considre the lattice in Figure 2.2. Then $(X, \wedge, \odot, \backsim, 1)$ is an $EQ$-algebra as Tables 1, 2 and 3.

Computations show that

$$Tr(\hookrightarrow,0)=\{0\}, Tr(\hookrightarrow,a)=\{0,a\}, Tr(\hookrightarrow,b)=\{0,a,b\}, Tr(\hookrightarrow,c)=\{0,a,c\}$$
$$\text{and } Tr(\hookrightarrow,d)=Tr(\hookrightarrow,1)=X.$$

**Theorem 2.3.** *Let* $(X,\wedge,\odot,\backsim,1)$ *be an* $EQ$*-algebra. Then* $\forall x,y,z\in X$ :

$(i)$ *If* $x\in Tr(\hookrightarrow,y)$, *then* $(x\wedge z)\in Tr(\hookrightarrow,y\wedge z)$.

$(ii)$ *If* $x\in Tr(\hookrightarrow,y)$, *then* $(x\backsim z)\in Tr(\hookrightarrow,y\backsim z)$.

*Proof.* $(i)$ Let $x\in Tr(\hookrightarrow,y)$, then $x\hookrightarrow y=1$. By Theorem 1.2, $x\hookrightarrow y\leq(x\wedge z)\hookrightarrow(y\wedge z)$, get that $(x\wedge z)\hookrightarrow(y\wedge z)=1$. Hence $(x\wedge z)\in Tr(\hookrightarrow,y\wedge z)$.
$(ii)$ Let $x\in Tr(\hookrightarrow,y)$, then $x\hookrightarrow y=1$. By Theorem 1.2, gain that $(x\backsim z)\backsim(y\backsim z)=1$. Using Theorem 1.2, $(x\backsim z)\backsim(y\backsim z)\leq(x\backsim z)\hookrightarrow(y\backsim z)$. Hence $(x\backsim z)\hookrightarrow(y\backsim z)=1$. Therefore $(x\backsim z)\in Tr(\hookrightarrow,y\backsim z)$. □

**Theorem 2.4.** *Let* $(X,\wedge,\odot,\backsim,1)$ *be an* $EQ$*-algebra. Then* $\forall x,y,z\in X$ :

$(i)$ $x\in Tr(\hookrightarrow,y)$ *iff* $x\in Tr(\hookrightarrow,x\wedge y)$.

$(ii)$ $x\in Tr(\hookrightarrow,y)$ *iff* $x\vee y\in Tr(\hookrightarrow,y)$.

$(iii)$ $x\in Tr(\hookrightarrow,y)$ *iff* $x\vee z\in Tr(\hookrightarrow,y\vee z)$.

*Proof.* $(i)$ By using Theorem 1.2, $x\hookrightarrow y=1$ iff $x\hookrightarrow(x\wedge y)=1$.
$(ii)$ By using Theorem 1.2, $x\hookrightarrow y=1$ iff $x\vee y\hookrightarrow y=1$.
$(iii)$ Applying Theorem 1.2, $x\hookrightarrow y=1$ iff $x\vee z\hookrightarrow y\vee z=1$. □

**Theorem 2.5.** *Assume* $(X,\wedge,\odot,\backsim,1)$ *be a good* $EQ$*-algebra and* $x,y,z\in X$. *Then*

$(i)$ $Tr(\hookrightarrow,y)\subseteq Tr(\hookrightarrow,x\hookrightarrow y)$.

$(ii)$ *If* $x\in Tr(\hookrightarrow,y\hookrightarrow z)$, *then* $x\odot y\in Tr(\hookrightarrow,z)$.

$(iii)$ $x\in Tr(\hookrightarrow,y\hookrightarrow z)$ *if and only if* $y\in Tr(\hookrightarrow,x\hookrightarrow z)$.

*Proof.* $(i)$ Let $z\in Tr(\hookrightarrow,y)$. Thus $z\hookrightarrow y=1$. Applying Theorem 1.3, $z\hookrightarrow(x\hookrightarrow y)=x\hookrightarrow(z\hookrightarrow y)=x\hookrightarrow 1=1$ and so $z\hookrightarrow(x\hookrightarrow y)=1$. Hence $z\in Tr(\hookrightarrow,x\hookrightarrow y)$ and so $Tr(\hookrightarrow,y)\subseteq Tr(\hookrightarrow,x\hookrightarrow y)$.
$(ii)$ Since $x\in Tr(\hookrightarrow,y\hookrightarrow z)$, so $x\hookrightarrow(y\hookrightarrow z)=1$. Applying Theorem 1.3, $x\odot y\hookrightarrow z=1$. Hence $x\odot y\in Tr(\hookrightarrow,z)$.
$(iii)$ By using Theorem 1.3, $x\hookrightarrow(y\hookrightarrow z)=1$ if and only if $y\hookrightarrow(x\hookrightarrow z)=1$. □

**Definition 2.6.** Let $(X,\wedge,\odot,\backsim,1)$ be an $EQ$-algebra and $x,y\in X$. Then $\Gamma(X)=(V=V(\Gamma(X))=X\setminus\{1\},E=E(\Gamma(X))$ is called an inverter graph of $X$, which for any two distinct vertices $x$ and $y,(x,y)\in E$ if and only if $Tr(\hookrightarrow,x)\cup Tr(\hookrightarrow,y)\neq Tr(\hookrightarrow,x\hookrightarrow y)$ and $(y,x)\in E$ if and only if $Tr(\hookrightarrow,y)\cup Tr(\hookrightarrow,x)\neq Tr(\hookrightarrow,y\hookrightarrow x)$. We emphasize that $Tr(\hookrightarrow,x)\cup Tr(\hookrightarrow,y)=Tr(\hookrightarrow,x\hookrightarrow y)$, if $(y,x)\in E$ or there is not any arc between vertices $x,y$.

It obvious that $\Gamma(X)=(V=V(\Gamma(X))=X\setminus\{1\},E=E(\Gamma(X)))$ has not loop and is not a simple graph, since for any $x,y\in V$, necessarily, $Tr(\hookrightarrow,x\hookrightarrow y)\neq Tr(\hookrightarrow,y\hookrightarrow x)$.

**Example 2.7.** Consider the $EQ$-algebra $(X,\wedge,\odot,\backsim,1)$ of Example 2.2. We gain the inverter graph $\Gamma(X)$ in Figure 2.7.

**Theorem 2.8.** *Let* $(X,\wedge,\odot,\backsim,1)$ *be an* $EQ$*-algebra and* $x,y\in X$.

$(i)$ *If* $(x,y)\notin E(\Gamma(X))$, *then* $Tr(x)\subseteq Tr(x\hookrightarrow y)$.

$(ii)$ *If* $(x,y),(y,x)\notin E(\Gamma(X))$, *then* $Tr(y\hookrightarrow x)=Tr(x\hookrightarrow y)$.

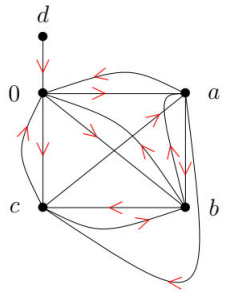


Fig. 2.

*Proof.* $(i)$ Assume $x, y \in X$. As $(x, y) \notin E(\Gamma(X))$, we get that $Tr(\hookrightarrow, x) \cup Tr(\hookrightarrow, y) = Tr(\hookrightarrow, x \hookrightarrow y)$ and so $Tr(\hookrightarrow, x) \subseteq Tr(\hookrightarrow, x \hookrightarrow y)$.
$(ii)$ By item $(i)$, $Tr(\hookrightarrow, x) \cup Tr(\hookrightarrow, y) = Tr(\hookrightarrow, x \hookrightarrow y)$ and $Tr(\hookrightarrow, x) \cup Tr(\hookrightarrow, y) = Tr(\hookrightarrow, y \hookrightarrow x)$. Thus $Tr(y \hookrightarrow x) = Tr(x \hookrightarrow y)$. □

**Corollary 2.9.** *Assume* $(X, \wedge, \odot, \backsim, 1)$ *be an* $EQ$*-algebra.*

$(i)$ *If* $\forall x, y \in X$, $Tr(x) \nsubseteq Tr(x \hookrightarrow y)$, *then* $\Gamma(X)$ *is a weakly connected graph.*

$(ii)$ *If* $\forall x, y \in X$, $Tr(x) \nsubseteq Tr(x \hookrightarrow y)$ *and* $Tr(y) \nsubseteq Tr(y \hookrightarrow x)$, *then* $\Gamma(X)$ *is a strongly connected graph.*

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# Exploring Bermudan Options in the CEV Framework

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**Abstract**

This paper presents a comprehensive analysis of Bermudan options using the Constant Elasticity of Variance (CEV) model, employing the Least Squares Monte Carlo (LSMC) method for pricing. We investigate the impact of varying elasticity parameters on the implied volatility smile and demonstrate how the CEV model effectively captures the stochastic nature of asset prices. Our numerical results include simulations of asset price paths, comparisons of implied volatility curves for different elasticity parameters, and sensitivity analyses of Bermudan option prices across various strike prices and maturities. These findings provide valuable insights for option traders and risk managers, enhancing their understanding of the behavior and valuation of Bermudan options under the CEV model.

## 1. Introduction

In the realm of financial derivatives, Bermudan options hold a unique position due to their hybrid exercise feature, which combines aspects of both American and European options. Unlike European options, which can only be exercised at maturity, and American options, which can be exercised at any time before maturity, Bermudan options can be exercised at specific intervals during their life. This exercise flexibility poses both opportunities and challenges in the valuation process.

The CEV model, introduced by Cox in 1975, extends the Black–Scholes model by incorporating a volatility component that is a function of the underlying asset price [2]. This feature allows the CEV model to capture the empirical phenomena of volatility smiles and leverage effects observed in financial markets [5]. Given the stochastic nature of volatility in real–world markets, the CEV model provides a more realistic framework for option pricing compared to its constant volatility counterparts [1, 3, 4, 6, 7].

This paper aims to explore the details of Bermudan option pricing under the CEV model, highlighting the theoretical underpinnings, computational methodologies, and practical implications. We will investigate a numerical technique employed to approximate the price of Bermudan options and demonstrate the effectiveness of the CEV model in capturing market dynamics. By doing so, we seek to provide a comprehensive understanding of the interplay between option exercise flexibility and stochastic volatility, thereby contributing to the existing body of knowledge in the field of financial derivatives.

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## 2. CEV Model

The CEV model is an extension of the Black-Scholes model used to price options. Unlike the Black-Scholes model, which assumes constant volatility, the CEV model assumes that the volatility of the underlying asset is a function of the asset price itself. This makes the CEV model particularly useful for capturing the observed market phenomena of volatility smiles and leverage effects.

### *2.1. Mathematical Formulation*

The CEV model assumes that the price of an asset follows a stochastic differential equation (SDE) of the form

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dW_t,$$

where $S_t$ is the asset price at time $t$, $r$ is the risk-free interest rate, $\sigma$ is the volatility, $\gamma$ is the elasticity parameter, which determines the relationship between the asset price and its volatility, and $W_t$ is a standard Brownian motion process.

### *2.2. Local Volatility Model*

The CEV model is a type of local volatility model. Local volatility models are a class of models where the volatility of the underlying asset is allowed to vary with both the asset price and time. In the case of the CEV model, the local volatility is a function of the asset price $S_t$, specifically:

$$\sigma_{local}(S_t) = \sigma S_t^{\gamma-1}.$$

This formulation means that the volatility changes with the asset price, allowing the CEV model to capture the empirical observation that volatility is not constant but varies with the level of the underlying asset.

### *2.3. Implied Volatility Curve*

The implied volatility curve, often referred to as the volatility smile or skew, represents the implied volatility of options across different strike prices or maturities. In markets, it is observed that options with different strike prices or maturities exhibit different implied volatilities. This phenomenon cannot be captured by the Black-Scholes model, which assumes constant volatility.

The CEV model, however, is capable of capturing the shape of the implied volatility curve due to its local volatility nature. By allowing volatility to be a function of the asset price, the CEV model can produce an implied volatility smile that aligns more closely with observed market data.

To visualize the implied volatility smile, we calculated the implied volatilities for options with varying strike prices under the CEV model. The algorithm includes the following steps:

**Step 1:** Set the initial parameters as follows:

- *Initial asset price* ($S_0 = \$100$)
- *Risk-free rate* ($r = 5\%$)
- *Time to maturity* ($T = 1$ year)
- *Strike prices* ($K = [80, 90, 100, 110, 120]$)
- *Market prices* corresponding to each strike price
- *Elasticity parameter of the CEV model* ($\gamma = 0.5$)

**Step 2:** For each strike price, simulate the asset paths using the discretized CEV model:

$$dS_t = rS_t dt + \sigma S_t^{\gamma} dW_t,$$

where $dW_t$ represents the Brownian Motion process increments.

The asset price at each time step was updated using:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sigma S_t^{\gamma}\sqrt{\Delta t}\epsilon_t,$$

with $\epsilon_t \sim \mathcal{N}(0,1)$.
Expected option payoff at maturity $T$ should be calculated as:

$$C_{\text{CEV}} = e^{-rT}\mathbb{E}\left[\max(S_T - K, 0)\right].$$

**Step 3:** Estimate the implied volatility using the Black-Scholes model with the option prices derived from the CEV model. The Black-Scholes formula for implied volatility is used as follows:

$$\text{implied\_vol} = \text{blsimpv}(S_0, K, r, T, C_{\text{CEV}})$$

**Step 4:** Plot the implied volatility smile curve by calculating the implied volatilities for each strike price and plotting them against the strike prices.
The algorithm can be summarized as follows:

1. Initialize parameters: $S_0$, $r$, $T$, market prices, strike prices $K$, and $\gamma$.
2. For each strike price:
   (a) Simulate asset price paths under the CEV model to compute $C_{\text{CEV}}$.
   (b) Estimate the implied volatility using the Black-Scholes formula.
3. Plot the implied volatilities versus strike prices.

The implied volatility smile curve obtained reflects how implied volatility varies with strike price under the CEV model, capturing the characteristic smile shape due to the model's ability to account for the asset's price-dependent volatility (see Figs 1 and 2).

## 3. Pricing Bermudan Options with LSMC in the CEV Model

Bermudan options, exercisable at specific intervals, present unique valuation challenges. The CEV model, which accounts for changing volatility with asset price, offers a realistic framework for pricing. The LSMC method combines Monte Carlo simulation with regression techniques, making it suitable for pricing Bermudan options under the CEV model. This section details the steps involved in defining model parameters, simulating asset paths, calculating payoffs, and applying regression to estimate the option's value accurately.

**Step 1:** Begin by specifying the parameters of the CEV model:

- Risk-free rate ($r$): The risk–free interest rate used for discounting.
- Volatility coefficient ($\sigma$): Represents the volatility of the underlying asset.
- Elasticity parameter ($\gamma$): Determines the relationship between the asset price and its volatility.
- Initial asset price ($S_0$): The starting price of the underlying asset.
- Time to maturity ($T$): The total time until the option expires.
- Exercise dates ($t_1, t_2, \cdots, t_n$): The specific dates on which the Bermudan option can be exercised.

**Step 2:** Simulate Asset Paths; to simulate asset paths, we need to discretize the continuous SDE. One common approach is to use the Euler–Maruyama method. For a small time step $\Delta t$, the discretized form of the SDE is:

$$S_{t+\Delta t} = S_t + rS_t\Delta t + \sigma S_t^{\gamma}\sqrt{\Delta t}\epsilon_t,$$

where $\epsilon_t$ is a standard normal random variable ($\epsilon_t \sim \mathcal{N}(0,1)$).
Simulation procedure is as follows:

1. *Initialize Parameters*:
   - Set the initial asset price $S_0$.
   - Define the number of time steps $N$ and the time step size $\Delta t = \frac{T}{N}$, where $T$ is the time to maturity.
   - Specify the number of simulation paths $M$.
2. *Generate Random Numbers*:
   - Generate $M \times N$ random normal variables $\epsilon_t$.
3. *Simulate Paths*:
   - For each path $j = 1, 2, \cdots, M$:
     - Initialize $S_0^j = S_0$.
     - For each time step $i = 0, 1, \cdots, N-1$:

$$S^j_{(i+1)\Delta t} = S^j_{i\Delta t} + rS^j_{i\Delta t}\Delta t + \sigma \left(S^j_{i\Delta t}\right)^{\gamma} \sqrt{\Delta t}\epsilon^j_i.$$

Generate multiple asset price paths and for each path, simulate the asset price at each of the exercise dates.
**Step 3:** For each simulated asset path, calculate the option payoff at each exercise date. The payoff depends on whether the option is a call or a put:

- *Call option payoff*: $\max(S_t - K, 0)$,
- *Put option payoff*: $\max(K - S_t, 0)$,

where $K$ is the strike price and $S_t$ is the asset price at time $t$.
**Step 4:** At each exercise date, estimate the continuation value using least squares regression. The continuation value is the expected discounted payoff if the option is not exercised at that date.

1. **For the last exercise date** $t_n$:
   - The continuation value is zero because there are no further exercise opportunities.
   - The option value is simply the payoff at $t_n$.
2. **For each preceding exercise date** $t_i$**, from** $t_{n-1}$ **to** $t_1$:
   - Use the simulated payoffs from $t_{i+1}$ to estimate the continuation value at $t_i$.
   - Perform least squares regression of the discounted payoffs on a set of basis functions (e.g., polynomials of the asset price) to approximate the continuation value.
   - The option value at $t_i$ is the maximum of the immediate exercise value and the continuation value.

**Step 5:** Once you have the option values at each exercise date for all simulated paths, estimate the Bermudan option price as the average of the discounted payoffs from the first exercise date.

## 4. Results and Discussion

In this section, we present the numerical results obtained from our analysis of Bermudan options under the CEV model. The figures illustrate the impact of varying model parameters on option prices and their corresponding implied volatility curves.
Fig. 1 shows the implied volatility smile curve calculated using the CEV model with an elasticity parameter ($\gamma$) set to 0.5. The curve illustrates how implied volatility varies with different strike prices. The smile shape indicates that implied volatility is higher for both deep in-the-money and out-of-the-money options compared to at-the-money options, reflecting market expectations of higher future volatility for these options.
Fig. 2 compares the implied volatility smile curves under the CEV model for $\gamma = 0.5$ and $\gamma = 0.7$. The red curve represents $\gamma = 0.5$, while the blue curve represents $\gamma = 0.7$. Higher gamma values result in a steeper smile, indicating

greater sensitivity of volatility to the asset price. The comparison shows how the choice of gamma affects the shape of the volatility smile.

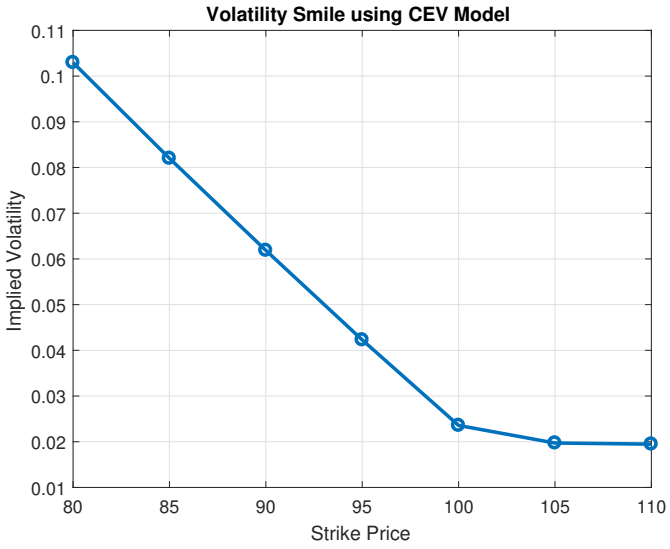


Fig. 1. Implied volatility smile curve under the CEV model with $\gamma = 0.5$.

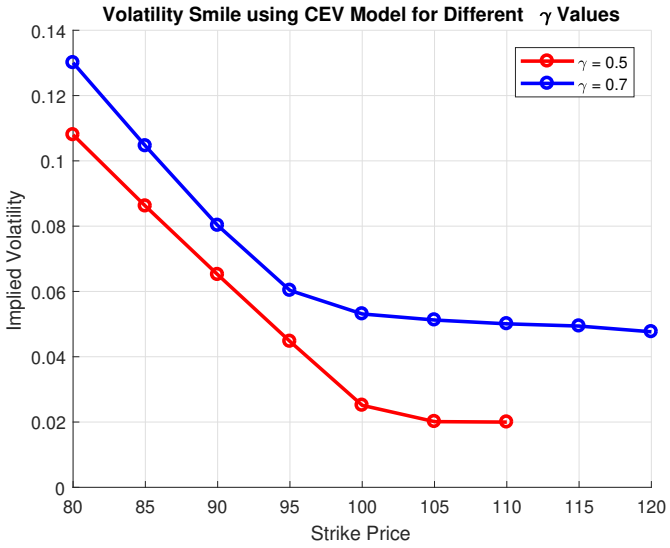


Fig. 2. Comparing implied volatility smile curve under the CEV model for $\gamma = 0.5$ and $\gamma = 0.7$.

Fig. 3 shows multiple simulated asset price paths over time under the CEV model with $\gamma = 0.5$. Each line represents a different simulated path, illustrating the evolution of asset prices and the variability introduced by the model's parameters.

Fig. 4 illustrates the relationship between Bermudan option prices, strike prices, and time to maturity. The Bermudan option price is plotted on the vertical axis, while the horizontal axes represent the strike price and time to maturity. As strike prices increase or decrease, and as time to maturity extends, the option price exhibits distinct patterns that help in understanding the option's sensitivity to these parameters.

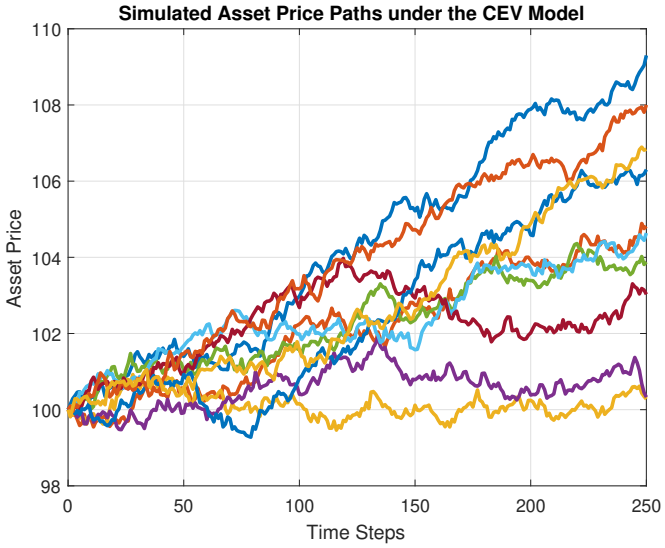


Fig. 3. Simulated asset price paths under the CEV model.

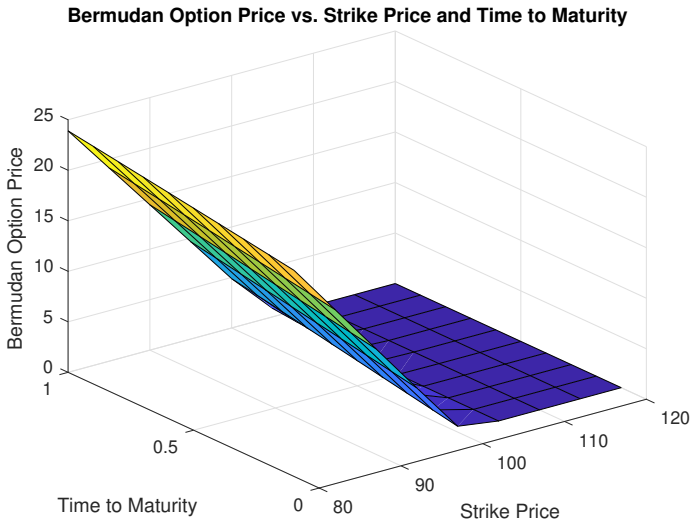


Fig. 4. Bermudan option price relative to strike price and time to maturity.

## 5. Conclusion

This paper presents a detailed examination of Bermudan options using the CEV model, employing the LSMC method for pricing. Our analysis demonstrates that the CEV model effectively captures the stochastic nature of asset prices and provides a more realistic representation of market dynamics compared to traditional models. Through numerical simulations, we have illustrated the impact of varying elasticity parameters on the implied volatility smile, highlighting the model's flexibility in reflecting different market conditions. Additionally, our sensitivity analysis of Bermudan option prices across various strike prices and maturities offers valuable insights for traders and risk managers. The findings of this study enhance the understanding of Bermudan option behavior under the CEV model and underscore

the importance of incorporating realistic volatility structures in option pricing models. These insights are crucial for improving risk management strategies and making informed trading decisions in financial markets.

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# A Special Property of Maximal Subgroups

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**Abstract**

Suppose that $M$ is a group. Let $K$ be a subgroup of $M$. If $K$ be a maximal subgroup $M$ then either $Z(M) \subseteq K$ or $K$ is a normal subgroup of $M$ containing $M'$.

## 1. Introduction

In mathematics, the term maximal subgroup is used to mean slightly different things in different areas of algebra.
In group theory, a maximal subgroup $H$ of a group $G$ is a proper subgroup, such that no proper subgroup $K$ contains $H$ strictly. In other words, $H$ is a maximal element of the partially ordered set of subgroups of $G$ that are not equal to $G$. Maximal subgroups are of interest because of their direct connection with primitive permutation representations of $G$. They are also much studied for the purposes of finite group theory: see for example Frattini subgroup, the intersection of the maximal subgroups.
In semigroup theory, a maximal subgroup of a semigroup $S$ is a subgroup (that is, a subsemigroup which forms a group under the semigroup operation) of $S$ which is not properly contained in another subgroup of $S$. Notice that, here, there is no requirement that a maximal subgroup be proper, so if $S$ is in fact a group then its unique maximal subgroup (as a semigroup) is $S$ itself. Considering subgroups, and in particular maximal subgroups, of semigroups often allows one to apply group-theoretic techniques in semigroup theory. There is a one-to-one correspondence between idempotent elements of a semigroup and maximal subgroups of the semigroup: each idempotent element is the identity element of a unique maximal subgroup. In algebra, a division ring, also called a skew field (or, occasionally, a sfield), is a nontrivial ring in which division by nonzero elements is defined. Specifically, it is a nontrivial ring in which every nonzero element $a$ has a multiplicative inverse. The structure of linear groups are now very well understood, but the structure of skew linear group is completely different. In some papers structure of maximal subgroup of general skew linear group is investigated. For some recently result see [1], [3], [6], [9], [20]. Situation of maximal subgroup of general linear group are investigated in several papers, for example in [1] and [13]. In addition in [1], the following

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conjecture appears:
**Conjecture.** Let $D$ be a division ring and $M$ a torsion maximal subgroup of $GL_n(D)$; then $D = F$, $CharF = p > 0$ and $F$ is algebraic over $\mathbb{F}_p$.
The following result is obtained in [9]:

Let $D$ be a division ring with center $F$ and $M$ be a maximal subgroup of $GL_n(D)$. If $D \neq F$ or $n > 1$, then $M/M \cap F^*$ cannot be locally finite unless $charF = p > 0$ and either:

1. $[D : F] = p^2, n = 1$, and $M \cup \{0\}$ is a maximal subfield of $D$, or
2. $D = F, n = p$, and $M \cup \{0\}$ is a maximal subfield of $M_p(F)$, or
3. $D = F$ and $F$ is locally finite.

Also, the question on the existence of non-cyclic free subgroups in linear groups over a field was studied by Tits in [25] which asserts that in the characteristic 0, every subgroup of the general linear group over a field $F$ either contains a non-cyclic free subgroup or is soluble-by-finite, and every finitely generated subgroup either contains a non-cyclic free subgroup or is soluble-by-finite in the case of prime characteristic. This result of Tits is now referred as the Tits Alternative. Lichtman in [18] showed that there exists a finitely generated group which is not soluble-by-finite and does not contain a non-cyclic free subgroup. See also the new result in [8], [10] and [11].
A good first reduction is to take into account the first isomorphism theorem, which tells us that the maximal subgroups containing a given normal subgroup $N$ of $G$ correspond, under the natural projection, to the maximal subgroups of the quotient group $G/N$.

In this article we study some special property of maximal subgroups.

## 2. Main Result

**Theorem 2.1.** *Suppose that $M$ is a group. Let $K$ be a subgroup of $M$. If $K$ be a maximal subgroup $M$ then either $Z(M) \subseteq K$ or $K$ is a normal subgroup of $M$ containing $M'$.*

*Proof.* First, assume that $M$ is a finite group. We prove our claim by induction over $| M |$. Consider that there exits a minimal normal subgroup $H$ of $M$ such that $H \subseteq K$. therefore $K/H$ is maximal subgroup of $M/H$. By induction:

$$Z(M/H) \subseteq K/H$$

or

$$(M/H)' = M'H/H \subseteq K/H.$$

But:

$$Z(M)H/H \subseteq Z(M/H).$$

Hence, we obtain that $Z(M) \subseteq K$ or $M' \subseteq K$, as we claimed. Now, in other case we suppose that $K$ does not have any nontrivial subgroup which is normal in $M$.

On the contrary, assume that $Z(M) \nsubseteq K$ and $M' \nsubseteq K$.
Then, by the maximality of $K$:

$$M = Z(M)K = M'K.$$

On the other hand, we have:

$$K \subseteq N_M(Z(M) \cap K)$$

and

$$Z(M) \subseteq N_M(M' \cap K).$$

In both cases, we conclude that $Z(M) \cap K$ and $M' \cap K$ are normal in $M$. Therefore, these subgroups are trivial. Hence:

$$M/Z(M) \cong K/(Z(M) \cap K)$$

and

$$K/(Z(M) \cap K) = K/(M' \cap K)$$

and

$$K/(M' \cap K) \cong M/M'.$$

Therefore, $M/Z(M)$ is abelian. So $M/Z(M)$ is nilpotent. Then $M$ is nilpotent. But $K$ is a maximal subgroup of the nilpotent group $M$. Thus $K \trianglelefteq M$. Consequently $M/K$ is a cyclic group of prime order and hence $M' \subseteq K$. This leads us to a contradiction.

In general case, assume that $Z(M)$ is not contained in $K$. We prove that $M' \subseteq K$.
So, $K \subsetneq KZ(M)$. But by maximality of $K$ we obtain that

$$M = KZ(M).$$

We know that $H = Z(M) \cap K$ is central and hence normal. Therefore,

$$M/H = K/H \times Z(M)/H.$$

Thus

$$(M/H)' = (K/H)'.$$

So,

$$M'H = K'H.$$

Now we conclude that:

$$M' \subseteq M'H = K'H = K'(K \cap Z(M)) \subseteq K,$$

as we desired.

□

Let $D$ be a division ring with center $F$ and $M$ be a maximal subgroup of $GL_n(D)$. If

$$F^* \nsubseteq M$$

, by previous theorem, we conclude that $SL_n(D) \subseteq M$. Thus $M$ is normal maximal subgroup.

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Also, let $D_n = |\mathcal{D}_n|$. The inclusion-exclusion principle gives

$$D(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Assume that $D_0 = 1$ and $D_1 = 0$. Then the number $D(n)$ is related to the numbers $D(n-1)$ and $D(n-2)$ (with $n \geq 2$) through a closed-form recursive relation

$$D(n) = (n-1)(D(n-1) + D(n-2)).$$

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In [9], authors started investigating a subclass of derangements on $n + r$ letters which restrict the first $r$ of these to be in distinct cycles. (Such derangements are called $r$-derangement.) The number of $r$-derangements is denoted by $D_r(n)$.

**Theorem 1.1** ([9],Theorem 4)**.** *Let $r \in \mathbb{N}^+$ and $s \in \{1, \dots, r\}$. Then for each $n \geq s$ we have*

$$D_r(n) = \sum_{j=s}^{n} \binom{j-1}{s-1} \frac{n!}{(n-j)!} D_{r-s}(n-j).$$

*In particular,*

$$D_r(n) = \sum_{j=r}^{n} \binom{j-1}{r-1} \frac{n!}{(n-j)!} D(n-j),\ n \geq r.$$

*Additionally, we have a closed formula for $r$-derangements numbers:*

$$D_r(n) = \sum_{j=r}^{n} \binom{j}{r} \frac{n!}{(n-j)!} (-1)^{(n-j)},\ n \geq r.$$

Derangements are arrangements of some number of objects into positions such that no object goes to its specified position. In the language of permutations, a derangement is a permutation in which none of the objects appear in their ”natural” (i.e., ordered) place. If we choose a random permutation, the probability that it is a derangement is close to $1/e$. Another version of the problem arises when we ask for the number of ways n letters, each addressed to a different person, can be placed in n pre-addressed envelopes so that no letter appears in the correctly addressed envelope. The derangement problem was formulated by P. R. de Montmort in 1708, and solved by him in 1713 (de Montmort 1713-1714). Nicholas Bernoulli also solved the problem using the inclusion-exclusion principle. The number of derangements of an n-element set is called the n -th derangement number or rencontres number, or the sub-factorial of n. Counting the derangements of a set amounts to what is known as the hat-check problem, in which one considers the number of ways in which n hats can be returned to n people such that no hat makes it back to its owner. This number satisfies the recurrences
More formally, let $S_{n+r}$ be the set of permutations in $[n + r]$ and consider the following:

$$\mathcal{D}(r,n) = \{\pi \in \mathcal{D}_{r+n} : \pi(i) \neq i \text{ for all } 1 \leq i \leq n + r \text{ and } \pi(i) \notin [r] \text{ for all } i \in [r]\}.$$

this is termed set as *block derangements*. Also, let $D(r,n) = \left|\mathcal{D}(r,n)\right|$. In this paper, we obtain a relation between block derangements and $r$-derangements (which is defined in [9]) on $S_{n+r}$.

**Remark 1.2.** It follows from the definition of block derangements that $n$ must be greater than or equal to $r$, i.e., For $n < r$, we have $D(r,n) = 0$. Now let $r = 0$. Then we have

$$D(0,n) = D(n) \text{ and } D(1,n) = D(n+1).$$

Thus the block derangement is a generalization of the derangement.

The interesting this is that the number e itself also has applications in probability theory, in a way that is not obviously related to exponential growth. Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Gordon and McMahon noted that the number of derangements in the hyperoctahedral group gives the rising 2-binomial transform of the derangement numbers for $S_n$. More generally, they shows that the cyclic derangement numbers give a mixed version of the rising r-binomial transform and falling $(r-1)$ binomial transform of $D - n$. This hybrid k-binomial transform may share many of the nice properties of Spivey and Steil’s transforms, including Hankel invariance and/or a simple description of the change in the exponential generating function. Further, it could be interesting to evaluate the expression for negative or even non-integer values of $k$. For instance, taking $k = 1/2$ gives the binomial mean transform which is of some interest.

We define a new special case of derangement, and also we obtain some relation on this subset of derangements. This special case of derangement is a subset of block derangement
A derangement $\pi \in S_{n+r}$ is called a Δ-derangement if $\pi(\Delta) \cap \Delta = \emptyset$. The number of Δ-derangements on $S_{n+r}$ is denoted by $athcalD(r,n)$

$$D(r,n) = |\{\pi \ : \ \pi \in D(n+r), \ \ \pi(\Delta) \cap \Delta = \emptyset\}|.$$
$$D(r,n) = |\{\pi \ : \ \pi \in S_{n+r}, \ \ \Pi(\Delta) \cap \Delta = \emptyset, x^{\pi} \neq x \text{ for all } x \in [n]\}|.$$

The Stirling number of the second kind $S(n,k)$ is the number of partitioning a set of $n$ elements into $k$ nonempty subsets. The Stirling numbers of the second kind can be defined by the exponential generating function

$$\frac{1}{n!}(e^x - 1)^n = \sum_{m=n}^{\infty} S(m,n)\frac{x^m}{m!}$$

Notice that the sum on the right hand side is now finite, as $S(m,n) = 0$ for $n > m$. By [4], for given power series

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$$

we have the series transformation formula

$$f(\frac{\mu}{\lambda}(e^{\lambda t} - 1)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\{\sum_{k=0}^{n} S(n,k)\lambda^{n-k}\mu^k a_k\} \tag{1}$$

where $\lambda$ and $\mu$ are parameters.

$$f\Big(\frac{\mu}{\lambda}\log(1+\lambda t)\Big) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Big\{\sum_{k=0}^{n} s(n,k)\lambda^{n-k}\mu^k a_k\Big\}$$

Let $m$ be a nonnegative integer. For the sequence of Stirling numbers $S(k,m)$ we have

$$\sum_{k=0}^{n} \binom{k}{m} S(k,m) = S(n+1,m+1). \tag{2}$$

Also, by [4]

$$\sum_{k=0}^{n} \binom{k}{m} S(k,m)x^k = \frac{1}{m!}\sum_{j=0}^{m} (-1)^{m-j}(1+jx)^n.$$

For $x = 1$, the left-hand side of this equation equals (2). Therefore,

$$S(n+1,m+1) = \frac{1}{m!}\sum_{j=0}^{m} (-1)^{m-j}(1+j)^n. \tag{3}$$

The integers $b_n = \sum_{k=0}^{n} S(n,k)$ are known as Bell numbers. They give the number of ways a set of n elements can be partitioned into nonempty disjoint subsets.
In this proved:

**Theorem 1.3.** *Suppose $r$ and $n$ are integers and $1 \leq r \leq n$. Then*

$$D(r,n) = r!\sum_{i=r}^{n} (-1)^{n-i}\binom{i}{r}\frac{n!}{(n-i)!}.$$

**Theorem 1.4.** *Suppose $r$ and $n$ are integers and $1 \leq r \leq n$. Then $D(r,n) = r!D_r(n)$.*

**Theorem 1.5.** *Suppose $r$ and $n$ are integers and $1 \leq r \leq n$. Then for each $r$ the sequence $\left\{\frac{D(r,n)}{(n+r)!}\right\}_{n\geq 1}$ tends to $1/e$.*

**Theorem 1.6.** *Let $k$ be an integer number. Then we have*

$$\sum_{k=0}^{n}(-1)^k S(n,k)D(r,k) = \sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{n-j+i+r}r!\binom{r}{i}(1+i)^{n-j}b_j$$

In [9, Theorem 6], it was shown that for each $r \in \mathbb{N}^+$, the sequence $\left\{\frac{D_r(n)}{(n+r)!}\right\}_{n\in\mathbb{N}}$ is convergent to $\frac{1}{r!e}$. Now, we can use Theorem 1.5, to establish asymptotic estimates for $r$-derangements in another way. Also, according to Theorem 1.4, it can be obtained, similar to the main results in [9], some conclusions about the number $D(r,n)$, so we avoid repeating them here. In Section **??**, we obtain the recursion formula for $D(r,n)$. Then we prove the main results of this section. Boyadzhiev gave an interesting combinatorial Stirling transform of derangement numbers in [3]. Similarly, at the end of section **??**, we obtain the Stirling transformation formula of $D(r,n)$.
The set
$\{1,2,\dots,n\}$ and the set $\{i,i+1,\dots,j\}$ are denoted by $[n]$ and $[i,j]$, respectively. Also, assume that a permutation $\pi \in S_n$ is represented as the word $\pi(1)\pi(2)\dots\pi(n)$. Assume that $\Delta = [r]$ and denote by $\pi(\Delta)$, the action of a permutation $\pi \in S_{r+n}$ on the set $\Delta$ by defining $\pi(\Delta) = \{\pi(x), x \in \Delta\}$. Obviously, A derangement $\pi \in S_{r+n}$ is a block derangement if and only if $\pi(\Delta) \cap \Delta = \emptyset$.

**Theorem 1.7.** *For a derangement $D(n) = nD(n-1) + (-1)^n$.*

*Proof.*

$$D(n+1) = nD(n) + nD(n-1)$$
$$= (n+1)D(n) - D(n) + nD(n+1)$$

Now, we use induction on $n$. □

$$nD(n-1) - D(n)$$
$$= (n-1)D(n-1) + D(n-1) -$$
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where $\lambda$ and $\mu$ are parameters.

$$f\Big(\frac{\mu}{\lambda}\log(1+\lambda t)\Big) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Big\{\sum_{k=0}^{n} s(n,k)\lambda^{n-k}\mu^k a_k\Big\}$$

Let $m$ be a nonnegative integer. For the sequence of Stirling numbers $S(k,m)$ we have

$$\sum_{k=0}^{n}\binom{k}{m}S(k,m) = S(n+1,m+1). \tag{2}$$

Also, by [4]

$$\sum_{k=0}^{n}\binom{k}{m}S(k,m)x^k = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}(1+jx)^n.$$

For $x = 1$, the left-hand side of this equation equals (2). Therefore,

$$S(n+1,m+1) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}(1+j)^n. \tag{3}$$

The integers $b_n = \sum_{k=0}^{n} S(n,k)$ are known as Bell numbers. They give the number of ways a set of n elements can be partitioned into nonempty disjoint subsets.
In this proved:

**Theorem 1.3.** *Suppose $r$ and $n$ are integers and $1 \leq r \leq n$. Then*

$$D(r,n) = r!\sum_{i=r}^{n}(-1)^{n-i}\binom{i}{r}\frac{n!}{(n-i)!}.$$

**Theorem 1.4.** *Suppose $r$ and $n$ are integers and $1 \leq r \leq n$. Then $D(r,n) = r!D_r(n)$.*

**Theorem 1.5.** *Suppose $r$ and $n$ are integers and $1 \leq r \leq n$. Then for each $r$ the sequence $\left\{\frac{D(r,n)}{(n+r)!}\right\}_{n\geq 1}$ tends to $1/e$.*

**Theorem 1.6.** *Let $k$ be an integer number. Then we have*

$$\sum_{k=0}^{n}(-1)^k S(n,k)D(r,k) = \sum_{j=0}^{n}\sum_{i=0}^{r}(-1)^{n-j+i+r}r!\binom{r}{i}(1+i)^{n-j}b_j$$

In [9, Theorem 6], it was shown that for each $r \in \mathbb{N}^+$, the sequence $\left\{\frac{D_r(n)}{(n+r)!}\right\}_{n\in\mathbb{N}}$ is convergent to $\frac{1}{r!e}$. Now, we can use Theorem 1.5, to establish asymptotic estimates for $r$-derangements in another way. Also, according to Theorem 1.4, it can be obtained, similar to the main results in [9], some conclusions about the number $D(r,n)$, so we avoid repeating them here. In Section **??**, we obtain the recursion formula for $D(r,n)$. Then we prove the main results of this section. Boyadzhiev gave an interesting combinatorial Stirling transform of derangement numbers in [3]. Similarly, at the end of section **??**, we obtain the Stirling transformation formula of $D(r,n)$.
The set
$\{1,2,\dots,n\}$ and the set $\{i,i+1,\dots,j\}$ are denoted by $[n]$ and $[i,j]$, respectively. Also, assume that a permutation $\pi \in S_n$ is represented as the word $\pi(1)\pi(2)\dots\pi(n)$. Assume that $\Delta = [r]$ and denote by $\pi(\Delta)$, the action of a permutation $\pi \in S_{r+n}$ on the set $\Delta$ by defining $\pi(\Delta) = \{\pi(x), x \in \Delta\}$. Obviously, A derangement $\pi \in S_{r+n}$ is a block derangement if and only if $\pi(\Delta) \cap \Delta = \emptyset$.

**Theorem 1.7.** *For a derangement $D(n) = nD(n-1) + (-1)^n$.*

*Proof.*

$$D(n+1) = nD(n) + nD(n-1)$$
$$= (n+1)D(n) - D(n) + nD(n+1)$$

Now, we use induction on $n$. □

$$nD(n-1) - D(n)$$
$$= (n-1)D(n-1) + D(n-1) -$$
$$((n-1)D(n-1) + (n-1)D(n-2))$$
$$D(n-1) - (n-1)D(n-2)$$
$$D(n-1) - (n-1)D(n-2) =$$

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# Continuity of $S$-Convex Functions

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**Abstract**

A condition which implies the continuity of a $s$-convex function is investigated. In fact any $s$-convex function bounded from above is continuous.

## 1. Introduction

It is known that a convex function defined on an open interval $(a, b) \subseteq \mathbb{R}$ is continuous and then integrable. The key point to prove the continuity of a convex function $F$ is to establish it's boundedness on closed subintervals of $(a, b)$ and then we are able to establish a Lipschitz condition and so we can conclude that $F$ is continuous (see [6]). A generalized convex function is not continuous necessarily and so there is no guarantee for it's integrability. Because of this, usually in the papers that we deal with integration of generalized convex functions, we use the sentence "assume that all considered integrals exist". Therefore one of the tools that guarantees the existence of integral for generalized convex functions is continuity. In this paper, we investigate on a condition that implies the continuity and then integrability of a $s$-convex function. We mainly use the structures stated for convex functions in [6] to obtain our results.
The concept of s-convexity in the second sense was considered in [2] as generalization of ordinary convex functions.
**(Convex function )**
Let $I \subseteq \mathbb{R}$ is an interval. A function $F : I \to \mathbb{R}$ is called convex, if for all $x, y \in I$ and $t \in [0, 1]$ we have

$$F\big(tx + (1 - t)y\big) \leq tF(x) + (1 - t)F(y).$$

**(s-Convex function in the second sense)**
A function $F : \mathbb{R}^+ \to \mathbb{R}^+$ is called Breckner $s$-convex (or briefly $s$-convex), if

$$F\big(tx + (1 - t)y\big) \leq t^s F(x) + (1 - t)^s F(y),$$

for every $x, y \in \mathbb{R}^+$, $t \in [0, 1]$ and fixed $s \in (0, 1]$. If we consider s = 1, then we recapture the usual convexity of $F$. In references [1, 3–5], we can find some basic results about $s$-convex functions.

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## 2. Continuity of $s$-convex functions

First, we give a result about the boundedness of a $s$-convex function on the whole of $(a,b)$. This fact is shown in the following proposition.

**Proposition 2.1.** *Suppose that $F$ is a $s$-convex function defined on $(a,b)$ such that it is bounded from above by $B$. Then $F$ is bounded from below by*

$$\frac{1}{(\frac{1}{2})^s}F\Big(\frac{a+b}{2}\Big)-B.$$

*Proof.* For lower bound of $F$ consider an arbitrary point in the form $\frac{a+b}{2}-t$ in $(a,b)$. Then

$$F\Big(\frac{a+b}{2}\Big)=F\Big(\frac{a+b}{4}+\frac{t}{2}+\frac{a+b}{4}-\frac{t}{2}\Big)=F\Big(\frac{1}{2}\big(\frac{a+b}{2}+t\big)+\frac{1}{2}\big(\frac{a+b}{2}-t\big)\Big)$$
$$\leq(\frac{1}{2})^s\Big[F\Big(\frac{a+b}{2}+t\Big)+F\Big(\frac{a+b}{2}-t\Big)\Big].$$

or

$$F\Big(\frac{a+b}{2}+t\Big)\geq\frac{1}{(\frac{1}{2})^s}F\Big(\frac{a+b}{2}\Big)-F\Big(\frac{a+b}{2}-t\Big).$$

Applying $B$ as the upper bound of $F$ we have,

$$-F\Big(\frac{a+b}{2}-t\Big)\geq-B.$$

So

$$F\Big(\frac{a+b}{2}+t\Big)\geq\frac{1}{(\frac{1}{2})^s}F\Big(\frac{a+b}{2}\Big)-B.$$

Since $t$ is arbitrary, then $\frac{1}{(\frac{1}{2})^s}F\Big(\frac{a+b}{2}\Big)-B$ can be chosen as the lower bound of $F$ on $(a,b)$. □

For continuity of a $s$-convex function we need the assumption of boundedness from above on $(a,b)$ and a certain condition.

**Lemma 2.2.** *Consider two nonnegative constants $\alpha,\beta$. Corresponding to any $\epsilon>0$, there exists $t\in(0,1)$ such that*

$$\frac{t^s}{(1-t)^s}\alpha+\frac{[(1-t)^s-1]}{(1-t)^s}\beta<\epsilon,$$

*and*

$$t^s\alpha+[(1-t)^s-1]\beta<\epsilon.$$

*Proof.* For any $\epsilon>0$ and any positive real constants $\alpha,\beta$ we have

$$\begin{cases}\exists\delta_1;\ \frac{t^s}{(1-t)^s}\alpha<\epsilon/2, & 0<t<\delta_1;\\ \exists\delta_2;\ \frac{(1-t)^s-1}{(1-t)^s}\beta<\epsilon/2, & 0<t<\delta_2.\end{cases}$$

By the choose of $0<t<\min\{\delta_1,\delta_2\}$ and the fact that $(1-t)^s<1$, we have the respected results. □

**Theorem 2.3.** *Suppose that $F$ is a $s$-convex function defined on $(a,b)$ such that it is bounded from above by $B$. Then $F$ is continuous on $(a,b)$.*

*Proof.* Suppose that $F$ is bounded from above by $B$ on $(a,b)$. Choose fixed $x_0 \in (a,b)$ and consider arbitrary $\epsilon > 0$ enough small such that $(x_0-\epsilon, x_0+\epsilon) \subset (a,b)$ as a neighborhood of $x_0$. We can find $\delta > 0$ $\big($open interval $(-\delta,\delta)\big)$ such that $(x_0-\delta, x_0+\delta) \subset (x_0-\epsilon, x_0+\epsilon)$. Consider $t \in (0,1)$ satisfying Lemma 2.2 with respect to real positive numbers $\alpha = B$ and $\beta = F(x_0)$. Now choose $x \in (x_0 - t\delta, x_0 + t\delta) \subset (x_0-\delta, x_0+\delta)$ and set

$$y = x - \frac{x-x_0}{t},$$

$$z = x_0 + \frac{x-x_0}{t}.$$

So

$$y - x_0 = \frac{1-t}{t}(x_0 - x) \in \frac{1-t}{t}(-t\delta, t\delta) \subset (-\delta,\delta),$$

and

$$z - x_0 = \frac{x-x_0}{t} \in \frac{1}{t}(-t\delta, t\delta) = (-\delta,\delta),$$

showing that $y, z \in (x_0-\delta, x_0+\delta)$.
Since $f$ is $s$-convex, then from relations

$$\begin{cases} x_0 = ty + (1-t)x, \\ x = tz + (1-t)x_0, \end{cases}$$

we have that

$$F(x_0) \le t^s F(y) + (1-t)^s F(x) \le t^s B + (1-t)^s F(x),$$

and

$$F(x) \le t^s F(z) + (1-t)^s F(x_0) \le t^s B + (1-t)^s F(x_0).$$

It follows that

$$F(x) - F(x_0) \ge -\frac{t^s}{(1-t)^s} B - \Big[1 - \frac{1}{(1-t)^s}\Big] F(x_0),$$

and

$$F(x) - F(x_0) \le t^s B + [(1-t)^s - 1] F(x_0).$$

Then we get

$$F(x) - F(x_0) > -\epsilon,$$

and

$$F(x) - F(x_0) < \epsilon.$$

Then for any $x \in (x_0-\delta, x_0+\delta)$, we have $|F(x) - F(x_0)| < \epsilon$, which implies that $F$ is continuous at $x_0$. Since $x_0$ is an arbitrary point in $(a,b)$, then we conclude the continuity of $F$ on $(a,b)$. □

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# A QSPR Study of Antibiotics Based on Degree-Based Topological Indices and Regression Modeling

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**Abstract**

This study utilized M-polynomials to calculate some topological indicators based on Degree for a collection of seventeen antibiotics commonly used to treat infections, including Amoxicillin, Penicillin G, Ampicillin, Piperacillin, Cefalexin, Cefuroxime, Ceftriaxone, Meropenem, Imipenem, Aztreonam, Azithromycin, Erythromycin, Clarithromycin, Gentamicin, Tobramycin, Amikacin, and Streptomycin. These indices were subsequently integrated into (QSPR) models to predict different kinds of chemicals in their physical or chemical state. Our analysis demonstrates that the regression models developed, consisting of fifty-six lines, successfully predict these physicochemical characteristics using the topological indices. The findings indicate that topological indices offer valuable insights for the design and optimization of antibiotics, supported by statistically significant findings ($p - value < 0.05$).

## 1. Introduction

There are many germs, known as bacteria, both within and beyond our bodies. While some bacteria are beneficial to humans, the majority are either harmless or harmful activity can lead to infections, for instance, strep throat and urinary tract infections [7]. Antibiotics are medicines that fight bacterial infections and stop their growth in people, animals, and craps [7]. Preventing and treating infections are crucial roles of Antibiotics and they are among the most potent drugs for treating life-threatening bacterial infections. Viruses, on the other hand, are distinct from bacteria. They lead to infections, including colds and the flu, many sinus infections, and some ear infections. Antibiotics are not effective for infections resulting from viruses.
(QSPR) analysis has become an essential computational tool in drug discovery and design. This has led to increased interest in chemical graph theory, which studies the relationships between molecular structures, properties, and activities. Molecular graphs represent molecules as networks of atoms (vertices) connected by chemical bonds (edges). Topological indices, mathematical descriptors of these graphs, are used to analyze the structure and properties of molecules [4, 5]. A key approach for investigating relationships between physicochemical properties and TIs is QSPR

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modeling. These models use regression analysis to establish connections between physicochemical features and their corresponding TI. Similarly, QSAR studies often use TIs to analyze drug activity [5, 6, 10]. This research aims to explore how various topological indices can help researchers understand the relevant physicochemical properties and potential chemical reactions associated with drugs used to treat infections. Contemporary research extensively utilizes (QSPR) and (QSAR) models, in conjunction with M-polynomial indices, to forecast the Physical or Chemical characteristics of compounds predicated on their connectivity descriptors. This procedure has found broad utilization in numerous scientific disciplines, incorporating cancer research, nanotube studies, and the investigation of M-polynomials. It effectively conveys the idea of recent research efforts focused on the relationship between M-polynomials and topological indices... [4, 7, 8, 11].

## 2. Graph-theoretical preliminaries

A molecular graph visually represents a chemical compound's structure using graph theory. Atoms are depicted as vertices, and bonds as edges in the simple, finite, and connected graph G (V, E). A vertex's degree, d(v), correlates with the compound's valence. For more details, consult references [1, 6, 10]. A connectivity index is a numerical descriptor that identifies the structural properties of a graph. It is calculated from the graph's topology, which refers to the arrangement of vertices and edges in the graph, independent of the spatial embedding of the graph. It offers a quantitative representation of the arrangement of atoms and their connectivity within a molecule and is employed in the expansion of (QSAR) and (QSPR) models. Three primary categories of structural descriptors exist: Based on Degree, Based on Distance, and Based on Spectral. Among these, degree-based connectivity indices have been extensively explored and exhibit a robust correlation with a multitude of properties of the molecules under investigation [9, 12, 13]. S. Klavžar and E. Deutsch introduced M-polynomial in 2015 [3].

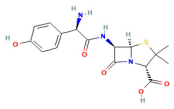
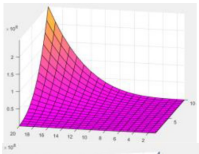
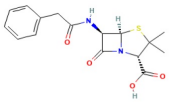
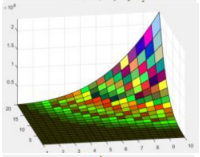
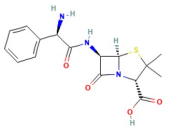
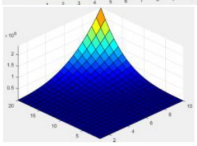
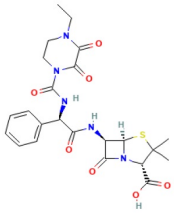
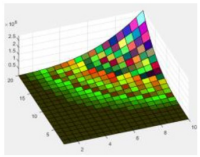
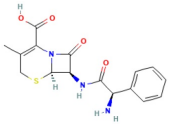
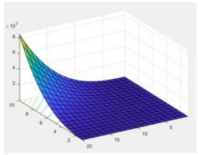
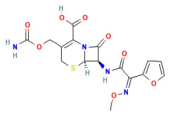
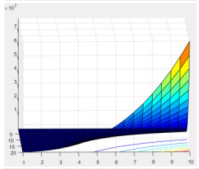
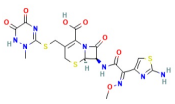
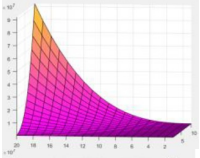
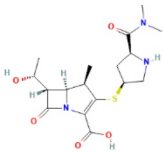
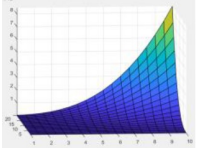
Table 1. Some connectivity descriptors are based on degree and their definitions.

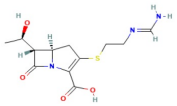
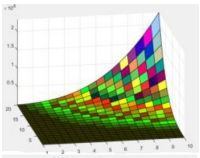
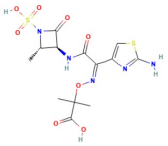
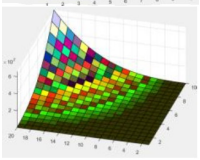
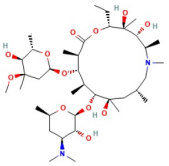
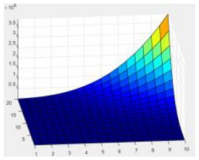
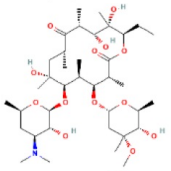
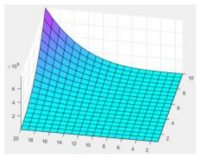
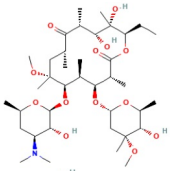
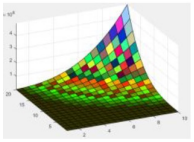
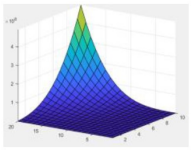
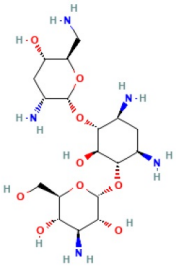
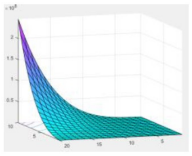
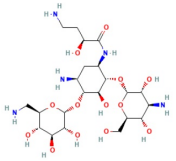
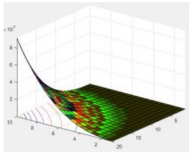
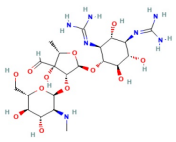
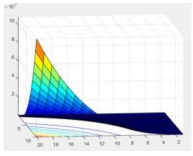
| TI Name | Notation | Formula of TI |
|---|---|---|
| First Zagreb | $M_1(G)$ | $M_1(G) = \sum_{uv \in E(G)} (d_u + d_v)$ |
| Second Zagreb | $M_2(G)$ | $M_2(G) = \sum_{uv \in E(G)} (d_u \cdot d_v)$ |
| Hyper Zagreb | $HM_1(G)$ | $HM_1(G) = \sum_{uv \in (G)} (d_u + d_v)^2$ |
| Randić | $R(G)$ | $R(G) = \sum_{uv \in (G)} \sqrt{\frac{1}{d_u \cdot d_v}}$ |
| Harmonic | $H(G)$ | $H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}$ |
| Sum-connectivity | $S(G)$ | $S(G) = \sum_{uv \in (G)} \sqrt{\frac{1}{d_u + d_v}}$ |
| Forgotten | $F(G)$ | $F(G) = \sum_{uv \in (G)} \left[(d_u)^2 + (d_v)^2\right]$ |
| Geometric Arithmetic | $GA(G)$ | $GA(G) = \sum_{uv \in (G)} \frac{2\sqrt{d_u \cdot d_v}}{d_u + d_v}$ |
| Atomic Bond Connectivity | $ABC(G)$ | $ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u \cdot d_v}}$ |
| M-polynomial | $M(G,x,y)$ | $M(G,x,y) = \sum_{i \leq j} m_j(G) x^i y^j$ |

## 3. Methodology

The polynomial values for each drug were computed according to the polynomial definition. The results, including Molecule Title, Construction, Structural Depiction, M-Polynomials, and their surface plots are demonstrated in Table 2.

Table 2. Antibiotics with their molecule title, construction, structural depiction.

| Molecule Title | Construction | Structural Depiction | M-Polynomial | Plot |
|---|---|---|---|---|
| Amoxicillin | $C_{16}H_{19}N_3O_5S$ |  | $6xy^3+2xy^4+2x^2y^2+7x^2y^3+x^2y^4+8x^3y^3+x^3y^4$ |  |
| Penicillin G | $C_{16}H_{17}N_2NaO_4S$ |  | $Xy^2+3xy^3+2xy^4+4x^2y^2+6x^2y^3+x^2y^4+6x^3y^3+x^3y^4$ |  |
| Ampicillin | $C_{16}H_{19}N_3O_4S$ |  | $5xy^3+2xy^4+4x^2y^2+5x^2y^3+x^2y^4+7x^3y^3+x^3y^4$ |  |
| Piperacillin | $C_{23}H_{27}N_5O_7S$ |  | $Xy^2+7xy^3+2xy^4+5x^2y^2+10x^2y^3+x^2y^4+12x^3y^3+x^3y^4$ |  |
| Cefalexin | $C_{16}H_{17}N_3O_4S$ |  | $6xy^3+4x^2y^2+6x^2y^3+10x^3y^3$ |  |
| Cefuroxime | $C_{16}H_{16}N_4O_8S$ |  | $Xy^2+6xy^3+6x^2y^2+9x^2y^3+9x^3y^3$ |  |
| Ceftriaxone | $C_{18}H_{18}N_8O_7S_3$ |  | $Xy^2+8xy^3+4x^2y^2+14x^2y^3+10x^3y^3$ |  |
| Meropenem | $C_{17}H_{25}N_3O_5S$ |  | $9xy^3+x^2y^2+5x^2y^3+10x^3y^3$ |  |

| | | | | |
|---|---|---|---|---|
| Imipenem | $C_{12}H_{17}N_3O_4S$ |  | $Xy^2 + 3xy^3 + 2xy^4 + 4x^2y^2 + 6x^2y^3 + x^2y^4 + 6x^3y^3 + x^3y^4$ |  |
| Aztreonam | $C_{13}H_{17}N_5O_8S_2$ |  | $Xy^2 + 5xy^3 + 4x^2y^2 + 2x^2y^3 + 8x^3y^3$ |  |
| Azithromycin | $C_{38}H_{72}N_2O_{12}$ |  | $6xy^3 + 5xy^4 + 2x^2y^2 + 7x^2y^3 + 7x^3y^3 + 2x^3y^4$ |  |
| Erythromycin | $C_{37}H_{67}NO_{13}$ |  | $2xy^2 + 13xy^3 + 5xy^4 + 17x^2y^3 + 3x^2y^4 + 10x^3y^3 + 4x^3y^4$ |  |
| Clarithromycin | $C_{38}H_{69}NO_{13}$ |  | $2xy^2 + 13xy^3 + 4xy^4 + 15x^2y^3 + 4x^2y^4 + 11x^3y^3 + 2x^3y^4$ |  |
| Gentamicin | $C_{21}H_{43}N_5O_7$ |  | $3xy^2 + 13xy^3 + 4xy^4 + 15x^2y^3 + 4x^2y^4 + 11x^3y^3 + 2x^3y^4$ |  |
| Tobramycin | $C_{18}H_{37}N_5O_9$ |  | $2xy^2 + 4xy^3 + 2xy^4 + 2x^2y^2 + 13x^2y^3 + x^2y^4 + 7x^3y^3 + x^3y^4$ |  |
| Amikacin | $C_{22}H_{43}N_5O_{13}$ |  | $2xy^2 + 8xy^3 + 14x^2y^3 + 10x^3y^3$ |  |
| Streptomycin | $C_{21}H_{39}N_7O_{12}$ |  | $3xy^2 + 10xy^3 + x^2y^2 + 14x^2y^3 + 12x^3y^3$ |  |

## 4. Topological descriptors and QSPR modeling

Nine topological indices, including ($M_1$) Index, ($M_2$) Index, (H) Index, (HM) Index, (F) Index, (R) Index, (SCI) Index, (GA) Index, (ABC) Index (which are identified in Table 1), were employed to model nine representative physical features: Boiling Point (BP), Enthalpy of Vaporization (EN), Index of Refraction (IR), Molar Refractivity (MR), Polar Surface Area (PSA), Polarizability (PO), Molar Volume (MV), Complexity (COM), Molar Weight (MW) with seventeen Antibiotics drugs: Amoxicillin, Penicillin G, Ampicillin, Piperacillin, Cephalexin, Cefuroxime, Ceftriaxone, Meropenem, Imipenem, Aztreonam, Azithromycin, Erythromycin, Clarithromycin, Gentamicin, Tobramycin, Amikacin, Streptomycin. Topological indices, derived from the degree of precision and experimentally measured values of characteristics of seventeen antibiotics (obtained from ChemSpider [2]), are presented in Tables 3 and 4. As shown in Table 5, these values exhibit a normal distribution, prompting the use of regression modeling (56 models) to analyze the data.

### *4.1. Regression models*

We used a linear regression model to analyze relationship between characteristics of antibiotics (P) and topological indices (TI):

$$P = B + A(TI)$$

While B represents a constant, A denotes the regression coefficient, and TI signifies the topological Descriptor. SPSS software used to analyze nine characteristics and nine topological indices for seventeen antibiotics. Using this equation, Types linear models were developed for different topological indices.

1. **First Zagreb Index** $[M_1(G)]$
   $BP = 511.664 + 1.319[M_1(G)]$, $IR = 1.838 + (-0.001)[M_1(G)]$, $MR = -7.022 + 0.717[M_1(G)]$,
   $MV = -125.794 + 2.586[M_1(G)]$, $COM = 46.195 + 4.217[M_1(G)]$, $MW = 11.768 + 2.708[M_1(G)]$
2. **Second Zagreb Index** $[M_2(G)]$
   $BP = 510.654 + 1.098[M_2(G)]$, $IR = 1.842 + (-0.001)[M_2(G)]$, $MR = -8.470 + 0.601[M_2(G)]$,
   $MV = -131.475 + 2.170[M_2(G)]$, $COM = 36.188 + 3.540[M_2(G)]$, $MW = 10.218 + 2.255[M_2(G)]$
3. **Harmonic Index** $[H(G)]$
   $BP = 489.693 + 17.407[H(G)]$, $EN = 39.720 + 4.955[H(G)]$, $IR = 1.829 + (-0.011)[H(G)]$,
   $MR = -11.491 + 8.926[H(G)]$, $MV = -135.447 + 31.768[H(G)]$, $COM = 13.818 + 53.005[H(G)]$,
   $MW = -2.587 + 33.681[H(G)]$
4. **Hyper-Zagreb Index** $[HM(G)]$
   $BP = 526.226 + 0.239[HM(G)]$, $IR = 1.840 + 0.000183[HM(G)]$, $MR = -2.532 + 0.134[HM(G)]$,
   $MV = -113.141 + 0.486[HM(G)]$, $COM = 75.823 + 0.783[HM(G)]$, $MW = 27.673 + 0.506[HM(G)]$
5. **Forgotten Topological Index** $[F(G)]$
   $BP = 538.695 + 0.421[F(G)]$, $IR = 1.837 + 0.000338[F(G)]$, $MR = 2.532 + 0.241[F(G)]$,
   $MV = -97.102 + 0.880[F(G)]$, $COM = 109.024 + 1.400[F(G)]$, $MW = 43.783 + 0.913[F(G)]$
6. **Randic Index** $[R(G)]$
   $BP = 499.773 + 15.708[R(G)]$, $EN = 42.710 + 4.464[R(G)]$, $IR = 1.831 + (-0.010)[R(G)]$,
   $MR = -8.146 + 8.193[R(G)]$, $MV = -126.167 + 29.321[R(G)]$, $COM = 38.535 + 48.336[R(G)]$,
   $MW = 7.859 + 31.004[R(G)]$
7. **Sum-Connectivity Index** $[SCI(G)]$MR
   $BP = 493.895 + 15.674[SCI(G)]$, $EN = 41.460 + 4.429[SCI(G)]$, $MR = -11.182 + 8.156[SCI(G)]$,
   $MV = -135.802 + 29.116[SCI(G)]$, $COM = 17.110 + 48.330[SCI(G)]$, $MW = -1.973 + 30.781[SCI(G)]$
8. **Geometric Arithmetic Index** $[GA(G)]$
   $BP = 489.431 + 7.607[GA(G)]$, $EN = 40.530 + 2.140[GA(G)]$, $MR = -13.515 + 3.953[GA(G)]$,
   $MV = -50.544 + 11.991[GA(G)]$, $COM = 0.547 + 23.500[GA(G)]$, $MW = -8.919 + 14.872[GA(G)]$
9. **Atomic Bond Connectivity Index** $[ABC(G)]$
   $BP = 506.880 + 9.284[ABC(G)]$, $EN = 45.785 + 2.599[ABC(G)]$, $IR = 1.835 + (-0.006)[ABC(G)]$,
   $MR = -7.650 + 4.969[ABC(G)]$, $MV = -126.611 + 17.866[ABC(G)]$, $COM = 42.900 + 29.242[ABC(G)]$,
   $MW = 8.741 + 18.804[ABC(G)]$

The physical properties of antibiotics utilized for treating infections are presented in the Table 3.

Table 3. Physical features of Antibiotics.

| Drugs | BP | EN | IR | MR | PSA | PO | MV | COM | MW |
|---|---|---|---|---|---|---|---|---|---|
| Amoxicillin | 743.2 | 113.7 | 1.702 | 91.5 | 158 | 36.3 | 236.2 | 590 | 365.4 |
| Penicillin G | – | – | – | – | 115 | – | – | 536 | 356.4 |
| Ampicillin | 683.9 | 5.4 | 1.675 | 89.9 | 138 | 35.7 | 239.3 | 562 | 349.4 |
| Piperacillin | – | – | 1.678 | 128.5 | 182 | 50.9 | 340.5 | 982 | 517.6 |
| Cephalexin | 727.4 | 111.5 | 1.7 | 89.4 | 138 | 35.4 | 231.3 | 600 | 347.4 |
| Cefuriaxone | – | – | 1.735 | 96.7 | 199 | 383 | 241 | 798 | 424.4 |
| Ceftriaxone | – | – | 1.889 | 130 | 288 | 515 | 281.7 | 1110 | 554.6 |
| Meropenem | 627.4 | 106.4 | 1.639 | 96.8 | 135 | 38.4 | 268.9 | 679 | 383.5 |
| Imipenem | 530.2 | 92.7 | 1.721 | 72.7 | 139 | 28.8 | 183.9 | 491 | 299.35 |
| Aztreonam | – | – | – | – | 243 | – | – | – | 435.4 |
| Azithromycin | 822.1 | 136 | 1.537 | 197.6 | 180 | 78.3 | 632.7 | 1150 | 749 |
| Erythromycin | 818.4 | 135.4 | 1.535 | 189.2 | 194 | 75 | 607.2 | 1180 | 733.9 |
| Clarithromycin | 805.5 | 133.4 | 1.526 | 194 | 183 | 76.9 | 631.9 | 1190 | 748 |
| Gentamicin | 669.4 | 112.6 | 1.583 | 122.6 | 200 | 48.6 | 366.9 | 636 | 477.6 |
| Tobramycin | 775 | 128.7 | 1.651 | 111.7 | 268 | 44.3 | 305.9 | 609 | 467.5 |
| Amikacin | 981.8 | 162.2 | 1.664 | 134.9 | 332 | 53.5 | 363.9 | 819 | 585.6 |
| Streptomycin | 872.9 | 144.1 | 1.762 | 121 | 331 | 48 | 293.5 | 940 | 581.6 |

Table 4. The calculated values of the topological indices for antibiotics under investigation.

| Drugs | $M_1(G)$ | $M_2(G)$ | $H(G)$ | $HM(G)$ | $F(G)$ | $R(G)$ | $SCI(G)$ | $GA(G)$ | $ABC(G)$ |
|---|---|---|---|---|---|---|---|---|---|
| Amoxicillin | 137 | 165 | 10.95 | 715 | 385 | 11.70 | 12.11 | 25.56 | 19.72 |
| Penicillin G | 127 | 156 | 10.25 | 669 | 357 | 10.80 | 11.26 | 23.91 | 17.97 |
| Ampicillin | 132 | 161 | 10.58 | 692 | 370 | 11.23 | 11.68 | 24.76 | 18.86 |
| Piperacillin | 196 | 239 | 16.08 | 1018 | 540 | 16.97 | 17.62 | 37.33 | 28.11 |
| Cephalexin | 128 | 155 | 10.9 | 650 | 340 | 11.41 | 11.85 | 25.07 | 18.67 |
| Cefuriaxone | 150 | 179 | 13.26 | 750 | 392 | 13.84 | 14.27 | 29.95 | 22.21 |
| Ceftriaxone | 192 | 231 | 16.33 | 972 | 510 | 17.11 | 17.77 | 37.56 | 28.00 |
| Meropenem | 142 | 175 | 11.4 | 742 | 392 | 12.14 | 12.58 | 26.67 | 20.29 |
| Imipenem | 102 | 123 | 9.033 | 516 | 270 | 9.485 | 9.684 | 20.21 | 15.07 |
| Aztreonam | 142 | 165 | 11.33 | 740 | 410 | 12.32 | 12.55 | 26.07 | 20.85 |
| Azithromycin | 269 | 323 | 21.70 | 1415 | 769 | 23.35 | 23.86 | 49.86 | 38.81 |
| Erythromycin | 270 | 324 | 21.64 | 1426 | 778 | 23.42 | 23.82 | 49.62 | 39.00 |
| Clarithromycin | 274 | 330 | 22.24 | 1446 | 786 | 23.98 | 24.36 | 50.71 | 39.55 |
| Gentamicin | 175 | 212 | 14.55 | 905 | 481 | 15.34 | 15.87 | 33.49 | 25.17 |
| Tobramycin | 168 | 202 | 14.26 | 856 | 452 | 15.08 | 15.49 | 32.53 | 24.51 |
| Amikacin | 206 | 247 | 17.83 | 1046 | 552 | 18.85 | 19.24 | 40.18 | 30.26 |
| Streptomycin | 209 | 251 | 17.61 | 1079 | 577 | 18.77 | 19.12 | 39.88 | 30.44 |

Table 5. Correlation coefficients associated with the physical properties of Antibiotics.

| TI | BP | EN | IR | MR | PSA | PO | MV | COM | MW |
|---|---|---|---|---|---|---|---|---|---|
| $M_1$ | 0.672 | 0.566 | -0.531 | 0.982 | 0.365 | 0.041 | 0.936 | 0.917 | 0.994 |
| $M_2$ | 0.666 | 0.560 | -0.536 | 0.982 | 0.349 | 0.033 | 0.937 | 0.917 | 0.99 |
| $H$ | **0.706** | **0.609** | -0.481 | 0.968 | 0.428 | 0.088 | 0.910 | 0.920 | 0.992 |
| $HM$ | 0.651 | 0.544 | -0.557 | **0.985** | 0.33 | 0.012 | **0.946** | 0.910 | 0.99 |
| $F$ | 0.639 | 0.531 | **-0.572** | **0.985** | 0.314 | -0.005 | 0.952 | 0.904 | 0.988 |
| $R$ | 0.697 | 0.601 | -0.499 | 0.972 | 0.415 | 0.068 | 0.919 | 0.917 | 0.995 |
| $SCI$ | 0.698 | 0.598 | -0.493 | 0.973 | 0.413 | 0.079 | 0.918 | 0.921 | 0.994 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $GA$ | 0.698 | 0.595 | -0.489 | 0.973 | 0.409 | -0.107 | 0.820 | **0.924** | 0.992 |
| $ABC$ | 0.682 | 0.579 | -0.518 | 0.979 | 0.387 | 0.055 | 0.931 | 0.919 | **0.996** |

## 5. Conclusion

Table 5 illustrates the correlation between the physicochemical characteristics of the compounds under investigation of Antibiotics and defined topological indices. In this paper, the Harmonic Index (H(G)) showed a positive and significant association with (BP) and (EN) with (r=0.706), (r=0.609), respectively. The Index of Refractivity (IR) property has an Inverse significant correlation with Forgotten Index (F(G)) and it has a high and significant correlation value (r=0.985). In addition, (MR) and (MV) are the best predictors of Haper-Zagreb Index (HM(G)) with (r=0.985), (r=0.945), respectively. Complexity property predicts a high value for (GA(G)) Index with (r=0.924). Atomic Bond Connectivity Index (ABC(G)) showed exhibited a high degree of correlation with (MW), (r=0.996). Polar Surface Area (PSA) and Polarizability (PO) did not have a significant correlation with any of the indices. The Index of Refractivity (IR) performed inverse and significant correlation with all of the indices. Molar Weight (MW), Complexity (COM), and Molar Refractivity (MR) showed a positive and significant correlation with all indices value over than (r=0.900). In summary, a strong relationship was established between the physicochemical features of antibiotics and the connectivity indicators. The results by the regression models in which All correlations exhibited statistical significance ($r > 0.5, p < 0.05$), demonstrating considerable linear prediction. Equations were chosen based on of minimum standard error (SE), the highest coefficient of determination ($R^2$), and the highest F-statistic (F). All physical and chemical properties were highly significant, highlighting the potential importance of these topological indices in QSPR analysis for Antibiotic drugs. The plotted regression lines (fifty-six lines) further support this conclusion. The findings could be used to improve the production, development, and effectiveness of Antibiotics drugs. Moreover, the methodology can be applied to investigate the structures of other drugs.

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# Special point of view in the introduction of formal local cohomology modules

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**Abstract**

Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. Let $M$ be a finitely generated $R$-module. There is a systematic study of the formal local cohomology modules $\varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)$, $i \in \mathbb{Z}$. We state their $R$-module structure and prove in particular that for any integer $i$, $i$th formal local cohomology module, occurs as the $i$th cohomology module of the $\mathfrak{a}$-adic completion from a $\check{C}$ech complex.

## 1. Introduction

Let $\mathfrak{a}$ denote an ideal of a local ring $(R,\mathfrak{m})$. For a finitely generated $R$-module $M$ let $H^i_{\mathfrak{a}}(M)$, $i \in \mathbb{N}$, denote the $i$th local cohomology module of $M$ with respect to $\mathfrak{a}$. We know that for an integer $i \in \mathbb{Z}$, the family of local cohomology modules $\{H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)\}_{n\in\mathbb{N}}$ by the natural homomorphisms form a projective system. their projective limit, $\varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^nM)$, is called the $i$th formal local cohomology module of $M$ with respect to $\mathfrak{a}$. there exists a relation between local cohomology, given by [1], and formal local cohomology, given by [4]. Not so much is known about these modules. The motivation of this work is precisely the formal local cohomology of P. Schenzel.
The main subject of the paper is to prove that $i$th formal local cohomology of $M$ with respect to $\mathfrak{a}$ occurs as the $i$th cohomology module of the $\mathfrak{a}$-adic completion of the $\check{C}$ech complex, $\check{C}_{\underline{x}} \otimes M$, where $\underline{x}$ denotes a system of elements of $R$ such that $\mathrm{Rad}(\underline{x}R) = \mathfrak{m}$.

## 2. On the definition of formal cohomology

Let $(R,\mathfrak{m},k)$ be a local noetherian ring. For an arbitrary $R$-complex $X : \cdots \longrightarrow X^n \xrightarrow{d^n_X} X^{n+1} \longrightarrow \cdots$ there is a complex $I$ of injective $R$-modules and a quasi-isomorphism $X \xrightarrow{\sim} I$. we call $I$ an injective resolution of $X$.

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Let $\underline{x} = x_1, x_2, \cdots, x_r$ be a sequence of elements of $R$ and $\mathfrak{c} = (x_1, x_2, \cdots, x_r)R$ the ideal generated by this elements. The local cohomology $R\Gamma_{\mathfrak{c}}(X)$ of $X$ with respect to $\mathfrak{c}$ in the derived category is defined by $\Gamma_{\mathfrak{c}}(I)$, where $X \xrightarrow{\sim} I$ denotes the injective resolution. For an integer $i \in \mathbb{Z}$ define $H^i_{\mathfrak{c}}(X) = H^i(\Gamma_{\mathfrak{c}}(I))$. Note that up to isomorphisms it is independent on $I$.
Moreover let $\check{C}_{\underline{x}}$ denote the Čech complex of $R$ with respect to $\underline{x}$. then there is a canonical isomorphism $\Gamma_{\mathfrak{c}}(I) \cong \check{C}_{\underline{x}} \otimes I$ for a complex of injective $R$-modules $I$ [5]. Because $\check{C}_{\underline{x}}$ is bounded $R$-complex of flat $R$-modules it induces the following isomorphism $\check{C}_{\underline{x}} \otimes X \cong \check{C}_{\underline{x}} \otimes I$. That is the local cohomology $H^i_{\mathfrak{c}}(X)$, $i \in \mathbb{Z}$, may be computed as the cohomology $H^i(\check{C}_{\underline{x}} \otimes X)$.
Let $\underline{x} = x_1, x_2, \cdots, x_r$ denote a system of elements of $R$ and $\mathrm{Rad}(\underline{x}R) = \mathfrak{b}$. Let $\check{C}_{\underline{x}}$ denote the Čech complex of $R$ with respect to $\underline{x}$. For an $R$-module $M$ and an ideal $\mathfrak{a}$ the projective system of $R$-modules $\{M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$ induces a projective system of $R$-complexes $\{\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M\}_{n\in\mathbb{N}}$. Its projective limit is the main object of our investigations.

**Definition 2.1.** For an integer $i \in \mathbb{Z}$ the cohomology module $H^i(\varprojlim_{n\in\mathbb{N}}(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M))$ is called the $i$th $\mathfrak{a}$-formal cohomology with respect to $\mathfrak{b}$. In the case of $\mathfrak{b} = \mathfrak{m}$ we speak simply about $i$th $\mathfrak{a}$-formal cohomology. By abuse of notaion we say also formal cohomology in case there will be no doubt on $\mathfrak{a}$.

In the following let $\Lambda^{\mathfrak{a}} = \varprojlim_{n\in\mathbb{N}}(- \otimes R/\mathfrak{a}^n)$ denote the $\mathfrak{a}$-adic completion. For an $R$-module $M$ it turns out that the complex $\varprojlim_{n\in\mathbb{N}}(\check{C}_{\underline{x}} \otimes M \otimes R/\mathfrak{a}^n)$ is isomorphic to $\Lambda^{\mathfrak{a}}(\check{C}_{\underline{x}} \otimes M)$. In the derived category this complex is isomorphic to $\Lambda^{\mathfrak{a}}(\Gamma_{\mathfrak{b}}(I))$, where $M \xrightarrow{\sim} I$ denotes an injective resolution of $M$.
As a result here there is a relation of the formal cohomology with respect to the projective limits of certain local cohomology modules. Here $\lim^1$ denotes the first right derived functor of the inverse limit functor.

**Theorem 2.2.** *With the previous notation there is the following short exact sequence*

$$0 \longrightarrow \varprojlim_{n\in\mathbb{N}}{}^1 H^{i+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M) \longrightarrow H^i(\varprojlim_{n\in\mathbb{N}}(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M)) \longrightarrow \varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{b}}(M/\mathfrak{a}^n M) \longrightarrow 0$$

*for all $i \in \mathbb{Z}$. In the case $\mathfrak{b} = \mathfrak{m}$ and a finitely generated $R$-module $M$ it provides isomorphisms*

$$H^i(\varprojlim_{n\in\mathbb{N}}(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M)) \cong \varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$$

*for all $i \in \mathbb{Z}$.*

*Proof.* The Čech complex $\check{C}_{\underline{x}}$ is a complex of flat $R$-modules. Whence the natural epimorphism $M/\mathfrak{a}^{n+1}M \longrightarrow M/\mathfrak{a}^n M$, $n \in \mathbb{N}$, induces an $R$-morphism of $R$-complexes

$$\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^{n+1}M \longrightarrow \check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M.$$

By the definition of the projective limit there is a short exact sequence of complexes

$$0 \longrightarrow \varprojlim_{n\in\mathbb{N}}(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M) \longrightarrow \prod(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M) \longrightarrow \prod(\check{C}_{\underline{x}} \otimes M/\mathfrak{a}^n M) \longrightarrow 0.$$

Now the long exact cohomology sequence provides the first part of the claim. To this end break it up into short exact sequences and take into account that homology commutes with direct products.
For the proof of the second part remember that $H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$, $i \in \mathbb{Z}$, is an Artiniaan $R$-module whenever $M$ is finitely generated [3]. So the corresponding projective system satisfies the Mittage-Leffler condition. that is, $\lim^1$ vanishes on the projective system of Artinian $R$-modules [2]. The proof is now a consequence of the first part.

□

Let $(\hat{R}, \hat{\mathfrak{m}})$ denote the $\mathfrak{m}$-adic completion of $(R, \mathfrak{m})$. An Artinian $R$-module $A$ has a natural structure of an $\hat{R}$-module such that the natural homomorphisms $A \longrightarrow \hat{A}$ and $A \longrightarrow A \bigotimes \hat{R}$ are isomorphisms.

**Proposition 2.3.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $M$ be a finitely generated $R$-module and $\mathfrak{a}$ be an ideal of $R$. Then $\varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$, $i \in \mathbb{Z}$, has a natural structure as an $\hat{A}$-module and there are isomorphisms*

$$\varprojlim_{n\in\mathbb{N}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong \varprojlim_{n\in\mathbb{N}} H^i_{\hat{\mathfrak{m}}}(\hat{M}/\mathfrak{a}^n \hat{M})$$

*for all $i \in \mathbb{Z}$.*

*Proof.* Let $N$ be a finitely generated $R$-module.Then is known that $H^i_{\mathfrak{m}}(N)$, $i \in \mathbb{Z}$, is an Artinian $R$-module [3]. Because of the previous remarks and the flatness of $\hat{R}$ over $R$ there are $R$-isomorphisms $H^i_{\mathfrak{m}}(N) \cong H^i_{\hat{\mathfrak{m}}}(\hat{N})$ for all $i \in \mathbb{Z}$. Now take $N = M/\mathfrak{a}^n M$ and pass to the projective limit. Then this prove the claim. □

The previous result has the advantage that one might assume the existence of a dualizing complex in order to consider the formal cohomology.
The formal local cohomology modules can be discussed more deeply. For example, for future stydies on the subject we can establish conditions for that formal local cohomology modules satisfies the conditions for to be an co-Cohen-Macualy $R$-module. Also, the existence of decompositions similar to what saw in [6] or isomorphisms which was stated in [7], can be checked.

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# Reduced Σ-spaces with exponential metric

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**Abstract**

In this paper, we consider the exponential metrics and we will study the Finsler Σ-spaces with these metrics. We prove that Finsler Σ-spaces with exponential metrics are Riemannian.

## 1. Introduction

In [2], Loos introduced the concept of Σ-spaces as a generalization of symmetric spaces. Recall that, for a $C^\infty$ manifold $N$ and smooth operation $*$, the space $(N,*)$ is called symmetric space if we have:

a) $x * x = x$,
b) $x * (x * y) = y$,
c) $x * (y * z) = (x * y) * (x * z)$,
d) In an appropriate neighborhood $U$ of $x \in N$, if $x * y = x$ for some $y \in U$, then $y = x$.

Also we note that a smooth manifold $N$ with a system of diffeomorphisms $\{s_x\}_{x\in N}$ is said to be a regular $s$-manifold if we have [3]:

a) $s_x x = x$,
b) $s_x \circ s_y = s_{s_x y} \circ s_x$,
c) $(s_x)_{*x} - Id_x$ is invertible.

For more details see [1, 4].

Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a norm iduced ba a Riemannian metric $\tilde{a}$ and $\beta(x,y) = b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold $N$. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{1}$$

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Now, let the function $F$ is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2}$$

where $\phi = \phi(s)$ is a positive $C^\infty$ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \le b < b_0. \tag{3}$$

Then $F$ is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in N$. A Finsler metric in the form (2) is called an $(\alpha, \beta)-$metric [4].
A Finsler space having the Finsler function:

$$F(x, y) = \alpha(x, y) \exp(\frac{\beta(x, y)}{\alpha(x, y)}) \tag{4}$$

is called a Finsler space with an exponential metric $(\alpha, \beta)-$metric.
The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space $T_x^*N$ such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on $T_x^*N$ induces a linear isomorphism between $T_x^*N$ and $T_xN$. Then the 1-form $\beta$ corresponds to a vector field $\tilde{X}$ on $N$ such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{5}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$ . Therefore we can write $(\alpha, \beta)-$metrics as follows:

$$F(x, y) = \alpha(x, y)\phi\Big(\frac{\tilde{a}(\tilde{X}(x), y)}{\alpha(x, y)}\Big), \tag{6}$$

where for any $x \in N$, $\sqrt{\tilde{a}(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < b_0$ [4].
In this paper, we consider the exponential metrics and we will study the Finsler Σ-spaces with these metrics. We prove that Finsler Σ-spaces with exponential metrics are Riemannian.

## 2. Preliminaries

Suppose that $N$ be a $C^\infty$ manifold with dimension $n$ and for any $a \in N$, $T_aN$ be the tangent space at point $a$. Denote by $TN = \cup_{a\in N} T_aN$ the tangent bundle of $N$ and by $T^*N = \cup_{a\in N} T_a^*N$ the cotangent bundle of $N$ where $T_a^*N$ is the cotangent space at $a$ and it is the dual space of $T_aN$. Now we can define the Finsler structure as follows:

**Definition 2.1.** A Finsler structure of $N$ is a function $F : TN \to [0, \infty)$ such that:

a) $F$ is $C^\infty$ on the $TN\backslash\{0\}$.
b) $F(x, \gamma y) = \gamma F(x, y)$, for all $\gamma > 0$.
c) The $n \times n$ Hessian matrix $(g_{ij}) = \Big(\big[\frac{F^2}{2}\big]_{y^i y^j}\Big)$ is positive-definite at every point of $TN\backslash\{0\}$.

For a manifold $N$ and Finsler structure $F$ as above, the pair $(N, F)$ is called a Finsler manifold.

**Definition 2.2.** Let $N$ be a $C^\infty$ connected manifold. Also assume that Σ be a Lie group and $\mu : N \times \Sigma \times N \to N$ be a smooth map. Then the triple $(N, \Sigma, \mu)$ is called a Σ-space if it satisfies the following conditions:

a) $\mu(x, \sigma, x) = x$,
b) $\mu(x, e, y) = y$,
c) $\mu\big(x, \sigma, \mu(x, \tau, y)\big) = \mu(x, \sigma\tau, y)$,
d) $\mu\big(x, \sigma, \mu(y, \tau, z)\big) = \mu\big(\mu(x, \sigma, y), \sigma\tau\sigma^{-1}, \mu(x, \sigma, z)\big)$,

where $x, y, z \in N, \sigma, \tau \in \Sigma$ and $e$ is the identity element of Σ.

Now for a fixed point $a \in N$ consider the following two functions:

$$\sigma_a : N \to N, \;\; \sigma_a(y) = \mu(a, \sigma, y), \tag{7}$$

$$\sigma^a : N \to N, \;\; \sigma^a(y) = \sigma_y(a). \tag{8}$$

Then in terms of these maps the above conditions can be written as follows:

a) $\sigma_a = a$,
b) $e_a = id_N$,
c) $\sigma_a \tau_a = (\sigma\tau)_a$,
d) $\sigma_a \tau_y \sigma_a^{-1} = (\sigma\tau\sigma^{-1})\sigma_a(y)$.

The best example of Σ-spaces is symmetric spaces. For each $x \in N$ by $\Sigma_x$ we denote the image of Σ under the map $\Sigma \to \Sigma_x$, $\sigma \to \sigma_x$. For each $\sigma \in \Sigma$ we define $(1,1)$ tensor field $S^\sigma$ on the Σ-space $N$ by

$$S^\sigma X_x = (\sigma_x)_* X_x, \;\; \forall x \in N, X_x \in T_x N. \tag{9}$$

We note that $S^\sigma$ is smooth.

**Definition 2.3.** A Σ-space $N$ is called a reduced Σ-space if for each $x \in N$ the following fulfilled:

1. $T_x N$ is generated by the set of all $\sigma^x(X_x)$, that is,

$$T_x N = gen\{(I - S^\sigma)X_x : X_x \in T_x N, \sigma \in \Sigma\}.$$

2. If $X_x \in T_x N$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$, then $X_x = 0$, and thus no non-zero vector in $T_x N$ is fixed by all $S^\sigma$.

We note that, a Finsler Σ-space $(N, \Sigma, F)$ is a reduced Σ-space together with a Finsler metric $F$ which is invariant under $\Sigma_p$ for $p \in N$.

## 3. Main Results

In this section, we give some results about reduced Σ-space with exponential metric.

**Lemma 3.1.** *Suppose that* $(N, \Sigma, F)$ *be a Finsler* Σ*-space with exponential metric* $F = \alpha \exp(\frac{\beta}{\alpha})$ *defined by the Riemannian metric* $\tilde{a}$ *and the vector field* $X$*. Then* $(N, \Sigma, \tilde{a})$ *is a Riemannian* Σ*-space.*

*Proof.* Assume that $\sigma_x$ be a diffeomorphism as below

$$\sigma_x : N \to N, \;\; \sigma_x(y) = \mu(x, \sigma, y).$$

Therefore for every $p \in N$ and for any $Y \in T_p M$ we have:

$$F(p, Y) = F(\sigma_x(p), d\sigma_x Y).$$

Applying equation (4) we get

$$\sqrt{\tilde{a}(Y,Y)} \exp(\frac{\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}}) = \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} \exp(\frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}). \tag{10}$$

Replacing $Y$ by $-Y$ in equation (10) we get

$$\sqrt{\tilde{a}(Y,Y)} \exp(\frac{-\tilde{a}(X_p, Y)}{\sqrt{\tilde{a}(Y,Y)}}) = \sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)} \exp(\frac{-\tilde{a}(X_{\sigma_x(p)}, d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y, d\sigma_x Y)}}). \tag{11}$$

Combining the above equations (10) and (11) we have

$$\exp(\frac{2\tilde{a}(X_p,Y)}{\sqrt{\tilde{a}(Y,Y)}}) = \exp(\frac{2\tilde{a}(X_{\sigma_x(p)},d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}}),$$

which implies

$$\frac{\tilde{a}(X_p,Y)}{\sqrt{\tilde{a}(Y,Y)}} = \frac{\tilde{a}(X_{\sigma_x(p)},d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}}. \tag{12}$$

From equation (10) and (12), we have

$$\tilde{a}(Y,Y) = \tilde{a}(d\sigma_x Y,d\sigma_x Y).$$

Therefore $\sigma_x$ is a isometry with respect to the Riemannian metric $\tilde{a}$. □

**Theorem 3.2.** *Assume that $(N,\Sigma,\tilde{a})$ be a Riemannian $\Sigma$-space. Also Let $F$ be an exponential metric defined by $\tilde{a}$ and a vector field $X$. Then $(N,\Sigma,F)$ is an exponential $\Sigma$-space if and only if $X$ is $\sigma_x-$invariant for all $x \in N$.*

*Proof.* Let $X$ be $\sigma_x-$invariant. Therefore for any $p \in N$, we have $X_{\sigma_x(p)} = d\sigma_x X_p$. Then for any $Y \in T_pN$ we have:

$$\begin{aligned} F(\sigma_x(p),d\sigma_x Y_p) &= \sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}\exp(\frac{\tilde{a}(X_{\sigma_x(p)},d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}}) \\ &= \sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}\exp(\frac{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}}) \\ &= \sqrt{\tilde{a}(Y,Y)}\exp(\frac{\tilde{a}(X_p,Y)}{\sqrt{\tilde{a}(Y,Y)}}) \\ &= F(p,Y). \end{aligned}$$

Conversely, let $F$ be a $\Sigma_N-$invariant. Then for any $p \in N$ and $Y \in T_pN$ we have $F(p,Y) = F(\sigma_x(p),d\sigma_x Y)$. Applying the Lemma 3 we get:

$$\frac{\tilde{a}(X_p,Y)}{\sqrt{\tilde{a}(Y,Y)}} = \frac{\tilde{a}(X_{\sigma_x(p)},d\sigma_x Y)}{\sqrt{\tilde{a}(d\sigma_x Y,d\sigma_x Y)}}, \tag{13}$$

which implies

$$\tilde{a}(Y,Y) = \tilde{a}(d\sigma_x Y,d\sigma_x Y). \tag{14}$$

From equation (13) and (14), we have:

$$\tilde{a}(X_x,Y) = \tilde{a}(X_{\sigma_x(p)},d\sigma_x Y).$$

Therefore $(d\sigma_x)_p X_p = X_{\sigma_x(p)}$. □

**Theorem 3.3.** *An exponential $\Sigma-$space must be Riemannian.*

*Proof.* Let $(N,\Sigma,F)$ be an exponential $\Sigma-$space with $F = \alpha\exp(\frac{\beta}{\alpha})$ defined by the Riemannian metric $\tilde{a}$ and the vector field $X$, and let $\sigma_x$ be a diffeomorphism defined by $\sigma_x(y) = \mu(x,\sigma,y)$. Then by Lemma 3, $(N,\Sigma,\tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x,d\sigma_x y) &= \sqrt{\tilde{a}(d\sigma_x y,d\sigma_x y)}\exp(\frac{\tilde{a}(X_x,d\sigma_x y)}{\sqrt{\tilde{a}(d\sigma_x y,d\sigma_x y)}}) \\ &= \sqrt{\tilde{a}(y,y)}\exp(\frac{\tilde{a}(X_x,y)}{\sqrt{\tilde{a}(y,y)}}) \\ &= F(x,y). \end{aligned}$$

Therefore $\tilde{a}(X_x,d\sigma_x y) = \tilde{a}(X_x,y)$, $\forall y \in T_xN$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of $T_xN$ having no nonzero fixed vectors. So we have $\tilde{a}(X_x,(S^\sigma - id)_x(y)) = 0$, $\forall y \in T_xN$. Since $(S-id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in N$. Hence $F$ is Riemannian. □

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# Some results on the butterfly lemma of crossed polymodules

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**Abstract**

A group $G$ may have no, only one or even more than one composition series. However, the Jordan-Holder theorem states that any two composition series of a group $G$ are equivalent. The Zassenhaus Lemma connects the subfactors appearing in Schreier's refinements. This Lemma is sometimes called the "Butterfly Lemma". In this paper, we show an analogous "Butterfly Lemma" for crossed polymodules. It is proof runs parallel to the proof of the classical versions.

## 1. Introduction

Crossed modules and its applications play very important roles in category theory, homology and cohomology of groups, homotopy theory, algebra, k-theory etc. Therefore, study crossed modules and its all kinds automorphisms at least through this is very important. This is in fact one of the motivations of recent half-century studies in this field. We recall some definitions and properties of the crossed module category. A crossed module $(T, G, \partial)$ consist of a group homomorphism $\partial : T \longrightarrow G$ together with an action $(g, t) \longrightarrow {}^{g}t$ of $G$ on $T$ satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(s)}t = sts^{-1}$, for all $g \in G$ and $s, t \in T$ [1, 6].

## 2. Crossed polymodules

The polygroup theory is a natural generalization of the group theory. Of course in a group the composition of two elements is an element, but in a polygroup the that, is a set. Polygroups have been applied in many area, such as

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lattices, geometry, color scheme and combinatorics. For study, there exists a rich bibliography, publication book within 2012 can be found in "Polygroup Theory and Related Systems" by B. Davvaz[5]. This book contains the principle definitions endowed with examples and the basic results of the theory. Applications of hypergroups appear in special subclasses like polygroups that they were studied by Comer[3], also see [4, 5]. Also, Comer and Davvaz developed the algebraic theory for polygroups. A polygroup is a completely regular, reversible in itself multigroup. According [3], a *polygroup* is a multi-valued system $\mathcal{M} =< P, \circ, e, {}^{-1} >$, with $e \in P$, ${}^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$, where the following axioms hold for all $x, y, z$ in $P$:

1. $(x \circ y) \circ z = x \circ (y \circ z)$
2. $e \circ x = x \circ e = x$
3. $x \in y \circ z$ implies $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Of course, $\mathcal{P}^*(P)$ is the set of all the non-empty subsets of $P$, and also if $x \in P$ and $A$, $B$ are non-empty subsets of $P$, then we have $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$, $x \circ B = \{x\} \circ B$ and $A \circ x = A \circ \{x\}$.

**Definition 2.1.** [2] A crossed polymodule $\chi = (C, P, \partial, \alpha)$ consists of polygroups $< C, *, e, {}^{-1} >$ and $< P, \circ, e, {}^{-1} >$ together with a strong homomorphism $\partial : C \longrightarrow P$ and a (left) action $\alpha : P \times C \longrightarrow \mathcal{P}^*(C)$ on $C$, statisfying the conditions:

1. $\partial({}^{p}c) = p \circ \partial(c) \circ p^{-1}$ for all $c \in C$ and $p \in P$,
2. ${}^{\partial(c)}c' = c * c' * c^{-1}$ for all $c, c' \in C$.

## 3. Butterfly lemma of crossed polymodules

A group $G$ may have no, only one or even more than one composition series. However, the Jordan-Holder theorem states that any two composition series of a group $G$ are equivalent. The Zassenhaus Lemma connects the subfactors appearing in Schreier's refinements. This Lemma is sometimes called the "Butterfly Lemma". In this section we show an analogous "Butterfly Lemma" for crossed polymodules. It is proof runs parallel to the proof of the classical versions.

**Theorem 3.1.** *If* $(P_1, P_2, \alpha, f)$, $(P_3, P_4, \beta, k)$, $(P_5, P_6, \beta', k')$ *be crossed ploymodules, and* $(P_3, P_4, \beta, k)$, $(P_5, P_6, \beta', k') \leq (P_1, P_2, \alpha, f)$ , *also* $P_5 \subseteq P_3, P_6 \subseteq P_4$, *then* $(P_5, P_6, \beta', k') \leq (P_3, P_4, \beta, k)$.

*Proof.* We have $P_3 \leq P_1, P_5 \leq P_1$ and $P_5 \subseteq P_3$, therefore $P_5 \leq P_3$. Also $P_4 \leq P_2, P_5 \leq P_2$ and $P_5 \subseteq P_4$, hence $P_6 \leq P_4$.
But for $p_5 \in P_5$, we have $p_5 k' = p_5 f = p_5 k \subseteq P_4$, and for $p_5 \in P_5$ and $p_6 \in P_6$, we have $p_5(p_6 \beta') = p_5(p_6 \alpha) = p_5(p_6 \beta)$.

□

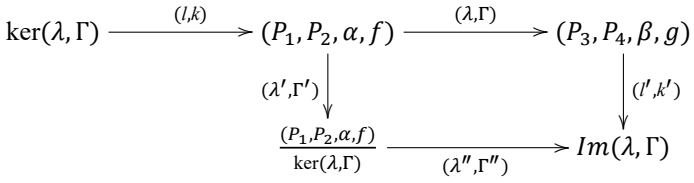
**Theorem 3.2.** *If* $(P_1, P_2, \alpha, f)$, $(P_3, P_4, \beta, k)$, $(P_5, P_6, \beta', k')$ *be crossed ploymodules, and* $(P_1, P_2, \alpha, f) \leq (P_3, P_4, \beta, k) \leq (P_5, P_6, \beta', k')$, *with* $(P_1, P_2, \alpha, f) \underline{\triangle} (P_5, P_6, \beta', k')$, *then*

$$(P_1, P_2, \alpha, f) \underline{\triangle} (P_3, P_4, \beta, k)$$

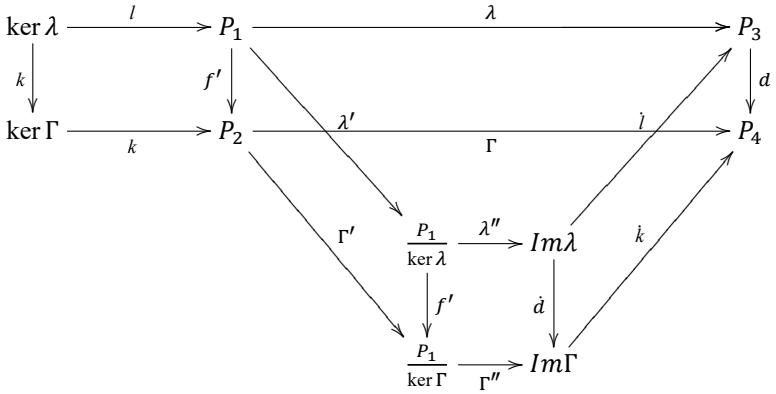
*Proof.* From $P_2 \leq P_4 \leq P_6$ and $P_2 \underline{\triangle} P_6$, we have $P_2 \underline{\triangle} P_4$. Also for $p_1 \in P_1, p_2 \in P_2, p_3 \in P_3, p_4 \in P_4$ we have $p_3^{-1} p_3^{p_2} \subseteq P_1$ and $p_1^{p_4} \subseteq N$, hence $(P_1, P_2, \alpha, f) \underline{\triangle} (P_5, P_6, \beta', k')$.

□

**Lemma 3.3.** *If $(P_1, P_2, \alpha, f)$, $(P_3, P_4, \beta, g)$, be crossed ploymodules, and given a crossed ploymodule morphism $(\lambda, \Gamma) : (P_1, P_2, \alpha, f) \to (P_3, P_4, \beta, g)$,*
*then there are the following comutative diagram,*




*In other words*




*Proof.* $\ker(\lambda, \Gamma) = (\ker\lambda, \ker\Gamma, \beta, f)$ and $Im(\lambda, \Gamma) = (Im\lambda, Im\Gamma, \dot{r}, \dot{d})$
and $\frac{(P_1,P_2,\alpha,f)}{\ker(\Lambda,\Gamma)}$ are crossed ploymodules by

$$\beta : \ker\Gamma \to Aut(\ker\lambda) \qquad p_1 \to (n \to (n)(p_1\beta) = (n)(p_1\alpha))$$

and $k = f\big|_{\ker\lambda}^{\ker\Gamma}$ be the restriction of f to $\ker\lambda$ and $\ker\Gamma$. Also, $\dot{d} = d|_{im\Gamma}^{im\lambda}$ be the restriction of d to $im\lambda$ and $im\Gamma$, and

$$\dot{r} : Im\Gamma \to Aut(im\lambda) \qquad p_2 \to (\dot{p_1} \to (\dot{p_1})(\dot{p_2}\dot{r}) = \dot{p_1}(\dot{p_2}\dot{r}))$$

We have the inctusion morphism $(l, k)$ and $(\dot{l}, \dot{k})$, also the crossed ploymodule morphism $(\lambda', \Gamma')$. We have bijective polygroup morphisms

$$\lambda'' : \frac{P_1}{\ker\lambda} \to Im\lambda \qquad p_1(\ker\lambda) \to p_1\lambda$$

and

$$\Gamma'' : \frac{P_2}{\ker\Gamma} \to Im\Gamma \qquad p_1(\ker\Gamma) \to p_2\Gamma$$

and $(\lambda'', \Gamma'')$ is a crossed ploymodule morphism.

Since, let $p_1(\ker\lambda) \subseteq \frac{P_1}{\ker\lambda}$ and $p_2(\ker\Gamma) \subseteq \frac{P_2}{\ker\Gamma}$.

$$(p_1(\ker\lambda))\lambda''\dot{d} = p_1\lambda\dot{d} = p_1\lambda d = p_1 f\Gamma = (p_1 f(\ker\Gamma))\Gamma'' = (p_1(\ker\lambda))f'\Gamma''$$

and

$$(p_1(\ker\lambda))^{p_2(\ker\Gamma)}\lambda'' = (p_1^{p_2}(\ker\lambda))\lambda'' = (p_1^{p_2})\lambda = (p_1\lambda)^{p_2\Gamma} = (p_1(\ker\lambda)\lambda'')^{p_2(\ker\Gamma)\Gamma''}$$

□

**Theorem 3.4.** *If $(P_1, P_2, \alpha, f)$, $(P_3, P_4, r, d)$ be crossed ploymodules, and given a surjective crossed ploymodule morphism $(\lambda, \Gamma) : (P_1, P_2, \alpha, f) \to (P_3, P_4, r, d)$, then there is a bijective crossed ploymodule morphism given by*

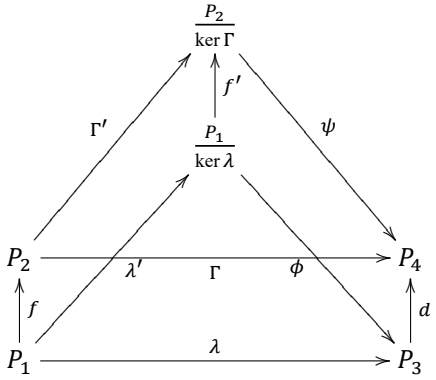
$$(\phi, \psi) : \frac{(P_1, P_2, \alpha, f)}{\ker(\lambda, \Gamma)} \longrightarrow (P_3, P_4, r, d)$$

*such that*

$$\phi : \frac{P_1}{\ker\lambda} \longrightarrow P_3 \qquad p_1(\ker\lambda) \to p_1\lambda$$

$$\psi : \frac{P_2}{\ker\Gamma} \longrightarrow P_4 \qquad p_2(\ker\Gamma) \to p_2\Gamma$$

*Also the following diagram is commutative,*




**Theorem 3.5.** *suppose we are give crossed ploymodules $(P_1, P_2, \alpha, f)$, $(P_3, P_4, \beta, k) \leq ((P_1, P_2, \alpha, f)$ and $(P_5, P_6, \beta', k') \leq ((P_1, P_2, \alpha, f)$*

*(i) Let $l := f|_{P_3\cap P_5}^{P_4\cap P_6}$ be the restriction of $f$ to $P_3 \cap P_5$ and $P_4 \cap P_6$.*

*Also the polygroup morphism*

$$\delta : P_4 \cap P_6 \longrightarrow Aut(P_3 \cap P_5) \qquad p \to (p_3 \to (p_3)(p\delta) = (p_3)(p\alpha))$$

*Then where is a crossed subpolymodule given by*

*$(P_3 \cap P_5, P_4 \cap P_6) = (P_3 \cap P_5, P_4 \cap P_6, \delta, l) \leq (P_1, P_2, \alpha, f)$,*

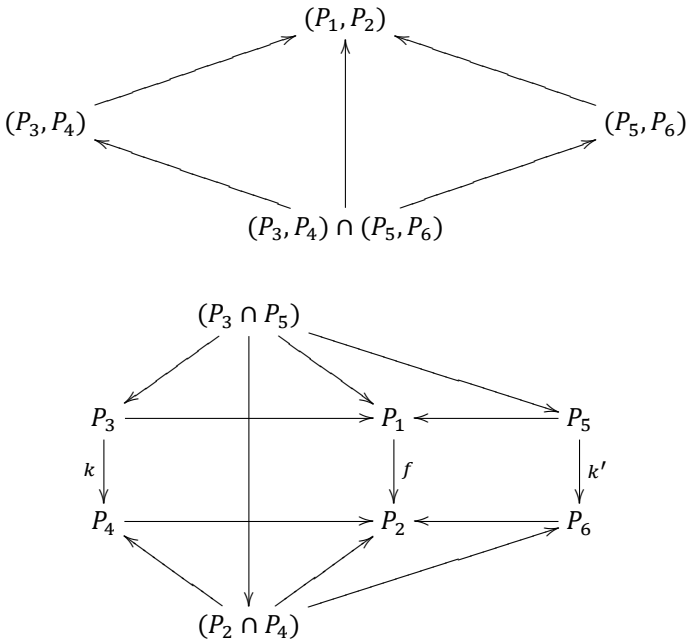
*and we write $(P_3, P_4) \cap (P_5, P_6) = (P_3 \cap P_5, P_4 \cap P_6)$.*

*In particular, $(P_3, P_4) \cap (P_5, P_6) \leq (P_3, P_4)$ and $(P_3, P_4) \cap (P_5, P_6) \leq (P_5, P_6)$.*

*(ii) If $(P_3, P_4) \underline{\triangle} (P_1, P_2)$ and $(P_5, P_6) \underline{\triangle} (P_1, P_2)$, then there are a normal crossed subpolymodule $(P_3, P_4) \cap (P_5, P_6) \underline{\triangle} (P_1, P_2)$.*

*In particular,* $(P_3, P_4) \cap (P_5, P_6) \underline{\triangle} (P_3, P_4)$ *and* $(P_3, P_4) \cap (P_5, P_6) \underline{\triangle} (P_5, P_6)$.

*Hence we have the following diagrams:*




**Theorem 3.6.** *Suppose* $(P_1, P_2, \alpha, f)$ *be a crossed ploymodule, and* $(P_5, P_6, \beta', k') \underline{\triangle} (P_1, P_2, \alpha, f)$ *, be a normal crossed subploymodule.*

*(i) If* $(P_3, P_4)$ *be a crossed subploymodule of* $(P_1, P_2, \alpha, f)$, $l : f|_{P_3 P_5}^{P_4 P_6}$

*be the restriction of f to* $P_3 P_5$ *and* $P_4 P_6$*, also we have the polygroup morphism*

$$\delta : P_4 P_6 \longrightarrow Aut(P_3 P_5) \qquad p \to (p' \to (p')(p\delta) := (p')(p\alpha))$$

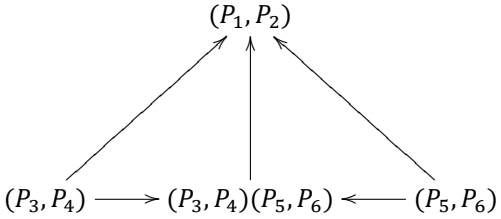
*then threr is crossed subploymodule given by* $(P_3 P_5, P_4 P_6) = (P_3 P_5, P_4 P_6 \gamma, l) \leq (P_1, P_2)$*, with* $(P_3, P_4)(P_5, P_6) = (P_3 P_5, P_4 P_6)$.

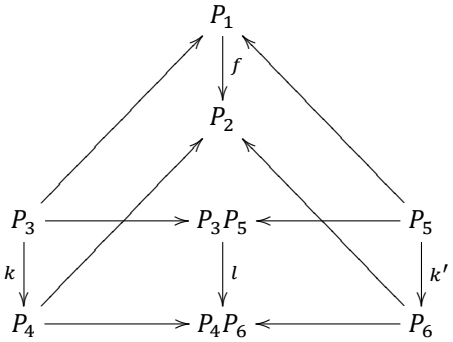
*In particular,* $(P_3, P_4) \leq (P_1 3 P_5, P_4 P_6)$ *and* $(P_5, P_6) \leq (P_3 P_5, P_4 P_6)$.

*(ii) If* $(P_3, P_4) \underline{\triangle} (P_1, P_2)$ *and* $(P_5, P_6) \underline{\triangle} (P_1, P_2)$*, then* $(P_3 P_5, P_4 P_6) \underline{\triangle} (P_1, P_2)$.

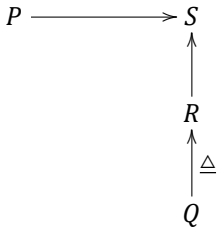
*In particular,* $(P_3, P_4) \underline{\triangle} (P_3 P_5, P_4 P_6)$ *and* $(P_5, P_6) \underline{\triangle} (P_3 P_5, P_4 P_6)$.

*Hence we have the following diagrams,*






**Theorem 3.7.** *Let $P = (P_1, P_2), Q = (Q_1, Q_2), R = (R_1, R_2), S = (S_1, S_2)$ be crossed ploymodules. Also suppose $P \leq S$ and $Q \underline{\triangle} R \leq S$. Namely we have the following situtation*

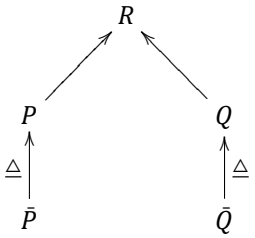



*Then (i) , (ii) are true:*

*(i) We have $P \cap Q \underline{\triangle} P \cap R$*

*(ii) If $P \underline{\triangle} S$, then $PQ \underline{\triangle} PR$*

**Lemma 3.8.** *(Butterfly Lemma). Let $P = (P_1, P_2), Q = (Q_1, Q_2)$ be a crossed subploymodules of a crossed ploymodule of $R = (R_1, R_2)$. Also $\bar{P} = (\bar{P_1}, \bar{P_2}) \underline{\triangle} P$, $\bar{Q} = (\bar{Q_1}, \bar{Q_2}) \underline{\triangle} Q$ be normal crossed subploymodules. On the other hand, if,*



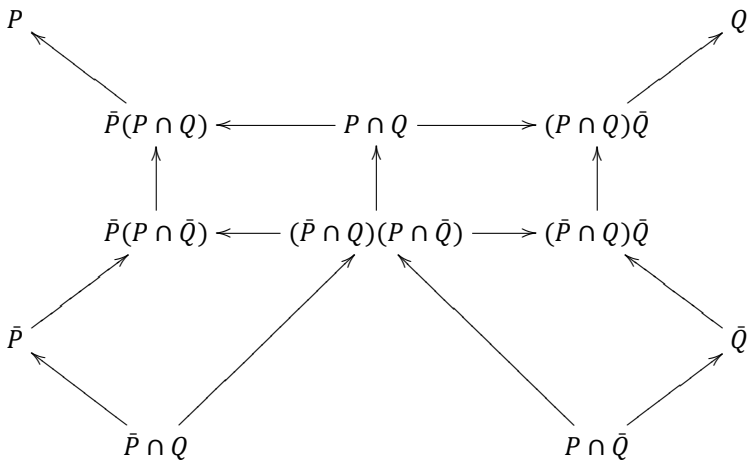

*then we have normal crossed subploymodules,*

$$\bar{P}(P \cap \bar{Q}) \underline{\triangle} \bar{P}(P \cap Q) \quad , \quad (\bar{P} \cap Q)\bar{Q} \underline{\triangle} (P \cap Q)\bar{Q},$$

*and isomorphics factor crossed ploymodules:*

$$\frac{\bar{P}(P \cap Q)}{\bar{P}(P \cap \bar{Q})} \cong \frac{P \cap Q}{(\bar{P} \cap Q)(P \cap \bar{Q})} \cong \frac{(P \cap Q)\bar{Q}}{(\bar{P} \cap Q)\bar{Q}}.$$

*On the other hand, then the butterfly becoms apparent:*




*Proof.* We have $\bar{Q} \underline{\triangle} P$ and by proposition 3.7, $P \cap \bar{Q} \underline{\triangle} P \cap Q \leq P$.

Also $\bar{P}(P \cap \bar{Q}) \underline{\triangle} \bar{P}(P \cap Q) \leq P$.

Now we consider the crossed ploymodule morphism

$$(\alpha_1, \alpha_2) : P \cap Q \longrightarrow \frac{\bar{P}(P \cap Q)}{\bar{P}(P \cap \bar{Q})}$$

such that

$$\alpha_1 : P_1 \cap Q_1 \longrightarrow \frac{\bar{P}_1(P_1 \cap Q_1)}{\bar{P}_1(P_1 \cap \bar{Q}_1)} \qquad p \to p(\bar{P}_1(P_1 \cap \bar{Q}_1))$$

and

$$\alpha_2 : P_2 \cap Q_2 \longrightarrow \frac{\bar{P}_2(P_2 \cap Q_2)}{\bar{P}_2(P_2 \cap \bar{Q}_2)} \qquad q \to q(\bar{P}_2(P_2 \cap \bar{Q}_2)).$$

The crossed ploymodule morphism$(\alpha_1, \alpha_2)$ is surjective, because, if $p \in \frac{\bar{P}_1(P_1 \cap Q_1)}{\bar{P}_1(P_1 \cap \bar{Q}_1)}$, then $p = \bar{p_1} p_1 \bar{P}_1(P_1 \cap \bar{Q}_1)$, such that $\bar{p_1} \in \bar{P}_1$ and $p_1 \in P_1 \cap Q_1$.

But we have

$$x = \bar{p_1} p_1 \bar{P}_1(P_1 \cap \bar{Q}_1) = p_1 \bar{p_1'}(\bar{P}_1(P_1 \cap \bar{Q}_1)) = p_1(\bar{P}_1(P_1 \cap \bar{Q}_1)) = p_1 \alpha_1.$$

As the same way, $\alpha_2$ is surjective.

In addition to, we have $\ker(\alpha_1, \alpha_2) = (\bar{P} \cap Q) \cap (P \cap \bar{Q})$

Actually, $\ker \alpha_1 = (\bar{P}_1 \cap Q_1)(P_1 \bar{Q}_1)$.

Because if $p \in \ker \alpha_1 \subseteq (P_1 \cap Q_!))$, then

$$p\alpha_1 = \bar{P}_1(P_1 \cap \bar{Q}_1) \Rightarrow p \in \bar{P}_1(P_1 \cap \bar{Q}_1) \Rightarrow p \in (P_1 \cap Q_1) \cap \bar{P}_1(P_1 \cap \bar{Q}_1)$$

Also, if $p \in (\bar{P}_1 \cap Q_1)(P_1 \cap \bar{Q}_1) \subseteq \bar{P}_1(P_1 \cap \bar{Q}_1)$, then $p\alpha_1 = p\bar{P}_1(P_1 \cap \bar{Q}_1)$.

As the same way $\ker \alpha_2 = (\bar{P}_2 \cap Q_2)(P_2 \cap \bar{Q}_2$.

Therefore by theorem 3.4, we have the isomorphism:

$$(\bar{\alpha}_1, \bar{\alpha}_2) : \frac{P \cap Q}{(\bar{P} \cap Q)(P \cap \bar{Q})} \longrightarrow \frac{\bar{P}(P \cap Q)}{\bar{P}(P \cap \bar{Q})}$$

such that

$$\bar{\alpha}_1 : \frac{P_1 \cap Q_1}{(\bar{P}_1 \cap Q_1)(P_1 \cap \bar{Q}_1)} \longrightarrow \frac{\bar{P}_1(P_1 \cap Q_1)}{\bar{P}_1(P_1 \cap \bar{Q}_1)}$$
$$p_1(\bar{P}_1 \cap Q_1)(P_1 \cap \bar{Q}_1) \longrightarrow p_1\bar{P}_1(P_1 \cap \bar{Q}_1)$$

and

$$\bar{\alpha}_2 : \frac{P_2 \cap Q_2}{(\bar{P}_2 \cap Q_2)(P_2 \cap \bar{Q}_2)} \longrightarrow \frac{\bar{P}_2(P_2 \cap Q_2)}{\bar{P}_2(P_2 \cap \bar{Q}_2)}$$
$$p_2(\bar{P}_2 \cap Q_2)(P_2 \cap \bar{Q}_2) \longrightarrow p_2\bar{P}_2(P_2 \cap \bar{Q}_2)$$

So

$$\frac{P \cap Q}{(\bar{P} \cap Q)(P \cap \bar{Q})} \cong \frac{\bar{P}(P \cap Q)}{\bar{P}(P \cap \bar{Q})}.$$

For this reason we have,

$$\frac{P \cap Q}{(\bar{P} \cap Q)(P \cap \bar{Q})} \cong \frac{(P \cap Q)\bar{Q}}{(P \cap \bar{Q})\bar{Q}}.$$

□

## 4. Conclusion

In this paper, we have shown, an analogous ”Butterfly Lemma” for crossed polymodules. It was proof runs parallel to the proof of the classical versions.

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# Stochastic Comparisons of Series Systems with additive Weibull Distributed Components having Archimedean Copulas

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**Abstract**

Stochastic comparison on order statistics from heterogeneous-dependent observations has been paid lots of attention recently. This work devotes to investigating the ordering properties of order statistics from dependent observations. We obtain the usual stochastic order for the smallest order statistic of samples having additive Weibull (AW) distribution and Archimedean survival copulas.

## 1. Introduction

Series and parallel systems are two basic systems which play prominent roles in various applications in reliability engineering. An $n$-component system with series (parallel) structure fails (works) if at least one of the components of the system fails (works). Let $X_1, X_2, ..., X_n$ denote the lifetimes of $n$ components that can be used to built up an $n$ component system. If $X_{1:n} \leq ... \leq X_{n:n}$ denote the ordered lifetimes of the components then it is known that $X_{1:n}$ and $X_{n:n}$ correspond to the lifetimes of series and parallel systems, respectively. Reliability and stochastic properties of series and parallel systems have been considered by various researchers under different scenarios. For example, stochastic comparisons of the lifetimes of series and parallel systems, in the case of heterogeneous component lifetimes with with exponentiated Weibull (EW) distributions, are considered in [2] and [3] and by [1] in the case of heterogeneous components with generalized exponential (GE) distributions.
The paper by [9] proposed the additive Weibull distribution(AWD) by combining the hazard rates of two Weibull distributions; one with increasing hazard rate and the other with decreasing hazard rate so as to incorporate the bathtub shape for its hazard rate function. The cumulative distribution function of a AWD with parameters $(\alpha, \beta, \gamma, \lambda)$, written as AWD(□, □, □, □), is given by

$$F(x) = 1 - e^{-\alpha x^{\beta} - \gamma x^{\lambda}}, \quad 0 < x, \alpha > 0, \beta > 0, \lambda > 0, \gamma \geq 0. \tag{1}$$

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This paper is devoted to investigating how heterogeneity of the sample impact order statistics. We study sample minimums from two dependent samples with AWD family of distributions. We derive the usual stochastic order of smallest order statistics.
The organization of the paper is laid out as follows: Section 2 introduces the required definitions and several useful lemmas which are used throughout the paper. Section 3 studies the usual stochastic order of the smallest order statistics from two AWD samples with Archimedean survival copulas, and Section 4 concludes the paper.

## 2. Preliminaries

There are many ways in which a random variable $X$ can be said to be smaller than another random variable $Y$. In the usual stochastic ordering case, a random variable $X$ with survival function $\bar{F} = 1 - F$ is stochastically smaller than a random variable $Y$ with survival function $\bar{G} = 1 - G$, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all $x$. For more details on various kinds of stochastic orders, one may refer to [8].
For a random vector $X = (X_1, \dots, X_n)$ with the joint survival function $\bar{F}$ and univariate survival functions $\bar{F}_1, \dots, \bar{F}_n$, if there exists some $\hat{C} : [0,1]^n \longrightarrow [0,1]$ such that, for all $x_i$ , $i = 1, \dots, n$,

$$\bar{F}(x_1, \dots, x_n) = \hat{C}(\bar{F}_1(x_1), \dots, \bar{F}_n(x_n)),$$

then $\hat{C}$ is called as the survival copula of $X$. A real function $\phi$ is $n$-monotone on $(a,b) \subseteq \mathbb{R}$ if $(-1)^{n-2}\phi^{(n-2)}$ is decreasing and convex in $(a,b)$ and $(-1)^k \phi^{(k)}(x) \geq 0$ for all $x \in (a,b), k = 0, 1, \dots, n-2$, in which $\phi^{(i)}(.)$ is the $i$th derivative of $\phi(.)$. For a $n$-monotone $(n \geq 2)$ function $\phi : [0, +\infty) \longrightarrow [0,1]$ with $\phi(0) = 1$ and $\lim_{x \to +\infty} \phi(x) = 0$, let $\psi = \phi,^{-1}$ be the right continuous inverse of $\psi$, then

$$C_\phi(u_1, \dots, u_n) = \phi(\psi(u_1) + \dots + \psi(u_n)), \quad \text{for all } u_i \in [0,1], i = 1, \dots, n,$$

is called an Archimedean copula with generator $\phi$. Archimedean copulas cover a wide range of dependence structures including the independence copula. For more detail on Archimedean copulas, readers may refer to [7].
Majorization orders are quite useful and powerful in establishing various inequalities. For preliminary notations and terminologies on majorization theory, see [5]. Let $x = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ be two real vectors anf $x_{(1)} \leq \dots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x}$.

**Definition 2.1.** The vector $\mathbf{x}$ is said to be

(i) weakly submajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \preceq_{\mathrm{w}} \mathbf{y}$) if $\sum_{i=j}^{n} x_{(i)} \leq \sum_{i=j}^{n} y_{(i)}$ for all $j = 1, \dots, n$,

(ii) weakly supermajorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\mathrm{w}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \dots, n$,

(iii) majorized by the vector $\mathbf{y}$ (denoted by $\mathbf{x} \overset{\mathrm{m}}{\preceq} \mathbf{y}$) if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$ for all $j = 1, \dots, n-1$.

**Definition 2.2.** A real valued function $\varphi$ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathcal{A}$ if

$$x \overset{\mathrm{m}}{\preceq} y \quad \text{on} \quad \mathcal{A} \Longrightarrow \varphi(x) \leq (\geq) \varphi(y).$$

Before proceeding to main results, let us present some lemmas to be utilized in the sequel.

**Lemma 2.3** ([5], Theorem 3.A.4)**.** Suppose $\mathbb{I} \subset \mathbb{R}$ is an open interval and $\Phi : \mathbb{I}^n \longrightarrow \mathbb{R}_+$ is continuously differentiable. Necessary and sufficient conditions for $\Phi$ to be Schur-convex (Schur-concave) on $\mathbb{I}^n$ are

(i) $\Phi$ is symmetric on $\mathbb{I}^n$,

(ii) for $i \neq j$ and all $z \in \mathbb{I}^n$,

$$(z_i - z_j)\left(\frac{\partial \Phi(z)}{\partial z_i} - \frac{\partial \Phi(z)}{\partial z_j}\right) \geq (\leq) 0,$$

where $\frac{\partial \Phi(z)}{\partial z_i}$ denotes the partial derivative of $\Phi$ with respect to its $i$-th argument.

The following lower orthant order on Archimedean copulas will also be utilized in the sequel.

**Lemma 2.4** ([4], Lemma A.1)**.** *For two $n$-dimensional Archimedean copulas $C_{\phi_1}(\mathbf{u})$ and $C_{\phi_2}(\mathbf{u})$, if $\psi_2 \circ \phi_1$ is super-additive, then $C_{\phi_1}(\mathbf{u}) \leq C_{\phi_2}(\mathbf{u})$ for all $\mathbf{u} \in [0,1]^n$.*

## 3. Main results

This section studies the usual stochastic order on the smallest order statistic from the AWD samples coupled by Archimedean survival copulas. The random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to follow the AWD if $X_i$ has the distribution function $F_i(x) = 1 - e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}}$ for $i = 1, \ldots, n$. Specifically, by $\mathbf{X} \sim \mathrm{AWD}(\alpha, \beta, \gamma, \lambda, \phi_1)$ we denote the sample having the Archimedean copula with generator $\phi_1$ and following a AWD.
In the following theorem, we consider two smallest order statistics that are formed from two different sets of random variables having different sets of shape parameters □ but the same set of parameters $\alpha, \gamma, \lambda$.

**Theorem 3.1.** Suppose, for $\mathbf{X} \sim \mathrm{AWD}(\alpha, \beta, \gamma, \lambda, \phi_1)$ and $X^* \sim \mathrm{AWD}(\alpha, \square^*, \gamma, \lambda, \phi_2)$, $\phi_1$ or $\phi_2$ is log-convex and $\psi_2 \circ \phi_1$ is super-additive, then $(\beta_1, \ldots, \beta_n) \overset{\mathrm{m}}{\geq} (\beta_1^*, \ldots, \beta_n^*)$ implies $X_{1:n} \leq_{\mathrm{st}} X_{1:n}^*$.

*Proof.* $X_{1:n}$ and $X_{1:n}^*$ have their respective survival functions, for $x \geq 0$,

$$\bar{F}_{X_{1:n}}(x) = p(X_k > x, 1 \leq k \leq n) = \phi_1\Big(\sum_{i=1}^{n} \psi_1(e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}})\Big) = J(\alpha, \beta, \gamma, \lambda, x, \phi_1), \tag{2}$$

$$\bar{F}_{X_{1:n}^*}(x) = p(X_k^* > x, 1 \leq k \leq n) = \phi_2\Big(\sum_{i=1}^{n} \psi_2(e^{-\alpha x^{\beta_i^*} - \gamma x^{\lambda}})\Big) = J(\alpha, \beta^*, \gamma, \lambda, x, \phi_2). \tag{3}$$

We only prove the case that $\phi_1$ is log-convex, and the other case can be finished similarly. The partial derivatives of $J(\alpha, \beta, \gamma, \lambda, x, \phi_1)$ with respect to $\beta_i$ are

$$\frac{\partial J(\alpha, \beta, \gamma, \lambda, x, \phi_1)}{\partial \beta_i} = -\alpha x^{\beta_i} \log(x) e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}} \frac{\phi_1'\big(\sum_{i=1}^{n} \psi_1(e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}})\big)}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}})\big)},$$

$$\text{for all} \quad x > 0.$$

To prove its Schur-concavety, it follows from Lemma 2.3 that we have to show that for $i \neq j$,

$$(\beta_i - \beta_j)\left(\frac{\partial J(\alpha, \beta, \gamma, \lambda, x, \phi_1)}{\partial \beta_i} - \frac{\partial J(\alpha, \beta^*, \gamma, \lambda, x, \phi_1)}{\partial \beta_j}\right) \leq 0,$$

For $i \neq j$, the decreasing $\phi_1$ implies

$$(\beta_i - \beta_j)\left(\frac{\partial J(\alpha, \beta, \gamma, \lambda, x, \phi_1)}{\partial \beta_i} - \frac{\partial J(\alpha, \beta^*, \gamma, \lambda, x, \phi_1)}{\partial \beta_j}\right) = -\alpha \log(x) \phi_1'\Big(\sum_{i=1}^{n} \psi_1(e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}})\Big)(\beta_i - \beta_j)$$

$$\left(\frac{x^{\beta_i} e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}})\big)} - \frac{x^{\beta_j} e^{-\alpha x^{\beta_j} - \gamma x^{\lambda}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_j} - \gamma x^{\lambda}})\big)}\right).$$

$$\overset{\mathrm{sgn}}{=} (\beta_i - \beta_j) \log(x)\left(\frac{x^{\beta_i} e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_i} - \gamma x^{\lambda}})\big)} - \frac{x^{\beta_j} e^{-\alpha x^{\beta_j} - \gamma x^{\lambda}}}{\phi_1'\big(\psi_1(e^{-\alpha x^{\beta_j} - \gamma x^{\lambda}})\big)}\right)$$

where $\overset{\mathrm{sgn}}{=}$ means that both sides have the same sign. Now, consider the following two cases.

(i) Let $x \geq 1$. Then, for $\beta_i \geq \beta_j$, we have $\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}) \geq \psi_1(e^{-\alpha x^{\beta_j}-\gamma x^{\lambda}})$. Further, the log-convexity of $\phi_1$ implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Then
$\frac{\phi_1(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}{\phi_1'(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}$ is decreasing in $\beta_i > 0$. Also the increasing property of $x^{\beta_i}$ implies that
$\frac{x^{\beta_i}\phi_1(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}{\phi_1'(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}$ is decreasing in $\beta_i > 0$. So, for $i \neq j$,

$$(\beta_i - \beta_j)\left(\frac{\partial J(\alpha,\beta,\gamma,\lambda,x,\phi_1)}{\partial \beta_i} - \frac{\partial J(\alpha,\beta,\gamma,\lambda,x,\phi_1)}{\partial \beta_j}\right) \leq 0,$$

(ii) Let $0 < x \leq 1$. Then, for $\beta_i \geq \beta_j$, we have $\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}) \leq \psi_1(e^{-\alpha x^{\beta_j}-\gamma x^{\lambda}})$. Further, the log-convexity of $\phi_1$ implies the decreasing property of $\frac{\phi_1}{\phi_1'}$. Then $\frac{\phi_1(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}{\phi_1'(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}$ is increasing in $\beta_i > 0$. Also the decreasing property of $x^{\beta_i}$ implies that $\frac{x^{\beta_i}\phi_1(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}{\phi_1'(\psi_1(e^{-\alpha x^{\beta_i}-\gamma x^{\lambda}}))}$ is increasing in $\beta_i > 0$. Using the fact $\log(x) \leq 0$ for $0 < x \leq 1$ we get

$$(\beta_i - \beta_j)\left(\frac{\partial J(\alpha,\beta,\gamma,\lambda,x,\phi_1)}{\partial \beta_i} - \frac{\partial J(\alpha,\beta,\gamma,\lambda,x,\phi_1)}{\partial \beta_j}\right) \leq 0,$$

Then Schur-concavety of $J(\alpha,\beta,\gamma,\lambda,x,\phi_1)$ follows from Lemma 2.3. So $(\beta_1,\dots,\beta_n) \overset{\text{m}}{\geq} (\beta_1^*,\dots,\beta_n^*)$ implies $J(\alpha,\beta,\gamma,\lambda,x,\phi_1) \leq J(\alpha,\beta^*,\gamma,\lambda,x,\phi_1)$. On the other hand, since $\psi_2 \circ \phi_1$ is super-additive by Lemma 2.4, we have $J(\alpha,\beta^*,\gamma,\lambda,x,\phi_1) \leq J(\alpha,\beta^*,\gamma,\lambda,x,\phi_2)$. So, it holds that

$$J(\alpha,\beta,\gamma,\lambda,x,\phi_1) \leq J(\alpha,\beta^*,\gamma,\lambda,x,\phi_1) \leq J(\alpha,\beta^*,\gamma,\lambda,x,\phi_2).$$

That is, $X_{1:n} \leq_{\text{st}} X_{1:n}^*$. □

**Example 3.2.** Suppose that $X$ and $X^*$ have either of the following two dependence structures. (i) Gumbel survival copulas with respective generators

$$\phi_1(x) = e^{-x^{\frac{1}{\beta_1}}}, \quad \phi_2(x) = e^{-x^{\frac{1}{\beta_2}}}, \beta_2 \geq \beta_1 \geq 1;$$

(ii) Archimedean survival copulas with respective generators

$$\phi_1(x) = (x^{\frac{1}{\beta_1}})^{-1}, \quad \phi_1(x) = (x^{\frac{1}{\beta_1}})^{-1}, \beta_2 \geq \beta_1 \geq 1.$$

It is easy to see that $\phi_i$ is log-convex for $i = 1, 2$. In view of $\psi_2(\phi_1(0)) = 0$ and the convexity of $\psi_2(\phi_1(x)) = x^{\frac{\beta_2}{\beta_1}}$, we conclude that $\psi_2(\phi_1(x))$ is super-additive by Proposition 21.A.11 in [5] .

[6] obtained the following theorem for the comparison of series systems under the usual stochastic order.

**Theorem 3.3.** Let $X$ and $X^*$ be two vectors of continuous random variables with $X_i \sim \bar{F}(x;\lambda_i)$ and $X_i^* \sim \bar{F}(x;\lambda_i^*)$, $i = 1,\dots,n$. Assume that their associated survival copulas are Archimedean with the respective generators $\phi_1$ and $\phi_2$. If $\phi_1$ or $\phi_2$ is log-convex, $\psi_2 \circ \phi_1$ is super-additive, and for any $x$ the function $\bar{F}(x;\lambda)$ is decreasing [increasing] and log-concave in $\lambda$, then $(\lambda_1,\dots,\lambda_n) \succeq_{\text{w}} [\overset{\text{w}}{\succeq}](\lambda_1^*,\dots,\lambda_n^*)$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

It should be noted that our results are different from the results given by [6], because $\bar{F}(x,\beta) = e^{-\alpha x^{\beta}-\gamma x^{\lambda}}$ is not decreasing (increasing) with respect to $\beta$.
The following corollary provides a comparison between the smallest order statistics arising from two sets of random variables, when the marginal distributions belong to the AWD distribution.

**Corollary 3.4.** Suppose, for $\mathbf{X} \sim \text{AWD}(\alpha,\beta,\gamma,\lambda,\phi_1)$ and $X^* \sim \text{AWD}(\alpha^*,\beta,\gamma,\lambda,\phi_2)$, $\phi_1$ or $\phi_2$ is log-convex and $\psi_2 \circ \phi_1$ is super-additive, then $(\alpha_1,\ldots,\alpha_n) \succeq_{\text{w}} (\alpha_1^*,\ldots,\alpha_n^*)$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

*Proof.* We can see that $\bar{F}(x,\alpha)$ is decreasing and log-concave in $\alpha$ . Hence, Theorem 3.3 immediately implies the result. □

In the following corollary, we compare two largest order statistics with respect to the usual stochastic order. Here, we assume that two sets of random variables have the same $\alpha,\beta,\lambda$ parameter but different $\gamma$ parameters.

**Corollary 3.5.** Suppose, for $\mathbf{X} \sim \text{AWD}(\alpha,\beta,\gamma,\lambda,\phi_1)$ and $X^* \sim \text{AWD}(\alpha,\beta,\gamma,\lambda,\phi_2)$, $\phi_1$ or $\phi_2$ is log-convex and $\psi_2 \circ \phi_1$ is super-additive, then $(\gamma_1,\ldots,\gamma_n) \succeq_{\text{w}} (\gamma_1^*,\ldots,\gamma_n^*)$ implies $X_{1:n} \leq_{\text{st}} X_{1:n}^*$.

*Proof.* We can see that $\bar{F}(x,\gamma)$ is decreasing and log-concave in $\gamma$ . Hence, Theorem 3.3 immediately implies the result. □

## 4. Concluding Remarks

This is the first try to study some stochastic comparisons of smallest order statistics from dependent and heterogeneous samples having additive Weibull family. We derived the usual stochastic order for the smallest order statistic of samples having additive Weibull family and Archimedean survival copulas.

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# Derangement Representation of Graphs

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**Abstract**

A derangement $k$-representation of a graph $G$ is a map $\pi$ of $V(G)$ to the symmetric group $S_k$, such that for any two vertices $v$ and $u$ of $V(G)$, $v$ and $u$ are adjacent if and only if $\pi(v)(i) \neq \pi(u)(i)$ for each $i \in \{1, 2, 3, ..., k\}$. The derangement representation number of $G$ denoted by $drn(G)$, is the minimum of $k$ such that $G$ has a derangement $k$-representation. Our previous work[1] proved that any graph has a derangement $k$-representation, and some lower and upper bounds for $drn(G)$, in terms of the basic parameters of $G$ were obtained. In this work, we determine the exact value or give a better bound of the derangement representation number of some classes of graphs. Also, we study the derangement spectra of $G$, denoted by $D_{spec}(G)$, which is the set of all positive integers $m$ such that $G$ has a derangement $m$-representation.

## 1. Introduction

All graphs we consider in this paper are simple, finite, and undirected. For a graph $G$, we denote its vertex and edge set by $V(G)$ and $E(G)$, respectively. Also, we use the notations $p(G)$, $q(G)$, $\omega(G)$ and $G^c$ for the order, the size, the maximum size of cliques, and the complement graph of $G$, respectively. The path and the cycle of order $n$ are denoted by $P_n$ and $C_n$, respectively. From now on, we use the notation $[n]$ and $\mathbb{N}_{\geq n}$ instead of $\{1, \dots, n\}$ and $\{m \in \mathbb{N} | \ m \geq n\}$, respectively. We mention some definitions that are referred to throughout this paper and for other necessary definitions and notation we refer the reader to a standard text-book [4].

A Cayley graph is a graph constructed from a group $G$ with connection set $S$ satisfying $1 \notin S$ and $S = S^{-1}$, denoted by $Cay(G, S)$, has vertices corresponding to the elements of $G$ and an edge exists between two vertice $g$ and $h$ if and only if $h = sg$ for some $s \in S$. Arthur Cayley introduced this concept in 1878 [5]. Cayley graphs are known to be vertex-transitive, making them a proper subfamily of all graphs. Babai and Sos demonstrated probabilistically that for any finite graph $G$, every sufficiently large group has a Cayley graph containing an induced subgraph isomorphic to $G$, precisely if $X$ is a finite graph of order $n$ and $G$ is a group of order at least $c_1 n^3$, then $X$ is isomorphic to an induced subgraph of $Cay(G, S)$ for some $S \subseteq G$ [2]. Additionally, Cayley's theorem in group theory states that every group $G$ is isomorphic to a subgroup of a symmetric group [5]. This highlights the significance of studying Cayley graphs on symmetric groups.

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In a previous study, we proved that any graph has a derangement $k$-representation. Also, we obtain some lower and upper bounds for $drn(G)$, in terms of the basic parameters of $G$. In this work, we determine the exact value or give a better bound of the derangement representation number of some classes of graphs. The derangement spectra of $G$, denoted by $D_{spec}(G)$, is the set of all positive integers $m$ such that $G$ has a derangement $m$-representation.

**Definition 1.1.** Let $G$ be a graph and $k \in \mathbb{N}$. We say $G$ is a *derangement $k$-representable* if there exists an injective map $\pi : V(G) \to S_k$, such that for any two vertices $v$ and $u$ of $G$, $v$ and $u$ are adjacent if and only if $\pi(v)(i) \neq \pi(u)(i)$ for all $i \in [k]$. In other words, $\pi(v)^{-1} \circ \pi(u) \in D_k$. In this case, $\pi$ is called a derangement $k$-representation of $G$. The derangement representation number of $G$, denoted by $drn(G)$, is the minimum of $k$ such that $G$ has a derangement $k$-representation.

**Example 1.2.** Let $G$ be a complete graph of order $n$, $drn(K_n)$=n

**Definition 1.3.** Let $G$ be a graph. The derangement spectra of $G$, denoted by $D_{spec}(G)$, is the set of all positive integers $m$ such that $G$ has a derangement $m$-representation. Equivalently, $D_{spec}(G) = \{m \in \mathbb{N} \mid G\ is\ an\ induced\ subgraph\ of\ Cay(S_m, D_$
Note that $drn(G) = \min D_{spec}(G)$.

**Example 1.4.** For complete graph $K_n$ we have $D_{spec}(K_n) = \mathbb{N}_{\geq n}$. In fact, for any $m \geq n$, we can choose a latin square $L$ of order $m$ and then remove the last $m - n$ rows of $L$ to achieve a derangement $m-$representation matrix of $K_n$.

**Example 1.5.** Let $n, k \in \mathbb{N}$ and $(k-1)! < n \leq k!$. Then $D_{spec}(\overline{K_n}) = \mathbb{N}_{\geq k+1}$.

*Proof.* for any $t \geq k+1$ the following matrix is a derangement $t$-representation matrix of $\overline{K_n}$:

$$L(\overline{K_n}) = \left[\begin{array}{c|c} A & \begin{array}{c} k+1\ k+2\ \cdots\ t \\ \vdots \\ k+1\ k+2\ \cdots\ t \end{array} \end{array}\right],$$

where $A = [a_{i,j}]_{n \times k}$ is a matrix that its rows represent $n$ permutations of $S_k$. □

**Theorem 1.6.** *Let $G$ be a graph of order $n$, $l \geq n$ and $k \in D_{spec}(G)$. Then $k + l \in D_{spec}(G)$.*

**Theorem 1.7.** *Let $G$ be a graph of order $n$, then $k \in D_{spec}(G)$ for any $k \geq drn(G) + n$.*

**Theorem 1.8.** *Let $G$ be a graph and $H$ be an induced subgraph of $G$, then $D_{spec}(G) \subset D_{spec}(H)$*

### *1.1. Structure of the paper*

After this introductory section where we established the background, purpose, and some basic definitions and theorems of the paper, we divide the paper into four sections. In Section 2, we prove Theorems 1.7, 1.8 and some basic lemmas and theorems and introduce the concept *Derangement spectra* of graphs. In Section 3, we determine the exact value of the derangement representation number for some nearly complete graphs and give better bounds of the parameter for some classes of graphs. In Section **??**, we present some computations performed by SageMath [11], and in the last section, we state some conjectures and open problems.

## 2. Proofs of Theorems 1.6, 1.7,and 1.8

**Proof of Theorem 1.6.** A *row latin rectangle* $L$ of order $n \times m$ with the set of $m$ symbols is a matrix in which no symbol occurs more than once in any row.
Let $L$ be a derangement $k$-representation matrix of $G$. Suppose that $L_1$ is a row latin rectangle of order $n \times l$ with symbols $\{k+1, k+2, \ldots, k+l\}$. Now consider the block matrix $L' = [L|L_1]$ of order $n \times (k+l)$, which obviously, is a derangement $(k+l)$-representation matrix of $G$. □

**Proof of Theorem 1.7.** Let $L$ be a derangement $k$-representation matrix of $G$. Now consider the block matrix $L' = [L \mid L \mid \cdots \mid L]$ of order $n \times (mk)$ (contains $m$ copy of $L$), which obviously, is a derangement $(mk)$-representation

matrix of $G$. □

**Proof of Theorem 1.8**. Suppose that $k \in D_{spec}(G)$. Hence $G$ is isomorphic to an induced subgraph of $Cay(S_k, D_k)$ such as Γ. Let $\alpha : V(G) \to V(\Gamma)$ be an isomorphism. Therefore, the subgraph of Γ induced by $\alpha(V(H))$ is isomorphic to $H$. So $H$ is isomorphic to an induced subgraph of $Cay(S_k, D_k)$. Thus $H$ is derangement $k$-representable and so $k \in D_{spec}(H)$. □

## 3. Improved results in Upper Bounds for Derangement number of Certain Graph Classes

In this section, we will calculate some better lower and upper bounds or the exact amount of derangement representation numbers for specific families of graphs.

**Definition 3.1.** The number of permutations $\pi$ on $[r+s]$ such that $\pi(r+i) \neq r+i$ for $1 \leq i \leq s$ is denoted by $d(r,s)$ and:

$$d(r,s) = \sum_{k=0}^{s} (-1)^k \binom{s}{k} (r+s-k)!$$

**Theorem 3.2.** *Suppose $G$ be a complete bipartite graph $G = K_{r,s}$ such that $r \leq s$;*
*If $d(r,s) = \sum_{k=0}^{s} (-1)^k \binom{s}{k} (r+s-k)!$ and $f_1(r) = min\{n \mid n! \geq r\}$, then*

$$drn(K_{r,s}) \leq 2f_1(r) + min\{k \mid d(f_1(r), k) \geq s\}.$$

*Proof.* Case 1: $r = s$:
Assume $f_1(r) = k$; let $A = \{1, 2, ..., k\}, B = \{k+1, k+2, ..., 2k\}$, then we define a matrix $L(G)_{2r.2k}$ . For vertices $v_1, ..., v_r$ from one part of $K_{r,s}$, we choose one permutation of $S(A)$ then we add $k+1, k+2, ..., 2k$ to the end of it, and for the vertices $u_1, ..., u_r$ from the other part of $K_{r,s}$, we choose one permutation of $S(B)$ and add $1, 2, ..., k$ to the end of it.

$$L(G) = \left[\begin{array}{c|cccc} & k+1 & k+2 & \cdots & 2k \\ S(A) & \vdots & & & \\ & k+1 & k+2 & \cdots & 2k \\ \hline & 1 & 2 & \cdots & k \\ S(B) & \vdots & & & \\ & 1 & 2 & \cdots & k \end{array}\right]$$

Case2: $r < s$:
Assume $f_1(r) = k$ and $t = min\{l \mid d(k,l) \geq s\}$. Let $A = \{1, 2, ..., k\}, B = \{k+1, k+2, ..., 2k+t\}$, then we define a matrix $L(G)_{(r+s).(2k+t)}$. for vertices $v_1, ..., v_r$ from one part of $K_{r,s}$, we choose one permutation of $S(A)$ and add $k+1, k+2, ..., 2k+t$ to the end of it, then for the vertices $u_1, ..., u_s$ from the other part of $K_{r,s}$, we choose one permutation of $S(B)$ and add $1, 2, ..., k$ to the end of it.

$$T = \left[\begin{array}{c|cccc} & k+1 & k+2 & \cdots & k+t \\ S(A) & \vdots & & & \\ & k+1 & k+2 & \cdots & k+t \end{array}\right]$$

$$L(G) = \left[\begin{array}{c|cccc} & k+t+1 & k+t+2 & \cdots & 2k+t \\ T & \vdots & & & \\ & k+t+1 & k+t+2 & \cdots & 2k+t \\ \hline & 1 & 2 & \cdots & k \\ S(B) & \vdots & & & \\ & 1 & 2 & \cdots & k \end{array}\right]$$

□

**Definition 3.3.** double star graph $ST(m,n)$ is a graph that is formed by two stars $ST(m)$ and $ST(n)$ via joining their centers by an edge.[**?** ]

**Theorem 3.4.** *Let G be a double star graph of order* $m+n$*, then* $drn(G) \leq 2p+3$ *where* $p = max\{f_1(m-1), f_1(n-1)\}$.

*Proof.* Let

$$V(G) = \{v_0, v_1, ..., v_{m-1}, u_0, u_1, ..., u_{n-1}\}.$$

$$E(G) = \{v_0v_1, v_0v_2, ..., v_0v_{m-1}, v_0u_0, u_0u_1, u_0u_2, ..., u_0u_{n-1}\}.$$

Consider $S_p$ an array consisting of $(m-1)$ permutations of $S(\{1, 2, \cdots, p\})$ and $T_P$ an array consisting of $(t-1)$ permutations of $S(\{p+1, p+2, \cdots, 2p\})$. We assign $S$ to the vertices of $\{v_1, ..., v_{m-1}\}$ and T to the vertices of $\{u_1, ..., u_{n-1}\}$.

$$S = \left[\begin{array}{c|c|cc} S_p & T_p & 2p+2 & 2p+3 \\ & & \vdots & \\ & & 2p+2 & 2p+3 \end{array}\right]$$

$$T = \left[\begin{array}{c|c|cc} T_p & S_p & 2p+1 & 2p+3 \\ & & \vdots & \\ & & 2p+1 & 2p+3 \end{array}\right]$$

$$A = [p+1, p+2, ..., 2p-1, 2p+3, 1, 2, ..., p-2, 2p+3, p-1 | 2p+1, 2p]$$

$$B = [1, 2, ..., p-2, 2p+3, p-1, p+1, p+2, ..., 2p-1, 2p+3 | 2p+2, p]$$

If we assign A to $v_0$ and B to $u_0$, then $L(G)$ is a $2p+3$-representation matrix for $G$.

$$L(G) = \left[\begin{array}{c} S \\ A \\ B \\ T \end{array}\right]$$

□

**Theorem 3.5.** *Let G be a complete* 3*-partite graph,* $K_{r,s,m}$ *such that* $r \leq s \leq m$*. If* $t' \geq k$ $k = min\{n \mid n! \geq r\}$, $t = min\{l \mid d(k,l) \geq s\}$, $t' = min\{p \mid d(2k+t, p) \geq m\}$*, and* $t' \geq k$ *then*

$$drn(K_{r,s,m}) \leq 3k + 2t + t'.$$

*Proof.* Assume $A = \{1, 2, ..., k\}$, $B = \{k+1, k+2, ..., k+t+t'\}$, and $C = \{2k+t, 2k+t+1, ..., 3k+2t+t'\}$. Also assume $C_1 = \{2k+t, 2k+t+1, ..., 3k+2t+t'\}$, $C_2 = \{2k+t, 2k+t+1, ..., 3k+2t+t'\}$, $C_3 = \{2k+t, 2k+t+1, ..., 3k+2t+t'\}$. Suppose $S'(B)$ be a subset of $S(B)$ such that it contains permutations $\pi$ such that $\pi(i) \neq i$ for $k+1 \leq i \leq k+t$. And suppose $S'(C)$ be a subset of $S(C)$ such that it contains permutations $\pi$ such that $\pi(i) \neq i$ for $3k+t+1 \leq i \leq 3k+2t+t'$. Then

$$M = \left[\begin{array}{c|ccc|ccc} S(A) & k+1 & \cdots & k+t & k+t+1 & \cdots & k+t+t' \\ & \vdots & & & \vdots & & \\ & k+1 & \cdots & k+t & k+t+1 & \cdots & k+t+t' \end{array}\right]$$

$$N = \left[\begin{array}{c|ccc} S'(B) & 3k+2t+1 & \cdots & 3k+2t+t' \\ & \vdots & & \\ & k+1 & \cdots & k+t \end{array}\right]$$

$$L(G) = \left[\begin{array}{c|c|c|c} M & C_1 & C_2 & C_3 \\ \hline N & C_2 & A & C_1 \setminus C_2 \\ \hline S'(C) & A & B_1 & B_2 \end{array}\right]$$

□

**Theorem 3.6.** *For* $G = \overline{K_r \square K_s}$ *:* $drn(G) \leq r + s$.

*Proof.* Let $A = L(K_r)$ and $B = L(K_s)$. Assume $C(v_i)$ be the row of $A$ for $v_i$ and $C(u_i)$ be the corresponding row of $B$ for $u_i$. Assume $V(G) = V(K_r) \times V(K_s)$, then for every vertex $(v_i, u_j)$ we assign $[C(v_i)|C(u_j)]$ for the corresponding row of $L(G)$. □

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# Split mixed equilibrium problems and their Levitin- Polyak well-Posedness

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**Abstract**

Here, our idea is to generalize the notion of Levitin-Polyak wellposedness to split mixed equilibrium problems in topological vector spaces. Hence, we obtain sufficient and necessary conditions for Levitin-Polyak wellposedness of mixed split equilibrium problems.

## 1. Introduction

The wellposedness notion has played a fundamental role in the field of stability analysis for equilibrium, optimization Problems, and variational inequality. Wellposedness ensures the convergence of some nets to exact solutions. This fact has convinced many authors to consider the wellposedness of equilibrium and optimization problems. The notion of a set was first defined by Tykhonov [7] for unconstrained optimization problems. Tykhonov considered a single-valued optimization problem to be wellposed if it has a unique solution and each minimizing sequence converges to the solution. An important characterization of Tykhonov's wellposedness is obtained in [8]. Another generalization of Tykhonov's well-posedness for variational inequalities, equilibrium, and optimization problems is the wellposedness by perturbations by Dontchev and Zolezzi [8]. Levitin and Polyak generalized the wellposed notion to the constrained case. Therefore, various concepts of Levitin-Polyak wellposed have been defined. In this paper, we generalized two notions of Levitin-Polyak wellposedness to split mixed equilibrium problems in topological vector spaces. Hence, we obtain the sufficient and necessary conditions for Levitin-Polyak well-posednes for mixed split equilibrium problems. In the sequel, we recall the classical notion of wellposedness posed by Tykhonov in [7], for a scalar optimization problem:

$$\min h(z), \text{s.t.} z \in Z$$

where $Z$ is a metric space and $h : Z \longrightarrow \mathbb{R}$. A sequence $(z_n) \subseteq Z$ is said to be a minimizing sequence for the above optimization problem, when $\lim_{n\longrightarrow+\infty} h(z_n) = \inf_Z h$.

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The above problem is said to be Tykhonov's wellposed iff it has a unique solution $\bar{z}$ and each minimizing sequence converges to $\bar{z}$. A momentous characterization of Tykhonov's wellposedness is gained in [8].

**Lemma 1.1.** *[8] The above optimization problem is Tykhonov's wellposed if and only if there exists $\bar{z} \in Z$ so as to each neighborhood $V$ of $z$ there exists $\varepsilon > 0$, that*

$$h(z) - h(\bar{z}) < \varepsilon \Longrightarrow z \in V.$$

We suppose that $X_1$ and $X_2$ are two topological vector spaces and denote by $B_\delta(x_0)$ be a closed ball centered at $x_0$ with positive radius $\delta > 0$ and $Y$ is a norm linear space, which we assumed to be ordered by a nonempty convex closed cone $Q \subseteq Y$ and $\preccurlyeq$ is defined as follows:

$$y_1 \preccurlyeq y_2 \Leftrightarrow y_2 - y_1 \in Q.$$

The convex cone $Q$ is called to be solid if $\text{int}Q \neq \emptyset$. Whenever $Q$ is solid, we have

$$y_1 \prec y_2 \Leftrightarrow y_2 - y_1 \in \text{int}Q.$$

Assume that denote by $\mathcal{P}(Y)$ the set consisting of all nonempty subsets of $Y$. For $C, D \in \mathcal{P}(Y)$, the lower set less order relation $\leq^l$ and upper set less order relation $\leq^u$ which was introduced in [6], is defined as follows:

$$C \leq^l D \text{ iff } D \subseteq C + P$$
$$\text{and}$$
$$C \leq^u D \text{ iff } C \subseteq D - P.$$

We say that $C <^l D$ iff $D \subseteq C + \text{int}Q$ and $C <^u D$ iff $C \subseteq D - \text{int}Q$. Obviously, $\leq^l$ and $\leq^u$ are preorder relations, but $<^l$ and $\leq^u$ are not necessarily a reflexive relation.
In this paper, we assume that $Q$ is a solid cone, $q \in \text{int}Q$, We assume we have equipped $\mathcal{P}(Y)$ with the preorder relation $\leq \in \{\leq^l, \leq^u\}$.
Let $j \in \{1, 2\}$, $C_j$ be nonempty closed and convex subsets of $X_j$ and $K_j : C_j \times C_j \longrightarrow 2^Y$ be a set-valued map with nonempty values. Hence, Let $\varphi_j : C_j \longrightarrow Y$ and $A : C_1 \longrightarrow C_2$ be a linear bounded operator. Split mixed equilibrium problem is defined as follows:

$$\text{(SMEP)} \quad \text{find } \bar{x} \in C_1 : \quad \varphi_1(\bar{x}) - \varphi_1(x) \leq K_1(\bar{x}, x) \quad \forall x \in C_1.$$

Where, $A\bar{x} = \bar{y}$ and

$$\varphi_2(\bar{y}) - \varphi_2(y) \leq K_2(\bar{y}, y) \quad \forall y \in C_2.$$

In the following, we conclude some special cases in other researchers, let $Y = \mathbb{R} \cup \{+\infty\}$ and $P$ be positive real number :
1) Let for all $j \in \{1, 2\}$, $X_j$ be a Hilbert space, $C_1 = C$, $C_2 = Q$ and $K_j := g_j$ where $g_j\tilde{} : X_j \times X_j \longrightarrow \mathbb{R}$, then $(SMEP)$ in [2] is deduced from $(SMEP)$.
2) Let $\emptyset \neq C_j \subseteq X_j$ that $X_j$ is a real Banach space and $K_1 := f$ and $K_2 := g$ where $f\tilde{} : X_1 \times X_1 \longrightarrow \mathbb{R}$ and $g : X_2 \times X_2 \longrightarrow \mathbb{R}$ and $\varphi_1 = \varphi_2$ are Constant functions, then $(SEP)(f, g, C, Q)$ in [4] is deduced from $(SMEP)$.
3) Let for all $j \in \{1, 2\}$, $X_j$ be a Hilbert space, $K_j$ is single-valued and $\varphi_1 = \varphi_2$ are Constant functions. Then $(SVI)$ in [5] is deduced from $(SMEP)$.
4) Let $X_1$ and $X_2$ be two real Hilbert spaces, $K_1 = f$ and $K_2 = g$ let for all $\bar{x} \in X_1$, $\bar{y} \in X_2$, $x \in C_1$ and $y \in C_2$ define $K_1(\bar{x}, x) := \langle f(\bar{x}), x - \bar{x} \rangle$ and $K_2(\bar{y}, y) := \langle g(\bar{y}), y - \bar{y} \rangle$. Then $(SVIP)$ in [1] is deduced from $(SMEP)$.

## 2. Main result

In this section, we define Levitin-Polyak (in short LP) approximating solution net for $(SMEP)$. In fact, we generalize Definition 6, in [5] and Definition 4, in [4].

**Definition 2.1.** A net $(x_\alpha, y_\alpha) \in X_1 \times X_2$ is said to be an LP approximating solution net for the $(SMEP)$ if there exists net $\varepsilon_\alpha \in \mathbb{R}^+$ such that $\varepsilon_\alpha \longrightarrow 0$ and

$$\begin{aligned} y_\alpha = A(x_\alpha) &\in B_{\varepsilon_\alpha}(x_\alpha), \\ \varphi_1(x_\alpha) - \varphi_1(x) &\leq K_1(x_\alpha, x) + \varepsilon_\alpha q \quad \forall x \in C_1, \\ \varphi_2(y_\alpha) - \varphi_2(y) &\leq K_2(y_\alpha, y) + \varepsilon_\alpha q \quad \forall y \in C_2. \end{aligned}$$

Now, we extend Definition 7, in [5] and Definition 5, in [4].

**Definition 2.2.** Suppose $S$ represents the solution set of the $(SMEP)$. Then we say that the $(SMEP)$ is

- LP wellposed if $S$ is a singleton set and every LP approximating solution net for the $(SMEP)$ converges to the unique solution.
- generalized LP wellposed, if $S$ is a nonempty set and every LP approximating solution net for the $(SMEP)$ has a subnet that converges to some $\bar{x} \in S$.

In the following, we define an approximate solution set to $(SMEP)$, then we obtain sufficient and necessary conditions for LP wellposed and generalized LP wellposed. Let $\varepsilon \in \mathbb{R}^+$, we defined approximate solution set to $(SMEP)$, as

$$S(\varepsilon) = \{(\bar{x}, \bar{y}) \in X_1 \times X_2 : \bar{y} = A(\bar{x}) \in B_\varepsilon(\bar{x});$$

$$\varphi_1(\bar{x}) - \varphi_1(x) \leq K_1(\bar{x}, x) + \varepsilon q \quad \forall x \in C_1, \;\; \varphi_2(y) - \varphi_2(\bar{y}) \leq K_2(\bar{y}, y) + \varepsilon q \quad \forall y \in C_2\}.$$

Note that $\cup_{\varepsilon>0} S(\varepsilon) = S$ and if $\varepsilon < \varepsilon_0$, then $S(\varepsilon) < S(\varepsilon_0)$.

**Theorem 2.3.** *The problem $(SMEP)$ is generalized LP wellposed, iff there exists compact and nonempty set $H \subseteq S$ so as to each neighborhood $V$ of $0$ there exists $\varepsilon > 0$ that*

$$x \in S(\varepsilon) \Longrightarrow x \in H + V. \tag{1}$$

*Proof.* Assume Problem $(SMEP)$ is LP wellposed in the general sense, then $S \neq \emptyset$. Let $H = S$, then $H$ is compact, because if $(z_\alpha)$ is a net in $H$, that is an LP approximating solution net. Therefore, there exists $z_0 \in S = H$ and a subnet $(z_\beta)$ of $(z_\alpha)$ that $z_\beta \longrightarrow z_0$ therefore, $H$ is compact. Now, We show that for each arbitrary neighborhood $V$ of $0$, there exists a $\varepsilon$ such that 1 holds. On the contrary, suppose that there exists a neighborhood $V$ of $0$ and a net $(\delta_\alpha) \subseteq \mathbb{R}^+$ such that $\delta_\alpha \longrightarrow 0$ and $(z_\alpha) \subseteq X$ such that $z_\alpha \in S(\delta_\alpha)$ and $z_\alpha \notin H + V$. Then, $(z_\alpha)$ is an LP approximating solution net, and by our assumption, there exists $z_0 \in S$ such that $z_\alpha \longrightarrow z_0$, which is a contradiction with $z_\alpha \notin H + V = S + V$. Conversely, we show that every LP approximating solution net of Problem $(SMEP)$ contains a subnet that converges to a solution of Problem $(SMEP)$. Let $(z_\alpha)$ be an LP approximating solution net for Problem $(SMEP)$, then there exists $(\varepsilon_\alpha) \subseteq \mathbb{R}^+$ such that $\varepsilon_\alpha \longrightarrow 0$ and for all $\alpha$, $z_\alpha \in S(\varepsilon_\alpha)$. In the sequel, we suppose that there exists a subnet of $(z_\alpha)$ that converges to a point $z_0 \in H$. Otherwise, for every $z \in H$, there exists a neighborhood $V_z$ of $0$ that $\{z\} + V_z$ does not contain any subnet of $z_\alpha$. On the other hand, for every $z \in H$, there exists a neighborhood $W_z$ of $0$ such that $W_z + W_z \subseteq V_z$. Since $H \subseteq \cup_{z \in H}\{z\} + W_z$ and $H$ is a compact set, there exists $n \in \mathbb{N}$ such that $H \subseteq \cup_{i=1}^n \{z_i\} + W_{z_i}$. Let $W = \cap_{i=1}^n W_{z_i}$, then from 1, there exists $\varepsilon_1 > 0$ such that

$$z \in S(\varepsilon_1) \quad \Longrightarrow z \in H + W.$$

On the other hand, there exists $\alpha_0$ such that for all $\alpha \geq \alpha_0$, $\varepsilon_\alpha < \varepsilon_1$, then for all $\alpha \geq \alpha_0$, $S(\varepsilon_\alpha) \subseteq S(\varepsilon_1)$ and $z_\alpha \in S(\varepsilon_\alpha) \subseteq S(\varepsilon_1)$. Therefore, $z_\alpha \in H + W$. But

$$\begin{aligned} H + W &\subseteq \cup_{i=1}^n (\{z_i\} + W_{z_i}) + W \subseteq \cup_{i=1}^n (\{z_i\} + W_{z_i} + W) \quad (2) \\ &\subseteq \cup_{i=1}^n (\{z_i\} + W_{z_i} + W_{z_i}) \subseteq \cup_{i=1}^n (\{z_i\} + V_{z_i}). \quad (3) \end{aligned}$$

From above relation and assumption $z_\alpha \in H + W$, we have $z_\alpha \in \cup_{i=1}^n (\{z_i\} + V_{z_i})$, which is a contradiction, because for very $z \in H$, $\{z\} + V_z$ does not contain any subnet of $z_\alpha$.

**Theorem 2.4.** *The problem $(SMEP)$ is LP wellposed, iff $S = \{x_0\}$ such that for each neighborhood $V$ of $0$, there exists $\varepsilon > 0$ such that*

$$x \in S(\varepsilon) \Longrightarrow x \in \{x_0\} + V. \tag{4}$$

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# Enhancements in the PGBMain Algorithm: Best Practices for Minimal Dickson Basis Selection

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**Abstract**

In this paper, we enhance the PGBMain algorithm [7], an efficient method for computing Gröbner bases of parametric polynomial ideals, or Gröbner systems. A key step in the PGBMain algorithm involves computing a minimal Dickson basis in each iteration, which is not unique. Our improvement introduces a novel sub-algorithm to select the optimal Minimal Dickson basis through a novel selection strategy. The Improved-PGBMain algorithm, incorporating this sub-algorithm, takes a parametric polynomial ideal and two monomial orderings as input and produces a Gröbner system of the ideal w.r.t. a compatible elimination product of the specified orderings. We have implemented the Improved-PGBMain algorithm in Maple and provide timing and memory usage data to compare its performance with the original PGBMain.

## 1. Introduction

Gröbner bases are specialized generating sets for polynomial ideals, serving as powerful computational tools in computer algebra with important applications in mathematics, science, and engineering. Introduced by Buchberger in his Ph.D. thesis, these bases come with the first algorithm for their computation [1]. This paper enhances the PGBMain algorithm [7] by introducing a new selection strategy for computing Gröbner systems—an extension of Gröbner bases for parametric polynomial ideals. A Gröbner system consists of a finite set of branches, each containing parametric constraints (including null and non-null parametric sets) and a corresponding set of polynomials. For any specialization, there exists a branch where the specialization fulfills its constraints, and the specialized polynomial set forms a Gröbner basis for the related parametric ideal. Weispfenning introduced the concept of Gröbner systems [10], proved their existence for any given parametric polynomial ideal [10, Proposition 3.4 and Theorem 2.7], and developed the first algorithm for their computation [10, Theorem 3.6]. Over the past three decades, numerous effective enhancements for computing Gröbner systems have been proposed [2–6, 8, 9]. In 2010, Kapur et al. designed an efficient algorithm, the PGBMain algorithm [7], which integrates several earlier algorithms. This algorithm computes a Gröbner basis $G$ over a polynomial ring with respect to both variables and parameters at each iteration, followed by computing a minimal Dickson basis for $G$, termed MDBasis. However, since the MDBasis is not unique, this step can be costly, resulting in a poorly calculated Gröbner basis. Additionally, the selected MDBasis may lead to numerous branches

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creating inefficiencies. To mitigate this issue, we introduce a sub-algorithm for selecting the optimal MDBasis and integrate it into the PGBMain algorithm, ultimately yielding a Gröbner system with fewer branches and more efficiency. Our new algorithm, termed Improved-PGBMain, along with the original PGBMain algorithm, has been implemented in `Maple`, and their efficiencies are evaluated across a variety of parametric polynomial ideals. We will now briefly review the basic notations and definitions related to Gröbner bases and Gröbner systems.

**Definition 1.1.** A polynomial ring in $n$ variables, denoted as $\mathcal{R} = \mathbb{K}[x_1, \dots, x_n]$, consists of polynomials constructed from the variables $x_1, \dots, x_n$ with coefficients from a field $\mathbb{K}$. A polynomial ideal is a subset of a polynomial ring closed under addition and multiplication by any polynomial from the ring. In other words, if $\mathcal{I}$ is an ideal in the polynomial ring $\mathcal{R} = \mathbb{K}[x_1, \dots, x_n]$, then for any $f, g \in \mathcal{I}$ (where $f$ and $g$ are polynomials in the ring) and any polynomial $h \in \mathcal{R}$, both $f + g \in \mathcal{I}$ and $h \cdot f \in \mathcal{I}$ must hold.

In this paper, we let $\mathcal{I} = \langle f_1, \dots, f_k \rangle \subset \mathcal{R}$ denote the ideal generated by the polynomials $f_i$. A monomial ordering $\prec$ is established for all monomials in $\mathcal{R}$. For any $f \in R$, the leading monomial $\mathrm{LM}_{\prec}(f)$ is the maximum monomial in $f$, with the leading coefficient $\mathrm{LC}_{\prec}(f)$, yielding the leading term $\mathrm{LT}_{\prec}(f) = \mathrm{LC}_{\prec}(f)\mathrm{LM}_{\prec}(f)$. The leading monomial ideal of $\mathcal{I}$ is defined as $\mathrm{LM}_{\prec}(\mathcal{I}) = \langle \mathrm{LM}_{\prec}(f) \mid f \in \mathcal{I} \rangle$. A finite set $\{g_1, \dots, g_m\} \subset \mathcal{I}$ forms a Gröbner basis for $\mathcal{I}$ with respect to $\prec$ if $\mathrm{LM}_{\prec}(\mathcal{I}) = \langle \mathrm{LM}_{\prec}(\mathrm{g}_1), \dots, \mathrm{LM}_{\prec}(\mathrm{g}_m) \rangle$.
Next, consider $\mathcal{S} = \mathbb{K}[\mathbf{a}, \mathbf{x}]$, where $\mathbb{K}$ is a field, $\mathbf{a} = a_1, \dots, a_m$ are parameters, and $\mathbf{x} = x_1, \dots, x_n$ are variables. We define monomial orders $\prec_{\mathbf{x}}$ and $\prec_{\mathbf{a}}$ for the variables and parameters, respectively. The product ordering $\prec_{\mathbf{x},\mathbf{a}}$ is defined such that for $\alpha, \beta \in \mathbb{N}^n$ and $\gamma, \delta \in \mathbb{N}^m$, we have $\mathbf{x}^{\alpha}\mathbf{a}^{\gamma} \prec_{\mathbf{x},\mathbf{a}} \mathbf{x}^{\beta}\mathbf{a}^{\delta}$ if either $\mathbf{x}^{\alpha} \prec_{\mathbf{x}} \mathbf{x}^{\beta}$ or $\mathbf{x}^{\alpha} = \mathbf{x}^{\beta}$ and $\mathbf{a}^{\gamma} \prec_{\mathbf{a}} \mathbf{a}^{\delta}$.
Consider the specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \overline{\mathbb{K}}$, where $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$. This allows the substitution of parameters in $f \in \mathbb{K}[\mathbf{a}]$ with elements from $\overline{\mathbb{K}}^m$. For a finite set $F \subset \mathcal{R}$, we define the variety of $F$ as $\mathbb{V}(F)$, the set of common zeros of $F$.
We define a Gröbner system for a parametric polynomial ideal as follows:

**Definition 1.2.** Let $F \subset \mathcal{S}$ and $\mathcal{G} = \{(G_i, N_i, W_i)\}_{i=1}^{\ell}$ be a finite set of triples, with $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$ and $G_i \subset \mathcal{S}$. The set $\mathcal{G}$ is a *Gröbner system* of $\langle F \rangle$ with respect to $\prec_{\mathbf{x},\mathbf{a}}$ on $V \subseteq \overline{\mathbb{K}}^m$ if for each $i$:

- For any specialization $\sigma$ satisfying $(N_i, W_i)$, the set $\sigma(G_i) \subset \overline{\mathbb{K}}[\mathbf{x}]$ forms a Gröbner basis for $\langle \sigma(F) \rangle$ with respect to $\prec_{\mathbf{x}}$. (Here, $\sigma$ satisfies $(N_i, W_i)$ if $\sigma(p) = 0$ for all $p \in N_i$ and $\sigma(q) \neq 0$ for some $q \in W_i$.)
- $V \subseteq \bigcup_{i=1}^{\ell} \mathbb{V}(N_i) \setminus \mathbb{V}(W_i)$.

Each $(N_i, W_i, G_i)$ represents a branch of the Gröbner system $\mathcal{G}$, where $N_i$ is the null condition set and $W_i$ the non-null condition set. Additionally, $\mathcal{G}$ is a Gröbner system of $F$ if $V = \overline{\mathbb{K}}^m$.

**Theorem 1.3.** *Every parametric polynomial ideal $F$ in $\mathcal{S}$ has a Gröbner system.*

*Proof.* See [10, Proposition 3.4 and Theorem 2.7]. □

**Example 1.4.** Let $F = \{(1-c)y - ax^2, x + by^2\} \subset \mathbb{K}[a, b, c, x, y]$, where $a, b, c$ are parameters and $x, y$ are variables. Using our implementation of the PGBMain algorithm in `Maple`, we compute the following CGS for $\langle F \rangle$ under the product ordering $y \prec_{lex} x$ and $c \prec_{lex} b \prec_{lex} a$:

$$\begin{cases} ([\,], & [ab^2], & [ab^2y^4 - y + cy, x + by^2]) \\ ([ab^2], & [c-1], & [cy - y, x + by^2]) \\ ([c-1, ab^2], & [\,], & [x + by^2]). \end{cases}$$

For instance, with $a = 2$, $b = 1$, and $c = 3$, the first branch corresponds to these parameter values. Therefore, $\{2y^4 - y + 3y, x + y^2\}$ constitutes a Gröbner basis for the ideal $\langle F \rangle_{|_{a=2,b=1,c=3}} = \langle -2y - 2x^2, x + y^2 \rangle$.

## 2. PGBMain Algorithm

In this section, we restate the PGBMain algorithm which is one of the most efficient algorithms for computing Gröbner systems. This algorithm receives two finite subsets $\mathbb{K}[\mathbf{a}]$ namely $N, W$, a finite subset of parametric polynomial ideal $F \subset \mathbb{K}[\mathbf{a}, \mathbf{x}]$ and its goal is to produce a Gröbner system of the input parametric polynomial ideal. Furthermore, we employ the MDBasis sub-algorithm which computes a *minimal Dickson basis* of a polynomial ideal with parametric coefficients.

**Definition 2.1.** Let $F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be a finite set of parametric polynomials. A finite set $G \subset F$ is a *minimal Dickson basis* of $F$, denoted as MDBasis($F$), if it satisfies:

1. $\langle \mathrm{LM}_{\prec_{\mathbf{x}}}(G) \rangle = \langle \mathrm{LM}_{\prec_{\mathbf{x}}}(F) \rangle$,
2. For any $p, q \in G$ we have $\mathrm{LM}_{\prec_{\mathbf{x}}}(p) \nmid \mathrm{LM}_{\prec_{\mathbf{x}}}(q)$ and $\mathrm{LM}_{\prec_{\mathbf{x}}}(q) \nmid \mathrm{LM}_{\prec_{\mathbf{x}}}(p)$.

A straightforward algorithm (not optimal) can be derived from the definition of a minimal Dickson basis.

**Algorithm 1** MDBasis (Minimal Dickson Basis)

**Input:** $F = \{f_1, \ldots, f_n\}$; a finite subset of $\mathbb{K}[\mathbf{a}, \mathbf{x}]$
**Output:** A minimal Dickson basis of $F$
  $F' :=$ NULL
  **for** $i$ **from** 1 to $n$ **do**
    flag:= false
    **for** $g \in \{F'\} \cup \{f_i, \ldots, f_n\}$ **while** flag= false **do**
      **if** LM($g$)| LM($f_i$) **then**
        flag:= true
      **end if**
    **end for**
    **if** flag= false **then**
      $F' := F', f_i$
    **end if**
  **end for**
  **Return** ($\{F'\}$)

The following simple example shows that a minimal Dickson basis may not be unique.

**Example 2.2.** Let us consider $F = \{ay^3 - z - b, bx^2 - y + z, by + 1, (a-1)x + y^2, (b-a)x + ay - c, (c+a)y - (b-2)z\} \subset \mathbb{K}[a, b, c][x, y, z]$. The following sets are minimal Dickson bases of $F$:

- $G_1 = \{(b-a)x + ay - c, (c+a)y - (b-2)z\}$
- $G_2 = \{by + 1, (a-1)x + y^2\}$
- $G_3 = \{by + 1, (b-a)x + ay - c\}$

Kapur et al.'s algorithm starts with a global variable PGB as an empty set, to which new branches are added in each iteration, ultimately forming a Gröbner system of the input ideal.

**Algorithm 2** PGBMain

**Input:** $N, W$; finite subsets of $\mathbb{K}[\mathbf{a}]$ and $F$; a finite subset of $\mathbb{K}[\mathbf{a}, \mathbf{x}]$
**Output:** A Gröbner system of $F$ on $\mathbb{V}(N) \setminus \mathbb{V}(W)$
  **if** $(N, W)$ is inconsistent **then**
    **Return** ($\emptyset$)
  **end if**
  $G :=$ ReducedGröbnerBasis($F \cup N, \prec_{\mathbf{x},\mathbf{a}}$)
  **if** $1 \in G$ **then**
    **Return** ($\{(N, W, \{1\})\}$)
  **end if**
  $G_r := G \cap \mathbb{K}[a]$
  **if** $(G_r, W)$ is inconsistent **then**
    **Return** (PGB)
  **else**
    $G_m :=$MDBasis($G \setminus G_r$)
    $h =$lcm$\{h_1, \ldots, h_k\}$ with $h_i = \mathrm{LC}_{\prec_{\mathbf{x}}}(g_i)$ for each $g_i \in G_m = \{g_1, \ldots, g_k\}$
    **if** $(G_r, W \times \{h\})$ is consistent **then**
      PGB:=PGB$\cup\{G_r, W \times \{h\}, G_m\}$
    **end if**
    **Return** PGB $\cup \bigcup_{h_i \in \{h_1, \ldots, h_k\}}$PGBMain$(G_r \cup \{h_i\}, W \times \{h_1 h_2 \cdots h_{i-1}\}, G \setminus G_r) \cup \{(\text{Other cases}, \{1\})\}$
  **end if**

## 3. IMPROVED-PGBMAIN Algorithm

The efficient PGBMAIN algorithm can compute Gröbner systems, but it has two main drawbacks. First, it may output a Gröbner system with multiple branches, resulting in a Gröbner basis of $\{1\}$. This issue can be addressed by merging branches, which reduces the consistency checks and significantly enhances algorithm performance. We incorporated this modification into the above PGBMAIN algorithm [?]. Second, since a minimal Dickson basis is not unique, the algorithm does not specify how to choose an optimal minimal Dickson basis for continuing Gröbner system computations. In this paper, we propose a sub-algorithm to compute an optimal minimal Dickson basis, improving the performance of PGBMAIN algorithm. The polynomial order in $F$ impacts the MDBasis output, as reflected in the algorithm's design. Therefore, selecting polynomials with specific properties is crucial. Poor choices can lead to unnecessary branches in the generating set before computing a Gröbner basis. Our strategy employs several simple heuristics as selection benchmarks:
*Diversity*: Choose the polynomial with the fewest distinct parameters in its coefficients among the input ideal's generators.
*Length*: Choose the polynomial with the fewest monomials from the input ideal generators.
*Degree*: Select the polynomial with the lowest total degree concerning the parametric coefficients.
*Position*: Pick the polynomial where the first parametric coefficient appears later in the input list.
These criteria inform an algorithm to sort polynomials in the input ideal or their coefficient matrix rows. The algorithm compares two parametric polynomials from the ideal $F$ (or two rows from the coefficient matrix) based on these criteria.

**Algorithm 3** COMPARISON

**Require:** $f_1, f_2 \in F \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}] = \mathbb{K}[a_1, \dots, a_m, x_1, \dots, x_n]$
**Ensure:** Returns either true or false
  $C_1 :=$ number of parameters in $f_1$
  $C_2 :=$ number of parameters in $f_2$
  $L_1 :=$ number of monomials in $f_1$
  $L_2 :=$ number of monomials in $f_2$
  $D_1 :=$ max degree of parametric coefficients in $f_1$
  $D_2 :=$ max degree of parametric coefficients in $f_2$
  $P_1 :=$ position of first parameter in $f_1$
  $P_2 :=$ position of first parameter in $f_2$
  **if** $C_2 < C_1$ **then**
    false
  **else if** $L_2 < L_1$ **then**
    false
  **else if** $D_2 < D_1$ **then**
    false
  **else if** $P_2 > P_1$ **then**
    false
  **else**
    true
  **end if**

We can sort a list of parametric polynomials $F$ using the COMPARISON function. By executing `sort(F,Comparison)`, we generate a sorted list of polynomials based on the specified criteria.

**Example 3.1.** Consider $F = [(a-1)x_1 + x_2 + (a+1)x_3 + (c^2-1)x_4, 2x_1 + 2x_2 - bx_3 + (c+1)x_4, (a+2)x_1 + 3x_2 - x_3 + (a-2)x_4, (a+b)x_1 + (c-3)x_2 + (-b-1)x_3 + 2x_4] \subset \mathbb{K}[a,b,c][x_1, x_2, x_3, x_4]$, where $a, b, c$ are parameters and $x_1, x_2, x_3, x_4$ are variables. Using our `Maple` implementation, we find:
`sort(F,Comparison)` $= [(a+2)x_1 + 3x_2 - x_3 + (a-2)x_4, 2x_1 + 2x_2 - bx_3 + (c+1)x_4, (a-1)x_1 + x_2 + (a+1)x_3 + (c^2-1)x_4, (a+b)x_1 + (c-3)x_2 + (-b-1)x_3 + 2x_4]$.

We can incorporate the COMPARISON algorithm into the MDBASIS procedure, creating the IMPROVED-MDBASIS algorithm. This enhancement optimizes the minimal Dickson basis, reducing additional branches and improving efficiency in memory and timing for Gröbner system computations. For this purpose, we first classify the input polynomials $F$ by their leading monomial, grouping those with the same leading monomial into cells. We then apply the COMPARISON command to these partitions. The collection of first polynomials in each cell forms an optimal minimal Dickson basis for $F$.

**Algorithm 4** IMPROVED-MDBASIS

**Input:** $F = \{f_1, \dots, f_n\}$; a finite subset of $\mathbb{K}[\mathbf{a}, \mathbf{x}]$
**Output:** An optimal minimal Dickson basis of $F$
  $F' :=$ NULL
  Classify $F$ into $\{F_i\}_{i=1}^{\ell}$ w.r.t. their leading monomial so that $F = \bigcup_{i=1}^{\ell} F_i$
  **for** $i$ **from** 1 to $\ell$ **do**
    $F_i' :=$sort$(F_i,$ COMPARISON$)$
    $f_i' :=$ the first polynomial of $F_i'$
    $F' := F', f_i'$
  **end for**
  **Return** $(\{F'\})$

We can now integrate the IMPROVED-MDBASIS algorithm into the PGBMAIN procedure, known as the IMPROVED-PGBMAIN algorithm. This integration will help prevent the unnecessary generation of extra branches, ultimately enhancing the efficiency of timing and memory utilization in Gröbner system computations.

**Algorithm 5** IMPROVED-PGBMAIN

**Input:** $N, W$; finite subsets of $\mathbb{K}[\mathbf{a}]$ and $F$; a finite subset of $\mathbb{K}[\mathbf{a}, \mathbf{x}]$
**Output:** A Gröbner system of $F$ on $\mathbb{V}(N) \setminus \mathbb{V}(W)$
  **if** $(N, W)$ is inconsistent **then**
    **Return** $(\emptyset)$
  **end if**
  $G :=$ ReducedGröbnerBasis$(F \cup N, \prec_{\mathbf{x},\mathbf{a}})$
  **if** $1 \in G$ **then**
    **Return** $(\{(N, W, \{1\})\})$
  **end if**
  $G_r := G \cap \mathbb{K}[a]$
  **if** $(G_r, W)$ is inconsistent **then**
    **Return** (PGB)
  **else**
    $G_m :=$IMPROVED-MDBASIS$(G \setminus G_r)$
    $h =$lcm$\{h_1, \dots, h_k\}$ with $h_i = \mathrm{LC}_{\prec_x}(g_i)$ for each $g_i \in G_m = \{g_1, \dots, g_k\}$
    **if** $(G_r, W \times \{h\})$ is consistent **then**
      PGB:=PGB$\cup\{G_r, W \times \{h\}, G_m\}$
    **end if**
    **Return** PGB $\cup \bigcup_{h_i \in \{h_1, \dots, h_k\}}$PGBMAIN$(G_r \cup \{h_i\}, W \times \{h_1 h_2 \cdots h_{i-1}\}, G \setminus G_r) \cup \{(\text{Other cases}, \{1\})\}$
  **end if**

**Theorem 3.2.** *The* IMPROVED-PGBMAIN *algorithm terminates in a finite number of steps and correctly computes a Gröbner system.*

*Proof.* The correction and termination of the IMPROVED-PGBMAIN algorithm are assured by the original PGBMAIN algorithm. □

**Example 3.3.** Let $F = [d(a-b)xy + y, bx^2 + cy] \subset \mathbb{K}[a, b, c, d][x, y]$, where $x, y$ are variables and $a, b, c, d$ are parameters. We use the monomial orderings $y \prec_{lex} x$ and $d \prec_{lex} c \prec_{lex} b \prec_{lex} a$. By applying our implementation

of the IMPROVED-PGBMAIN algorithm in `Maple`, we obtain the following Gröbner system for the ideal $\langle F \rangle$ with respect to the elimination product ordering of these monomial orderings:

$$\begin{cases} ([\,], [bcd^2(a-b)^2], & [a^2cd^2y^2 - 2abcd^2y^2 + b^2cd^2y^2 + by, -acdy^2 + bcdy^2 + bxy, bx^2 + cy]), \\ ([a^2cd^2 - 2abcd^2 + b^2cd^2], [b], & [by, bx^2]), \\ ([b], [c], & [cy]), \\ ([c, b], [a, d], & [adxy + y]), \\ ([c, b, ad], [\,], & [y]). \end{cases}$$

The computed Gröbner system with the IMPROVED-PGBMAIN algorithm has five branches, whereas the one derived from our `Maple` implementation of the PGBMAIN algorithm has eight branches—three more than that produced by the IMPROVED-PGBMAIN algorithm:

$$\begin{cases} ([\,], [d^2bac(a-b)^2], & [a^2cd^2y^2 - 2abcd^2y^2 + b^2cd^2y^2 + by, y - acd^2y^2 + bcd^2y^2 + adxy, bx^2 + cy]), \\ ([a], [b, c, d], & [bcd^2y^2 + y, cdy^2 + xy, bx^2 + cy]), \\ ([bcd^2, a], [b], & [y, bx^2]), \\ ([ad], [a, b], & [ay, bx^2 + cy]), \\ ([a^2cd^2 - 2abcd^2 + b^2cd^2], [a, b, d], & [by, bx^2]), \\ ([b], [c], & [cy]), \\ ([c, b], [a, d], & [adxy + y]), \\ ([c, b, ad], [\,], & [y]). \end{cases}$$

The PGBMAIN algorithm using MDBASIS generates new branches when facing an undecidable leading coefficient, resulting in unnecessary branches and decreased efficiency. This can be mitigated by applying the IMPROVED-MDBASIS sub-algorithm, which employs selection strategies to sort polynomials according to the COMPARISON function.

## 4. Examples and Comparison

This section compares the performance of the IMPROVED-PGBMAIN algorithm with the PGBMAIN algorithm, using implementations in `Maple 18`. We selected specific parametric ideals in the ring $\mathcal{S} = \mathbb{K}[a, b, c, d, m, n, r, t][x, y, z, u, v, w]$ and aimed to compute a Gröbner system for each ideal with respect to the orderings $w \prec_{lex} v \prec_{lex} u \prec_{lex} z \prec_{lex} y \prec_{lex} x$ and $t \prec_{lex} r \prec_{lex} n \prec_{lex} m \prec_{lex} d \prec_{lex} c \prec_{lex} b \prec_{lex} a$.

- EX.1 = $[rx^5 + (ab - c)z - n, cy^3 + acx + dn, z^3 - (c - t)y]$
- EX.2 = $[by^2 - a - y, rx^2 + (ab - c)x, acz + cy + dn, z^2 - (c - t)y]$
- EX.3 = $[(dm - 1)x^2y + by, y^2 + c - 1, -nx + y^2 + y, az^4 - 1]$
- EX.4 = $[d(a - b)xy + y, bx^2 + cmy]$
- EX.5 = $[abxy + ay^3 - cz + 1, ax^2 + ax + cuu, tuu^3 + tuu, bz^3 + mnx - bz]$
- EX.6 = $[(1 - a)y^2 - bx^2, d(a - b)xy + y - t, bx^2 + cmy, (a - b)xy + ay^2, (a - b)xy + (b - a - c)x^2, y^2 + (b - c)x^2]$
- EX.7 = $[(c - d)(a - b)xy + y, bx^2 + cy, (d - c)y^2 - y - 1]$
- EX.8 = $[(t - a)xz + (1 - a)z, (t - a)x + (m - b)xy, (a - 2)x^2 + uu, bcx^2 + my]$
- EX.9 = $[(a - b)xy + ay, bx^3 + cy - 1, z^2 - n + y, mnz^3 - z]$

The results are shown in the following table, with timings recorded on a personal computer featuring a Ryzen 6800 processor, 8 GB RAM, and a 64-bit Windows 10 operating system. The third and fourth columns represent CPU time (in seconds) and memory usage (in gigabytes) for each procedure. The last column indicates the number of branches in the computed Gröbner system. Also, in the above table "–-" means that the PGBMAIN algorithm can not compute a Gröbner system within 300 seconds.

| Example | Method | Time (Sec) | Used Memory (GB) | Branch |
|---|---|---|---|---|
| EX.1 | Improved-PGBMain | 27.85 | 5.56 | 14 |
| | PGBMain | — | — | — |
| EX.2 | Improved-PGBMain | 39.78 | 8.76 | 15 |
| | PGBMain | — | — | — |
| EX.3 | Improved-PGBMain | 0.52 | 0.03 | 5 |
| | PGBMain | 0.69 | 0.0 | 7 |
| EX.4 | Improved-PGBMain | 0.59 | 0.04 | 5 |
| | PGBMain | 0.78 | 0.05 | 8 |
| EX.5 | Improved-PGBMain | 45.83 | 12.84 | 20 |
| | PGBMain | 240.16 | 86.3 | 22 |
| EX.6 | Improved-PGBMain | 1.47 | 0.28 | 9 |
| | PGBMain | 1.82 | 0.35 | 11 |
| EX.7 | Improved-PGBMain | 0.49 | 0.04 | 2 |
| | PGBMain | 3.91 | 0.78 | 6 |
| EX.8 | Improved-PGBMain | 121.03 | 35.87 | 39 |
| | PGBMain | — | — | — |
| EX.9 | Improved-PGBMain | 31.28 | 10.05 | 8 |
| | PGBMain | 36.14 | 11.01 | 10 |

A comparison of the timing columns and our tests across various examples highlights the efficiency of our implemented Improved-PGBMain algorithm.

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# $\lambda$-pure derived category

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**Abstract**

Let $\mathcal{A}$ be a Grothendieck category and $\lambda = \lambda_{\mathcal{A}}$ is a unique minimal regular cardinal. In this paper, we introduce a model structure on $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$ the exact category of all complexes with the degree-wise $\lambda$-pure exact structure. Our result is based on the Gillespie's Theorem by introducing two compatible cotorsion pairs on this category. We call this model structure the $\lambda$-pure projective model structure on $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$ and its homotopy category is $\mathbb{D}_{\lambda\text{-pur}}(\mathcal{A})$, the $\lambda$-pure derived category.

## 1. introduction

In [4], Crawley-Boevey demonstrated that locally finitely presented additive categories serve as a natural framework for developing a robust notion of purity theory. A locally finitely presented additive category $\mathcal{A}$ is defined as an additive category with direct limits, where every object in $\mathcal{A}$ can be expressed as a direct limit of finitely presented objects, and the class of finitely presented objects is skeletally small. In this category a sequence $0 \to X \to Y \to Z \to 0$ is pure if $0 \to \mathcal{A}(G, X) \to \mathcal{A}(G, Y) \to \mathcal{A}(G, Z) \to 0)$ is exact for each finitely presented object $G \in \mathcal{A}$. Using this definition, we can construct an exact structure on $\mathcal{A}$ , referred to as the pure exact structure, which gives rise to the pure derived category $\mathbb{D}_{pur}(\mathcal{A})$. This category has been studied by authors such as Christensen and Hovey [3], Krause [11], and Šťovíček [13]. Recently in [7] it has been shown that this pure derived category can be obtained as the homotopy category of two model category structures by using the concept of pure projectives and the pure injectives. One of the model structures constructed in the category of complexes was introduced by Estrada, Gillespie, and Odabasi in [5]. They consider a degree-wise $\otimes$-pure exact structure on $\mathbb{C}(\mathcal{G})$ the category of complexes on a Grothendieck category $\mathcal{G}$, and introduced a Hovey pair so as to construct a model structure in which its homotopy category is the pure derived category.
In this paper, we will introduce a model structure on $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$ the exact category of complexes with the degree-wise $\lambda$-pure exact structure where $\mathcal{A}$ is a Grithendieck category and $\lambda$ is a regular cardinal such that $\mathcal{A}$ is a locally

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$\lambda$-presentable category. Our main result is based on the Gillespie's Theorem in [6] by introducing two compatible cotorsion pairs on this category.
The paper is organized as follows. In Section 2 we provide any background information needed through this paper such as Exact category and Purity. Our main result appears in Section 3 as Theorem 3.4.

## 2. Purity

In this section we define proper exact structure on a category to introduce an exact category that we need it in the next section. First, we recall the notion of exact category from [2] and Gillespie's Theorem from [6].
An exact category is a pair $(\mathcal{A},\mathcal{E})$ where $\mathcal{A}$ is an additive category and $\mathcal{E}$ is a distinguished class of diagrams of the form $X \xrightarrow{i} Y \xrightarrow{d} Z$ called *conflation*, satisfying certain axioms which make conflations behave similar to short exact sequences in an abelian category. A map such as $i$ in the language of exact categories is called an *inflation*(denoted by $\rightarrowtail$) while $d$ is called a *deflation* (denoted by $\twoheadrightarrow$).
Now let $(\mathcal{A},\mathcal{E})$ be an exact category. The axioms of exact category allow us to define Yoneda Ext groups with usual properties. The abelian group $\mathrm{Ext}^1_{\mathcal{E}}(X,Y)$ is the group of equivalence classes of short exact sequences $Y \rightarrowtail Z \twoheadrightarrow X$. In particular, $\mathrm{Ext}^1_{\mathcal{E}}(X,Y) = 0$ if and only if every short exact sequence $Y \rightarrowtail Z \twoheadrightarrow X$ is isomorphic to the split exact sequence $Y \rightarrowtail Y \bigoplus X \twoheadrightarrow X$.
Let $(\mathcal{A},\mathcal{E})$ be an exact category. A pair $(\mathcal{F},\mathcal{D})$ of full subcategories of $\mathcal{A}$ is called a cotorsion pair provided that

$$\mathcal{F} = {}^{\perp}\mathcal{D} \qquad \text{and} \qquad \mathcal{F}^{\perp} = \mathcal{D},$$

where $^{\perp}$ is taken with respect to the functor $\mathrm{Ext}^1_{\mathcal{E}}$. The cotorsion pair $(\mathcal{F},\mathcal{D})$ is said to have enough projectives if for every $X \in \mathcal{A}$ there is a short exact sequence $D \rightarrowtail F \twoheadrightarrow X$ with $D \in \mathcal{D}$ and $F \in \mathcal{F}$. We say that it has enough injectives if it satisfies the dual statement. If both of these hold we say the cotorsion pair is complete.
Hovey defined in [10], Definition 2.1, the notion of an abelian model structure. In order to define exact model structure on exact category $(\mathcal{A},\mathcal{E})$ we need to specify cofibrant, fibrant and trivial object in this category. Note that for any $X \in \mathcal{A}$, $0 \rightarrowtail X$ is an inflation and $X \twoheadrightarrow 0$ is a deflation.
Now suppose $(\mathcal{A},\mathcal{E})$ has a model structure as defined in Definition 1.1.3 of [9]. An object $W \in \mathcal{A}$ is said to be a trivial if $0 \rightarrowtail W$ is a weak equivalence. An object $A \in \mathcal{A}$ is said to be a cofibrant (resp. trivially cofibrant) if $0 \rightarrowtail A$ is a cofibration (resp. trivially cofibration). Dually $B \in \mathcal{A}$ is fibrant (resp. trivially fibrant) if $B \twoheadrightarrow 0$ is fibration (resp. trivial fibration).

**Definition 2.1.** Let $(\mathcal{A},\mathcal{E})$ be an exact category. An exact model structure on $(\mathcal{A},\mathcal{E})$ is a model structure in the sense of Definition 1.1.3 of [9] in which each of the following hold.

(1) A map is a (trivial) cofibration if and only if it is an inflation with a (trivially) cofibrant cokernel.

(2) A map is a (trivial) fibration if and only if it is a deflation with a (trivially) fibrant kernel.

The next theorem is a result due to Hovey [10] which is described by Gillespie in the sense of exact category, see [6]. We just recall that a class of objects $\mathcal{W} \in \mathcal{A}$ is a *thick subcategory* of $\mathcal{A}$ if it is closed under direct summands and if two out of three of the terms in a short exact sequence are in $\mathcal{W}$, then so is the third.

**Theorem 2.2.** *[6, Theorem 3.3] Let $(\mathcal{A},\mathcal{E})$ be an exact category with an exact model structure. Let $\mathcal{C}$ be the class of cofibrant objects, $\mathcal{F}$ be the class of fibrant objects and $\mathcal{W}$ be the class of trivial objects. Then $\mathcal{W}$ is a thick subcategory of $\mathcal{A}$ and both $(\mathcal{C},\mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$ are complete cotorsion pairs in $\mathcal{A}$. If we further assume that $(\mathcal{A},\mathcal{E})$ is weakly idempotent complete then the converse holds. That is, given two compatible cotorsion pairs $(\mathcal{C},\mathcal{W}\cap\mathcal{F})$ and $(\mathcal{C}\cap\mathcal{W},\mathcal{F})$, each complete and with $\mathcal{W}$ a thick subcategory, then there is an exact model structure on $\mathcal{A}$ where $\mathcal{C}$ is the class of cofibrant objects, $\mathcal{F}$ is the class of fibrant objects and $\mathcal{W}$ is the class of trivial objects.*

**Definition 2.3.** [1, Definition 1.13] Let $\mathcal{A}$ be a category and $\lambda$ be an regular cardinal. An object $A \in \mathcal{A}$ is called $\lambda$-presentable if the functor $\mathcal{A}(A,-) : \mathcal{A} \to \mathbf{Ab}$ commutes with $\lambda$-directed colimits in $\mathcal{A}$, i.e. for each $\lambda$-direct system $(Y_i \mid i \in I)$ the canonical map

$$\varinjlim \mathcal{A}(A,Y_i) \longrightarrow \mathcal{A}(A,\varinjlim Y_i)$$

is an isomorphism.

**Definition 2.4.** [1, Page 68, 2.1] If a category $\mathcal{A}$ has $\lambda$-direct colimits and there exists a small set $\mathcal{S}$ of $\lambda$-presentable objects such that every $X \in \mathcal{A}$ can be expressed as a $\lambda$-direct colimit of objects from $\mathcal{S}$, then $\mathcal{A}$ is called $\lambda$-accessible. If moreover, $\mathcal{A}$ is cocomplete, it is called locally $\lambda$-presentable.

**Lemma 2.5.** *[8, Lemma 2.5] Let $\mathcal{A}$ be a Grothendieck category. Then there exists a unique minimal regular cardinal $\lambda_{\mathcal{A}}$ such that $\mathcal{A}$ is locally $\lambda_{\mathcal{A}}$-presentable.*

Throughout this paper, we assume that $\mathcal{A}$ is a Grothendieck category and $\lambda = \lambda_{\mathcal{A}}$ is such a unique minimal regular cardinal.

**Definition 2.6.** A short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ is defined to be $\lambda$-pure if for any $\lambda$-presentable object $F$, the sequence $0 \to \mathcal{A}(F, A') \to \mathcal{A}(F, A) \to \mathcal{A}(F, A'') \to 0$ is an exact sequence of abelian groups.

**Definition 2.7.** An object $P \in \mathcal{A}$ is defined to be $\lambda$-pure projective if $P$ is projective with respect to $\lambda$-pure exact sequences. We denote the class of all $\lambda$-pure projective objects by $\lambda$-PPrj$\mathcal{A}$.

**Definition 2.8.** The exact structure on $\mathbb{C}(\mathcal{A})$ the category of complexes whose conflations are defined by property that they are $\lambda$-pure exact in $\mathcal{A}$ in each degree, will be called the induced $\lambda$-pure exact structure on $\mathbb{C}(\mathcal{A})$. We denote the Ext groups with respect to this exact structure by $\mathrm{Ext}^i_{\mathbb{C}(\mathcal{A})}$.

The second exact structure which is often considered on $\mathbb{C}(\mathcal{A})$ is degree-wise split exact structure, whose conflations are those sequences of complexes which are split exact in each degree. we denote the Ext groups with respect to this exact structure by $\mathrm{Ext}^i_{\mathrm{dw}}$.

**Definition 2.9.** A complex $\mathbf{X} \in \mathbb{C}(\mathcal{A})$ is $\lambda$-pure acyclic if each $0 \to Z^i(\mathbf{X}) \to X^i \to Z^{i+1}(\mathbf{X}) \to 0$ is $\lambda$-pure exact sequence in $\mathcal{A}$. The class of $\lambda$-pure acyclic complexes denoted by $\mathbb{C}_{\lambda\text{-pac}}(\mathcal{A})$.

## 3. $\lambda$-pure derived category

In this section we assume $\mathcal{A}$ is a Grothendieck category and $\lambda = \lambda_{\mathcal{A}}$ is such a unique minimal regular cardinal. We now define the following class in $\mathbb{C}(\mathcal{A})$:

$$\mathrm{dg}\lambda\text{-PPrj}\mathcal{A} = \{L \in \mathbb{C}(\mathcal{A}) \mid L^n \in \lambda\text{-PPrj}\mathcal{A} \text{ and each map } L \to E \text{ is homotopic to } 0, \forall\ E \in \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A})\}$$

**Proposition 3.1.** *Let $\mathcal{A}$ be a Grothendieck category. Then $(\mathbb{C}(\lambda\text{-PPrj}\mathcal{A}), \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}))$ is a complete cotorsion pair in $\mathbb{C}(\mathcal{A})$ with the induced $\lambda$-pure exact structure.*

*Proof.* By [7, Theorem 4.6] we can say that $(\mathrm{dg}\lambda\text{-PPrj}\mathcal{A}, \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}))$ is a complete cotorsion pair. So it is enough to show that $\mathbb{C}(\lambda\text{-PPrj}\mathcal{A}) = \mathrm{dg}\lambda\text{-PPrj}\mathcal{A}$. According to the definition of $\mathrm{dg}\lambda\text{-PPrj}\mathcal{A}$ we can clearly say that $\mathrm{dg}\lambda\text{-PPrj}\mathcal{A} \subseteq \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$. Now let $\mathbf{Q} \in \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$. According to [12, Theorem 1.3] we have $\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(\mathbf{Q}, \mathbf{X}) = 0$ for all $\mathbf{X} \in \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A})$. Moreover, it is to see that $\Sigma^{-1}\mathbf{Q}$ belongs to $\mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$. Hence

$$\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(\Sigma^{-1}\mathbf{Q}, \mathbf{X}) = \mathrm{Ext}^1_{\mathrm{dw}}(\mathbf{Q}, \mathbf{X}) = 0$$

Since $\mathbf{Q} \in \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$ we have

$$\mathrm{Ext}^1_{\mathbb{C}(\mathcal{A})}(\mathbf{Q}, \mathbf{X}) = \mathrm{Ext}^1_{\mathrm{dw}}(\mathbf{Q}, \mathbf{X}) = 0$$

Therefore, this implies that $\mathbf{Q} \in {}^{\perp}(\mathbb{C}_{\lambda\text{-pac}}(\mathcal{A})) = \mathrm{dg}\lambda\text{-PPrj}\mathcal{A}$. So we can say that $\mathbb{C}(\lambda\text{-PPrj}\mathcal{A}) \subseteq \mathrm{dg}\lambda\text{-PPrj}\mathcal{A}$ and we are done. □

Recall that a complex $\mathbf{X}$ in $\mathbb{C}(\mathcal{A})$ is contractible if the identity map on $\mathbf{X}$ is null-homotopic. Now we have the following interesting corollary:

**Corollary 3.2.** *Suppose that $\mathbf{X} \in \mathbb{C}(\mathcal{A})$ is a $\lambda$-pure acyclic complex. If all components of $\mathbf{X}$ are $\lambda$-pureprojective then $\mathbf{X}$ is contractible.*

*Proof.* Assume that $\mathbf{X} \in \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}) \cap \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$. By Proposition 3.1 we have $\mathrm{Ext}^1_{\mathcal{A}}(\Sigma\mathbf{X}, \mathbf{X}) = 0$. In addition, since $\mathbf{X} \in \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$ we can say that

$$\mathrm{Hom}_{\mathbb{K}(\mathcal{A})}(\mathbf{X}, \mathbf{X}) = \mathrm{Ext}^1_{dw}(\Sigma\mathbf{X}, \mathbf{X}) = \mathrm{Ext}^1_{\mathbb{C}(\mathcal{A})}(\Sigma\mathbf{X}, \mathbf{X}) = 0$$

This means that $1_{\mathbf{X}} \sim 0$, hence $\mathbf{X}$ is contractible. □

Now we can introduced our main result:

**Definition 3.3.** Consider the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$. Let $\mathbb{K}_{\lambda\text{-ac}}(\mathcal{A})$ be a full subcategory of $\mathbb{K}(\mathcal{A})$ consisting of $\lambda$-pure acyclic complexes. Notice that $\lambda$-pure acyclic complexes are closed under homotopy equivalences, so $\mathbb{K}_{\lambda\text{-ac}}(\mathcal{A})$ is well defined. If $f : \mathbf{X} \to \mathbf{Y}$ is morphism between $\lambda$-pure acyclic complexes, then $\mathrm{Con}(f)$ is again $\lambda$-pure acyclic. Thus $\mathbb{K}_{\lambda\text{-ac}}(\mathcal{A})$ is a triangulated subcategory of $\mathbb{K}(\mathcal{A})$. Because $\lambda$-pure acyclic complexes are closed under direct summands, $\mathbb{K}_{\lambda\text{-ac}}(\mathcal{A})$ is a thick subcategory of $\mathbb{K}(\mathcal{A})$. Then by the Verdier's correspondence, we get the $\lambda$-pure derived category

$$\mathbb{D}_{\lambda\text{-pur}}(\mathcal{A}) := \mathbb{K}(\mathcal{A})/\mathbb{K}_{\lambda\text{-ac}}(\mathcal{A})$$

**Theorem 3.4.** *There is a model structure on the exact category $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$ in which every object is fibrant and the cofibrant objects are all complexes which are $\lambda$-pure projective on each degree. The trivial objects are the $\lambda$-pure acyclic complexes. We call this model structure the $\lambda$-pure projective model structure on $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$ and its homotopy category is $\mathbb{D}_{\lambda\text{-pur}}(\mathcal{A})$, the $\lambda$-pure derived category.*

*Proof.* We claim that $(\mathbb{C}(\lambda\text{-PPrj}\mathcal{A}), \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}), \mathbb{C}(\mathcal{A}))$ is a Hovey triple. First, we note that by Corollary 3.2 $\mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}) \cap \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$ is equal to the class of all contractible complexes whit $\lambda$-pure projective components. So it can be said that this class is equal to all projective objects in $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$, i.e. we have

$$\mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}) \cap \mathbb{C}(\lambda\text{-PPrj}\mathcal{A}) = \text{Prj-}\mathbb{C}(\mathcal{A})$$

Clearly $(\text{Prj-}\mathbb{C}(\mathcal{A}), \mathbb{C}(\mathcal{A}))$ is a complete cotorsion pair in $(\mathbb{C}(\mathcal{A}), \mathcal{E}_\lambda)$. On the other hand by Proposition 3.1 $(\mathbb{C}(\lambda\text{-PPrj}\mathcal{A}), \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A}))$ is a complete cotrsion pair. Moreover, by [7, Lemma 4.4] $\mathbb{C}_{\lambda\text{-pac}}(\mathcal{A})$ is a thick subcategory. So if we set $\mathcal{C} = \mathbb{C}(\lambda\text{-PPrj}\mathcal{A})$, $\mathcal{W} = \mathbb{C}_{\lambda\text{-pac}}(\mathcal{A})$, and $\mathcal{F} = \mathbb{C}(\mathcal{A})$ then by Theorem 2.2 we can say that $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a Hovey triple. Note that by [7, Corollary 4.7] the homotopy category of this model category is $\mathbb{D}_{\lambda\text{-pur}}(\mathcal{A})$. □

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# Finiteness of Generalized Local Homology Modules

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**Abstract**

Let $R$ be a commutative Noetherian ring and $\mathfrak{a}$ be an ideal of $R$. Suppose $M$ is a finitely generated $R$-module and $N$ is an Artinian $R$-module. We define the concept of filter coregular sequence to determine the infimum of integers $i$ such that the generalized local homology $\mathrm{H}_i^{\mathfrak{a}}(M,N)$ is not finitely generated as an $\widehat{R}^{\mathfrak{a}}$-module, where $\widehat{R}^{\mathfrak{a}}$ denotes the $\mathfrak{a}$-adic completion of $R$. In particular, if $R$ is a complete semi-local ring, then $\mathrm{H}_i^{\mathfrak{a}}(M,N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all non-negative integers $i$ if and only if $(0 :_N \mathfrak{a} + \mathrm{Ann}(M))$ has finite length.

## 1. Introduction

In this paper, we consider a commutative Noetherian ring $R$ with non-zero identity, and an ideal $\mathfrak{a} \subseteq R$, as well as two $R$-modules $M$ and $N$. We denote the $\mathfrak{a}$-adic completion of $N$ by $\Lambda_{\mathfrak{a}}(N)$, and note that the $\mathfrak{a}$-adic completion functor $\Lambda_{\mathfrak{a}}(\cdot)$ is an additive covariant functor on the category of $R$-modules. We use $L_i^{\mathfrak{a}}(\cdot)$ to denote the $i$-th left derived functor of $\Lambda_{\mathfrak{a}}(\cdot)$. However, since the tensor functor is not left exact and the inverse limit is not right exact on the category of $R$-modules, computing the left-derived functors of $\Lambda_{\mathfrak{a}}(\cdot)$ is generally difficult. Moreover, it is important to note that $L_0^{\mathfrak{a}}(\cdot) \ncong \Lambda_{\mathfrak{a}}(\cdot)$.
Matlis studied $L_i^{\mathfrak{a}}(\cdot)$ in the case where $\mathfrak{a}$ is generated by a regular sequence and $R$ is a local ring in [9, 10], and proved some duality between this functor and the local cohomology functor. Recently, Divaani-Aazar et al. in [4] studied the containment of $L_i^{\mathfrak{a}}(\cdot)$ in a Serre class of $R$-modules up to a given upper bound $s \geq 0$.
Cuong and Nam in [2] defined the $i$-th *local homology* $\mathrm{H}_i^{\mathfrak{a}}(N)$ of $N$ with respect to $\mathfrak{a}$ as follows:

$$\mathrm{H}_i^{\mathfrak{a}}(N) := \varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_i^R(R/\mathfrak{a}^n, N).$$

They also showed that $L_i^{\mathfrak{a}}(N) \cong \mathrm{H}_i^{\mathfrak{a}}(N)$ when $N$ is Artinian. Similarly, the $i$-th *generalized local homology* $\mathrm{H}_i^{\mathfrak{a}}(M,N)$ of $M$ and $N$ with respect to $\mathfrak{a}$ is defined by

$$\mathrm{H}_i^{\mathfrak{a}}(M,N) := \varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_i^R(M/\mathfrak{a}^n M, N);$$

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see [12, 13] for basic properties and more details.
Matlis in [8] introduced the concept of cosequence (or coregular sequence) as a dual of the concept of regular sequence (see [14] and [16] for more details and basic properties). If $N$ is Artinian and $(0 :_N \mathfrak{a}) \neq 0$, then all maximal coregular $N$-sequences in $\mathfrak{a}$ have the same length, denoted by width$(\mathfrak{a}, N)$, where $(0 :_N \mathfrak{a})$ denotes the set of all elements $x \in N$ such that $rx = 0$ for all $r \in \mathfrak{a}$. Moreover,

$$\text{width}(\mathfrak{a}, N) = \inf\{i \in \mathbb{Z} : \text{H}^{\mathfrak{a}}_i(N) \neq 0\}$$

(see [3, Theorem 4.11]).
The filter regular sequences can be used to study the Artinianess of local cohomology modules of finitely generated $R$-modules (see [5, Sec. 3]). In this paper as a dual of the concept of filter regular sequence, we introduce the concept of filter coregular sequence to study the finiteness of local homology modules of Artinian $R$-modules.
Let Cosupp$(N)$ denote the set of all prime ideals of $R$ containing Ann$(N)$. A sequence $x_1, \dots, x_n$ of elements of $\mathfrak{a}$ is called a filter coregular $N$-sequence (of length $n$) in $\mathfrak{a}$ if

$$\text{Cosupp}\left((0 :_N (x_1, \dots, x_{i-1})R)/x_i(0 :_N (x_1, \dots, x_{i-1})R)\right) \subseteq \text{Max}(R)$$

for all $1 \leq i \leq n$, where Max$(R)$ denotes the set of all maximal ideals of $R$.
Assuming that $M$ is finitely generated and $N$ is Artinian, we prove that if there exists a filter coregular $N$-sequence in $\mathfrak{a}$ of infinite length, then every filter coregular $N$-sequence in $\mathfrak{a}$ can be extended to a filter coregular $N$-sequence in $\mathfrak{a}$ of infinite length, and in this case we set f-width$(\mathfrak{a}, N) = \infty$. Now suppose that all filter coregular $N$-sequences in $\mathfrak{a}$ have finite length. Then all maximal filter coregular $N$-sequences in $\mathfrak{a}$ are of the same length, denoted by f-width$(\mathfrak{a}, N)$. We show (see Theorem 2.6 and Remark 2.7) that:

$$\text{f-width}(\text{Ann}(M), N) = \inf\{i \in \mathbb{N}_0 : \text{Tor}^R_i(M, N) \text{ has infinite length as an } R\text{-module}\}$$

and

$$\text{f-width}(\mathfrak{a} + \text{Ann}(M), N) = \inf\{i \in \mathbb{N}_0 : \text{H}^{\mathfrak{a}}_i(M, N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\}.$$

In particular,

$$\text{f-width}(\mathfrak{a}, N) = \inf\{i \in \mathbb{N}_0 : \text{H}^{\mathfrak{a}}_i(N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\}.$$

We also show in Corollary 2.9 that if $\text{H}^{\mathfrak{a}}_i(M, N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$, then $(0 :_N \mathfrak{a}+\text{Ann}(M))$ has finite length. The converse statement is true when $R$ is a semi-local ring that is complete with respect to its Jacobson radical.

## 2. Main Results

We shall use the following notations and terminologies. Let $\mathfrak{a}$ be an ideal of $R$ and $N$ be an $R$-module. The radical of $\mathfrak{a}$ will be denoted by $\sqrt{\mathfrak{a}}$; also, Ann$(N)$ will denote the ideal $\{r \in R : rx = 0 \text{ for all } x \in N\}$ of $R$; and $(0 :_N \mathfrak{a})$ will denote the submodule $\{x \in N : rx = 0 \text{ for all } r \in \mathfrak{a}\}$ of $N$. We denote by V$(\mathfrak{a})$ the set of all prime ideals of $R$ containing $\mathfrak{a}$; and we use Cosupp$(M)$ to denote V$(\text{Ann}(M))$. The symbol $\mathbb{N}$ (respectively $\mathbb{N}_0$) will denote the set of positive (respectively non-negative) integers. We refer the reader for any unexplained terminology or notation to [1, 11, 15].

**Definition 2.1.** Let $N$ be an $R$-module. We say a prime ideal $\mathfrak{p}$ of $R$ is an *attached prime* of $N$, if there exists a submodule $M$ of $N$ such that $\mathfrak{p} = \text{Ann}(N/M)$. We denote by Att$(N)$ the set of all attached primes of $N$.

For an $R$-module $N$, it is clear that Att$(N) \subseteq$ Cosupp$(N)$ (we refer the reader to [14] for basic properties and more details of these notations). When $N$ has a *secondary representation* in the sense of [7], our definition of Att$(N)$ coincides with that of Macdonald (see [1, Exercise 7.2.5]). In particular, the set of attached primes of an Artinian module is a finite set.

**Proposition 2.2** ([6]). *Let $x_1, \dots, x_n$ be elements of $R$, and let $N$ be an Artinian $R$-module. The following conditions are equivalent:*

(i) $x_1, \dots, x_n$ *is a filter coregular $N$-sequence;*
(ii) $(0 :_N (x_1, \dots, x_{i-1})R)/x_i(0 :_N (x_1, \dots, x_{i-1})R)$ *has finite length for all* $1 \leq i \leq n$*;*
(iii) $\operatorname{Att}((0 :_N (x_1, \dots, x_{i-1})R)/x_i(0 :_N (x_1, \dots, x_{i-1})R)) \subseteq \operatorname{Max}(R)$ *for all* $1 \leq i \leq n$*; and*
(iv) $x_i \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(0:_N (x_1,\dots,x_{i-1})R) \setminus \operatorname{Max}(R)} \mathfrak{p}$ *for all* $1 \leq i \leq n$.

**Proposition 2.3** ([6]). *Let $M$ and $N$ be $R$-modules, and let $x_1, \dots, x_n$ be elements of $R$. For each $i \in \mathbb{N}_0$, there are the following inclusions:*

$$\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, (0 :_N (x_1, \dots, x_n)R))\right) \subseteq \left(\bigcup_{j=i}^{i+n} \operatorname{Cosupp}\left(\operatorname{Tor}_j^R(M, N)\right)\right) \cup \left(\bigcup_{k=1}^{n} \bigcup_{j=i+2}^{i+2+n-k} \operatorname{Cosupp}\left(\operatorname{Tor}_j^R\left(M, \frac{(0 :_N (x_1, \dots, x_{k-1})R)}{x_k(0 :_N (x_1, \dots, x_{k-1})R)}\right)\right)\right);$$

*and if, in addition, $x_1, \dots, x_n$ belong to* $\operatorname{Ann}(M)$*, then*

$$\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \subseteq \operatorname{Cosupp}\left(\operatorname{Tor}_{i-n}^R(M, (0 :_N (x_1, \dots, x_n)R))\right) \cup \left(\bigcup_{k=1}^{n} \bigcup_{j=i+1-k}^{i+2-k} \operatorname{Cosupp}\left(\operatorname{Tor}_j^R\left(M, \frac{(0 :_N (x_1, \dots, x_{k-1})R)}{x_k(0 :_N (x_1, \dots, x_{k-1})R)}\right)\right)\right).$$

**Corollary 2.4** ([6]). *Let $M$ and $N$ be $R$-modules, and let $x_1, \dots, x_n$ be a filter coregular $N$-sequence in* $\operatorname{Ann}(M)$*. Then*

$$\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \subseteq \operatorname{Max}(R)$$

*for all $i < n$, and*

$$\operatorname{Cosupp}\left(\operatorname{Tor}_n^R(M, N)\right) \cup \operatorname{Max}(R) = \operatorname{Cosupp}(M \otimes_R (0 :_N (x_1, \dots, x_n)R)) \cup \operatorname{Max}(R).$$

**Lemma 2.5** ([6]). *Let $M, N$ and $L$ be $R$-modules such that $M$ and $L$ are finitely generated, and let $n \in \mathbb{N}$. If* $\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \subseteq \operatorname{Max}(R)$ *for all $i < n$ and* $\operatorname{Supp}(L) \subseteq \operatorname{Supp}(M)$*, then*

$$\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(L, N)\right) \subseteq \operatorname{Max}(R)$$

*for all $i < n$.*
*In particular,* $\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(L, N)\right) \subseteq \operatorname{Max}(R)$ *for all $i < n$ if and only if* $\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \subseteq \operatorname{Max}(R)$ *for all $i < n$ whenever* $\operatorname{Supp}(L) = \operatorname{Supp}(M)$.

**Theorem 2.6** ([6]). *Let $\mathfrak{a}$ be an ideal of $R$, and let $M$ and $N$ be $R$-modules such that $M$ is finitely generated and $N$ is Artinian. For each $n \in \mathbb{N}$, the following conditions are equivalent:*

(i) *there is a filter coregular $N$-sequence in $\mathfrak{a}$ of length $n$;*
(ii) *any filter coregular $N$-sequence in $\mathfrak{a}$ of length less than $n$ can be extended to a filter coregular $N$-sequence in $\mathfrak{a}$ of length $n$;*
(iii) $\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(R/\mathfrak{a}, N)\right) \subseteq \operatorname{Max}(R)$ *(or equivalently* $\operatorname{Tor}_i^R(R/\mathfrak{a}, N)$ *has finite length) for all $i < n$;*
(iv) *if* $\operatorname{Supp}(M) = \operatorname{V}(\mathfrak{a})$*, then* $\operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \subseteq \operatorname{Max}(R)$ *(or equivalently* $\operatorname{Tor}_i^R(M, N)$ *has finite length) for all $i < n$; and*
(v) *if* $\operatorname{Ann}(M) \subseteq \mathfrak{a}$*, then* $\operatorname{H}_i^{\mathfrak{a}}(M, N)$ *is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i < n$.*

**Remark 2.7.** Let $\mathfrak{a}$ be an ideal of $R$, and let $N$ be an Artinian $R$-module. When there exists a filter coregular $N$-sequence in $\mathfrak{a}$ of infinite length, then, by the equivalence of (i) and (ii) in Theorem 2.6, any filter coregular $N$-sequence in $\mathfrak{a}$ can be extended to a filter coregular $N$-sequence in $\mathfrak{a}$ of arbitrary length, and in this case we set f-width$(\mathfrak{a}, N) = \infty$. Now assume that all filter coregular $N$-sequences in $\mathfrak{a}$ have finite length. Again, by the equivalence of (i) and (ii) in Theorem 2.6, we can extend any filter coregular $N$-sequence in $\mathfrak{a}$ to a maximal one, and all maximal filter coregular $N$-sequences in $\mathfrak{a}$ are of the same length which we denote this common length by f-width$(\mathfrak{a}, N)$. Moreover, if $M$ is a finitely generated $R$-module such that $\operatorname{Supp}(M) = \mathrm{V}(\mathfrak{a})$, then, by Theorem 2.6, we have

$$\begin{aligned}\text{f-width}(\mathfrak{a}, N) &= \inf\{i \in \mathbb{N}_0 : \operatorname{Cosupp}\left(\operatorname{Tor}_i^R(M, N)\right) \nsubseteq \operatorname{Max}(R)\}\\ &= \inf\{i \in \mathbb{N}_0 : \operatorname{Tor}_i^R(M, N) \text{ has infinite length as an } R\text{-module}\}\\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^{\mathfrak{a}}(N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\} \qquad (*)\end{aligned}$$

(we note that $\mathrm{H}_i^{\mathfrak{a}}(R, N) = \mathrm{H}_i^{\mathfrak{a}}(N)$). Also, for an arbitrary finitely generated $R$-module $L$, since $\mathrm{H}_i^{\mathfrak{a}}(L, N) \cong \mathrm{H}_i^{\mathfrak{a}+\operatorname{Ann}(L)}(L, N)$, if we replace $\mathfrak{a}$ by $\mathfrak{a} + \operatorname{Ann}(L)$ in Theorem 2.6, then the equivalence of (ii) and (v) in Theorem 2.6 yields

$$\text{f-width}(\mathfrak{a} + \operatorname{Ann}(L), N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_i^{\mathfrak{a}}(L, N) \text{ is not a finitely generated } \widehat{R}^{\mathfrak{a}}\text{-module}\}.$$

Finally, since $\mathrm{V}(\mathfrak{a}) = \mathrm{V}\left(\sqrt{\mathfrak{a}}\right)$, it follows from the first equality in the equation $(*)$ that f-width$(\mathfrak{a}, N) =$ f-width$(\sqrt{\mathfrak{a}}, N)$.

**Proposition 2.8** ([6]). *Let $\mathfrak{a}$ be an ideal of $R$, and let $N$ be an Artinian $R$-module. If* f-width$(\mathfrak{a}, N) = \infty$, *then $(0 :_N \mathfrak{a})$ has finite length. The converse statement holds whenever $R$ is a semi-local ring which is complete with respect to its Jacobson radical.*

**Corollary 2.9** ([6]). *Let $\mathfrak{a}$ be an ideal of $R$, and let $M$ and $N$ be $R$-modules such that $M$ is finitely generated and $N$ is Artinian.*

(i) *If $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \in \mathbb{N}_0$, then $(0 :_N \operatorname{Ann}(M))$ has finite length.*
(ii) *If $\mathrm{H}_i^{\mathfrak{a}}(M, N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$, then $(0 :_N \mathfrak{a} + \operatorname{Ann}(M))$ has finite length. In particular, $(0 :_N \mathfrak{a})$ has finite length whenever $\mathrm{H}_i^{\mathfrak{a}}(N)$ is a finitely generated $\widehat{R}^{\mathfrak{a}}$-module for all $i \in \mathbb{N}_0$.*

*Moreover, the converse statements hold when $R$ is a complete semi-local ring.*

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# SOMBOR AND ZAGREB INDICES OF THE SUBDIVISION OF S[G,t]

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**Abstract**

Sierpiński gasket graphs have many applications in diverse areas including dynamical systems and chemistry. In this paper, we determine the Sombor index and the Zagreb indices of the generalized Sierpiński gasket of some famous families of graphs including paths, complete graphs and cycles.

## 1. Introduction

Throughout this paper, we consider simple connected graphs. Let $G = (V, E)$ be a graph with $n = |V(G)|$ vertices and $m = |E(G)|$ edges. The subdivision of an edge is the insertion of a new vertex in the middle of that edge accompanied by the joining of the original edge endpoints with the new vertex to form new edges. Therefore, the subdivision of G, denoted by $S_G$, converts an edges of $G$ into a path of length 2. In mathematical chemistry and chemical graph theory, a topological index is a numerical parameter (a real number) that is measured based on the molecular graph of a chemical constitution [3]. Two important topological indices introduced about forty years ago by Ivan Gutman and Trinajstic [3] are the first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$ which are defined as:

$$M_1(G) = \sum_{v \in V(G)} (\deg_G(v))^2$$

$$M_2(G) = \sum_{uv \in E(G)} \deg(u) \deg(v).$$

Also the Forgotten topological index is defined as [2]:

$$F(G) = \sum_{v \in V(G)} (\deg_G(v))^3.$$

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The generalized first Zagreb index of a graph G is defined as [7]:

$$Z^{\alpha}(G) = \sum_{x \in V(G)} \deg_G^{\alpha}(x) = \sum_{uv \in E(G)} \deg_G(u)^{\alpha-1} + \deg_G(v)^{\alpha-1},$$

where $\alpha \in R, \alpha \neq 0, \alpha \neq 1$. If $\alpha = 3$, then generalized first Zagreb index becomes Forgotten index. For more details on these topological indices we refer the reader to [7] and [8]. For a connected graph $G$, two related graphs are defined as follows [2]. Also, Gutman in [4] defined a new vertex-degree-based graph invariant, named "Sombor index" for a graph $G$, denoted by $SO(G)$, as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{\deg(u)^2 + \deg(v)^2}$$

Mathematical properties and applications of the Sombor index were established in [4]. Decomposition into special substructures that inherit remarkable features is an important method used for the investigation of some mathematical structures, specifically when the regarded structures have self-similarity features. In these cases, we usually only need to study the substructures and the way that they are related to each other. Klavžar et al. for the first time, introduced the Sierpiński graph $S(K_n, t)$, see [5] and [6]. One of the most important families of these self-similar graphs is the family of Sierpiński gasket graphs, see [9] for more details.

**Definition 1.1.** [1] Let $G = (V, E)$ be a graph of order $n \geq 2$ and $t$ be a positive integer. If $l$ is adjacent to $j$ in $G$, then by contracting the new edge between two copies $l$ and $j$ (the linking edge) in the generalized Sierpiński graph, the generalized Sierpiński gasket graph is obtained. In other words, when $j$ is adjacent to $l$ in $G$, the vertex $\mathbf{u} = v_1 v_2 \dots v_r jl \dots l$ is adjacent to $\mathbf{v} = v_1 v_2 \dots v_r lj \dots j$, in $S(G, t)$, $0 \leq r \leq t-2$, the edge **uv** will be contracted in $S[G, t]$, and this new vertex will be denoted by $v_1 v_2 \dots v_r \{j, l\}_{t-r}$ or shortly by $v_{(r)}\{j, l\}_{t-r}$, see Figure 1.

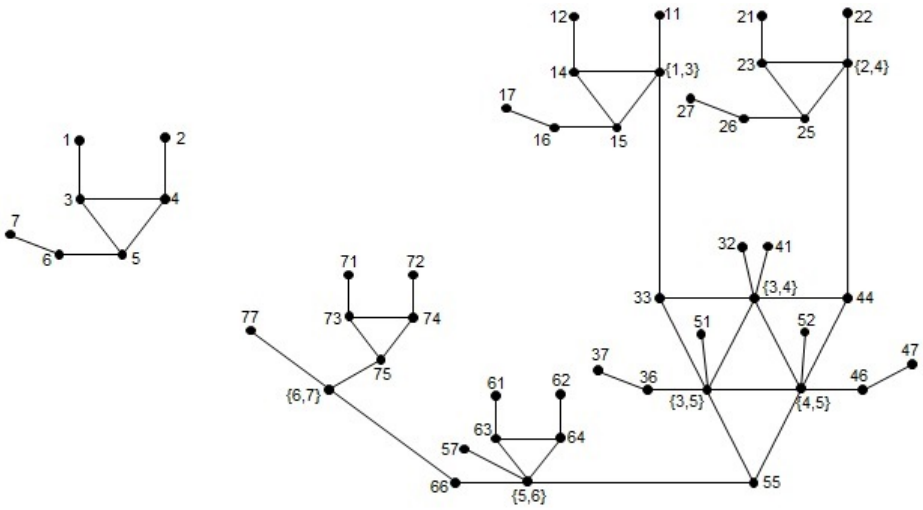


Fig. 1. A graph $G$ and its generalized Sierpiński gasket $S[G, 2]$.

**Remark 1.2.** Similar to the structure of the generalized Sierpiński graph $S(G, t)$, $S[G, t]$ is constructed inductively by inserting a copy of $S[G, t-1]$ instead of each vertex of $G$ ($S_i[G, t]$ for $i \in V(G)$) and then by contracting the new $|E(G)|$ linking edges (of $S(G, t)$). More precisely, when $i$ is adjacent to $j$ in the graph $G$, then the linking edge between $ijj \dots j$ and $jii \dots i$ is contracted and the new vertex is shown by $\{i, j\}_t$ in $S[G, t]$. Note that the vertex $\{i, j\}_t$ is the unique common shared vertex between two copies $S_i[G, t]$ and $S_j[G, t]$.

In [1], the degree sequence, Hamiltonianity, and the first general Zagreb index of the generalized Sierpiński gasket graph $G$ at step $t$ is determined.

**Theorem 1.3.** *[1] Let $G$ be a simple graph of order $n \geq 2$. For each integer $\alpha \geq 0$, the general first Zagreb index of the generalized Sierpiński gasket graph $S[G,t]$, $t \geq 1$ is given by*

$$Z_\alpha(S[G,t]) = n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{k=2}^{2\Delta(G)} a_k \cdot k^\alpha.$$

## 2. Main results

In this section, we determine some of the topological indices of the generalized Sierpiński gasket graph for special graphs like $P_n$, $C_n$ and $K_n$.

**Theorem 2.1.** *The first Zagreb index of the subdivision of the generalized Sierpiński gasket graph $G$ at step $t$ is given by*

$$M_1(sub(S[G,t])) = M_1(S[G,t]) + 4|E(S[G,t])|.$$

The following results are directly obtained from Theorem 2.1.

**Corollary 2.2.** *For each $t \geqslant 2$, the first Zagreb index of the subdivision of the generalized Sierpiński gasket graph $K_n$ at step $t$ is*

$$M_1(sub(S[K_n,t])) = M_1(S[K_n,t]) + 4|E(S[K_n,t])|.$$

**Corollary 2.3.** *For the first Zagreb index of $sub(S[C_n,t])$ we have*

$$M_1(sub(S[C_n,t])) = M_1(S[C_n,t]) + 4|E(S[C_n,t])|.$$

**Theorem 2.4.** *The second Zagreb index and the Sombor index of the subdivision of the generalized Sierpiński gasket graph of complete graph $K_n$ in step $t$ is given by*

$$\begin{aligned} M_2(sub(S[K_n,t])) &= 32(n-1)^3 m\frac{n^{t-1}-1}{n-1} + 4n(n-1)^3, \\ &and \\ SO(sub(S[K_n,t])) &= 2(n-1)m\frac{n^{t-1}-1}{n-1}\sqrt{2^2+4(n-1)^2} + 2m\sqrt{2^2+(n-1)^2}. \end{aligned}$$

In the following Theorem two topological indices of the subdivision generalized Sierpiński gasket graph $S[C_n,t]$ is determined, see Figure 2

**Theorem 2.5.** *If $n \geqslant 4$ and $t \geqslant 2$, then the second Zagreb index and the Sombor index of the subdivision of $S[C_n,t]$) is obtained*

$$\begin{aligned} M_2(sub(S[C_n,t])) &= 256m\frac{n^{t-1}-1}{n-1} + 16\left(2mn^{t-1} - 4m\frac{n^{t-1}-1}{n-1}\right), \\ &and \\ SO(sub(S[C_n,t])) &= 8\sqrt{5}m\frac{n^{t-1}-1}{n-1} + 4\sqrt{2}\left(mn^{t-1} - 2m\frac{n^{t-1}-1}{n-1}\right). \end{aligned}$$

**Theorem 2.6.** *If $n \geqslant 4$, then the second Zagreb index and the Sombor index of subdivision of the generalized Sierpiński gasket graph of the path $P_n$ in step 2 is determined by*

$$\begin{aligned} M_2(sub(S[P_n,2])) &= 8mn + 12n - 32, \\ &and \\ SO(sub(S[P_n,2])) &= (10n-26)\sqrt{5} + (4mn-12n+16)\sqrt{2} + 6\sqrt{13}. \end{aligned}$$

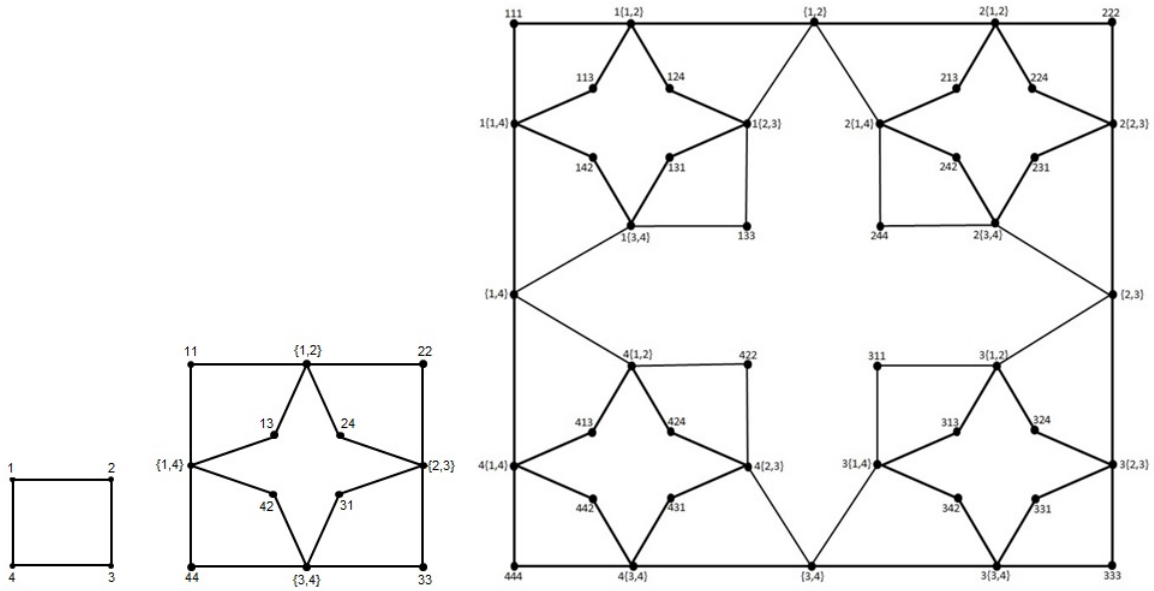


Fig. 2. Generalized Sierpiński gasket graphs $C_4$, $S[C_4, 2]$ and $S[C_4, 3]$.

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# The Rational character table and quasi-permutation representations for the special orthogonal groups

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**Abstract**

For a given finite group $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $Q$ and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of $G$. The purpose of this paper is to calculate above quantities for the group $SO(3,q)$. Also we will give the character table of the irreducible rational representations of $SO(3,q)$.

## 1. Introduction

In [10] and [11], Wong defined a quasi-permutation group of degree $n$, to be a finite group $G$ of automorphisms of an $n$-dimensional complex vector space such that every element of $G$ has non-negative integral trace. The terminology drives from the fact that if $G$ is a finite group of permutations of a set $\Omega$ of size $n$ and we think of $G$ as acting on the complex vector space with basis $\Omega$, then the trace of an element $g \in G$ is equal to the number of points of $\Omega$ fixed by $g$. In [10] and [11] Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. Then in 1994 Hartley with their colleague investigated further the analogy between permutation groups and quasi-permutation groups by studying the relation between the minimal degree of a faithful permutation representation of a given finite group $G$ and the minimal degree of a faithful quasi-permutation representation. They also worked over the rational field and found some interesting results. We shall often prefer to work over the rational field rather than the complex field.
By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. Thus every permutation matrix over $C$ is a quasi-permutation matrix. For a given finite group $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $Q$ and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices.
By a rational valued character we mean a character $\chi$ corresponding to a complex representation of $G$ such that $\chi(g) \in Q$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued

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character is in fact integer valued. A quasi-permutation representation of $G$ is then simply a complex representation of $G$ whose character valuses are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from $G$ to $GL(n,Q)$ a rational representation of $G$ and its corresponding character will be called a rational character of $G$. Let $r(G)$ denote the minimal degree of a faithful rational valued character of $G$. It is easy to see that for a finite group $G$ the following inequalities hold

$$r(G) < c(G) \leq q(G).$$

It is easy to see that if $G$ is a symmetric group of degree 6, then $r(G) = 5$ and $c(G) = q(G) = 6$. If $G$ is the quaternion group of order 8, then $r(G) = 2, c(G) = 4$ and $q(G) = 8$.
Finding the above quantities have been carried out in some papers, for example in [3], [4] and [5] we found these for the groups $GL(2,q), SU(3,q^2), PSU(3,q^2), SL(3,q)$ and $PSl(3,q)$. Our principal aim in this paper is to investigate these quantities and inequalities further. In this paper we will give the character table of the irreducible rational representations of $SO(3,q)$, by using the character table and the Schur indices of these groups.

## 2. Notation and preliminary results

Let $G$ be a finite group and $\chi$ be an irreducible complex character of $G$. Let $m_Q(\chi)$ denote the Schur index of $\chi$ over $Q$. Let $\Gamma(\chi)$ be the Galois group $Q(\chi)$ over $Q$. It is known that

$$\sum_{\alpha\in\Gamma(\chi)} m_Q(\chi)\chi^\alpha \qquad (*)$$

is a character of an irreducible $Q(G)$-module [[8], Corollary 10.2(b)]. So by knowing the character table of a group $G$ and Suchr indices of each of the irreducible characters of $G$, we can find the irreducible rational characters of $G$.

Let $\chi$ be a character of $G$ such that, for all $g \in G, \chi(g) \in Q$ and $\chi(g) \geq 0$. Then we say that $\chi$ is a non-negative rational valued character.

**Definition 2.1.** Let $\Gamma(\chi)$ be the Galois group of $Q(\chi)$ over $Q$.
Let $G$ be a finite group and $\chi$ be an irreducible complex character of $G$. Then we define

**1)** $d(\chi) = |\Gamma(\chi)|\chi(1)$

**2)** $m(\chi) = \begin{cases} 0 & \chi = 1_G \\ |\min\{\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha(g) : g \in G\}| & otherwise \end{cases}$

**3)** $c(\chi) = \sum_{\alpha\in\Gamma(\chi)}\chi^\alpha + m(\chi)1_G$.

**Lemma 2.2.** *Let $\chi$ be a character of $G$. Then $Ker\chi = Ker\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$. Moreover $\chi$ is faithful if and only if $\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$ is faithful.*

**Lemma 2.3.** *Let $\chi \in Irr(G)$ ,then $\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$ is a rational valued character of $G$. Moreover $c(\chi)$ is a non-negative rational valued character of $G$ and $c(\chi)(1) = d(\chi) + m(\chi)$.*

**Lemma 2.4.** *Let $\chi \in Irr(G), \chi \neq 1_G$. Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.*

**Lemma 2.5.** *Let $\chi \in Irr(G)$. Then*

**(1)** $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

**(2)** $c(\chi)(1) \leq 2d(\chi)$. *Equality occurs if and only if $Z(\chi)/ker\chi$ is of even order.*

Now according to Corollary 3.11 of [1] and above statements the following corollary is useful for calculation of $r(G), c(G)$ and $q(G)$.

**Corollary 2.6.** *Let $G$ be a finite group with a unique minimal normal subgroup. Then*

**1)** $r(G) = \min\{d(\chi) : \chi$ *is a faithful irreducible complex character of* $G\}$

**2)** $c(G) = \min\{c(\chi)(1) : \chi$ *is a faithful irreducible complex character of* $G\}$

**3)** $q(G) = \min\{m_Q(\chi)c(\chi)(1) : \chi$ *is a faithful irreducible complex character of* $G\}$.

If the Schur index of each non-principal irreducible character of $G$ over $Q$ is equal to $m$ , then from [1] Corollary 3.15 we have $q(G) = mc(G)$.

The proof of the following facts may be found in [2].

Let $\epsilon$ be a primitive $n$-th root of unity in $C$. Then $\epsilon + \epsilon^{-1}$ is rational if and only if $n = 1, 2, 3, 4, 6$.

Also $\epsilon^j + \epsilon^{-j}, 1 \le j \le n$ is rational if and only if $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$.

In this case if $i \in Z$ and $d_i = (i, n)$, and $n > 2d_i$, then $[Q(\epsilon^i + \epsilon^{-i}) : Q] = \frac{1}{2}\varphi(\frac{n}{d_i})$, and if $n \neq d_i, 2d_i$, then

$$\sum_{\alpha \in \Gamma_i} (\epsilon^i + \epsilon^{-i})^\alpha = \mu(\frac{n}{d_i})$$

where $\Gamma_i = (Q(\epsilon^i + \epsilon^{-i} : Q)$ and $\mu$ is the Möbius function.

With the above assumption if we set $\Gamma = (Q(\epsilon + \epsilon^{-1}) : Q)$, then

$$\sum_{\alpha \in \Gamma} (\epsilon^i + \epsilon^{-i})^\alpha = \frac{\varphi(n)}{\varphi(\frac{n}{d_i})}\mu(\frac{n}{d_i})$$

where $d_i = (i, n)$.

Now by [8] we have three important lemmas as follows:

**Lemma 2.7.** *Let $G$ be a finite group and let $\chi \in Irr(G)$ .Then $m_Q(\chi) \mid \chi(1)$. Moreover when $\chi$ is linear we have $m_Q(\chi) = 1$.*

**Lemma 2.8. a)** *Let $V_i (i = 1, 2)$ be $KG$-modules . Then the tensor product $V_1 \otimes_K V_2$ over $K$ obviously becomes a $K[G_1 \times G_2]$ module by*

$$(v_1 \otimes v_2)(g_1, g_2) = v_1 g_1 \otimes v_2 g_2$$

*For $v_i \in V_i, g_i \in G$ .*
*If $\chi_i$ is the character of $G_i$ on $V_i$ , then the character $\tau$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ is given by*

$$\tau((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$$

*For $g_i \in G_i$ .*

**b)** *Let $\chi_1, \ldots, \chi_h$ be the irreducible characters of $G_1$ over $C$ and $\psi_1, \ldots, \psi_k$ be the irreducible characters of $G_2$ over $C$ . Then the $t_{ij}$ defined by $t_{ij}((g_1, g_2)) = \chi_i(g_1)\psi_j(g_2)$ where $i = 1, \ldots, h$ and $j = 1, \ldots, k$ are all the irreducible characters of $G_1 \times G_2$ .*

**Lemma 2.9.** *Let $G = H \times K$ and $\psi \in Irr(H)$ and $\theta \in Irr(K)$. Let $\chi = \psi \times \theta$ and let $F \subseteq C$.*

**a)** *$m_F(\chi)$ divides $m_F(\psi)m_F(\theta)$.*

**b)** *Equality occurs in $(a)$ provided $(m_F(\psi), \theta(1)|F(\theta) : F|) = 1$ and $(m_F(\theta), \psi(1)|F(\psi) : F|) = 1$*

## 3. Main results

The orthogonal group of degree $n$ over a field $F$ (write as $O(n, F)$) is the group of $n$-by-$n$ orthogonal matrices with entries from $F$, with the group operation that of matrix multiplication. This is a subgroup of the general linear group $GL(n, F)$ given by

$$O(n, F) = \{Q \in GL(n, F) \mid Q^T Q = QQ^T = I\}.$$

where $Q^T$ is the transpose of $Q$. The classical orthogonal group over the real numbers is usually just written $O(n)$. More generally the orthogonal group of a non-singular quadratic form over $F$ is the group of matrices preserving the form. The Cartan-Dieudonne theorem describes the structure of the orthogonal group. Every orthogonal matrix has determinant either 1 or $-1$. The orthogonal $n$-by-$n$ matrices with determinant 1 form a normal subgroup of $O(n, F)$ known as the special orthogonal group $SO(n, F)$. If the characteristic of $F$ is 2, then $1 = -1$, hence $O(n, F)$ and $SO(n, F)$ coincide; otherwise the index of $SO(n, F)$ in $O(n, F)$ is 2.

Now let $L = \{\begin{pmatrix} ad+bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{pmatrix} \mid ad - bc = 1 \text{ and } a, b, c, d \in GF(q)\}$
.

Dickson [6 , Theorem 178] shows that $L$ is a normal subgroup of index 2 of $SO(3, q)$ and is isomorphic to $PSL(2, q) = SL(2, q)/\{\pm I\}$ through the isomorphism

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} ad+bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{pmatrix}, \qquad (**)$$

for each $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, q)$. Furthermore ,

$$SO(3, q) = L \cup \begin{pmatrix} 1 & & \\ & \omega^{-1} & \\ & & \omega \end{pmatrix} \cdot L$$

is isomorphic to

$$PGL(2, q) = PSL(2, q) \cup \begin{pmatrix} 1 & \\ & \omega \end{pmatrix} \cdot PSL(2, q)$$

by extending the isomorphism in $(**)$ to $PGL(2, q)$ by defining

$$\begin{pmatrix} 1 & \\ & \omega \end{pmatrix} \mapsto \begin{pmatrix} 1 & & \\ & \omega^{-1} & \\ & & \omega \end{pmatrix}.$$

Since $SO(3, q)$ and $PGL(2, q)$ are isomorphic, therefore they have the same conjugacy class structures .
From the conjugacy class structure of $PGL(2, q)$ in Steinberg [9], we know that $SO(3, q)$ has $q + 2$ conjugacy classes, namely, 1 class of size 1

1 class of size $q^2 - 1$

$\frac{q-3}{2}$ classes of size $q(q+1)$
$\frac{q-1}{2}$ classes of size $q(q-1)$
1 class of size $\frac{q(q+1)}{2}$
1 class of size $\frac{q(q-1)}{2}$.

Define $T(x) = tr(x) + 1, \forall x \in SO(3,q)$. Then we have

$$T = \begin{cases} (a+d)^2 & if\ x = \begin{pmatrix} ad+bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{pmatrix} \in L \\ \omega^{-1}(a+\omega d)^2 & if\ x = \begin{pmatrix} 1 & & \\ & \omega^{-1} & \\ & & \omega \end{pmatrix}\begin{pmatrix} ad+bc & ac & bd \\ 2ab & a^2 & b^2 \\ 2cd & c^2 & d^2 \end{pmatrix} \in SO(3,q)\backslash L \end{cases}$$

Thus $T$ is a surjective map from $SO(3,q)$ onto $GF(q)$ and for all $x$ in $SO(3,q)$ such that $T(x) \neq 0, x \in L$ if and only if $T(x)$ is a square in $GF(q)$.
Since two elements of $SO(3,q)$ which are conjugate in $SO(3,q)$ necessarily have the same trace and hence the same value of $T$, by counting the number of elements in $SO(3,q)$ with the same value of $T$ and taking into account that every conjugacy class of $SO(3,q)$ is either contained in $L$ or disjoint from $L$, we have the conjugacy classes of $SO(3,q)$ as follows :

**Table (1)**

| | conjugacy classes | class size |
|---|---|---|
| $A_1$ | $D = \{I\}$ | 1 |
| $A_2$ | $F = \{x \in L \mid T(x) = 4, x \neq I\}$ | $q^2 - 1$ |
| | $[\theta_i] = \{x \in L \mid T(x) = \theta_i\}, 1 \leq i \leq \frac{q-5}{4}$ | $q(q+1)$ |
| $A_3$ | $[\gamma_j] = \{x \in SO(3,q)\backslash L \mid T(x) = \gamma_j\}, 1 \leq j \leq \frac{q-1}{4}$ | $q(q+1)$ |
| | $[0] = \{x \in L \mid T(x) = 0\}$ | $\frac{q(q+1)}{2}$ |
| | $[\pi_k] = \{x \in L \mid T(x) = \pi_k\}, 1 \leq k \leq \frac{q-1}{4}$ | $q(q-1)$ |
| $B_1$ | $[\xi_l] = \{x \in SO(3,q)\backslash L \mid T(x) = \xi_l\}, 1 \leq l \leq \frac{q-1}{4}$ | $q(q-1)$ |
| | $[0]' = \{x \in SO(3,q)\backslash L \mid T(x) = 0\}$ | $\frac{q(q-1)}{2}$ |

where $A_1, A_2, A_3$ and $B_1$ are the notations of the types of conjugacy classes of $PGL(2,q)$ used in Steinberg [9].

Therefore , from the character table of $PGL(2,q)$ , we can determine the character table of $SO(3,q)$ because these two groups are isomorphic . Thus we have the character table of $SO(3,q)$ as follows.

**Table (2)**
**Irreducible characters of $SO(3,q)$**

| | $\rho_1$ | $\rho_q$ | $\rho_1'$ | $\rho_q'$ | $\rho_{q+1}^{(n)}$ $1 \le n \le \frac{q-3}{2}$ | $\rho_{q-1}^{(m)}$ $1 \le m \le \frac{q-1}{2}$ |
|---|---|---|---|---|---|---|
| $D$ | 1 | $q$ | 1 | $q$ | $q+1$ | $q-1$ |
| $F$ | 1 | 0 | 1 | 0 | 1 | $-1$ |
| $[\theta_i]$ | 1 | 1 | 1 | 1 | $\varepsilon^{(2i)n}+\varepsilon^{-(2i)n}$ | 0 |
| $[\gamma_j]$ | 1 | 1 | $-1$ | $-1$ | $\varepsilon^{(2j-1)n}+\varepsilon^{-(2j-1)n}$ | 0 |
| $[0]$ | 1 | 1 | 1 | 1 | $2(-1)^n$ | 0 |
| $[\pi_k]$ | 1 | $-1$ | 1 | $-1$ | 0 | $-(\delta^{(2k)m}+\delta^{-(2k)m})$ |
| $[\xi_l]$ | 1 | $-1$ | $-1$ | 1 | 0 | $-(\delta^{(2l-1)m}+\delta^{-(2l-1)m})$ |
| $[0]'$ | 1 | $-1$ | $-1$ | 1 | 0 | $-2(-1)^m$ |

where $\varepsilon^{q-1}=1=\delta^{q+1}$.

Now , since $O(3,q)\cong\{\pm I\}\times SO(3,q)$, hence the negative of each conjugacy class in $SO(3,q)$ is a conjugacy class in $O(3,q)$, so we have a complete characterization of the conjugacy classes of $O(3,q)$. Furthermore ,by Lemma 2.8 the irreducible characters of $G$ are all the possible products of an irreducible character of the cyclic group of order two , $\{\pm I\}$, with an irreducible character of $SO(3,q)$. Thus we have the character table of $O(3,q)$ as follows.

## 4. Character tables of irreducible rational representations of the groups $SO(3,q)$

**Lemma 4.1.** *Let $G=SO(3,q)$, $\varepsilon$ be a primitive $(q-1)$-th root of unity , $\delta$ be a primitive $q+1$-th root of unity, $n=\frac{q-1}{4}$ and $m=\frac{q+1}{4}$. Then*

**1)** $\varepsilon^{(2i)n}+\varepsilon^{-(2i)n}=\varepsilon^{(2j-1)n}+\varepsilon^{-(2j-1)n}=\begin{cases}-2 & if\ i\ (j)\ is\ odd\\ 2 & if\ i\ (j)\ is\ even\end{cases}$

**2)** $-(\delta^{(2k)m}+\delta^{-(2k)m})=-(\delta^{(2l-1)m}+\delta^{-(2l-1)m})=\begin{cases}2 & if\ k\ (l)\ is\ odd\\ -2 & if\ k\ (l)\ is\ even\end{cases}$

*Proof.* We know that $\varepsilon^{(2i)n}+\varepsilon^{-(2i)n}$ , $\varepsilon^{(2j-1)n}+\varepsilon^{-(2j-1)n}$ , $-(\delta^{(2k)m}+\delta^{-(2k)m})$ and $-(\delta^{(2l-1)m}+\delta^{-(2l-1)m})$ are rationals. (See section 2)

Now since $(1),(2)$ have similar proofs , we will prove only $(1)$ .

$\varepsilon^{(2i)\frac{q-1}{4}}+\varepsilon^{-(2i)\frac{q-1}{4}}=\varepsilon^{i\frac{q-1}{2}}+\varepsilon^{-i\frac{q-1}{2}}=(-1)^i+(-1)^{-i}=\begin{cases}-2 & if\ i\ is\ odd\\ 2 & if\ i\ is\ even\end{cases}$

and

$\varepsilon^{(2j-1)\frac{q-1}{4}}+\varepsilon^{-(2j-1)\frac{q-1}{4}}=\varepsilon^{(2j)\frac{q-1}{4}}+\varepsilon^{-(2j)\frac{q-1}{4}}+\varepsilon^{\frac{-(q-1)}{4}}+\varepsilon^{\frac{q-1}{4}}=$

$\begin{cases}-2 & if\ j\ is\ odd\\ 2 & if\ j\ is\ even\end{cases}+(-1)^{\frac{-1}{2}}+(-1)^{\frac{1}{2}}$

but $(-1)^{\frac{-1}{2}}+(-1)^{\frac{1}{2}}=0$ and so the result follows. □

**Corollary 4.2.** *Let $G=SO(3,q)$, then the irreducible characters $\rho_{q+1}^{(\frac{q-1}{4})}$ and $\rho_{q-1}^{(\frac{q+1}{4})}$ are rationals.*

*Proof.* This follows from Lemma 4.1. □

**Lemma 4.3.** *Let $G = SO(3,q)$, then the Galois orbit sums in $Irr(G)$ are as follows:*
**a)** $\rho_{q+1}^{(e)} = \sum_{\alpha\in\Gamma}(\rho_{q+1}^{(n)})^{\alpha}$
*where:* $e = (n, q-1)$ *and* $1 \le n \le \frac{q-3}{2}$ *and* $\Gamma = \Gamma(Q(\rho_{q+1}^{(n)}) : Q)$.

**b)** $\rho_{q-1}^{(f)} = \sum_{\alpha\in\Gamma}(\rho_{q-1}^{(m)})^{\alpha}$
*where:* $f = (m, q+1)$ *and* $1 \le m \le \frac{q-1}{2}$ *and* $\Gamma = \Gamma(Q(\rho_{q-1}^{(m)}) : Q)$.

**c)** $\rho_{q+1}^{(\frac{1}{4}(q-1))}$ *and* $\rho_{q-1}^{(\frac{1}{4}(q+1))}$.

**d)** $\rho_1$, $\rho_q$, $\rho_1'$, $\rho_q'$.

*Proof.* Since (a) and (b) have similar proofs, we will prove only (a). Fix an integer $n$, $1 \le n \le \frac{q-3}{2}$. Recall that $\varepsilon$ is a primitive $(q-1)$-th root of unity. Since $\Gamma(Q(\rho_{q+1}^{(n)}) : Q) = \Gamma(Q(\varepsilon^n + \varepsilon^{-n}) : Q)$ and $\varepsilon^n$ is a primitive $\frac{q-1}{e}$-th root of unity where $e = (n, q-1)$, so , $\sum_{\alpha\in\Gamma}(\rho_{q+1}^{(n)})^{\alpha} = \sum_{i\in A}(\rho_{q+1}^{(i)})$ where $A = \{i : e = (i, q-1)\}$ and $1 \le i \le \frac{q-3}{2}$.
**c)** Follows from Lemma 4.1 .
**d)** Follows from the character table of $SO(3,q)$.

□

By Lemma 2.5 we know :

If $\rho_{q+1}^{(n)}, \rho_{q-1}^{(m)}$ are irreducible characters of $G$, then $\sum_{\alpha\in\Gamma_1}(\rho_{q+1}^{(n)})^{\alpha}$ and $\sum_{\alpha\in\Gamma_2}(\rho_{q-1}^{(m)})^{\alpha}$ are rational valued characters of $G$ , where $\Gamma_1 = \Gamma(Q(\rho_{q+1}^{(n)}) : Q)$ and $\Gamma_2 = \Gamma(Q(\rho_{q-1}^{(m)}) : Q)$.

**Corollary 4.4.** *Let $G = SO(3,q)$ ,then characters of an irreducible $Q(G)$-module are as follows:*

$$\rho_{q+1}^{(e)} \text{ , } \rho_{q-1}^{(f)} \text{ , } \rho_{q+1}^{(\frac{1}{4}(q-1))} \text{ , } \rho_{q-1}^{(\frac{1}{4}(q+1))} \text{ , } \rho_1 \text{ , } \rho_q \text{ , } \rho_1' \text{ , } \rho_q'.$$

*Proof.* By [7] and [8, Lemma 2.22] , Schur index of each irreducible characters is 1 . So by $(*)$ and Lemma 4.3 the corollary follows. □

**Lemma 4.5.** *Let $G = SO(3,q)$, then*

**1)** *$e$ and $e'$ divisors of $q-1$ such that $e \le \frac{q-3}{2}$ and $e' \le \frac{q-3}{2}$.*

**2)** *$f$ and $f'$ denote divisors of $q+1$ such that $f \le \frac{q-1}{2}$ and $f' \le \frac{q-1}{2}$.*

**3)** *$\varepsilon_e$ is a primitive $\frac{q-1}{e}$-th root of unity.*

**4)** *$\delta_f$ is a primitive $\frac{q+1}{f}$-th root of unity.*

**5)** $S_1(e,e') = \sum_{\alpha\in\Gamma}(\varepsilon_e^{e'} + \varepsilon_e^{-e'})^{\alpha} = \frac{\varphi(\frac{q-1}{e})}{\varphi(\frac{\frac{q-1}{e}}{(\frac{q-1}{e},e')})}\mu(\frac{\frac{q-1}{e}}{(\frac{q-1}{e},e')})$.

**6)** $S_2(f,f') = \sum_{\alpha\in\Gamma}(\delta_f^{f'} + \delta_f^{-f'})^\alpha = \frac{\varphi(\frac{q+1}{f})}{\varphi(\frac{\frac{q+1}{f}}{(\frac{q+1}{f},f')})}\mu(\frac{\frac{q+1}{f}}{(\frac{q+1}{f},f')})$.

*where* $\Gamma = \Gamma(Q(\rho_{q-1}^{(f)}) : Q)$.*(See Section 2)*

**Lemma 4.6.** *Let* $G = SO(3,q)$ *, then for the irreducible characters* $\rho_{q+1}^{(e)}, \rho_{q-1}^{(f)}$ *of rational representations of* $G$ *we have*

**a)** $\rho_{q+1}^{(e)}(D) = \frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$
$\rho_{q+1}^{(e)}(F) = \frac{1}{2}\varphi(\frac{q-1}{e})$
$\rho_{q+1}^{(e)}([\vartheta_i]) = S_1(e,e')$
$\rho_{q+1}^{(e)}([\gamma_j]) = S_1(e,e')$
$\rho_{q+1}^{(e)}([0]) = (-1)^e\varphi(\frac{q-1}{e}) = \begin{cases} \varphi(\frac{q-1}{e}) & if\ e\ is\ even \\ -\varphi(\frac{q-1}{e}) & if\ e\ is\ odd \end{cases}$
$\rho_{q+1}^{(e)}([\pi_k]) = \rho_{q+1}^{(e)}([\xi_l]) = \rho_{q+1}^{(e)}([0]') = 0$

**b)** $\rho_{q-1}^{(f)}(D) = \frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$
$\rho_{q-1}^{(f)}(F) = -\frac{1}{2}\varphi(\frac{q+1}{f})$
$\rho_{q-1}^{(f)}([\vartheta_i]) = \rho_{q-1}^{(f)}([\gamma_j]) = \rho_{q-1}^{(f)}([0]) = 0$
$\rho_{q-1}^{(f)}([\pi_k]) = -S_2(f,f')$
$\rho_{q-1}^{(f)}([\xi_l]) = -S_2(f,f')$
$\rho_{q-1}^{(f)}([0]') = \begin{cases} -\varphi(\frac{q+1}{f}) & if\ f\ is\ even \\ \varphi(\frac{q+1}{f}) & if\ f\ is\ odd \end{cases}$

*Proof.* These follows from Lemmas 4.1, 4.3 , Notation 4.5 and Table(2) .

□

Now in the above notation we have :

**Theorem 4.7.** *The character table of the irreducible rational representation of* $SO(3,q)$ *is as follow:*

**Table (3)**
**The irreducible rational characters of** $SO(3,q)$

| | $\rho_1$ | $\rho_q$ | $\rho_1'$ | $\rho_q'$ | $\rho_{q+1}^{(\frac{q-1}{4})}$ | $\rho_{q-1}^{(\frac{q+1}{4})}$ | $\rho_{q+1}^{(e)}$ | $\rho_{q-1}^{(f)}$ |
|---|---|---|---|---|---|---|---|---|
| $D$ | 1 | $q$ | 1 | $q$ | $q+1$ | $q-1$ | $\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$ | $\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$ |
| $F$ | 1 | 0 | 1 | 0 | 1 | $-1$ | $\frac{1}{2}\varphi(\frac{q-1}{e})$ | $-\frac{1}{2}\varphi(\frac{q+1}{f})$ |
| $[\theta_i]$ | 1 | 1 | 1 | 1 | $2(-1)^i$ | 0 | $S_1(e,e')$ | 0 |
| $[\gamma_j]$ | 1 | 1 | $-1$ | $-1$ | $2(-1)^i$ | 0 | $S_1(e,e')$ | 0 |
| $[0]$ | 1 | 1 | 1 | 1 | $2(-1)^{\frac{q-1}{4}}$ | 0 | $(-1)^e\varphi(\frac{q-1}{e})$ | 0 |
| $[\pi_k]$ | 1 | $-1$ | 1 | $-1$ | 0 | $-2(-1)^k$ | 0 | $-S_2(f,f')$ |
| $[\xi_l]$ | 1 | $-1$ | $-1$ | 1 | 0 | $-2(-1)^l$ | 0 | $-S_2(f,f')$ |
| $[0]'$ | 1 | $-1$ | $-1$ | 1 | 0 | $-2(-1)^{\frac{q+1}{4}}$ | 0 | $(-1)^{f+1}\varphi(\frac{q+1}{f})$ |

.

## 5. Quasi-permutation representations

In this section we calculate $r(G), c(G)$ and $q(G)$ for the group $SO(3,q)$ in two ways.

**Theorem 5.1.** *Let $G = SO(3,q)$, then*

**a)** $r(G) = q - 1$

**b)** $c(G) = q(G) = q + 1$

**c)** $Lim_{q\to\infty} \frac{c(G)}{r(G)} = 1.$

*Proof.* By [7] and [8, Lemma 2.22] Schur index of each irreducible characters is 1 and so we have $c(G) = q(G)$.
By Corollary 4.2 we know that the characters $\rho_{q-1}^{(\frac{q+1}{4})}$ and $\rho_{q+1}^{(\frac{q-1}{4})}$ are rational valued characters , now by definition of $d(\chi)$ and $c(\chi)$ and Table (2) we have :
$d(\rho_{q-1}^{(m)}) = |\Gamma|\rho_{q-1}^{(m)}(1) \geq q - 1$ where $\Gamma = \Gamma(Q(\rho_{q-1}^{(m)}) : Q)$ and $m(\rho_{q-1}^{(m)}) \geq 2$ and so $c(\rho_{q-1}^{(m)})(1) \geq q + 1$ and equality hold if $m = \frac{q+1}{4}$.
And $d(\rho_{q+1}^{(n)}) \geq q + 1$ and $m(\rho_{q+1}^{(n)}) \geq 2$ and so $c(\rho_{q+1}^{(m)})(1) \geq q + 3$ and equality hold if $n = \frac{q-1}{4}$.
For other characters we have $d(\rho_q) = d(\rho_q') = q$ and $m(\rho_q) = m(\rho_q') = 1$ and so $c(\rho_q)(1) = c(\rho_q')(1) = q + 1$.

Now we have the following table:

**Table (4)**

| $\chi$ | $d(\chi)$ | $c(\chi)(1)$ |
|---|---|---|
| $\rho_q$ | $q$ | $q+1$ |
| $\rho_q'$ | $q$ | $q+1$ |
| $\rho_{q+1}^{(n)}$ | $\geq q+1$ | $\geq q+3$ |
| $\rho_{q-1}^{(m)}$ | $\geq q-1$ | $\geq q+1$ |

Now by Definition 2.1 and and Table (4) we have :

$\min\{d(\chi) : ker\chi = 1\} = q - 1$ and $\min\{c(\chi)(1) : ker\chi = 1\} = q + 1$.

We can prove the above statement with another way :
We know that $\frac{1}{2}\varphi(\frac{q-1}{e})$ , $\frac{1}{2}\varphi(\frac{q+1}{f}) \geq 1$. Thus the degrees of faithful rational characters of $G$ are at least $q - 1$ .
Also the degree of $\rho_{q-1}^{(\frac{q+1}{4})}$ is exactly $q - 1$, therefore by definition of $r(G)$ and Table(4) we have $r(G) = q - 1$. As $min\{\rho_{q-1}^{(\frac{q+1}{4})}(g) \mid g \in G\} = -2$ and $min\{\rho_q(g) \mid g \in G\} = min\{\rho_q'(g) \mid g \in G\} = -1$ , so by definition of $c(G)$ and Table (4) we have $c(G) = (q - 1) + 2$ or $c(G) = q + 1$.

□

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# A Cauchy inequality for bilinear mappings of operators

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**Abstract**

The Cauchy inequality is one of the most well-known properties of positive sesquilinear forms on vector spaces. We present a Cauchy inequality for bilinear mappings of Hilbert space operators.

## 1. Introduction

Assume that $\mathcal{B}(\mathcal{H})$ is the $C^*$-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. An operator $T$ is called positive and is denoted by $T \geq 0$ if $\langle Tx, x\rangle \geq 0$ for all vectors $x \in \mathcal{H}$. This provides a partial order on the set of all Hermitian operators as $T \leq S$, when $S - T \geq 0$. The set of all positive operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})^+$.
A mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is said to be positive if $\Phi(T) \geq 0$ for every $T \geq 0$. If $\Phi(I) = I$, then $\Phi$ is called unital, where we denote the identity operator on both $\mathcal{H}$ and $\mathcal{K}$ by $I$.
Positive linear mappings are important objects in the study of operator algebras as well as in application of operator theory in mathematical physics.
If $\mathfrak{A}$ is a $C^*$-algebra, then $\mathbb{M}_n(\mathfrak{A})$ is the $C^*$-algebra including $n \times n$ matrices with entries in $\mathfrak{A}$.
Every linear map $\Phi$ induces the mappings $\Phi_n : \mathbb{M}_n(\mathcal{B}(\mathcal{H})) \to \mathbb{M}_n(\mathcal{B}(\mathcal{K}))$ for every $n \in \mathbb{N}$ by

$$\Phi_n([T_{ij}]) = [\Phi(T_{ij})].$$

If $\Phi_n$ is positive, then $\Phi$ is said to be $n$-positive. Typical example of $n$-positive maps are maps of the form $T \mapsto V^*TV$. To see more information and examples about $n$-positive maps see [1, 4].

We are interested in multilinear mappings here. A multilinear mapping $\Phi\colon \mathcal{B}(\mathcal{H})^m \to \mathcal{B}(\mathcal{K})$ is called positive if

$$T_i \in \mathcal{B}(\mathcal{H})^+, \ (i = 1, \dots, m) \quad \Longrightarrow \quad \Phi(T_1, \dots, T_m) \in \mathcal{B}(\mathcal{K})^+.$$

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Typical example of positive multilinear mappings are tensor products of operators:

$$(T_1, \dots, T_m) \mapsto T_1 \otimes \cdots \otimes T_m.$$

As another example consider $\Phi : \mathcal{B}(\mathcal{H})^2 \to \mathcal{B}(\mathcal{H})$ defined by $\Phi(T,S) = \langle Tx, x\rangle S$, where $x \in \mathcal{H}$. To see more information see [2].
Let $\mathcal{X}$ be a vector space. It is known that every positive sesquilinear form $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ satisfies a Cauchy inequality as

$$|\sigma(x,y)|^2 \le \sigma(x,x)\sigma(y,y) \qquad (x,y \in \mathcal{X}).$$

We intend to present a variant of this inequality for multilinear mappings.

## 2. Main result

We say that a multilinear mapping $\Phi : \mathcal{B}(\mathcal{H})^m \to \mathcal{B}(\mathcal{K})$ is $n$-positive if the induced map

$$\Phi_n : \mathbb{M}_n(\mathcal{B}(\mathcal{H}))^m \to \mathbb{M}_n(\mathcal{B}(\mathcal{K}))$$

defined by $\Phi_n([T_{ij}]_1, \cdots, [T_{ij}]_m) = [\Phi(T_{ij}^1, \cdots, T_{ij}^m)]$ is positive.

We give some examples. Consider the bilinear mapping $\Phi : \mathbb{M}_m \times \mathbb{M}_k \to \mathbb{M}_{mk}$ defined by $\Phi(A,B) = A \otimes B$. We claim that $\Phi$ is 2-positive. To see this, assume that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ are positive matrices in $\mathbb{M}_2(\mathbb{M}_m)$ and $\mathbb{M}_2(\mathbb{M}_k)$, respectively. It is known that [1, 3] the block matrix operator $\begin{bmatrix} T & R \\ R^* & S \end{bmatrix}$ is positive in $\mathbb{M}_2(\mathcal{B}(\mathcal{H}))$ if and only if $T, S \ge 0$ and $T \ge RS^{-1}R^*$.
By this, we learn that $A_{11} \ge A_{12}A_{22}^{-1}A_{21}$ and $B_{11} \ge B_{12}B_{22}^{-1}B_{21}$. Since $\Phi$ is a positive bilinear map, we obtain

$$\begin{aligned} A_{11} \otimes B_{11} = \Phi(A_{11}, B_{11}) &\ge \Phi(A_{12}A_{22}^{-1}A_{21}, B_{12}B_{22}^{-1}B_{21}) \\ &= (A_{12}A_{22}^{-1}A_{21}) \otimes (B_{12}B_{22}^{-1}B_{21}) \\ &= (A_{12} \otimes B_{12})(A_{22} \otimes B_{22})^{-1}(A_{21} \otimes B_{21}) \end{aligned}$$

and this is just the positivity of

$$\Phi_2([A_{ij}], [B_{ij}]) = \begin{bmatrix} A_{11} \otimes B_{11} & A_{12} \otimes B_{12} \\ A_{21} \otimes B_{21} & A_{22} \otimes B_{22} \end{bmatrix}$$

as we claimed.

Now we present our main result. To this end, we need the following lemma.

**Lemma 2.1.** *[3] Let $T, S \in \mathcal{B}(\mathcal{H})^+$ and let $M = \begin{bmatrix} T & R \\ R^* & S \end{bmatrix}$. Then $M$ is positive if and only if*

$$|\langle x, Ry\rangle|^2 \le \langle x, Tx\rangle\langle y, Sy\rangle$$

*for all $x, y \in \mathcal{H}$.*

Our result gives a Cauchy inequality for positive bilinear maps.

**Theorem 2.2.** *If $\Phi : \mathcal{B}(\mathcal{H})^2 \to \mathcal{B}(\mathcal{K})$ is a 2-positive bilinear map, then*

$$\left\|\Phi(A^*B, C^*D)\right\| \le \left\|\Phi(A^*A, C^*C)\right\|^{1/2} \left\|\Phi(B^*B, D^*D)\right\|^{1/2}$$

*for all operators $A, B, C, D \in \mathcal{B}(\mathcal{H})$. The equality holds if $\Phi(T,S) = T \otimes S$.*

Some special cases of Theorem 2.2 are of special interest. Assume that $\varphi$ is a positive linear functional on $\mathcal{B}(\mathcal{H})$ so that $\Phi(T,S)=\varphi(T)S$ is a positive bilinear map. Applying Theorem 2.2 we conclude

$$|\varphi(A^*B)|\,\|C^*D\| \le \varphi(A^*A)^{1/2}\varphi(B^*B)^{1/2}\|C\|\|D\|,$$

which by $C=D=I$ gives the well-known Cauchy inequality for positive linear functionals. On the other hand, if $\Phi\colon \mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ is a positive linear mapping, then $\Psi(T,S)=\langle Tx,x\rangle\Phi(S)$ is a positive bilinear map, where $x\in\mathcal{H}$ is a unit vector. Applying Theorem 2.2 gives

$$|\langle Bx,Ax\rangle|\|\Phi(C^*D)\| \le \|Ax\|\|Bx\|\|\Phi(C^*C)\|^{1/2}\|\Phi(D^*D)\|^{1/2}$$

whence we derive a Cauchy inequality for positive linear mappings on $\mathcal{B}(\mathcal{H})$ with $A=B=I$.

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# On a Heinz norm inequality

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**Abstract**

The Heinz norm inequality gives a refinement of the arithmetic-geometric mean inequality in the setting of matrices. We present a complementary inequality to a result of F. Kittaneh, which employed the integrals to refine the Heinz norm inequality.

## 1. Introduction

Let $\mathbb{M}_n$ is the algebra of all $n \times n$ matrices with complex entries and $I$ denotes the identity matrix. A positive matrix is a Hermitian matrix with non-negative eigenvalues. We denote by $\mathbb{M}_n^+$ the set of all positive matrices. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if it satisfies $|||UAV||| = |||A|||$ for every $A \in \mathbb{M}_n$ and all unitary matrices $U$ and $V$. As an example, it is known that the class of Schatten $p$-norms defined by

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j(A)^p\right)^{1/p} \qquad (1 \le p < \infty),$$

are unitarily invariant norms, where $s_1(A) \ge \ldots \ge s_n(A)$ are singular values of $A$, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$.
As a matrix version of the arithmetic-geometric mean inequality, we know that

$$2|||A^{1/2}XB^{1/2}||| \le |||AX + XB|||$$

holds for all $A, B \in \mathbb{M}_n^+$ and every $X \in \mathbb{M}_n$. However, the scalar arithmetic-geometric mean inequality has an interpolation as $\sqrt{ab} \le H_t(a,b) \le \frac{a+b}{2}$ in which $H_t(a,b) = \frac{1}{2}\left(a^t b^{1-t} + a^{1-t}b^t\right)$ is called the Heinz mean. The famous matrix version reads as follows:

$$2|||A^{1/2}XB^{1/2}||| \le |||A^tXB^{1-t} + A^{1-t}XB^t||| \le |||AX + XB|||.$$

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This is known as the Heinz norm inequality in the literature. Considering the function $f(t) = |||A^tXB^{1-t}+A^{1-t}XB^t|||$, one can see that $f$ is a convex function on $[0,1]$ with minimum at $t = 1/2$ and the Heinz norm inequality is just $f(1/2) \le f(t) \le f(0)$.
Another related inequality is

$$|||A^{1/2}XB^{1/2}|||^2 \le |||A^tXB^{1-t}|||\,|||A^{1-t}XB^t||| \le |||AX|||\,|||XB|||.$$

F. Kittaneh [2, Theorem 1] showed that

$$\begin{aligned}\frac{1}{2}\,|||A^{1/2}XB^{1/2}||| &\le \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{1-t}XB^t + A^tXB^{1-t}|||\;dt\right| \\ &\le |||A^{1-\mu}XB^{\mu} + A^{\mu}XB^{1-\mu}|||\,, \qquad (0 \le \mu \le 1).\end{aligned} \tag{1}$$

holds for every $\mu \in [0, 1/2]$.

## 2. Main Result

In this paper, we present some complementary inequalities to (1).

**Theorem 2.1.** *Let $A, B \in \mathbb{M}_n^+$. Then inequality*

$$\begin{aligned}&|||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||^2 \\ &\le \frac{2}{3}\frac{1}{1-\mu}\int_{2\mu-1}^{\mu} \|A^tXB^{1-t} + A^{1-t}XB^t\|^2\;dt + \frac{1}{3}\,\|AX+XB\|^2 \\ &\le (8-8\mu)\left\|A^{1/2}XB^{1/2}\right\|^2 + (2\mu-1)\,\|AX+XB\|^2\end{aligned} \tag{2}$$

*holds for every $X \in \mathbb{M}_n$.*

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# A Carleman-Knopp type Inequality for pseudo-integral

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**Abstract**

pseudo-analysis is a generalization of classical analysis with two pseudo-multiplication and pseudo-addition operations on the real interval $[a, b]$ of $[-\infty, \infty]$. In recent years, the generalization of inequalities from the framework of classical analysis to some integrals that include classical analysis as special cases has been an interesting topic. One of the famous mathematical inequalities is Carlmann-Knopp's inequality. In this paper, some new generalizations of Carleman-Knopp's type inequality via pseudo-integrals are studied.

## 1. Introduction

We know that the classical Carleman's inequality [6] is as follows:

$$\int_0^{+\infty} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))dt\right)dx \leq e\int_0^{+\infty} f(x)dx, \tag{1}$$

where $f : [0, +\infty) \to [0, +\infty)$ is Riemann integrable function which $\int_0^{+\infty} f(x)dx < \infty$. We also know that if the improper Riemann integral $\int_0^{+\infty} f(x)dx$ converges, and $f \geq 0$ on $[0, +\infty)$, then $f$ is Lebesgue integrable on $[0, +\infty)$ and the integrals are the same[3], i.e.

$$\int_{[0,+\infty)} f d\mu = \int_0^{+\infty} f(x)dx, \tag{2}$$

where the integral applied on the left side is the standard Lebesgue integral.
The inequality (1) is often called Knopp's inequality, referring to [13]. Also, inequality (1) has been used in several mathematics and physics fields [4, 11]. Further related references, information and generalizations can be found in [6, 10–12, 17, 21, 23].

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**Theorem 1.1.** *[15] Let $f, F : [0, +\infty) \to [0, +\infty)$ be strictly increasing functions and $\int_0^{+\infty} f(x)dx < \infty$. Then we have*

$$\oint_0^{+\infty} F\left(\frac{1}{x}\int_0^x F^{-1}(f(t))dt\right)d\mu \le \oint_0^{+\infty} f(x)d\mu. \tag{3}$$

We note that in inequality (3) the outer integrals are in the Sugeno sense [7, 8, 24], whereas the inner integral is a Riemann integral. In 2013, Ma et al. proved the above extension of the inequality (1) in the fuzzy context. Also, in 2020 RomPán-Flores et al. proved the following extension of inequality (1) to the fuzzy context, where all integrals involved in its formulation are in the Sugeno sense.

**Theorem 1.2.** *[22] (Carleman-Knopp's fuzzy integral inequality). Let $f : [0, +\infty) \to [1, +\infty)$ be a Sugeno-integrable function with respect to the Lebesgue measure. Then*

$$\oint_0^{+\infty} \exp\left(\frac{1}{x}\oint_0^x \ln(f(t))dt\right)dx \le e\oint_0^{+\infty} f(x)dx.$$

## 2. Pseudo-integral

Let $[a, b] \subseteq [-\infty, +\infty]$ be a closed (in some cases semiclosed) interval and $\preceq$ be a total order on $[a, b]$. Pseudo-addition $\oplus$ is defined as follows.

**Definition 2.1.** [5, 20] A binary operation $\oplus$ on $[a, b]$ is pseudo-addition if for all $x, y, z \in [a, b]$,

1. $x \oplus y = y \oplus x$ (commutativity);
2. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (associativity);
3. If $x \preceq y$, then $x \oplus z \preceq y \oplus z$ (monotonicity);
4. $\mathbf{0}_\oplus \oplus x = x$, where $\mathbf{0}_\oplus \in [a, b]$ is a neutral (zero) element (boundary condition).

A pseudo-addition $\oplus$ is said to be continuous if it is a continuous function in $[a, b]^2$, i.e. if $\lim_{n\to+\infty} x_n$ and $\lim_{n\to+\infty} y_n$ exist and are finite, then $\lim_{n\to+\infty}(x_n \oplus y_n) = \lim_{n\to+\infty} x_n \oplus \lim_{n\to\infty} y_n$. A pseudo-addition $\oplus$ is called strict if $\oplus$ is continuous and strictly monotone. Let $[a, b]_+ = \{x : x \in [a, b], \mathbf{0}_\oplus \preceq x\}$. Pseudo-multiplication $\otimes$ associated with the pseudo-addition $\oplus$ is defined as follows.

**Definition 2.2.** [5, 20] Let $\oplus$ be a given pseudo-addition on $[a, b]$. A binary operation $\otimes$ on $[a, b]$ is pseudo-multiplication if for all $x, y, z \in [a, b]$ and $w \in [a, b]_+$,

1. $x \otimes y = y \otimes x$ (commutativity);
2. $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ (associativity);
3. If $x \preceq y$, then $x \otimes w \preceq y \otimes w$ (monotonicity);
4. $(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$ (left distributivity);
5. $\mathbf{1}_\otimes \otimes x = x$, where $\mathbf{1}_\otimes \in [a, b]$ is a neutral (unit) element (boundary condition).

A pseudo-multiplication $\otimes$ is said to be continuous if it is a continuous function in $[a, b]^2$. A binary operation $*$ on $[a, b]$ is said to be pseudo-operation if it is commutative, associative, nondecreasing and has a neutral element. The pseudo-addition $\oplus$ and the pseudo-multiplication $\otimes$ are pseudo-operations on the interval $[a, b]$. The pseudo-operation $* : [a, b]^2 \to [a, b]$ is idempotent if for any $x \in [a, b]$, $x * x = x$ holds. It is easy to see that the structure $([a, b], \oplus, \otimes)$ is a semiring, see [14].

In this paper, we will consider only special semiring with the following continuous operations: Both operations $\oplus$ and $\otimes$ are not idempotent. The pseudo-operations are generated by a strictly monotone and continuous function $g$ [18]. In this case, we will consider only strict pseudo-addition $\oplus$.

By Aczel's representation theorem [2] for each strict pseudo-addition $\oplus$ there exists a strictly monotone and continuous surjective function $g$ (generator for $\oplus$), $g : [a, b] \to [0, +\infty]$ such that $g(\mathbf{0}_\oplus) = 0$ and

$$x \oplus y := g^{-1}(g(x) + g(y)).$$

Using a generator $g$ of a strict pseudo-addition $\oplus$ we can define a pseudo-multiplication $\otimes$ by

$$x \otimes y := g^{-1}(g(x)g(y)).$$

with the convention $0 \times (+\infty) := 0$. This is the only way to define a pseudo-multiplication $\otimes$, which is distributive with respect to a given pseudo-addition $\oplus$ generated $g$. If the zero element for the pseudo-addition is $a$, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = +\infty$. If the zero element for the pseudo-addition is $b$, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = +\infty$.

**Definition 2.3.** [16, 19] Let $X$ be a non-empty set and $\mathcal{E}$ be a $\sigma$-algebra of the subsets of $X$. The set function $m : \mathcal{E} \to [a, b]_+$ is a $\sigma$-$\oplus$-measure if

1. $m(\emptyset) = \mathbf{0}_{\oplus}$;
2. For any sequence $(E_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{E}$,

$$m(\bigcup_{i=1}^{+\infty} E_i) = \bigoplus_{i=1}^{+\infty} m(E_i) := \lim_{n\to+\infty} \bigoplus_{i=1}^{n} m(E_i)$$

If pseudo-addition $\oplus$ is idempotent, then condition (1) and pairwise disjointedness of sets can be left out.

**Definition 2.4.** [16, 19] Suppose that $X$ is a non-empty set, $\mathcal{E}$ is a $\sigma$-algebra of the subsets of $X$ and $m : \mathcal{E} \to [a, b]_+$ is a $\sigma$-$\oplus$-measure. The pseudo-integral of a bounded measurable function $f : X \to [a, b]$, where the pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, is defined by

$$\int_X^{\oplus} f(x) \otimes \mathrm{d}m := g^{-1}\left(\int_X (g \circ f)\, \mathrm{d}(g \circ m)\right).$$

If $X \subseteq [-\infty, +\infty]$ is a closed (semiclosed) interval, $\mathcal{E} = \mathcal{B}_X$ is $\sigma$-algebra of Borel subsets of $X$ and $m = g^{-1} \circ \mu$ where $\mu$ is the standard Lebesgue measure on $X$, then the pseudo-integral for the function $f$ has the form

$$\int_X^{\oplus} f(x) \otimes \mathrm{d}m = g^{-1}\left(\int_X g\big(f(x)\big)\mathrm{d}\mu\right), \tag{4}$$

where the integral applied on the right side is the standard Lebesgue integral and is called the $g$-integral of the function $f$.

In order to present the pseudo-analytic exponential $Exp(x)$, it is necessary to introduce the pseudo-power. For $x \in [a, b]_+$ and $p \in (0, \infty)$, the pseudo-power $x_{\otimes}^{(p)}$ is defined in the following way in few steps.

- for $n, m \in \mathbb{N}$ and $r = \frac{m}{n}$

  $x_{\otimes}^{(n)} := \underbrace{x \otimes x \otimes \ldots \otimes x}_{n-times}$, $x_{\otimes}^{(0)} := \mathbf{1}_{\otimes}$, $x_{\otimes}^{(\frac{1}{n})} := \sup\left\{y \;\middle|\; y_{\otimes}^{(n)} \leqslant x\right\}$, $x_{\otimes}^{(r)} = x_{\otimes}^{(\frac{m}{n})} = \left(x_{\otimes}^{(\frac{1}{n})}\right)^{(m)}$.

  Note that $x^r$ is well defined for all rational $r \in (0, \infty)$, independently of the representation of $r$,

- if $p$ is not rational, then according to the continuity of $\otimes$

$$x_{\otimes}^{(p)} := \sup\left\{x_{\otimes}^{r} \mid r \in ]0, p[, r \in Q\right\}.$$

Obviously, if $x \otimes y = g^{-1}(g(x).g(y))$, then

$$x_{\otimes}^{(p)} = g^{-1}(g^p(x)).$$

On the other hand, if $\otimes$ is idempotent, then $x_{\otimes}^{(p)} = x$ for any $x \in [a, b]_+$ and $p \in (0, \infty)$.
In this paper, similar to Paper [9] we suppose that the generator function $g : [0, \infty] \to [0, \infty]$ is strictly monotone, onto, $g(\mathbf{0}_{\oplus}) = 0$, $g'(x) \neq 0$ for all $x$ , $g \in C^2$ and $g^{-1} \in C^2$. By applying this function, we shall introduce some new operations as follows: for all $x, y \in [a, b]$, and $n \in \mathbb{R}$

- Pseudo-division:

$$x \oslash^{-1} y := g^{-1}\left(\frac{g(x)}{g(y)}\right),$$

provided $y \neq \mathbf{0}_{\oplus}$.

- Pseudo-analytic exponential:

$$Exp_{\oplus}(x) := \sum_{n=0}^{+\infty} g^{-1}\left(\frac{1}{n!}\right) \otimes x_{\otimes}^{(n)},$$

i.e.,

$$Exp_{\oplus}(x) = g^{-1}(\exp(g(x))),$$

where $\exp(g(x))$ is the standard exponential function.

- Pseudo-logarithm:

$$Ln_{\oplus}(x) := g^{-1}(\ln(g(x))),$$

where $\ln(g(x))$ is the standard logarithm function.

## 3. Main results

In this section, we prove two Carleman type inequalities which are derived from (1) for pseudo-integral.

**Theorem 3.1.** *(Pseudo Carleman-Knopp's inequality, increasing case) Let* $([0,+\infty), \oplus, \otimes)$ *be a semiring and the generator* $g : [0,+\infty) \to [0,+\infty)$ *of the pseudo-addition* $\oplus$ *and the pseudo-multiplication* $\otimes$ *be a strictly increasing and surjective function. Then, for any* $\sigma$-$\oplus$*-measure* $m$*, the following inequality*

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(g^{-1}(1) \oslash^{-1} g^{-1}(x)\right) \otimes g^{-1}\left(\int_0^x \ln(f(t))dt\right)\right) \otimes dm \leq g^{-1}(e) \otimes g^{-1}\left(\int_0^{+\infty} f(x)dx\right), \quad (5)$$

*holds true for any nonnegative Riemann integrable function* $f$ *on* $[0,+\infty)$ *which* $\int_0^{+\infty} f(x)dx < \infty$.

*Proof.* By the definition of pseudo-division, using the equality $x \otimes y = g^{-1}(g(x)g(y))$, by applying the definition

of pseudo-analytic exponential and using the equation (4) of Definition 2.4, respectively, we have

$$\begin{aligned}
&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(g^{-1}(1)\otimes^{-1} g^{-1}(x)\right)\otimes g^{-1}\left(\int_0^x \ln(f(t))dt\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(g^{-1}\left(\frac{g(g^{-1}(1))}{g(g^{-1}(x))}\right)\otimes g^{-1}\left(\int_0^x \ln(f(t))dt\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(g^{-1}\left(\frac{1}{x}\right)\otimes g^{-1}\left(\int_0^x \ln(f(t))dt\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(g^{-1}\left(g(g^{-1}(\frac{1}{x})).g(g^{-1}(\int_0^x \ln(f(t))dt))\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(g^{-1}\left(\frac{1}{x}\int_0^x \ln(f(t))dt\right)\right)\otimes dm \qquad (6)\\
=&\int_{[0,+\infty)}^{\oplus} g^{-1}\left(\exp\left(g\left(g^{-1}(\frac{1}{x}\int_0^x \ln(f(t))dt\right)\right)\right)\right)\otimes dm\\
=&\int_{[0,+\infty)}^{\oplus} g^{-1}\left(\exp\left(\frac{1}{x}\int_0^x \ln(f(t))dt\right)\right)\otimes dm\\
=&\ g^{-1}\left(\int_{[0,+\infty)} g\left(g^{-1}\left(\exp(\frac{1}{x}\int_0^x \ln(f(t))dt)\right)\right)d\mu\right)\\
=&\ g^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))dt\right)d\mu\right).
\end{aligned}$$

Now by the equation (2), using the fact that $g^{-1}$ is increasing and applying the classical Carleman's inequality (1), we see that we will have the inequality

$$\begin{aligned}
g^{-1}\left(\int_{[0,+\infty)} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))dt\right)d\mu\right) &= g^{-1}\left(\int_0^{+\infty} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))dt\right)dx\right)\\
&\le g^{-1}\left(e\int_0^{+\infty} f(x)dx\right) \qquad (7)\\
&= g^{-1}\left(g\left(g^{-1}(e)\right).g\left(g^{-1}(\int_0^{+\infty} f(x)dx)\right)\right)\\
&= g^{-1}(e)\otimes g^{-1}(\int_0^{+\infty} f(x)dx).
\end{aligned}$$

Hence, combining (6) and (7) yields inequality (5). The proof of Theorem 3.1 is completed. □

**Example 3.2.** Let $[a,b)=[0,+\infty]$,using Theorem 3.1 we get the Carleman's type inequalities.

**a** For $g(x)=x$. The corresponding pseudo-operations are $x\oplus y=x+y$ and $x\otimes y=xy$. The inequality (5) obtains the following form

$$\int_0^{+\infty} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))\right)dt \le e\int_0^{+\infty} f(x)dx,$$

which is the same as classical Carleman's inequality (1).

**b** For $g(x)=x^{\alpha}$, $\alpha\in(1,+\infty)$. The corresponding pseudo-operations are $x\oplus y=\sqrt[\alpha]{x^{\alpha}+y^{\alpha}}$ and $x\otimes y=xy$.

The inequality (5) obtains the following form

$$\sqrt[\alpha]{\int_0^{+\infty} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))\right)dt} \leq \sqrt[\alpha]{e\int_0^{+\infty} f(x)dx}.$$

**c** For $g(x) = \ln(x+1)$. The corresponding pseudo-operations are $x \oplus y = (x+1)(y+1) - 1$ and $x \otimes y = e^{\ln(x+1)\ln(y+1)-1}$. The inequality (5) obtains the following form

$$e^{\int_0^{+\infty} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))\right)dt} \leq e^{e\int_0^{+\infty} f(x)dx}.$$

**Theorem 3.3.** (Pseudo Carleman-Knopp's inequality, decreasing case) Let $([0,+\infty), \oplus, \otimes)$ be a semiring and the generator $g : [0,+\infty) \to [0,+\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\otimes$ be a strictly decreasing and surjective function. Then, for any $\sigma$-$\oplus$-measure $m$, the following inequality

$$\int_{[0,+\infty)}^{\oplus} Exp_{\oplus}\left(\left(g^{-1}(1)\otimes^{-1} g^{-1}(x)\right) \otimes g^{-1}\left(\int_0^x \ln(f(t))dt\right)\right) \otimes dm \geq g^{-1}(e) \otimes g^{-1}\left(\int_0^{+\infty} f(x)dx\right),$$

holds true for any nonnegative Riemann integrable function $f$ on $[0,+\infty)$ which $\int_0^{+\infty} f(x)dx < \infty$.

*Proof.* It is similar to the proof of Theorem 3.1, with the difference that $g$ is a decreasing function and reverses the direction of inequality (7). □

**Example 3.4.** Let $[a,b) = [0,+\infty)$ and $g(x) = \frac{1}{e^x}$. The corresponding pseudo-operations are $x \otimes y = x + y$ and $x \oplus y = \ln(\frac{e^{x+y}}{e^x+e^y})$. Using Theorem 3.3 we get the following inequality

$$\ln\left(\frac{1}{\int_0^{+\infty} \exp\left(\frac{1}{x}\int_0^x \ln(f(t))\right)dt}\right) \geq \ln\left(\frac{1}{e\int_0^{+\infty} f(x)dx}\right).$$

## 4. Conclusion

In this paper, we have proved two Karlmann-Knopp type inequalitie's for quasi-integrals. For further investigation, we consider other integral inequalities for quasi-integrals similar to [1].

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# On the classes of hereditarily $\ell_1$ Banach spaces without the Schur property

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**Abstract**

We study some properties of the classes of Banach spaces constructed by Azimi and Hagler. First we show that these spaces contain subspaces which are weakly sequentially complete with an unconditional basis $(u_i)$ such that $u_i \to 0$ weakly but not in norm. For bounded linear operators on the spaces we also investigate complemented subspace of the spaces.

## 1. Definitions and Preliminaries

We review the classes of Banach sequence spaces constructed by Azimi and Hagler as examples of hereditarily $\ell_1$ Banach spaces failing the Schur property. In this paper we study some more geometric and topological properties of the spaces. In particular, a further subspace structure of the spaces are investigated. Definitions and notations are standard, but we give some of these here. we also refer to the book of Lindenstruss and Tzafriri [4] for further material in Banach spaces.

**Definition 1.1.** An infinite dimensional Banach space $X$ is said to contains $\ell_1$ hereditarily if every infinite dimensional subspace of $X$ contains a subspace isomorphic to $\ell_1$.

**Definition 1.2.** A subset Y of X is complemented in X if there is a bounded projection $P : X \longrightarrow Y$ such that $PX = Y$.

**Definition 1.3.** A Banach space $X$ is said to have the Schur property if weak convergence of the sequences in $X$ imply their norm convergence.

$[x_n]$ is closed linear span of $(x_n)$. If $(x_n)$ is a Schauder basis for X, then bi-orthogonal functionals on X, associated to $(x_n)$, is denoted by $(x_n^*)$ and $P_n$ is the natural projections associated with the unit vector basis of $X$.

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Now we go through the construction of the $X_\alpha$ spaces. These class of spaces were presented and studied by Azimi and Hagler [2]. First, by a block we mean an interval (finite or infinite) of integers. For any block $F$ and $x = (t_1, t_2, ...)$ a finitely non-zero sequence of scalars, we let $< x, F >= \sum_{j\in F} t_j$. A sequence of blocks $F_1, F_2, ...$ is admissible if $maxF_i < minF_{i+1}$ for each $i$. Finally, consider a nonnegative sequence $(\alpha_i)$ of reals which satisfies the following properties:

1) $\alpha_1 = 1$ and $\alpha_{i+1} \leq \alpha_i$ for $i = 1, 2, ...$
2) $\lim_{i\to\infty} \alpha_i = 0$
3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

We now define a norm which uses the $\alpha_i$'s and admissible sequences of blocks in its definition. Let $x = (t_1, t_2, ...)$ be a finitely non-zero sequence of reals, define

$$||x|| = \max \sum_{i=1}^{n} \alpha_i | < x, F_i > |$$

where the max is taken over all $n$, and admissible sequences $F_1, F_2, ... F_n$. The Banach space $X_\alpha$ is the completion of the finitely non-zero sequence of scalers in this norm. Here is the basic properties of the $X_\alpha$ spaces. For more information see [1], [2].

## 2. Results

Our first result is the following.

**Theorem 2.1.** *Let $u_i = e_{2i} - e_{2i-1}$ and $\tilde{X}_\alpha = [u_i]$. Then*

*1) $(u_i)$ is an unconditional basis,*

*2) $u_i \longrightarrow 0$ weakly if $i \to \infty$,*

*3) $\tilde{X}_\alpha$ is weakly sequentially complete.*

*Proof.* 1) $(u_i)$ is an unconditional basic sequence. This follow from the fact that, for any sequence $(t_i)$ of scalars, and any $j$, $||\sum_{i\neq j} t_i u_i|| \leq ||\sum_i t_i u_i||$. See [4] (Proposition 1.c.6, page 18)
2) By the theorem 1.4 $(e_i)$ is a weakly Cauchy sequence, so $(u_i)$ is weakly Cauchy. Now assume that $(u_i)$ does not converge weakly to 0. Then, there exists an $f \in X^*$, $||f|| = 1$, and a $\delta > 0$ such that (passing to a subsequence of $(u_i)$ and not renaming) $f(u_i) > \delta$ for all $i$. On the other hand, since $(u_i)$ is unconditional and not equivalent to the usual basis of $\ell_1$, there are an $N$ and non-negative scalars $t_1, t_2, ..., t_N$ such that $\sum_{i=1}^{N} t_i = 1$ and $||\sum_{i=1}^{N} t_i v_i|| < \frac{\delta}{2}$. Thus,

$$\delta < \sum_{i=1}^{N} t_i f(v_i) < f(\sum_{i=1}^{N} t_i v_i) < \frac{\delta}{2}$$

which contradicts the assumption that $(u_i)$ does not converge weakly to 0. This completes the proof. The definition of the norm implies that $||u_i|| = 1 + \alpha_2$, so $(u_i)$ does not converge to 0 in norm.
3) Since $(u_i)$ is an unconditional basic sequence and since $\tilde{X}_\alpha$ contains no isomorphic copy of $c_0$, it follows that $\tilde{X}_\alpha$ is weakly sequentially complete. See [4] (Theorem 1.c.10, page 22) □

For $x \in X$, put $s(x) = \max | < x, G > |$ where the max is taken over all blocks $G$. We need the following lemma from [2] in proof of the next theorem.

**Lemma 2.2.** *Let $(u_i)$ be a sequence of norm one vector in $X_\alpha$ and $(G_i)$ an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$. For each i put $s_i = s(u_i)$. If $\lim s_i = 0$ then a subsequence $(v_k)$ of $(u_k)$ is equivalent to usual basis of $\ell_1$.*

From the definition of the norm of $X_\alpha$, we can see that the unit vector basis is spreading (equivalent to each of its subsequence) and bi-monotone. That is for each $x \in X_\alpha$ and $n < m$ , $||(P_m - P_n)x|| \leq ||x||$. Observe each block $F$ defines a functional which is bounded on $X_\alpha$. In fact $< x, F >= \sum_{i\in F} e_i^*(x)$. Further, if $(e_{i_k})$ is a subsequence of $(e_n)$, then $[(e_{i_k})]$ is complemented. Indeed if $\{F_i\}$ is a sequence of blocks without gaps ($maxF_i + 1 = minF_{i+1}$) such that $i_k \in F_k$, then $[\{e_{i_k}\}]$ is complemented by the projection

$$Px = \sum_{i=1}^{\infty} < x, F_k > .e_{i_k}$$

Since $(F_i)$ has no gaps, any estimate of $||Px||$ is also an estimate of $||x||$, so $||P|| = 1$.
The following lemma is extracted from the proof of theorem 1 of [2].

**Lemma 2.3.** *1) If $(x_i) \subset X_\alpha$ converges $weak^*$ to $x^{**} \in X_\alpha^{**}$, then $x^{**} = x + \theta$ where $x \in X_\alpha$ and $e_i^*(\theta) = 0$ for all $i$. 2) If $(x_i) \subset X_\alpha$ is weakly Cauchy, then $(x_i)$ converges $weak^*$ to $x+\alpha\theta_0$ where $x \in X_\alpha$ and $\alpha = lim_{i\to\infty} < x_i - x, \mathbb{N} >$, and $\theta_0$ is the $weak^*$ limit of $(e_i)$.*

**Lemma 2.4.** *Let $(x_i)$ be a block basic sequence of $(e_i)$, let $F = \{M+1, M+2, ...\} \subset N$ and suppose $< x_i, F >= \gamma > 0$ for all $i$. Then for any scalars sequence $(a_i)$,*

$$\gamma||\sum a_i e_i|| \leq ||\sum a_i x_i||$$

*Proof.* Let $(a_i)$ be a sequence of scalars and let $x = \sum_1^n a_i e_i$, $y = \sum_i^n a_i x_i$. Since $< x_i, F >= \gamma$, there exist an admissible sequence of blocks $(F_i)$ such that $< x_i, F_i >= \gamma$ and $supp\ x_i \subset F_i$ for all $i$. Let $F_i = [f_i, g_i]$. Let $(G_k)$ be a admissible sequence with $||x|| = \sum_k \alpha_k| < x, G_k > |$, and for each $k$, let $G_k' = [n_k, m_k]$ where $n_k = \min\{f_i : i \in G_k\}$, $m_k = \max\{g_i : i \in G_k\}$. Then $(G_k)$ is admissible and

$$\begin{aligned} ||y|| &\geq \sum \alpha_k| < y, G_k' > | \\ &= \sum \alpha_k\gamma| < x, G_k > | \\ &= \gamma||x|| \end{aligned}$$

□

We state the following corollary from theorem 1.4. which is important in our further studies.

**Corollary 2.5.** *Let $X_\alpha = A \oplus B$. Then exactly one of $A$ or $B$ is weakly sequentially complete and the other is of codimension one in its first Baire class.*

**Theorem 2.6.** *If $T : X_\alpha \longrightarrow X_\alpha$ is bounded linear operator, then either $TX_\alpha$ or $(I - T)X_\alpha$ contains a complemented isomorphic of $X_\alpha$.*

*Proof.* Let $Z = (I - T)X_\alpha$. The sequences $(Te_i)$ and $((I - T)e_i)$ are weakly Cauchy. Since $Z$ is weakly sequentially complete, lemma2.3 implies that

$$(I - T)e_i \xrightarrow{w^*} y \in X_\alpha \quad (1)$$

and

$$Te_i \xrightarrow{w^*} x + \alpha\theta_0 \qquad (2)$$

Now $e_i \xrightarrow{w^*} \theta_0 \in X_\alpha^{**} - X_\alpha$ and $e_i = (I - T)e_i + Te_i$, so $(Te_i)$ and $((I - T)e_i)$ cannot both have weak-star limit in $X_\alpha$. Hence $\alpha = \lim < Te_i - x, N >\neq 0$. In fact by standard perturbation argument we may assume there exist $M \in N$ such that $P_M y = y$ and $P_M x = x$, where $x, y$ are as in (1), (2) and $P_M$ is the natural projection. Then with $F = \{M + 1, M + 2, ...\}$,

$$1 =< e_i, F >=< (I - T)e_i, F > + < Te_i, F >$$

So $\lim < Te_i, F >= 1$. Applying lemma 2.3 part 1, passing to a subsequence $(e_i)$ and perturbing, we may assume that $Te_{i_k} = v_k = x + w_k$ with (M and F as above)

$$P_M x = x$$

$$< w_k, F >= 1 \quad for\ all\ \ k$$

and $supp\ w_k \subset G_k$ where $(G_k)$ is an admissible sequence without gaps. Then for any scalar sequence $(a_k)$

$$\begin{aligned} ||\sum a_k v_k|| &= ||(\sum a_k)x + \sum a_k w_k|| \\ &\geq ||\sum a_k w_k||\ \ ((e_i)\ is\ \ bi-monotone) \\ &\geq ||\sum a_k e_k|| \end{aligned}$$

by lemma2.5. Since $||\sum a_k v_k|| \leq ||T||||\sum a_k e_k||$, the sequence $(v_k)$ is equivalent to $(e_k)$, and hence $TX_\alpha$ contains isomorphic of $X_\alpha$. A projection onto $[v_k]$ is defined by

$$Qz = \sum_{k=1}^{\infty} < z, G_k > v_k$$

$Q$ is bounded, since

$$\begin{aligned} ||Qz|| &= ||\sum < z, G_k > v_k|| \\ &\leq ||T||||\sum < z, G_k > e_k|| \\ &\leq ||T||||z||, \end{aligned}$$

since $(G_k)$ has no gaps.

□

**Definition 2.7.** A Banach space $X$ is said to have the Dunford-Pettis property (DPP) if for every weakly null sequence $(x_n)$ in $X$ and $(x_n^*)$ in $X^*$, then we have $\lim_n x_n^*(x_n) = 0$.

**Theorem 2.8.** *The Banach spaces $X_\alpha$ fail the DPP.*

*Proof.* Let $u_i = e_{2i} - e_{2i-1}$ and $f_i : X_\alpha \to \mathbb{R}$ such that for any $x = (t_1, t_2, \ldots) \in X_\alpha$, we have $f_i(x) = t_i$ for integer $i$. Then for $g_n = f_{2n} - f_{2n-1}$, we have $g_n(u_n) = 2$. To complete the proof we need to show that $u_n \to 0$ weakly, and $g_n \to 0$ weakly. The first one follows from Lemma 2.2. We claim that $g_n \to 0$ weakly. If not there are $F \in X_\alpha^{**}$ with $||F|| = 1$, $\delta > 0$ and a subsequence $(g_{n_k})$ such that $F(g_{n_k}) > \delta$ for all integer $k$. So for integer $N$, we have $\sum_{k=1}^{N} F(g_{n_k}) > N\delta$ and hence

$$\frac{||\sum_{k=1}^{N} g_{n_k}||}{N} > \delta.$$

This implies that for any integer $N$, there exist $x = (t_1, t_2, \ldots) \in X_\alpha$ such that

$$\frac{1}{N}\sum_{k=1}^{N} g_{n_k}(x) > \delta.$$

Then $\lim_n t_n = 0$ for integer $N$ and corresponding $x = (t_1, t_2, \ldots)$, since $\sum_{i=1}^{\infty} \alpha_i = \infty$. Therefore,

$$\begin{aligned} |\frac{1}{N}\sum_{k=1}^{N} g_{n_k}(x)| &= \frac{1}{N}|\sum_{k=1}^{N}(t_{2n_k} - t_{2n_k-1})| \\ &\leq \frac{1}{N}\sum_{k=1}^{N}|t_{2n_k}| + \frac{1}{N}\sum_{k=1}^{N}|t_{2n_k-1}| \to 0 \end{aligned}$$

as $N \to \infty$ which is a contradiction. □

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# Some special graphs and second inverse sum indeg index

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**Abstract**

In this paper, we introduce second inverse sum indeg index. Extremal values of this index for some special graph are determined. In the present paper, we consider the graph with $p$ pendent vertices and we get attractive upper and lower bounds. All graphs considered in this report are simple graph and connected and without loops. Topological indices are important tools in prediction of chemical phenomena, that's why several types of topological indices have been defined.

## 1. Main result

The second inverse sum indeg (henceforth, $ISI_2$) index of a graph $G$ defined as [1] $ISI_2(G) = \sum_{uv \in E(G)} \frac{n_u n_v}{n_u + n_v}$, where $n_u$ is the number of vertices of graph $G$ lying closer to $u$ and $n_v$ is the number of vertices of graph $G$ lying closer to $v$. For more study on this family of topological indices we can see [2, 3].

**Theorem 1.1.** *Let $G$ be a connected graph of order $n$ with $m$ edges and $p$ pendent vertices, then*

$$ISI_2(G) \leq \frac{mn^2}{8} + p\left(1 - \frac{1}{n} - \frac{n^2}{8}\right), \tag{1}$$

*with equality if and only if graph $G \cong K_2$.*

**Theorem 1.2.** *Let $G$ be a connected graph of order $n$ with $m$ edges and $p$ pendent vertices, then*

$$ISI_2(G) \geq \frac{m}{n} + p\left(1 - \frac{2}{n}\right), \tag{2}$$

*with equality if and only if graph $G \cong K_2$.*

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# Application of the Ritz method for the identification of a time-dependent heat source in a parabolic equation with Caputo's fractional derivative

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**Abstract**

In this paper, An inverse heat equation with Caputo's fractional derivative is studied to retrieve a time-dependent heat coefficient. The Dirichlet boundary conditions along with nonlinear initial condition are treated by applying the satisfier function and the governing equation is discretized using a linear combination of Legendre's basis functions. We employ the weighted residual approach and change the main problem to the solution of a linear system of algebraic equations. Effectiveness of this method is shown by providing a numerical example.

## 1. Introduction

In the last few years, fractional calculations have had a good reflection in science and engineering, and considerable work has been done in the field of numerical solution of fractional equations. With the expansion of human knowledge, various applications of fractional-boundary initial value problems have been discovered by scientists. Unlike direct fractional initial value problems that have been studied recently, inverse fractional initial value problems are still in their infancy [1–4].
In this paper, we used Caputo's fractional derivatives and integrals, which are formally described as follows:

$$\frac{\partial^{\alpha} v(y,t)}{\partial t^{\alpha}} = (D_t^{\alpha} v)(y,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \left( \frac{\partial^n v(y,s)}{\partial t^n} \right) ds & \alpha > 0, \quad n-1 < \alpha < n, \\ \frac{\partial^n v(y,t)}{\partial t^n} & \alpha = n \in N. \end{cases} \tag{1}$$

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$$I_y^{\alpha} v(y) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (y-s)^{n-\alpha-1} v(s) ds. \tag{2}$$

According to definition (1) we solve the following system of equations

$$\frac{\partial^{\alpha} v}{\partial t^{\alpha}} - \frac{\partial^2 v}{\partial y^2} = p(y)q(t) \quad 0 < y < 1, \quad 0 < t \leq T, \tag{3}$$

with boundary conditions

$$v(0,t) = h_1(t), \quad v(1,t) = h_2(t), \quad 0 \leq t \leq T, \tag{4}$$

Nonlocal initial condition

$$v(y,0) = \sum_{j=1}^{N} \lambda_j(y) v(y,T_j) + \psi(y), \quad 0 \leq y \leq 1, \quad 0 < T_1 < T_2 < \ldots < T_N = 1, \tag{5}$$

and additional condition

$$v(y,T_*) = v_*(y), \tag{6}$$

where $\psi(y), h_1(t), h_2(t), p(y), \lambda_j(y)$ are given functions and the temperature distribution $v(y,t)$ and heat source q are unknown.

Recently, articles have been presented regarding the solution of inverse parabolic differential equations. In [5], the author investigated fractional-time parabolic inverse problems with non-local initial and boundary conditions along with specified conditions at a specific point in the domain space, and with the help of the system of linear equations resulting from the Euler approximation, which has a pseudo-trigonal coefficient matrix, presented an efficient method for solving the linear system of the predictive-corrector method for calculating the solution and updating the unknown coefficient of the problem. In [6], the authors considered the two-dimensional generalization of the analog parabolic equation using the Fourier method and investigated the solvability of inverse problems with Dirichlet conditions. In [7],the authors investigated the inverse heat equation problems with unknown heat source and non-local data. Their approach shows that having only one data set at a point guarantees the existence of a weak solution to the inverse problem. Moreover, if there is an additional data at the additional point, it leads to a special formula for the time-dependent source coefficient. In [8], the authors solved inverse parabolic equations based on basic polynomials and using the finite difference and least squares method, and by interpolating at each point T, they obtained an approximate solution for this group of equations. In [9], the authors investigated two-dimensional fractional-time equations with Dirichlet boundary conditions and using the Petrov-Galerkin method and applying an additional integral condition presented an efficient calculation method to solve this class of problems.

In this paper, we use the Ritz technique to provide the approximate solution of the FDEs with nonlocal initial condition given by equations (3)-(5) for the first time. As a great advantage of our method is the flexibility of satisfying all the input data accurately.

In section(2) provides an explanation of the approximation method and finally in Section(3) evaluates the efficiency and effectiveness of the method through the presentation of numerical examples.

## 2. Methodology

First, by integrating both sides of equation (3) and applying the Caputo's integral operator, we have:

$$v(y,t) - v(y,0) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \frac{\partial^2 v(y,s)}{\partial y^2} \right) ds + q(t) \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) ds \right). \tag{7}$$

Here we consider the unknown coefficient $q(t)$ as a linear combination of legendre functions, it's mean:

$$q(t) = \sum_{j=0}^{N} d_j \phi_j(t). \tag{8}$$

Therefore, by substituting the nonlocal initial condition and approximation $q(t)$ in equation (7), we have:

$$v(y,t) = \sum_{j=1}^{N} \lambda_j(y) v(y,T_j) + \psi(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \frac{\partial^2 v(y,s)}{\partial y^2} \right) ds \tag{9}$$

$$+ \left( \sum_{j=0}^{N} d_j \phi_j(t) \right) \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) ds \right).$$

And also, we consider the satisfier function with the help of boundary conditions (4) as follows:

$$s(y,t) = h_1(t) + (h_2(t) - h_1(t))y, \tag{10}$$

We define the Ritz approximation of equations (9) and (4) as follows:

$$\bar{z}(y,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} y(y-1) c_{i,j} \phi_i(y) \phi_j(t) + s(y,t). \tag{11}$$

Now, by deriving the equation (11) with respect to y twice, we have:

$$\bar{z}_{yy}(y,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} \left( 2\phi_i(y) + (4y-2)\phi_i'(y) + (y^2-y)\phi_i''(y) \right) c_{i,j} \phi_j(t). \tag{12}$$

And then we define the residual functions as follows:

$$Res_1(y,t) = v(y,t) - \sum_{j=1}^{N} \lambda_j(y) v(y,T_j) + \psi(y) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \frac{\partial^2 v}{\partial y^2} + p(x) q(s) \right) ds, \tag{13}$$

$$Res_2(y) = v(y,T_*) - v_*(y), \tag{14}$$

Therefore, by substituting the approximations $\bar{z}_{yy}(y,t)$, $v(y,t)$ and $q(t)$ in $Res_1(y,t)$, we have:

$$Res_1(y,t) \simeq \bar{z}(y,t) - \sum_{j=1}^{N} \lambda_j(y) \bar{z}(y,T_j) + \psi(y) \tag{15}$$

$$- \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{M} \sum_{j=0}^{N} c_{i,j} \left( 2\phi_i(y) + (4y-2)\phi_i'(y) + (y^2-y)\phi_i''(y) \right)$$

$$\int_0^t (t-s)^{\alpha-1} \left( \phi_j(s) + \left( \sum_{j=0}^{N} d_j \phi_j(t) \right) p(s) \right) ds.$$

Then, we collocating the residual function (15) at points

$$y_i = \frac{i}{M+2},\ t_j = \frac{jT}{N+2},\ i = 0,\dots,M,\ j = 0,\dots,N, \tag{16}$$

and the residual function (14) at t points

$$y_i = \frac{i}{M+2},\ i = 1,\dots,M+1, \tag{17}$$

And we solve the following system of algebraic equations:

$$Res_1(y_i,t_j) = 0,\ i = 0,\dots,M,\ j = 0,\dots,N, \tag{18}$$

$$Res_2(y) = 0,\ i = 1,\dots,M+1. \tag{19}$$

That the system of linear equations (18) and (19) for the elements $c_{i,j}$ and $d_j$ are solved using Newton's method and finally we get an approximate solution for the $q(t)$ and $v(y,t)$ functions.

## 3. Numerical example

To show the efficiency and effectiveness of this method, we examine the following example.

**Example 3.1.** We consider (3)-(5) with N=7 and let

$$v(y,0) = v(y,1) + cos(y)(1-e^2),$$
$$h_1(t) = e^{2t}, \quad h_2(t) = (cos(1))e^{2t},$$
$$v_*(y) = (cos(0.25))e^{2t},$$
$$p(y) = cos(y),$$

The exact solution of the coefficient $q(t)$ of this problem is:

$$q(t) = 3e^{2t}. \tag{20}$$

To solve this example, we consider the unknown coefficient $q(t)$ as

$$q(t) = \sum_{j=0}^{N} d_j \phi_j(t), \tag{21}$$

where $\phi_j(t)$ are Legendre polynomials in [0,1], and then we solve the problem with the help of the Ritz technique. In table (1) shows the approximate solutions of $q(t)$ for $N = 7$ with $\alpha = 0.75, 0.85, 0.95, 1$ and different $y$, and we show the convergence of this method in figure (1).

Table 1. Approximate values of function $v(y,t)$ and $ResL^2norm$ function with different values of $alpha$.

| y=t | $\alpha = 0.75$ | $\alpha = 0.85$ | $\alpha = 0.95$ | $\alpha = 1$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 0.1 | 1.2153 | 1.2153 | 1.2153 | 1.2153 |
| 0.2 | 1.4620 | 1.46209 | 1.46209 | 1.46209 |
| 0.3 | 1.74074 | 1.74074 | 1.74074 | 1.74074 |
| 0.4 | 2.04986 | 2.04986 | 2.04986 | 2.04986 |
| 0.5 | 2.38552 | 2.38552 | 2.38552 | 2.38552 |
| 0.6 | 2.74021 | 2.74021 | 2.74021 | 2.74021 |
| 0.7 | 3.10159 | 3.10159 | 3.10159 | 3.10159 |
| 0.8 | 3.45081 | 3.45081 | 3.45081 | 3.45081 |
| 0.9 | 3.76052 | 3.76052 | 3.76052 | 3.76052 |
| 1 | 3.99232 | 3.99232 | 3.99232 | 3.99232 |
| $ResL^2norm$ | 0.0503032 | 0.0468953 | 0.0130527 | $1.04161 \times 10^{-9}$ |

## 4. Conclusion

In this article, we used an efficient method to solve the inverse heat equation of fractional order with the Caputo's fractional derivative numerically. To discretize the problem, we first considered the approximation of $q(t)$ as a linear combination of the Legendre basis functions such that the initial and boundary conditions are satisfied. By solving a numerical example, we showed that the approximate solutions are in good agreement with exact solutions.

Table 2. Approximate values of the function $q(t)$ with different values of $alpha$.

| t | $\alpha = 0.75$ | $\alpha = 0.85$ | $\alpha = 0.95$ | $\alpha = 1$ |
|---|---|---|---|---|
| 0 | 1.5083 | 1.88457 | 2.52619 | 3.00006 |
| 0.1 | 2.69888 | 3.05056 | 3.45037 | 3.66421 |
| 0.2 | 3.53104 | 3.87983 | 4.26676 | 4.47548 |
| 0.3 | 4.49407 | 4.86537 | 5.26138 | 5.46636 |
| 0.4 | 5.62307 | 6.02761 | 6.45495 | 6.67662 |
| 0.5 | 6.97783 | 7.43147 | 7.90847 | 8.15485 |
| 0.6 | 8.62603 | 9.14435 | 9.68431 | 9.96035 |
| 0.7 | 10.5914 | 11.1837 | 11.8206 | 12.1656 |
| 0.8 | 13.2266 | 14.0023 | 14.6762 | 14.859 |
| 0.9 | 18.472 | 20.0604 | 20.0046 | 18.1482 |
| 1 | 32.4595 | 37.032 | 33.1135 | 22.1634 |

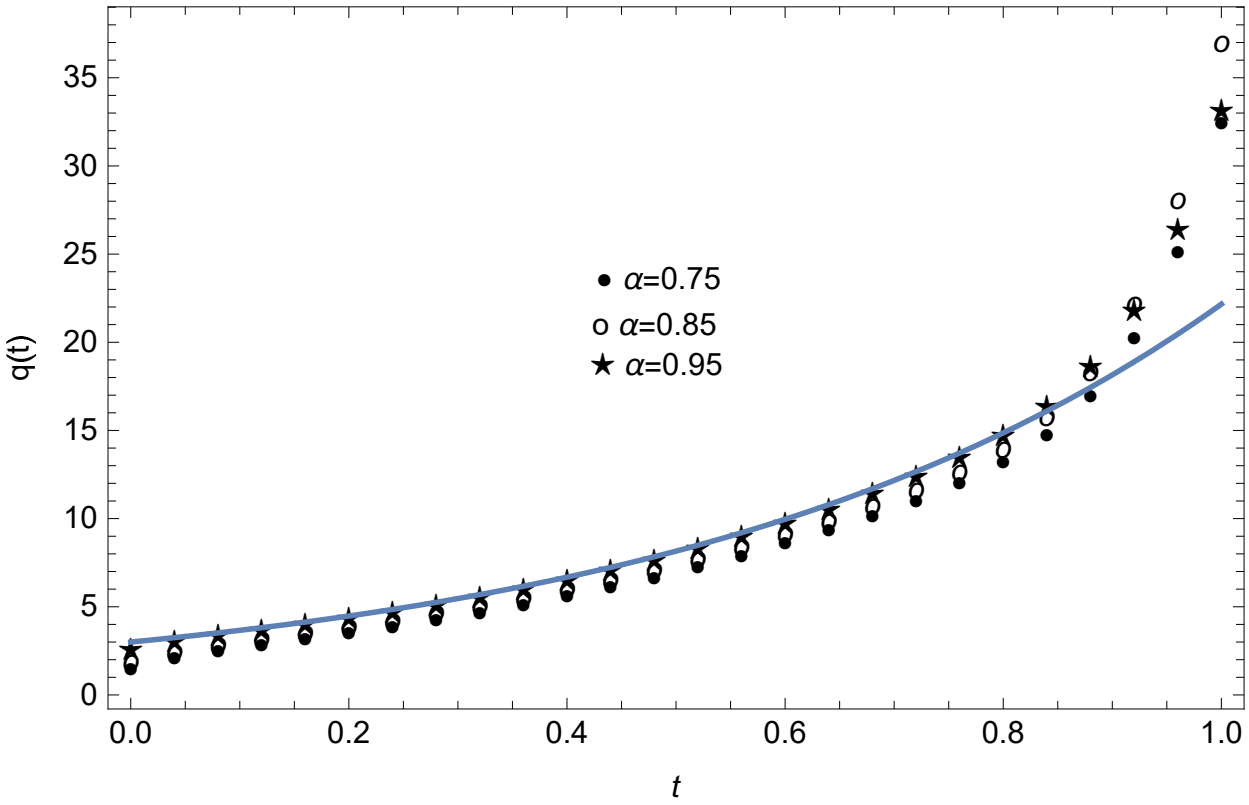


Fig. 1. This figure presents the approximate solutions and exact solution (blue curve) of the problem described in Example with $N = 7$ and different values of parameter α.

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# The Role of Some Graphs in the Study of Lattices

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**Abstract**

The study of algebraic structures, using the properties of graph theory, has an exciting research topic in the last decade. This helps us to study some properties of algebraic structures by associated graphs. The purpose of this note is the study of Cayley graph associated with a lattice. In this note first, we associate a Cayley graph to every lattice and then we study the properties of this graph.

## 1. Introduction

The first time, the definition of the Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups being described by a set of generators. In the last 50 years, the theory of Cayley graphs has been growing into a substantial branch in algebraic theory [9]. Let $G$ be an abelian additive group, $C$ be a subset of $G$. Whenever $0 \notin C$ and $-C = \{-c : c \in C\} \subseteq C$, then the Cayley graph $Cay(G, C)$ is the graph with vertex set $G$ and edge set $\{\{a, b\} : a - b \in C\}$. We refer the reader to [7] for general properties of Cayley graphs.
In recent years, for a ring $R$, the Cayley graph of the abelian group $(R, +)$ with respect to subsets of $R$ has received much attention in the literature (see [2, 10, 12]). Ramin and Abbasi in [11], introduced the torsion-unitary Cayley graph of an $R$-module $M$, denoted by $\Gamma_R(M)$. It is a simple graph with vertex set $M \times R$ and two elements $(m, r), (n, s) \in M \times R$ are adjacent if $(m, r) - (n, s) \in \Gamma(M) \times U(R)$, where $\Gamma(M) = \{m \in M : rm = 0 for\ some\ 0 \neq r \in R\}$ is the set of torsion elements of $M$ and $U(R)$ is the set of unit elements of $R$. They obtained some properties of this graph. Barati [3] associated a simple graph $\Gamma_S(R)$ to a multiplicatively closed subset $S$ of a commutative ring $R$ with all elements of $R$ as vertices and two distinct vertices $x, y$ are adjacent if $x + y \in S$. Afkhami [1] introduce the same graph structure on a lattice. They considered a lattice $L$ and defined a graph $\Gamma_S(L)$ with all elements of $L$ as vertices and two distinct $x, y \in L$ are adjacent if $x \vee y \in S$ where $S$ is a subset of $L$ which is closed under the operation $\wedge$.
In this paper, we introduce the Cayley graph of a lattice and investigate some properties of this graph.
Let us recall notions of lattices, defined in [5].
Let $P$ be a set. An order (or partial order) on $P$ is a binary relation $\leq$ on $P$ such that for all $x, y, z \in P$, we have

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(i) $x \leq x$;
(ii) $x \leq y$ and $y \leq x$ implies $x = y$;
(iii) $x \leq y$ and $y \leq z$ implies $x \leq z$.

These conditions are referred to, respectively, as reflexivity, antisymmetry, and transitivity. A set $P$ equipped with an order relation $\leq$ is said to be an order set. The ordered set $p$ is a chain if, for all $x, y \in P$, either $x \leq y$ or $y \leq x$.
An algebra $L = (L, \vee, \wedge)$ is called a lattice if for all $a, b, c \in L$, we have

(i) $a \wedge a = a$, $a \vee a = a$;
(ii) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$;
(iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$;
(iv) $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

There is an equivalent definition for a lattice (see for example [6]). To do this, for a lattice $L$, one can define an order $\leq$ on $L$ as follows: For any $a, b \in L$, we say $a \leq b$ if $a \wedge b = a$. Then $(L \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let $P$ be an ordered set such that, for every pair $a, b \in P$, $g.l.b.(a, b) \in P$ and $l.u.b.(a, b) \in P$. For each $a$ and $b$ in $P$, we define $a \wedge b := g.l.b.(a, b)$ and $a \vee b := l.u.b(a, b)$. Clearly $(P, \vee, \wedge)$ is a lattice.
The lattice $L$ is said to be bounded if there are elements 0 and 1 such that,

$$a \wedge 1 = a \text{ and } a \vee 0 = a,$$

for all $a \in L$. A nonempty subset $S$ of $L$ is called a multiplicatively closed subset of $L$ if $x \wedge y \in S$, for all $x, y \in S$. Also, we say that $S$ is a sublattice of $L$ if $a, b \in S$ implies $a \vee b \in S$ and $a \wedge b \in S$.
Throughout this paper, $L$ is a bounded lattice.
Let $L$ and $S$ be two lattices. A map $f : L \to S$ is called a homomorphism if for all $x, y \in L$, the following conditions hold:

$$f(x \vee y) = f(x) \vee f(y) \text{ and } f(x \wedge y) = f(x) \wedge f(y).$$

A homomorphism $f$ is an isomorphism if $f$ is both injective and surjective. We write $L \cong S$ if $L$ is isomorphic to $S$.

**Definition 1.1.** (i) Let $P$ be an ordered set and $x, y \in P$. We say that $x$ is covered by $y$ (or $y$ covers $x$) and write $x- < y$ or $y > -x$, if $x < y$ and $x \leq z < y$ implies $z = x$.
(ii) An element $x$ in $L$ is called an atom if $0- < x$. We denote the set of all atoms in a lattice $L$ by $Atom(L)$.

**Definition 1.2.** Let $P$ and $Q$ be (disjoint) ordered sets. The linear sum $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$: $x \leq y$ if and only if

$$\begin{aligned} &x, y \in P \text{ and } x \leq y \text{ in } P, \\ &\text{or } x, y \in Q \text{ and } x \leq y \text{ in } Q, \\ &\text{or } x \in P \text{ and } y \in Q. \end{aligned}$$

**Lemma 1.3.** *Let $L$ be a lattice. Then for all $a, b, c \in L$, we have*

$$a \leq b \text{ implies } a \vee c \leq b \vee c \text{ and } a \wedge c \leq b \wedge c.$$

**Definition 1.4.** Let $L$ and $S$ be two lattices. Define product $L \times S$ to be the lattice whose $\vee$ and $\wedge$ cordinatewise on $L \times S$ for $(l_1, s_1), (l_2, s_2) \in L \times S$, as follows

$$(l_1, s_1) \vee (l_2, s_2) = (l_1 \vee l_2, s_1 \vee s_2),\ (l_1, s_1) \wedge (l_2, s_2) = (l_1 \wedge l_2, s_1 \wedge s_2).$$

We now recall the definition of congruence on lattices. We say that an equivalence relation $\theta$ on a lattice $L$ is compatible with $\vee$ and $\wedge$ if for all $a, b, c, d \in L$, we have

$$a \equiv b \ (mod\ \theta) \text{ and } c \equiv d \ (mod\ \theta)$$

imply

$$a \vee c \equiv b \vee d(mod\ \theta) \text{ and } a \wedge c \equiv b \wedge d(mod\ \theta).$$

**Lemma 1.5.** *Let $L$ and $S$ be two lattices and $f : L \to S$ be a homomorphism. Then the equivalence relation $\theta$ defined on $L$, for $a, b \in L$, by*

$$a \equiv b(mod\ \theta) \Leftrightarrow f(a) = f(b).$$

*is compatible with $\vee$ and $\wedge$.*

**Definition 1.6.** An equivalence relation on a lattice $L$ which is compatible with both $\vee$ and $\wedge$ is called a congruence on $L$.

If $L$ and $S$ are two lattices and $f : L \to S$ is a homomorphism, then the associated congruence $\theta$ on $L$, defined in Lemma 1.5, is known as the Kernel of $f$ and is denoted by $Ker\, f$.

Given a congruence $\theta$ on a lattice $L$. Let $a \in L$ and $\theta[a]$ be the equivalence class of $a$ and $\frac{L}{\theta} = \{\theta[a] : a \in L\}$ the set of all equivalence classes of elements of $L$. Define the operations $\vee$ and $\wedge$ as follows:

$$\theta[a] \vee \theta[b] = \theta[a \vee b],\ \theta[a] \wedge \theta[b] = \theta[a \wedge b].$$

**Lemma 1.7.** *Let $\theta$ be a congruence on the lattice $L$. Then $(\frac{L}{\theta}, \vee, \wedge)$ is a lattice.*

Now, we recall some key definitions of graph theory which are needed in this note.
Let $X$ be a graph with the vertex set $V(X)$. For a graph $X$, the degree of a vertex $v$ in $X$ is the number of vertices adjacent to $v$ and denoted by $deg(v)$. A graph $X$ is said to be connected if, for each pair of distinct vertices $v$ and $w$, there is a finite sequence $v = v_1, \cdots v_n = w$ of distinct vertices where each pair $\{v_i, v_{i+1}\}$ is an edge. Otherwise, $X$ is called disconnected. Such a sequence is said to be a path. If $v = w$, then a path is called a cycle. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph and the notation $K_n$ is used for complete graphs with $n$ vertices. A connected graph is called a tree if contains no cycle. A tree of order $n$ is called a star if it has a vertex of degree $n - 1$ and the rest of the vertices are of degree 1. The complement of $X$ is a graph denoted by $\bar{X}$ with the same vertex set as $X$ and two vertices in $\bar{X}$ are adjacent if and only if they are not adjacent in $X$. The complement of the complete graph $K_n$ is called the null graph on $n$ vertices (see [4] for more details)
Recall that a subgraph $Y$ of a graph $X$ is a graph whose set of vertices and the set of edges are both subsets of $X$. A clique of a graph is a complete subgraph of $X$ and the number of vertices in the largest clique of graph $X$, denoted by $\omega(X)$, is called the clique number of $X$. We now recall some graph operations [8]. Suppose $X$ and $Y$ are graphs with disjoint vertex sets. The disjoint union $X + Y$ is a graph with $V(X + Y) = V(X) \cup V(Y)$ and $E(X + Y) = E(X) \cup E(Y)$. The tensor product (or direct product) $X \times Y$ of graphs $X$ and $Y$ is the graph whose vertex set is $V(X) \times V(Y)$ in such a way that vertices $(x, y)$ and $(x', y')$ are adjacent if $x, x' \in E(X)$ and $y, y' \in E(Y)$.

## 2. Main Results

First, we introduce the notion torsion element and after using it, we associate a Cayley graph with a lattice.
Let $L$ be a lattice. Suppose that $\Gamma(L) = \{x \in L : x \wedge y = 0\ for\ some\ y \neq 0\}$ be the set of torsion elements.

**Definition 2.1.** We introduce the Cayley graph of $L$, denoted by $\Gamma\Gamma(L)$, as a graph with vertex set $L$ and two elements $x, y$ are adjacent if $\{x, y\} \subseteq \Gamma(L)$ and is denoted by $xEy$.

Since for all $y \in L$, $1 \wedge y = y$, so $1 \notin \Gamma(L)$.
If we do not consider vertex 1, we have the following Theorem.

**Theorem 2.2.** $\Gamma\Gamma(L)$ *is complete if and only if* $\Gamma(L) = L - \{1\}$.

**Lemma 2.3.** *If* $\Gamma(L) = L - \{1\}$, *then* $deg(x) = |L| - 1$.

**Proposition 2.4.** *Let* $|L| \geq 3$. *Then*

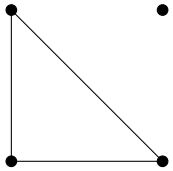
*(i) If* $|\Gamma(L)| \geq 3$, *then there exists a cycle with length* 3.
*(ii) If* $|\Gamma(L)| = 1$, *then* $\Gamma\Gamma(L)$ *is disconnected.*

In the following Proposition, we present a lower bound for the clique number of $\Gamma\Gamma(L)$.

**Proposition 2.5.** $\omega(\Gamma\Gamma(L)) \geq max\{|A| : A\ is\ a\ chain\ in\ \Gamma(L)\}$.

**Lemma 2.6.** *If* $|\Gamma(L)| \geq 2$, *then* $|\Gamma(L) \cap Atom(L)| \geq 1$.

**Example 2.7.** Let $X = \{0, 1\}$. Then by [5], $(P(X), \cup, \cap)$ is a lattice. It is routine to check $\Gamma(P(X)) = \{\{0\}, \{1\}, \emptyset\}$ and graph $\Gamma\Gamma(P(X))$ is as in follows:



Moreover, If we do not consider vertex $\{0, 1\}$, then we have $\Gamma\Gamma(P(X))$ is a complete and we saw that $\Gamma(P(X)) = P(X) - \{0, 1\}$. Also, we have a cycle with length 3 and finally $\Gamma(P(X)) \cap Atom(P(X)) = \{\{0\}, \{1\}\}$.

We remark that the properties of Cayley graph a lattice is completely different to graph defined in [1], for example in Example 2.7, in Cayley graph $\Gamma\Gamma(P(X))$, $\{0\}E\{1\}$ but in graph $\Gamma_{\Gamma(P(X))}(P(X))$, there is no edge between $\{0\}$ and $\{1\}$ because $\{0\} \cup \{1\} = \{0, 1\} \notin \Gamma(P(X))$.

**Lemma 2.8.** *Let* $S$ *be a sublattice of* $L$. *Then* $\Gamma(S) \subseteq \Gamma(L)$.

**Corollary 2.9.** *Let* $S$ *be a sublattice of* $L$. *Then* $\Gamma\Gamma(S)$ *is a subgraph of* $\Gamma\Gamma(L)$.

In the graph $\Gamma\Gamma(L)$, since vertices that in the set $\Gamma(L)$ are adjacent and the vertices that are not in $\Gamma(L)$ are not adjacent, the set of vertices can be considered as a disjoint union of two sets of $\Gamma(L)$ and $L - \{\Gamma(L)\}$,we have the following Theorem.

**Theorem 2.10.** *Let* $L$ *be a lattice. Then* $\Gamma\Gamma(L) = K_{|\Gamma(L)|} + \bar{K}_{|L-\{\Gamma(L)\}|}$.

**Lemma 2.11.** *Let* $f : L \to S$ *be a homomorphism and* $\alpha$ *is a corresponding congruence relation with it. Then there exists a homomorphism from* $\Gamma\Gamma(\frac{L}{\alpha})$ *to* $\Gamma\Gamma(S)$.

**Theorem 2.12.** *Let* $L$ *and* $S$ *be two lattices and* $L \cong S$. *Then* $\Gamma\Gamma(L) \cong \Gamma\Gamma(S)$.

**Theorem 2.13.** *Let* $L$ *and* $S$ *be two lattices. Then* $\Gamma\Gamma(L \times S) \cong \Gamma\Gamma(L) \times \Gamma\Gamma(S)$.

**Theorem 2.14.** *Let* $L$ *and* $S$ *be two lattices and* $L \cap S = \{0\}$. *Then there exists a homomorphism from* $\Gamma\Gamma(L) + \Gamma\Gamma(S)$ *to* $\Gamma\Gamma(L \oplus S)$.

## 3. Conclusion

In this note, by considering the notion of algebraic properties of lattice, a graph associated with a lattice was introduced and some related results were obtained.

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# Numerically option pricing using a time Caputo–Hadamard fractional Black–Scholes model

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**Abstract**

In this paper, a Black-Scholes model with a fractional derivative of the Caputo-Hadamard type is solved numerically. An L1-type approximation is obtained for the Caputo-Hadamard derivative approximation. The collocation method with cubic B-spline bases is also used to approximate the spatial derivatives. The numerical results indicate the accuracy of the method. The execution speed of the implemented algorithm is also relatively high, which indicates the efficiency of the presented method.

## 1. Introduction

The Caputo-Hadamard derivative is a fractional derivative that combines the concepts of the Caputo and Hadamard derivatives. It has gained significant attention in recent years due to its ability to model complex systems with memory and non-local properties. The Caputo-Hadamard derivative is particularly useful in fields such as control theory, signal processing, and viscoelastic materials.

The concept of fractional calculus, which includes the Caputo-Hadamard derivative, has been extensively studied over the past few decades. The Caputo derivative, introduced by Michele Caputo in 1967, is known for its applicability in solving differential equations with initial conditions. On the other hand, the Hadamard derivative, proposed by Jacques Hadamard in 1892, is used in problems involving boundary conditions. The combination of these two derivatives results in the Caputo-Hadamard derivative, which offers a more comprehensive approach to modeling real-world phenomena.

Recent studies have focused on the theoretical aspects of the Caputo-Hadamard derivative, exploring its properties and potential applications. For instance, a study by Diethelm and Ford [1] investigated the stability and convergence of numerical methods for the Caputo-Hadamard derivative. Another study by Liu et. al. [2] explored the application of the Caputo-Hadamard derivative in viscoelastic material modeling, demonstrating its effectiveness in capturing the material's behavior over time.

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The Caputo-Hadamard derivative has found applications in various fields due to its ability to model systems with memory and non-local effects. In control theory, it is used to design controllers for systems with fractional-order dynamics. In signal processing, it helps in the analysis and synthesis of signals with fractional characteristics. Additionally, in viscoelastic material modeling, the Caputo-Hadamard derivative provides a more accurate representation of the material's response to stress and strain over time.
Recent research has also highlighted the potential of the Caputo-Hadamard derivative in biomedical engineering, particularly in modeling the behavior of biological tissues and systems. A study by Karami et. al. [3] demonstrated the use of the Caputo-Hadamard derivative in optimal control problems.
Fractional derivatives in the Hadamard type are often defined from the generalization of the derivative operator $\left(t\frac{d}{dt}\right)^n$, $n \in \mathbb{N}$ through logarithmic kernel functions.

**Definition 1.1.** The Caputo-type Hadamard fractional derivative of order $0 < \alpha < 1$ of the function $f(t)$ on $[0,T]$ is defined as

$$ {}_a^{CH}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\ln\left(\frac{t}{s}\right)\right)^{-\alpha} f'(s)ds. \tag{1}$$

The Black-Scholes model, introduced by Fischer Black and Myron Scholes in 1973 [4], has been a cornerstone in financial mathematics for option pricing. The model assumes that the price of the underlying asset follows a geometric Brownian motion with constant volatility. However, empirical evidence has shown that financial markets exhibit features such as long memory and heavy tails, which are not captured by the standard Black-Scholes model.
To address these limitations, researchers have extended the Black-Scholes model by incorporating fractional derivatives, which can better capture the memory and anomalous diffusion observed in financial markets. Fractional calculus, a generalization of ordinary differentiation and integration to non-integer orders, has been applied to various fields, including finance, to model complex systems with memory effects.
Recent studies have shown that the Black-Scholes model with fractional derivatives provides a more accurate description of market behavior. For instance, a study by Bian et al. [5], demonstrated that incorporating fractional Brownian motion into the Black-Scholes framework significantly improves the pricing of options with long memory. Another research [6] explored the impact of fractional derivatives on hedging strategies and found that fractional models offer better risk management compared to the classical Black-Scholes model.
Moreover, advancements in numerical methods have facilitated the implementation of fractional derivatives in option pricing models. A recent paper by Zhang et al. [7], proposed a novel algorithm for efficiently computing the prices of options under the fractional Black-Scholes model, which has been validated through extensive simulations.
In this work, our objective is to investigate the approximate solution of the fractional Black-Scholes model derived by [8]

$$ {}_0^{CH}D_t^{\alpha} V(S,t) + \frac{\sigma^2}{2}S^2\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \quad 0 < \alpha < 1,\ (S,t) \in (0,+\infty) \times (0,T), \tag{2}$$

with the following terminal and boundary conditions

$$ V(S,T) = \eta(S), \tag{3}$$

and

$$ V(0,t) = \gamma(t), \quad V(\infty,t) = \zeta(t), \tag{4}$$

respectively. By substituting $\tau = T - t$, $x = \ln(S)$ and assuming $V(S,t) = u(x,\tau)$ the relations (2)–(4) convert to

$$ {}_0^{CH}D_t^{\alpha} u(x,\tau) = \frac{\sigma^2}{2}u_{xx}(x,\tau) + (r - \frac{\sigma^2}{2})u_x(x,\tau) - ru(x,\tau) + f(x,\tau), \quad (x,\tau) \in (a,b) \times (0,T), \tag{5}$$

with the following initial and boundary conditions

$$ u(x,0) = u_0(x), \tag{6}$$

and

$$u(a,\tau)=\gamma(\tau),\quad u(b,\tau)=\zeta(\tau). \tag{7}$$

In order to use the numerical solution, the previously infinite domain in (5) is restricted to a finite domain. A source term $f(x,\tau)$ is also added to obtain an exact solution for testing the numerical method.

## 2. Numerical Method

Let $t_n = n\Delta t$, $n = 0,1,\dots,N$, be a uniform partition of $[0,T]$ where $\Delta t = \frac{T}{N}$ denotes the temporal step size. To determine a semi-discrete formulation for (1), the fractional derivative can be estimated using the L1 method as follows:

$$\begin{aligned}
{}_a^{CH}D_t^{\alpha}f(t) &= \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(\ln t_n-\ln s)^{-\alpha}f'(s)ds\\
&= \frac{1}{\Delta t\Gamma(1-\alpha)}\sum_{i=0}^{n-1}(f(t_{i+1})-f(t_i))J_{i,n}+O(\Delta t)\\
&\approx \frac{1}{\Delta t\Gamma(1-\alpha)}\sum_{i=0}^{n-2}(f(t_{i+1})-f(t_i))J_{i,n}+\frac{1}{\Delta t\Gamma(1-\alpha)}(f(t_n)-f(t_{n-1}))J_{n-1,n},
\end{aligned} \tag{8}$$

where

$$J_{i,n}=\int_{t_i}^{t_{i+1}}(\ln t_n-\ln s)^{-\alpha}ds = t_n\left(\Gamma(1-\alpha,\ln(\frac{n}{i+1}))-\Gamma(1-\alpha,\ln(\frac{n}{i}))\right),$$

and $\Gamma(a,x)$ denotes the incomplete Gamma function $\Gamma(a,x)=\frac{1}{\Gamma(a)}\int_x^{\infty}t^{a-1}e^{-t}dt$ available numerically in MATLAB. It is worth noting that $\Gamma(a,0)=1$ and $\Gamma(a,\infty)=0$.
By applying the Crank-Nicholson parametric method to equation (5), we have

$$\begin{aligned}
&\frac{1}{\Delta t\Gamma(1-\alpha)}\sum_{i=0}^{n-1}(u(x,\tau_{i+1})-u(x,\tau_i))J_{i,n+1}+\frac{1}{\Delta t\Gamma(1-\alpha)}(u(x,\tau_{n+1})-u(x,\tau_n))J_{n,n+1}\\
&=\theta\left(\frac{\sigma^2}{2}u_{xx}(x,\tau_{n+1})+(r-\frac{\sigma^2}{2})u_x(x,\tau_{n+1})-ru(x,\tau_{n+1})\right)\\
&\quad+(1-\theta)\left(\frac{\sigma^2}{2}u_{xx}(x,\tau_n)+(r-\frac{\sigma^2}{2})u_x(x,\tau_n)-ru(x,\tau_n)\right)\\
&\quad+\theta f(x,\tau_{n+1})+(1-\theta)f(x,\tau_n)+O(\Delta t),
\end{aligned} \tag{9}$$

From relation (9), the following semi-discrete scheme is obtained

$$\begin{aligned}
&A_nu(x,\tau_{n+1})-Bu_{xx}(x,\tau_{n+1})-Cu_x(x,\tau_{n+1})\\
&=-\sum_{i=0}^{n-1}(u(x,\tau_{i+1})-u(x,\tau_i))J_{i,n+1}+D_nu(x,\tau_n)+Eu_{xx}(x,\tau_n)+Fu_x(x,\tau_n)+h_n(x),
\end{aligned} \tag{10}$$

where

$$A_n=\Delta t\Gamma(1-\alpha)\theta r+J_{n,n+1},\quad B=\Delta t\Gamma(1-\alpha)\theta\frac{\sigma^2}{2},\quad C=\Delta t\Gamma(1-\alpha)\theta(r-\frac{\sigma^2}{2}),$$

$$D_n=J_{n,n+1}-\Delta t\Gamma(1-\alpha)(1-\theta)r,\quad E=\Delta t\Gamma(1-\alpha)(1-\theta)\frac{\sigma^2}{2},\quad F=\Delta t\Gamma(1-\alpha)(1-\theta)(r-\frac{\sigma^2}{2}),$$

$$h_n(x)=\Delta t\Gamma(1-\alpha)\left(\theta f(x,\tau_{n+1})+(1-\theta)f(x,\tau_n)\right).$$

In this approach, the space derivatives are approximated by using cubic B-spline method. A mesh Ω which is equally divided by knots $x_i$ into $M$ subintervals $[x_i, x_{i+1}]$, $i = 0, 1, \dots, M-1$ such that $\Omega : a = x_0 < x_1 < \dots < x_M = b$ is used. Also, let $S_4(\Omega)$ be the space of cubic splines on Ω. The corresponding set of cubic B-splines $\{B_{-1}, B_0, \dots, B_{M+1}\}$, where is a basis for $S_4(\Omega)$, are defined as follows [9]

$$B_j(x) = \begin{cases} \frac{(x-x_j)^3}{(x_{j+1}-x_j)(x_{j+2}-x_j)(x_{j+3}-x_j)}, x \in [x_j, x_{j+1}), \\ \frac{(x-x_j)^2(x_{j+2}-x)}{(x_{j+2}-x_j)(x_{j+2}-x_{j+1})(x_{j+3}-x_j)} \\ + \frac{(x-x_j)(x_{j+3}-x)(x-x_{j+1})}{(x_{j+3}-x_j)(x_{j+3}-x_{j+1})(x_{j+2}-x_{j+1})} \\ + \frac{(x_{j+4}-x)(x-x_{j+1})^2}{(x_{j+2}-x_{j+1})(x_{j+3}-x_{j+1})(x_{j+4}-x_{j+1})}, x \in [x_{j+1}, x_{j+2}), \\ \frac{(x-x_j)(x_{j+3}-x)^2}{(x_{j+3}-x_j)(x_{j+3}-x_{j+1})(x_{j+3}-x_{j+2})} \\ + \frac{(x-x_{j+1})(x_{j+3}-x)(x_{j+4}-x)}{(x_{j+3}-x_{j+1})(x_{j+3}-x_{j+2})(x_{j+4}-x_{j+1})} \\ + \frac{(x_{j+4}-x)^2(x-x_{j+2})}{(x_{j+4}-x_{j+1})(x_{j+4}-x_{j+2})(x_{j+3}-x_{j+2})}, x \in [x_{j+2}, x_{j+3}), \\ \frac{(x_{j+4}-x)^3}{(x_{j+4}-x_{j+1})(x_{j+4}-x_{j+2})(x_{j+4}-x_{j+3})}, x \in [x_{j+3}, x_{j+4}), \\ 0, o.w., \end{cases}$$

for $j = -3, -2, \dots, M-1$. Now, we consider a linear combination of B-splines as an approximation of $u(x, \tau_n)$, $u_x(x, \tau_n)$ and $u_{xx}(x, \tau_n)$, as follows

$$u(x, \tau_n) = \sum_{k=-1}^{M+1} c_k^n B_k(x), \tag{11}$$

$$u(x, \tau_n) = \sum_{k=-1}^{M+1} c_k^n B_k'(x), \tag{12}$$

$$u(x, \tau_n) = \sum_{k=-1}^{M+1} c_k^n B_k''(x). \tag{13}$$

By substituting (11)–(13) into (10), we have

$$\sum_{j=-3}^{M-1} c_j^{n+1} P_j^n(x) = R^n(x), \tag{14}$$

where

$$\begin{aligned} R^n(x) &= J_{0,n+1} u(x, \tau_0) - \sum_{i=1}^{n-1} (J_{i-1,n+1} - J_{i,n+1}) u(x, \tau_i) \\ &\quad - D_n u(x, \tau_n) + E u_{xx}(x, \tau_n) + F u_x(x, \tau_n) + h_n(x), \\ P_j^n(x) &= A_n B_j(x) - B B''_j(x) - C B'_j(x). \end{aligned}$$

By placing the node points $x_k$, $k = 0, 1, \dots, M$ in relations (14) and using (7), we have:

$$\begin{aligned}
&\sum_{j=-3}^{M-1} c_j^{n+1} P_j^n(x_k) = R^n(x_k), \quad k = 0, 1, \dots, M, \\
&\sum_{j=-3}^{M-1} c_j^{n+1} B_j(x_0) = \gamma(\tau_{n+1}) \\
&\sum_{j=-3}^{M-1} c_j^{n+1} B_j(x_M) = \zeta(\tau_{n+1}).
\end{aligned} \tag{15}$$

Using relations (15), the following system is obtained for calculating the coefficients $c_j^{n+1}$'s:

$$\mathbf{M}^n C^{n+1} = \mathbf{R}^n, \tag{16}$$

where

$$C^{n+1} = \left(c_0^{n+1}, c_1^{n+1}, \dots, c_M^{n+1}\right)^T, \quad \mathbf{R}^n = \left(\gamma(\tau_{n+1}), R^n(x_0), R^n(x_1), \dots, R^n(x_M), \zeta(\tau_{n+1})\right)^T.$$

$$\mathbf{M}^n = \begin{pmatrix}
B_{-3}(x_0) & B_{-2}(x_0) & B_{-1}(x_0) & \cdots & B_{M-2}(x_0) & B_{M-1}(x_0) \\
P_{-3}(x_0) & P_{-2}(x_0) & P_{-1}(x_0) & \cdots & P_{M-2}(x_0) & P_{M-1}(x_0) \\
P_{-3}(x_1) & P_{-2}(x_1) & P_{-1}(x_1) & \cdots & P_{M-2}(x_1) & P_{M-1}(x_1) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
P_{-3}(x_M) & P_{-2}(x_M) & P_{-1}(x_M) & \cdots & P_{M-2}(x_M) & P_{M-1}(x_M) \\
B_{-3}(x_M) & B_{-2}(x_M) & B_{-1}(x_M) & \cdots & B_{M-2}(x_M) & B_{M-1}(x_M)
\end{pmatrix}.$$

It is worth noting that the values of $u_x$ and $u_{xx}$ at $t_0$ are obtained using the initial condition (6).

## 3. Numerical results

**Example 3.1.** Let us consider $\sigma = 0.25$, $\theta = 1$, $r = 0.05$, $T = 1$, $a = 0$, $b = 1$, $\gamma(\tau) = \zeta(\tau) = 0$ with

$$\begin{aligned}
f(x,t) &= (t+1)^2\Big(-\frac{\sigma^2}{2}(2-6x) - (r-\frac{\sigma^2}{2})(2x-3x^2) + rx^2(1-x)\Big) \\
&\quad + \frac{2x^2(1-x)}{\Gamma(1-\alpha)}(t^2 2^{\alpha-1} + t). \\
u_0(x) &= x^2(1-x),
\end{aligned}$$

From this assumptions the exact solution can be calculated by $u(x,t) = x^2(1-x)(1+t)^2$ [9].
The figure of the approximate solution and the absolute error at all node points with $N = 1000$ and $M = 100$ are shown in Fig. 1. The versatility and the accuracy of the method is measured using the maximum absolute error norm $L_\infty$, defined by

$$L_\infty(M,N) = \max_{0 \le i \le M} \left|U(x_i, t_N) - U_{exact}(x_i, t_N)\right|,$$

where $U(x_i, t_N)$ and $U_{exact}(x_i, t_N)$ are numerical and exact solutions at $(x_i, t_N)$ point. The time convergence orders are defined as follows

$$\text{Rate}^t = \frac{\log\left(L_\infty(M, N_1)\right) - \log\left(L_\infty(M, N_2)\right)}{\log(N_2) - \log(N_1)}.$$

Table 1 shows the method error along with the temporal order error and the algorithm execution time in seconds.

According to Table 1, it can be seen that the temporal order of the method is about 1.5.

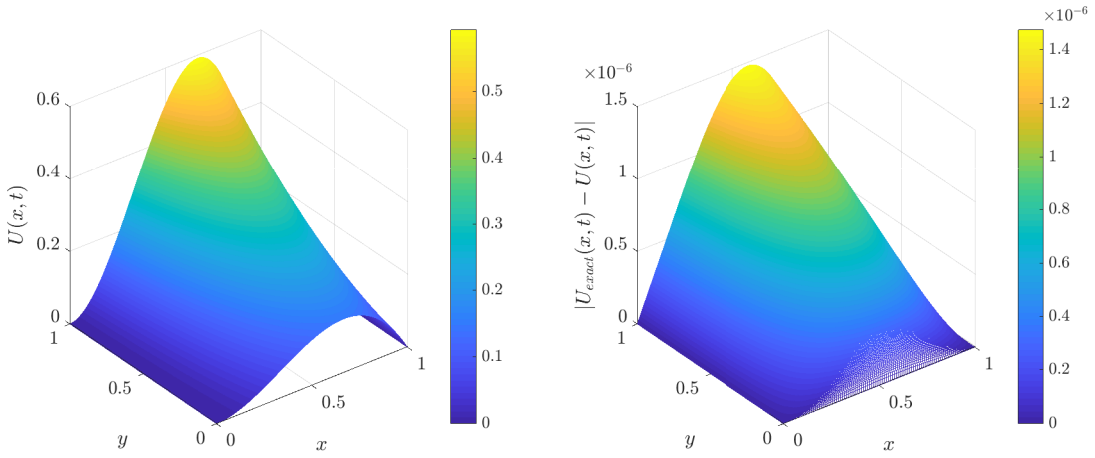


Fig. 1. The graph of the approximate solution (left) and the absolute error at all node points (right) for example 3.1 with $N = 1000$ and $M = 100$.

Table 1. The maximum-norm errors and temporal convergence orders with $M = 100$ for Example 3.1.

| $N$ | $L_\infty(M,N)$ | Rate$^t$ | CPU time(S) |
|---|---|---|---|
| 4 | $5.215266e-03$ | – | 0.15 |
| 8 | $1.925455e-03$ | 1.4375 | 0.24 |
| 16 | $6.986236e-04$ | 1.4626 | 0.49 |
| 32 | $2.509957e-04$ | 1.4769 | 1.14 |
| 64 | $8.965026e-05$ | 1.4853 | 3.25 |
| 128 | $3.190647e-05$ | 1.4905 | 12.55 |
| 256 | $1.132981e-05$ | 1.4937 | 20.73 |
| 512 | $4.017290e-06$ | 1.4958 | 64.96 |
| 1024 | $1.423080e-06$ | 1.4972 | 243.57 |
| 2048 | $5.037922e-07$ | 1.4981 | 944.37 |

## 4. Conclusion

This paper presents a novel approach to solving the Black-Scholes model using fractional derivatives of the Caputo-Hadamard type. Using the L1-type approximation for the Caputo-Hadamard derivative and the collocation method with cubic B-spline bases has proven to be effective in approximating the spatial derivatives. The numerical results demonstrate the accuracy of the proposed method, with the implemented algorithm exhibiting high execution speed. These findings indicate that the presented method is not only accurate but also efficient, making it a promising tool for option pricing and other financial applications. Future research could explore further enhancements to the algorithm and its applicability to more complex financial models.

It is notable that, we perform all of the computations by MATLAB® R2019a software (V9.6.0.1072779, 64-bit (win64), License Number: 968398, MathWorks Inc., Natick, MA) running on a Sony VAIO Laptop (Intel® Core™ i5-2410M Processor 2.30 GHz with Turbo Boost up to 2.90 GHz, 8 GB of RAM, 64-bit) PC.

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# Covering mappings in Heyting algebra models and their applications in intuitionistic logic

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**Abstract**

This study examines the intrinsic connections between covering spaces and Heyting algebras, presenting significant theoretical and practical advancements. A notable finding reveals that covering maps serve as a fundamental model for constructive operations in intuitionistic logic. Additionally, intuitive negation in topology operates through the interaction between the base space and its universal cover, effectively demonstrated in covering structures, emphasizing the interplay between local and global perspectives. This paper highlights the importance of the deck group as a pivotal tool for assessing the preservation of topological properties during continuous transformations, thereby showcasing the broader applicability of the covering space theory. Ultimately, this research establishes a vital link between algebraic topology and intuitionistic logic, paving the way for future explorations in mathematics and interdisciplinary fields.

## 1. Introduction

Algebraic topology, particularly through the use of groups, is a branch of mathematics that studies topological spaces and their properties under continuous transformations. It seeks to understand the properties of spaces that remain invariant under such transformations [5].

*Definitions and Properties of Covering Maps*

**Definition 1.1.** A **covering map** $p : \tilde{X} \to X$ is a continuous function that satisfies the following two conditions:

1. **Uniform Covering:** Every point $x \in X$ has an open neighborhood $U$ such that $p^{-1}(U) = \bigsqcup_{i\in I} V_i$, where each $V_i$ is an open subset of $X$ and $p|_{V_i} : V_i \to U$ is a homeomorphism.
2. **Discrete Fibers:** For every $x \in X$, the fiber $p^{-1}(x)$ is a discrete set in $X$ (i.e., each point in the fiber has a distinct open neighborhood).

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*Key Properties*

1. **Path Lifting:** For any path $\gamma : [0,1] \to X$ and any point $\tilde{x}_0 \in p^{-1}(\gamma(0))$, there exists a unique path $\tilde{\gamma} : [0,1] \to X$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$.
2. **Deck Transformation Group:** The group of automorphisms of $X$ that commute with $p$ is called the **deck group**, denoted $\mathrm{Deck}(\tilde{X}/X)$. This group is isomorphic to the fundamental group $\pi_1(X)$.
3. **Universal Cover:** If $X$ is simply connected (i.e., $\pi_1(X) = 0$), then $X$ is called the **universal cover**. This cover is unique and can be used to derive all other covers (up to homeomorphism).

*Classic Example*

The map $p : \mathbb{R} \to S^1$ defined by $p(t) = e^{2\pi i t}$ is a universal cover of the unit circle $S^1$. Every point in $S^1$ has a neighborhood $U$ such that $p^{-1}(U)$ is a disjoint union of open intervals in $\mathbb{R}$.(See [6].)

*The Role of Covering Maps in Algebraic Topology*

Covering maps are essential tools for studying homotopy groups and their global structures.

1. **Connection with the Fundamental Group:** If $p : \tilde{X} \to X$ is a universal cover, then the deck group $\mathrm{Deck}(\tilde{X}/X)$ is isomorphic to $\pi_1(X)$.
2. **Lifting of Homotopies:** Any homotopy in $X$ can be uniquely lifted to $\tilde{X}$. This property is useful for proving theorems such as the homotopy lifting theorem and is also relevant in the context of category theory.
3. **Computation of Higher Homotopy Groups:** For covering spaces, the higher homotopy groups $\pi_n(X)$ coincide with $\pi_n(\tilde{X})$.

*Historical Connection with Mathematical Logic through duality (e.g., Stone and Esakia)*

Intuitionistic logic, which emphasizes constructive proofs, can be traced back to classical duality. These duality often contrast classical logic principles, such as the law of the excluded middle, with intuitionistic views that reject such principles [1]. This historical context is essential for understanding the evolution of logical frameworks.

The work of philosophers like Brouwer, Heyting, and Kolmogorov laid the groundwork for intuitionistic logic, which fundamentally differs from classical logic. Their interpretations highlight the importance of constructive methods in proving the existence of mathematical objects, marking a significant shift in logical thought [1].

The duality of Stone and Esakia illustrate the tension between classical and intuitionistic approaches, emphasizing how different perspectives on truth and proof have shaped mathematical logic. Their contributions have influenced the development of various logical systems [1].

*Heyting Algebra and Intuitionistic Logic*

Heating Algebra act as algebraic structures that encapsulate the principles of intuitionistic logic, providing a framework for constructive reasoning. This aligns with the fundamental ideas of intuitionism, which prioritize constructive proofs over non-constructive methods in classical logic [3].

These Algebra can be viewed as models that reflect the behavior of logical connectives in intuitionistic logic, offering a structured system for exploring how these connectives interact. This interaction is crucial for a deeper understanding of the logical implications and their constructive nature [3].

The concept of Heyting Algebra emphasizes continuity and transformation in logical reasoning, paralleling the intuitionistic approach to truth. This perspective highlights the dynamic nature of logical operations, in contrast to the static interpretations often found in classical logic [3].

*Problem and Objectives*

Topological coverings as tools for analyzing and transferring information in Heyting algebra models.
Topological coverings can serve as a framework for understanding the relationships between different Heyting algebra models and facilitate the transfer of information between them. This is particularly relevant in the context of intuitionistic logic, where the structure of the models can significantly influence the interpretation of logical statements [4].
The use of coverings allows for the exploration of how the local properties of Heyting Algebra can impact global structures, providing insights into the behavior of logical operations within these models. This can enhance our understanding of how information is preserved or transformed during logical inference [4].
Analyzing coverings in Heyting algebra models can reveal the underlying topological properties that govern interactions between different logical structures, offering a more nuanced perspective on the semantics of intuitionistic logic . This approach may lead to new methods for reasoning about logical consequences and their representations.

*Connection between Covering Theory and Intuitionistic Logic*

Coverings in topology, such as Ã©tale coverings, can be understood through intuitionistic logic, which emphasizes constructive proofs and the existence of mathematical objects. This perspective aligns with the need for explicit structures in covering spaces, where one must demonstrate the existence of local entities [2].
The law of the excluded middle, often used in classical logic, contrasts with intuitionistic logic, which requires a more constructive approach. In the context of coverings, this means that one must provide examples with explicit structures rather than relying on non-constructive existence proofs. This is particularly important in the study of Galois coverings, where the relationships between field extensions must be explicitly defined [2].

*Heyting Algebra and Intuitionistic Logic*

A Heyting algebra is an algebraic structure that serves as a mathematical model for intuitionistic logic. In intuitionistic logic, unlike classical logic, the law of the excluded middle ($A \vee \neg A = T$) does not necessarily hold. In other words, there are propositions that cannot be proven or disproven. This idea is reflected in the definition of logical operations in Heyting algebras, where conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$) are defined structurally without necessarily satisfying $\neg\neg A = A$.

*Difference from Boolean Algebras*

In Boolean algebras, every element $A$ has a unique complement $\neg A$ such that $A \vee \neg A = T$ and $A \wedge \neg A = F$. However, in Heyting algebras, the complement $\neg A$ is defined as the largest element that does not intersect with $A$. This definition means that $\neg\neg A$ does not necessarily equal $A$.

**Example 1.2.** The set of open sets $\mathrm{Open}(X)$ of a topological space $X$ forms a Heyting algebra. Here:

- Disjunction ($\vee$) corresponds to the union of open sets.
- Conjunction ($\wedge$) corresponds to the intersection of the open sets.
- Intuitionistic negation ($\neg$) corresponds to the interior of the complement, i.e., $\neg U = \mathrm{Int}(X \setminus U)$.

For example, if $U$ is an open set, $\neg U$ consists of points that have no neighborhood entirely contained in $U$.

*Connection between Covering Maps and Heyting Algebras*

This continuous map covers $X$ locally and uniformly through sheets ($\tilde{X}$). The Heyting algebra associated with $\tilde{X}$, $\mathrm{Open}(\tilde{X})$, can provide richer information than $\mathrm{Open}(X)$.

*Inverse Map $p^{-1}$: Extending Structures*

The inverse map $p^{-1} : \text{Open}(X) \to \text{Open}(\tilde{X})$ is a homomorphism that preserves the following operations:

1. $p^{-1}(U \vee V) = p^{-1}(U) \vee p^{-1}(V)$
2. $p^{-1}(U \wedge V) = p^{-1}(U) \wedge p^{-1}(V)$
3. $p^{-1}(\neg U) = \neg p^{-1}(U)$

However, $p^{-1}$ is not a complete homomorphism on negation because $p^{-1}(\neg U)$ may be larger than $\neg p^{-1}(U)$. This difference reflects the fact that $\tilde{X}$ may have a more complex topological structure than $X$.

*Representation of Logical Formulas in the Universal Cover*

If $X$ is simply connected (i.e., it has no non-trivial covers), then the universal cover $\tilde{X}$ allows the representation of all logical formulas expressible in $\text{Open}(X)$. This is due to two key properties:

1. **Absence of Loops:** The universal cover "unfolds" all loops in $X$, converting them into paths in $\tilde{X}$.
2. **Localness:** Any logical statement in $X$ can be locally represented in $\tilde{X}$, as $\tilde{X}$ is locally homeomorphic to $X$.

*Classic Example: $X = S^1$ and $\tilde{X} = \mathbb{R}$*

Consider a statement like "the point is in the upper half of the circle" for $S^1$. In $\mathbb{R}$, $p^{-1}(U)$ is the union of intervals $(k - \frac{1}{4}, k + \frac{1}{4})$ for all $k \in \mathbb{Z}$. While this statement is global in $S^1$ (since the circle is compact), it is locally defined in $\mathbb{R}$ for each interval.

The intuitionistic negation $\neg U$ in $S^1$ corresponds to the interior of the complement, i.e., $\neg U = \text{Int}(S^1 \setminus U)$, which is a smaller interval in the lower half of the circle. In $\mathbb{R}$, $p^{-1}(\neg U)$ consists of points that have no neighborhood entirely contained in $p^{-1}(U)$. This set is more complex than $\neg p^{-1}(U)$, reflecting the linear structure of $\mathbb{R}$.

## 2. Role of the Universal Cover in Complete Logical Representation

The universal cover $\tilde{X}$ resolves all the topological ambiguities in $X$, allowing the Heyting algebra $\text{Open}(\tilde{X})$ to:

- Represent all possible paths in $X$ as unique paths in $\tilde{X}$.
- Encode all possible states of open sets in $X$ across different sheets of $\tilde{X}$.
- Define intuitionistic negation more precisely, as there are no unexpected connections in $\tilde{X}$.

*Complex Example*

If $X$ is a hyperbolic space (e.g., the upper half-plane $\mathbb{H}^2$), then the universal cover $\tilde{X}$ depends on the hyperbolic geometry of $X$, allowing for a simpler representation of the geometric structures.

*Summary: Topology, Logic, and the Transfer of Structures*

Covering maps act as a bridge between topology and logic. The Heyting algebra $\text{Open}(X)$ provides a local model for reasoning in $X$, while $\text{Open}(\tilde{X})$ enables global inferences. The universal cover maximizes the topological information, reflecting all logical operations in $X$.

This deep connection unifies seemingly disparate areas of mathematics, such as algebraic topology and logic, highlighting the profound interplay between abstract structures and practical applications.

### *2.1. Impact of Coverings on the Logical Properties*

#### *2.1.1. Enhancing Consistency and Completeness in Intuitive Systems*

Local systems can utilize homomorphisms to adapt to environmental changes, making intuitive systems more responsive and dynamic. This adaptability is crucial for maintaining functionality across various contexts [4].

#### *2.1.2. Improving Model Accuracy and Reliability*

The consistency of Boolean models enhances the accuracy of complex propositions, ensuring reliability in various applications [3].

#### *2.1.3. Integration of Components*

The use of homomorphisms in the system structures facilitates the integration of different components, creating a more cohesive system. This integration addresses new challenges and requirements, leading to more robust solutions [2].

### *2.2. Conclusion*

This research highlights the key role of covering maps and Heyting Algebra in analyzing topological structures and modeling intuitionistic logic. The findings demonstrate how covering maps facilitate the transfer of information between local and global spaces, providing an efficient framework for modeling constructive operations in intuitionistic logic. The connection between the deck group and the fundamental group of spaces offers a better understanding of how topological properties are preserved under continuous transformations. Furthermore, the representation of Heyting algebras in universal covers (e.g., $\mathbb{R} \to S^1$) strengthens the interplay between local and global concepts in intuitionistic logic, showing how intuitive negation is reflected in topological structures.

### *2.3. Implications of the Results*

These results have broad implications for mathematics, logic, and computer science. In intuitionistic logic, the use of covering aids in developing more constructive reasoning systems, where mathematical objects are explicitly defined based on local structures. In computer science, this approach can improve the design of formal verification algorithms and enhance uncertainty management in machine learning through topological modeling. The unification of the algebraic topology and Heyting Algebra opens new horizons for studying the interaction between geometry and logic.

### *2.4. Future Research Directions*

1. **Generalization to Multi-valued and Fuzzy Logic Systems:** Exploring how topological coverings can be generalized to model more complex logical systems with multi-valued and fuzzy truth values.
2. **Applications in Artificial Intelligence:** Using topological concepts to improve the interpretability of deep learning models and manage sparse data.
3. **Connecting Coverings with Topos Theory:** Investigating the relationship between coverings and topos theory to develop more comprehensive frameworks for intuitionistic logic.
4. **Studying Non-Euclidean Coverings:** Analyzing coverings with non-Euclidean deck groups and their impact on the structure of Heyting Algebra in non-Euclidean spaces.

This research not only bridges algebraic topology and intuitionistic logic but also provides a theoretical foundation for future innovations in mathematics, emphasizing the enduring importance of the interaction between abstract structures and practical applications in advanced technologies for solving real-world problems.

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# Statistical inference for the Burr X distribution under various progressive Type-II censoring schemes

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**Abstract**

Among the various families of Burr distributions introduced by Burr (1942), the Burr X and Burr XII distributions have received the most attention for modeling data across multiple fields. The Burr X distribution, also known as the generalized Rayleigh distribution, has been widely used for modeling lifetime data under both complete and censored conditions.
In recent life-testing experiments, progressive Type-II censoring has been extensively applied to various statistical problems. However, a major drawback of this method is the potentially long duration of the experiment. To address this issue, researchers have introduced progressive Type-II hybrid censoring and, more recently, adaptive progressive Type-II censoring.
This study aims to evaluate the advantages and disadvantages of these censoring methods under the Burr X distribution. To achieve this, we will derive the Maximum Likelihood and Bayesian estimators for the unknown parameters using progressively Type-II, progressive Type-II hybrid, and adaptive progressive Type-II censored data. The efficiency of these estimators will be assessed through a comprehensive simulation study. Additionally, we will construct asymptotic, percentile bootstrap, and highest posterior density (HPD) intervals for the unknown parameters and evaluate their effectiveness through another simulation study.

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# Two empirical goodness-of-fit tests for Rayleigh distribution based on progressively first-failure-censored data

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**Abstract**

In this article, we introduce two goodness-of-fit test statistics for the Rayleigh distribution, specifically designed for data obtained from a life testing plan known as the progressive first-failure censoring scheme, as proposed by Wu and Kuş (2009). In this censoring scheme, the experimenter organizes the test units into several sets and conducts tests on all units simultaneously. The life test concludes after observing $m$ failures among all the units involved in the test.

When the first failure occurs, $R_1$ groups, chosen randomly, along with the group that experiences the first failure, are withdrawn from the test. When the second failure occurs, $R_2$ groups, chosen randomly, along with the group that experiences the second failure, are withdrawn from the test, and so on. Finally, after the $m$-th failure is recorded, all remaining groups are eliminated from the test, concluding the life test. Note that the values of $m$ and $(R_1, \ldots, R_m)$ are pre-determined.

The first proposed goodness-of-fit test statistic is based on the sample correlation coefficient between the observed lifetime data and the logarithm of their Kaplan-Meier estimator of the survival function. The second test statistic is derived from the sample correlation between the observed lifetime data and their Nelson-Aalen estimator of the cumulative hazard function.

The performance of these test statistics will be assessed against various alternative distributions through a Monte Carlo simulation study. The results demonstrate that both test statistics exhibit good performance in detecting deviations from the Rayleigh distribution under the alternative hypotheses.

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# Stress-strength reliability estimation for Burr Type X distribution in the presence of one outlier

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**Abstract**

The stress-strength parameter $\xi = P(X < Y)$, first proposed by Birnbaum (1956), where the random variables $X$ and $Y$ represent stress and strength, respectively, is widely recognized in statistical research as an indicator of system efficiency. The estimation of $\xi$ for various families of probability distributions has been extensively studied in the literature. While existing methods for estimating $\xi$ offer practical advantages, they often lack stability and robustness in the presence of outliers and extreme values.

Practical studies in reliability and stress-strength contexts reveal that outliers can contaminate the variables $X$ and $Y$, particularly due to noise in life testing processes. In this work, we address the challenge of estimating $\xi$ when the random variables $X$ and $Y$ are independent and subject to outliers. Recent advancements have introduced several methods and statistical models for outlier detection and robust inference. We will focus on the well-known Dixit model (Dixit, 1987) as a powerful approach for modelling outliers.

Moreover, because of the lack of the estimation of $\xi$ under the Burr Type X model, the estimation of $\xi$ under this model is the other main motivation of the present work. We assume that $X$ follows a contaminated Burr Type X distribution with one outlier, while $Y$ adheres to an uncontaminated Burr Type X distribution. In other words, $Y$ has the Burr type X distribution with parameter $\alpha$ and $X$ has the Burr type X distribution with presence of one outlier with parameters $\beta$ and $\gamma$.

We will discuss the maximum likelihood estimator for $\xi$ using conventional techniques and propose a Bayesian estimator with independent gamma priors. Bayesian estimators will be derived using both symmetric and asymmetric loss functions. Finally, a Monte Carlo simulation study will be conducted to evaluate the performance of the proposed estimators.

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# Advanced Topics in Covering Maps: Semicoverings and Their Fundamental Groups

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**Abstract**

This paper explores and develops concepts of coverings and semicoverings in algebraic topology. A key issue in this field is understanding the structure of topological fundamental groups of topological spaces and their relationship with generalized classical coverings. The paper focuses on semicoverings and the properties of topological fundamental groups ($\pi_1^{qtop}$ and $\pi_1^{sc}$), introducing tools for analyzing non-homotopically Hausdorff spaces. The methods used include generalizing classical coverings to semicoverings and studying fundamental groups with appropriate topologies applied to them. Concepts such as wep spaces and small loop transfer properties are utilized to analyze subgroups and their relationships with the existence of semicoverings. The main results include identifying necessary and sufficient conditions for the existence of semicoverings on various spaces and extending covering theory to non-homotopically Hausdorff spaces. This research innovates by generalizing classical covering concepts to a broader class of topological spaces. Applications of these results are seen in solving problems related to shape groups and generalized universal coverings. Additionally, this study can provide practical applications in the analysis of complex topological spaces and their fundamental group properties.

## 1. Introduction

Algebraic topology studies topological properties using algebraic methods, and covering maps play a crucial role in understanding spaces and fundamental groups. Covering maps, defined as local homeomorphisms, facilitate the exploration of fundamental groups through their relationship with universal covering spaces and the interplay between topology and algebra. However, the conditions for the existence of covering maps may sometimes not be met, leading to the introduction of semicoverings as a generalization. Semicoverings, introduced by Brazas et al., help explore fundamental groups in unusual spaces due to their flexible base space requirements, aiding in solving problems in algebraic topology and fostering a deeper understanding of spatial topological relationships.

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### *1.1. Definition of Covering Maps and Their Importance in Algebraic Topology*

Covering maps and their importance in algebraic topology are defined, with semicoverings serving as generalizations applicable to non-locally connected spaces. A semicovering $p : \tilde{X} \to X$ allows for local estimations of paths in $X$ under weaker separation conditions than traditional covering maps. This property aids in understanding local topological behavior [2]. Covering maps significantly contribute to the study of fundamental groups ($\pi_1(X)$), facilitating the lifting of paths and homotopies for analyzing algebraic structures in topological contexts [2]. The concept of local homeomorphism in covering maps enhances the decomposition of topological spaces and structural analysis [2]. Covering maps find applications in group theory, differential geometry, and physics, illuminating interactions between groups and spatial actions and modeling complex systems [2].

### *1.2. Introduction to Semicoverings (Semicoverings)*

Semicoverings offer a novel approach in algebraic topology for studying complex topological spaces. Unlike classical covering maps restricted to locally connected spaces, semicoverings include non-connected spaces, exemplified by the ”Hawaiian earring” space, emphasizing the necessity of semicoverings [3]. Furthermore, semicoverings advance the study of generalized fundamental groups, with the semicovering fundamental group ($\pi_1^{sc}(X)$) extending classical groups to include ”semicovering paths,” potentially generating non-abelian groups even in simpler spaces [5]. Semicoverings have broad implications in group theory and geometry, aiding in the analysis of fundamental groups of complex spaces, especially in cases where traditional methods are insufficient for infinite-loop spaces [7, 8]. In differential geometry, semicoverings are relevant for studying non-smoothly connected manifolds [7].

### *1.3. Objective of the Paper*

The primary objective of this paper is to examine the structure of semicovering fundamental groups and investigate the necessary and sufficient conditions for their existence. This exploration is critical for understanding the algebraic properties of topological spaces [2]. The focus will be on the applications of semicoverings in group theory and geometry, providing insights into how these concepts can be utilized across various mathematical fields [4]. The paper aims to present new findings regarding semicoverings and compare these results with previous research to highlight advancements in covering theory [6].

## 2. Theoretical Foundations

A covering map is formally defined as a surjective continuous function $p : \tilde{X} \to X$ such that for every point $x \in X$, there exists an open neighborhood $U$ where the preimage $p^{-1}(U)$ consists of disjoint sets in $\tilde{X}$, each homeomorphic to $U$ [2]. Key features of covering maps include:

- **Local Isomorphisms:** Every point in the base space has a neighborhood evenly covered by the covering map, facilitating local homeomorphisms [1, 10].
- **Discrete Fibers:** The preimage of any point in the base space is discrete, indicating that fibers are both countable and separable [6].
- **Connections to Fundamental Groups:** Covering maps allow for the lifting of paths and homotopies, essential for analyzing the fundamental group $\pi_1(X, x_0)$ [6, 8].

Examples include:

- The covering map from $\mathbb{R}$ to $S^1$, where the real line wraps around the circle [6].
- The covering map from $S^n$ to $\mathbb{RP}^n$, showing how spheres can cover projective spaces [6].

*Fundamental Groups and Their Relationship to Covering Maps*

The fundamental group $\pi_1(X, x_0)$ is defined as the set of equivalence classes of loops based at $x_0$ under concatenation and possesses properties such as being a topologically unimodular group [6]. Fundamental groups play a central role in classifying covering maps, providing insights into the structural properties of a space through its loops and paths [6]. A classical theorem asserts that every covering map corresponds to a normal subgroup of the fundamental group, establishing a direct correlation between the algebraic structure of the group and the topological properties of the space [2, 9].

Studying covering maps has long been a cornerstone of algebraic topology, bridging local and global topological features. Classical covering maps impose strict conditions on the base space, such as local path-connectedness and semi-local simple connectivity. These conditions limit their applicability in more complex spaces like the Hawaiian earring or spaces lacking local path-connectedness. Semicoverings, introduced by J. Brazas [6][1], generalize covering maps by reducing the requirement for uniformly covered neighborhoods while preserving key lifting properties. This report combines recent advances in semicovering theory, focusing on their relationship with fundamental groups, classification via subgroups, and applications in non-classical spaces.

*Formal Definition of Semicoverings*

A semicovering $p : Y \to X$ is defined as a local homeomorphism admitting unique lifts of paths and homotopies relative to the basepoint [2, 8]. Formally:

- **Local Homeomorphism:** For every $y \in Y$, there exists an open neighborhood $U \subseteq Y$ of $y$ such that $p|_U : U \to p(U)$ is a homeomorphism.
- **Unique Path Lifting:** For every path $\alpha : I \to X$ starting at $p(y)$, there exists a unique lift $\widetilde{\alpha} : I \to Y$ starting at $y$.
- **Unique Homotopy Lifting:** For every homotopy $H : I \times I \to X$ with $H(0, t) = p(y)$, there exists a unique lift $\widetilde{H} : I \times I \to Y$ starting at $y$.

Unlike classical covering maps, semicoverings do not require uniformly covered neighborhoods, allowing them to apply in spaces where classical coverings fail. For example, the Hawaiian earring $\mathbb{HE}$, a compact, path-connected but not semi-locally simply connected space, admits semicoverings that are not coverings [1][3].

*Relationship to Covering Maps*

Every covering map is a semicovering, but the converse holds only under additional conditions. If $X$ is locally path-connected and semi-locally simply connected, semicoverings coincide with coverings [6][1]. Compositions of semi-coverings remain semicoverings, unlike coverings, which may lose covering properties under composition [1]. This closure under composition highlights the flexibility of semicoverings in managing complex mappings.

*Fundamental Groups and Classification of Semicoverings*

**Quasi-Topological Fundamental Group:** The quasi-topological fundamental group $\pi_1^{qtop}(X, x_0)$, equipped with the quotient topology inherited from the space of loops, plays a central role in classifying semicoverings. A subgroup $H \subseteq \pi_1^{qtop}(X, x_0)$ is open if it corresponds to a semicovering $p : Y \to X$ via an isomorphism $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ [6][4]. This classification mirrors classical covering theory but extends to spaces lacking local path-connectedness.

**Spanier Groups and Open Subgroups:** For spaces lacking local path-connectedness, Spanier groups generalize classical Spanier groups by considering open covers of the path space $(PX)_{x_0}$. Given an open cover $\mathcal{U}$ of $(PX)_{x_0}$, the Spanier group $\pi^{ps}(\mathcal{U}, x_0)$ is generated by classes $[\alpha \cdot \beta^{-1}]$ where $\alpha, \beta \in U$ for some $U \in \mathcal{U}$ [6]. These groups identify open subgroups of $\pi_1^{qtop}(X, x_0)$, enabling classification of semicoverings in non-standard settings.

**Theorem ([6]):** Suppose $X$ is locally wep-connected (locally "weakly eventually path-connected"). A subgroup $H \subseteq \pi_1(X, x_0)$ is a semicovering subgroup if and only if $H$ is open in $\pi_1^{qtop}(X, x_0)$.

This theorem emphasizes the duality between semicoverings and open subgroups, akin to the Galois correspondence for coverings.

*Structure of Semicovering Fundamental Groups*

The semicovering fundamental group, denoted $\pi_1^{sc}(X)$, acts as an extension of the classical fundamental group, incorporating paths associated with semicoverings and facilitating comprehensive exploration of loop structures in topological spaces [6]. A notable feature of $\pi_1^{sc}(X)$ is its potential stability even in elementary spaces, indicating that the configuration of the fundamental group can exhibit flexibility under specific conditions [3]. A deep connection exists between $\pi_1^{sc}(X)$ and divisible groups, revealing valuable insights into the algebraic properties of these groups [3].

## 3. Applications of Semicoverings

*Applications in Group Theory*

Semicoverings play a pivotal role in group theory, particularly in analyzing fundamental groups of complex topological spaces, facilitating a deeper understanding of behaviors exhibited by these groups under various topological conditions. This is especially useful when dealing with spaces that evade representation through simpler structures [6]. For instance, analyzing fundamental groups related to spaces with infinitely many loops, such as the Hawaiian earring, exemplifies this application. The inherent complexity of such spaces necessitates the use of semicoverings to skillfully investigate their fundamental group configurations [6].

*Applications in Geometry*

In differential geometry and manifold analysis, semicoverings can illuminate local properties of spaces that may not conform to manifold characteristics. This is particularly useful in scenarios where standard covering maps may be insufficient due to the intrinsic complexity of the space [6]. A representative example involves studying semicoverings within non-manifold spaces that exhibit local connections. Such spaces may display behaviors challenging established geometric paradigms, and semicoverings provide a suitable framework for evaluating their properties [6].

## 4. Conclusion

This paper examines and develops concepts of coverings and semicoverings in the context of algebraic topology. Key findings include identifying necessary and sufficient conditions for the existence of semicoverings on various topological spaces, extending covering theory to non-homotopically Hausdorff spaces, and investigating the relationship between topological fundamental groups ($\pi_1^{qtop}$ and $\pi_1^{sc}$) and semicovering structures. Additionally, the paper explores the role of wep spaces in generalizing classical covering theory to a broader class of topological spaces. The results demonstrate that semicoverings can serve as powerful tools for analyzing the fundamental group structures of complex spaces.

Limitations and Future Directions: While this research advances covering theory significantly for non-homotopically Hausdorff spaces, further investigation is needed. For example, a more detailed examination of semicovering behavior in specific spaces or extending this theory to higher-dimensional contexts remains open. Exploring practical applications of semicoverings in knot theory or topological space classification could lead to new discoveries. By developing semicovering concepts and analyzing fundamental group structures in topological spaces, this paper contributes meaningfully to the advancement of covering theory and its applications in algebraic topology, paving the way for further research in this field.

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# On the Cayley Graph of Symmetric Groups

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**Abstract**

Let $S_n$ be the symmetric group of degree $n$. We classify non-isomorphic Cayley graphs of $S_4$ of valency 3. We verify that there are only 10 non-isomorphic valency 3 Cayley graphs of $S_4$.

## 1. Introduction

The Cayley graph of the symmetric group $S_n$ and alternating group $A_n$ have been studied in several papers. In [3] the authors have shown that there are exactly 22 non-isomorphic Cayley graphs of $A_4$. The number of undirected Cayley graphs of $S_n$ and $A_n$ have determined by Adiga and Ariamanesh in [1]. They also have shown that there are only 8 Cayley graphs of $S_3$ and 4 Cayley graphs of $S_4$ of valency 2, up to isomorphism.
We investigate the Cayley graphs of symmetric group and specially the valency 3 Cayley graphs of $S_4$. Moreover, we study the CI-graphs of the Cayley graphs of $S_4$ of valency 3. Also, we find a relation between the Cayley graph of $S_{n+1}$ and the Cayley graph of $S_n$ for a fixed subset $S$.
Suppose that $G$ is a finite group with the identity $1_G$. Also, suppose that $S$ is a subset of the group $G$, such that $1_G \notin S$ and $S$ is an inverse-closed subset, i.e. $S = S^{-1}$, where $S^{-1} := \{s^{-1} | s \in S\}$. Then, the associated Cayley graph, denoted by $Cay(G,S)$, is a graph such that its vertices are the members of $G$ and two distinct vertices $g, h \in G$ are adjacent if and only if $gh^{-1} \in S$. The number $|S|$ is called the valency of the Cayley graph. Since $S$ is an inverse-closed subset of G, so $Cay(G,S)$ is an undirected graph.

**Proposition 1.1.** *[2] Suppose that $S$ is a subset of the group $G$, such that $1_G \in S$ and $S = S^{-1}$ and $|S| = m$. Then $Cay(G,S)$ is an $m-$regular graph.*

It is easy to see that a Cayley graph $Cay(G,S)$ is connected if and only if $G =< S >$, i.e. $S$ is a generating set of the group $G$. So, if $G \neq< S >$, then $Cay(G,S)$ is disconnected.

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**Proposition 1.2.** *[2] Suppose that $S$ is a subset of the group $G$, such that $1_G \notin S$ and $S = S^{-1}$. Then all components of $Cay(G, S)$ are isomorphic to $Cay(< S >, S)$.*

The elements of $S_4$ with the cycle type are:

$$\{(1); (12); (13); (14); (23); (24); (34); (123); (132); (124); (142); (134); (143); (234);$$
$$(1234); (1432); (1243); (1342); (1324); (1423); (12)(34); (13)(24); (14)(23)\}$$

## 2. Main Results

**Lemma 2.1.** *All possible subsets $S$ of $S_4$, such that $|S| = 3$, $(1) \in S$ and $S = S^{-1}$, up to isomorphism, are as following:*

| *No.* | *The subset $S$ up to isomorphism* |
|---|---|
| *1* | $\{(ij), (il), (jl)\}$ |
| *2* | $\{(12)(34), (14)(23), (13)(24)\}$ |
| *3* | $\{(ij), (lk), (ij)(lk)\}$ |
| *4* | $\{(ij)(lk), (iljk), (ikjl)\}$ |
| *5* | $\{(ij), (il)(kj), (ik)(lj)\}$ |
| *6* | $\{(ij), (iljk), (ikjl)\}$ |
| *7* | $\{(ij)(lk), (ijlk), (iklj)\}$ |
| *8* | $\{(ij), (ijl), (ilj)\}$ |
| *9* | $\{(ij), (ij)(lk), (il)(jk)\}$ |
| *10* | $\{(ij)(lk), (ijl), (ilj)\}$ |
| *11* | $\{(ij), (il), (jk)\}$ |
| *12* | $\{(ij), (ijlk), (iklj)\}$ |
| *13* | $\{(ij), (ilk), (ikl)\}$ |
| *14* | $\{(ij), (il), (il)(jk)\}$ |
| *15* | $\{(ij), (jl), (ij)(lk)\}$ |
| *16* | $\{(ij), (il), (ik)\}$ |

Fig. 1. possible subsets $S$ for $S_4$.

**Proposition 2.2.** *Let $S$ be a subset of $S_4$, such that $S = \{(ij), (il), (jl)\}$, where $i$, $j$ and $l$ are distinct elements in $\{1, 2, 3, 4\}$. Then $Cay(S_4, S)$ is isomorphic to $\cup_1^4 K_{3,3}$.*

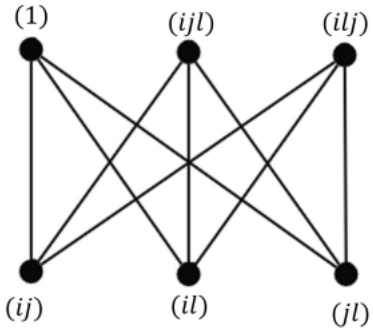


Fig. 2. The Graph $Cay(< S >, S) \cong K_{3,3}$.

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# Existence of three weak solutions for a non-local elliptic problem

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**Abstract**

In this note, we investigate the existence of weak solutions for a nonlocal elliptic problem involving a control parameter. Using variational method, we establish the existence of at least three distinct weak solutions under appropriate assumptions on the nonlinearity and the parameter. Our approach relies on critical point theory, particularly the three-critical-points theorem, to derive the sufficient conditions for the existence of multiple solutions.

## 1. Introduction

Consider the operator

$$L(y) := -(\theta(x)y'(x))' + \tau\eta(x)\int_{\alpha}^{\beta}\eta(x)y(x)\,dx$$

where $0 \le \alpha < \beta$, $\theta \in C^1([\alpha,\beta])$ with $\theta^- := \operatorname{ess\,inf}_{x\in[\alpha,\beta]}\theta(x) > 0$, $\eta \in L^2([\alpha,\beta])$ with $\eta \not\equiv 0$, and $\tau \in \mathbb{R}$ satisfies

$$\tau > -\frac{\pi^2\theta^-}{4(\beta-\alpha)^2\|\eta\|^2_{L^2[\alpha,\beta]}}. \tag{1}$$

This operator arises in a variety of mathematical models describing physical and biological phenomena. Notable applications include the analysis of microsensor thermistors in gas environments, superconductivity, plasma reactions, thermal processes, and population dynamics, among others (see [1, 6] for further details).
In this note, we study three weak solutions for the following non-local elliptic problem

$$\begin{cases} L(y) = \lambda\psi(x,y(x)) & \text{in } (\alpha,\beta), \\ y(\alpha) = y'(\beta) = 0, \end{cases} \tag{2}$$

where $\lambda > 0$ is a parameter and $\psi : [\alpha,\beta]\times\mathbb{R} \to \mathbb{R}$ is an $L^1$-Carathéodory function.

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The study of equations such as (2), which arise naturally from the mathematical modeling of diverse physical and biological phenomena, has long been a subject of significant interest in the research community, as evidenced by works such as [4, 5, 7].
The function space required to analyze this problem is defined as:

$$E = \left\{ y \in W^{1,2}([\alpha,\beta]) : y(\alpha) = 0 \right\}.$$

It is easy to show that the following three norms on this space are equivalent:

$$\|y\| = \left( \int_\alpha^\beta |y'(x)|^2 \, dx \right)^{1/2} + \left( \int_\alpha^\beta |y(x)|^2 \, dx \right)^{1/2},$$

$$\|y\|_0 = \|y'\|_{L^2} = \left( \int_\alpha^\beta |y'(x)|^2 \, dx \right)^{1/2}$$

and

$$\|y\|_E = \sqrt{\int_\alpha^\beta \theta(x)|y'(x)|^2 dx + \tau \left( \int_\alpha^\beta \eta(x) y(x) \, dx \right)^2}.$$

Indeed, for every $y \in E$ the following inequalities hold:

$$\|y\|_0 \le \|y\| \le (1 + \beta - \alpha)\|y\|_0$$

and

$$c_1\|y\|_0 \le \|y\|_E \le c_2\|y\|_0, \tag{3}$$

where $c_1, c_2$ with $0 < c_1 \le c_2$, are defined as

$$c_1 = \left( \theta^- + \min\left\{ 0, \frac{4\tau(\beta-\alpha)^2 \|\eta\|_{L^2}^2}{\pi^2} \right\} \right)^{1/2},$$

$$c_2 = \left( \theta^+ + \max\left\{ 0, \frac{4\tau(\beta-\alpha)^2 \|\eta\|_{L^2}^2}{\pi^2} \right\} \right)^{1/2}.$$

**Remark 1.1.** In light of Proposition 2.1 from [2] and inequality (3), it follows that

$$\max_{x\in[\alpha,\beta]} |y(x)| \le \frac{\sqrt{\beta-\alpha}}{c_1} \|y\|_E$$

for every $y \in E$.

The following lemma from [3] serves as our primary tool in demonstrating our main result.

**Lemma 1.2.** *Let the functionals $J$ and $I$ be defined on $E$, both belonging to the class $C^1$ on $E$, with $J$ being coercive. Also, following conditions hold:*

*(i) $J(0)=I(0)=0=\inf_E J$ and $J$ is convex;*

*(ii) for each $\lambda > 0$ and local minima $y_1, y_2 \in E$ of $J-\lambda I$ with $I(y_1) \ge 0$ and $I(y_2) \ge 0$, one has $\inf_{0\le\xi\le1} I(\xi y_1 + (1-\xi)y_2) \ge 0$.*

*Moreover, suppose there exist* $r_1, r_2 > 0$ *and* $\bar{e} \in E$ *with* $r_1 < J(\bar{e}) < \frac{r_2}{2}$, *such that*

*(iii)*

$$\frac{\sup_{y \in J^{-1}(-\infty, r_1)} I(y)}{r_1} < \frac{1}{2} \frac{I(\bar{e})}{J(\bar{e})};$$

*(iv)*

$$\frac{\sup_{y \in J^{-1}(-\infty, r_2)} I(y)}{r_2} < \frac{1}{4} \frac{I(\bar{e})}{J(\bar{e})}.$$

*Then, for all* $\lambda \in \left( \frac{2J(\bar{e})}{I(\bar{e})}, \min\left\{ \frac{r_1}{\sup_{y \in J^{-1}(-\infty, r_1)} I(y)}, \frac{r_2/2}{\sup_{y \in J^{-1}(-\infty, r_2)} I(y)} \right\} \right)$, *the functional* $J - \lambda I$ *has at least three distinct critical points* $y_1$, $y_2$, $y_3$ *such that* $y_1 \in J^{-1}(-\infty, r_1)$, $y_2 \in J^{-1}\left(r_1, \frac{r_2}{2}\right)$ *and* $y_3 \in J^{-1}(-\infty, r_2)$.

A function $y \in E$ is said to be a *weak solution* of problem (2) if the following identity holds for all test functions $v \in E$

$$\int_\alpha^\beta \theta(x) y'(x) v'(x)\, dx + \tau \int_\alpha^\beta \eta(x) y(x) dx \int_\alpha^\beta \eta(x) v(x) dx - \lambda \int_\alpha^\beta \psi(x, y(x)) v(x)\, dx = 0.$$

## 2. Main result

In this section, the principal results are stated and established. By imposing two algebraic sign conditions on the nonlinear term, we prove the existence of at least three non-negative weak solutions to Problem (2).

**Theorem 2.1.** *Suppose that* $\psi : [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ *be an* $L^1-$*Caratheodory function such that*

$$\Psi(x, \xi) := \int_0^\xi \psi(x, s)\, ds \geq 0$$

*for all* $(x, \xi) \in [\alpha, \beta] \times \mathbb{R}$. *Additionally, assume that condition* (1) *is true and that there exist three positive constants* $v_1$, $v_2$ *and* $d$, *with* $v_1 < \sqrt{2}\, d$ *and* $4c_2\, d < c_1\, v_2$ *such that*

(i1)

$$\frac{\int_\alpha^\beta \max_{|\xi| \leq v_1} \Psi(x, \xi) dx}{v_1^2} < \frac{c_1^2}{16 c_2^2} \frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x, d) dx}{d^2};$$

(i2)

$$\frac{\int_\alpha^\beta \max_{|\xi| \leq v_2} \Psi(x, \xi) dx}{v_2^2} < \frac{c_1^2}{32 c_2^2} \frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x, d) dx}{d^2}.$$

*Then, for each* $\lambda \in \Omega$, *where*

$$\Omega := \left[ \frac{8c_2^2}{\beta - \alpha} \frac{d^2}{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x, d) dx}, \frac{c_1^2}{2(\beta - \alpha)} \min\left\{ \frac{2v_1^2}{\int_\alpha^\beta \max_{|\xi| \leq v_1} \Psi(x, \xi) dx}, \frac{v_2^2}{\int_\alpha^\beta \max_{|\xi| \leq v_2} \Psi(x, \xi) dx} \right\} \right[,$$

*problem* (2) *admits at least three distinct non-negative weak solutions* $y_1$, $y_2$, $y_3$ *such that* $|y_i(x)| < v_2$ *for* $x \in [\alpha, \beta]$ *and* $i = 1, 2, 3$.

*Proof.* To apply Lemma 1.2, we introduce the functionals $J, I : E \to \mathbb{R}$ for every $y \in E$ as

$$J(y) := \frac{1}{2}\|y\|_E^2, \qquad I(y) := \int_\alpha^\beta \Psi(x, y(x))\, dx.$$

Additionally, let $r_1 = \frac{c_1^2 v_1^2}{2(\beta-\alpha)}$ and $r_2 = \frac{c_1^2 v_2^2}{2(\beta-\alpha)}$. Now, define the function $\overline{\phi} \in E$ as

$$\overline{\phi}(x) = \begin{cases} 4d\frac{x-\alpha}{\beta-\alpha}, & x \in \left[\alpha, \alpha + \frac{\beta-\alpha}{4}\right[, \\ d, & x \in \left[\alpha + \frac{\beta-\alpha}{4}, \beta - \frac{\beta-\alpha}{4}\right], \\ 4d\frac{\beta-x}{\beta-\alpha}, & x \in \left]\beta - \frac{\beta-\alpha}{4}, \beta\right]. \end{cases}$$

It can be easily confirmed that

$$\frac{4c_1^2 d^2}{\beta-\alpha} \le J(\overline{\phi}) \le \frac{4c_2^2 d^2}{\beta-\alpha}. \tag{4}$$

Given the conditions $v_1 < \sqrt{2}\, d$ and $4c_2 d < c_1 v_2$, we derive the inequality $r_1 < J(\overline{\phi}) < \frac{r_2}{2}$. From Remark 1.1 for every $y \in E$ satisfying $J(y) \le r_1$, it follows that

$$\max_{x\in[\alpha,\beta]} |y(x)| \le v_1,$$

and for all $y \in E$ with $J(y) \le r_2$, one has

$$\max_{x\in[\alpha,\beta]} |y(x)| \le v_2.$$

So, we obtain

$$\frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_1} = \frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} \int_\alpha^\beta \Psi(x,y(x))dx}{r_1} \le \frac{\int_\alpha^\beta \max_{|\xi|\le v_1} \Psi(x,\xi)dx}{r_1},$$

as well as

$$\frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_2} = \frac{\sup_{y\in J^{-1}(]-\infty,r_2[)} \int_\alpha^\beta \Psi(x,y(x))dx}{r_2} \le \frac{\int_\alpha^\beta \max_{|\xi|\le v_2} \Psi(x,\xi)dx}{r_2}.$$

On the other hand, we have

$$\frac{1}{2}\frac{I(\bar{\phi})}{J(\bar{\phi})} \ge \frac{1}{2}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{\frac{4c_2^2 d^2}{\beta-\alpha}} = \frac{\beta-\alpha}{8c_2^2}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{d^2} > \frac{2(\beta-\alpha)}{c_1^2}\frac{\int_\alpha^\beta \max_{|\xi|\le v_2} \Psi(x,\xi)dx}{v_1^2} \ge \frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_1},$$

and also,

$$\frac{1}{4}\frac{I(\bar{\phi})}{J(\bar{\phi})} \geq \frac{1}{4}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{\frac{4c_2^2d^2}{\beta-\alpha}} = \frac{\beta-\alpha}{16c_2^2}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{d^2}$$

$$> \frac{2(\beta-\alpha)}{c_1^2}\frac{\int_\alpha^\beta \max_{|\xi|\leq \nu_2} \Psi(x,\xi)dx}{\nu_2^2} \geq \frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_2}.$$

Therefore, according to Lemma 1.2, we arrive at the desired result. □

**Remark 2.2.** In the Theorem 2.1, it was proven that there are three weak solutions to the (2), the range all of which lie within the interval $[0,\nu_2]$. Therefore, the function $\Psi$ can be non-negative on the $[\alpha,\beta]\times[0,\nu_2]$. If we also apply condition $\psi(x,0)\neq 0$ for all $x\in[\alpha,\beta]$, the existence of three positive solutions to the problem is guaranteed.

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# Analyzing a second-order differential equation using critical point theory

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**Abstract**

In this paper, we study the existence of weak solutions for a second-order differential equation using critical point theory. By applying a critical point theorem, we identify specific intervals for the control parameter $\lambda$ such that the problem admits at least three weak solutions in a suitable function space.

## 1. Introduction

In this study, we rigorously analyze and demonstrate the existence of at least three distinct weak solutions for a class of second-order differential equations. By employing advanced variational methods and critical point theory, we derive our results for the following boundary value problem:

$$\begin{cases} -(\theta(x)y'(x))' + \tau\eta(x)\int_\alpha^\beta \eta(x)y(x)\,dx = \lambda\psi(x,y(x)) + h(y(x)) & \text{in } (\alpha,\beta), \\ y(\alpha) = y'(\beta) = 0, \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter and $\psi : [\alpha,\beta]\times\mathbb{R} \to \mathbb{R}$ is an $L^1$-Carathéodory function. Also, $0 \le \alpha < \beta$, $\theta \in C^1([\alpha,\beta])$ with $\theta^- := \operatorname{ess\,inf}_{x\in[\alpha,\beta]}\theta(x) > 0$, $\eta \in L^2([\alpha,\beta])$ with $\eta \not\equiv 0$, and $\tau \in \mathbb{R}$ satisfies

$$\tau > -\frac{\pi^2\theta^-}{4(\beta-\alpha)^2\|\eta\|^2_{L^2[\alpha,\beta]}}. \tag{2}$$

Moreover, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant $L > 0$, meaning

$$|h(s_1) - h(s_2)| \le L|s_1 - s_2|$$

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for all $s_1, s_2 \in \mathbb{R}$, with $h(0) = 0$.
These types of problems arise in a wide range of physical and biological phenomena, making them a subject of significant interest in various research fields. Notable applications include:
• The analysis of microsensor thermistors operating in gas environments,
• Modeling superconductivity and phase transitions in materials,
• Describing plasma reactions and their dynamics,
• Studying thermal processes and heat transfer mechanisms,
• Analyzing population dynamics and ecological systems.
Due to their relevance in these and other areas, such problems have been extensively studied in numerous scientific works, highlighting their importance in both theoretical and applied contexts. For further insights, see [1, 4–6, 8]. The lemma below, taken from [3] acts as the primary tool in establishing our main result.

**Lemma 1.1.** *Let $E$ be a reflexive real Banach space; $J : E \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $E^*$, $I : E \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_E J = J(0) = I(0) = 0\,.$$

*Assume that there exist two positive constants $r_1, r_2 > 0$ and $\bar{x} \in X$ with $2r_1 < J(\bar{x}) < \frac{r_2}{2}$ such that*

*(j)* $\frac{\sup_{J(x)\le r_1} I(x)}{r_1} < \frac{2}{3}\frac{I(\bar{x})}{J(\bar{x})}$,

*(jj)* $\frac{\sup_{J(x)\le r_2} I(x)}{r_2} < \frac{1}{3}\frac{I(\bar{x})}{J(\bar{x})}$,

*(jjj) for each $\lambda$ in*

$$\Lambda^*_{r_1,r_2} := \left]\frac{3}{2}\frac{J(\bar{x})}{I(\bar{x})}, \min\left\{\frac{r_1}{\sup_{J(x)\le r_1} I(x)}, \frac{r_2}{2\sup_{J(x)\le r_2} I(x)}\right\}\right[$$

*and for every $x_1, x_2 \in E$, which are local minima for the functional $J - \lambda I$, and such that $I(x_1) \ge 0$ and $I(x_2) \ge 0$, one has $\inf_{t\in[0,1]} I(tx_1 + (1-t)x_2) \ge 0$.*

*Then for each $\lambda \in \Lambda^*_{r_1,r_2}$ the functional $J - \lambda I$ has at least three distinct critical points which lie in $J^{-1}(]-\infty, r_2[)$.*

We now consider the space

$$E = \left\{y \in W^{1,2}([\alpha,\beta]) : y(\alpha) = 0\right\},$$

on which we define the following equivalent norms:

$$\|y\|_0 = \|y'\|_{L^2} = \left(\int_\alpha^\beta |y'(x)|^2\,dx\right)^{1/2}$$

and

$$\|y\|_E = \sqrt{\int_\alpha^\beta \theta(x)|y'(x)|^2 dx + \tau\left(\int_\alpha^\beta \eta(x)y(x)\,dx\right)^2}.$$

It can be shown that for every $y \in E$ the following inequalities hold:

$$c_1\|y\|_0 \le \|y\|_E \le c_2\|y\|_0, \tag{3}$$

where $c_1, c_2$ are defined as

$$c_1 = \left(\theta^- + \min\left\{0, \frac{4\tau(\beta-\alpha)^2\|\eta\|^2_{L^2}}{\pi^2}\right\}\right)^{1/2},$$

$$c_2 = \left(\theta^+ + \max\left\{0, \frac{4\tau(\beta-\alpha)^2\|\eta\|_{L^2}^2}{\pi^2}\right\}\right)^{1/2}.$$

**Proposition 1.2.** *Let the operator $T : E \to E^*$ is defined as follows:*

$$T(y)(\phi) = \int_\alpha^\beta \theta(x)y'\,\phi'\,dx + \tau\int_\alpha^\beta \eta(x)y\,dx\int_\alpha^\beta \eta(x)\phi\,dx - \int_\alpha^\beta h(y)\phi dx$$

*for all $y, \phi \in X$. If $L(\beta-\alpha)^2 < \pi^2 c_1^2$, then, $T$ is invertible, and both $T$ and $T^{-1}$ are continuous on $E$ and $E^*$, respectively.*

*Proof.* For any $y, \phi \in E$, one has

$$\sup_{w\in E,\,\|w\|\le 1} |T(y)(w) - T(\phi)(w)| \le \frac{1}{c_1^2}\left(\theta^+ + |\tau|\|\eta\|_{L^2}^2\frac{(\beta-\alpha)^2}{\pi^2} + \frac{L(\beta-\alpha)^2}{\pi^2}\right)\|y-\phi\|_E,$$

via following Poincaré type inequality (see, for instance, [7, Lemma 2.3]):

$$\|y\|_{L^2([\alpha,\beta])}^2 \le \frac{(\beta-\alpha)^2}{\pi^2}\|y\|_0^2, \tag{4}$$

for all $y \in E$. Hence, the operator $T$ on the space $E$ is continuous.
Also, since for each $y \in E \setminus \{0\}$ the inequality

$$T(y)y \ge \left(1 - \frac{L(\beta-\alpha)^2}{\pi^2 c_1^2}\right)\|y\|_E^2,$$

holds, one has $T$ is coercive. Furthermore, for $y, \phi \in E$, we have

$$\langle T(y) - T(\phi), y-\phi\rangle \ge \left(1 - \frac{L(\beta-\alpha)^2}{\pi^2 c_1^2}\right)\|y-\phi\|_E^2, \tag{5}$$

which shows that the operator $T$ is uniformly monotonic. By invoking Theorem 26.A(d) from [9], we can conclude that $T$ is invertible. □

**Remark 1.3.** By Proposition 2.1 of [2], one has

$$\max_{x\in[\alpha,\beta]} |y(x)| \le \sqrt{\beta-\alpha}\|y\|_0$$

for any $y \in E$.

A *weak solution* to problem (1) is a function $y \in E$ that satisfies the following identity for all test functions $v \in E$:

$$\int_\alpha^\beta \theta(x)y'(x)v'(x)\,dx + \tau\int_\alpha^\beta \eta(x)y(x)dx\int_\alpha^\beta \eta(x)v(x)dx$$
$$-\int_\alpha^\beta h(u(x))v(x)dx - \lambda\int_\alpha^\beta \psi(x,y(x))v(x)\,dx = 0.$$

## 2. Main result

Here, the main result is presented and proven. By introducing two algebraic sign conditions on the nonlinear term, we demonstrate the existence of at least three non-negative weak solutions to Problem (1).

**Theorem 2.1.** *Assume that* $\psi : [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ *be an* $L^1-$*Caratheodory function satisfying*

$$\Psi(x, \xi) := \int_0^{\xi} \psi(x, s)\, ds \geq 0$$

*for all* $(x, \xi) \in [\alpha, \beta] \times \mathbb{R}$. *Moreover, let the condition* (2) *holds and that there exist three positive constants* $\nu_1$, $\nu_2$ *and* $d$, *with* $\nu_1 < \sqrt{2}\, d$ *and* $4d < \sqrt{\frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{\pi^2 c_2^2 + L(\beta-\alpha)^2}}\, \nu_2$ *such that the following inequalities are satisfied:*

(i1)

$$\frac{\int_\alpha^\beta \max_{|\xi| \leq \nu_1} \Psi(x, \xi) dx}{\nu_1^2} < \frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{12(\pi^2 c_2^2 + L(\beta-\alpha)^2)} \frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x, d) dx}{d^2};$$

(i2)

$$\frac{\int_\alpha^\beta \max_{|\xi| \leq \nu_2} \Psi(x, \xi) dx}{\nu_2^2} < \frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{24(\pi^2 c_2^2 + L(\beta-\alpha)^2)} \frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x, d) dx}{d^2}.$$

*Then, for any* $\lambda \in ]\lambda_1, \lambda_2[$, *where*

$$\lambda_1 := \frac{12(\pi^2 c_2^2 + L(\beta-\alpha)^2) d^2}{\pi^2 (\beta-\alpha) \int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x, d) dx},$$

$$\lambda_2 := \frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{2\pi^2(\beta-\alpha)} \min\left\{ \frac{2\nu_1^2}{\int_\alpha^\beta \max_{|\xi| \leq \nu_1} \Psi(x, \xi) dx}, \frac{\nu_2^2}{\int_\alpha^\beta \max_{|\xi| \leq \nu_2} \Psi(x, \xi) dx} \right\},$$

*problem* (1) *possesses at least three distinct non-negative weak solutions* $y_1$, $y_2$, $y_3$ *such that* $|y_i(x)| < \nu_2$ *for* $x \in [\alpha, \beta]$ *and* $i = 1, 2, 3$.

*Proof.* In order to apply Lemma 1.1, we define the functionals $J, I : E \to \mathbb{R}$ for every $y \in E$ as

$$J(y) := \frac{1}{2}\|y\|_E^2 - \int_\alpha^\beta H(y(x)) dx, \qquad I(y) := \int_\alpha^\beta \Psi(x, y(x))\, dx.$$

In addition, suppose $r_1 = \frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{2\pi^2(\beta-\alpha)} \nu_1^2$ and $r_2 = \frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{2\pi^2(\beta-\alpha)} \nu_2^2$. Now, consider the function $\overline{\phi} \in E$ as

$$\overline{\phi}(x) = \begin{cases} 4d\frac{x-\alpha}{\beta-\alpha}, & x \in \left[\alpha, \alpha + \frac{\beta-\alpha}{4}\right[, \\ d, & x \in \left[\alpha + \frac{\beta-\alpha}{4}, \beta - \frac{\beta-\alpha}{4}\right], \\ 4d\frac{\beta-x}{\beta-\alpha}, & x \in \left]\beta - \frac{\beta-\alpha}{4}, \beta\right]. \end{cases}$$

A simple computations shows that

$$\frac{4d^2(\pi^2 c_1^2 - L(\beta-\alpha)^2)}{\pi^2(\beta-\alpha)} \leq J(\overline{\phi}) \leq \frac{4d^2(\pi^2 c_2^2 + L(\beta-\alpha)^2)}{\pi^2(\beta-\alpha)}. \tag{6}$$

Taking into account the conditions $\nu_1 < \sqrt{2}\, d$ and $4d < \sqrt{\frac{\pi^2 c_1^2 - L(\beta-\alpha)^2}{\pi^2 c_2^2 + L(\beta-\alpha)^2}}\, \nu_2$, one has $2r_1 < J(\overline{\phi}) < \frac{r_2}{2}$. According to Remark 1.3 if $y \in E$ satisfies $J(y) \leq r_1$, then

$$\max_{x \in [\alpha, \beta]} |y(x)| \leq \nu_1.$$

Similarly, for any $y \in E$ with $J(y) \leq r_2$, we have

$$\max_{x\in[\alpha,\beta]} |y(x)| \leq \nu_2.$$

Thus, we deduce the following inequalities:

$$\frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_1} = \frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} \int_\alpha^\beta \Psi(x,y(x))dx}{r_1} \leq \frac{\int_\alpha^\beta \max_{|\xi|\leq\nu_1} \Psi(x,\xi)dx}{r_1},$$

and similarly,

$$\frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_2} = \frac{\sup_{y\in J^{-1}(]-\infty,r_2[)} \int_\alpha^\beta \Psi(x,y(x))dx}{r_2} \leq \frac{\int_\alpha^\beta \max_{|\xi|\leq\nu_2} \Psi(x,\xi)dx}{r_2}.$$

At the same time, we observe that

$$\frac{2}{3}\frac{I(\bar{\phi})}{J(\bar{\phi})} \geq \frac{2}{3}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{\frac{4d^2(\pi^2c_2^2+L(\beta-\alpha)^2)}{\pi^2(\beta-\alpha)}} = \frac{\pi^2(\beta-\alpha)}{6(\pi^2c_2^2+L(\beta-\alpha)^2)}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{d^2}$$
$$> \frac{2\pi^2(\beta-\alpha)}{\pi^2c_1^2-L(\beta-\alpha)^2}\frac{\int_\alpha^\beta \max_{|\xi|\leq\nu_1} \Psi(x,\xi)dx}{\nu_1^2} \geq \frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_1},$$

and also,

$$\frac{1}{3}\frac{I(\bar{\phi})}{J(\bar{\phi})} \geq \frac{1}{3}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{\frac{4d^2(\pi^2c_2^2+L(\beta-\alpha)^2)}{\pi^2(\beta-\alpha)}} = \frac{\pi^2(\beta-\alpha)}{12(\pi^2c_2^2+L(\beta-\alpha)^2)}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{d^2}$$
$$> \frac{2\pi^2(\beta-\alpha)}{\pi^2c_1^2-L(\beta-\alpha)^2}\frac{\int_\alpha^\beta \max_{|\xi|\leq\nu_2} \Psi(x,\xi)dx}{\nu_2^2} \geq \frac{\sup_{y\in J^{-1}(]-\infty,r_1[)} I(y)}{r_2},$$

so, by applying Lemma 1.1, we achieve the desired conclusion. □

In problem (1) if $h = 0$, Theorem 2.1 can be expressed in the following form:

**Theorem 2.2.** *Suppose that $\psi : [\alpha,\beta] \times \mathbb{R} \to \mathbb{R}$ be an $L^1-$Caratheodory function such that*

$$\Psi(x,\xi) := \int_0^\xi \psi(x,s)\,ds \geq 0$$

*for all $(x,\xi) \in [\alpha,\beta] \times \mathbb{R}$. Additionally, assume that condition (2) is true and that there exist three positive constants $\nu_1, \nu_2$ and $d$, with $\nu_1 < \sqrt{2}\,d$ and $4d < \frac{c_1}{c_2}\nu_2$ such that*

(i1)

$$\frac{\int_\alpha^\beta \max_{|\xi|\leq\nu_1} \Psi(x,\xi)dx}{\nu_1^2} < \frac{c_1^2}{12c_2^2}\frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{d^2};$$

(i2)

$$\frac{\int_{\alpha}^{\beta} \max_{|\xi|\le \nu_2} \Psi(x,\xi)dx}{\nu_2^2} < \frac{c_1^2}{24c_2^2} \frac{\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}{d^2}.$$

*Then, for each*

$$\lambda \in \left[ \frac{12c_2^2 d^2}{(\beta-\alpha)\int_{\frac{3\alpha+\beta}{4}}^{\frac{\alpha+3\beta}{4}} \Psi(x,d)dx}, \frac{c_1^2}{2(\beta-\alpha)} \min\left\{ \frac{2\nu_1^2}{\int_{\alpha}^{\beta} \max_{|\xi|\le \nu_1} \Psi(x,\xi)dx}, \frac{\nu_2^2}{\int_{\alpha}^{\beta} \max_{|\xi|\le \nu_2} \Psi(x,\xi)dx} \right\} \right[,$$

*the following problem*

$$\begin{cases} -(\theta(x)y'(x))' + \tau\eta(x)\int_{\alpha}^{\beta} \eta(x)y(x)\,dx = \lambda\psi(x,y(x)) & \text{in } (\alpha,\beta), \\ y(\alpha) = y'(\beta) = 0, \end{cases}$$

*admits at least three distinct non-negative weak solutions* $y_1, y_2, y_3$ *such that* $|y_i(x)| < \nu_2$ *for* $x \in [\alpha,\beta]$ *and* $i = 1, 2, 3$.

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# Hypergraphs in role of studying some modules

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**Abstract**

Let $M$ be a module over a ring $R$. The intersection hypergraph assigned to $M$ which is denoted by $\mathcal{IH}_R(M)$ will be studied. Vertices of this hypergraph are all nontrivial submodules of $M$ and a set of vertives $E$ forms a hyperedge in case each two elements of $E$ has zero intersection and $E$ is maximal with respect to this property.

## 1. Introduction

In Berge [4, 5], the author introduced hypergraphs as a means to generalize the graph approach. As given in Berge [4, 5], a hypergraph $\mathcal{H} = (V; E)$ on a finite set of vertices (or nodes) $V = \{v_1, \dots, v_n\}$ is defined as a family of hyperedges $E = \{e_j \mid 1 \leq j \leq m\}$ where each hyperedge is a non-empty subset of $V$ and such that $\cup_{j=1}^{m} e_j = V$. It means that in a hypergraph, a hyperedge links one or more vertices.
Hypergraphs can be viewed as incidence structures. In particular, there is a bipartite "incidence graph" or "Levi graph" corresponding to every hypergraph, and conversely, most, but not all, bipartite graphs can be regarded as incidence graphs of hypergraphs.
Finding the complete subgraphs of a graph can be viewed as a an important issue in graph theory. Intersection and co-intersection graphs on modules were studied in [1] and [2]. To find the complete subgraphs of $\Gamma(M)$ (co-intersection graph on submodules defined on a module $M$), we shall define a hypergraph on $M$. In this paper, motivated by [2] and works done about hypergraphs, we introduce a new hypergraph assigned to a right $R$-module $M$. We define $\mathcal{IH}_R(M)$ as a hypergraph where the vertices are all nontrivial submodules of $K$ and a $n$-subset ($n \geq 2$) $E_i$ of the set all nontrivial submodules of $K$ is a hyperedge of $\mathcal{IH}_R(M)$ provided for each two $N, L \in E_i$, $N \cap L = 0$ and $E_i$ is maximal to this property. Connecting to $\Gamma(M)$, we can say $E_i$ is a hyperedge in $\mathcal{IH}_R(M)$ if and only if $E_i$ is a maximal subset of $V$ with respect to the property that the elements of $E_i$ form a complete subgraph of $\Gamma(M)$.
It should be noted that a submodule $N$ of a module $M$ is called *essential*, provided $N \cap L \neq 0$ for all nonzero submodules $L$ of $M$. Also, $M$ is said to be *uniform* in case every nonzero submodule of $M$ is essential in $M$. The submodule

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$Soc(M)$ is defined to be the intersection of all essential submodules of $M$, equivalently $Soc(M)$ is the sum of all simple submodules of $M$. The radical of $M$, denoted by $Rad(M)$ is the intersection of all maximal submodules of $M$. Any unexplained terminologies related to modules and rings can be found in [8] and we refer the readers to [7] for more information about graphs and related concepts.

## 2. Properties of intersection hypergraph of submodules of a module

Let $S$ be a set. By a $n$-subset of $S$, we mean a subset of $S$ with $n$ elements.
We start with the definition.

**Definition 2.1.** Let $M$ be an $R$-module. The intersection hypergraph $\mathcal{IH}_R(M)$ on $M$ is a hypergraph where the vertices are all nontrivial submodules of $M$ and a $n$-subset $(n \geq 2)$ $E_i$ of the set all nontrivial submodules of $M$ is a hyperedge of $\mathcal{IH}_R(M)$ provided for each two $N, L \in E_i$, $N \cap L = 0$ and $E_i$ is maximal with respect to this property.

We shall deal with some properties of the intersection hypergraph of a module. We prove that a module $M$ is semisimple if and only if for every vertex $N$ in $\mathcal{IH}_R(M)$ we have $deg_{\mathcal{IH}_R(M)}(N) \neq 0$.

**Proposition 2.2.** For a right $R$-module $M$, if $\delta(\mathcal{IH}_R(M)) \geq 1$, then the intersection hypergraph is connected and its diameter is at last three.

**Lemma 2.3.** A nontrivial submodule $N$ of $M$ is essential if and only if $deg_{\mathcal{IH}_R(M)}(N) = 0$.

**Proposition 2.4.** The intersection hypergraph of an $R$-module $M$ is null if and only if $M$ is uniform.

As a consequence of Proposition 2.4, the intersection hypergraph of the $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is null.
The following is a characterization for a module such that its intersection hypergraph has just one hyperedge containing all nontrivial submodules of that module.

**Theorem 2.5.** Let $M$ be a finite $R$-module with at least two nontrivial submodules. Then $\mathcal{IH}_R(M)$ has only a hyperedge containing all nontrivial submodules of $M$ if and only if $M$ can be written as a direct sum of each two nontrivial submodules.

As examples of modules satisfying in the condition of Theorem 2.5, we can consider the $\mathbb{Z}$-modules $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and in general $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for a prime number $p$.

**Proposition 2.6.** Suppose that $M$ is a finite module. Then $M$ is semisimple if and only if every nontrivial submodule of $N$ of $M$, we have $deg_{\mathcal{IH}_R(M)}(N) \neq 0$.

**Corollary 2.7.** Suppose that $M$ is a finite $R$-module with exactly three nontrivial submodules $H_1$, $H_2$ and $H_3$. Then $\mathcal{IH}_R(M)$ either has just one hyperedge of the form $\{H_1, H_2, H_3\}$ or $\{H_1, H_2\}$, or $\mathcal{IH}_R(M)$ is a null hypergraph.

The following includes examples of modules satisfying conditions in Theorem **??**.

**Example 2.8.** (1) Let $F$ be a field. Consider the ring $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F\}$. Then the bimodule ${}_RR_R = M$ has exactly three nontrivial submodules $H = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F\}$, $L = \{\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in F\}$ and $N = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in F\}$. Then $H$ and $L$ are maximal submodules of $M$ and $N = H \cap L = Rad(M) = Soc(M)$. Also, $M$ is uniform and $\mathcal{IH}_R(M)$ is null.
(2) Let $R = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a \in \mathbb{Z}_4, b, c \in \mathbb{Z}_2\}$. Consider the $R$-module $M = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_4, b \in \mathbb{Z}_2\}$. Then $M$ has just three nontrivial submodules $N = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a = 0, 2\}$, $L = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b = 0, 1\}$ and $H = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a = 0, 2,\ b = 0, 1\}$. Then $N \cap L = 0$, $N, L \subseteq H$, $N \oplus L = H = Rad(M) = Soc(M)$.

We next show that there does not exist a finite module $K$ with submodules $N, T, L, H$ such that $N \cap T = 0$, $L \cap H = 0$ and $N \cap L(or H) \neq 0$ and $T \cap L(or H) \neq 0$.

**Proposition 2.9.** Let $M$ be an $R$-module with four nontrivial submodules $T_1$,..., $T_4$. Then $E(\mathcal{IH}_R(M)) \neq \{\{T_i, T_j\}, \{T_s, T_t\}\}$, for $i \neq j \neq s \neq t \in \{1, \dots, 4\}$.

**Proposition 2.10.** Suppose that $M$ is a finite $R$-module with exactly four nontrivial submodules. Then $M$ is semisimple if and only if $M$ can be written as a direct sum of each two nontrivial distinct submodules. In this case, $\mathcal{IH}_R(M)$ has just one hyperedge $\{H_1, \dots, H_4\}$, for nontrivial distinct submodules $H_1$, $H_2$, $H_3$ and $H_4$ of $M$.

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# Some results on F-contractions in b-metric spaces

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**Abstract**

F-contraction has been a widely investigated problem in the Fixed Point Theory during the last decade. In this paper, we analyze, generalize and correct some recent results on F-contractions within b-metric spaces. There are different results regarding generalizations and modifications of F-contraction in various settings, along with the results concerning application of those concepts, mostly in the area of differential and difference equations, fractional calculus, etc. In this way, the wider class of F-contraction is formed and, for this new type of contraction, called simple F-contraction or sF-contraction, we prove the existence and uniqueness of the fixed point on a b-metric space.

## 1. Introduction and preliminaries

In 2012, Wardowski [15] introduced a new type of contraction called F-contraction (also called Wardowski contraction [4]) and proved a fixed point theorem concerning F-contractions. Since then much work has been done on the fixed point theory of F-contraction mappings and their extensions (refer to [4, 5, 13]).

**Definition 1.1.** ([15]) Let $F : (0,\infty) \to \mathbb{R}$ be a map satisfying the following conditions:

- (F1) F is strictly increasing.
- (F2) For each sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
- (F3) There exists $k \in (0,1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

For a metric space $(X,d)$, a mapping $T : X \to X$ is said to be a Wardowski F-contraction if there exists $\tau > 0$ such that $d(Tx,Ty) > 0$ implies

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y))$$

for all $x, y \in X$.

**Theorem 1.2.** *[15] Let $(X,d)$ be a complete metric space and let $T : X \to X$ be a Wardowski F-contraction. Then $T$ has a unique fixed point $x^* \in X$. On the other hand, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to $x^*$ for every $x \in X$.*

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In 1989, Bakhtin [1] introduced the notion of b-metric spaces, which was formally defined by Czerwik [3] in 1993 with a view of generalizing Banach contraction principle. There are many authors who have worked on the generalization of fixed point theorems in b-metric spaces (refer to [6, 7, 10, 14] and the references therein). However, unlike the normal metric the b-metric $d$ is not continuous in the topology generated by it (for instance, refer to Example 2.6 of [12]). In this paper, we present some results on fixed point theory in b-metric spaces considering a new type of mapping which is a combination of F-contraction by Wardowski [15] as well as Kannan contraction [9] mappings. We also try to develop a fixed point existence results for such type of expanding mappings on b-metric spaces. We start by defining some of the terms used in this paper.

**Definition 1.3.** ([1, 3]) Let $X$ be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$; and
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

The triplet $(X, d, s)$ is called a b-metric space with coefficient $s$.

The problems that arise in proving fixed point results due to the possible discontinuity of the b-metric can be fortunately managed with the following lemma.

**Lemma 1.4.** *Let $(X, d, s)$ be a b-metric space and $\{x_n\}$ be a convergent sequence in X with $\lim_{n\to\infty} x_n = x$. Then for all $y \in X$*

$$s^{-1}d(x, y) \leq \lim_{n\to\infty} \inf d(x_n, y) \leq \lim_{n\to\infty} \sup d(x_n, y) \leq sd(x, y).$$

Further, in ([13], Definition 2.7.) authors introduced the following condition.

(F4): If $\inf F = -\infty$ and $x, y, z \in (0, +\infty)$ and $s \geq 1$ are such that $\tau + F(s \cdot x) \leq F(y)$ and $\tau + F(s \cdot y) \leq F(z)$, then $\tau + F(s^2 \cdot x) \leq F(s \cdot y)$. Authors in [13] denote by Fs$\tau$ the family of all functions $F : (0, +\infty) \to (-\infty, +\infty)$ which satisfy (F1), (F3), and (F4).

We make two important remarks. First, if $\inf F \neq -\infty$ then (Fs$\tau$) is satisfied, for all $s \geq 1$ and $\tau > 0$. Second, when $s = 1$ and $\tau > 0$ is arbitrary, condition (Fs$\tau$) is a tautology, hence in this case the family Fs$\tau$ is F.

In [2] the authors introduce the following condition (F4):
(F's$\tau$) if $(\alpha_n)n \in \mathbb{N} \subseteq (0, \infty)$ is a sequence such that $\tau + F(s\alpha_n) \leq F(\alpha_{n-1})$, for all $n \in \mathbb{N}$ and for some $\tau > 0$, then $\tau + F(s^n\alpha_n) \leq F(s^{n-1}\alpha_{n-1})$, for all $n \in \mathbb{N}$.
The equivalence of these two conditions is proven in proposition of 2.8. in [13].

For the functions from Fs$\tau$ authors in ([13], Definition 3.1., Theorem 3.2.) introduced and proved the following:

**Definition 1.5.** Let $(X, d, s \geq 1)$ be a b-metric space and $T : X \to X$ be an operator. If there exist $\tau > 0$ and $F \in Fs\tau$ such that for all $x, y \in X$ the inequality $d(Tx, Ty) > 0$ implies

$$\tau + F(s.d(Tx, Ty)) \leq F(d(x, y)), \tag{1}$$

then $T$ is called an F-contraction.

**Theorem 1.6.** *Let $(X, d, s \geq 1)$ be a complete b-metric space and $T : X \to X$ be an F−contraction, then T has a unique fixed point $x^*$. Furthermore, for any $x_0 \in X$, the sequence $x_{n+1} = T_{x_n}$ is convergent and $\lim_{n\to+\infty} x_n = x^*$.*

Also, for the functions from Fs$\tau$ authors in ([13], Definition 4.1., Theorem 4.2.) introduced and proved the next:

**Definition 1.7.** $(X, d, s \geq 1)$ be a b-metric space and $T : X \to X$ be an operator. If there exists $\tau > 0$ and $F \in Fs\tau$ such that for all $x, y \in X$ the inequality $d(Tx, Ty) > 0$ implies

$$\tau + F(s.d(Tx, Ty)) \leq F\Big(\max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\Big\}\Big)$$

then $T$ is called an F-weak contraction.

**Theorem 1.8.** *Let $(X, d, s \geq 1)$ be a complete b-metric space and $T : X \to X$ be an F−weak contraction, then T has a unique fixed point and for any $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ is convergent in X. Furthermore, if either T or F is continuous then T has a unique fixed point $x^*$ and for all $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to $x^*$.*

As mentioned, this result was modified in several different manners. We will discuss only on the characterization of the family F and we will prove that it is possible to omit some conditions in the definition of class F without losing any of the already obtained results regarding existence and uniqueness of a fixed point.

## 2. Main Results

In this part of the paper we shall use only the condition (F1) for the proof of all Theorems from Section 1, Introduction and preliminaries.

**Definition 2.1.** A set $F^\natural$ is the set of all strictly increasing functions $F : (0, \infty) \to \mathbb{R}$.

Therefore, $F^\natural$ the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying (F1) and F, $Fs\tau \subseteq F^\natural$. In accordance with newly defined class of functions, we define a simple F-contraction or sF-contraction.

**Definition 2.2.** Let $(X, d, s)$ be a b-metric space and $T : X \to X$ a mapping. If there exist $F \in F^\natural$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(s.d(Tx, Ty)) \leq F(d(x, y)), \tag{2}$$

then a mapping $T$ is called a simple F-contraction (sF-contraction).

**Lemma 2.3.** *Any F-contraction is a sF -contraction.*

**Lemma 2.4.** *Any Fsτ-contraction is a sF-contraction.*

The main result is dedicated to the existence and uniqueness of a fixed point for a simple F-contraction.

**Theorem 2.5.** *Let $(X, d, s)$ be a complete b-metric space and $T : X \to X$ a simple F-contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ converges to $x^*$.*

*Proof.* Since the function F is strictly increasing 2 yields

$$d(Tx, Ty) < \frac{1}{s}.d(x, y), \tag{3}$$

for all $x, y \in X$ with $x \neq y$. Now condition 3 directly implies that the mapping $T$ is continuous and that its possible fixed point is unique. We did not use the function F as in [13] to prove uniqueness. The proof further goes on as in [8] and [11] □

Our next result is a generalization and correction of Theorem 1.8 from Section 1, Introduction and preliminaries of this paper.

**Definition 2.6.** $(X, d, s \geq 1)$ be a b-metric space and $T : X \to X$ be an operator. If there exists $\tau > 0$ and $F \in F^\natural$ such that for all $x, y \in X$ the inequality $d(Tx, Ty) > 0$ implies

$$\tau + F(s.d(Tx, Ty)) \leq F\Big(\max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\Big\}\Big)$$

then $T$ is called a simple F-weak contraction.

**Theorem 2.7.** *Let $(X, d, s \geq 1)$ be a complete b-metric space and $T : X \to X$ be a simple F−weak contraction, then T has a unique fixed point and for any $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ is convergent in X. Furthermore, if either T or F is continuous then T has a unique fixed point $x^*$ and for all $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to $x^*$.*

**Remark 2.8.** Mapping $T : X \to X$ on a b-metric space $(X, d, s)$ is called contractive mapping if

$$s.d(Tx, Ty) < d(x, y).$$

If $T$ is a sF-contraction, then

$$F(s.d(Tx, Ty)) \leq F(d(x, y)) - \tau < F(d(x, y))$$

along with the fact that F is strictly increasing leads to the conclusion that $T$ is a contractive mapping.

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# Some Properties of Generalized r-submodules

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**Abstract**

In this paper, we introduce and study the properties of $(1,r)$-submodules and $(2,r)$-submodules, which generalize the concepts of $r$-submodules, $sr$-submodules and prime submodules [1–3]. We extend several key results related to $(1,r)$-submodules, $(2,r)$-submodules, and prime submodules to a more general framework. Finally, we present some of the results concerning these concepts and investigate their behavior in the context of fractional modules and Cartesian products.

## 1. Introduction

Throughout this paper, $R$ is a non-trivial commutative ring with an identity element, and $M$ is a unitary $R$-module. The main purpose of this paper is to introduce and investigate some properties of the concept of $(1,r)$-submodules and $(2,r)$-submodules, which generalize $r$-submodules, $sr$-submodules and prime submodules. One of the methods used to generalize prime and primary submodules is the use of $Z(M)$, which is defined as the set of zero divisors of $M$. It is denoted as follows.

$$Z_R(M) = \{r \in R \mid \exists\, 0 \neq m \in M,\ rm = 0_M\} = \{r \in R \mid \operatorname{ann}_M(r) \neq 0\}$$

In 2015, Mohammadian introduced the concept of $r$-ideals in commutative rings [2]. A proper ideal $I$ is called an $r$-ideal if, for $ab \in I$ with $\operatorname{ann}(a) = 0$, it follows that $b \in I$. Note that $\operatorname{ann}(a) = 0$ if and only if $a \notin Z(R)$. Thus, if $ab \in I$ with $a \notin Z(R)$, then $b \in I$. In 2018, Koç defined the concept of $r$-submodules [1]. A proper submodule $N$ of $M$ is called an $r$-submodule if, for $a \in R$ and $m \in M$, $am \in N$ with $\operatorname{ann}_M(a) = 0_M$ implies $m \in N$. In this work, we present a different generalization of $r$-submodules and $sr$-submodules, focusing on the choice of elements. We say that a proper submodule $N$ of $M$ is a $(1,r)$-submodule of $M$, whenever $abm \in N$ for two non-unit elements $a, b \in R$ and $m \in M$, then $a \in Z(M)$ or $bm \in N$. Also, we say a proper submodule $N$ of $M$ is a $(2,r)$-submodule of $M$, if $abm \in N$ for elements $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in Z_R(M)$. In the first part, we provide some necessary definitions, and in the second part, we establish basic properties of these submodules. Through examples, we demonstrate that $(1,r)$-submodules generalize $r$-submodules and $(2,r)$-submodules, while $(2,r)$-submodules generalize prime submodules.

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In the section 2.1, we investigate the $(1,r)$-submodules in the direct product of modules and fraction modules. Also, we explore how $(1,r)$-submodules and (2,r)-submodules are preserved under isomorphism. To this end, we present some results concerning $(1,r)$- submodules in the direct product of modules. It is noteworthy that if $N_i$ is $(1,r)$-submodule of $M_i$, for i = 1, 2, then $N_1 \times N_2$ is not necessarily a $(1,r)$-submodule. See Example 2.14 for more details. Similarly, in the final section, we introduce the concept of $(2,r)$-submodules, and we explore some of its properties, such as its behavior under intersection, isomorphism, direct product.

## 2. (1,r)-submodules

First we introduce the concept $(1,r)$-submodule and then we examine some of its basic properties.

**Definition 2.1.** Let $N$ be a proper submodule of $M$. Then $N$ is called a $(1,r)$-submodule of $M$, whenever $abm \in N$ for two non-unit elements $a, b \in R$ and $m \in M$, then $a \in Z(M)$ or $bm \in N$.

**Example 2.2.** The following statements hold:
(1)$N = (0)$ is a $(1,r)$-submodule.
(2) Every $r$-submodule is a $(1,r)$-submodule.

In the following, we give an example of $(1,r)$-submodules, that is not $r$-submodule.

**Example 2.3.** Let $M = \mathbb{Z}$ be as a $\mathbb{Z}$-module and $N = 3\mathbb{Z}$ be a submodule of $M$. It is obvious that the set of nonunit elements of $\mathbb{Z}$ is equal to $\mathbb{Z} - \{1\}$. Let $a, b \in \mathbb{Z} - \{1\}$ and $m \in M$ such that $abm \in 3\mathbb{Z}$, hence $3 \mid a \vee 3 \mid b \vee 3 \mid m$. If $a = 0$ or $b = 0$, we have $a \in Z(M)$ or $b \in Z(M)$. Now suppose that $a, b \neq 0$, hence $a, b \notin Z(M) = 0$. If $3 \mid a$, then $am \in N$ and if $3 \mid b$, then $bm \in N$. Also if $3 \mid m$, then $am \in N$ and $bm \in N$. Therefore $N$ is a $(1,r)$-submodule of $M$. Later we show that $N$ is not a $r$-submodule. Suppose that $a \neq 0$ and $am \in N$ such that $3 \mid a$ and $3 \nmid m$. Hence nither $a \in Z(M)$ nor $m \in N$, so $N$ is not a $r$-submodule.

### 2.1. *Some Properties of (1,r)-Submodules*

**Proposition 2.4.** (1) *Let $M$ be a finitely generated $R$-module and $\{N_i : i \in I\}$ be a directed set of $(1,r)$-submodules of $M$. Then the submodule $N = \cup_{i\in I} N_i$ is a $(1,r)$-submodule of $M$.*
(2) *Let $\{N_i : i \in I\}$ be a set of $(1,r)$-submodules of $M$. Then the submodule $N = \cap_{i\in I} N_i$ is a $(1,r)$-submodule of $M$.*

*Proof.* (1) Since $M$ is a finitely generated $R$-module, so $N$ is a proper submodule of $M$. Suppose that for two non-unit $a, b \in R$ and $m \in M$ we have $abm \in N = \cup_{i\in I} N_i$ and $a \notin Z(M)$. Then there exists $i \in I$ such that $abm \notin N_i$. Since $a \notin Z(M)$, by assumption, $bm \in N_i \subseteq \cup_{i\in I} N_i = N$.
(2) Let two non-unit elements $a, b \in R$ and $m \in M$ such that $abm \in \cap_{i\in I} N_i$ and $a \notin Z(M)$. Then for each $i \in I$, $abm \in N_i$. Since for each $i \in I$, $N_i$ is $(1,r)$-submodule and $a \notin Z(M)$, so $bm \in \cap_{i\in I} N_i$. □

**Proposition 2.5.** *Let $N$ be a proper submodule of $M$. Then for every two non-unit elements $a, b \in R$, the following statements are equivalent:*
(1) *$N$ is a $(1,r)$-submodule of $M$.*
(2) *$(N :_M ab) = (N :_M b)$ for every $a \notin Z(M)$.*
(3) *$N \cap abM \subseteq aN$ for every $a \notin Z(M)$.*

*Proof.* (1)⇒(2) Let $m \in (N :_M ab)$, so $abm \in N$. Then by (1), $bm \in N$, hence $m \in (N :_M b)$. Also we know $(N :_M b) \subseteq (N :_M ab)$. Then (2) is true.
(2)⇒(3) Let $x \in N \cap abM$, then $x = abm$ for some $m \in M$ and $x = abm \in N$. Since $a \notin Z(M)$ and $m \in (N :_M ab)$, so $m \in (N :_M b)$. Therefore $x = abm \in aN$.
(3)⇒(1) Let $abm \in N$ for two non-unit elements $a, b \in R$ and $m \in M$ such that $a \notin Z(M)$. Then $abm \in N \cap abM$, so by (3), $abm \in aN$. Hence $bm \in N$ and $N$ is a $(1,r)$-submodule of $M$. □

**Proposition 2.6.** *Suppose that $M$ is an $R$-module such that every proper submodule of $M$ is a $(1,r)$-submodule. Then for every submodule $N$ of $M$ and two non-unit elements $a,b \in R$ such that $a \notin Z(M)$, $abN = bN$.*

*Proof.* Let $N$ be a submodule of $M$ and $a \notin Z(M)$. We show that $abN = bN$. Case1: If $N = M$ and $abM \neq bM$, then there exists a $m \in M$ such that $bm \notin abM$. Hence by assumption, $abM$ is a $(1,r)$-submodule. Since $abm \in abM$ and $a \notin Z(M)$, so $bm \in abM$, which is a contradiction. Therefore $abM = bM$. Case2: If $N$ be a proper submodule of $M$, then $abN \subseteq N \subsetneq M$ and $abN$ is a $(1,r)$-submodule. Since for every $n \in N$, $abn \in abN$ and $a \notin Z(M)$. Therefore $bn \in abN$ for every $n \in N$ and $bN \subseteq abN$, so $abN = bN$. □

In the next, we present an alternative definition of $(1,r)$-submodules based on ideals of $R$ and submodules of $M$.

**Theorem 2.7.** *Let $M$ be an $R$-module and $N$ be a proper submodule of $M$. The following statements are equivalent:*
*(1) $N$ is a $(1,r)$-submodule of $M$.*
*(2) For every proper ideal $I$ and $J$ of $R$ and every submodule $K$ of $M$ we have $IJK \subseteq N$ such that $I \nsubseteq Z(M)$ then $JK \subseteq N$.*

*Proof.* (1)⇒(2) Let $IJK \subseteq N$ such that $I \nsubseteq Z(M)$. Hence there is $a \in I$ such that $a \notin Z(M)$. Also, for every $j \in J$ and $k \in K$ we have $ajk \in N$. Since $a \notin Z(M)$, so by (1), $jk \in N$ for every $j \in J$ and $k \in K$. Therefore $JK \subseteq N$.
(2)⇒(1) Suppose that $abm \in N$ for two non-unit elements $a,b \in R$ and $m \in M$, such that $a \notin Z(M)$. Let $I = (a)$, $J = (b)$ and $K = (m)$. Hence $IJK \subseteq N$ and $I \nsubseteq Z(M)$, so by assumption, $JK \subseteq N$. Hence $bm \in N$. □

**Proposition 2.8.** *Let $N$ be a proper submodule of $M$. If $N$ is a $(1,r)$-submodule, then $(N :_M t)$ is a $(1,r)$-submodule for every $t \notin Z(M)$ .*

*Proof.* Let $abm \in (N :_M t)$ for non-unit elements $a,b \in R$ and $m \in M$ such that $a \notin Z(M)$. We show that $bm \in (N :_M t)$. Let $t$ be non-unit, then $abmt \in N$, $t \notin Z(M)$. Since $N$ is $(1,r)$-submodule hence $abm \in N$. Since $a \notin Z(M)$, so $bm \in N$. Hence $bmt \in N$ and $bm \in (N :_M t)$. If $t$ be unit, then $abm \in N$. So $bm \in N$ and $bmt \in N$. □

**Proposition 2.9.** *Let $N$ be a $(1,r)$-submodule of $M$ and $S$ be a non-empty subset of $R$ such that $S \nsubseteq (N : M)$, then $(N :_M S)$ is a $(1,r)$-submodule of $M$. In general if $S \nsubseteq (N :_R M)$, then $Ann_M(S)$ is a $(1,r)$-submodule of $M$.*

*Proof.* Let $abm \in (N :_M S)$ for non-unit elements $a,b \in R$ and $m \in M$. Then $abmS \subseteq N$, so $a \in Z(M)$ or $bmS \subseteq N$. Hence $a \in Z(M)$ or $bm \in (N :_M S)$. □

**Lemma 2.10.** *Let $M$ be an $R$-module and $S$ be a multiplicatively closed subset of $R$. Then:*
1. *$Z_R(M)$ is a ideal of $R$ if and only if $Z_R(M)$ is closed under summation.*
2. *Suppose that $Z(M)$ is a subring of $R$ with $Z_R(M) \cap S = \emptyset$, then $S^{-1}Z_R(M) = Z_{S^{-1}R}(S^{-1}M)$.*
3. *$Z_{R_1 \times R_2}(M_1 \times M_2) = \big(Z_{R_1}(M_1) \times R_2\big) \cup \big(R_1 \times Z_{R_2}(M_2)\big)$*

**Theorem 2.11.** *Let $N$ be a $(1,r)$-submodule of $M$ and $S$ be a multiplicatively closed subset. Then $S^{-1}N$ is a $(1,r)$-submodule of $S^{-1}R$-module $S^{-1}M$.*

*Proof.* Suppose that $\frac{a}{t}, \frac{b}{s}$ are two non-unit elements of $S^{-1}R$ and $\frac{m}{r} \in S^{-1}M$ such that $\frac{a}{t}\frac{b}{s}\frac{m}{r} \in S^{-1}M$ for $t,s,r \in S$. Since $\frac{a}{t}$ is non-unit, so $a \in R$ is non-unit. by asumption, there is $u \in S$ such that $uabm \in N$. Since $N$ is a $(1,r)$-submodule, so $a \in Z(M)$ or $ubm \in N$. If $a \in Z(M)$, then there is a $0 \neq m'$ such that $am' = 0$. Hence $\frac{a}{t}\frac{m'}{1} = \frac{0}{1}$ and $\frac{a}{t} \in Z(S^{-1}M)$. Also if $ubm \in N$, then $\frac{b}{s}\frac{m}{r} = \frac{ubm}{usr} \in S^{-1}N$. Thenfore $S^{-1}N$ is a $(1,r)$-submodule of $S^{-1}M$. □

**Theorem 2.12.** *Let $f : M \to M'$ be a $R$-homomorphism.*
*(1) Let $f$ be a epimorphism and $N'$ be a $(1,r)$-submodule of $M'$ such that $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a $(1,r)$-submodule of $M$.*
*(2) Let $f$ be a isomorphism and $N$ be a $(1,r)$-submodule of $M$ such that $Ker(f) \subseteq N$, then $f(N)$ is a $(1,r)$-submodule of $M'$.*

*Proof.* Let $abm \in f^{-1}(N')$, for two non-unit elemets $a, b \in R$ and $m \in M$. Hence $abf(m) = f(abm) \in N'$. Since $N'$ is a $(1,r)$-submodule, so $a \in Z(M')$ or $bf(m) \in N'$. If $a \in Z(M')$, then there is $0 \neq m' \in M'$ such that $am' = 0$. Let $m' = f(m)$, then $am \in Ker(f)$. Thus $a \in Z(M)$. If $bf(m) \in N'$, then $bm \in f^{-1}(N')$. (2) The proof proceeds similarly. □

Suppose that $R = R_1 \times R_2$ is a direct product of commutative rings. Let $M_i$ be an $R_i$-module for $i = 1, 2$. Then $M = M_1 \times M_2$ can be viewed as an $R$-module using scalar multiplication $(r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$.

**Theorem 2.13.** *Let the situation be as described. Let $M = M_1 \times M_2$ and $N_1$ be a proper submodule of $M_1$. The following statements are equivalent:*
(1) *$N_1$ is a $(1,r)$-submodule of $M_1$.*
(2) *$N_1 \times M_2$ is a $(1,r)$-submodule of $M$.*

*Proof.* (1)⇒(2) Suppose that $(a_1, a_2)(b_1, b_2)(m_1, m_2) \in N_1 \times M_2$ such that $(a_1, a_2), (b_1, b_2)$ are non-unit and $(m_1, m_2) \in M_1 \times M_2$. We note that at most two members can be the unit. Also we know that $a_1 b_1 m_1 \in N_1$ and $a_2 b_2 m_2 \in M_2$. Case i If $a_1$ and $b_1$ be unit, then $a_2$ and $b_2$ are non-unit. Hence $b_1 m_1 \in N_1$ and $b_2 m_2 \in M_2$, so in this case $(b_1 m_1, b_2 m_2) \in N_1 \times M_2$. Case ii If only $b_1$ be a unit, then $(a_1 m_1, a_2 m_2) \in N_1 \times M_2$. Case iii If $a_1$ and $b_2$ be unit, then $(b_1 m_1, b_2 m_2) \in N_1 \times M_2$. Case iv If all of elemens be non-unit, then by lemma 2.10, the statemet is true.
(2)⇒(1) Suppose that $a_1, b_1$ be two non-unit elemens of $R_1$ and $m_1 \in M_1$ such that $a_1 b_1 m_1 \in N_1$. Then $(a_1, 1)(b_1, 1)(m_1, 0) \in N_1 \times M_2$, so by (2), either $(a_1, 1) \in Z(M_1 \times M_2)$ or $(b_1 m_1, 0) \in N_1 \times M_2$. Casei There is $(0,0) \neq (m_1', m_2') \in M_1 \times M_2$ such that $(a_1, 1)(m_1', m_2') = (0,0)$. Thus $a_1 m_1' = 0$ and $m_2' = 0$. Hence $m_1' \neq 0$ and $a_1 \in Z(M_1)$. So $N$ is a $(1,r)$-submodule. Case ii If $(b_1 m_1, 0) \in N_1 \times M_2$, then $b_1 m_1 \in N_1$. Hence $N_1$ is a $(1,r)$-submodule. □

We note that in the above theorem it is not possible to replace $M_2$ with a proper submodule of it.

**Example 2.14.** Let $M = \mathbb{Q}$ be as a $\mathbb{Z}$-module. Then $N_1 = N_2 = 3\mathbb{Z}$ are $(1,r)$-submodules of $M$. We know $(1,3).(3,1).(2,2) \in 3\mathbb{Z} \times 3\mathbb{Z}$ for $(1,3), (3,1) \in \mathbb{Z} \times \mathbb{Z}$ and $(2,2) \in \mathbb{Q} \times \mathbb{Q}$ but neither $(1,3), (3,1) \in Z(M)$ nor $(1,3)(2,2), (3,1)(2,2) \in N_1 \times N_2$. So $N_1 \times N_2$ is not a $(1,r)$-submodule of $M_1 \times M_1$.

Next, we will introduce and study the concept of $(2,r)$-submodules as another generalization of $r$-submodules.

## 3. (2,r)-submodules

**Definition 3.1.** Let $N$ be a proper submodule of $M$. Then $N$ is called a $(2,r)$-submodule of $M$, whenever $abm \in N$ for elements $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in Z_R(M)$.

**Example 3.2.** The following statements hold:
(1) Every prime submodule is a $(2,r)$-submodule.
(2) Every $(2,r)$-submodule is a $(1,r)$-submodule.

*Proof.* (1) Let $abm \in N$ for $a, b \in R$ and $m \in M$. Then $(bm) \in N$ or $a \in (N : M)$, so $bm \in N$ or $am \in N$. Hence $N$ is a $(2,r)$-submodule.
(2) Let $abm \in N$ for two non-unit elements $a, b \in R$ and $m \in M$, so by assumption, $am \in N$ or $bm \in N$, that implies the statement is true or $ab \in Z(M)$. In this case there is $0 \neq m \in M$ such that $abm = 0$. If $bm = 0$, then $b \in Z(M)$ which implies $N$ is a $(1,r)$-submodule. If $bm \neq 0$, then $a \in Z(M)$, that we conclude $N$ is a $(1,r)$-submodule. □

### 3.1. *Some Properties of $(2,r)$-submodules*

**Proposition 3.3.** *Let $N$ be a proper submodule of $M$. Then $N$ is $(2,r)$-submodule if and only if for every $a, b \in R$, $(N : ab) = (N : a)$ or $(N : ab) = (N : b)$ or $(N : ab) \cap Ann_M(ab) \neq 0$.*

*Proof.* (⇒) Let $m \in (N : ab)$, then $abm \in N$. So $am \in N$ or $bm \in N$ or $ab \in Z(M)$. Casei $m \in (N : a)$ or $m \in (N : b)$, which implies $(N : ab) \subseteq (N : a)$ or $(N : ab) \subseteq (N : b)$. We know that $(N : a) \subseteq (N : ab)$ and $(N : b) \subseteq (N : ab)$. Therefore, the above two equalities hold. Caseii There is $0 \neq m' \in M$ such that $abm' = 0 \in N$. Hence $0 \neq m' \in (N : ab) \cap Ann_M(ab)$.
(⇐) Let $abm \in N$ for $a, b \in R$ and $m \in M$, then $m \in (N : ab)$. So by assumption, $m \in (N : a)$ or $m \in (N : b)$, which implies $am \in N$ or $bm \in N$. Since $(N : ab) \cap Ann_M(ab) \neq 0$, then there is a $0 \neq m' \in M$ such that $abm' = 0$. Hence $ab \in Z(M)$. □

**Proposition 3.4.** *Let $M$ be a finitely generated $R$-module and $\{N_i : i \in I\}$ be a directed set of $(2,r)$-submodules of $M$. Then the submodule $N = \cup_{i\in I} N_i$ is a $(2,r)$-submodule of $M$.*

*Proof.* Since $M$ is a finitely generated $R$-module, so $N$ is a proper submodule of $M$. Suppose that for $a, b \in R$ and $m \in M$ we have $abm \in N = \cup_{i\in I} N_i$ and $ab \notin Z(M)$. Then there exist $i \in I$ such that $abm \in N_i$. Since $ab \notin Z(M)$, so by assumption, $am \in N_i \subseteq \cup_{i\in I} N_i = N$ or $bm \in N_i \subseteq \cup_{i\in I} N_i = N$. □

**Theorem 3.5.** *Let $N$ be a $(2,r)$-submodule of $M$ and $S$ be a multiplicatively closed subset. Then $S^{-1}N$ is a $(2,r)$-submodule of $S^{-1}R$-module $S^{-1}M$.*

*Proof.* Suppose that $\frac{a}{t}, \frac{b}{s} \in S^{-1}R$ and $\frac{m}{r} \in S^{-1}M$ such that $\frac{a}{t}\frac{b}{s}\frac{m}{r} \in S^{-1}M$ for $t, s, r \in S$. So there is $u \in S$ such that $ab(um) \in N$. Since $N$ is a $(2,r)$-submodule, so $ab \in Z(M)$ or $a(um) \in N$ or $b(um) \in N$. If $ab \in Z(M)$, then there is a $0 \neq m'$ such that $abm' = 0$. Hence $\frac{a}{t}\frac{b}{s}\frac{m'}{1} = \frac{0}{1}$ and $\frac{a}{t}\frac{b}{s} \in Z(S^{-1}M)$. Also if $a(um) \in N$ or $b(um) \in N$, then $\frac{a}{t}\frac{m}{r} = \frac{uam}{utr} \in S^{-1}N$ or $\frac{b}{s}\frac{m}{r} = \frac{ubm}{usr} \in S^{-1}N$. Thenfore $S^{-1}N$ is a $(2,r)$-submodule of $S^{-1}M$. □

**Theorem 3.6.** *Let $M = M_1 \times M_2$ and $N_1$ be a proper submodule of $M_1$. The following statements are equivalent:*
*(1) $N_1$ is a $(2,r)$-submodule of $M_1$ with $R_2 \neq Z(M_2)$.*
*(2) $N_1 \times M_2$ is a $(2,r)$-submodule of $M$.*

*Proof.* (1)⇒(2) Suppose that $(a_1, a_2)(b_1, b_2)(m_1, m_2) \in N_1 \times M_2$ and $(m_1, m_2) \in M_1 \times M_2$, so $a_1b_1m_1 \in N_1$ and $a_2b_2m_2 \in M_2$. Hence $a_1m_1 \in N_1$ or $b_1m_1 \in N_1$ or $a_1b_1 \in Z(M_1)$. Hence $(a_1m_1, a_2m_2) \in N_1 \times M_2$ or $(b_1m_1, b_2m_2) \in N_1 \times M_2$ or $(a_1b_1, a_2b_2) \in Z(M_1) \times R_2 \subseteq Z(M_1 \times M_2)$.
(2)⇒(1) Suppose that $a_1, b_1 \in R_1$ and $m_1 \in M_1$ such that $a_1b_1m_1 \in N_1$. Then $(a_1, 1)(b_1, 1)(m_1, 0) \in N_1 \times M_2$, so by (2), either $(a_1, 1)(m_1, 0) \in N_1 \times M_2$ or $(b_1m_1, 0) \in N_1 \times M_2$ or $(a_1b_1, 1) \in Z(M_1 \times M_2)$. Hence $a_1m_1 \in N_1$ or $b_1m_1 \in N_1$ which implies $N_1$ is a $(2,r)$-submodule. Also in last cases $(a_1b_1, 1) \in Z(M_1) \times R_2$, which implies $a_1b_1 \in Z(M_1)$ ( hence $N_1$ is a $(2,r)$-submodule) or $(a_1b_1, 1) \in R_1 \times Z(M_2)$ which is a contradiction with $1 \in Z(M_2) \neq R_2$. □

In the future, we want to investigate more theorems about this submodule. For example, we also examine them in multiplicative modules, fraction modules and amalgamation modules. It is also possible to use $T(M)$ instead of $Z(M)$, and provide another different generalization of $r$-submodules and prime submodules.

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# Relation between strongly clean and uniquely clean elements

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**Abstract**

It is well known that every unit regular ring is clean. At first in one example, we will inspect how to write an element of a unit regular ring as unit regularity and cleanness form. Moreover, we provided an example to show that uniquely clean element maybe not strongly clean. Also, we obtain necessary and sufficient condition for strongly cleanness of uniquely clean elements.

## 1. Introduction

Nicholson and Zhou [4] proved that each idempotent in any uniquely clean ring is central so uniquely clean rings are strongly clean but we show that if ring R isn't uniquely clean then uniquely clean element maybe not strongly clean. In the following, we have several definitions.

**Definition 1.1.** [4] An element $a \in R$ is clean if we can write $a = u + e$, where $u \in U(R)$ is a unit and $e \in R$ is an idempotent of $R$. If $eu = ue$ we say $a$ is strongly clean.

The ring $R$ is called clean ring (strongly clean) if every element of $R$ is clean (strongly clean).

**Definition 1.2.** [4] An element is uniquely clean if it has exactly one representation as the sum of an idempotent and a unit. A ring $R$ is called a uniquely clean ring if every element is uniquely clean.

**Definition 1.3.** [2] An element in a ring $R$ is called unit-regular if it can be expressed as a product of a unit and an idempotent. The ring $R$ is called a unit regular ring if every element is unit-regular.

In [1], Camillo and Khurana proved that every unit-regular ring is clean. We show $M_2(\mathbb{R})$ is unit regular and so clean ring and we will how represent its construction elements in the form unit regularity and cleanity, and we will see that this ring is not uniquely and strongly clean ring. As [4], we know that uniquely clean rings are always strongly clean. We will search when this result holds element-wise.

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**Lemma 1.4.** *In any ring every idempotent is clean element.*

*Proof.* If $e$ is an idempotent then $e = (1-e) + (2e-1)$ in which $(1-e)$ is an idempotent and $2e-1$ is a unit with inverse $2e-1$. □

**Example 1.5.** Let $A \in M_2(\mathbb{R})$. If $A$ is a unit then $A = AI$ and $A = A + 0$ is respectively unit regular and clean presentations of $A$.

If $A$ is not a unit of $M_2(\mathbb{R})$ then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad = bc$. If $a$ and $b$ is not zero, let $r = \frac{c}{a} = \frac{d}{b}$, then $A = \begin{bmatrix} 1 & 0 \\ r & 0 \end{bmatrix}\begin{bmatrix} a & b \\ x & y \end{bmatrix}$, we see $\begin{bmatrix} 1 & 0 \\ r & 0 \end{bmatrix}$ is idempotent and we can choose $x$ and $y$ such a way $\begin{bmatrix} a & b \\ x & y \end{bmatrix}$ become a unit element.

Now, we show $A$ is clean element. Since $A$ is not unit then we have $ad = bc$. Now, if $a \neq 1$ and $d \neq 0$ then we obtain

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a-1 & b \\ 0 & d \end{bmatrix},$$

where $\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$ is an idempotent and $\begin{bmatrix} a-1 & b \\ 0 & d \end{bmatrix}$ is a unit element.

If $a \neq 1$ and $d = 0$ then yields

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a-1 & b \\ c & -1 \end{bmatrix}$$

is a clean decomposition.

If $a = 1$ and $d \neq 0$ then we have

$$\begin{bmatrix} 1 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & b \\ c & d-1 \end{bmatrix}$$

is a clean decomposition. Finally if $a = 1$ and $d = 0$ then $\begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$ but in each case $A$ is an idempotent. Hence, by using Lemma 1.4, we conclude that $A$ is clean.

In the following, we show there exist a ring $R$ and an element $a$ in $R$ such that $a$ is uniquely clean but $a$ isn't strongly clean, then we will show in any ring, a uniquely clean element under certain condition will be strongly clean.

**Example 1.6.** Let $R = M_2(\mathbb{Z})$, and $\begin{bmatrix} 14 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + U$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Idm(R), U \in U(R)$. In view of [3], we have

$$\det \begin{bmatrix} 14-a & 5-b \\ -c & -d \end{bmatrix} = \pm 1$$

and $ad - bc = 0$ and also $d - 1 \in c\mathbb{Z}$. So we obtain $5c - 14d = \pm 1$. First, we will solve $5c - 14d = 1$. This diophantine's equation answers in the form $c = 14r + 3$, $d = 5r + 1$, now because $d - 1 \in c\mathbb{Z}$, there exist $s \in \mathbb{Z}$ such that $5r = s(14r + 3)$ so $5r$ divisible by $s$ and $3s$ divisible by $r$. so there exists $n$ and $m$ in $\mathbb{Z}$ such that $sn = 5r$, $rm = 3s$, therefore $rsnm = 15rs$ and this implied $r = 0$ or $nm = 15$. If $r = 0$ then $c = 3$, $d = 1$. If $nm = 15$ then $n = \pm 1; \pm 15; \pm 3; \pm 5$. But by inspection, in these cases has no integer solution except $c = 3$; $d = 1$.

Now, if

$$5c - 14d = -1 \tag{1}$$

then $c = 14r + 11$, $d = 5r + 4$ and by [3] $d - 1 \in c\mathbb{Z}$ therefore there exist $s \in \mathbb{Z}$ such that $5r + 3 = s(14r + 11)$, if $r > 0$, $14r + 11 > 5r + 3$ therefore the Eq. (1) has no solution. If $r \leq -1$ then $14r + 11 < 5r + 3$ so $s(14r + 11) \leq 5r + 3$ and therefore the Eq. (1) has no solution. Hence, it is evident that

$$\begin{bmatrix} 14 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 3 & 1 \end{bmatrix} + U.$$

Since $\begin{bmatrix} a & b \\ 3 & 1 \end{bmatrix} \in Idm(R)$ then we obtain that $a = b = 0$. So, we conclude that

$$\begin{bmatrix} 14 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 14 & 5 \\ -3 & -1 \end{bmatrix}$$

therefore $\begin{bmatrix} 14 & 5 \\ 0 & 0 \end{bmatrix}$ is a uniquely clean and not strongly clean.

**Lemma 1.7.** *Let $R$ be a noncommutative ring with identity and $e \in R$ is an idempotent then for any $r \in R$, $(e + er - ere)$, $(e - er + ere)$, $(e + re - ere)$ and $(e - re + ere)$ are idempotents in $R$ and $(1 + er - ere)$ is a unit whit inverse $(1 - er + ere)$, and $(1 + re - ere)$ is a unit whit inverse $(1 - re + ere)$.*

*Proof.* The result is easily obtained by multiplying. □

In [4] Nicholson and Zhou proved in any uniquely clean ring each idempotent is central, therefore any uniquely clean ring is strongly clean. Example 1.6 provided that each uniquely clean element maybe not strongly clean. In the following theorem, we provide a necessary and sufficient condition for it.

**Theorem 1.8.** *Let $R$ be a arbitrary ring and $a \in R$ is a uniquely clean element such that $a = e + u$, $e \in Idm(R)$, $u \in U(R)$, then $a$ is strongly clean if and only if $e \in Z(R)$.*

*Proof.* If $a \in R$ is uniquely and strongly clean then there exist uniquely $e \in Idm(R)$, $u \in U(R)$ such that $a = e + u$ and $eu = ue$, now by using Lemma 1.7 for any $r \in R$, $(1 + er - ere)$ is a unit, let $r = u^{-1}x$ so $1 + eu^{-1}x - eu^{-1}xe$ is a unit of $R$, since $eu = ue$ then $eu^{-1} = u^{-1}e$. Therefore, the assertion

$$1 + eu^{-1}x - eu^{-1}xe = 1 + u^{-1}ex - u^{-1}exe = u^{-1}(u + ex - exe)$$

is a unit. Thus, $(u + ex - exe)$ is a unit of $R$. Also by applying Lemma 1.7, we have $(e - ex + exe)$ is an idempotent of $R$. Hence, we have

$$a = e + u = (e - ex + exe) + (u + ex - exe).$$

But $a$ is uniquely clean so $e = (e - ex + exe)$. Thus, $ex = exe$. Similar the pervious method, we have $xe = exe$, thus $ex = xe$. Hence $e \in Z(R)$. The sufficient condition of Theorem 1.8 is obvious. This complete the proof of Theorem 1.8. □

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# Solving inverse spectral problems for a conformable Sturm-Liouville operator by two given spectra

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**Abstract**

In this paper, we study a spectral Sturm-Liouville boundary value problem with the conformable derivative. We prove the uniqueness solution of the inverse problem for this operator by two spectra.

## 1. Introduction

Inverse spectral problems for various types of Sturm-Liouville operators are a part of research in mathematical physics that arise in various fields of physics, quantum and mechanics [4]. Inverse problems for Sturm-Liouville equations with the ordinary derivation have been studied in [1]. The conformable fractional derivation instead of the ordinary derivation has attracted much attention for investigating of these problems. Therefore inverse problems for Sturm-Liouville equations with the conformable derivative have been investigated by researchers recently [3]. In this paper, we study the inverse spectral problem for the conformable Sturm-Liouville equation with the spectral boundary conditions. We prove the uniqueness of the solution of the inverse problem for the conformable fractional Sturm-Liouville operators by two spectra.
Consider the conformable fractional Sturm-Liouville equation

$$-C_x^\alpha C_x^\alpha y(x,k) + p(x)y(x,k) = \lambda y(x,k), \quad x \in (0,\pi), \tag{1}$$

with the spectral boundary condition at $x = 0$,

$$C_x^\alpha y(0,k) - hky(0,k) = 0, \tag{2}$$

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and with the boundary condition at $x = \pi$,

$$y(\pi, k) = 0. \tag{3}$$

Here $C_x^\alpha$ is the conformable fractional derivative of order $\alpha \in (0, 1]$. The coefficient $h$ is complex numbers and $\lambda = k^2$ is a spectral parameter. The complex-valued function $p(x)$ is continuous on $[0, \pi]$.
We denote the boundary value problem (1)-(3) by $L_\alpha := L_\alpha(p, h)$. We also remark that some definitions and properties of the conformable fractional calculus can be found in [5].

## 2. Preliminaries

Let $A(x, k)$ be the solution of Eq. (1) under the initial conditions

$$A(0, k) = 1, \quad C_x^\alpha A(0, k) = hk.$$

For each fixed $x$, this function and its derivative with respect to $x$ are entire in $k$. From [3], we have the following asymptotic formulae for sufficiently large $|k|$ and $x \in [0, \pi]$,

$$A(x, k) = \sqrt{1 + h^2} \cos\left(\frac{k}{\alpha} x^\alpha - \sigma\right) + O\left(\frac{1}{k} \exp\left(\frac{|\Im k|}{\alpha} x^\alpha\right)\right), \tag{4}$$

$$C_x^\alpha A(x, k) = -k\sqrt{1 + h^2} \sin\left(\frac{k}{\alpha} x^\alpha - \sigma\right) + O\left(\exp\left(\frac{|\Im k|}{\alpha} x^\alpha\right)\right), \tag{5}$$

where $\sigma = \frac{1}{2i} ln \frac{i-h}{i+h}$.
Denote the entire function $\Delta_\alpha(k)$ (entire in $k$),

$$\Delta_\alpha(k) = A(\pi, k),$$

as the characteristic function of $L_\alpha$. Therefore by using (4) and (5), we can give for sufficiently large $|k|$,

$$\Delta_\alpha(k) = \sqrt{1 + h^2} \cos\left(\frac{k}{\alpha} \pi^\alpha - \sigma\right) + O\left(\frac{1}{k} \exp\left(\frac{|\Im k|}{\alpha} \pi^\alpha\right)\right). \tag{6}$$

Let $\delta > 0$ be fixed and $C_\delta > 0$ be a constant. Put $G_\delta := \{k; | \, k - k_n \, | \geq \delta, \ \forall n\}$. Taking (6) and the known technique [6], one gets

$$|\Delta_\alpha(k)| \geq C_\delta \exp\left(\frac{|\Im k|}{\alpha} \pi^\alpha\right), \quad k \in G_\delta.$$

By the Rouche's theorem [2] and the known technique [6], one can give that the roots of the characteristic function $\Delta_\alpha(k)$ have the asymptotics

$$k_n = \frac{\alpha}{\pi^\alpha} \left(n\pi + \frac{\pi}{2} + \sigma\right) + o\,(1)\,,$$

for large enough $n$.
We consider the boundary value problem $L_{1\alpha}$ with the boundary condition $y(0, k) = 0$ instead of (2) in $L_\alpha$. We assume that $\mu_n$ be the eigenvalues of the boundary value problem $L_{1\alpha}$ which are roots of the characteristic function

$$\Delta_{1\alpha}(k) := B(0, k). \tag{7}$$

Let $B(x, k)$ and $\mathbf{B}(x, k)$ be the solution of Eq. (1) under the initial conditions

$$B(\pi, k) = 0, \quad C_x^\alpha B(\pi, k) = 1,$$
$$\mathbf{B}(0, k) = 0, \quad C_x^\alpha \mathbf{B}(0, k) = 1.$$

Define the meromorphic function

$$\phi(x,k) = -\frac{B(x,k)}{\Delta_\alpha(k)}, \tag{8}$$

which is called the Weyl solution of the boundary value problem $L_\alpha$. Considering the initial conditions at $x = 0$, we can also give

$$\phi(x,k) = \mathbf{B}(x,k) + M(k)A(x,k),$$

in which

$$M(k) := \phi(0,k), \tag{9}$$

and is called the Weyl function of the boundary value problem $L_\alpha$.
By virtue of the results of Ref. [3], we have the following theorem that is necessary for us to prove the main result.

**Theorem 2.1.** *Let $M(k) = \widetilde{M}(k)$. Then $p(x) = \widetilde{p}(x)$, a.e. on $(0,\pi)$ and $h = \widetilde{h}$.*

## 3. Result

Here we study the inverse problem for the boundary value problem $L_\alpha$. The inverse problem is formulated as follows: Given two spectra $\{\lambda_n, \mu_n\}$, find the coefficients of the boundary value problem $L_\alpha$.
To show the uniqueness theorem in this section, alongside $L_\alpha := L_\alpha(p,h)$, a boundary value problem $\tilde{L}_\alpha := L_\alpha(\widetilde{p},,\widetilde{h})$ of the similar form (1)-(3) is considered. We suppose that if $\alpha$ signifies an object relevant to $L$, then $\widetilde{\alpha}$ will signify the similar object relevant to $\tilde{L}$.

**Theorem 3.1.** *Let $\lambda_n = \tilde{\lambda}_n$ and $\mu_n = \widetilde{\mu}_n$. Then $p(x) = \widetilde{p}(x)$ a.e. on $(0,\pi)$ and $h = \widetilde{h}$.*

**Proof.** The equality $\lambda_n = \tilde{\lambda}_n$ gives that $h = \widetilde{h}$. On the other hand, the entire functions $\Delta_\alpha(\lambda)$ and $\Delta_{1\alpha}(\mu)$ can be determined by their zeros on the base of the Hadamard's factorization theorem as

$$\Delta_\alpha(\lambda) = \frac{\pi^{2\alpha-2}}{\alpha^2}\prod_{n=0}^{\infty}\frac{\lambda_n - \lambda}{(n+\frac{1}{2}+\sigma)^2}, \quad \Delta_{1\alpha}(\mu) = \frac{\pi^{2\alpha-2}}{\alpha^2}\prod_{n=0}^{\infty}\frac{\mu_n - \mu}{n^2}.$$

Thus, by the hypothesis of the theorem $\Delta_\alpha(\lambda) = \widetilde{\Delta}_\alpha(\lambda)$ and $\Delta_{1\alpha}(\mu) = \widetilde{\Delta}_{1\alpha}(\mu)$. Together with (7), (8) and (9), this yields that $M(k) = \widetilde{M}(k)$ and so Theorem 2.1 completes the proof. □

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# Analytical solutions for the nonlinear partial differential equations using an algebraic method

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**Abstract**

In this paper, we apply an algebraic method to study the exact solutions of nonlinear partial differential equations. This approach is characterized by its simplicity and it can be applied to many nonlinear differential equations.

## 1. Introduction

Many nature phenomena can be modeled by nonlinear partial differential equations(PDEs) such as signal processing, fluid mechanics, engineering, hydrodynamics, chemistry, optics, control theory, biology, etc.[1, 3, 4]. Finding exact solutions of most of PDEs is not easy, but studying exact solutions for these equations is a continuing investigation. Many powerful methods for obtaining exact solutions of PDEs have been presented such as Lie group method[5], a numerical method using Bernstein polynomials[2], exp-function method[1] and so on.
In this paper, we study the Maccari equation:

$$\psi'' - \frac{1}{\omega^2(2\kappa-\rho)}\psi^3 - \frac{(\kappa^2+\lambda)}{\omega^2}\psi = 0, \tag{1}$$

## 2. Description of method

In this section, we consider the following ODE

$$\tilde{R}(\Psi, \Psi', \Psi'', \Psi''', \ldots) = 0. \tag{2}$$

Here $\Psi^{(n)} = \frac{d^n\Psi}{d\xi^n}$. Suppose the solution of Eq. (2) be:

$$\Psi(\xi) = \sum_{j=0}^{n} A_j s^{j\,\Lambda(\xi)}, \tag{3}$$

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where the positive integer $n$ can be calculated by considering the homogeneous balance between the highest order derivatives and the highest nonlinear terms of $\Phi(\xi)$ in equation (2), and $A_j$ ($A_n \neq 0$) are constants to be determined later. Here $\Lambda = \Lambda(\xi)$ satisfies the following ODE

$$(s^{\Lambda(\xi)})' = m + n\, s^{\Lambda(\xi)} + r\, s^{2\Lambda(\xi)} \tag{4}$$

or

$$\Lambda'(\xi) = \frac{1}{\ln(s)}(m\, s^{-\Lambda(\xi)} + n + r\, s^{\Lambda(\xi)}), \tag{5}$$

where $m, n$ and $r$ are constants and which has the following special solutions.

Case1: For $n^2 - mr < 0$ and $r \neq 0$,

$$s^{\Lambda(\xi)} = [\frac{-n}{r} + \frac{\sqrt{-(n^2 - mr)}}{r} \tan(\frac{\sqrt{-(n^2 - mr)}}{2}\xi)], \tag{6}$$

$$s^{\Lambda(\xi)} = [\frac{-n}{r} + \frac{\sqrt{-(n^2 - mr)}}{r} \cot(\frac{\sqrt{-(n^2 - mr)}}{2}\xi)]. \tag{7}$$

Case2: For $n^2 - mr > 0$ and $r \neq 0$,

$$s^{\Lambda(\xi)} = [\frac{-n}{r} - \frac{\sqrt{n^2 - mr}}{r} \tanh(\frac{\sqrt{n^2 - mr}}{2}\xi)], \tag{8}$$

$$s^{\Lambda(\xi)} = [\frac{-n}{r} - \frac{\sqrt{n^2 - mr}}{r} \coth(\frac{\sqrt{n^2 - mr}}{2}\xi)]. \tag{9}$$

Case3: For $n^2 + m^2 > 0,\ r \neq 0,$ and $r = -m$,

$$s^{\Lambda(\xi)} = [\frac{n}{m} + \frac{\sqrt{n^2 + m^2}}{m} \tanh(\frac{\sqrt{n^2 + m^2}}{2}\xi)], \tag{10}$$

$$s^{\Lambda(\xi)} = [\frac{n}{m} + \frac{\sqrt{n^2 + m^2}}{m} \coth(\frac{\sqrt{n^2 + m^2}}{2}\xi)]. \tag{11}$$

Case4: For $n^2 + m^2 < 0,\ r \neq 0$ and $r = -m$,

$$s^{\Lambda(\xi)} = [\frac{n}{m} + \frac{\sqrt{-(n^2 + m^2)}}{m} \tan(\frac{\sqrt{-(n^2 + m^2)}}{2}\xi)], \tag{12}$$

$$s^{\Lambda(\xi)} = [\frac{n}{m} + \frac{\sqrt{-(n^2 + m^2)}}{m} \cot(\frac{\sqrt{-(n^2 + m^2)}}{2}\xi)]. \tag{13}$$

Case5: For $n^2 - m^2 < 0$ and $r = m$,

$$s^{\Lambda(\xi)} = [-\frac{n}{m} + \frac{\sqrt{-(n^2 - m^2)}}{m} \tan(\frac{\sqrt{-(n^2 - m^2)}}{2}\xi)], \tag{14}$$

$$s^{\Lambda(\xi)} = [-\frac{n}{m} + \frac{\sqrt{-(n^2 - m^2)}}{m} \cot(\frac{\sqrt{-(n^2 - m^2)}}{2}\xi)]. \tag{15}$$

Case6: For $n^2 - m^2 > 0$ and $r = m$,

$$s^{\Lambda(\xi)} = [-\frac{n}{m} + \frac{\sqrt{n^2 - m^2}}{m} \tanh(\frac{\sqrt{n^2 - m^2}}{2}\xi)], \tag{16}$$

$$s^{\Lambda(\xi)} = [-\frac{n}{m} + \frac{\sqrt{n^2 - m^2}}{m} \coth(\frac{\sqrt{n^2 - m^2}}{2}\xi)]. \tag{17}$$

Case7: For $m\,r < 0$, $r \neq 0$ and $n = 0$,

$$s^{\Lambda(\xi)} = [\sqrt{\frac{-m}{r}} \tanh(\frac{\sqrt{-m\,r}}{2}\xi)], \tag{18}$$

$$s^{\Lambda(\xi)} = [\sqrt{\frac{-m}{r}} \coth(\frac{\sqrt{-m\,r}}{2}\xi)]. \tag{19}$$

Case8: For $n = 0$ and $m = -r$.

$$s^{\Lambda(\xi)} = [\frac{-(1 + e^{2m\,\xi}) \pm \sqrt{2(e^{4m\,\xi} + 1)}}{e^{2m\,\xi} - 1}], \tag{20}$$

$$s^{\Lambda(\xi)} = [\frac{-(1 + e^{2m\,\xi}) \pm \sqrt{e^{4m\,\xi} + 6e^{2m\,\xi} + 1}}{2e^{2m\,\xi}}]. \tag{21}$$

Case9: For $m = r = 0$,

$$s^{\Lambda(\xi)} = [\frac{-(1 + e^{2n\,\xi}) \pm \sqrt{2(e^{4n\,\xi} + 1)}}{e^{2n\,\xi} - 1}], \tag{22}$$

$$s^{\Lambda(\xi)} = [\frac{-(1 + e^{2n\,\xi}) \pm \sqrt{e^{4n\,\xi} + 6e^{2n\,\xi} + 1}}{2e^{2n\,\xi}}]. \tag{23}$$

Case10: For $n^2 = m\,r$,

$$s^{\Lambda(\xi)} = [\frac{-m(n\,\xi + 2)}{n^2\,\xi}]. \tag{24}$$

Case11: For $n = k$, $m = 2k$ and $r = 0$,

$$s^{\Lambda(\xi)} = [e^{k\,\xi} - 1]. \tag{25}$$

Case12: For $n = k$, $r = 2k$ and $m = 0$,

$$s^{\Lambda(\xi)} = [\frac{e^{k\,\xi}}{1 - e^{k\,\xi}}]. \tag{26}$$

Case13: For $2n = m + r$,

$$s^{\Lambda(\xi)} = [\frac{1 - me^{\frac{1}{2}(m-r)\,\xi}}{1 - re^{\frac{1}{2}(m-r)\,\xi}}], \tag{27}$$

$$s^{\Lambda(\xi)} = [\frac{me^{\frac{1}{2}(m-r)\,\xi} + 1}{-re^{\frac{1}{2}(m-r)\,\xi} - 1}]. \tag{28}$$

Case14: For $-2n = m + r$,

$$s^{\Lambda(\xi)} = [\frac{e^{\frac{1}{2}(m-r)\xi} + m}{e^{\frac{1}{2}(m-r)\xi} + r}]. \tag{29}$$

Case15: For $m = 0$,

$$s^{\Lambda(\xi)} = [\frac{ne^{n\xi}}{1 + \frac{r}{2}e^{n\xi}}]. \tag{30}$$

Case16: For $n = m = r \neq 0$,

$$s^{\Lambda(\xi)} = [\frac{-(m\xi + 2)}{m\xi}]. \tag{31}$$

Case17: For $n = r = 0$,

$$s^{\Lambda(\xi)} = [\frac{m}{2}\xi]. \tag{32}$$

Case18: For $n = m = 0$,

$$s^{\Lambda(\xi)} = [\frac{-2}{r\xi}]. \tag{33}$$

Case19: For $n = 0$ and $m = r$,

$$s^{\Lambda(\xi)} = [\tan(\frac{m\xi + c_1}{2})]. \tag{34}$$

Case20: For $r = 0$,

$$s^{\Lambda(\xi)} = [e^{n\xi} - \frac{m}{2n}]. \tag{35}$$

where $c_1$ is an arbitrary constant. Now, the exact solutions of FPDEs can be derived as follows. In equation (2), we replace (3) with Eq.(5). Upon performing this substitution, we derive a polynomial in terms of $s^{\Lambda(\xi)}$. Within this polynomial, we group together the terms with identical powers of $s^{\Lambda(\xi)}$ and proceed to equate all coefficients to zero. This procedure results in a set of algebraic equations involving $A_j$ for $j = 0, 1, 2, ..., n$, $\varpi$, and $p$. By solving this system of algebraic equations and subsequently substituting the obtained results along with the general solutions of equations (6)-(35) into (3), the solutions for (2) are determined.

## 3. Application of the method

In this section, we study the Maccari equation:

$$\psi'' - \frac{1}{\omega^2(2\kappa - \rho)}\psi^3 - \frac{(\kappa^2 + \lambda)}{\omega^2}\psi = 0, \tag{36}$$

Balancing $\psi''$ with $\psi^3$ in (36) gives n=1. Therefore, the exact solution of Eq. (36) can be expressed in the form:

$$\psi(\xi) = A_0 + A_1 s^{\Lambda(\xi)}, \quad A_1 \neq 0, \tag{37}$$

where $\Lambda$ is the solution of equation (5). Substituting Eq. (37) along Eq. (5) into Eq. (36) and collecting all the terms of the same power $s^{\Lambda(\xi)}$ and equating them to zero, we obtain a system of algebraic equations for $A_0, A_1, \omega, \kappa, \rho, \lambda, m, n$ and $r$. Solving obtained system using *Mathematica*, we obtain

$$\bullet A_0 = 0,\ A_1 = \frac{\sqrt{r(2\kappa-\rho)(\kappa^2+\lambda)}}{\sqrt{m}}, \qquad n = 0,\ m \neq 0,\ r \neq 0. \tag{38}$$

By using of the (37), (38) and cases (6)-(17) and (20)-(21) respectively, we get

$$\psi_1(x,y,t) = \sqrt{(2\kappa-\rho)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \tan\Big(\frac{\sqrt{mr}}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_2(x,y,t) = \sqrt{(2\kappa-\rho)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \cot\Big(\frac{\sqrt{mr}}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_3(x,y,t) = -\sqrt{(2\kappa-\rho)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \tanh\Big(\frac{\sqrt{-mr}}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_4(x,y,t) = -\sqrt{(2\kappa-\rho)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \coth\Big(\frac{\sqrt{-mr}}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_5(x,y,t) = \sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \tanh\Big(\frac{m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_6(x,y,t) = \sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \coth\Big(\frac{m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_7(x,y,t) = -\sqrt{(2\kappa-\rho)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \tan\Big(\frac{i\,m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_8(x,y,t) = -\sqrt{(2\kappa-\rho)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \cot\Big(\frac{i\,m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_9(x,y,t) = \sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \tan\Big(\frac{m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_{10}(x,y,t) = \sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \cot\Big(\frac{m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_{11}(x,y,t) = -\sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \tanh\Big(\frac{i\,m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_{12}(x,y,t) = -\sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \coth\Big(\frac{i\,m}{2}(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))\Big),$$

$$\psi_{13}(x,y,t) = \sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \Big[\frac{-(1+e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))})}{e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}-1} \pm \frac{\sqrt{2(e^{4m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}+1)}}{e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}-1}\Big],$$

$$\psi_{14}(x,y,t) = \sqrt{(\rho-2\kappa)(\kappa^2+\lambda)}\, e^{i\theta+\sigma\beta(t)-\frac{\sigma^2}{2}t} \Big[\frac{-(1+e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))})}{2e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}} \pm \frac{\sqrt{e^{4m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}+6e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}+1}}{2e^{2m\,(\omega(\frac{x^\gamma}{\gamma}+\rho y-2\kappa t))}}\Big],$$

## Conclusion

In this study, an algebraic method has been used to obtain the analytical solutions of Maccari equation. Our study demonstrates that this method is highly versatile and can be used to study a wide range of nonlinear fractional differential equations.

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# A Game Theory Approach to Taxi driver-Passenger Interactions

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**Abstract**

Using a game theory concept - Bayesian Nash Equilibrium (BNE) - we can frame the interactions between taxi drivers and passengers into a daily occurrence. to train the model on the uncertainty and hidden implicated in these interactions. Consider how you'd feel: As a customer hopping into a taxi, you want to have faith that the driver will find the shortest path or that you will pay a fair fare. Similarly, the driver has no sense whether you're going to be a nice passenger who tips well or a complication of a ride. These types of unknowns put both sides in a position where they are making decisions based on who they think the other side is.

We did so by building a model of this dynamic, in which passengers want to get a good deal - a fair price, a speedy trip to where they want to go, and safe transport – and drivers want to make a good living while preserving their reputation. We also considered things like trust, transparency and how people signal their intentions. A driver might, for example, use GPS to show that they're choosing the best route, and a passenger might post a positive review to indicate that they're happy with the service. What we found out is that when there's anywhere you can add more transparency - whether it's transparent pricing or real-time tracking or rating systems -both drivers and passengers feel more powerful in the transaction. This minimizes misunderstandings and mistrust and makes rides smoother and outcomes happier all round." It illustrates how practical these concepts are, with parts of the ride-sharing experience, like Uber and Lyft, already implementing similar concepts to create trust and improve experiences.

From here, we explore whether trust evolves over time via the repeated rides, considering if different kinds of drivers and passengers treat one another better or worse, and how the context of traffic or the local scene contributes to whether people trust or not. This study is an example of how game theory can be used in a practical context to generate insights that can lead to better policies and systems for everyone involved with these real world issues.

In short, this paper illustrates how an understanding of the "hidden rules" that dictate the dynamic between drivers and passengers can be very important for building trust, strengthening passengers and drivers, and developing a more equitable and effective ride-sharing system.

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# Analysis of Spousal Challenge Using Game Theory

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**Abstract**

This paper demonstrates how game theory can be applied to one of the most common social dilemmas faced by almost every couple in the early stages of their relationship – making mutually agreeable decisions during a situation in which both parties have something to gain or lose. We frame the problem as a dynamic game and determine the strategies for both individuals - derived from the idea of Bayesian Nash equilibrium- that maximize satisfaction and minimize conflicts. The analysis takes into account, among other factors, one partner's uncertainty about the other's preferences - whether he/she will strongly favor his/her preference over theirs or is more likely to give in -and attaches probabilities to this based on prior assumptions. Results of the equilibrium illustrate how strategic proposals can serve the interests of both partners, and lead to mutual desirable outcomes.

Based on their theoretical findings, the study proposes concrete recommendations for practitioners, including randomized methods of decision making, attention to logistical factors when assigning a schedule, planning lists that promote collaboration, and scheduling with a priority on generating balance. The following are practical tips and insights into how young couples can navigate decision-making stumbles out of the gate. Additionally, this cross-disciplinary method also shows the practical applications of math in solving real life dilemma, thereby stressing the importance of math in day-to-day life. This not only highlights the practical relevance of mathematics but also opens up new paths for engaging with mathematical concepts, thus providing valuable insights on how to improve interest and literacy in declining math fields in academia and society at large. This is an example of how the aesthetic and accessible presentations of game theory can be used to tackle a variety of challenges, and how mathematical thinking can be friendly and meaningful.

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# Game Theory Analysis of Husbands Hindering Wives' Education

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**Abstract**

The article explores the problem of husbands preventing wives from pursuing further studies and analyzes it from psychological, legal, and game theory perspectives. The psychological causes of this are economic factors, cultural values, and patriarchal ideology that lead to increased illiteracy and psychological problems among women. Law-wise, education of women is welcomed according to Iranian law, but the husband can oppose his wife's working or continuing studies if he feels that it is against the interests of the family. Dynamically, using game theory, the battle is simulated in the research and there exist two Nash equilibria: the wife capitulates to her husband's opposition or she proceeds and continues studies despite. The study recommends publicity campaigns, financial aid, professional counseling services for schools and legal aid to mitigate this issue. It recommends measures such as lessening the economic burden on the students and creating educational and family counseling centers so that students will not drop out and there would be better mutual understanding between wives and husbands.

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# Rainbow Connection Number of Cartesian Product of Graphs

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**Abstract**

A path in an edge-colored graph $G$ is a rainbow path if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of $G$ is the smallest integer $k$ for which there exists a (not necessarily proper) $k$-edge-coloring of $G$ such that every pair of distinct vertices of $G$ is connected by a rainbow path. In this paper, we study the rainbow connection number of Cartesian products of paths, cycles, complete graphs, and stars.

## 1. Introduction

Edge coloring of a graph is a function from its edge set to the set of natural numbers (called colors). A path in an edge-colored graph with no two edges sharing the same color is called a rainbow path. An edge-colored graph is said to be rainbow connected if every pair of vertices is connected by at least one rainbow path. Such a coloring is called a rainbow coloring of the graph. The minimum number of colors required to rainbow color a connected graph is called its rainbow connection number, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a star is its number of leaves. For a basic introduction to the topic, see Chapter 11 in [4] and for a comprehensive treatment of the area see the recent monograph by Li and Sun [6]. The concept of rainbow coloring was introduced in [3].

Rainbow coloring is used in the modeling of the problem of message transmission with complete security. Suppose that the graph $G$ represents a network. We wish to send messages between any two vertices in a network and we need that each link on the path between the vertices (each edge on the path) be assigned a distinct channel (e.g., a distinct frequency). The color of each edge indicates the frequency used to transmit the message. Clearly, we want to minimize the number of different channels we use in our network. In this paper, we study the rainbow connection number of Cartesian products of some classes of graphs.

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## 2. Preliminary definitions and theorems

In this section, we collect the definitions and the concepts which are used in this paper. In this paper, all graphs are finite, simple, connected, and undirected. The distance between two vertices $u$ and $v$ in $G$ is the length of the shortest path between them and is denoted by $d_G(u,v)$.
*Eccentricity* of a vertex $v$ of graph $G$ is the maximum distance between a vertex $v$ to all other vertices of $G$, denoted by $ecc_G(v)$. The *diameter* of a connected graph $G$, denoted by $diam(G)$, is the maximum eccentricity of its vertices. The *radius* of $G$, denoted by $rad(G)$, is the smallest eccentricity of its vertices.

**Definition 2.1.** [4] Given two graphs $G$ and $H$, the Cartesian product of $G$ and $H$, denoted by $G\square H$, is a graph defined as follows: $V(G\square H) = V(G) \times V(H)$. Two distinct vertices $(g_1, h_1)$ and $(g_2, h_2)$ of $G\square H$ are adjacent if and only if either $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$ or $h_1 = h_2$ and $\{g_1, g_2\} \in E(G)$.

**Definition 2.2.** [4] The $n$-dimensional hypercube is a graph whose vertex set is $\{1,0\}^n$ (i.e., there are exactly $2^n$ vertices, each labeled with a distinct $n$-bit string), and with an edge between two vertices if and only if they differ in exactly one bit position, and denoted by $Q_n$. By the definition of Cartesian product of graphs, we have $Q_n = Q_{n-1}\square K_2 = \square_{i=1}^{n} K_2$.

**Definition 2.3.** [4] The star graph of order $n$ is denoted by $S_n$, is a tree on $n$ nodes with one node having vertex degree $n-1$ and the other $n-1$ having vertex degree 1.

**Proposition 2.4.** *[9] Let $G$ and $H$ be two graphs, then the Cartesian product $G\square H$ is connected if and only if both $G$ and $H$ are connected, also $diam(G\square H) = diam(G) + diam(H)$.*

**Lemma 2.5.** *Let $G$ and $H$ be connected graphs, then $rc(G\square H) \leq rc(G) + rc(H)$*

*Proof.* Suppose that $c_G$ and $c_H$ are rainbow colorings of graphs $G$ and $H$, respectively, with distinct color sets $A$ and $B$. We define an edge coloring $c_{G\square H}$ of $G\square H$ as follows:

$$c_{G\square H}\{(g,h),(g',h')\} = \begin{cases} c_G\{g,g'\} \text{ if } gg' \in E(G) \text{ and } h = h', \\ c_H\{h,h'\} \text{ if } hh' \in E(H) \text{ and } g = g'. \end{cases}$$

For every two vertices $(x,y)$ and $(x',y')$ of $G\square H$ there are two cases:
(1) If $(x,y)$ and $(x',y')$ are in same copy of $G$ or $H$. Then, according to the rainbow coloring of the graphs $G$ and $H$, this copy has a rainbow path between those two vertices.
(2) If $(x,y)$ and $(x',y')$ are not in the same copy of $G$ or $H$. Then, We consider the path

$$P : (x,y) \cdots (x',y) \cdots (x',y').$$

such that the first part of the path $P$ from $(x,y)$ to $(x',y)$ is in a copy of $G$ and the second part of the path from $(x',y)$ to $(x',y')$ is in a copy of $H$. Therefore, according to the rainbow coloring of graph $G$, there is a rainbow colored path between $(x,y)$ and $(x',y)$ with colors from $A$. Similarly, by a rainbow coloring of the graph $H$ between $(x',y)$ and $(x',y')$, there is a rainbow path with colors from $B$. Then, there is a rainbow path between $(x,y)$ and $(x',y')$. As a result, $c_{G\square H}$ is a rainbow coloring of $G\square H$. Then, $rc(G\square H) \leq rc(G) + rc(H)$. □

**Remark 2.6.** For example, suppose that $G = H = P_2$. We have $rc(G) = rc(H) = 1$. Clearly, $G\square H = C_4$ and $rc(G\square H) = rc(G) + rc(H) = 2$.

The last example shows that the upper bound in Lemma 2.5 can be reached.
Note that, for any connected graph $G$ by definition of rainbow coloring, we have, $rc(G) \geq diam(G)$.

**Corollary 2.7.** *Let $G$ and $H$ be connected graphs. Then*

$$diam(G\square H) \leq rc(G\square H) \leq rc(G) + rc(H).$$

**Proposition 2.8.** *Let $G$ and $H$ be graphs, $rc(G) = diam(G)$ and $rc(H) = diam(H)$. Then $rc(G\square H) = diam(G\square H) = rc(G) + rc(H)$*

*Proof.* By 2.4 and 2.7, we have

$$\begin{aligned} diam(G) + diam(H) &= diam(G\square H) \\ &\leq rc(G\square H) \\ &\leq rc(G) + rc(H) \\ &= diam(G) + diam(H). \end{aligned}$$

Then $rc(G\square H) = rc(G) + rc(H)$. □

In the next section, we study the rainbow connection number of the Cartesian product of some classes of graphs and find their upper and lower bounds.

## 3. Main results

Let $P_n$, $C_n$, $S_n$, and $K_n$ be the path, cycle, star ($S_n = K_{1,n-1}$), and complete graph of order $n$, respectively. In addition, let $\mathcal{P}$, $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{K}$ be the class of paths, cycles, stars and complete graphs consequently. A trivial graph is a graph of order 1.
Since $rc(K_n) = diam(K_n) = 1$ , $rc(P_n) = diam(P_n) = n - 1$, and $rc(C_{2n}) = diam(C_{2n}) = n$, and according to Proposition 2.8, we have the following corollaries.

**Corollary 3.1.** *$rc(G\square H) = diam(G\square H)$ for any two graphs $G$ and $H$ from $\mathcal{K} \cup \mathcal{P}$.*

**Corollary 3.2.** *Let $n, m \in \mathbb{N}$. Then*

1. $rc(P_n\square C_{2m}) = diam(P_n\square C_{2m}) = n - 1 + m$,
2. $rc(K_n\square C_{2m}) = diam(K_n\square C_{2m}) = m + 1$,
3. $rc(C_{2n}\square C_{2m}) = diam(C_{2n}\square C_{2m}) = n + m$

**Corollary 3.3.** *Let $Q_n$ be a $n$-hypercube graph. Then*

$$rc(Q_n) = diam(Q_n) = n.$$

In [2] the authors proved that, for any two connected graphs $G$ and $H$, $rc(G\square H) \leq 2rad(G\square H)$.

**Theorem 3.4.** *Let $n, m \in \mathbb{N}$, $n$ is odd and $G = K_m\square C_n$. Then*

$$rc(G) = diam(K_m\square C_n) = \lfloor \frac{n}{2} \rfloor + 1.$$

*Proof.* We know that $rc(K_m\square C_n) \geq diam(K_m\square C_n) = \lfloor \frac{n}{2} \rfloor + 1$.
Conversely, suppose that $V(C_n) = \{v_1, ..., v_n\}$ and $V(K_m) = \{u_1, ..., u_m\}$. Then,

$$V(C_n\square K_m) = \{(v_i, u_j) | 1 \leq i \leq n; 1 \leq j \leq m\}$$

is a vertex set of $G$. So there are $m$ copies of $C_n$ and $n$ copies of $K_m$ in $G$ that denoted by $K_m^i$ for $(1 \leq i \leq n)$. We consider the edge partitioning of the graph $G$ as follows

$$E_i = \{(v_i, u_j)(v_i, u_{j'}) | 1 \leq i \leq n, 1 \leq j, j' \leq m\},$$

$$E_i' = \{(v_i, u_j)(v_{i+1}, u_j) | 1 \leq i \leq n, 1 \leq j \leq m, v_{n+1} = v_1\}.$$

Let $E_i$ be every edge in a complete graph $K_m^i$ for $(1 \leq i \leq n)$ and $E_i'$ be any edge between $K_m^i$ and $K_m^{i+1}$ $(1 \leq i \leq n)$ such that $K_m^{n+1} = K_m^1$.

Suppose that $k = diam(K_m \square C_n) = \lfloor \frac{n}{2} \rfloor + 1$.
Now, we define the coloring $c$ of the graph $G$ as follows:

$$c(E_i') = \begin{cases} i & 1 \le i \le k-1, \\ i-k+1 & k \le i \le n. \end{cases}$$

$$c(E_i) = \begin{cases} i+1 & 1 \le i \le k-1, \\ i-k+2 & k \le i < n, \\ 1 & i = n. \end{cases}$$

We show that there is a rainbow path between any two vertices in the graph $G$. Let $(x, y)$ and $(x', y')$ be arbitrary vertices in $G$.
(i) If $x = x'$ or $(y = y')$, then $(x, y)$ and $(x', y')$ are in the same copy of $K_m$ or $C_n$ so by the coloring $c$, there is a rainbow path between $(x, y)$ and $(x', y')$ of length at most of $k-1$.
(ii) If $x \neq x'$ and $y \neq y'$ then $(x, y)$ and $(x', y')$ are not in the same copy of $K_m$ or $C_n$. So there is a path between those two vertices of length at most $k$. Therefore, there is a rainbow path between $(x, y)$ and $(x', y)$ like as $P$ of length at most $k-1$ in a copy of $C_n$. Then, $P$ passes through $k$ copies of $K_m$.
According to the coloring $c$, at least the color of the edges in one copy among $k$ copies of $K_m$ differs from the colors of the edges of the path $P$. Let edge $\{(x'', y), (x'', y')\}$ has different color from the colors of path $P$. Therefore, by using the path $(x, y), \ldots, (x'', y), (x'', y'), \ldots, (x', y')\}$ between $(x, y)$ and $(x', y')$. So the rainbow path of length $k$ is obtained. Then, $(x, y)$ and $(x', y')$ are rainbow connected.
As a result, every two vertices of $G$ are connected by a rainbow path of maximum length $k$. Therefore, $c$ is a rainbow coloring of $G$ with $k$ colors and so $rc(C_n \square K_m) = diam(C_n \square K_m) = \lfloor \frac{n}{2} \rfloor + 1$. □

**Theorem 3.5.** *Let $n, m \in \mathbb{N}$, $n$ is odd and $G = P_m \square C_n$. Then $rc(G) = diam(G) = \lfloor \frac{n}{2} \rfloor + m - 1$.*

*Proof.* Suppose that $V(G) = \{v_{j,i} | 1 \le j \le m,\ 1 \le i \le n\}$ is the vertex set of $G$ and partition the edges of graph G as follows.

$$E_{j,i} = \{v_{j,i}v_{j,i+1} | 1 \le i \le n,\ 1 \le j \le m,\ v_{j,n+1} = v_{j,1}\}$$
$$E'_{j,i} = \{v_{j,i}v_{j+1,i} | 1 \le i \le n,\ 1 \le j \le m-1\}$$

Such that $E_{j,i}$ is the set of edges of a cycle $j$th $C_n^j$ and $E'_{j,i}$ is the set of edges between vertices of cycle $C_n^j$ and vertices of cycle $C_n^{j+1}$. Since $n$ is odd, there exists $k = \lfloor \frac{n}{2} \rfloor$ such that $n = 2k + 1$.
We know that $rc(G) \ge diam(G) = \lfloor \frac{n}{2} \rfloor + m - 1$.
Now, we define an edge coloring of $G$, $c : E(G) \to \{1, 2, \ldots, k + m - 1\}$ of the edges of $G$ as follows

$$c(e_{j,i}) = \begin{cases} i & 1 \le i \le k, \\ i-k & k+1 \le i \le n. \end{cases}$$

$$c(e'_{j,i}) = \begin{cases} i & 1 \le i \le k \text{ and } j = 1, \\ k+1 & k < i \le n \text{ and } j = 1, \\ k+j & 1 < j \le m-1. \end{cases}$$

We will show that $c$ is a rainbow connected coloring for $G$. We must show that every two vertices of $G$ are connected by a rainbow path.
Suppose that $x = v_{j,i}, y = v_{j',i'}$ are two arbitrary vertices of $G$; there are two cases.
(1) If $i = i'$ or $j = j'$ so $x$ and $y$ are in same copy of $P_m$ or $C_n$ then $x$ and $y$ are connected by rainbow path of length

at most $m-1$ or $k$.
(2) If $i \neq i'$ and $j \neq j'$ $(j < j')$, We consider two cases.
a) Suppose that $j = 1$ then there is at least two paths between the vertices $x$ and $y$ of length $d(x,y) \leq diam(G) = k+m-1$ like as

$$P : v_{1,i}, v_{2,i}, \dots, v_{j',i}, \dots, v_{j',i'}$$

$$Q : v_{1,i}, \dots, v_{1,i'}, v_{2,i'}, \dots, v_{j,i'}$$

According to the coloring $c$, at least one of the paths $P$ and $Q$ is the rainbow path.
b) Suppose that $2 \leq j \leq j'$, in this case there is a rainbow path between $x$ and $y$ as follows

$$P : v_{j,i}, \dots, v_{j',i}, \dots, v_{j',i'}$$

Thus, $c$ is a rainbow coloring of $G$ and $rc(G) \leq k+m-1$ and so $rc(G) = k+m-1$. □

**Lemma 3.6.** *Let $T_m$ $(m \geq 4)$ be a tree that has at least three pendant vertices. then,*

$$rc(K_n \square T_m) \leq rc(T_m) = m-1$$

*Proof.* We have $rc(T_m) = m-1$. Assign $m-1$ distinct colors to all edges of $T_m$. Define the coloring of the edges of $K_n \square T_m$ as follows. For every two adjacent vertices $x$ and $y$ from $T_m$, in $K_n \square T_m$ there exist two adjacent copies of $K_n$ that are opposite of $x$ and $y$, so assign the color of edge $xy$ to all edges between these two copies of $K_n$.
In tree $T_m$, we consider $P$ the path of length $diam(T_m) < m-1$ and suppose that $u, v \in P$ and $d(u) = d(v) = 1$ ($u$ and $v$ are end-vertices of the path $P$) in graph $K_n \square T_m$, since $d(u,v) \leq m-2$. Then, there is a color among $m-1$ colors such as $a$, which is different from colors on the edges of path $P$. So, by assigning color $(a)$ to all edges in two copies of $K_n$ opposite of $u$ and $v$, and assigning every color among $m-1$ color without color $(a)$ arbitrarily assigns to the all edges in every copy of $K_n$. Since every two vertices of $K_n \square T_m$ are connected by a rainbow path. Therefore, $rc(K_n \square T_m) \leq m-1$. □

**Example 3.7.** Consider a complete graph $K_n$ and tree $T_7$ with three pendant vertices. Let $K_n \square T_7$ the Cartesian product of $K_n$ and $T_7$.
By coloring the edges of $K_n \square T_7$ with colors $1, \dots, 6$ (as you see in Figure 1), there is a rainbow path between every two vertices in $rc(K_n \square T_7)$, then $rc(K_n \square T_7) \leq 6$.
On the other hand $rc(K_n \square T_7) \geq diam(K_n \square T_7) = 6$. Therefore, $rc(K_n \square T_7) = 6$.

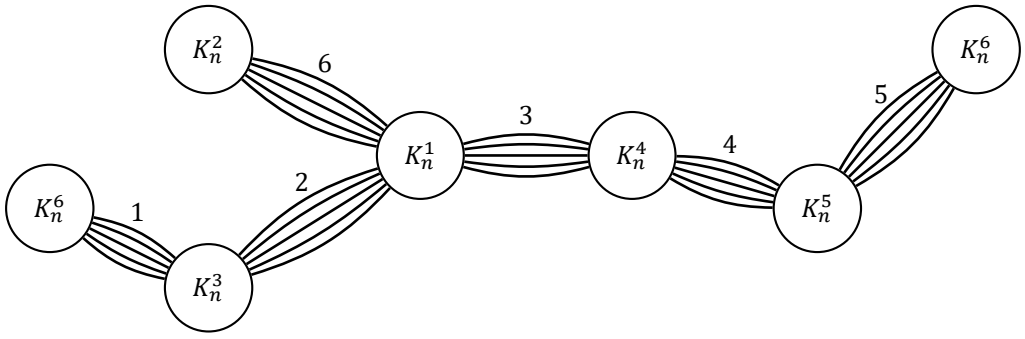


Fig. 1. rainbow coloring of ($K_n \square T_7$)

Let $S_m = K_{1,m-1}$ be a star graph from order $m$, with vertex set $V(S_m) = \{v_1, v_2, \dots, v_m\}$ such that the vertex $v_1$ of degree $m-1$ and the other vertices of degree 1. Since $S_m$ is a tree, so $rc(S_m) = m-1$.

**Theorem 3.8.** *Let $n, m \in \mathbb{N}$ and $n > 1$. Then, $rc(K_n \square S_m) = 4$ when $m \geq 5$ and $rc(K_n \square S_4) = 3$.*

*Proof.* We have $diam(K_n \square S_m) = 3$ when $m \geq 5$. Assume, to the contrary, that there exists a rainbow connected coloring $c$ of $K_n \square S_m$, using at most three colors.
In every copy of $S_m$, between any two vertices (without center vertex) there is one path of length two and many paths of length 4. Since coloring $c$ has at most three colors and $m \geq 5$, there exists at least two vertices in $K_n \square S_m$ which are not connected by a rainbow path. So that $rc(K_n \square S_m) \geq 4$.
On the other hand, let $V(S_m) = \{u_1, u_2, \dots, u_m\}$ and $V(K_n)\{v_1, v_2, \dots, v_n\}$ be the vertex-sets of $S_m$ and $K_n$. Then the vertex set of $K_n \square S_m$ is

$$V(K_n \square S_m) = \{(v_i, u_j) | 1 \leq i \leq n, 1 \leq j \leq m\}$$

Partition the edges of $K_n \square S_m$ into two sets $E_1$ and $E_2$ as follows

$$E_1 = \{(v_i, u_j)(v_{i'}, u_j) |\ 1 \leq i, i' \leq n, 1 \leq j \leq m\}$$
$$E_2 = \{(v_i, u_1)(v_i, u_j) |\ 1 \leq i \leq n, 2 \leq j \leq m\}$$

Now, we define an edge coloring with 4 colors for graph $K_n \square S_m$ as follows

$$c(E_1) = \begin{cases} 1 & j = 1, \\ 2 & j \geq 2. \end{cases}$$
$$c(E_2) = \begin{cases} 3 & i = 1, \\ 4 & i \neq 1. \end{cases}$$

For every two non-adjacent vertices $x = (v_i, u_j)$ and $y = (v_{i'}, u_{j'})$ of $K_n \square S_m$, if one of $j$ or $j'$ equal to 1, then there is a rainbow path of length 2 from $x$ to $y$.
Otherwise, we have the following cases.
If $i = i' = 1$, so there exists the rainbow path $P = (v_1, u_j), (v_1, u_1), (v_2, u_1), (v_2, u_{j'}), (v_1, u_{j'})$ between $x$ and $y$ of length 4.
We now consider the case, if at least one of $i$ and $i'$ is not equal to 1, then the path

$$P' = (v_i, u_j), (v_i, u_1), (v_1, u_1), (v_1, u_{j'}), (v_{i'}, u_{j'})$$

is a rainbow path of length at most 4 from $x$ to $y$.
Then, there exists rainbow coloring of $K_n \square S_m$ uses four colors. Therefore, $rc(K_n \square S_m) = 4$.
Clearly, by 3.6, we have $rc(K_n \square S_4) = 3$. □

**Theorem 3.9.** *Let $n, m \in \mathbb{N}$ such that $n, m \geq 3$. Then $rc(S_n \square S_m) = 4$.*

*Proof.* Suppose that $V(S_n) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ such that $d(v_0) = n - 1$ and the vertices $v_i$, $(1 \leq i \leq n - 1)$ are pendant vertices and $V(S_m) = \{u_0, u_1, u_2, \dots, u_{m-1}\}$ such that $d(u_0) = m - 1$ and the vertices $u_j$, $(1 \leq j \leq m - 1)$ are pendant vertices. Now define the edge-coloring $c$ of the graph $S_n \square S_m$ by

$$c(e) = \begin{cases} 1 & e = (v_0, u_0)(v_0, u_j),\ 1 \leq j \leq m - 1, \\ 2 & e = (v_0, u_0)(v_i, u_0),\ 1 \leq i \leq n - 1, \\ 3 & e = (v_0, u_j)(v_i, u_j),\ 1 \leq i \leq n - 1,\ 1 \leq j \leq m - 1, \\ 4 & e = (v_i, u_0)(v_i, u_j),\ 1 \leq i \leq n - 1,\ 1 \leq j \leq m - 1. \end{cases}$$

Clearly, in the coloring $c$, between any two non-adjacent vertices $x = (v_i, u_j)$ and $y = (v_{i'}, u_{j'})$ of $S_n \square S_m$, we have the path $P = (v_i, u_j), (v_i, u_0), (v_0, u_0), (v_0, u_{j'}), (v_{i'}, u_{j'})$ which is a rainbow path of length at most 4. Thus $rc(S_n \square S_m) \leq 4$. Since $diam(S_n \square S_m) = 4$. Then, $rc(S_n \square S_m) = 4$. □

Let $S_\infty$ be an infinite complete bipartite graph with vertex set $V(S_\infty) = \mathbb{N} \cup \{0\}$ and edge set $E(S_\infty) = \{\{0, i\} \mid i \in \mathbb{N}\}$. We call this graph an infinite star. We know that $S_\infty$ does not have a finite rainbow connection number.

**Theorem 3.10.** *Let $n, m \in \mathbb{N}$ and $n > 1$. Then $rc(P_n \square S_m) = n + 1$ when $n, m \geq 3$ (We remove the other cases because $K_2 = S_2 = P_2$).*

*Proof.* Graph $P_n \square S_m$ contains $n$ copies of $S_m$, $(S_m^1, S_m^2, \dots, S_m^n)$. Now, let $\{v_0^i, v_1^i, v_2^i, \dots, v_{m-1}^i\}$ be the vertices of $S_m^i$ for $(1 \leq i \leq n)$ such that $v_0^i$ is the center of $S_m^i$ and $d(v_0^i) = m - 1$. Since $diam(P_n \square S_m) = diam(P_n) + diam(S_m) = n + 1$ then $rc(P_n \square S_m) \geq n + 1$.
Now, we define an edge-coloring $c$ of $P_n \square S_m$ as follows. for $1 \leq i \leq n$ and $1 \leq j \leq m - 1$

$$c(e) = \begin{cases} 1 & ; \quad e = v_0^i v_j^i \quad \text{and} \quad i \equiv 1(mod\ 2), for 1 \leq i \leq n \text{ and } 1 \leq j \leq m-1, \\ 2 & ; \quad e = v_0^i v_j^i \quad \text{and} \quad i \equiv 0(mod\ 2), for 1 \leq i \leq n \text{ and } 1 \leq j \leq m-1, \\ i+2 & ; \quad e = v_0^i v_0^{i+1} \quad \text{for} \quad 1 \leq i \leq n-2, \\ n+2-i & ; \quad e = v_j^i v_j^{i+1} \quad \text{for} \quad 1 \leq i \leq n-2 \text{ and } 1 \leq j \leq m-1, \\ 3 & ; \quad e = v_0^{n-1} v_0^n, \\ n+1 & ; \quad e = v_j^{n-1} v_j^n \quad \text{for} 1 \leq j \leq m-1. \end{cases}$$

For any two non-adjacent vertices $x = v_j^i$ and $y = v_{j'}^{i'}$ of the graph $P_n \square S_m$, we have
(i) If $x$ and $y$ on same copy of $S_m$ then $i = i'$ and so the path $P : x = v_j^i, v_0^i, v_0^{i+1}, v_{j'}^{i+1}, v_{j'}^i = y$ is a rainbow path between $x$ and $y$ in $P_n \square S_m$.
(ii) If $x$ and $y$ on distinct copies of $S_m$, suppose that $i < i'$ then there is the path

$$P' : x = v_j^i, v_0^i, v_0^{i+1}, v_{j'}^{i+1}, v_{j'}^{i+2}, \dots, v_{j'}^{i'} = y$$

is a rainbow path between $x$ and $y$ in $P_n \square S_m$.
Indeed, every two vertices of $P_n \square S_m$ are connected by a rainbow path of length at most $n + 1$. It follows that the coloring $c$ is rainbow coloring for $P_n \square S_m$. So $rc(P_n \square S_m) \leq n + 1$.
Therefore, $rc(P_n \square S_m) = n + 1$. □

By Theorem 3.10, we have the following proposition.

**Proposition 3.11.** *For any* $4 \leq r \in \mathbb{N}$. *There is a graph* $G$ *with rainbow connection number* $r$.

In the following, we show that there exists an infinite graph $G$ with finite diameter such that $rc(G) = \infty$ but $rc(G \square G)$ is finite.

**Definition 3.12.** Let $S_\infty$ (infinite star) be a connected graph with vertex set $V(S_\infty) = \mathbb{N}$ and edge set $E(S_\infty) = \{\{1, j\} | 1 \leq j \in \mathbb{N}\}$. . Clearly, $diam(S_\infty) = 2$ and $rc(S_\infty) = \infty$.

Similar to the proof of Theorem 3.9, we can conclude the following proposition.

**Proposition 3.13.** $rc(S_\infty \square S_\infty) = 4 = diam(S_\infty \square S_\infty)$.

By the definition of $S_\infty$ and the Proposition 3.11 we have the following corollary.

**Corollary 3.14.** *Let* $4 \leq r \in \mathbb{N}$. *Then an infinite graph* $G$ *with rainbow connection number* $r$ *exists.*

We consider the graph $P_{r-1} \square S_\infty$ for $4 \leq r \in \mathbb{N}$. Then, $rc(P_{r-1} \square S_\infty) = r$.

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# Estimation of Univariate Density Function with Measurement Error Using Global Thresholding and Blocking

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**Abstract**

The problem of estimating a probability density function (PDF) $f$ from $X_1, X_2, \cdots, X_n$ , given by $t = f * u$ , is analyzed. We develop both linear and non-linear wavelet estimators utilizing Mayer-type wavelets. These estimators are demonstrated to be asymptotically optimal and adaptive when $f$ belongs to the Sobolev space $H^{\alpha}$ . Additionally, a comparison is made between the block thresholding method applied to supersmooth and non-supersmooth density functions

## 1. Introduction

Consider $\gamma \sim f$ and $\epsilon \sim u$ are independent random variables , where $f$ is the unknown density function and $u$ is the known density function. We observe a n sample of $X_i$ as follows:

$$X_i = \gamma_i + \epsilon_i \tag{1}$$

In this article, the aim is to estimate the density function of $f$. $X_i \sim t$ is the convolution of u and f:

$$t(x) = \int_{-\infty}^{\infty} u(x-\gamma) f(\gamma) d\gamma \tag{2}$$

So the issue of f function estimation in this context is known as deconvolution. This issue appears in very application for example Desouza[4], and over the past decade, extensive research has been conducted on this topic. The prevalent

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method involves estimating $t(X)$ using a kernel estimator, followed by solving equation 1 through the application of a Fourier transform. For example Carroll and Hall [3] and others. Fan [12] demonstrated that the estimators of $f(\gamma)$are asymptotically optimal both locally and globally , provided the kernel has a limited bandwidth,meaning the Fourier transform of the kernel has bounded support. These estimators and similar methods have been examined in various contexts. For example Fan[13] and others.

This paper addresses the estimation of a deconvolution density through wavelet method . The core concept involves representing $f(\gamma)$using a wavelet expansion and then to estimate the coefficients using a deconvolution algorithm. This approach builds on orthogonal series methods for estimating prior densities [see Walter [33],Penkaya[28] and recent advancements in wavelet techniques for curve estimation [ see Antoniadis, Gregoire , and McKeague[2]and others].

The density estimation of $f(\gamma)$ is performed within Sobolev space $H^{\alpha}$. The estimation of the density function is investigated in two parts: when the error distribution $\varepsilon$ is supersmooth (i.e., the Fourier transform $\tilde{u}$ of f shows exponential decay) and when $\tilde{u}$ has polynomial decay. In the supersmooth part, even without knowing $\alpha$, the proposed linear wavelet estimator adjusts its parameter choices to achieve the best possible convergence rate. For polynomial decay, where the linear wavelet estimator falls short if $\alpha$ is unknown, a non-linear adaptive wavelet estimator is developed, achieving the best possible rate of convergence. The estimators utilize Meyer-type wavelets instead of wavelets with bounded support. Meyer-type wavelets belong to the subset of band-limited wavelets, which allow for prompt deconvolution. Notably, the non-linear wavelet estimator uses ”global thresholding,” different from the ”block thresholding” method by Hall et al. [17], where ”global thresholding” involves simultaneous thresholding of all coefficients at the same level, whereas ”block thresholding” solely aggregates a limited set of coefficients. Estimators that employ Meyer-type wavelets are asymptotically optimal. This means that for $f \in H^{\alpha}$, the rates of convergence for the mean squared error (MSE) cannot be further enhanced [see Fan [14]]. Additionally, the estimators developed in this article automatically adjust when $f(\gamma)$ is supersmooth. In such cases, both the linear and non-linear wavelet estimators attain enhanced convergence rates without altering any parameters. Specifically, if both $f(\gamma)$ and u(X) are supersmooth, the linear wavelet estimator attains a polynomial rate of convergence, which is superior to the logarithmic rate of convergence for $f \in H^{\alpha}$. If $f(\gamma)$ is supersmooth and $\tilde{u}$ has polynomial decay, the MSE of the non-linear wavelet estimator is $O(n^{-1}ln^{\nu}n)$ with $\nu > 0$ as $n \to \infty$.

## 2. Mayer Wavelets

We use an orthonormal wavelet, which is produced by dilation and transmission of a ”father” wavelet of Mayer-type $\varphi$ and a ”mother” wavelet of Mayer-type $\zeta$. The principal characteristic of such wavelets are:

- 1- Wavelets are smooth and frequency band-limited, i.e. The Fourier transforms $\varphi$ and $\zeta$ have compact supports with
$$supp(F(\varphi)) \subset [-4\pi 3^{-1}, 4\pi 3^{-1}]$$
and
$$supp(F(\zeta)) \subset [-8\pi 3^{-1}, -2\pi 3^{-1}] \cup [2\pi 3^{-1}, 8\pi 3^{-1}]$$
. where supp denotes the support.

- 2- If the Fourier transforms of $\pi$ and $\zeta$ are also in $C^m$ for a chosen $m \in N$ , then it can be easily shown that $\varphi$ and $\zeta$ obey
$$|\pi(t)| = O((1+|t|)^{-m-1}), \quad |\zeta(t)| = O((1+|t|)^{-m-1})$$
for every $t \in R$.

- 3- The function $(\varphi, \zeta)$ is differentiable for all degree of differentiation. because their Fourier transform has a compact support, and $\zeta$ has an infinite number of vanishing moments . that is, for each $v \in N$ , $\int_{-\infty}^{\infty} x^v dx = 0$

In this paper, we employ the Meyer wavelet. For each $x$, any integer $j$, and any $h \in \{0, \ldots, 2^j - 1\}$, ,allow

$$\varphi_{j,h} = 2^{j/2}\varphi(2^j x - h), \quad \zeta_{j,h} = 2^{j/2}\zeta(2^j x - h) \tag{3}$$

are the elements of the wavelet basis.
Consider $j_c \in Z$, which that $j_c > j_*$ . A function $f \in H^\alpha$ it would be developed into a series as

$$f(\gamma) = \sum_{h=0}^{2^{j_c}-1} \alpha_{j_c,h}\varphi_{j_c,h}(\gamma) + \sum_{j=j_c}^{\infty}\sum_{h=0}^{2^j-1} \beta_{j,h}\zeta_{j,h}(\gamma) \tag{4}$$

where

$$\alpha_{j,h} = \int_0^1 f(\gamma)\phi_{j,h}d\gamma, \quad \beta_{j,h} = \int_0^1 f(\gamma)\zeta_{j,h}d\gamma \tag{5}$$

The coefficients $\alpha_{j_c,h}$ and $\beta_{j,h}$ may be interpreted like the expected values of functions $\tau_{j_c,h}$ and $\nu_{j,h}$

$$\alpha_{j_c,h} = \int_{-\infty}^{\infty} \tau_{j_c,h}t(x)dx \qquad \beta_{j,h} = \int_{-\infty}^{\infty} \nu_{j,h}t(x)dx \tag{6}$$

This condition is met if $\tau_{j_c,h}$ and $\nu_{j,h}$ satisfy the following equations:

$$\int_{-\infty}^{\infty} u(x-\gamma)\tau_{j_c,h}(x)dx = \phi_{j_c,h}(\gamma), \qquad \int_{-\infty}^{\infty} u(x-\gamma)\nu_{j,h}(x)dx = \zeta_{j,h}(\gamma)$$

Applying the Fourier transform to both sides, we derive $\tau_{j_c,h}(x) = 2^{j_c/2}\Upsilon_{j_c}(2^{j_c}x - h), \nu_{j,h}(x) = 2^{j/2}V_j(2^j x - h)$, where $\Upsilon_{j_c}(.)$ and $V_j(.)$ are the inverse Fourier transform of the functions

$$\tilde{\Upsilon}_{j_c}(\omega) = \tilde{\phi}(\omega)/\tilde{u}(-2^{j_c}\omega) \qquad \tilde{V}_j(\omega) = \tilde{\zeta}(\omega)/\tilde{u}(-2^{j_c}\omega) \tag{7}$$

respectively. Therefore, estimating $\alpha_{j_c,h}$ and $\beta_{j,h}$ by:

$$\hat{\alpha}_{j_c,h} = n^{-1}\sum_{i=1}^{n} 2^{j_c/2}\Upsilon_{j_c}(2^{j_c}X_i - h) \qquad \hat{\beta}_{j,h} = n^{-1}\sum_{i=1}^{n} 2^{j/2}V_j(2^{j_c}X_i - h) \tag{8}$$

and truncating the relation 4, we derive a linear wavelet estimator

$$\hat{f}_n^{(L)}(\gamma) = \sum_{h=0}^{2^{j_c}-1} \hat{\alpha}_{j_c,h}\varphi_{j_c,h}(\gamma) \tag{9}$$

and a non-linear wavelet estimator of $f(\gamma)$,

$$\hat{f}_n^{(N)}(\gamma) = \sum_{h=0}^{2^{j_c}-1} \hat{\alpha}_{j_c,h}\varphi_{j_c,h}(\gamma) + \sum_{j=j_c}^{m+r}[\sum_{h=0}^{2^{j_c}-1} \hat{\beta}_{j,h}\zeta_{j,h}(\gamma)]I(\sum_{h=0}^{2^{j_c}-1})\hat{\beta}_{j,h}^2 > \delta_{j,n}^2 \tag{10}$$

It's important to note that the block thresholding method used in estimator 10 differs from the approach taken by Hall, Penev, Kerkyacharian, and Picard [17] and Hall, Kerkyacharian, and Picard [15]. Those studies focused on estimating a pdf from direct observations using wavelets with bounded support, and they partitioned the coefficients $\beta_{j,h}$ into blocks $A = \{\beta_{j,h} : (j-1)l < h < jl\}$ of length l, thresholding all coefficients in a block simultaneously. In contrast, in this article, all coefficients $\beta_{j,h}$ for $h \in Z$ are thresholded together. Initially, the estimators 9 and 10 may appear

computationally daunting due to their reliance on infinite series calculations. However, under rather lenient conditions, these infinite series estimators can be substituted with finite series estimators.

$$\hat{f}_n^{(LF)}(\gamma) = \sum_{|h|\le K_n} \hat{\alpha}_{j_c,h}\varphi_{j_c,h}(\gamma) \tag{11}$$

$$\hat{f}_n^{(NF)}(\gamma) = \sum_{|h|\le M_n} \hat{\alpha}_{j_c,h}\varphi_{j_c,h}(\gamma) + \sum_{j=j_c}^{m+r}[\sum_{|h|\le L_n} \hat{\beta}_{j,h}\zeta_{j,s}(\gamma)]I(\sum_{|h|\le L_n} \hat{\beta}_{j,h}^2 > \delta_{j,n}^2) \tag{12}$$

without compromising the convergence rate.

## 3. Asymptotic properties of the wavelet estimator

To explore the asymptotic characteristics of the estimators 9 and (2.8), we assume that the $pdf$ is a member of the following class:

$$F_\alpha(C_\alpha) = \{f \in H^\alpha : ||f||_\alpha \le C_\alpha, \alpha > 0\} \tag{13}$$

where $||f||_\alpha$ is the norm in $H^\alpha$,

$$||f||_\alpha = \left\{\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2(\omega^2 + 1^\alpha)d\omega\right\}^{1/2} < \infty$$

We assess the effectiveness of an estimator $f_n(\gamma)$ through the Mean Squared Error (MSE):

$$MSE(f_n) = E\Big(f_n(\gamma) - f(\gamma)\Big)^2 \tag{14}$$

Define:

$$\Gamma_1(j_c) = \int_{-\infty}^{\infty} |\tilde{\phi}(\omega)|^2 |\tilde{u}(2^{j_c}\omega)|^{-2}d\omega \tag{15}$$
$$\Gamma_k(j) = \int_{-\infty}^{\infty} |\tilde{\zeta}(\omega)|^k |\tilde{u}(2^{j}\omega)|^{-k}d\omega \qquad k = 2,4$$

The theorem below determines the upper bound for the MSE of the linear wavelet estimator 9 uniformly over the specified class $F_\alpha(C_\alpha)$ defined in 13.

**Theorem 3.1.**

$$sup_{F_\alpha} MSE(\hat{f}_n^L) \le 2\pi^{-1}(2\pi/3)^{-2\alpha}||\zeta||^2 C_\alpha^2 2^{-2j_c\alpha} + n^{-1}2^{j_c+1}\Gamma_1(j_c) \tag{16}$$

**CorolLary1**. If $|\tilde{u}(\omega)| \ge C_0(\omega^2+1)^{-s/2}exp\{-B|\omega|^\theta\}$ and $j_c$ is such that

$$2^{j_c} = \begin{cases} n^{1/(2\alpha+2s+1)} & B = 0 \\ \left[\left(2B(4\pi/3)^\theta + \Delta\right)^{-1} ln(n)\right]^{1/\theta} & B > 0 \end{cases} \tag{17}$$

for $\Delta > 0$. then

$$sup_{f\in F_\alpha} MSE(\tilde{f}_n^{(L)}) = \begin{cases} O(n^{-2\alpha/(2\alpha+2s+1)}) & B = 0 \\ O((lnn)^{-2\alpha/\theta}) & B > 0 \end{cases} \tag{18}$$

Notice that the convergence rates in 18 match The best possible rate of convergence [as indicated by Fan [14]]. In the context of exponential decay of $\tilde{u}(\omega)$, the linear wavelet estimator demonstrates adaptability., meaning The parameter $j_c$ does not rely on the unknown smoothness $\alpha$ of the pdf $f(\gamma)$. Nevertheless, when dealing with polynomial descent, the estimator 9 does not achieve The best possible rate of convergence if α is not known. This issue may be resolved with employing the non-linear estimator 10.

**Theorem 3.2.** *Consider* $|\tilde{u}(\omega)| \geq C_0(\omega^2+1)^{-s/2}$.*Let* $\hat{f}_b^N$ *be the estimator 10 with* $j_c = (2+\epsilon)log_2(lnn)$ *where* $\epsilon > 0$, $j_c + r = (2s+1)^{-1}log_2 n$ *and* $\delta_{j,n} = 2^{j(s+0.5)}\delta_n$.*If* $\delta_n = \delta_0 n^{-1/2}$ *with* $\delta_0 \geq 2\sqrt{2}K_2$ *and* $\Gamma_4(j)/\Gamma_2^2(j) \leq D_0$ *for any* $j$, *then*

$$sup_{f\in F_\alpha} MSE(\tilde{f}_n^{(N)}) = O(n^{-2\alpha/(2\alpha+2s+1)}) \tag{19}$$

where $K_2$ is a fixed constant.
The logic behind Theorem 3.2 goes like this: If $\alpha$'s value were determined, the optimal selection for $j_c$ in the linear estimator 9 Could be approximately $j_{opt} \sim (2\alpha+2s+1)^{-1}\log_2 n$. Given that $\alpha$ is not known, we merely ascertain such that for all $\lambda$, the best value of m falls among $\lambda log_2(lnn)$ and $(2s+1)^{-1}log_2 n$. Consequently, we construct the non-linear estimator 10with $j_c = (2+\epsilon)log_2(lnn)$, which is less than the optimal value $j_{opt}$, and $j_c + r = (2s+1)^{-1}log_2 n$. This approach includes all terms with $j \leq (2+\epsilon)log_2(lnn)$ and excludes terms with $j > (2s+1)^{-1}log_2 n$. The terms with $(2+\epsilon)log_2(lnn) < j \leq (2s+1)^{-1}log_2 n$ are only included if $\sum_{k\in Z}\hat{\beta}_{j,h}^2 \geq \delta_{j,h}^2$, where $\delta_{j,h}^2 \sim n^{-1}\sum_{h\in Z} Var\hat{\beta}_{j,h}$. This ensures that only terms with a variance not exceeding $O(n^{-2\alpha/(2\alpha+2s+1)})$ are included, thereby guaranteeing the most efficient rate of convergence. Note that to ensure $j_c < j_{opt}$ for finite values of $n$, $\epsilon$ should remain small; ideally,$\epsilon$ should be less than a predefined $\epsilon_0$. To substitute the estimators 9 and 10 with In comparison to their finite series equivalents, we assume that $f$ exhibits a specific rate of decline as $|\gamma| \to \infty$. Specifically, let's consider...

$$F_\alpha^*(C_\alpha, C_f) = \{f : f \in F_\alpha(C_\alpha), sup[|\gamma|f(\gamma)] \leq C_f\} \tag{20}$$

where $F_{C_\alpha}$ is defined in 13, and let's assume that $f \in F_\alpha^*(C_\alpha, C_f)$. It's important to keep in mind that the condition $sup[|\gamma|f(\gamma)] < \infty$is quite unconstrained and is valid for each known probability density function (p.d.f.). The next theorem demonstrates that the convergence rate of 11 and 12 uniformly over $F_\alpha^*(C_\alpha, C_f)$ is equivalent to the convergence rate of 9 and 10) uniformly over $F_\alpha(C_\alpha)$.

**Theorem 3.3.** *Let that the assumption of corollary 3 and Theorem3.2 are hold and* $K_n$ , $M_n$ *and* $L_n$ *are such that*

$$lim_{n\to\infty} nK_n^{-1} = 0, \qquad lim_{n\to\infty} nM_n^{-1} = 0, \qquad lim_{n\to\infty} n^{(2s+2)/(2s+1)}L_n^{-1} = 0$$

Consequently, the estimator 11) and 12 with the same chose of parameters $j_c$ , $r$ and $\delta_{j,n}$ as in Corollary 1 and Theorem 2 , exhibit the following rates of convergence uniformly over $F_\alpha^*(C_\alpha, C_f)$:

$$sup_{f\in F_\alpha^*} MSE(\hat{f}_n^{LF}) = O((lnn)^{-2\alpha/\theta}) \qquad if B > 0 \tag{21}$$

$$sup_{f\in F_\alpha^*} MSE(\hat{f}_n^{LF}) \sim sup_{f\in F_\alpha^*} MSE(\hat{f}_n^{NF}) = O((lnn)^{-2\alpha/(2\alpha+2s+1)}) \qquad if B = 0 \tag{22}$$

## 4. Estimation in the context of supersmooth conditions $f(\gamma)$

The asymptotic results presented with Theorems 3.1-3.3 are rather discouraging: if $u(x)$ is supersmooth, the estimator exhibits a logarithmic convergence rate. However, this is the best possible outcome if $f$ belongs to $H^\alpha$. However is the situation always this bleak? It may be intuitively guessed which is an improvement convergence rate may be accomplished if $f \in H^\alpha$ ; for instance, if $f$ is supersmooth on its own. Let's consider...

$$G_{\alpha,\eta,\rho}(C_\alpha) = \{f : \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2(\omega^2+1)^\alpha exp\{2\rho|\omega|^\eta\}d\omega \leq C_\alpha\} \tag{23}$$

$$G_{\alpha,\eta,\rho}^*(C_\alpha, C_f) = \{f : f \in G_{\alpha,\eta,\rho}(C_\alpha), sup_\gamma[|\gamma|f(\gamma)] \leq C_\alpha\} \tag{24}$$

and consider $f \in G_{\alpha,\eta,\rho}^*(C_\alpha, C_f)$ with positive $\rho$ and $\eta$. Observe that $G_{\alpha,\eta,\rho}(C_\alpha) \subseteq F_\alpha(C_\alpha)$ , $G_{\alpha,\eta,\rho}^*(C_\alpha, C_f) \subseteq F_{(C_\alpha,C_f)}^*$ and $\rho = 0$ the sets coincide:$G_{\alpha,\eta,\rho}(C_\alpha) = F_\alpha(C_\alpha)$ , $G_{\alpha,\eta,\rho}^*(C_\alpha, C_f) = F_{(C_\alpha,C_f)}^*$.
The benefit of Meyer-type wavelet estimators lies in their ability to automatically adapt to the smoothness of $f(\gamma)$. This implies that the estimators11 and 12, using the same parameter choices for $j_c$,$r$ and $\delta_j$ as previously, can achieve improved convergence rates when $f(\gamma)$ is supersmooth.

**Theorem 4.1.** *If the corollary 3 are vaild , consequently, the estimator 11 with $j_c$ as specified in 17 achieves the following convergence rate:*

$$sup_{f\in G^*_{\alpha,\eta,\rho}} MSE(\hat{f}_n^{LF}) = \begin{cases} O(n^{-1}(lnn)^{(2s+1)/\eta}) & if \quad B=0 \\ O(n^{-\mu}(lnn)^{\xi}) & if \quad B>0 \quad and \quad \eta \geq \beta \\ O((lnn)^{-2\alpha/\beta} exp\{-\lambda(lnn)^{\eta/\beta}\}) & if \quad B>0 \quad and \quad \eta < \beta \end{cases} \tag{25}$$

provided $lim_{n\to\infty} K_n^{-1} n^{\sigma} = 0$. Here $\eta > 0$ and $rho > 0$ , $\sigma = 1$ if $B$ is positive and $\sigma = \frac{2s+2}{2s+1}$ if $B = 0$ ; $\lambda = [2B(4\pi/3)^{\beta} + \Delta]^{-1} 2\rho(2\pi/3)^{\eta}$.If $\beta = \eta$, consequently $\xi = \frac{2s+1}{\beta} I(\Lambda \geq 2\rho(2\pi/3)^{\beta}) - 2\alpha\beta^{-1} I(\Lambda < 2\rho(2\pi/3)^{\beta})$ and $\xi = [2\beta(4\pi/3)^{\beta} + \Lambda]^{-1}$ min $(\Lambda, 2\rho((2\pi/3)^{\beta}))$. If $\beta < \eta$, consequently $\xi = \frac{2s+1}{\beta}$ and $\mu = [2\beta(4\pi/3)^{\beta} + \Lambda]\Lambda$.

**Theorem 4.2.** *Assuming the conditions of Theorem 3.2 are satisfied, let $L_n$ and $M_n$ be defined as follows:*

$$lim_{n\to\infty} n M_n^{-1} = 0 \qquad lim_{n\to\infty} n^{(2s+2)/(2s+1)} L_n^{-1} = 0$$

*Then*

$$sup_{f\in G_{\alpha,\eta,\rho}} MSE(\hat{f}_n^{NF}) = O((lnn)^{m(2s+1)} n^{-1}) \tag{26}$$

with $m = \eta^{-1}$ if $\eta < 0.5$ and $m = 2 + a$ if $\eta \geq 0.5$. Here $a$ ia an arbitrary constant.

## 5. Conclusian

In this paper, we developed both linear and non-linear estimators for a deconvolution density $f(\gamma)$ using Meyer type wavelets. We demonstrated that these estimators are asymptotically optimal for $f \in H^{\alpha}$ . Furthermore, we showed that the linear wavelet estimator is globally adaptive when $B > 0$ , and the non-linear wavelet estimator is globally adaptive when $B = 0$ . Also, according to the table **??** , it can be observed that for the non-linear wavelet estimator in the block method, the MSE is less compared to the linear estimator.In the second case, the non-linear wavelet estimator using the block method also has a lower MSE compared to the linear estimator. Therefore, in both cases, the non-linear estimator has a optimal convergence rate compared to the linear estimator. However, when comparing estimators for the two types of statistical distributions, Normal(supersmooth) and Laplace, the Laplace estimator has a better convergence rate.

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# What is Data Engineering?

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**Abstract**

Data engineering is a critical discipline within the realm of data science and analytics, focusing on the design, construction, and management of robust data systems. It encompasses a wide range of activities, including data acquisition, transformation, storage, and accessibility, ensuring that high-quality data is readily available for analysis and decision-making. As organizations increasingly rely on data-driven insights to drive strategic initiatives, the role of data engineers has become pivotal in building scalable data pipelines, optimizing data architectures, and implementing best practices for data governance. This paper explores the fundamental principles of data engineering, a brief history of the field and a case in of the the big tech companies.

## 1. History of Data Engineering

In this section we will see a brief history and timeline of data engineering field.

1. **Early Beginnings (1960s - 1980s)** Mainframe Era: The roots of data engineering can be traced back to the 1960s with the advent of mainframe computers. Data management primarily revolved around batch processing and hierarchical databases. Database Management Systems (DBMS): The development of relational databases in the 1970s, particularly with the introduction of SQL (Structured Query Language) and systems like IBM's DB2, laid the groundwork for structured data management. The focus was on how to store, retrieve, and manage data efficiently.
2. **Rise of Data Warehousing (1980s - 1990s)** Data Warehousing Concepts: By the late 1980s and early 1990s, the concept of data warehousing emerged, popularized by figures like Bill Inmon and Ralph Kimball. Data engineers began to specialize in extracting, transforming, and loading (ETL) data into warehouses, enabling organizations to analyze historical data for decision-making. Business Intelligence (BI): This period saw the rise of BI tools that relied on data warehouses, emphasizing the need for data engineering to ensure data quality and accessibility.
3. **The Big Data Revolution (2000s)** Explosion of Data: The 2000s marked a significant shift with the explosion of unstructured data from sources like social media, IoT devices, and web applications. This necessitated new approaches to data storage and processing. Introduction of NoSQL: Technologies such as NoSQL databases (e.g., MongoDB, Cassandra) emerged to handle large volumes of unstructured data, allowing data engineers to

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manage diverse data types more effectively. Hadoop Ecosystem: The introduction of Apache Hadoop in 2006 revolutionized data processing. It provided a framework for distributed storage and processing of large datasets, enabling data engineers to build scalable data pipelines.

4. **Cloud Computing and Modern Data Engineering (2010s - Present)** Cloud-Based Solutions: The rise of cloud computing platforms (e.g., AWS, Google Cloud, Azure) transformed data engineering by providing scalable resources for storage and processing. Data engineers began leveraging cloud services for building data lakes, stream processing, and serverless architectures. DataOps and Automation: The concept of DataOps emerged, focusing on automating data workflows and enhancing collaboration between data engineering and data science teams. This led to the development of tools for continuous integration and deployment in data pipelines. Real-Time Data Processing: Technologies like Apache Kafka and Apache Spark enabled real-time data processing, allowing organizations to make timely decisions based on streaming data.
5. **Current Trends and Future Directions** Focus on Data Governance and Ethics: As data privacy concerns grow, data engineering emphasizes data governance, compliance with regulations (like GDPR), and ethical data usage. Integration of AI and Machine Learning: Data engineers increasingly work alongside data scientists to build infrastructure that supports machine learning models, focusing on data quality and availability. Emergence of Data Mesh and Data Fabric: New paradigms like data mesh and data fabric are gaining traction, promoting decentralized data management and integration across different business units.

## 2. Case Study: Data Engineering at Netflix

Netflix, a leading streaming service with millions of subscribers globally, relies heavily on data engineering to manage and utilize the vast amounts of data generated by its users. The company leverages data engineering to enhance user experience, personalize content recommendations, and optimize streaming quality.

### *2.1. Challenge*

As Netflix's user base grew, so did the volume and complexity of the data it collected. The company faced challenges in:

- **Scalability**: Handling petabytes of data generated daily from user interactions, viewing habits, and content metadata.
- **Real-time Processing**: Delivering personalized recommendations in real-time based on user behavior.
- **Data Quality**: Ensuring the accuracy and consistency of data collected from various sources.

### *2.2. Data Engineering Solutions*

And now we can see the solution that Data Engineering brought to the scientists attention at Netflix:

1. **Data Pipeline Development**: Netflix built robust data pipelines using Apache Kafka for real-time data streaming and Apache Spark for batch processing. This allowed them to ingest data from various sources, including user activity, system logs, and content metadata, efficiently and at scale.
2. **Data Warehousing**: The company utilizes a data lake architecture on Amazon S3, which allows for the storage of both structured and unstructured data. They employ data warehousing solutions like Amazon Redshift to perform analytics and reporting, facilitating complex queries and insights generation.
3. **Machine Learning Integration**: Data engineers at Netflix collaborate with data scientists to build and deploy machine learning models that analyze user behavior and predict content preferences. These models provide personalized recommendations, significantly enhancing user engagement and satisfaction.
4. **Monitoring and Governance**: Netflix implemented data governance practices to ensure data integrity and security. They developed monitoring tools to track data quality and pipeline performance, allowing them to identify and resolve issues proactively.

## 3. Final Thoughts

Data engineering has emerged as a foundational pillar in the landscape of data science and analytics, playing a crucial role in the effective management and utilization of data. As organizations increasingly rely on data-driven insights to inform their strategies and operations, the importance of skilled data engineers continues to grow. They are tasked with building robust data architectures, developing efficient data pipelines, and ensuring data quality and accessibility—all of which are essential for supporting advanced analytics and machine learning initiatives.

The evolution of data engineering, from its early beginnings to the current focus on real-time processing and cloud-based solutions, highlights its adaptability in the face of changing technological landscapes. Furthermore, the integration of automation, governance, and ethical considerations into data engineering practices underscores the discipline's commitment to not only efficiency but also responsible data usage.

As we look to the future, data engineering will remain critical in helping organizations navigate the complexities of big data, enabling them to harness the power of information to drive innovation, enhance customer experiences, and ultimately achieve their business objectives. Investing in data engineering capabilities will be essential for organizations aiming to thrive in an increasingly data-centric world.

If you are interested in Data Engineering, you can consult one the these references: [1–3]

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# What is MLOps?

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**Abstract**

As machine learning (ML) technologies become integral to various industries, the need for effective management and deployment of ML models has grown. This paper introduces Machine Learning Operations (MLOps), a framework that facilitates the seamless integration of ML development and operations. We examine key MLOps components, including version control, CI/CD pipelines, and performance monitoring, and present case studies demonstrating MLOps' impact on collaboration, model reliability, and deployment speed.

## 1. Introduction

The advent of artificial intelligence (AI) and machine learning (ML) has revolutionized numerous sectors, from healthcare and finance to retail and transportation. As organizations increasingly leverage ML models to drive decision-making and improve operational efficiency, the complexity of deploying and maintaining these models in production environments has become a pressing challenge. Traditional software development practices often fall short in addressing the unique requirements of ML, such as handling vast amounts of data, ensuring model accuracy, and adapting to evolving business needs.

Machine Learning Operations (MLOps) has emerged as a vital discipline that bridges the gap between data science and IT operations. By adopting MLOps practices, organizations can enhance collaboration between data scientists and operations teams, automate workflows, and ensure the reliability and scalability of ML systems. This paper aims to provide a comprehensive overview of MLOps, exploring its core components and their significance in managing the ML lifecycle effectively.

## 2. History of MLOps

In this section we will see a brief history and timeline of MLOps field.

The roots of MLOps can be traced back to the broader DevOps movement, which sought to improve collaboration between software development and IT operations. As organizations began to recognize the potential of machine learning, it became evident that similar methodologies were needed to address the unique challenges associated with

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ML workflows. The term "MLOps" was popularized in the late 2010s, as practitioners and researchers began to define processes and best practices specifically tailored for ML projects.
Early MLOps efforts focused on standardizing workflows, establishing version control for datasets and models, and implementing continuous integration and continuous deployment (CI/CD) pipelines. As the field matured, practitioners began to incorporate automation, monitoring, and governance into MLOps frameworks, allowing organizations to effectively manage the lifecycle of ML models from development through deployment and beyond.
Today, MLOps encompasses a wide range of practices and tools that facilitate the efficient management of ML systems, enabling organizations to harness the full potential of their data-driven initiatives. As businesses continue to invest in AI technologies, the adoption of MLOps frameworks will be crucial for ensuring the successful deployment and sustainability of machine learning solutions.

## 3. One case study of MLOps

A leading healthcare organization aimed to enhance patient outcomes by leveraging machine learning models to predict patient readmissions. The organization had a team of data scientists working on various predictive models, but they faced significant challenges in deploying these models into production and ensuring their reliability over time. The absence of a structured MLOps framework led to inefficiencies, including lengthy deployment cycles, difficulties in monitoring model performance, and challenges in collaboration between data scientists and IT operations teams.
The primary objective was to establish an MLOps framework that would streamline the deployment and management of predictive models, improve collaboration among stakeholders, and ensure the models' continuous performance monitoring and retraining based on new data.

### *3.1. MLOps implementation Steps*

1. **Assessment and Planning:** The organization conducted an assessment of its existing ML processes and identified bottlenecks in the model development lifecycle, including manual deployment processes and lack of performance monitoring.
2. **Establishing Version Control:** The team implemented version control systems (e.g., Git) for both code and model artifacts. This allowed them to track changes in datasets, feature engineering scripts, and model parameters, facilitating collaboration among data scientists.
3. **Building CI/CD Pipelines:** Continuous Integration and Continuous Deployment (CI/CD) pipelines were created using tools like Jenkins and Docker. These pipelines automated the testing and deployment of ML models, significantly reducing deployment time from weeks to days.
4. **Monitoring and Performance Evaluation:** The organization integrated monitoring tools (e.g., Prometheus and Grafana) to track model performance in real-time. Key metrics such as accuracy, precision, and recall were continuously monitored, allowing the team to detect model drift and performance degradation quickly.
5. **Automated Retraining:** An automated retraining pipeline was established, where models would be retrained periodically using the latest data. This ensured that the models remained accurate and relevant as patient data evolved over time.
6. **Collaboration and Governance:** Regular communication was established between data scientists, operations teams, and healthcare professionals. They held bi-weekly meetings to discuss model performance, gather feedback, and make necessary adjustments.

## Conclusion

In conclusion, MLOps is not merely a set of tools or practices; it is a transformative approach that can significantly enhance the effectiveness of machine learning initiatives. By embracing MLOps, organizations can harness the full potential of their data, drive better business outcomes, and create a sustainable framework for future AI developments. As the field continues to mature, the lessons learned from MLOps implementations will serve as valuable guidance for organizations striving to navigate the complexities of machine learning at scale.
If you are interested in this fascinating topic you can see references [1–4].

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# A note on Gorenstein projective modules

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**Abstract**

Let $(S, \mathfrak{m})$ be a commutative noetherian local ring and $\omega \in \mathfrak{m}$ be a non-zero divisor element of $S$. Let $f : P \to Q$ and $g : Q \to P$ be two morphisms of finitely generated projective $S$-modules such that the compositions $fg$ and $gf$ are multiplications by $\omega$. The aim of this paper is to show that the cokernels of $f$ and $g$ are Gorenstein projective over the factor ring $S/(\omega)$.

## 1. Introduction

A (finitely generated) module $M$ is said to be Gorenstein projective, if it is a syzygy of a totally acyclic complex of projective modules. Gorenstein projective modules, which are a refinement of projective modules, was defined by Enochs and Jenda in [2]. The corresponding classes of Gorenstein injective and Gorenstein flat modules were defined in [2] and [3]. The origin of Gorenstein projective modules even goes back to Auslander and Bridger [1], who introduced the G-dimension of a finitely generated module over a commutative noetherian ring to characterize Gorenstein local rings: a commutative noetherian local ring is Gorenstein if and only if its residue field has finite G-dimension. Since then, Gorenstein homological algebra has developed rapidly during the past several years. The theory of modules of finite Gorenstein dimensions has found some interesting applications in representation theory; these include the structure of the stable category of Cohen-Macaulay modules, the Auslander-Reiten theory, the existence of Serre duality at the level of perfect complexes and the theory of singularities
Assume that $(S, \mathfrak{m})$ is a commutative noetherian local ring and $\omega \in \mathfrak{m}$ is a non-zero divisor element of $S$. Assume that $f : P \to Q$ and $g : Q \to P$ are two morphisms of finitely generated projective $S$-modules such that $fg = \omega.\mathrm{id}_Q$ and $gf = \omega.\mathrm{id}_P$. In this paper we show that $\mathrm{coker} f$ as well as $\mathrm{coker} g$ are Gorenstein projective $S/(\omega)$-modules.

## 2. Results

Let us begin this section by recalling the definition of Gorenstein projective modules. Throughout the paper, $S$ is a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and $R$ is the factor ring $S/(\omega)$.

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Gorenstein projective modules. An acyclic complex of projective $S$-modules;

$\mathbf{P}_\bullet : \cdots \longrightarrow P_{n+1} \overset{d_{n+1}}{\longrightarrow} P_n \overset{d_n}{\longrightarrow} P_{n-1} \overset{d_{n-1}}{\longrightarrow} \cdots$ is called *totally acyclic*, if the acyclicity is preserved by $\mathrm{Hom}_S(-,P)$ for every projective $S$-module $P$. An $S$-module $M$ is said to be *Gorenstein projective*, if it is a syzygy of a totally acyclic complex of projective modules. Clearly, every projective module is Gorenstein projective.
It is known that over a Gorenstein ring $S$, every acyclic complex is totally acyclic, and also $d$-th syzygy of any $S$-module is Gorenstein projective, where $d = \dim S$; see [2, Theorem 10.2.14]. Finitely generated Gorenstein projective modules are equal to the maximal Cohen-Macaulay modules. In what follows, $f : P \to Q$ and $g : Q \to P$ are two morphisms of finitely generated projective $S$-modules such that $fg = \omega.\mathrm{id}_Q$ and $gf = \omega.\mathrm{id}_P$.

**Remark 2.1.** Take an object $x \in P$ such that $f(x) = 0$. So $gf(x) = \omega.x = 0$. Now since $\omega$ is non-zero divisor, we have $x = 0$, meaning that $f$ is a monomorphism. Similarly, $g$ is a monomorphism. Next take an arbitrary object $\alpha \in \mathrm{coker}f$. So there exists an object $y \in Q$ such that $\alpha = \mathrm{Im}f + x$, and then, $\omega.\alpha = \mathrm{Im}f + \omega.y = \mathrm{Im}f + fg(y) = 0$. Consequently, $\omega\mathrm{coker}f = 0$, and in particular, $\mathrm{coker}f$ is an $R$-module.

**Proposition 2.2.** $0 \to \mathrm{coker}f \to P/\omega P \overset{\bar{f}}{\to} Q/\omega Q \to \mathrm{coker}f \to 0$ *is an exact sequence of $R$-modules, where* $\bar{f} = f \otimes_S R$.

*Proof.* Since $f : P \to Q$ is a morphism of $S$-modules, $\bar{f} : P/\omega P \to Q/\omega Q$ is a morphism of $R$-modules. This gives us the exact sequence of $R$-modules $0 \to \ker\bar{f} \to P/\omega P \overset{\bar{f}}{\to} Q/\omega Q \to \mathrm{coker}\bar{f} \to 0$ . So it remains to show the validity of the equalities $\ker\bar{f} = \mathrm{coker}f = \mathrm{coker}\bar{f}$. Asthe functor $- \otimes_S R$ is right exact, $\mathrm{coker}\bar{f} = \mathrm{coker}f \otimes_S R = \mathrm{coker}f \otimes_S S/\omega = \mathrm{coker}f/\omega\mathrm{coker}f \cong \mathrm{coker}f$, because by Remark 2.1, $\omega\mathrm{coker}f = 0$. Next, we show the equality $\ker\bar{f} = \mathrm{coker}f$. To see this, one may apply the functor $- \otimes_S R$ to the short exact sequence of $S$-modules $0 \to P \overset{f}{\to} Q \to \mathrm{coker}f \to 0$, and get the exact sequence of $R$-modules

$$0 \to \mathrm{Tor}_1^S(\mathrm{coker}f, R) \to P/\omega P \overset{\bar{f}}{\to} Q/\omega Q \to \mathrm{coker}f \to 0.$$

Thus it is enough to show that $\mathrm{Tor}_1^S(\mathrm{coker}f, R) \cong \mathrm{coker}f$. Tothis ends, we apply the functor $\mathrm{coker}f \otimes_S -$ to the short exact sequence of $S$-modules; $0 \to S \overset{\omega}{\to} S \to R \to 0$ and obtain the exact sequence of $R$-modules

$$0 \to \mathrm{Tor}_1^S(\mathrm{coker}f, R) \to \mathrm{coker}f \overset{\omega}{\to} \mathrm{coker}f \to \mathrm{coker}f \otimes_S R \to 0.$$

Now since $\mathrm{coker}f$ is annihilated by $\omega$, $\mathrm{Tor}_1^S(\mathrm{coker}f, R) \cong \mathrm{coker}f$. So the proof is finished. □

**Theorem 2.3.** $\mathrm{coker}f$ *and* $\mathrm{coker}g$ *are Gorenstein projective $R$-modules.*

*Proof.* By the similarity, we only deal with $\mathrm{coker}f$. According to Proposition 2.2, there exists exact sequence of $R$-modules $0 \to \mathrm{coker}f \to P/\omega P \overset{\bar{f}}{\to} Q/\omega Q \to \mathrm{coker}f \to 0$. Also, in the same way, $0 \to \mathrm{coker}g \to Q/\omega Q \overset{\bar{g}}{\to} P/\omega P \to \mathrm{coker}g \to 0$ will be an exact sequence of $R$-modules. Since $S$ is a local ring, $P, Q$ are free $S$-modules, implying that $P/\omega P$ and $Q/\omega Q$ are free $R$-modules, as well. All of these yield that ${}_R(\mathrm{coker}f) = \mathrm{coker}g$ and ${}_R(\mathrm{coker}g) = \mathrm{coker}f$. In particular, we get the following acyclic complex of projective $R$-modules

$$\cdots \to P/\omega P \overset{\bar{f}}{\to} Q/\omega Q \overset{\bar{g}}{\to} P/\omega P \overset{\bar{f}}{\to} Q/\omega Q \overset{\bar{f}}{\to} \cdots.$$

Since the projective dimensions of $\mathrm{coker}f$ and $\mathrm{coker}f$ over $S$ are at most one, one has the equalities $\mathrm{Ext}_S^i(\mathrm{coker}f, S) = 0 = \mathrm{Ext}_S^i(\mathrm{coker}g, S)$ for all $i \geq 2$. Moreover, as $\omega\mathrm{coker}f = 0$, applying [4, Lemma 2(i), page 140] gives us the isomorphism $\mathrm{Ext}_S^{i+1}(\mathrm{coker}f, S) \cong \mathrm{Ext}_R^i(\mathrm{coker}f, R)$ for each $i \geq 1$. Similarly, one will have the isomorphism $\mathrm{Ext}_S^{i+1}(\mathrm{coker}f, S) \cong \mathrm{Ext}_R^i(\mathrm{coker}f, R)$ for each $i \geq 1$. Consequently, $\mathrm{Ext}_R^i(\mathrm{coker}f, R) = 0 = \mathrm{Ext}_R^i(\mathrm{coker}g, R)$ for all $i \geq 1$. Hence, the latter acyclic complex is totally acyclic, and so, $\mathrm{coker}f$ and $\mathrm{coker}g$ are Gorenstein projective $R$-modules. So the proof is completed. □

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# On the derivation of reliability bounds for coherent systems

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**Abstract**

The concept of system signature provides a powerful tool for the analysis and reliability assessment of engineering systems. In this paper, we derive bounds for the survival function of coherent systems under the assumption that component lifetimes are independent and identically distributed. These bounds are established by leveraging the system signature framework and properties of order statistics associated with component lifetimes. The results offer valuable insights into the reliability behavior of coherent systems and provide a foundation for further theoretical and practical applications in reliability engineering.

## 1. Introduction

Coherent systems have been widely used in engineering reliability, particularly in electronic and electrotechnical designs (see e.g., Billinton and Allan [1]). A system is considered coherent if every component is relevant (meaning that the failure of any component affects the system's failure) and the system's structure function is monotone in each component (i.e., replacing a failed component with a working one does not degrade the system's performance). For further details, refer to Barlow and Proschan [2]. Recently, the concept of system signature, introduced by Samaniego [3], has been extensively utilized to analyze the stochastic and aging properties of coherent systems. For instance, Kochar *et al.* [4] employed system signatures to compare coherent systems with independent and identically distributed (iid) component lifetimes. Extending this work, Navarro *et al.* [5] applied signatures to compare coherent systems with exchangeable component lifetimes. More specifically, let $T$ represent the lifetime of a coherent system composed of $n$ iid component lifetimes $X_1, \cdots, X_n$, each with the common cumulative distribution function (cdf) $F$. Assuming that $P(T \in \{X_{1:n}, \cdots, X_{n:n}\}) = 1$, Samaniego [6] demonstrated that the system's reliability can be expressed in terms of its signature as follwos:

$$\bar{F}_T(t) := P(T > t) = \sum_{i=1}^{n} p_i \bar{F}_{i:n}(t), \ \forall\, t > 0, \tag{1}$$

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where $X_{i:n}$ $(i = 1, \cdots, n)$ stands for the $i$-th order statistics among $X_1, \cdots, X_n$ with the survival function (SF)

$$\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} [F(t)]^j [\bar{F}(t)]^{n-j},\ t \geq 0,\ 1 \leq i \leq n, \tag{2}$$

which $\bar{F}(t) = 1 - F(t)$. The vector $\mathbf{p} = (p_1, \cdots, p_n)$ is called signature of system where $p_i = P(T = X_{i:n})$ is the probability that the lifetime coincides with the $i$-th order statistic of the component lifetime $(i = 1, \cdots, n)$. Note that $\sum_{i=1}^{n} p_i = 1$. A comprehensive discussion about the applications of the system signature can be found in Samaniego [6].

Due to the complexity of system configurations, computing the reliability structure function of engineering systems is often challenging and, in some cases, impossible. In such scenarios, deriving bounds for system reliability can be highly beneficial. The literature has explored various bounds for the reliability function of coherent systems. In this paper, we provide bounds for the SF of the coherent system. The results are further extended to coherent systems with exchangeable component lifetimes.

## 2. Reliability Bounds

Hereafter, we present the main results of this paper, which focus on deriving bounds for the reliability function of coherent systems using the concept of system signatures. These bounds not only simplify the analysis but also provide critical insights into the performance and failure characteristics of coherent systems. Hereafter, we give the main results of this paper. If $T$ is the lifetime of a coherent system with $n$ iid components and signature $\mathbf{p} = (0, \cdots, 0, p_r, \cdots, p_k, 0, \cdots, 0)$ where $1 \leq r \leq k \leq n$, and $p_r > 0,\ p_k > 0$, then from (1), we get

$$\bar{F}_{r:n}(t) \leq \bar{F}_T(t) \leq \bar{F}_{k:n}(t),\ t > 0. \tag{3}$$

Navarro and Rychlik [7] showed that the representation (1) also holds for the lifetime of coherent systems with exchangeable component lifetimes. Notice that in this case,

$$\bar{F}_{i:n}(t) = P(X_{i:n} > t) = \sum_{j=0}^{i-1} \binom{n}{j} P(X_1 < t, \cdots, X_j < t, X_{j+1} \geq t, \cdots, X_n \geq t). \tag{4}$$

They also provided bounds for the SF of the coherent system. Suppose $G_{\mathbf{P}}, S_{\mathbf{P}} : [0,1] \to R$ denote the greatest convex and smallest concave functions, respectively, satisfying $G_{\mathbf{P}}(0) = S_{\mathbf{P}}(0) = 0$, and

$$G_{\mathbf{p}}(j/n) \leq \sum_{i=1}^{j} p_i \leq S_{\mathbf{p}}(j/n),\ j = 1, \cdots, n.$$

Navarro and Rychlik [7] proved that the system reliability with exchangeable component lifetimes satisfy

$$1 - S_{\mathbf{p}}(1 - \bar{F}(t)) \leq \bar{F}_T(t) \leq 1 - G_{\mathbf{p}}(1 - \bar{F}(t)). \tag{5}$$

From (3) and (5), the reliability bounds for the system lifetime can be improved as:

$$\bar{F}_L(t) \leq \bar{F}_T(t) \leq \bar{F}_U(t),\ \forall t > 0. \tag{6}$$

where

$$\bar{F}_L(t) = \max\left(\bar{F}_{r:n}(t), 1 - S_{\mathbf{p}}(1 - \bar{F}(t))\right),$$

and

$$\bar{F}_U(t) = \min\left(\bar{F}_{k:n}(t), 1 - G_{\mathbf{p}}(1 - \bar{F}(t))\right).$$

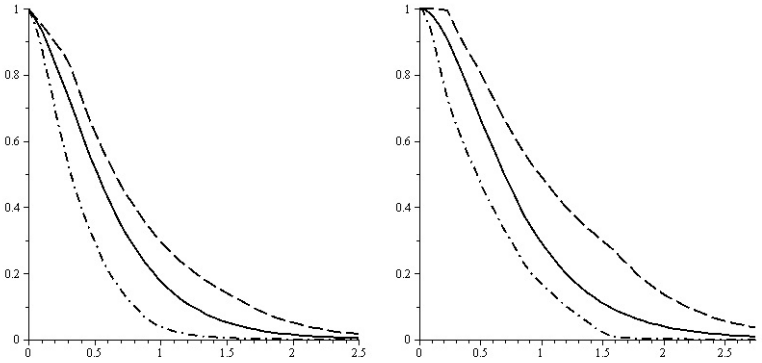

Fig. 1. Reliability bounds for the bridge system with exchangeable (left) and iid (right) component lifetimes.

**Remark 2.1.** The improved expectation bounds for the lifetime of a given coherent system can be obtained from (3) and the expectation bounds in Corollary 1 duo to Navarro and Rychlik [7] as follows:

$$\max\left\{\mu_{r:n}, \int_0^1 F^{-1}(x)S'_{\mathbf{p}}(x)dx\right\} \leq \mu_T \leq \min\left\{\mu_{k:n}, \int_0^1 F^{-1}(x)G'_{\mathbf{p}}(x)dx\right\}, \tag{7}$$

where $F^{-1}$ is the quantile function of $F$ and $S'_{\mathbf{p}}(x)$ and $G'_{\mathbf{p}}(x)$ are the derivative of functions $S_{\mathbf{p}}$ and $G_{\mathbf{p}}$, respectively.

To illustrate the above results, consider the following example.

**Example 2.2.** Let $\mathbf{p} = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$ be the signature of the well-known Bridge system consisting $n = 5$ components with the common cdf $F$. From (3), $\bar{F}_{2:5}(t) \leq \bar{F}_T(t) \leq \bar{F}_{4:5}(t)$ for $t > 0$, and after some algebraic calculations, we have

$$\bar{F}_L(t) = \max\left\{\frac{4\bar{F}(t)-1}{3}, \frac{5\bar{F}(t)-1}{5}, \bar{F}_{2:5}(t)\right\}, \tag{8}$$

$$\bar{F}_U(t) = \min\left\{\frac{5\bar{F}(t)+1}{5}, \frac{4}{3}\bar{F}(t), \bar{F}_{4:5}(t)\right\}. \tag{9}$$

From (8) and (9), the SF of the system satisfies

$$\max\left\{\frac{4\bar{F}(t)-1}{3}, \frac{5\bar{F}(t)-1}{5}, \bar{F}_{2:5}(t)\right\} \leq \bar{F}_T(t) \leq \min\left\{\frac{5\bar{F}(t)+1}{5}, \frac{4}{3}\bar{F}(t), \bar{F}_{4:5}(t)\right\}.$$

(i) In iid case, assuming the standard exponential distribution for lifetimes of components. The proposed bounds as well as the exact value of reliability are depicted in Figure 1 (right diagram). It is easy to see that

$$0.555 = \max(0.450, 0.555) \leq \mu_T \leq \min(1.283, 1.280) = 1.280.$$

Notice that $\mu_T = 0.816$. Here we see that the proposed expectation bounds coincide with the bounds obtained by Navarro and Rychlik [7].

(ii) In exchangeable case, assume that

$$\begin{aligned}\bar{F}(x_1,x_2,x_3,x_4,x_5) &= P(X_1 > x_1, X_2 > x_2, X_3 > x_3, X_4 > x_4, X_5 > x_5)\\ &= \exp\left(-\sum_{i=1}^{5} x_i - 0.5 \max_{1\leq i\leq 5}(x_i)\right),\ \forall\, x_i \geq 0,\ 1 \leq i \leq 5,\end{aligned} \tag{10}$$

which is known as the Marshall and Olkin's multivariate exponential distribution; see, e.g., Kotz *et al.* [**?** ], p. 391. From (4) and (10), we have

$$\bar{F}_{2:5}(t) = 5e^{-4.5t} - 4e^{-5.5t}, \quad (11)$$

$$\bar{F}_{4:5}(t) = 10e^{-2.5t} - 20e^{-3.5t} + 15e^{-4.5t} - 4e^{-5.5t}. \quad (12)$$

Also component lifetimes have the common marginal survival function $\bar{F}(t) = e^{-1.5t}$. The proposed bounds as well as the exact value of reliability are depicted in Figure 1 (left diagram). We see that bounds in two cases are work similar. Also, we have

$$0.383 = \max(0.383, 0.370) \leq \mu_T \leq \min(0.891, 0.853) = 0.853.$$

Notice that $\mu_T = 0.623$. In this case we see that the proposed expectation bounds is better than the bounds obtained by Navarro and Rychlik [7].

## 3. Conclusions

In this paper, we have modified and extended the bounds for the reliability of coherent systems originally derived by Navarro and Rychlik [7]. By refining these bounds, we have provided a more versatile framework for analyzing the reliability of coherent systems, particularly those with exchangeable component lifetimes. This extension broadens the applicability of the results to a wider range of system configurations, offering valuable tools for reliability assessment in complex engineering scenarios. To demonstrate the practical utility of our findings, we have included an illustrative example that highlight the effectiveness of the proposed bounds in predicting system behavior and reliability.

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# Reliability Bounds for Coherent Systems Under Sequential Order Statistics

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**Abstract**

This research focuses on deriving bounds for the survival function of coherent systems with component lifetimes following sequential order statistics. Utilizing the system signature, particularly its dynamic extension to handle partial failure information, we provide a more refined approach to reliability analysis. An illustrative example demonstrates the practical value of our bounds in assessing system dependability.

## 1. Introduction

Engineering reliability analyses, particularly in electronic and electrotechnical designs, extensively utilize coherent systems. A system is defined as coherent if all its components are essential, meaning component failure does not directly lead to system success, and its structure function exhibits monotonicity, implying that replacing a failed component with a functional one never degrades system performance. For a comprehensive treatment, refer to Barlow and Proschan [1]. The system signature, introduced by Samaniego [2], has become a prevalent tool for investigating the stochastic and aging characteristics of coherent systems. Notably, Kochar et al. [3] employed the system signature to facilitate comparisons between coherent systems with independent and identically distributed (iid) component lifetimes. Building upon this, Navarro et al. [4]-[6] extended the application of the signature to compare coherent systems featuring exchangeable component lifetimes. Specifically, given a coherent system with lifetime T and n iid component lifetimes $X_1, \cdots, X_n$ each with a common cumulative distribution function (CDF) F, Samaniego [2] demonstrated that the system reliability can be determined, provided $P(T \in \{X_{1:n}, \cdots, X_{n:n}\}) = 1$, as follows:

$$\bar{F}_T(t) := P(T > t) = \sum_{i=1}^{n} p_i \bar{F}_{i:n}(t), \ \forall\, t > 0, \tag{1}$$

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where $X_{i:n}$ $(i = 1, \cdots, n)$ stands for the $i$-th order statistics among $X_1, \cdots, X_n$ with the survival function (SF)

$$\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} [F(t)]^j [\bar{F}(t)]^{n-j},\ t \geq 0,\ 1 \leq i \leq n, \tag{2}$$

which $\bar{F}(t) = 1 - F(t)$. The vector $\mathbf{p} = (p_1, \cdots, p_n)$ is defined as the system signature, where $p_i = P(T = X_{i:n})$ represents the probability that the system lifetime T coincides with the i-th order statistic $X_{i:n}$ of the component lifetimes, for $i = 1, \cdots, n$. It is important to note that $\sum_{i=1}^{n} p_i = 1$. A detailed exploration of the applications of the system signature is available in Samaniego [7].

The intricate nature of system configurations often makes calculating the structure function of engineering systems challenging or, in some instances, infeasible. In such scenarios, determining bounds for system reliability proves advantageous. This study presents bounds for the SF of coherent systems, which are employed to forecast the system lifetime behavior. These findings are further applied to coherent systems with components exhibiting exchangeable lifetimes.

## 2. Reliability Bounds for Coherent systems with SOS components

Here we extend the proposed bounds in preceding sections when lifetimes of components follow sequential order statistics (SOS) introduced by Kamps [8]. Specifically, if a component in working system fails, then the remaining components might be affected by the failure. Let $F_i,\ (i = 1, \cdots, n)$ denote the common cdf of independent component lifetimes $X_1^{(i)}, \cdots, X_{n-i+1}^{(i)}$ when $n - i + 1$ components work in the system. Then the $i$th failure time of a component is given by

$$X_{i:n}^{\star} = \min\{X_1^{(i)}, \cdots, X_{n-i+1}^{(i)}\}.$$

The resulting ordered lifetimes $X_{1:n}^{\star} \leq \cdots \leq X_{n:n}^{\star}$ are called sequential order statistics (SOS) based on $F_1, \cdots, F_n$; see e.g., Kamps [8] and Navarro and Burkschat [9] for more details. Recently, the idea underlying SOS model has been extended to coherent systems. In order to describe component lifetimes in a system, where failures might change the lifetimes of remaining components, a particular dependence model for components is assumed. The vector of component lifetimes $X_1^{\star}, \cdots, X_n^{\star}$ is chosen to be an exchangeable coincides with the $X_{1:n}^{\star} \leq \cdots \leq X_{n:n}^{\star}$ based on $F_1, \cdots, F_n$. Let $T^{\star}$ be the lifetime of a coherent system with exchangeable component lifetimes $X_1^{\star}, \cdots, X_n^{\star}$. Navarro and Burkschat [9] showed that

$$\bar{F}_T^{\star}(t) = P(T^{\star} > t) = \sum_{i=1}^{n} s_i P(X_{i:n}^{\star} > t), \tag{3}$$

where $X_{1:n}^{\star} \leq \cdots \leq X_{n:n}^{\star}$ are SOS coming from $F_1, \cdots, F_n$. If $T^{\star}$ is the lifetime of a coherent system with $n$ iid components and signature $\mathbf{p} = (0, \cdots, 0, p_r, \cdots, p_k, 0, \cdots, 0)$ where $1 \leq r \leq k \leq n$, and $p_r > 0,\ p_k > 0$, then from (3), we have

$$P(X_{r:n}^{\star} > t) \leq \bar{F}_T^{\star}(t) \leq P(X_{k:n}^{\star} > t), \tag{4}$$

for all $t > 0$. Moreover, we have the following another bound.

**Proposition 2.1.** *Let $T^{\star}$ be the lifetime of a coherent system with signature $\boldsymbol{p}$ and n exchangeable component lifetimes $X^{\star} = (X_1^{\star}, \cdots, X_n^{\star})$ constructed based on a permutation of a SOS coming from $F_1, \cdots, F_n$ and a given marginal distribution function $F^{\star}$. If $G_{\boldsymbol{P}}, S_{\boldsymbol{P}} : [0,1] \to R$ denote the greatest convex and smallest concave functions, respectively, satisfying $G_{\boldsymbol{P}}(0) = S_{\boldsymbol{P}}(0) = 0$ and*

$$G_p(j/n) \leq \sum_{i=1}^{j} p_i \leq S_p(j/n),\ j = 1, \cdots, n.$$

*Then*

$$G_p(F^{\star}(t)) \leq \bar{F}_T^{\star}(t) \leq S_p(F^{\star}(t)),\ t \in R, \tag{5}$$

*where $\bar{F}^{\star}(t) = \frac{1}{n} \sum_{i=1}^{n} \bar{F}_{i:n}^{\star}(t)$.*

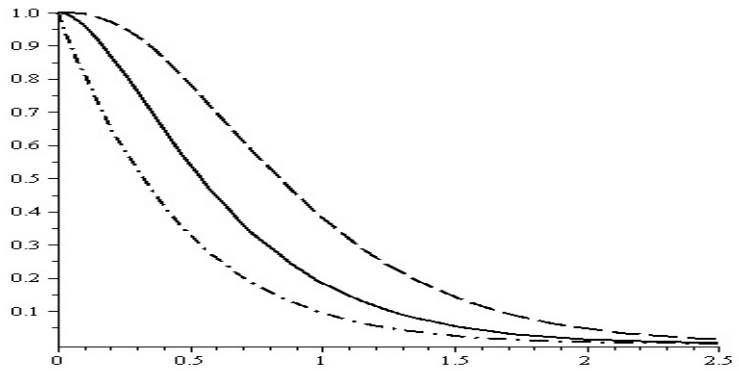


Fig. 1. Reliability bounds for the coherent system with SOS components

*Proof.* From Definition 2.2 of Navarro and Burkschat [9], there exists an exchangeable random vector $X_1^\star, \cdots, X_n^\star$ with common marginal distribution function $F^\star$ such that $T^\star = \phi(X_1^\star, \cdots, X_n^\star)$ where $\phi$ is the structure function of the system. Hence Theorem 1 due to Navarro and Rychlik [5] completes the proof. □ □

Thus, relations (4) and (5), the proposed bounds can be obtained as follows:

$$\max\left(\bar{F}_{r:n}^\star(t), 1 - S_{\mathbf{p}}(1 - \bar{F}^\star(t))\right) \leq \bar{F}_T^\star(t) \leq \min\left(\bar{F}_{k:n}^\star(t), 1 - G_{\mathbf{p}}(1 - \bar{F}^\star(t))\right). \tag{6}$$

**Example 2.2.** The proposed expectation bounds for the lifetime of a coherent system with SOS component lifetimes can as follows:

$$\max\left\{\mu_{r:n}^\star, \int_0^1 {F^\star}^{-1}(x) S_{\mathbf{p}}'(x)dx\right\} \leq \mu_T \leq \min\left\{\mu_{k:n}^\star, \int_0^1 {F^\star}^{-1}(x) G_{\mathbf{p}}'(x)dx\right\}, \tag{7}$$

where ${F^\star}^{-1}$ is the quantile function of $F^\star$. Notice that the explicit computation of $F^\star$ is not easy. In general, numerical methods must be employed to compute the expectation bounds.

**Example 3. (Burkschat and Navarro [10])** Consider a coherent system with lifetime $T^\star$ and signature $\mathbf{p} = (0, \frac{2}{3}, \frac{1}{3})$. They showed that

$$P(X_{1:3}^\star > t) = e^{-3t}, \tag{8}$$
$$P(X_{2:3}^\star > t) = 16e^{-3t} - 15e^{-3.2t}, \tag{9}$$

and

$$P(X_{3:3}^\star > t) = 136e^{-3t} - 255e^{-3.2t} + 120e^{-3.4t}. \tag{10}$$

Therefore we have

$$\bar{F}^\star(t) = \frac{1}{3}\sum_{i=1}^{3} \bar{F}_{i:n}^\star(t) = 51e^{-3t} - 90e^{-3.2t} + 40e^{-3.4t}.$$

After some calculation, bounds for the SF of the coherent system is

$$\max\left\{\bar{F}^\star(t), \bar{F}_{2:3}^\star(t)\right\} \leq \bar{F}_T^\star(t) \leq \min\left\{\frac{3}{2}\bar{F}^\star(t), \bar{F}_{3:3}^\star(t)\right\}.$$

In Figure 3, bounds are depicted for reliability of the system lifetime with SOS components.

## 3. Conclusions

In this paper, we have successfully derived novel bounds for the reliability of coherent systems, leveraging the power of system signature analysis, specifically when component lifetimes exhibit sequential order statistics. This approach provides a valuable tool for accurately assessing system dependability in complex engineering scenarios where component failures are not independent. The illustrative examples presented not only validate the theoretical framework but also demonstrate the practical utility of our bounds in providing tighter estimates of system reliability. This work contributes to the growing body of literature on system reliability analysis, particularly in the context of dependent component lifetimes.

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# Fuzzy regression: water quality modeling based on effective parameters

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**Abstract**

One of the most important issues in human life is the knowledge and enjoyment of health and hygiene in nutrition, particularly regarding water. In this regard, researchers and related organizations were trying to determine a criterion for achieving different ranges of water quality and presenting it as a default model based on effective variables such as minerals and salts in water. However, since the variables affecting water quality are different in different water resource beds, the models presented will be different. On the other hand, the ambiguous nature of quality forces us to use fuzzy structures. In this article, based on a fuzzy regression model, using different data from different beds and based on ten variables present in water and affecting its quality (which have been measured in all these beds), we will determine the effectiveness coefficients of each variable to determine different levels of water quality.

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