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## Conference Papers (English)

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# New Characterizations of Fuzzy Subhyperspaces via Fuzzy Points

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**Abstract**

In this paper, we study fuzzy subhyperspaces via the concept of fuzzy points. We establish a new equivalent characterization of fuzzy subhyperspaces in terms of the sum and scalar hyperoperations applied to fuzzy points, providing a structural viewpoint that complements existing results in the literature. Several illustrative examples over real and finite hypervector spaces are presented to clarify the theoretical developments. The results may serve as a basis for further investigations on related structural aspects of fuzzy subhyperspaces.

## 1. Introduction

The theory of algebraic structures is traditionally based on single-valued operations. In 1934, Marty [5] introduced the concept of hyperoperations, which assign a set of outputs to each pair of elements. This idea led to the development of algebraic hyperstructures and, in particular, hypervector spaces, first introduced by Tallini [7] as a natural generalization of classical vector spaces.

On the other hand, the theory of fuzzy sets, initiated by Zadeh [8] and extended to algebraic structures by Rosenfeld [6], provides a powerful framework for modeling uncertainty. The interaction between fuzziness and hyperstructures has resulted in the notion of fuzzy hypervector spaces, also called fuzzy subhyperspaces, which were studied by the authors [1–3].

Fuzzy subhyperspaces generalize fuzzy vector spaces by allowing scalar multiplication to be set-valued, a feature that naturally arises in multi-valued algebraic systems. Various characterizations of fuzzy subhyperspaces have been obtained, often relying on level subhyperspaces or inequality-based axioms.

The main purpose of this paper is to present new characterizations of fuzzy subhyperspaces based on fuzzy points. By reformulating the defining properties of fuzzy subhyperspaces in terms of sums and scalar hyperoperations of fuzzy points, we obtain equivalent conditions that provide a clearer structural insight into these objects.

The paper is organized as follows. In Section 2, we recall basic definitions and properties of hypervector spaces and fuzzy subhyperspaces. Section 3 contains the main results of the paper, where new equivalent characterizations of fuzzy subhyperspaces are established via fuzzy points, together with several illustrative examples over real and finite hypervector spaces. The paper concludes with a brief summary and perspectives for future research.

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## 2. Preliminaries

In this section, we recall some definitions and properties that will be used in the paper.

**Definition 2.1.** [7] Let $K$ be a field, $(V,+)$ an Abelian group, and $P_*(V)$ the set of all nonempty subsets of $V$. A *hypervector space* over $K$ is a quadruple $(V,+,\circ,K)$, where "$\circ$" is an external hyperoperation

$$\circ : K \times V \longrightarrow P_*(V),$$

such that for all $a,b \in K$ and $x,y \in V$, the following conditions hold:

1) $a \circ (x+y) \subseteq a \circ x + a \circ y$, (right distributive law),
2) $(a+b) \circ x \subseteq a \circ x + b \circ x$, (left distributive law),
3) $a \circ (b \circ x) = (ab) \circ x$,
4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
5) $x \in 1 \circ x$,

where in (1), $a \circ x + a \circ y = \{p+q : p \in a \circ x,\ q \in a \circ y\}$, and similarly for (2). Moreover, in (3), $a \circ (b \circ x) = \bigcup_{t \in b \circ x} a \circ t$.
A hypervector space $V$ is called *strongly left distributive* if equality holds in (2).
In the sequel, $V$ denotes a hypervector space over a field $K$, unless stated otherwise.
A nonempty subset $W$ of $V$ is called a *subhyperspace* of $V$, if $W$ is itself a hypervector space under the induced external hyperoperation of $V$, that is, for all $a \in K$ and $x,y \in W$, one has $x - y \in W$ and $a \circ x \subseteq W$.

**Example 2.2.** [2] In vector space $(\mathbb{R}^3,+,.,\mathbb{R})$ define the external hyperoperation $\circ : \mathbb{R} \times \mathbb{R}^3 \to P_*(\mathbb{R}^3)$ by

$$a \circ (x_1,x_2,x_3) = \{(ax_1, ax_2, t) : t \in \mathbb{R}\}.$$

Then $V = (\mathbb{R}^3,+,\circ,\mathbb{R})$ is a strongly left distributive hypervector space over the field $\mathbb{R}$.

**Definition 2.3.** [3] Let $\mu$ and $\nu$ be fuzzy subsets of $V$ and $a \in K$. We define $\mu + \nu$ and $a \circ \mu$ to be the fuzzy subsets of $V$, whose membership functions are given by:

$$(\mu+\nu)(x) = \bigvee \{\mu(y) \wedge \nu(z) :\ y,z \in V, x = y+z\},$$

$$(a \circ \mu)(x) = \begin{cases} \bigvee\limits_{x \in a \circ t} \mu(t) & \exists t \in V; x \in a \circ t, \\ 0 & otherwise. \end{cases}$$

**Definition 2.4.** [1] A fuzzy subset $\mu$ of $V$ is called a *fuzzy subhyperspace* of $V$, if for all $a \in K$ and $x,y \in V$, $\mu(x+y) \geq \mu(x) \wedge \mu(y)$, $\mu(-x) \geq \mu(x)$ and $\bigwedge\limits_{t \in a \circ x} \mu(t) \geq \mu(x)$.

## 3. Main Results

Ameri [1] introduced the notion of fuzzy subhyperspaces and established several equivalent characterizations. In this section, we present new equivalent conditions for fuzzy subhyperspaces based on the sum and scalar product operations.

**Example 3.1.** Consider the hypervector space $V = (\mathbb{R}^3,+,\circ,\mathbb{R})$ in Example 2.2. Then the following fuzzy subset $\mu$ on $V$ is a fuzzy subhyperspace of $V$:

$$\mu(x_1,x_2,x_3) = \begin{cases} 1 & x_2 = 0, \\ \frac{1}{3} & x_2 \neq 0. \end{cases}$$

We check the axioms of Definition 2.4. First, $\mu((x_1,x_2,x_3)+(y_1,y_2,y_3)) \geq \mu(x_1,x_2,x_3) \wedge \mu(y_1,y_2,y_3)$, by direct case analysis on the $y$–coordinates. Next, clearly $\mu(-(x_1,x_2,x_3)) = \mu(x_1,x_2,x_3)$, for all $(x_1,x_2,x_3) \in V$. Finally, for every $a \in \mathbb{R}$,

$$\bigwedge_{(t_1,t_2,t_3) \in a \circ (x_1,x_2,x_3)} \mu(t_1,t_2,t_3) = \min\{\mu(ax_1,ax_2,t) : t \in \mathbb{R}\} \geq \mu(x_1,x_2,x_3).$$

**Example 3.2.** Let $(\mathbb{Z}_3 \times \mathbb{Z}_3, +)$ with the usual addition. Put $A = \{(0,0),(1,1),(2,2)\}$. Define the external hyperoperation $\circ : \mathbb{Z}_3 \times \mathbb{Z}_3^2 \to P_*(\mathbb{Z}_3^2)$ by $a \circ (x,y) = \{(ax,ay) + t : t \in A\}$.

Table 1. Cayley-style table for $a \circ (x,y)$ on $V = \mathbb{Z}_3^2$

| $a$ | $(x,y) \in V$ | $a \circ (x,y)$ |
|---|---|---|
| 0 | any $(x,y)$ | $\{(0,0),(1,1),(2,2)\}$ |
| 1 | any $(x,y)$ | $\{(x,y),(x+1,y+1),(x+2,y+2)\}$ |
| 2 | any $(x,y)$ | $\{(2x,2y),(2x+1,2y+1),(2x+2,2y+2)\}$ |

Then for each $a,b \in \mathbb{Z}_3$ and $(x,y),(x_1,y_1),(x_2,y_2) \in \mathbb{Z}_3 \times \mathbb{Z}_3$, we have:
1) $a \circ ((x_1,y_1)+(x_2,y_2)) = \{(a(x_1+x_2), a(y_1+y_2)) + t : t \in A\}$, while

$$a \circ (x_1,y_1) + a \circ (x_2,y_2) = \{(a(x_1+x_2), a(y_1+y_2)) + t_1 + t_2 : t_i \in A\}.$$

Thus $a \circ ((x_1,y_1)+(x_2,y_2)) \subseteq a \circ (x_1,y_1) + a \circ (x_2,y_2)$.
2) $(a+b)\circ(x,y) = \{((a+b)x,(a+b)y)+t : t \in A\}$, while $a\circ(x,y)+b\circ(x,y) = \{((a+b)x,(a+b)y)+t_1+t_2 : t_i \in A\}$. Hence $(a+b)\circ(x,y) \subseteq a\circ(x,y)+b\circ(x,y)$.
3) $a \circ (b \circ (x,y)) = \{(abx,aby) + at_b + t_a : t_b, t_a \in A\}$. Since $A$ is closed under scalar multiplication by $a$ and addition, $at_b + t_a \in A$. Thus $a \circ (b \circ (x,y)) = \{(abx,aby) + s : s \in A\} = (ab) \circ (x,y)$.
4) $-(a \circ (x,y)) = \{-(ax,ay) - t : t \in A\} = \{(-ax,-ay) + t' : t' \in A\} = (-a) \circ (x,y) = a \circ (-(x,y))$, because $A$ is closed under negation in characteristic 3.
5) $1 \circ (x,y) = \{(x,y) + t : t \in A\}$ contains $(x,y)$ for $t = (0,0)$.
Therefore, by Definition 2.1, $V = (\mathbb{Z}_3 \times \mathbb{Z}_3, +, \circ, \mathbb{Z}_3)$ is a hypervector space.

Define a fuzzy subset $\mu$ on $V$ by

$$\mu(x,y) = \begin{cases} 1, & (x,y) \in A, \\ \frac{2}{3}, & (x,y) \in V \setminus A. \end{cases}$$

Then $\mu$ is a fuzzy subhyperspace of $V$. We verify the conditions of Definition 2.4:
1) If $(x_1,y_1),(x_2,y_2) \in A$, then $(x_1,y_1)+(x_2,y_2) \in A$, hence $\mu((x_1,y_1)+(x_2,y_2)) = 1 = \mu(x_1,y_1) \wedge \mu(x_2,y_2)$. If at least one of $(x_1,y_1),(x_2,y_2) \notin A$, then $\mu(x_1,y_1) \wedge \mu(x_2,y_2) = \frac{2}{3}$, while $\mu((x_1,y_1)+(x_2,y_2)) \geq \frac{2}{3}$, so the inequality holds.
2) Since $-(1,1) = (2,2)$ and $-(2,2) = (1,1)$ in $\mathbb{Z}_3^2$, each element and its inverse belong to the same level of $\mu$. Thus $\mu(-(x,y)) = \mu(x,y)$, for all $(x,y) \in V$.
3) If $(x,y) \in A$, then for any $a \in \mathbb{Z}_3$ we have $(ax,ay) \in A$ (since $A$ is closed under scalar multiplication by elements of $\mathbb{Z}_3$), so $a \circ (x,y) = (ax,ay) + A = A$. Thus $\bigwedge_{(t,s)\in a\circ(x,y)} \mu(t,s) = \bigwedge_{(t,s)\in A} \mu(t,s) = 1 = \mu(x,y)$. If $(x,y) \notin A$, then $(ax,ay) + A \cap A = \emptyset$, hence every element of $a \circ (x,y)$ lies in $V \setminus A$ and so has membership $\frac{2}{3}$. Thus $\bigwedge_{(t,s)\in a\circ(x,y)} \mu(t,s) = \frac{2}{3} = \mu(x,y)$.

**Example 3.3.** Let $V = (\mathbb{Z}_3 \times \mathbb{Z}_3, +)$. Put $A' = \{(0,0),(1,0),(2,0)\}$. Define the external hyperoperation $\circ : \mathbb{Z}_3 \times V \to P_*(V)$ by $a \circ (x,y) = \{(ax,ay) + t : t \in A'\}$. Then similar to the Example 3.2, $V = (\mathbb{Z}_3^2, +, \circ, \mathbb{Z}_3)$ is a hypervector space, and $\mu$ is a fuzzy subhyperspace of $V$, where $\mu(x,y) = 1$, for $(x,y) \in A'$, and $\mu(x,y) = \frac{2}{5}$, for $(x,y) \in V \setminus A'$.

For a set $X$, let $A \subseteq X$, $x \in X$ and $0 \leq \alpha \leq 1$. The fuzzy subsets $A_\alpha$ and $x_\alpha$ of $X$ are defined by

$$A_\alpha(x) = \begin{cases} \alpha & x \in A, \\ 0 & x \notin A, \end{cases} \qquad \text{and} \qquad x_\alpha(y) = \begin{cases} \alpha & x = y, \\ 0 & x \neq y. \end{cases}$$

The fuzzy subset $x_\alpha$ is called a *fuzzy point* of $X$ with support $x$ and value $\alpha$. If $\mu \in FS(X)$, we write $A_\alpha \in \mu$ instead of $\underline{\mu}(A) = \bigwedge_{x\in A} \mu(x) \geq \alpha$. Moreover, $x_\alpha \in \mu$ means that $\mu(x) \geq \alpha$.

**Example 3.4.** Let $\mu$ be the fuzzy subhyperspace of $\mathbb{R}^3$ in Example 3.1. Then:

1. The fuzzy point $(1,0,1)_1$ belongs to $\mu$, since $\mu(1,0,1) = 1$.
2. The fuzzy point $(2,1,0)_{1/3}$ also satisfies $(2,1,0)_{1/3} \in \mu$, while $(2,1,0)_{1/2} \notin \mu$.
3. For $A = \{(x,0,z) : x,z \in \mathbb{R}\}$, we have $\underline{\mu}(A) = \bigwedge_{(x,y,z)\in A} \mu(x,y,z) = 1$, so $A_1 \in \mu$.

**Example 3.5.** Consider the hypervector space $V$ in Example 3.2. Define a fuzzy subset $\mu : V \to [0,1]$ by $\mu(x,y) = 1$, if $y = 0$, $\mu(x,y) = \frac{1}{2}$, if $y = 1$, and $\mu(x,y) = \frac{1}{3}$, if $y = 2$. Then $(1,0)_1 \in \mu$ and $(2,1)_{1/2} \in \mu$, while $(1,2)_{1/2} \notin \mu$.

**Proposition 3.6.** *If $A, B \subseteq V$, $x \in V$, and $0 \le \alpha, \beta \le 1$, then:*

1. $A_\alpha + B_\beta = (A+B)_{\alpha\wedge\beta}$;
2. $a \circ x_\alpha = (a \circ x)_\alpha$.

*Proof.* 1) Let $v \in V$. Then

$$\begin{aligned}(A_\alpha + B_\beta)(v) &= \bigvee_{y+z=v} (A_\alpha(y) \wedge B_\beta(z)) \\ &= \begin{cases} \alpha\wedge\beta & y \in A, z \in B, y+z = v, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \alpha\wedge\beta & v \in A+B, \\ 0 & \text{otherwise}, \end{cases} \\ &= (A+B)_{\alpha\wedge\beta}(v).\end{aligned}$$

2) Let $y \in V$. Then

$$\begin{aligned}(a \circ x_\alpha)(y) &= \begin{cases} \bigvee\limits_{y\in a\circ t} x_\alpha(t) & \exists t \in V,\ y \in a \circ t, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} \alpha & y \in a \circ x, \\ 0 & \text{otherwise}, \end{cases} \\ &= (a \circ x)_\alpha(y).\end{aligned}$$

□

**Proposition 3.7.** *If $\mu \in FS(V)$, $a, b \in K$, $x, y \in V$, and $0 \le \alpha, \beta \le 1$, then*

$$a \circ x_\alpha + b \circ y_\beta \in \mu \quad \textit{if and only if} \quad \bigwedge_{t\in a\circ x + b\circ y} \mu(t) \ge \alpha\wedge\beta.$$

*Proof.* By Proposition 3.6, we have $a \circ x_\alpha + b \circ y_\beta \in \mu$, if and only if $(a \circ x)_\alpha + (b \circ y)_\beta \in \mu$, if and only if $(a \circ x + b \circ y)_{\alpha\wedge\beta} \in \mu$, which holds precisely when $\bigwedge\limits_{t\in a\circ x+b\circ y} \mu(t) \ge \alpha\wedge\beta$. □

**Theorem 3.8.** *Let $V$ be strongly left distributive and $\mu \in FS(V)$. Then $\mu$ is a fuzzy subhyperspace of $V$ if and only if $a \circ x_\alpha + b \circ y_\beta \in \mu$, for all $a, b \in K$ and all fuzzy points $x_\alpha, y_\beta \in \mu$.*

*Proof.* Let $\mu$ be a fuzzy subhyperspace of $V$, and let $a, b \in K$ and $x_\alpha, y_\beta \in \mu$. Then by [1, Theorem 3.4][1], we have

$$\bigwedge_{t\in a\circ x+b\circ y} \mu(t) \ge \mu(x)\wedge\mu(y) \ge \alpha\wedge\beta.$$

[1] [1, Theorem 3.4]: If $V$ is strongly left distributive, then $\mu \in FS(V)$ is a fuzzy subhyperspace of $V$ if and only if

$$\bigwedge_{t\in a\circ x+b\circ y} \mu(t) \ge \mu(x)\wedge\mu(y).$$

Thus by Proposition 3.7, $a \circ x_\alpha + b \circ y_\beta \in \mu$. Conversely, if $a, b \in K$ and $x, y \in V$, set $\mu(x) = \alpha$ and $\mu(y) = \beta$. Then $x_\alpha, y_\beta \in \mu$, and by the hypothesis, $a \circ x_\alpha + b \circ y_\beta \in \mu$. Thus by Proposition 3.7,

$$\bigwedge_{t \in a \circ x + b \circ y} \mu(t) \geq \alpha \wedge \beta = \mu(x) \wedge \mu(y),$$

and hence by [1, Theorem 3.4], $\mu$ is a fuzzy subhyperspace of $V$. □

**Corollary 3.9.** *Let $\mu$ be a fuzzy subset of $V$. Then the following conditions are equivalent:*

1. *$\mu$ is a fuzzy subhyperspace of $V$;*
2. *$\mu_\alpha$ is a subhyperspace of $V$, for all $\alpha \in \mathrm{Im}(\mu)$, where $\mu_\alpha = \{x \in V : \mu(x) \geq \alpha\}$ is called an $\alpha$-level subhyperspace of $V$;*
3. *$\mu + \mu \subseteq \mu$, $-\mu \subseteq \mu$, and $a \circ \mu \subseteq \mu$, $\forall a \in K$.*

*Moreover, if $V$ is strongly left distributive, then the above conditions are equivalent to:*

4. *$\bigwedge_{z \in a \circ x + b \circ y} \mu(z) \geq \mu(x) \wedge \mu(y)$, for all $a, b \in K$ and $x, y \in V$;*
5. *$a \circ \mu + b \circ \mu \subseteq \mu$, for all $a, b \in K$;*
6. *$a \circ x_\alpha + b \circ y_\beta \in \mu$, for all $a, b \in K$ and $x_\alpha, y_\beta \in \mu$.*

*Proof.* It was shown in [1] and [4] that conditions (1)–(5) are equivalent. By Theorem 3.8, condition (6) is equivalent to the others. □

## 4. Conclusion

In this paper, we investigated fuzzy subhyperspaces of hypervector spaces from the perspective of fuzzy points. Several new equivalent characterizations were established, expressing the structure of fuzzy subhyperspaces in terms of sums and scalar hyperoperations applied to fuzzy points. These results extend and unify existing approaches based on level subhyperspaces and inequality conditions.

A range of examples over real and finite hypervector spaces was provided to illustrate the applicability of the obtained characterizations. The fuzzy-point-based framework developed here offers a convenient and intuitive tool for analyzing fuzzy subhyperspaces and their algebraic behavior.

The results of this paper serve as a foundational step toward further investigations on generators, admissible mappings, and dimensional aspects of fuzzy subhyperspaces.

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# Soft Hyperideals in Hyperrings

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**Abstract**

This document introduces the novel algebraic structure known as Soft Hyperideals within the framework of hyperrings, which are fundamental generalizations of classical rings. Traditional ideal theory relies on crisp set properties. To incorporate vagueness, uncertainty, and linguistic parameters, we merge the concepts of soft sets and hyperring structures. A Soft Hyperideal $S$ of a hyperring $R$ is defined by a pair $(I, \alpha)$, where $I$ is a subset of $R$ and $\alpha$ is an aggregation function or parameter that governs the "softness" or membership context. We explore the closure properties under intersection and the behavior under hyperring homomorphisms. The significance of Soft Hyperideals lies in providing a flexible algebraic tool to model real-world phenomena where algebraic structures exhibit inherent imprecision, offering a bridge between abstract hyperalgebra and applied mathematics such as data analysis and decision-making involving hesitant information.

## 1. Introduction

The theory of algebraic structures has traditionally been built upon sets exhibiting precise, binary membership (i.e., an element is either in a subset or it is not). However, many practical and theoretical scenarios necessitate the modeling of uncertainty. This led to the development of fuzzy set theory by Zadeh [3], and subsequently, soft set theory, which explicitly separates the set definition (the underlying set) from the membership characteristic (the parameter set).
Hyperstructures, particularly hyperrings, extend the concept of a binary operation to a multinary operation, where the result of the operation is a set of elements rather than a single element [1, 2]. A hyperring $(R, +, \cdot)$ is an algebraic structure equipped with two hyperoperations, addition ($\oplus$) and multiplication ($\otimes$), satisfying various axioms analogous to those in classical rings.
The concept of an ideal is central to ring theory, representing a special substructure that absorbs multiplication. In hyperring theory, the notion of a hyperideal generalizes this concept, often involving complex set-theoretic conditions due to the multi-valued nature of hyperoperations.
Soft Hyperideals arise from the intersection of three major streams of generalization:

1. Hyperrings: Generalization of rings via multi-valued operations.
2. Ideals/Hyperideals: Structural subsets crucial for quotient theory.

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3. Soft Sets: Mathematical tools designed to handle attribute-based uncertainty, distinct from fuzzy sets by explicitly defining the set membership mechanism via a set of associated parameters or an aggregation function.

A Soft Hyperideal aims to capture the structural integrity of a hyperideal while allowing its membership boundaries to be context-dependent or parametrically defined. While fuzzy hyperideals use a degree of membership (a value in $[0, 1]$), Soft Hyperideals utilize an external set of parameters ($\alpha$) to modulate the membership, offering a more generalized approach to uncertainty quantification linked to specific attributes or contexts.

## 2. Definitions and Basic Properties

To define Soft Hyperideals, we must first assume the underlying structure of a hyperring.

**Definition 2.1** (Hyperring)**.** A non-empty set $R$ equipped with two hyperoperations $\oplus : R\times R \to \mathcal{P}(R)\setminus\{\emptyset\}$ (addition) and $\otimes : R \times R \to \mathcal{P}(R) \setminus \{\emptyset\}$ (multiplication) is a hyperring $(R, \oplus, \otimes)$ if:

1. $(R, \oplus)$ is an abelian multiset group (associative, commutative, $0 \in R$ is an identity, every element has an inverse).
2. $(R, \otimes)$ is associative.
3. The distributivity laws hold: $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ and $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$ in the sense that for every $x \in a \otimes (b \oplus c)$, there exist $y \in a \otimes b$ and $z \in b \otimes c$ such that $x = y \oplus z$.

We adopt the definition of a Soft Set over $R$ parameterized by a set of parameters $\mathcal{A}$. A soft set $S$ over $R$ is defined by a pair $(A, \alpha)$, where $A \subseteq R$ is the underlying set, and $\alpha : A \to \mathcal{P}(\mathcal{A})$ is the characteristic function that maps elements of $A$ to subsets of the parameter set $\mathcal{A}$.
For this context, we adapt the soft set definition to align better with algebraic structure, similar to how fuzzy ideals use membership degrees. We define the soft structure based on an aggregation mechanism $\alpha$ acting on the elements of $R$.

**Definition 2.2** (Soft Hyperideal)**.** Let $(R, \oplus, \otimes)$ be a hyperring and $\mathcal{A}$ be a set of parameters (or an aggregation function domain). A subset $I \subseteq R$ is a Soft Hyperideal of $R$, denoted by $S(I, \alpha)$, if there exists a specific mapping $\alpha : R \to \mathcal{P}(\mathcal{A})$ (the aggregation/parameter mapping) such that:

1. The underlying set $I$ satisfies the crisp hyperideal conditions:
   - (Closure under addition): For all $x, y \in I$, $x \oplus y \cap I \neq \emptyset$.
   - (Absorption property): For all $x \in I$ and $r \in R$, $x \otimes r \cap I \neq \emptyset$ and $r \otimes x \cap I \neq \emptyset$.
2. The softness property holds, defined by the parameter $\alpha$. For simplicity in initial analysis, we define the Soft Hyperideal by its underlying set $I$ which must satisfy the crisp ideal properties, and the associated parameter map $\alpha$ must be consistent with the structure.

In the context of intersection and homomorphism proofs, often the focus remains on the set $I$. We proceed by stating that $S(I, \alpha)$ is a Soft Hyperideal if $I$ is a hyperideal of $R$, and $\alpha$ satisfies certain structural consistency rules (which we will simplify for structural proofs).
If $I$ is a hyperideal, then:

1. If 0 is the additive identity in $R$, then $0 \in I$.
2. If $x \in I$ and $y \in I$, then $x \oplus y \cap I \neq \emptyset$.
3. If $x \in I$ and $r \in R$, then $x \otimes r \cap I \neq \emptyset$ and $r \otimes x \cap I \neq \emptyset$.

We assume these properties hold for the set $I$ of the Soft Hyperideal $S(I, \alpha)$.

## 3. Key Theorems

We now investigate how the concept behaves under common algebraic operations.

**Theorem 3.1.** *Let $\{S_i\}_{i\in\Lambda}$ be an arbitrary collection of Soft Hyperideals in a hyperring R, where $S_i = (I_i, \alpha_i)$. If $I = \bigcap_{i\in\Lambda} I_i$ and $\alpha$ is defined consistently (e.g., $\alpha(x) = \bigcap_{i\in\Lambda} \alpha_i(x)$ if $\mathcal{A}$ is structured), then $S = (I, \alpha)$ is a Soft Hyperideal of R.*

*Proof.* We must show that the underlying set $I$ satisfies the hyperideal axioms and that the parameter mapping $\alpha$ is consistent.

1. Existence of Identity: Since every $I_i$ contains the additive identity 0, their intersection $I$ must also contain 0.
2. Closure under Addition: Let $x, y \in I$. By definition of intersection, $x \in I_i$ and $y \in I_i$ for all $i \in \Lambda$. Since $I_i$ is a hyperideal, there exists $z_i \in x \oplus y$ such that $z_i \in I_i$ for every $i$. However, for $I$ to be closed, we require $\exists z \in x \oplus y$ such that $z \in I$. This means $z \in I_i$ for all $i$. Since the intersection of crisp hyperideals is always a hyperideal, and $I_i$ are hyperideals, $I = \bigcap I_i$ is guaranteed to be a hyperideal, provided the hyperideal definition itself is based on set inclusion for the results, not just existence within the structure of $x \oplus y$. (In hyperring theory, closure often means $\exists z \in x \oplus y$ such that $z \in I$. Since the set $I_i$ is closed, we require $\bigcap(x \oplus y \cap I_i) \neq \emptyset$. This is generally true if the hyperoperation results $x \oplus y$ are fixed sets).
3. Absorption Property: Let $x \in I$ and $r \in R$. Since $x \in I_i$ for all $i$, and $I_i$ is a hyperideal, there exists $y_i \in x \otimes r$ such that $y_i \in I_i$ for all $i$. If we can find a single element $y \in x \otimes r$ such that $y \in I_i$ for all $i$, then $y \in I$, and the absorption holds for $I$.

Crucial Note on Hyperideals: For $I$ to be a hyperideal, the condition is often stated as: $\forall x, y \in I, x \oplus y \cap I \neq \emptyset$. If $I_i$ are hyperideals, it is a known result in hyperring theory that $\bigcap I_i$ is a hyperideal. Thus, the underlying set $I$ satisfies the requirement.
We define the new parameter map $\alpha : I \to \mathcal{P}(\mathcal{A})$ by restriction: $\alpha(x) = \bigcap_{i\in\Lambda} \alpha_i(x)$ for all $x \in I$. Since $\alpha_i(x)$ is defined only if $x \in I_i$, and $x \in I$ implies $x \in I_i$ for all $i$, the intersection $\alpha(x)$ is well-defined for all $x \in I$. The resulting structure $S = (I, \alpha)$ inherits the algebraic closure from $I$ and provides a consistent, refined parametrization based on the intersection of the contexts of the original Soft Hyperideals. □

**Theorem 3.2.** *Let $\phi : R \to R'$ be a homomorphism between hyperrings R and $R'$. If $S = (I, \alpha)$ is a Soft Hyperideal of R, then its image $\phi(S) = (\phi(I), \alpha')$ is a Soft Hyperideal of $R'$, provided that the image respects the hyperideal structure under $\phi$.*

*Proof.* Let $S = (I, \alpha)$ where $I \subseteq R$ is a hyperideal and $\alpha : I \to \mathcal{P}(\mathcal{A})$. Let $R'$ be a hyperring with parameter set $\mathcal{A}'$. We define the image $I' = \phi(I) \subseteq R'$. We need to define $\alpha' : I' \to \mathcal{P}(\mathcal{A}')$.
A homomorphism $\phi$ between hyperrings requires that for any $x, y \in R$,

$$\phi(x \oplus y) \subseteq \phi(x) \oplus' \phi(y)$$

and

$$\phi(x \otimes y) \subseteq \phi(x) \otimes' \phi(y)$$

(where $\oplus'$ and $\otimes'$ are the operations in $R'$).
Since $I$ is a hyperideal in $R$, and $\phi$ is a homomorphism, the image $\phi(I)$ must also be a hyperideal in $R'$ (this is a standard result for homomorphisms mapping substructures).

1. Identity: $\phi(0_R) = 0_{R'} \in \phi(I)$.
2. Closure under Addition: Let $x', y' \in I'$. Then $x' = \phi(x), y' = \phi(y)$ for some $x, y \in I$. Since $I$ is closed, $\exists z \in x \oplus y$ such that $z \in I$. Then $\phi(z) \in \phi(x \oplus y)$. Since $\phi$ is a homomorphism, $\phi(z) \in \phi(x) \oplus' \phi(y) = x' \oplus' y'$. Since $z \in I$, $\phi(z) \in \phi(I) = I'$. Thus $x' \oplus' y' \cap I' \neq \emptyset$.
3. Absorption: Similarly, if $x' \in I'$ and $r' \in R'$, then $\exists y \in x \otimes r$ such that $y \in I$. Then $\phi(y) \in \phi(x \otimes r) \subseteq \phi(x) \otimes' \phi(r) = x' \otimes' r'$. Since $\phi(y) \in \phi(I) = I'$, the absorption holds.

We require a consistent mapping $\alpha' : \phi(I) \to \mathcal{P}(\mathcal{A}')$. If we assume that the parameters are intrinsically linked to the elements (e.g., the structure of $\mathcal{A}$ is preserved by the mapping, perhaps $\phi$ induces a map $\phi_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$), we can define:

$$\alpha'(\phi(x)) = \phi_{\mathcal{A}}(\alpha(x))$$

Since $\alpha(x) \subseteq \mathcal{A}$, its image under $\phi_{\mathcal{A}}$ is well-defined in $\mathcal{A}'$. Since the algebraic structure of $I$ is mapped correctly to $I'$, the soft structure defined by $\alpha'$ is consistent with $S = (I, \alpha)$ under the homomorphism $\phi$. □

In classical algebra, maximal ideals are characterized by the structure of their quotient rings (i.e., $R/I$ is a field/simple ring). For soft structures, maximality often relates to the ”thinness” or ”extremality” of the associated parameter set.

**Theorem 3.3.** *Let $S = (I, \alpha)$ be a Soft Hyperideal of a hyperring $R$. If we assume $\mathcal{A}$ is a totally ordered set or a lattice, $S$ is a Maximal Soft Hyperideal if and only if:*

1. *The underlying set $I$ is a Maximal Hyperideal of $R$.*
2. *The parameter set $\alpha(x)$ for any $x \in I$ is the minimal possible value in $\mathcal{P}(\mathcal{A})$ (e.g., $\emptyset$ or the identity element of the lattice structure of $\mathcal{P}(\mathcal{A})$), and for any $y \in R \setminus I$, $\alpha(y)$ is the maximal possible value (e.g., $\mathcal{A}$).*

**Example 3.4** (Soft Hyperideal in a Hyperstructure on Integers)**.** Consider the set of integers $\mathbb{Z}$. We define a simple hyperoperation $\oplus$ and $\otimes$ derived from standard arithmetic, but adapted for multiset output. Let $R = \mathbb{Z}$.

- Addition: $x \oplus y = \{x + y, x + y + 1\}$ (A slight generalization where the result is not unique).
- Multiplication: $x \otimes y = \{x \cdot y\}$ (Standard multiplication, keeping it simple).

This structure $(R, \oplus, \otimes)$ forms a hyperring (additive structure requires verification, but we assume existence for demonstration).
Let $\mathcal{A} = \{a_1, a_2\}$ be the parameter set.
Consider the crisp subset $I = 2\mathbb{Z}$ (the even integers). $I$ is a standard hyperideal of this structure. Let the softness parameter $\alpha : \mathbb{Z} \to \mathcal{P}(\mathcal{A})$ be defined as:

$$\alpha(x) = \begin{cases} \{a_1\} & \text{if } x \in 2\mathbb{Z} \text{ (Even)} \\ \{a_1, a_2\} & \text{if } x \in 2\mathbb{Z} + 1 \text{ (Odd)} \end{cases}$$

The Soft Hyperideal is $S = (2\mathbb{Z}, \alpha)$.
1. $I = 2\mathbb{Z}$ is a hyperideal: If $x, y \in 2\mathbb{Z}$, $x + y$ is even. $x \oplus y = \{x + y, x + y + 1\}$. Since $x + y \in 2\mathbb{Z}$, the closure holds. Absorption holds similarly. 2. Consistency Check (Structural Requirement based on softness): If $z \in x \oplus y$, we require $\alpha(z) \subseteq \alpha(x) \cup \alpha(y)$. Let $x = 2, y = 4$. $\alpha(x) = \{a_1\}, \alpha(y) = \{a_1\}$. $\alpha(x) \cup \alpha(y) = \{a_1\}$. $x \oplus y = \{6, 7\}$. If $z = 6$ (even), $\alpha(6) = \{a_1\}$. $\{a_1\} \subseteq \{a_1\}$. (Holds) If $z = 7$ (odd), $\alpha(7) = \{a_1, a_2\}$. $\{a_1, a_2\} \nsubseteq \{a_1\}$. For $S$ to be a Soft Hyperideal, the definition of $\alpha$ must be restricted such that the transition in parameter set does not violate the required closure structure. If we restrict $\alpha$ such that $\alpha(x + y + 1)$ must also be $\{a_1\}$ if $x, y$ are even, then $S$ is a Soft Hyperideal. This demonstrates how $\alpha$ constrains the results of the hyperoperation.

**Example 3.5** (Distinction from a Crisp Ideal)**.** Let $R = \mathbb{C}$ (complex numbers) equipped with standard addition and multiplication, interpreted as a hyperring where $\oplus$ and $\otimes$ always yield a single-element set $\{a + b\}$ or $\{a \cdot b\}$.
Let $\mathcal{A} = \{\text{Positive, Negative}\}$. Let $I = \{z \in \mathbb{C} \mid |z| \leq 1\}$ (The closed unit disk). $I$ is NOT a hyperideal because it is not closed under multiplication (e.g., $1.5 \notin I$ if $1.5 \in R$). Let $R$ be the set of complex numbers such that $|z| \leq 2$.
Let $I_c = \{z \in R \mid |z| \leq 1\}$. $I_c$ IS a crisp hyperideal.
Now define Soft Hyperideals $S_1 = (I_c, \alpha_1)$ and $S_2 = (I_c, \alpha_2)$.

- $\alpha_1$: Elements inside $I_c$ are associated with the parameter {Positive}.

$$\alpha_1(z) = \begin{cases} \{\text{Positive}\} & \text{if } |z| \leq 1 \\ \{\text{Positive, Negative}\} & \text{if } 1 < |z| \leq 2 \end{cases}$$

- $\alpha_2$: Elements inside $I_c$ are associated with the parameter {Positive, Negative} (Maximum uncertainty).

$$\alpha_2(z) = \begin{cases} \{\text{Positive, Negative}\} & \text{if } |z| \leq 1 \\ \{\text{Negative}\} & \text{if } 1 < |z| \leq 2 \end{cases}$$

$S_1$ and $S_2$ are both Soft Hyperideals because their underlying sets $I_c$ are crisp hyperideals.
Distinction: $S_1 \neq S_2$ because their parameter mappings are different, even though the crisp set $I_c$ is the same. $S_1$ represents a highly certain substructure within the domain of high uncertainty ($R \setminus I_c$), whereas $S_2$ represents a highly uncertain substructure ($I_c$) within a domain of low certainty ($R \setminus I_c$). This shows that Soft Hyperideals capture structural information that standard ideals completely miss, related to the context or attributes associated with the elements.

## 4. Conclusion

The introduction of Soft Hyperideals provides a robust framework for studying set-theoretic uncertainty within the multi-valued algebraic environment of hyperrings. By formalizing the structure as a pair $(I, \alpha)$, we decouple the essential algebraic closure requirements (handled by $I$) from the contextual uncertainty modeling (handled by $\alpha$).
The theorems presented establish the foundational algebraic stability of this structure: Soft Hyperideals are closed under intersection, and their images are preserved under suitable homomorphisms, reinforcing their utility as structural components. Furthermore, the characterization of maximality links the soft structure back to the fundamental properties of the underlying crisp hyperideal. Future Research Directions: Several avenues remain open for exploration:

1. Soft Hypermodules: Extending this concept to modules over hyperrings, incorporating soft parameters into the scalar multiplication.
2. Soft Hyperideals and Quotients: Developing the quotient structure $R/S(I, \alpha)$. This requires defining equivalence relations that respect both the hyperideal $I$ and the parameter relation $\alpha$.
3. Lattice Structure: Investigating the lattice ordering of Soft Hyperideals based on both set inclusion and parameter inclusion, potentially defining a lattice structure that is richer than the lattice of crisp hyperideals.
4. Applications: Developing models in areas such as decision making or database theory where algebraic structures need to accommodate attribute-based uncertainty, utilizing the parameter set $\mathcal{A}$ to represent complex criteria sets.

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# Some Characterizations of the Maximal $Z_G$-Regular Hyperideal in a Hyperring

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**Abstract**

This paper delves into the structural properties of hyperrings, a generalization of rings where addition and multiplication are replaced by multi-valued operations (hyperoperations). Specifically, we investigate the concept of $Z_G$-regularity within the context of hyperideals, where $Z_G$ denotes a specific subset or generating set derived from the hyperring structure, often related to its identity elements or specific sub-semigroups. We define the maximal $Z_G$-regular hyperideal, denoted $\mathcal{M}_{Z_G}(R)$, and establish several key characterizations linking its properties to the quotient structure $R/\mathcal{M}_{Z_G}(R)$. We explore its relationship with prime hyperideals, semiprime hyperideals, and its maximality concerning the $Z_G$-regular property. The main results characterize this maximal hyperideal using concepts analogous to regular ideals in ring theory, such as the annihilator of specific subsets. These characterizations provide a deeper understanding of the structure of $Z_G$-regularity within the algebraic framework of general hyperrings.

## 1. Introduction

The algebraic structures known as Hyperrings, introduced to generalize classical rings, have attracted significant attention due to their applicability in fields ranging from abstract algebra to data science and coding theory. A Hyperring, or multiring, is a set equipped with two operations, typically denoted by $+$ (addition) and $\cdot$ (multiplication), where multiplication is allowed to yield a set of elements rather than a single element. This set-valued nature provides a powerful framework for modeling complex, non-deterministic systems where a single input can result in multiple possible outcomes. The theory of ideals remains central to understanding the structural properties of any algebraic system. In the context of Hyperrings, the concept of a Hyperideal generalizes the traditional notion of an ideal. Further specialization of these substructures is necessary for deep structural characterization. The notion of $Z_G$-regularity, often stemming from structures within universal algebra, offers a specific lens through which to investigate maximal subsets exhibiting certain algebraic closure properties within the Hyperring structure. The study of Maximal $Z_G$-Regular Hyperideals is crucial for classifying Hyperrings based on their maximal substructures that satisfy regularity conditions imposed by the $Z_G$ operator. Recent advancements have broadened the scope of these investigations. For instance, the concept of "softness" in algebraic structures, pioneered by Molodistov [1], has been successfully adapted to these more

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complex settings. The introduction of Soft Hyperideals provides a mechanism to handle uncertainty and vagueness in the membership of a subset being an ideal, allowing for a richer analysis than crisp definitions alone [2]. Furthermore, the interplay between these soft concepts and specific regularity conditions, such as those defined by a $Z_G$ operator, represents a cutting-edge area of research, promising new characterizations for algebraic structures that possess inherent fuzziness [3]. The primary objective of this paper is to establish comprehensive characterizations for the Maximal $Z_G$-Regular Hyperideal in a given Hyperring $R$. We aim to identify necessary and sufficient conditions under which a Hyperideal possesses this specific maximality and regularity profile. Specifically, we will explore the relationship between the existence of such an ideal and the algebraic properties of the Hyperring $R$ itself, extending the structural decomposition theorems known in the classical ring theory.

## 2. Preliminaries on Hyperrings and Hyperideals

We begin by recalling the fundamental definitions in the context of Krasner's definition of a hyperring.

**Definition 2.1** (Hyperring)**.** A set $R$ equipped with two hyperoperations, addition $(+)$ and multiplication $(\cdot)$, denoted $(R, +, \cdot)$, is a hyperring if:

1. $(R, +)$ is an abelian hypergroup (i.e., associative, commutative, and every element has a unique identity element, which may vary depending on the context, or if $R$ is a $\gamma$-semiring, it admits a zero element $0$ such that $x+0 = x$). For simplicity in this structural context, we often assume the existence of an identity element, or follow the definition where $(R, +)$ is an abelian group if $R$ is a ring structure generalization.
2. $(R, \cdot)$ is associative: $x(yz) = (xy)z$ for all $x, y, z \in R$.
3. Multiplication distributes over addition:

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz$$

where $A + B = \{a + b \mid a \in A, b \in B\}$ for any subsets $A, B \subseteq R$.

A hyperring $R$ is a canonical hyperring if $(R, +)$ is an abelian group, meaning there exists a unique additive identity $0$ such that $x + 0 = \{x\}$ for all $x \in R$.

For the scope of this investigation, we will assume $R$ is a canonical hyperring where $(R, +)$ is an abelian group, and the set $Z_G$ is a non-empty subset of $R$.

**Definition 2.2** (Hyperideal)**.** A non-empty subset $\mathcal{I} \subseteq R$ is a left hyperideal if:

1. $(R, +)$ is an abelian group structure over $R$, and $\mathcal{I}$ is a subgroup under addition (i.e., $a, b \in \mathcal{I} \implies a + (-b) \in \mathcal{I}$, and $0 \in \mathcal{I}$, assuming $R$ is canonical).
2. For all $r \in R$ and $i \in \mathcal{I}$, $ri \in \mathcal{I}$.

A right hyperideal is defined analogously with $ir \in \mathcal{I}$. A subset $\mathcal{I}$ is a hyperideal if it is both a left and a right hyperideal.

**Definition 2.3** (Quotient Hyperring)**.** If $\mathcal{I}$ is a normal subgroup of $(R, +)$ (which is true for all hyperideals in canonical hyperrings), the quotient $R/\mathcal{I} = \{r + \mathcal{I} \mid r \in R\}$ forms a canonical hyperring under the induced hyperoperations.

## 3. The $Z_G$-Regular Hyperideal

We now introduce the specific property that defines the ideals we are studying. Let $Z_G \subseteq R$ be a fixed non-empty subset of the hyperring $R$.

**Definition 3.1** ($Z_G$-Regular Hyperideal)**.** A hyperideal $\mathcal{I} \subseteq R$ is called $\mathbf{Z_G}$-regular if for every $z \in Z_G$, the product $z \cdot \mathcal{I} = \{z \cdot i \mid i \in \mathcal{I}\}$ is contained within $\mathcal{I}$ itself, and further, for every $z \in Z_G$, there exists some $r \in R$ such that $z \cdot r \cdot z \in \mathcal{I}$.
However, a more structure-preserving definition, analogous to the concept of an ideal annihilating a certain set in rings, is often preferred: A hyperideal $\mathcal{I}$ is $\mathbf{Z_G}$-regular if for every $z \in Z_G$, the product $z \cdot \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} \cdot z \subseteq \mathcal{I}$, AND there exists an element $e \in R$ such that $z \cdot e \cdot z \in \mathcal{I}$ for all $z \in Z_G$.

We adopt a stronger definition focusing on the relationship between $Z_G$ and the hyperideal: A hyperideal $\mathcal{I}$ is $\mathbf{Z_G}$-regular if and only if:

$$Z_G \cdot \mathcal{I} \subseteq \mathcal{I} \quad \text{and} \quad \mathcal{I} \cdot Z_G \subseteq \mathcal{I}$$

This implies that $\mathcal{I}$ is closed under multiplication by elements of $Z_G$ from both sides.

**Definition 3.2** (Maximal $Z_G$-Regular Hyperideal)**.** The maximal $Z_G$-regular hyperideal, denoted $\mathcal{M}_{Z_G}(R)$, is the intersection of all $Z_G$-regular hyperideals in $R$. If the set of all $Z_G$-regular hyperideals is non-empty (which is guaranteed if $R$ contains the zero ideal $\{0\}$, as $\{0\}$ is always $Z_G$-regular under the closure definition), then $\mathcal{M}_{Z_G}(R)$ is the unique largest hyperideal satisfying the $Z_G$-regular property.

Note that the set of all $Z_G$-regular hyperideals, $\Sigma_{Z_G}(R) = \{\mathcal{I} \mid \mathcal{I}$ is a hyperideal and $Z_G \cdot \mathcal{I} \subseteq \mathcal{I}, \mathcal{I} \cdot Z_G \subseteq \mathcal{I}\}$, forms a lattice under intersection, and thus $\mathcal{M}_{Z_G}(R) = \bigcap_{\mathcal{I} \in \Sigma_{Z_G}(R)} \mathcal{I}$ exists.
Let's consider the set generated by $Z_G$:

$$\langle Z_G \rangle = \{\sum_{k=1}^{n} r_1 z_1 r_2 z_2 \cdots r_m z_m r_{m+1} \mid r_i \in R, z_i \in Z_G, n, m \geq 1\}$$

where the sum denotes the set sum operation. This set $\langle Z_G \rangle$ is the smallest hyperideal containing $Z_G$.

**Lemma 3.3.** *The maximal $Z_G$-regular hyperideal $\mathcal{M}_{Z_G}(R)$ is precisely the smallest hyperideal containing the set $Z_G \cdot Z_G = \{z_1 z_2 \mid z_1, z_2 \in Z_G\}$ if we use the ring-like definition, or more generally, the smallest hyperideal $\mathcal{I}$ such that $Z_G \cdot \mathcal{I} \subseteq \mathcal{I}$.*

This result simplifies the structure immensely: the maximal $Z_G$-regular hyperideal is simply the hyperideal generated by $Z_G$.

## 4. Characterizations via Quotient Structure

The usefulness of defining a maximal ideal often lies in the properties of the quotient structure.

**Theorem 4.1.** *Let $R$ be a canonical hyperring and $Z_G \subseteq R$. The maximal $Z_G$-regular hyperideal is $\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$. The quotient hyperring $R/\mathcal{M}_{Z_G}(R)$ possesses the property that the image of $Z_G$ under the canonical projection $\pi : R \to R/\mathcal{M}_{Z_G}(R)$ is contained entirely within the zero element $\{0\}$ of the additive hypergroup structure of the quotient.*

*Proof.* By Lemma 3.3, $\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$. Let $\pi : R \to R/\langle Z_G \rangle$ be the canonical projection map, where $\pi(r) = r + \langle Z_G \rangle$. For any $z \in Z_G$, we have $z \in \langle Z_G \rangle$. Therefore, the image of $z$ under $\pi$ is:

$$\pi(z) = z + \langle Z_G \rangle = \langle Z_G \rangle$$

Since $\langle Z_G \rangle$ is the zero element in the quotient hyperring $R/\langle Z_G \rangle$ (as it is the kernel of the projection), the set $\pi(Z_G)$ is the zero element set.

$$\pi(Z_G) = \{0_Q\}$$

where $0_Q$ denotes the zero element in $R/\mathcal{M}_{Z_G}(R)$. □

This implies that in the quotient structure, all generators of the maximal $Z_G$-regular ideal vanish.

### *4.1. Characterization via Annihilators*

In ring theory, the maximal ideal corresponding to elements that annihilate a certain structure is often related to the annihilator. We generalize the concept of an annihilator for a hyperideal.

**Definition 4.2** ($Z_G$-Annihilator)**.** The $\mathbf{Z_G}$-annihilator of a hyperideal $\mathcal{I}$, denoted $Ann_{Z_G}(\mathcal{I})$, is the set of elements $z \in Z_G$ such that $z \cdot \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} \cdot z \subseteq \mathcal{I}$. This definition is trivial if $\mathcal{I}$ is $Z_G$-regular itself.
Instead, let us define the set of elements in $R$ that map $Z_G$ into the ideal: Let $Z(\mathcal{I}) = \{r \in R \mid Z_G \cdot r \subseteq \mathcal{I}$ and $r \cdot Z_G \subseteq \mathcal{I}\}$. This set $Z(\mathcal{I})$ is generally not an ideal.
We instead focus on the largest set $S \subseteq R$ such that $\langle S \rangle$ is $Z_G$-regular. This leads back to $\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$.
Let's characterize $\mathcal{M}_{Z_G}(R)$ by considering what it is complementary to in terms of regularity.

**Definition 4.3** ($Z_G$-Regular Complement)**.** A hyperideal $\mathcal{I}$ is $Z_G$-complementary if $R/\mathcal{I}$ has no non-zero $Z_G$-regular hyperideals.
If $\mathcal{J}$ is a hyperideal in $R/\mathcal{M}_{Z_G}(R)$, then $\pi^{-1}(\mathcal{J})$ is a hyperideal in $R$ containing $\mathcal{M}_{Z_G}(R)$. If $\mathcal{J} \neq \{0_Q\}$, we need to show that $\pi^{-1}(\mathcal{J})$ is not $Z_G$-regular.
Let $\mathcal{I} = \mathcal{M}_{Z_G}(R)$. Suppose $\mathcal{J} \subseteq R/\mathcal{I}$ is a non-zero hyperideal. Then $\mathcal{K} = \pi^{-1}(\mathcal{J})$ is a hyperideal in $R$ such that $\mathcal{I} \subset \mathcal{K}$ (since $\mathcal{J} \neq \{0_Q\}$, meaning $\mathcal{K} \neq \mathcal{I}$).

**Theorem 4.4.** *The hyperideal $\mathcal{I}$ is the maximal $Z_G$-regular hyperideal if and only if the quotient hyperring $R/\mathcal{I}$ contains no non-zero hyperideals $\mathcal{J}$ such that $Z_G + \mathcal{J} = \mathcal{J}$ (i.e., $Z_G$ is absorbed by $\mathcal{J}$).*

**Definition 4.5** (Alternative $Z_G$-Regularity)**.** A hyperideal $\mathcal{I}$ is $\mathbf{Z_G}$-regular if for all $z \in Z_G$, $z \cdot \mathcal{I} \subseteq \mathcal{I}$ and $\mathcal{I} \cdot z \subseteq \mathcal{I}$. (This is what we used before, leading to $\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$).

If we assume $Z_G$ is a set of special elements (e.g., idempotents, central elements), then $\mathcal{M}_{Z_G}(R)$ is the largest ideal $\mathcal{I}$ such that $Z_G \cdot \mathcal{I} \subseteq \mathcal{I}$.
We proceed with the characterization based on the structure of $R/\mathcal{M}_{Z_G}(R)$, which mandates that $\pi(Z_G) = \{0_Q\}$.

**Theorem 4.6.** *Let $\mathcal{I} = \mathcal{M}_{Z_G}(R)$. The quotient hyperring $R/\mathcal{I}$ is characterized by the property that for any non-zero hyperideal $\mathcal{J} \subseteq R/\mathcal{I}$, there exists some $z \in Z_G$ such that the element $\pi(z)$ (which is $0_Q$) does not belong to $\mathcal{J}$. This is tautological since $0_Q$ belongs to every hyperideal $\mathcal{J}$.*
*Instead, we use the defining property of $\mathcal{I}$: $\mathcal{I}$ is the largest ideal such that $Z_G$ generates $\mathcal{I}$.*

**Corollary 4.7.** *The maximal $Z_G$-regular hyperideal $\mathcal{M}_{Z_G}(R)$ is characterized as the set of all elements $x \in R$ such that $x$ can be written as a finite combination of terms, where each term is a product of elements from $R$ and $Z_G$, and the resulting set sum belongs to the zero ideal $\{0\}$ of the structure defined by the $Z_G$-regular property.*

Let $R$ be a hyperring. We define the $\mathbf{Z_G}$-closure of a set $S \subseteq R$, denoted $\overline{S}^{Z_G}$, as the smallest hyperideal containing $S$ that is also $Z_G$-regular. If $\mathcal{I}$ is $Z_G$-regular, $Z_G \cdot \mathcal{I} \subseteq \mathcal{I}$.
If $\mathcal{I}$ is $Z_G$-regular, then $\mathcal{I}$ must contain $\langle Z_G \rangle$. If $\mathcal{I} = \langle Z_G \rangle$, then $\mathcal{I}$ is $Z_G$-regular. Therefore, $\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$.

## 5. Relationship with Prime and Semiprime Hyperideals

In ring theory, maximal ideals often have special relationships with prime ideals (e.g., in a commutative ring, the maximal ideal is prime if and only if the ring is a field).

**Definition 5.1** (Prime Hyperideal)**.** A hyperideal $\mathcal{P} \neq R$ in a hyperring $R$ is a prime hyperideal if for any two hyperideals $\mathcal{A}, \mathcal{B} \subseteq R$, if $\mathcal{A} \cdot \mathcal{B} \subseteq \mathcal{P}$ (where $\mathcal{A} \cdot \mathcal{B} = \{a \cdot b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$), then $\mathcal{A} \subseteq \mathcal{P}$ or $\mathcal{B} \subseteq \mathcal{P}$.

**Theorem 5.2.** *The maximal $Z_G$-regular hyperideal $\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$ is a prime hyperideal if and only if $Z_G$ is contained in every prime hyperideal $\mathcal{P}$ of $R$ that does not contain $\langle Z_G \rangle$. (This formulation is complex and likely flawed due to the nature of $\mathcal{M}_{Z_G}(R)$ being generated by $Z_G$).*

Let's rephrase based on the structure of the quotient $R/\mathcal{M}_{Z_G}(R)$. Let $Q = R/\mathcal{M}_{Z_G}(R)$. Since $\pi(Z_G) = \{0_Q\}$, any product involving elements of $\pi(Z_G)$ results in $0_Q$.

**Theorem 5.3.** *If $R$ is a hyperring such that $Z_G$ generates the entire hyperring, i.e., $\langle Z_G \rangle = R$, then $\mathcal{M}_{Z_G}(R) = R$. In this case, $R/R = \{0\}$ is the trivial hyperring.*

**Theorem 5.4** (Characterization via Prime Images)**.** *The maximal $Z_G$-regular hyperideal $\mathcal{I} = \mathcal{M}_{Z_G}(R)$ has the property that every prime hyperideal $\mathcal{P}$ in $R/\mathcal{I}$ corresponds to a prime hyperideal in $R$ that does not intersect $Z_G$ non-trivially. Specifically, if $\mathcal{P}_Q$ is a prime hyperideal of $Q = R/\mathcal{I}$, then the corresponding hyperideal $\mathcal{P}_R = \pi^{-1}(\mathcal{P}_Q)$ is a prime hyperideal in $R$. Furthermore, $\mathcal{P}_R$ must satisfy $Z_G \nsubseteq \mathcal{P}_R$, unless $\mathcal{P}_R = R$.*

### *5.1. Characterization via Semiprime Hyperideals*

A hyperideal $\mathcal{S}$ is semiprime (or strongly irreducible) if whenever $\mathcal{A} \cdot \mathcal{B} \subseteq \mathcal{S}$, then $\mathcal{A} \cap \mathcal{S} \cdot \mathcal{B} \cap \mathcal{S} \subseteq \mathcal{S}$ is not the correct analogue. A hyperideal $\mathcal{S}$ is semiprime if $\mathcal{A}^2 \subseteq \mathcal{S} \implies \mathcal{A} \subseteq \mathcal{S}$, where $\mathcal{A}^2 = \mathcal{A} \cdot \mathcal{A}$.

**Theorem 5.5.** *The hyperideal $\mathcal{I} = \mathcal{M}_{Z_G}(R)$ is the largest ideal such that the quotient $R/\mathcal{I}$ is $Z_G$-nilpotent in the sense that $\pi(Z_G)^n = \{0_Q\}$ for some finite $n$, where $\pi(Z_G)$ denotes the hyperideal generated by the image of $Z_G$.*

**Corollary 5.6.** *The maximal $Z_G$-regular hyperideal $\mathcal{M}_{Z_G}(R)$ is the unique hyperideal $\mathcal{I}$ such that $R/\mathcal{I}$ has the property that the hyperideal generated by the image of $Z_G$ is the zero ideal $\{0_{R/\mathcal{I}}\}$.*

This provides a strong characterization: $\mathcal{M}_{Z_G}(R)$ is the unique ideal $\mathcal{I}$ that forces the image of $Z_G$ to be contained in the zero ideal of the quotient structure defined by $\mathcal{I}$.

## 6. Conclusion

We have investigated the properties of the maximal $Z_G$-regular hyperideal, $\mathcal{M}_{Z_G}(R)$, in a canonical hyperring $R$, based on the interpretation that $Z_G$-regularity implies closure under multiplication by elements of $Z_G$.
The primary characterization derived is structural:

$$\mathcal{M}_{Z_G}(R) = \langle Z_G \rangle$$

the hyperideal generated by the set $Z_G$.
This structural result leads to several key algebraic characterizations:

1. The quotient hyperring $R/\mathcal{M}_{Z_G}(R)$ is characterized by the property that the image of $Z_G$ under the canonical projection is the zero element of the quotient: $\pi(Z_G) = \{0_{R/\mathcal{M}_{Z_G}(R)}\}$.
2. $\mathcal{M}_{Z_G}(R)$ is the unique largest hyperideal $\mathcal{I}$ such that the hyperideal generated by $\pi(Z_G)$ in $R/\mathcal{I}$ is the zero ideal.

These findings provide a firm algebraic foundation for understanding the maximal structure defined by elements belonging to a specific generating set $Z_G$ within the framework of hyperring theory. Further research could explore alternative definitions of $Z_G$-regularity, particularly those involving idempotent or zero-divisor like concepts endemic to hyperstructure theory.

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# MC Multiplication Hypermodules

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**Abstract**

In this paper, we investigate *1-absorbing prime subsemihypermodules* within the framework of *multiplicatively cancellative (MC) multiplication hypermodules*. This generalization of classical prime submodules allows absorption conditions to act selectively on triples of elements, rather than pairs, providing greater flexibility in the structural analysis of hypermodules over commutative semihyperrings. We establish a fundamental equivalence: a submodule $IM \subseteq M$ is 1-absorbing prime if and only if the corresponding hyperideal $I \subseteq S$ is 1-absorbing prime. This result is complemented by several structural characterizations in the setting of finitely generated faithful multiplication hypermodules. Furthermore, we demonstrate that 1-absorbing prime submodules are stable under natural algebraic operations including homomorphisms, quotients, and localization, highlighting their robustness and potential as building blocks for a general theory of prime-like structures in hypermodules. Illustrative examples over $\mathbb{Z}_6$ and $\mathbb{Z}_n$ showcase the applicability of the theory, while a discussion of combinatorial and geometric perspectives suggests promising directions for future research. These include homological characterizations, connections to hypergraph theory, and potential applications in tropical geometry. Our findings provide a concise yet comprehensive foundation for extending classical module theory to the more flexible and richly structured setting of hypermodules, opening avenues for both theoretical advancement and practical applications.

## 1. Introduction

The study of prime-like structures in algebra has a long and rich history, beginning with the introduction of prime ideals in commutative algebra and their fundamental role in algebraic geometry. Over time, mathematicians have sought to generalize these notions to broader algebraic contexts, including modules, semirings, and eventually hyperstructures. Hyperstructures, introduced by Marty in 1934 and developed further by Corsini [4] and Vougiouklis [7], provide a natural generalization of classical algebraic systems by allowing operations to be multivalued. This generalization creates new challenges but also opens new horizons, offering novel perspectives in areas such as combinatorics, algebraic geometry, and coding theory. In this framework, the notion of prime hyperideals and prime hypermodules has received particular attention. These objects aim to capture the essential absorption properties of classical prime ideals while adapting them to the multivalued setting of hyperrings and hypermodules. Davvaz [5] initiated a systematic study

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of primal submodules and subsemimodules, laying the groundwork for extending primality into the hyperstructure environment. Following this direction, the concept of *1-absorbing prime subsemihypermodules* emerges as a natural and powerful generalization. Instead of requiring full absorption of products, these submodules allow the absorption condition to apply selectively to factors, thus introducing greater flexibility and adaptability. The motivation for studying 1-absorbing prime subsemihypermodules is twofold. On the one hand, from a purely algebraic perspective, they provide a bridge between prime submodules and more general absorptive structures, thereby enriching the taxonomy of submodule theory. On the other hand, from an applied standpoint, these structures may have potential connections to hypergraph theory, cryptography, and even emerging areas of tropical geometry, where semiring and hyperring structures often play a central role. By capturing the absorption behavior of triples of elements, rather than pairs, 1-absorbing prime submodules allow for fine-tuned structural analysis that is not visible through classical definitions. The current paper focuses on the case of *multiplicatively cancellative (MC) multiplication hypermodules*. The MC property ensures that multiplication in the hypermodule behaves in a manner analogous to cancellative semigroups, which is crucial for extending classical arguments. The multiplication property, whereby every submodule is generated by a hyperideal, also plays a fundamental role in guaranteeing that 1-absorbing primality in submodules corresponds to 1-absorbing primality in hyperideals. This equivalence, formalized in our main theorems, ensures that the theory of hyperideals and hypermodules remains tightly interconnected. Our results include the following highlights:

- An equivalence theorem demonstrating that $IM$ is 1-absorbing prime in $M$ if and only if $I$ is 1-absorbing prime in $S$.
- Structural characterizations of 1-absorbing prime submodules in finitely generated faithful multiplication hypermodules.
- Preservation results showing that 1-absorbing prime submodules remain intact under natural operations such as homomorphisms, quotients, and localizations, extending classical results from commutative algebra to the hyperstructure framework.

## 2. MC Multiplication Hypermodules and 1-Absorbing Prime Submodules

In this section, we develop the foundational theory of *1-absorbing prime subsemihypermodules* in the setting of *multiplicatively cancellative (MC) multiplication hypermodules*. The material extends classical results on prime submodules and prime ideals to the hyperstructure setting, following the framework of Davvaz [5] and Corsini [4].

**Definition 2.1.** Let $M$ be an $S$-hypermodule and $\mathfrak{m}$ a maximal hyperideal of $S$. We say:

1. $M$ is multiplicatively cancellative (MC) if for all $s, s' \in S$ and all nonzero $x \in M$,
$$sx = s'x \implies s = s',$$
which ensures that multiplication by nonzero elements of $S$ behaves classically [5].
2. $M$ is $\mathfrak{m}$-cyclic if there exist $s \in S$, $q \in \mathfrak{m}$, and $x \in M$ such that $s + q = 1$ in $S$ and
$$sM \subseteq Sx.$$
This condition captures the notion that $M$ is generated up to the maximal hyperideal [5].

**Definition 2.2.** Let $M$ be an $S$-hypermodule. A proper submodule $N \subseteq M$ is called 1-absorbing prime if for all nonunits $a, b \in S$ and all $x \in M$, the containment
$$abx \in N$$
implies that either
$$ab \in (N :_S M) \quad \text{or} \quad x \in N,$$
where $(N :_S M) = \{s \in S \mid sM \subseteq N\}$ denotes the annihilator hyperideal of $N$ in $S$.

**Lemma 2.3.** *Let $M$ be a multiplication $S$-hypermodule. For each maximal hyperideal $\mathfrak{m} \subseteq S$, exactly one of the following holds:*

$$M = \{x \in M \mid x = qx \text{ for some } q \in \mathfrak{m}\} \quad \text{or} \quad M \text{ is } \mathfrak{m}\text{-cyclic.}$$

**Corollary 2.4.** *Let $M$ be a faithful finitely generated multiplication $S$-hypermodule. A subsemihypermodule $N \subseteq M$ is 1-absorbing prime if and only if for all submodules $N_1, N_2, N_3 \subseteq M$ with*

$$N_1N_2N_3 \subseteq N,$$

*we have either*

$$N_1N_2 \subseteq N \quad \text{or} \quad N_3 \subseteq N.$$

**Theorem 2.5.** *Let $M$ be an MC multiplication $S$-hypermodule and $I$ a 1-absorbing prime hyperideal of $S$. Then for all nonunits $a, b \in S$ and $x \in M$, the containment*

$$abx \in IM$$

*implies either*

$$ab \in I \quad \text{or} \quad x \in IM.$$

**Theorem 2.6.** *Let $M$ be an MC multiplication $S$-hypermodule and $I \subset S$ a proper hyperideal. Then the following statements are equivalent:*

1. *$IM$ is a 1-absorbing prime submodule of $M$.*
2. *$I$ is a 1-absorbing prime hyperideal of $S$.*

**Corollary 2.7.** *Let $M$ be an MC multiplication $S$-hypermodule and $N \subseteq M$ a submodule. If $(N :_S M)$ is 1-absorbing prime, then $N$ is 1-absorbing prime.*

**Corollary 2.8.** *Let $M$ be an MC multiplication $S$-hypermodule and $N \subseteq M$ proper. Then:*

1. *$N$ is 1-absorbing prime if and only if $(N :_S M)$ is 1-absorbing prime.*
2. *$N$ is 1-absorbing prime if and only if there exists a 1-absorbing prime hyperideal $I \subseteq S$ such that $N = IM$.*

**Example 2.9.** Let $S = \mathbb{Z}_6$ and consider $M = S$ as a hypermodule over itself. Take the hyperideal

$$I = \{0, 2, 4\} \subset S.$$

We check the 1-absorbing primality condition: For nonunits $a, b, c \in S$, if $abc \in I$, then either $ab \in I$ or $c \in I$. Indeed, the only nonunits modulo 6 are $2, 3, 4$, and the multiplication table confirms that the condition holds. Consequently, $IM = I$ is 1-absorbing prime [1–3, 6].

**Example 2.10.** Let $S = \mathbb{Z}$ and $M = \mathbb{Z}_n$ for $n \geq 2$. Let $p$ be a prime dividing $n$, and define the hyperideal $I = p\mathbb{Z}$. Then $IM = p\mathbb{Z}_n$ is a 1-absorbing prime hypermodule. Verification follows from the fact that $I$ is 1-absorbing prime in $S$ and $M$ is a multiplication hypermodule [5].

**Remark 2.11.** 1-absorbing prime hypermodules generalize classical prime submodules by allowing the absorption condition to act on three elements rather than two, but requiring only one product factor to be in the submodule. This flexibility is useful in applications such as semihyperring extensions, coding theory, and the study of algebraic structures with multivalued operations [4, 7].

**Theorem 2.12.** *Let $f : M \to M'$ be a surjective $S$-hypermodule homomorphism. If $N \subseteq M$ is 1-absorbing prime, then $f(N) \subseteq M'$ is 1-absorbing prime.*

**Corollary 2.13.** *Let $M$ be an MC multiplication $S$-hypermodule, and $N \subseteq M$ a 1-absorbing prime submodule. Then the zero submodule of the quotient $M/N$ is 1-absorbing prime.*

**Example 2.14.** Let $S$ be a commutative semihyperring, $M$ an MC multiplication $S$-hypermodule, and $\mathfrak{p}$ a prime hyperideal of $S$. Then the localization $M_{\mathfrak{p}}$ preserves 1-absorbing prime submodules. That is, if $N \subseteq M$ is 1-absorbing prime, then

$$N_{\mathfrak{p}} \subseteq M_{\mathfrak{p}}$$

is 1-absorbing prime. This follows from standard localization arguments adapted to the hypermodule setting [5].

**Theorem 2.15.** *Let $M$ be a faithful finitely generated multiplication $S$-hypermodule. Then every 1-absorbing prime submodule $N \subseteq M$ is of the form*

$$N = IM$$

*for some 1-absorbing prime hyperideal $I \subseteq S$.*

**Theorem 2.16.** *Let $M$ be an MC multiplication $S$-hypermodule and let $\{N_\lambda \mid \lambda \in \Lambda\}$ be a nonempty family of 1-absorbing prime submodules of $M$ such that*

$$\bigcap_{\lambda\in\Lambda} N_\lambda \neq M.$$

*Then the intersection*

$$N = \bigcap_{\lambda\in\Lambda} N_\lambda$$

*is a 1-absorbing prime submodule of $M$.*

**Remark 2.17.** As an example of the applications of theorem, the Intersection Property of 1-absorbing prime submodules can be applied in the decomposition of complex submodules into simpler, prime-like components:

**Example 2.18.** Consider the hypermodule $M = \mathbb{Z}_6$ over the semiring $S = \mathbb{Z}_6$, and let $N_1 = \{0, 2, 4\}$ and $N_2 = \{0, 3\}$. Both $N_1$ and $N_2$ are 1-absorbing prime submodules. Their intersection $N = N_1 \cap N_2 = \{0\}$ remains 1-absorbing prime, which allows us to analyze the submodule structure of $M$ by studying simpler components. This method is useful in computational algebra for simplifying calculations of radicals and for identifying minimal generating sets of complex hypermodules.

**Theorem 2.19.** *Let $f : M \to M'$ be a surjective $S$-hypermodule homomorphism. If $N \subseteq M$ is 1-absorbing prime, then the image $f(N) \subseteq M'$ is also a 1-absorbing prime submodule.*

**Remark 2.20.** As an example of the applications of theorem, the preservation of 1-absorbing primality under surjective homomorphisms facilitates the classification of hypermodules via quotient structures:

**Example 2.21.** Let $M = \mathbb{Z} \times \mathbb{Z}_4$ be a multiplication hypermodule over $\mathbb{Z}$, and define $N = 2\mathbb{Z} \times \{0, 2\}$. Let $f : M \to \mathbb{Z}_4$ be the projection onto the second component. Then $f(N) = \{0, 2\}$ is a 1-absorbing prime submodule of $\mathbb{Z}_4$. This application is significant in homological algebra for studying extensions and exact sequences, as it guarantees that prime-like properties are preserved in factor modules, aiding in structural analysis.

**Theorem 2.22.** *Let $S$ be a commutative semihyperring, $M$ an MC multiplication $S$-hypermodule, and $\mathfrak{p}$ a prime hyperideal of $S$. If $N \subseteq M$ is a 1-absorbing prime submodule, then its localization $N_{\mathfrak{p}} \subseteq M_{\mathfrak{p}}$ is also 1-absorbing prime.*

**Remark 2.23.** As an example of the applications of theorem, Localization Stability allows the study of 1-absorbing prime submodules in a local context, which is crucial for linking algebraic properties with combinatorial or geometric structures:

**Example 2.24.** Let $S = \mathbb{Z}[x]$ and $M = S/(x^2)$, and let $N = (x)M$ be a 1-absorbing prime submodule. Localizing at the prime ideal $\mathfrak{p} = (x)$ yields $M_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$, where $N_{\mathfrak{p}}$ remains 1-absorbing prime. This technique is particularly useful in tropical geometry and hypergraph representations of algebraic structures, where local properties at prime ideals correspond to combinatorial or geometric invariants, enabling precise local analysis of complex hypermodules.

An important aspect of algebraic structures is their stability under endomorphisms and linear transformations. Consider a linear operator $T : M \to M$.

**Definition 2.25.** A substructure $\mathcal{S} \subseteq M$ is said to be *stable under* $T$ if

$$T(\mathcal{S}) \subseteq \mathcal{S}.$$

This property is nontrivial in the hypermodule context, because the image of a multivalued sum or product under $T$ may produce new elements outside $\mathcal{S}$.

**Example 2.26.** Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ and $T(x) = Ax$ for a fixed matrix $A$. Then $\mathcal{S}$ is stable under $T$ precisely when $\|Ax\| \leq 1$ for all $\|x\| \leq 1$, which is equivalent to the operator norm condition $\|A\| \leq 1$. This analogy suggests that similar stability criteria could be formulated for hypermodules over semihyperrings.

**Definition 2.27.** A subset $C \subseteq M$ is said to be maximally connected if for any $x, y \in C$, there exists a finite sequence $\{x_i\}_{i=0}^{k} \subseteq C$ with $x_0 = x$, $x_k = y$, and each $x_{i+1}$ is related to $x_i$ by the adjacency relation

$$x_i \sim x_{i+1} \iff \exists a, b \in S \setminus U(S) : abx_i = x_{i+1}.$$

**Example 2.28.** In the additive graph of integers modulo $n$, with adjacency defined by $x \sim y$ if $|x - y| \equiv 1 \pmod{n}$, each connected component is maximally connected. Such adjacency relations can be defined in hypermodules as well, opening the possibility of studying graph-theoretic invariants attached to hypermodules.

## 3. Conclusion

We have extended the classical notion of prime submodules and prime ideals to the broader context of hypermodules over semihyperrings, focusing on *1-absorbing prime submodules*. Our main results establish their equivalence with 1-absorbing prime hyperideals in MC multiplication hypermodules, and we demonstrated their robustness under standard module-theoretic operations such as homomorphisms, quotients, and localizations. Examples over $\mathbb{Z}_6$ and $\mathbb{Z}_n$ illustrate the theory concretely. The study of 1-absorbing prime hypermodules not only enriches the structure theory of hypermodules but also opens avenues for applications in combinatorics, algebraic geometry, and hyperring extensions.

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# On weakly uniserial modules

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**Abstract**

An $R$-module $M$ is said to be weakly uniserial if for every pair $N, K$ of submodules of $M$, $\mathrm{Hom}_R(N,K)$ or $\mathrm{Hom}_R(K,N)$ contains an injective element. This concept first introduced and studied in [11]. Here we give more properties of this class of modules. For instance, we determine over the ring $\mathbb{Z}$, when uniform modules, finitely generated modules and a direct sum of cyclic modules are weakly uniserial. It is well known that a ring $R$ is simple Artinan if and only if every (2-generated) right $R$-module is weakly uniserial. In here we show that this fact is not true for cyclic modules. Finally, we characterize quasi-continuous, continuous and quasi-injective $\mathbb{Z}$-modules that are weakly uniserial.

## 1. Examples and some properties of weakly uniserial modules

In this note $R$ is a ring with nonzero identity element and $M$ is a unitary right $R$-module. We use the notation $N \leq M$, when $N$ is a submodule of $M$. Also $N \leq M$ is called essential, if for any nonzero submodule $K$ of $M$, $K \cap N \neq 0$. We write $E(M)$ for the injective hull of $M$. Also, if $X$ is a subset of $M$, the annihilator of $X$ in $R$ is the right ideal $ann_r(X) = \{r \in R | Xr = 0\}$. For any two $R$-modules $M, N$ if there exists a $R$-monomorphism from $M$ to $N$, we use the notation $M \hookrightarrow N$ otherwise we write $M \not\hookrightarrow N$. For an $R$-module $M$, the notation $M^{(\lambda)}$ means direct sum of copies of $M$ indexed over $\lambda$ , where $\lambda$ is a set. In this note we have used [1, 3–5, 8–10].

**Example 1.1.** (a) Any semisimple module is not weakly uniserial in general. For example in $\mathbb{Z}_6$, we have $2\mathbb{Z}_6 \not\hookrightarrow 3\mathbb{Z}_6$ and $3\mathbb{Z}_6 \not\hookrightarrow 2\mathbb{Z}_6$. But any homogenuous seimisimple module is weakly uniserial. In partigular, any vector space is a weakly uniserial.

(b) $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^\infty}$ are weakly uniserial $\mathbb{Z}$-modules where $p$ is a prime number and $n \geq 0$ is an integer number. In general, any uniserial module is weakly uniserial.

(c) $M = \bigoplus_{i \in I} \mathbb{Z}$ is weakly uniserial as a $\mathbb{Z}$-module, where $I$ is an arbitrary index set. Because evrey submodule of $M$ is again free and hence it is isomorphic to $\bigoplus_{i \in J} \mathbb{Z}$, where $|J| \leq |I|$. In fact it is easy to see that for any two free $\mathbb{Z}$-modules $F_1$ anf $F_2$ we have $F_1 \hookrightarrow F_2$ or $F_2 \hookrightarrow F_1$.

(d) For any prime number $p$, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is not weakly uniserial. Because $\mathbb{Z}_p \oplus p\mathbb{Z}_{p^2} \not\hookrightarrow \mathbb{Z}_{p^2}$ and $\mathbb{Z}_{p^2} \not\hookrightarrow \mathbb{Z}_p \oplus p\mathbb{Z}_{p^2}$

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For any prime number $p$, the subgroup of $\mathbb{Q}$ generated by $1/p^n$ (for $n = 1, 2, \ldots$), is denoted by $G_p$.

**Proposition 1.2.** *A uniform $\mathbb{Z}$-module $M$ is weakly uniserial if and only is it is isomorphic to either $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}$ or $G_p$, where $p$ is a prime number and $n \geq 1$.*

**Proposition 1.3.** *A finitely generated $\mathbb{Z}$-module is weakly uniserial if and only if it is isomorphic to either $\mathbb{Z}^n$, $\bigoplus_{i=1}^{n}\mathbb{Z}_p$ or $\mathbb{Z}_{p^n}$, where $p$ is a prime number and $\lambda$ is a set.*

**Proposition 1.4.** *Let $G$ be a direct sum of cyclic $\mathbb{Z}$-modules. Then $G$ is weakly uniserial if and only if $G \cong \mathbb{Z}_p^{(\lambda)}$ or $G \cong \mathbb{Z}^{(\lambda)}$, where $p$ is a prime number and $\lambda$ is a set.*

*Proof.* Note that for any two distinct prime numbers $p, q$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^n}, \mathbb{Z}_{q^m}) = 0$ and $\mathbb{Z}_{p^2} \bigoplus \mathbb{Z}_p$ is not weakly uniserial, because $\mathbb{Z}_{p^2} \ncong \mathbb{Z}_p \bigoplus \mathbb{Z}_p$.

□

[12, Theorem 1.4], the authors proved that a ring $R$ is simple Artinian if and only if every right $R$-module is weakly uniserial, if and only if every 2-generated right $R$-module is weakly uniserial. This fact is not true for cyclic modules. In the following we give a simple noncommutative ring that is neither right (left) Artinian nor right (left) uniserial, however every cyclic right $R$-module is weakly uniserial.

Let $K$ be a field with derivation $D$ and let $K[x, D]$ denote the ring of differential polynomials in the indeterminate $x$ with coefficients in $K$, i.e., the additive group of $K[x, D]$ is the additive group of the ring of polynomials in the indeterminate $x$ with coefficients in $K$, and multiplication in $K[x, D]$ is defined by: $xa = ax + D(a)$ for all $a \in K$. Let $f = \sum_{i=1}^{n} a_i x^i \in K[x, D]$, $a_n \neq 0$. Define the degree of $f$, $\delta(f) = n$. The following properties are immediate:
(1) $\delta(fg) = \delta(f) + \delta(g)$, for all $f, g \in K[x, D]$.
(2) For $f, g \in k[x, D]$, there exist $h, r \in k[x, D]$ such that $f = gh + r$, where $r = 0$ or $\delta(r) < \delta(g)$.

Now assume $K$ is a field of characteristic 0 and $D$ is a derivation of $K$. A result due to Kolchin asserts the existence of a field $K \subseteq U$ and a derivation $\bar{D}$ of $U$ extending $D$ such that the equation $p(x, \bar{D}(x), \cdots, \bar{D}^{(n)}(x)) = 0$, $n$ arbitrary, has a solution in $U$ for all $p(X) \in U[X_1, \cdots, X_{n+1}] \backslash U$. Further more, every homogeneous linear differential equation in $\bar{D}$ over $U$ has a nontrivial solution in $U$. Such a field $U$ is called a *universal extension* of $K$ or a *universal differential field* (see [6]).

**Example 1.5.** [7, Theorem 1.4]. Let $R = K[x, D]$, where $K$ is a universal differential field with derivation D. Then
$(a)$ $R$ is a principal right and left ideal domain,
$(b)$ $R$ is simple,
$(c)$ $R$ is a $V$-ring, i.e., every simple right $R$-module is injective,
$(d)$ $R$ is not a field,
$(e)$ $R$ has a unique simple right $R$-module, up to isomorphism.

We show that every cyclic right $R$-module is weakly uniserial, but $R$ is not right uniserial. Let $I$ be a nonzero right ideal of $R$. Since $R$ is right $V$-ring, we have $I = \cap_{i=1}^{n}\mathfrak{m}_i$, where $\mathfrak{m}_i$ is a maximal right ideal of $R$. Thus by Chinese Reminder Theorem, $\frac{R}{I} = \frac{R}{\cap_{i=1}^{n}\mathfrak{m}_i} \cong \frac{R}{\mathfrak{m}_1} \oplus \ldots \oplus \frac{R}{\mathfrak{m}_n} \cong \bigoplus_{i=1}^{n}\frac{R}{\mathfrak{m}}$, for some maximal right ideal $\mathfrak{m}$ (because $R$ has only one simple right $R$-module). Therefore $\frac{R}{I}$ is a homogeneous semisimple right $R$-module and hence it is weakly uniserial. Also we note that $R$ is a right weakly uniserial ring because every domain is right (left) weakly uniserial. Thus every cyclic right $R$-module is weakly uniserial. In here note that the conditions of Chinese Reminder Theorem hold here. For instance, let $\mathfrak{m}_1 = (x - \alpha_1)R$, $\mathfrak{m}_2 = (x - \alpha_2)R$ and $\mathfrak{m}_3 = (x - \alpha_3)R$ be three distinct maximal right ideals. Then it is easy to see that $\mathfrak{m}_3 \nsubseteq \mathfrak{m}_1 \cap \mathfrak{m}_2$. We show that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \nsubseteq \mathfrak{m}_3$. By Division Algorithm, $(x - \alpha_1)(x - \alpha_2) = (x - \alpha_2)f + r$, for some $f \in R$ and $r \in R$. By setting $x = \alpha_2$, we get $r = 0$. Thus $(x - \alpha_1)(x - \alpha_2) \in \mathfrak{m}_1 \cap \mathfrak{m}_2$. If $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subseteq \mathfrak{m}_3$, then $(x - \alpha_1)(x - \alpha_2) = (x - \alpha_3)f$, for some $f \in R$. This implies that $(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2) = 0$ and so $\alpha_3 = \alpha_1$ or $\alpha_3 = \alpha_2$, a contradiction. Thus $\mathfrak{m}_1 \cap \mathfrak{m}_2 \nsubseteq \mathfrak{m}_3$ and so $\mathfrak{m}_3$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2$ are coprime, i.e., $\mathfrak{m}_3 + \mathfrak{m}_1 \cap \mathfrak{m}_2 = R$.

A module $M$ over an arbitrary ring $R$ is said to be quasi-injective if, for any submodule $L$ of $M$, any $f \in Hom_R(L, M)$ can be extended to an endomorphism of $M$. Now, consider the following conditions on an $R$-module $M$:

$(C1)$ Every submodule of $M$ that has no proper essential extension within $M$ is a direct summand of $M$;
$(C2)$ Every submodule isomorphic to a direct summand of $M$ itself is a direct summand;
$(C3)$ Every sum of two independent direct summands of $M$ is again a direct summand.
If $M$ satisfies conditions $(C1)$ and $(C2)$, then $M$ is called continuous and if it satisfies conditions $(C1)$ and $(C3)$, then $M$ is called quasi-continuous. It is not hard to check that $(C2) \Rightarrow (C3)$. Thus

$$\text{Injective} \Longrightarrow \text{Quasi-injective} \Longrightarrow \text{Continuous} \Longrightarrow \text{Quasi-continuous}.$$

**Lemma 1.6.** *Let $M = \mathbb{Z}^{(n)}$, where $n$ is a natural number. Then:*
$(1)$ *$M$ is not continuous,*
$(2)$ *$M$ is quasi-continuous if and only if $n = 1$.*

*Proof.* $(1)$. If $n = 1$, then $M = \mathbb{Z}$ and $2\mathbb{Z} \cong \mathbb{Z} \leq_{\oplus} \mathbb{Z}$, but $2\mathbb{Z} \not\leq_{\oplus} \mathbb{Z}$. Thus $M = \mathbb{Z}$ is not continuous. Now for $n = 2$, we show that $M = \mathbb{Z} \oplus \mathbb{Z}$ is not continuous. Note that every submodule of $M$ is isomorphic to either $(0)$, $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Thus if $M$ is continuous, then every submodule of $M$ is a direct summand of $M$, i.e., $M$ is semisimple, a contradiction, because $\mathbb{Z}$ is not semisimple. Now we proceed by induction. Let $M = \mathbb{Z}^{(n)}$, where $n \geq 3$ and we suppose that $\mathbb{Z}^{(k)}$ is not continuous, for any $k \leq n-1$. Thus there exist $N_1, N_2 \leq \mathbb{Z}^{(n-1)}$ such that $N_1 \cong N_2$ and $N_1 \leq_{\oplus} \mathbb{Z}^{(n-1)}$ but $N_2 \not\leq_{\oplus} \mathbb{Z}^{(n-1)}$. Then $N_1 \oplus \mathbb{Z} \cong N_2 \oplus \mathbb{Z}$ and $N_1 \oplus \mathbb{Z} \leq_{\oplus} \mathbb{Z}^{(n)}$. Now if $M = \mathbb{Z}^{(n)}$ is continuous, then $N_2 \oplus \mathbb{Z} \leq_{\oplus} M$ and so $N_2 \leq_{\oplus} M$. Since $N_2 \leq \mathbb{Z}^{(n-1)} \leq \mathbb{Z}^{(n)} = M$, we must have $N_2 \leq_{\oplus} \mathbb{Z}^{(n-1)}$, a contradiction.
$(2)$. For $n = 1$, we note that $M = \mathbb{Z}$ is quasi-continuous. Because the only direct summands of $\mathbb{Z}$ are $(0)$, $\mathbb{Z}$ and $(0) \oplus \mathbb{Z}$ is again a direct summand of $\mathbb{Z}$. For $n = 2$, we show that $M = \mathbb{Z} \oplus \mathbb{Z}$ is not quasi-continuous. It is easy to see that $(1,1)\mathbb{Z} \oplus (2,1)\mathbb{Z} = (1,1)\mathbb{Z} \oplus (2,3)\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$. Now if $M$ is quasi-continuous, then $(2,1)\mathbb{Z} \oplus (2,3)\mathbb{Z}$ is a direct summand of $M$. But since $\text{u.dim}(\mathbb{Z} \oplus \mathbb{Z}) = 2 = \text{u.dim}((2,1)\mathbb{Z} \oplus (2,3)\mathbb{Z})$, we maust have $\mathbb{Z} \oplus \mathbb{Z} = (2,1)\mathbb{Z} \oplus (2,3)\mathbb{Z}$. But it is easy to see that there are not $n, m \in \mathbb{Z}$ such that $(1,0) = (2,1)n + (2,3)m$. Thus $M = \mathbb{Z} \oplus \mathbb{Z}$ is not quasi-continuous. Now by induction, we show that $\mathbb{Z}^{(n)}$ is not quasi-continuous and assume that $\mathbb{Z}^{(k)}$ is not quasi-continuous, for any $k < n$. Since $\mathbb{Z}^{(n-1)}$ is not quasi-continuous, there exist $N_1, N_2 \leq \mathbb{Z}^{(n-1)}$ such that $N_1 \leq_{\oplus} \mathbb{Z}^{(n-1)}$, $N_2 \leq_{\oplus} \mathbb{Z}^{(n-1)}$, $N_1 \cap N_2 = 0$, but $N_1 \oplus N_2 \not\leq_{\oplus} \mathbb{Z}^{(n-1)}$. Since $N_1 \leq_{\oplus} \mathbb{Z}^{(n-1)} \leq_{\oplus} \mathbb{Z}^{(n)}$, we have $N_1 \leq_{\oplus} \mathbb{Z}^{(n)}$. Similarly $N_2 \leq_{\oplus} \mathbb{Z}^{(n)}$. Now if $M = \mathbb{Z}^{(n)}$ is quasi-continuous, we have $N_1 \oplus N_2 \leq_{\oplus} \mathbb{Z}^{(n)}$. But Since $N_1 \oplus N_2 \leq \mathbb{Z}^{(n-1)} \leq \mathbb{Z}^{(n)}$, we conclude that $N_1 \oplus N_2 \leq_{\oplus} \mathbb{Z}^{(n-1)}$, a contradiction. □

**Proposition 1.7.** *Let $M$ be a nonzero quasi-continuous $\mathbb{Z}$-module. Then $M$ is weakly uniserial if and only if $M$ is isomorphic to either $\mathbb{Z}_{p^n}$, $\mathbb{Z}_p{}^{(\lambda)}$, $\mathbb{Z}_{p^\infty}$, $\mathbb{Z}$ or $G_p$, where $p$ is a prime number, $n \geq 1$ and $\lambda$ is a set.*

**Corollary 1.8.** *Let $M$ be a nonzero continuous $\mathbb{Z}$-module. Then $M$ is weakly uniserial if and only if $M$ is isomorphic to either $\mathbb{Z}_{p^n}$, $\mathbb{Z}_p{}^{(\lambda)}$ or $\mathbb{Z}_{p^\infty}$, where $p$ is a prime number, $n \geq 1$ and $\lambda$ is a set.*

**Corollary 1.9.** *Let $M$ be a nonzero quasi-injective $\mathbb{Z}$-module. Then $M$ is weakly uniserial if and only if $M$ is isomorphic to either $\mathbb{Z}_{p^n}$, $\mathbb{Z}_p{}^{(\lambda)}$ or $\mathbb{Z}_{p^\infty}$, where $p$ is a prime number, $n \geq 1$ and $\lambda$ is a set.*

**Corollary 1.10.** *Let $M$ be a nonzero weakly uniserial $\mathbb{Z}$-module. Then $M$ is continuous if and only if it is quasi-injective.*

**Corollary 1.11.** *Let $M$ be a nonzero injective $\mathbb{Z}$-module. Then $M$ is weakly uniserial if and only if $M$ is isomorphic to $\mathbb{Z}_{p^\infty}$, where $p$ is a prime number.*

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# On co-weakly uniserial modules

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**Abstract**

An $R$-module $M$ is said to be co-weakly uniserial if for every pair $N, K$ of submodules of $M$, $\mathrm{Hom}_R(M/N, M/K)$ or $\mathrm{Hom}_R(M/K, M/N)$ contains a surjective element. This concept is the dual of weakly unisrial. Here we give some properties of this class of modules. For instance, we show that a commutative ring $R$ is uniserial if and only if it is co-weakly uniserial, if and only if every cyclic $R$-module is co-weakly uniserial. Also we show that if $R$ is not commutative, the above result may be not true. Then we prove that a co-weakly uniserial $\mathbb{Z}$-module is continuous if and only if it is quasi-continuous.

## 1. Examples and some properties of co-weakly uniserial modules

In this note $R$ is a ring with nonzero identity element and $M$ is a unitary right $R$-module. We use the notation $N \leq M$, when $N$ is a submodule of $M$. Also $N \leq M$ is called essential, if for any nonzero submodule $K$ of $M$, $K \cap N \neq 0$. We write $E(M)$ for the injective hull of $M$. Also, if $X$ is a subset of $M$, the annihilator of $X$ in $R$ is the right ideal $ann_r(X) = \{r \in R | Xr = 0\}$. For any two $R$-modules $M, N$ if there exists an $R$-epimorphism from $M$ to $N$, we use the notation $M \twoheadrightarrow N$ otherwise we write $M \not\twoheadrightarrow N$. For an $R$-module $M$, the notation $M^{(\lambda)}$ means direct sum of copies of $M$ indexed over $\lambda$ , where $\lambda$ is a set. In this note we have used [1, 3–5, 8–10].

**Example 1.1.** (a) Any semisimple module is not co-weakly uniserial in general. For example in $\mathbb{Z}_6$, we have $\mathbb{Z}_6/2\mathbb{Z}_6 \not\twoheadrightarrow \mathbb{Z}_6/3\mathbb{Z}_6$ and $\mathbb{Z}_6/3\mathbb{Z}_6 \not\twoheadrightarrow \mathbb{Z}_6/2\mathbb{Z}_6$. But any homogenuous seimisimple module is co-weakly uniserial. In partigular, any vector space is a co-weakly uniserial.

(b) $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^\infty}$ are co-weakly uniserial $\mathbb{Z}$-modules where $p$ is a prime number and $n \geq 0$ is an integer number. In general, any uniserial module is co-weakly uniserial.

(c) A commutative ring is co-weakly uniserial if and only if it is uniserial. Because if $A$ and $B$ are two ideals in a co-weakly uniserial ring $R$, then we have $R/A \twoheadrightarrow R/B$ or $R/B \twoheadrightarrow R/A$ and hence $A = ann_r(R/A) \subseteq ann_r(R/B) = B$ or $B = ann_r(R/B) \subseteq ann_r(R/A) = A$. Conversely, if $R$ is a uniserial ring, for any two ideals $A$ and $B$ of $R$, we may assume that $A \subseteq B$. Then the natural map $f : R/A \to R/B$ by $f(r + A) = r + B$ is an epimorphism.

(d) It is not hard to check that a finitely generated $\mathbb{Z}$-module that contains a torsion-free element is not co-weakly uniserial. For example, $\mathbb{Z} \oplus \mathbb{Z}_2$ is not co-weakly uniserial.

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**Proposition 1.2.** *For any uniform $\mathbb{Z}$-module $M$, the following are equivalent:*
(1) *$M$ is uniserial.*
(2) *$M$ is co-weakly uniserial.*
(3) *$M$ is isomorphic to either $\mathbb{Z}_{p^n}$ or $\mathbb{Z}_{p^\infty}$, where $p$ is a prime number and $n \geq 1$.*

*Proof.* (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious.
(2) $\Rightarrow$ (3). Since $M$ is uniform, $E(M)$ is also uniform and so it is indecomposable. It is well known that every injective indecomposable $\mathbb{Z}$-module is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$, where $p$ is a prime number. If $E(M) \cong \mathbb{Q}$, then $M$ is isomorphic to a submodule of $\mathbb{Q}$. But by [2, Theorem 3],$\mathbb{Q}$ contains no co-weakly uniserial submodule. Thus $M$ is a submodules of $\mathbb{Z}_{p^\infty}$ and hence $M$ is isomorphic to either $\mathbb{Z}_{p^n}$ or $\mathbb{Z}_{p^\infty}$, for some prime number $p$ and $n \geq 1$. □

**Proposition 1.3.** *Let $G$ be a direct sum of cyclic $\mathbb{Z}$-modules. Then $G$ is co-weakly uniserial if and only if $G \cong \mathbb{Z}_p^{(\lambda)}$ where $p$ is a prime number and $\lambda$ is a set.*

*Proof.* Note that for any two distinct prime numbers $p, q$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^n}, \mathbb{Z}_{q^m}) = 0$ and $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ is not co-weakly uniserial, because $\mathbb{Z}_{p^2} \ncong \mathbb{Z}_p \oplus \mathbb{Z}_p$.

□

**Theorem 1.4.** *For a commutative ring $R$, the following are equivalent:*
(1) *$R$ is uniserial.*
(2) *$R$ is co-weakly uniserial.*
(3) *Every cyclic $R$-module is co-weakly uniserial.*

*Proof.* (1) $\Leftrightarrow$ (2). It is obvouis by Example 1.1, $(c)$.
(2) $\Rightarrow$ (3). Suppose that $R$ is co-weakly uniserial and consider cyclic $R$-module $\frac{R}{I}$, where $I$ is an ideal of $R$. Also assume that $\frac{J_1}{I}$ and $\frac{J_2}{I}$ are two ideals of $\frac{R}{I}$. Since $R$ is co-weakly uniserial, we can assume that $\frac{R}{J_1} \twoheadrightarrow \frac{R}{J_2}$. Then

$$\frac{R}{J_1} \cong \frac{\frac{R}{I}}{\frac{J_1}{I}} \twoheadrightarrow \frac{\frac{R}{I}}{\frac{J_2}{I}} \cong \frac{R}{J_2}$$

and this shows that $\frac{R}{I}$ is co-weakly uniserial.
(3) $\Rightarrow$ (1). Since $R$ is co-weakly uniserial, for any two ideals $I$ and $J$ of $R$, we may assume that $\frac{R}{I} \twoheadrightarrow \frac{R}{J}$. Then $I = \mathrm{ann}_R(\frac{R}{I}) \subseteq \mathrm{ann}_R(\frac{R}{J}) = J$ and so $R$ is uniserial. □

The commutativity of the ring is a necessary condition in the above theorem (see the following example). Let $K$ be a field with derivation $D$ and let $K[x, D]$ denote the ring of differential polynomials in the indeterminate $x$ with coefficients in $K$, i.e., the additive group of $K[x, D]$ is the additive group of the ring of polynomials in the indeterminate $x$ with coefficients in $K$, and multiplication in $K[x, D]$ is defined by: $xa = ax + D(a)$ for all $a \in K$. Let $= \sum_{i=1}^{n} a_i x^i \in K[x, D]$, $a_n \neq 0$. Define the degree of $f$, $\delta(f) = n$. The following properties are immediate:
(1) $\delta(fg) = \delta(f) + \delta(g)$, for all $f, g \in K[x, D]$.
(2) For $f, g \in k[x, D]$, there exist $h, r \in k[x, D]$ such that $f = gh + r$, where $r = 0$ or $\delta(r) < \delta(g)$.

Let $K$ be a field of characteristic 0 and $D$ be a derivation of $K$. A result due to Kolchin asserts the existence of a field $K \subseteq U$ and a derivation $\bar{D}$ of $U$ extending $D$ such that the equation $p(x, \bar{D}(x), \cdots, \bar{D}^{(n)}(x)) = 0$, $n$ arbitrary, has a solution in $U$ for all $p(X) \in U[X_1, \cdots, X_{n+1}] \backslash U$. Further more, every homogeneous linear differential equation in $\bar{D}$ over $U$ has a nontrivial solution in $U$. Such a field $U$ is called a *universal extension* of $K$ or a *universal differential field* (see [6]). Let $K$ be a universal differential field with derivation D. For the remainder of this section, we shall always denote $k[x, D]$ by $R$.

**Example 1.5.** [7, Theorem 1.4]. Let $R = K[x, D]$, where $K$ is a universal differential field with derivation D. Then:
$(a)$ $R$ is a principal right and left ideal domain,
$(b)$ $R$ is simple,
$(c)$ $R$ is a $V$-ring, i.e., every simple right $R$-module is injective,
$(d)$ $R$ is not a field,
$(e)$ $R$ has a unique simple right $R$-module, up to isomorphism.

We show that every cyclic right $R$-module is co-weakly uniserial, but $R$ is not right uniserial. Let $I$ be a non-zero right ideal of $R$. Since $R$ is right $V$-ring, we have $I = \cap_{i=1}^{n}\mathfrak{m}_i$, where $\mathfrak{m}_i$ is a maximal right ideal of $R$. Thus by Chinese Reminder Theorem, $\frac{R}{I} = \frac{R}{\cap_{i=1}^{n}\mathfrak{m}_i} \cong \frac{R}{\mathfrak{m}_1} \oplus \ldots \oplus \frac{R}{\mathfrak{m}_n} \cong \oplus_{i=1}^{n}\frac{R}{\mathfrak{m}}$, for some maximal right ideal $\mathfrak{m}$ (because $R$ has only one simple right $R$-module). Therefore $\frac{R}{I}$ is a homogeneous semisimple right $R$-module and hence it is co-weakly uniserial. On the other hand, since $R$ is right principal ideal domain, every two right ideals are isomorphic and hence $R$ is right co-weakly uniserial. Thus every cyclic right $R$-module is co-weakly uniserial. But clearly, $R$ is not right uniserial.
Note that the conditions of Chinese Reminder Theorem hold here. For instance, let $\mathfrak{m}_1 = (x-\alpha_1)R$, $\mathfrak{m}_2 = (x-\alpha_2)R$ and $\mathfrak{m}_3 = (x-\alpha_3)R$ be three distinct maximal right ideals. Then it is easy to see that $\mathfrak{m}_3 \nsubseteq \mathfrak{m}_1 \cap \mathfrak{m}_2$. We show that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \nsubseteq \mathfrak{m}_3$. By Division Algorithm, $(x-\alpha_1)(x-\alpha_2) = (x-\alpha_2)f + r$, for some $f \in R$ and $r \in R$. By setting $x = \alpha_2$, we get $r = 0$. Thus $(x-\alpha_1)(x-\alpha_2) \in \mathfrak{m}_1 \cap \mathfrak{m}_2$. If $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subseteq \mathfrak{m}_3$, then $(x-\alpha_1)(x-\alpha_2) = (x-\alpha_3)f$, for some $f \in R$. This implies that $(\alpha_3-\alpha_1)(\alpha_3-\alpha_2) = 0$ and so $\alpha_3 = \alpha_1$ or $\alpha_3 = \alpha_2$, a contradiction. Thus $\mathfrak{m}_1 \cap \mathfrak{m}_2 \nsubseteq \mathfrak{m}_3$ and so $\mathfrak{m}_3$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2$ are coprime, i.e., $\mathfrak{m}_3 + \mathfrak{m}_1 \cap \mathfrak{m}_2 = R$.

A module $M$ over an arbitrary ring $R$ is said to be quasi-injective if, for any submodule $L$ of $M$, any $f \in Hom_R(L, M)$ can be extended to an endomorphism of $M$. Now, consider the following conditions on an $R$-module $M$:
$(C1)$ Every submodule of $M$ that has no proper essential extension within $M$ is a direct summand of $M$;
$(C2)$ Every submodule isomorphic to a direct summand of $M$ itself is a direct summand;
$(C3)$ Every sum of two independent direct summands of $M$ is again a direct summand.
If $M$ satisfies conditions $(C1)$ and $(C2)$, then $M$ is called continuous and if it satisfies conditions $(C1)$ and $(C3)$, then $M$ is called quasi-continuous. It is not hard to check that $(C2) \Rightarrow (C3)$. Thus

$$\text{Injective} \Longrightarrow \text{Quasi-injective} \Longrightarrow \text{Continuous} \Longrightarrow \text{Quasi-continuous}.$$

**Proposition 1.6.** *[2, Theorem 4]. Let $M$ be a nonzero quasi-continuous $\mathbb{Z}$-module. Then $M$ is co-weakly uniserial if and only if $M$ is isomorphic to either $\mathbb{Z}_{p^n}$, $\mathbb{Z}_{p^\infty}^{(\lambda)}$ or $\mathbb{Z}_p^{(\lambda)}$, where $p$ is a prime number, $n \geq 1$ and $\lambda$ is a set.*

**Corollary 1.7.** *Let $M$ be a co-weakly uniserial $\mathbb{Z}$-module. Then $M$ is quasi-continuous if and only if it is continuous.*

*Proof.* Suppose that $M$ is quasi-continuous. Then by Proposition 1.6, either $M \cong \mathbb{Z}_{p^n}$ or $M \cong \mathbb{Z}_{p^\infty}^{(\lambda)}$ or $M \cong \mathbb{Z}_p^{(\lambda)}$, where $p$ is a prime number, $n \geq 1$ and $\lambda$ is a set. If $M \cong \mathbb{Z}_{p^n}$, then since $M$ is uniserial, $(0)$ is the only submodule of $M$ that has no proper essential extension in $M$. Clearly $(0)$ is a direct summand of $M$ and hence $M$ satisfies condition $(C1)$. On the other hand, $(0)$ and $M$ are the only submodules of $M$ that are direct summands of $M$. Clearly every submodule that is isomorphic to either $(0)$ or $M$, again is a direct summand of $M$. Thus in this case $M$ satisfies condition $(C2)$ and so $M$ is continuous. If $M \cong \mathbb{Z}_p^{(\lambda)}$, then $M$ is semisimple and hence satisfies conditions $(C1)$ and $(C2)$. Thus in this case $M$ is again continuous. Finally, if $M \cong \mathbb{Z}_{p^\infty}^{(\lambda)}$, then $M$ is injective and cleary every injective module is conditions. Therefore in any case $M$ is continuous. The converse is clear.

□

**Corollary 1.8.** *Let $M$ be a nonzero injective $\mathbb{Z}$-module. Then $M$ is co-weakly uniserial if and only if $M$ is isomorphic to either $\mathbb{Z}_{p^\infty}^{(\lambda)}$, where $p$ is a prime number.*

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# Weakly Δ-clean Rings

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**Abstract**

In this paper, we introduce the notion of weakly Δ-clean rings where every element $r$ can be written as $r = d + e$ or $r = d - e$, where $d \in \Delta(R)$ and $e \in Id(R)$. We study various properties of weakly Δ-clean rings. We prove that, if $R$ is a ring and $\sigma$ be a ring endomorphism of $R$, then $R$ is a weakly Δ-clean ring $\Leftrightarrow$ The formal power series ring $R[[x]]$ of $R$ is a weakly Δ-clean ring $\Leftrightarrow$ The skew power series ring $R[[x;\sigma]]$ of $R$ is a weakly Δ-clean ring.

## 1. Introduction

Let $R$ be an associative ring with identity. An element $r$ in a ring $R$ is called clean if $r$ is the sum of an idempotent and a unit in $R$, and $R$ is called a clean ring if every element is clean [2, 9]. A ring $R$ is said to be weakly clean ring if every element $r$ of $R$ can be expressed as $r = u + e$ or $r = u - e$, where $u \in U(R)$, $e \in Id(R)$ [3, 6, 7]. The set $\Delta(R)$ defined as

$$\Delta(R) = \{x \in R \mid 1 - xu \in U(R) \; for \; all \; u \in U(R)\}.$$

A ring $R$ is said to be $J$-clean if every element of $R$ is sum of an idempotent from $R$ and an element from $J(R)$ [4, 5]. A ring $R$ is said to be strongly Δ-clean ring if every element of $R$ is the sum of an idempotent from $R$ and an element from $\Delta(R)$ that commute with each other [8].

In this paper, we introduce the notion of a weakly Δ-clean ring as a new generalization of a strongly Δ-clean or $J$-clean ring. Let $R$ be a ring. Then an element $r \in R$ is called weakly Δ-clean if there exist $d \in \Delta(R)$ and $e \in Id(R)$ such that $r = d + e$ or $r = d - e$. A ring $R$ is called weakly Δ-clean if every element of $R$ is weakly Δ-clean. We study various properties of weakly Δ-clean rings. We show that, every homomorphic image of a weakly Δ-clean ring is weakly Δ-clean (Lemm 2.5). We prove that, if $R$ is a ring and $I$ be an ideal of $R$ such that $I \subseteq J(R)$, then $R$ is weakly Δ-clean if and only if $R/I$ is weakly Δ-clean. (Lemma 2.8).

Finally, we show that, if $R$ is a ring and $\sigma$ be a ring endomorphism of $R$, then $R$ is a weakly Δ-clean ring $\Leftrightarrow$ The formal power series ring $R[[x]]$ of $R$ is a weakly Δ-clean ring $\Leftrightarrow$ The skew power series ring $R[[x;\sigma]]$ of $R$ is a weakly Δ-clean ring. (Lemma 2.12).

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## 2. Main Results

**Definition 2.1.** A ring $R$ is said to be $J$-clean if every element of $R$ is sum of an idempotent from $R$ and an element from $J(R)$ [**?** ].

**Definition 2.2.** A ring $R$ is said to be strongly $\Delta$-clean ring if every element of $R$ is the sum of an idempotent from $R$ and an element from $\Delta(R)$ that commute with each other [8].

Let $R = \mathbb{Z}_5$. Since $Id(\mathbb{Z}_5) = \{0,1\}$ and $\Delta(\mathbb{Z}_5) = \{0,4\}$, $R$ is not strongly $\Delta$-clean. Every $J$-clean ring is strongly $\Delta$-clean. Every strongly $\Delta$-clean ring is strongly clean [8][Lemma 2.4].
In the following, we define the weakly $\Delta$-clean rings, then we study some of the basic properties of weakly $\Delta$-clean rings.

**Definition 2.3.** Let $R$ be a ring. Then an element $r \in R$ is called weakly $\Delta$-clean if there exist $d \in \Delta(R)$ and $e \in Id(R)$ such that $r = d + e$ or $r = d - e$. A ring $R$ is called weakly $\Delta$-clean if every element of $R$ is weakly $\Delta$-clean.

Every strongly $\Delta$-clean or $J$-clean ring is weakly $\Delta$-clean.

**Proposition 2.4.** *Let $R$ be a ring and $r \in R$. Then $r$ is weakly $\Delta$-clean if and only if $r = d + e$, where $d \in \Delta(R)$ or $1 + d \in \Delta(R)$.*

*Proof.* Suppose that $r$ is weakly $\Delta$-clean. Hence $r = d + e$ or $r = d - e$ for some $d \in \Delta(R)$ and $e \in Id(R)$. If $r = d + e$, then $d \in \Delta(R)$. If $r = d - e$, then $r = (d - 1) + (1 - e)$, where $1 + (d - 1) = d \in \Delta(R)$. Conversely, is clear. □

**Lemma 2.5.** *Homomorphic image of weakly $\Delta$-clean ring is weakly $\Delta$-clean.*

*Proof.* Since homomorphic images of every element of $\Delta(R)$ and $Id(R)$ are again in $\Delta(R)$ and $Id(R)$, respectively, the assertion holds. □

**Lemma 2.6.** *Let $\{R_\alpha\}$ be a family of rings. Then the direct product $R = \prod_\alpha R_\alpha$ is weakly $\Delta$-clean if and only if each $R_\alpha$ is weakly $\Delta$-clean.*

*Proof.* Suppose that every $R_\alpha$ is weakly $\Delta$-clean and $r = (r_\alpha) \in R$. Hence $r_\alpha = d_\alpha + e_\alpha$ or $r_\alpha = f_\alpha - e_\alpha$ for some $f_\alpha \in \Delta(R_\alpha)$ and $e_\alpha \in Id(R_\alpha)$. Then $r = d + e$ such that $d = (d_\alpha) \in \Delta(R)$ and $e = (e_\alpha) \in Id(R)$, and so $R$ is weakly $\Delta$-clean. Conversely, follows from Lemma 2.5. □

**Lemma 2.7.** *Every weakly $\Delta$-clean ring $R$ is weakly clean.*

*Proof.* Suppose that $R$ is a weakly $\Delta$-clean ring and $r \in R$. Hence $r = d + e$ or $r = d - e$ for some $d \in \Delta(R)$ and $e \in Id(R)$. Hence $r = (2e - 1 + d) + (1 - e)$ or $r = (-2e + 1 + d) - (1 - e)$. Since $(2e - 1)^2 = 1 = (1 - 2e)^2$, $2e - 1 + d, 1 - 2e + d \in U(R)$. Therefore $R$ is weakly clean. □

*Let $I$ be an ideal of a ring $R$. Then we say idempotents can be lifted modulo $I$ if, given $r \in R$ with $r - r^2 \in I$, there exists $e \in Id(R)$ such that $e - r \in Id(R)$ [10].*

**Lemma 2.8.** *Let $R$ be a ring and $I$ be an ideal of $R$ such that $I \subseteq J(R)$. Then $R$ is weakly $\Delta$-clean if and only if $R/I$ is weakly $\Delta$-clean.*

*Proof.* Suppose that $R$ is weakly $\Delta$-clean. Hence $R/I$ is weakly $\Delta$-clean, by Lemma 2.5. Conversely, assume that $R/I$ is weakly $\Delta$-clean and $r \in R$. Hence $r+I = (d+I)+(e+I)$ or $r+I = (d+I)-(e+I)$ with $(d+I)+(u+I) = u'+I$ for some $u, u' \in U(R)$. Since idempotents can be lifted modulo $I$, $e$ is an idempotent and $r = d+b+e$ or $r = d+b-e$ for some $b \in I$. Since $(d + I) + (u + I) = u' + I$, $(d + b + u) + I = u' + I$. Therefore, $d + b \in \Delta(R)$. Then $R$ is weakly $\Delta$-clean. □

**Lemma 2.9.** *Let $R$ be a ring. Then $R$ is weakly $\Delta$-clean if and only if for every $r \in R$ there exist $f \in Id(R)$ and $d \in \Delta(R)$ such that $fr = fd$ and $(f-1)(r-1) = (f-1)d$ or $fr = fd$ and $(f-1)(r-1) = (f-1)d + 2(1-f)$.*

*Proof.* Suppose that $R$ is weakly $\Delta$-clean and $r \in R$. Hence $r = d + e$ or $r = d - e$ for some $e \in Id(R)$ and $d \in \Delta(R)$. Assume $f = 1 - e$. If $r = d + e$, then $fr = f(d + e) = fd$ and $(f - 1)(r - 1) = (f - 1)d$. If $r = d - e$, then $fr = f(d + e) = fd$ and $(f - 1)(r - 1) = (f - 1)d + 2(1 - f)$. Conversely, assume that for every $r \in R$ there exist $f \in Id(R)$ and $d \in \Delta(R)$ such that $fr = fd$ and $(f - 1)(r - 1) = (f - 1)d$. Then $fd - d = fr - f - r + 1$, and so $r = d + (1 - f)$. If for every $r \in R$ there exist $f \in Id(R)$ and $d \in \Delta(R)$ such that $fr = fd$ and $(f - 1)(r - 1) = (f - 1)d + 2(1 - f)$, then $fd - d + 2(1 - f) = fr - f - r + 1$, and so $r = d - (1 - f)$. Therefore $R$ is weakly $\Delta$-clean. □

**Lemma 2.10.** *Let $R$ be a ring and $R[\alpha] = R + R\alpha + \cdots + R\alpha^n$ with $\alpha^{n+1} = 0$. Then $R$ is weakly $\Delta$-clean if and only if $R[\alpha]$ is weakly $\Delta$-clean.*

*Proof.* Suppose that $R$ is weakly $\Delta$-clean. Since $J(R[\alpha]) = J(R) + \langle\alpha\rangle$,

$$R[\alpha]/J(R[\alpha]) \cong R/J(R).$$

Then $R[\alpha]/J(R[\alpha])$ is weakly $\Delta$-clean, by Lemma 2.5. Hence, $R[\alpha]$ is weakly $\Delta$-clean, by Lemma 2.8. Conversely, suppose that $R[\alpha]$ is weakly $\Delta$-clean. Since $R[\alpha]/J(R[\alpha]) \cong R/J(R)$, $R/J(R)$ is weakly $\Delta$-clean. Therefore $R$ is weakly $\Delta$-clean, by Lemma 2.8. □

Let $R$ be a ring and ${}_RM_R$ be an $R$-$R$-bimodule which is a ring possibly without a unity in which $(mn)r = m(nr)$, $(mr)n = m(rn)$ and $(rm)n = r(mn)$ held for all $m, n \in M$ and $r \in R$. The ideal extension of $R$ by $M$ is defined to be the additive abelian group $I(R, M) = R \oplus M$ with multiplication $(r, m)(s, n) = (rs, rn + ms + mn)$.

**Lemma 2.11.** *Let $R$ be a weakly $\Delta$-clean and ${}_RM_R$ be an $R$-$R$-bimodule such that for any $m \in M$, there exist $n \in M$ and $u' \in U(R)$ such that $mu' + mn = 0$. Then the ideal-extension $I(R, M)$ of $R$ by $M$ is weakly $\Delta$-clean.*

*Proof.* Suppose that $(r, m) \in I(R, M)$. Hence $r = d + e$ or $r = d - e$ for some $e \in Id(R)$ and $d \in \Delta(R)$. Then $(r, m) = (d, m) + (e, 0)$ or $(r, m) = (d, m) - (e, 0)$. It is clear that $(e, 0) \in Id(I(R, M))$. Hence there exist $u, u' \in U(R)$ and $n \in M$ such that $d + u = u'$ and $mu' + mn = 0$. Then $(d + u, m)(u', n) = (1, 0)$, and so $(d, m) \in \Delta(I(R, M))$. Therefore $Id(I(R, M)$ is weakly $\Delta$-clean. □

Let $R$ be a ring and $\sigma$ be a ring endomorphism of $R$. Then the skew power series ring $R[[x; \sigma]]$ of $R$ is the ring obtained by giving the formal power series ring over $R$ with the new multiplication $xr = \sigma(r)x$ for all $a \in R$. In particular, $R[[x]] = R[[x; 1_R]]$.

**Lemma 2.12.** *Let $R$ be a ring and $\sigma$ be a ring endomorphism of $R$. Then the following statements are equivalent.*

*(i) $R$ is a weakly $\Delta$-clean ring.*

*(ii) The formal power series ring $R[[x]]$ of $R$ is a weakly $\Delta$-clean ring.*

*(iii) The skew power series ring $R[[x; \sigma]]$ of $R$ is a weakly $\Delta$-clean ring.*

*Proof.* $(ii) \Longrightarrow (i)$ Suppose $R[[x]]$ is a weakly $\Delta$-clean ring. Since $R$ is a homomorphic image of $R[[x]]$, $R$ is weakly $\Delta$-clean, by Lemma 2.5.
$(iii) \Longrightarrow (i)$ Suppose $R[[x; \sigma]]$ is a weakly $\Delta$-clean ring. Since $R$ is a homomorphic image of $R[[x; \sigma]]$, $R$ is weakly $\Delta$-clean, by Lemma 2.5.
$(i) \Longrightarrow (iii)$ Suppose $R$ is a weakly $\Delta$-clean ring and $g = r_0 + r_1x + \cdots \in R[[x; \sigma]]$. Then $r_0 = d_0 + e_0$ or $r_0 = d_0 - e_0$ for some $d_0 \in \Delta(R)$ and $e_0 \in Id(R)$. If $r_0 = d_0 + e_0$ and $g' = g - e_0 = d_0 + r_1x + \cdots$ such that $(1 - d_0u_0) \in U(R)$ for every $u_0 \in U(R)$, then $g' \in \Delta(R[[x; \sigma]])$, and $g = g' + e_0$ with $e_0 \in Id(R[[x; \sigma]])$. If $r_0 = f_0 - e_0$ and $g' = g + e_0 = d_0 + r_1x + \cdots$ such that $(1 - d_0u_0) \in U(R)$ for every $u_0 \in U(R)$, then $g' \in \Delta(R[[x; \sigma]])$, and $g = g' - e_0$ with $e_0 \in Id(R[[x; \sigma]])$. Therefore $R[[x; \sigma]]$ is weakly $\Delta$-clean.
$(i) \Longrightarrow (ii)$ Suppose $R$ is a weakly $\Delta$-clean ring. Since $R[[x]] = R[[x; 1_R]]$, the proof is similar to $(i) \Longrightarrow (iii)$. □

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# A Counterexample to the Converse Implication: $N(p) = N(q)$ Does Not Imply $\mu_p(G) = \mu_q(G)$ in Prime Graphs

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**Abstract**

Let $G$ be a finite group. The prime graph of $G$, whose idea was first introduced by Grünberg and Kegel, is denoted by $\Gamma(G)$. It is the simple graph with vertex set $\pi(G)$, where two distinct primes $p$ and $q$ in $\pi(G)$ are adjacent if and only if $G$ contains an element of order $pq$. We denote the set of primes adjacent to $p$ in the prime graph $\Gamma(G)$ by $N(p)$, and the set of maximal element orders whose order is divisible by $p$ as $\mu_p(G)$. It is a known result that the equality of maximal element order sets, $\mu_p(G) = \mu_q(G)$, implies identical adjacency in the prime graph, i.e., $N(p) = N(q)$. We claim that the converse of this result does not hold. We put $G = (C_p \rtimes C_q) \times C_r$, and suppose that $q \mid (p-1)$. Then $N(p) = N(q) = \{r\}$. However, the sets of maximal element orders containing $p$ and $q$ are different, $\mu_p(G) = \{pr\}$ and $\mu_q(G) = \{qr\}$. Thus $\mu_p(G) \neq \mu_q(G)$.

Let $p, q, r$ be distinct primes with $q \mid (p-1)$, and set $G = (C_p \rtimes C_q) \times C_r$. Since $q \mid (p-1)$, the semidirect product $C_p \rtimes C_q$ contains no element of order $pq$. On the other hand, since $C_r$ is a direct factor of $G$, the group contains elements of orders $pr$ and $qr$. Therefore $N(p) = N(q) = \{r\}$. However, the sets of maximal element orders containing $p$ and $q$ are different. Indeed, $pr$ is a maximal element order divisible by $p$, whereas $qr$ is a maximal element order divisible by $r$. Hence $\mu_p(G) = \{pr\}$ and $\mu_q(G) = \{qr\}$. Thus $\mu_p(G) \neq \mu_q(G)$. By constructing an explicit counterexample group $G$, we showed that the equality of neighborhoods alone is not sufficient to guarantee the conclusion $\mu_p(G) = \mu_q(G)$. The relationship between the set of maximal element orders, $\mu_p(G)$, and the adjacency structure $N(p)$ within the prime graph $\Gamma(G)$ is crucial for group recognition. However, the non-validity of the converse ($N(p) = N(q) \not\Rightarrow \mu_p(G) = \mu_q(G)$) highlights the limit of $\Gamma(G)$'s descriptive power [1, 3], distinguishing groups that are isospectral (same graph) but possess different maximal order characteristics.

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# A Partial Converse: When Identical Prime Graph Adjacencies Imply Coincident Maximal Element Orders

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**Abstract**

Let $\Gamma(G)$ be the prime graph of a finite group $G$ (see [2]). Denote by $\omega(G)$ the set of element orders of $G$, which is the partially ordered under divisibility. We define the set of maximal element orders of $G$ by

$$\mu(G) = \{m \in \omega(G) \mid m \text{ is maximal with respect to divisibility}\},$$

and for a prime $p$, we define $\mu_p(G) = \{m \in \mu(G) \mid p|m\}$, also we put $N(p) = \{r \in \pi(G) \mid pr \in \omega(G)\}$. The set of maximal element orders reflects the local structure determining the adjacency of $p$ in $\Gamma(G)$ (see [1, 3, 4]. It is known that identical neighborhoods, $N(p) = N(q)$, do not generally imply the coincidence of these maximal order sets, i.e., $\mu_p(G) = \mu_q(G)$. In this note, we record a simple condition under which the converse implication does hold.

Let $G$ be a finite group and let $p, q$ be two distinct primes dividing $|G|$. Assume that the vertices $p$ and $q$ have identical neighborhoods ($N(p) = N(q)$) in the prime graph $\Gamma(G)$. Suppose further that $G$ has no element whose order is divisible by three distinct primes, and no nontrivial semidirect product occurs between subgroups of Sylow corresponding to vertices with identical neighborhoods in $\Gamma(G)$. Then the sets of maximal element orders containing $p$ and $q$ coincide; that is $\mu_p(G) = \mu_q(G)$.

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# $z$-submodules of a module over a commutative ring

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**Abstract**

In this article, we introduce and study the concept of $z$-submodules as a generalization of $z$-ideals. Let $M$ be a module over a commutative ring with identity $R$. A proper submodule $N$ of $M$ is called a $z$-submodule if for any $x \in M$ and $y \in N$ such that every maximal submodule of $M$ containing y also contains $x$, then $x \in N$ as well. We investigate the properties of $z$-submodules, particularly considering their stability with respect to various module constructions.

## 1. Introduction

For the frst time, the term of $z$-ideal was used by Kohls [kohls] in the study of rings of continuous functions. Then, Mason [mason] was initiated the study of $z$-ideals in any arbitrary commutative rings. Recently, the authors have introduced and studied the concept of $z$-submodules of a module, generalizing $z$-ideals. Let $R$ be a commutative ring with identity $1 \neq 0$, and $M$ an $R$-module. We let $\mathrm{Max}(M)$ denote the set of maximal submodules of $M$. For any submodule $N$ of $M$, we set

$$\mathcal{H}(N) := \{K \in \mathrm{Max}(M) \mid N \subseteq K\}.$$

A proper submodule $N$ of $M$ is said to be a $z$-submodule if, for any $x \in M$ and $y \in N$, $\mathcal{H}(x) \supseteq \mathcal{H}(y)$ implies that $x \in N$. It is immediate that every maximal submodule of any $R$-module $M$ is a $z$-submodule, and that the intersection of an arbitrary familly of $z$-submodules is again a $z$-submodule. Moreover, for any submodule $N$ of $M$,

$$N_z = \cap\{K \supseteq N \mid K \text{ is a } z\text{-submodule}\}$$

is a $z$-submodule of $M$ containing $N$. Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is called a *prime submodule* if for $\mathfrak{p} = (P : M)$, whenever $rm \in P$ for $r \in R$ and $m \in M$, we have $r \in \mathfrak{p}$ or $m \in P$. The radical of a submodule $N$ of $M$, denoted $\mathrm{rad}(N)$, is defined as the intersection of all prime submodules of $M$ containing $N$ or, in case there are no such prime submodules, $\mathrm{rad}(N)$ is $M$. For an ideal $I$ of a ring $R$, we assume that $\sqrt{I}$ denotes the radical of $I$. A submodule $N$ of $M$ is called a *radical submodule* if $\mathrm{rad}(N) = N$. For a submodule $N$ of $M$, we take

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$(N : M) := \{r \in R \mid rM \subseteq N\}$ which is an ideal of $R$. Note that for any submodule $N$ of $M$, $(N : M)_z \subseteq (N_z : M)$ and $\sqrt{(N : M)} \subseteq (N_z : M)$. It follows that, if $N$ is a $z$-submodule of $M$, then $(N : M)$ is a $z$-ideal of $R$ and so $(N : M)$ is a radical ideal of $R$.

## 2. Main results

We begin by stating some facts about the sum of $z$-submodules. Although for any family $\{N_i\}_{i\in I}$ of submodules of $M$, we have $(\sum_{i\in I} N_i)_z = (\sum_{i\in I}(N_i)_z)_z$, is a finite sum of $z$-submodules of an $R$-module $M$ is not necessarily a $z$-submodule of $M$. Of course, any fnite sum of $z$-ideals of $C(X)$ is a $z$-ideal of $C(X)$ [2, p. 198]. Next, we give a similar result for $z$-submodules. The following lemma collects some basic facts about $N_z$'s.

**Lemma 2.1.** *Let $N$ and $L$ submodules of an $R$-module $M$ and $\{N_i\}_{i\in I}$ a collection of submodules of $M$. Then*

*(1) $(\cap_{i\in I} N_i)_z \subseteq \cap_{i\in I}(N_i)_z$. In particular, any intersection of z-submodules of $M$ is a z-submodule.*

*(2) $(\sum_{i\in I} N_i)_z = (\sum_{i\in I}(N_i)_z)_z$.*

*(3) $(\mathrm{rad}(N))_z = N_z$*

*(4) If $M$ be a finitely generated $R$-module. Then $N_z = M$ if and only if $N = M$.*

**Proposition 2.2.** *Every z-submodule of a multiplication $R$-module $M$ is a radical submodule of $M$.*

An $R$-module $M$ is called a *multiplication* $R$-module, if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. It can be observed that $M$ is a multiplication $R$-module if and only if for each submodule $N$ of $M$, $N = (N : M)M$. Cyclic $R$-modules, ideals of a Dedekind domain $R$ and ideals of any regular ring $R$ are the well-known examples of multiplication $R$-modules. Since every $z$-ideal is a radical ideal, we can rewrite [6, Proposition 8] as follows.

**Lemma 2.3.** *Let $M$ be a finitely generated $R$-module. Then for each z-ideal $I$ of $R$, $(IM : M) = I$ if and only if $I \supseteq \mathrm{Ann}(M)$.*

The following can be compared to a result concerning radical ideals provided in [8, Exercise 9.23].

**Theorem 2.4.** *Let $M$ be a finitely generated $R$-module and let $I$ be an ideal of $R$. Then*

$$(IM : M)_z = (I + \mathrm{Ann(M)})_z.$$

The following lemma provides an elementwise interpretation for $N_z$ in an special case.

**Proposition 2.5.** *Let $M$ be an $R$-module and $N$ be a submodule of $M$. If $\{\mathcal{H}(x) \mid x \in N\}$ is closed under finite intersection, then*

$$N_z = \{x \in M \mid \exists y \in N \ \ s.t. \ \mathcal{H}(y) \subseteq \mathcal{H}(x)\}.$$

**Theorem 2.6.** *Let $M$ be an $R$-module. For any ideal $I$ of $R$, $(IM)_z = (I_z M)_z$. In particular, if $M$ is a multiplication $R$-module, then for each submodule $N$ of $M$, $N_z = ((N : M)_z M)_z$.*

In the theorem below, we use a result that states if $m$ is a maximal ideal of a ring $R$ and $M$ is a finitely generated faithful multiplication $R$-module, then $mM$ is a maximal submodule of $M$ (see [4, Theorem 3.1]). In particular, for any maximal ideal $m$ of $R$ and any non-zero finitely generated ideal $I$ of the real-valued continuous functions $C(X)$ on a completely regular Hausdorff space $X$, $mI$ is a maximal subideal of $I$.

**Theorem 2.7.** *Let $R$ be a ring and $M$ be a finitely generated $R$-module. Then:*

*(1) If $M$ is a faithful multiplication $R$-module and $I$ is a z-ideal of $R$, then $IM$ is a z-submodule of $M$;*

*(2) If $M$ is a faithful $R$-module and $IM$ is a z-submodule of $M$, then $I$ is a z-ideal of $R$.*

**Corollary 2.8.** *Let $M$ is a finitely generated faithful multiplication $R$-module and $P$ is a prime submodule of $M$. Then either $P$ is a z-submodule or the maximal z-submodules contained in $P$ are prime z-submodules.*

**Theorem 2.9.** *Every z-submodule of $M$ contracts to a z-submodule of $M'$ if and only if every maximal submodule of $M$ contracts to a z-submodule of $M'$.*

**Theorem 2.10.** *Let $M$ and $M'$ be $R$-modules, let $f : M \longrightarrow M'$ be a surjective $R$-module homomorphism and* $\operatorname{Ker} f$ *is contained in each maximal submodules of $M$. Then*

*(1) If $M$ is a finitely generated $R$-module and $N$ is a z-submodule of $M'$, then $f^{-1}(N)$ is a z-submodule of $M$.*

*(2) If $M'$ is a finitely generated $R$-module and $N + Kerf$ a z-submodule of $M$, then $f(N)$ is a z-submodule of $M'$.*

**Lemma 2.11.** *Let $M$ is a finitely generated faithful multiplication $R$-module and $P$ is a prime submodule of $M$. Then either $P$ is a z-submodule or the maximal z-submodules contained in $P$ are prime z-submodules.*

**Theorem 2.12.** *Let $M$ be a multiplication $R$-module. If $P$ is a minimal among prime submodules of $M$, then $(P : M)$ is a z-ideal of $R$. If, in addition, $M$ is a finitely generated faithfull $R$-module, then $P$ is a z-submodule of $M$.*

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# On certain mappings between lattices of submodules and $z$-submodules

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**Abstract**

Let $R$ be a commutative ring and $M$ be a unital $R$-module. In this paper, we investigate when the mapping $\eta_M : \mathcal{L}(M) \to \mathcal{Z}(M)$, from the lattice of submodules of $M$ to the lattice of $z$-submodules of $M$ defined by $\eta_M(N) = N_z$ is a lattice homomorphism. We show that if $M$ is a multiplication module (i.e., if $\lambda$ is surjective), then $\eta_M$ is a homomorphism.

## 1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be unitary. Let $R$ be a ring and $M$ be an $R$-module. It is well-known that the set of all submodules of $M$ forms a lattice with the following operations:

$$L \vee N = L + N \quad \text{and} \quad L \wedge N = L \cap N,$$

for all submodules $L$ and $N$ of $M$. We shall denote this lattice by $\mathcal{L}(_RM)$ (or simple $\mathcal{L}(M)$) and if $M = R$ by $\mathcal{L}(R)$. As in [5], we consider the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}(M)$ given by $\lambda(I) = IM$, for ideals $I$ of $R$, and the mapping $\mu : \mathcal{L}(M) \to \mathcal{L}(R)$ given by $\mu(N) = (N : M)$, for all submodules $N$ of $M$, where $(N : M) := \{r \in R \mid rM \subseteq N\}$. It is easily seen that $\lambda(I \vee J) = \lambda(I) \vee \lambda(J)$ and $\mu(N \wedge L) = \mu(N) \wedge \mu(L)$. An $R$-module $M$ is called a $\lambda$-module (resp. $\mu$-module), if $\lambda$ (resp. $\mu$) is a lattice homomorphism [5].

In 1957, Kohls [1] was the first to use the concept of $z$-ideals in the study of the rings of continuous functions. He observed that they could be characterized purely algebraically. Modifying that characterization, it was Mason [2] who initiated the study of $z$-ideals of commutative rings in earnest.

The authors have introduced and studied the notion of a z-submodule as a generalization of $z$-ideals [4]. Considering $\mathrm{Max}(M)$ as the set of maximal submodules of an $R$-module $M$ and putting

$$\mathcal{M}(x) = \{K \in \mathrm{Max}(M) \mid x \in K\}$$

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for all $x \in M$, a submodule $N$ of $M$ is a $z$-submodule if for any $x \in M$ and $y \in N$, $\mathcal{M}(x) \supseteq \mathcal{M}(y)$ implies that $x \in N$. Clearly, whenever $M = R$, then the notion of a $z$-submodule coincide with that of a $z$-ideal. For any submodule $N$ of $M$, we will denote the intersection of all $z$-submodule of $M$ containing $N$ by $N_z$. It is clear that any maximal submodule of any $R$-module $M$ is a $z$-submodule, and that the intersection of an arbitrary collection of $z$-submodules is again a $z$-submodule.
Let $\mathcal{Z}(_RM)$ (or simple $\mathcal{Z}(M)$) be the set of $z$-submodules of an $R$-module $M$. It should be noted that by [3, Example 2.3], the finite sum of $z$-ideals of a ring $R$ is not necessarily a $z$-ideal, and so $\mathcal{Z}(_RM)$ is not in general a sublattice of the usual lattice $\mathcal{L}(_RM)$ consisting of all submodules of $M$. However $\mathcal{Z}(M)$ forms a lattice with respect to the following operations:

$$L \vee N = (L + N)_z \quad \text{and} \quad L \wedge N = L \cap N.$$

In [3], the authers have examined the properties of the mapping $\phi : \mathcal{Z}(R) \to \mathcal{Z}(M)$ defined by $\phi(I) = (IM)_z$ for every ideal $I$ of $R$ and the mapping $\psi : \mathcal{Z}(M) \to \mathcal{Z}(R)$ defined by $\psi(N) = (N : M)$ for every submodule $N$ of $M$. In particular, they have considered when these mapping are lattice homomorphisms. In this paper, we are going to explore the relationship between the lattices $\mathcal{L}(M)$ and $\mathcal{Z}(M)$ via the mapping $\eta_M : \mathcal{L}(M) \to \mathcal{Z}(M)$ defined by $\eta_M(N) = N_z$. It is shown that if $M$ is amultiplication module ( i.e., if $\lambda$ is surjective), then $\eta_M$ is a homomorphism (Theorem 2.5).

## 2. Main results

Let $M$ be an $R$-module and $N$ be a submodule of $M$. Recall that $\mathcal{M}(x)$ denotes the set of all maximal submodules of $M$ containing $x$. It is evident that $N$ is a $z$-submodule of $M$ if and only if $N_z = N$. In this case, we write $N \leq_z M$. If $M$ has no maximal submodule, then we put $N_z = M$.
The following lemma collects some frequently used facts on $z$-taking of submodules.

**Lemma 2.1.** *[4, Lemma 2.3] Let $N$ and $L$ be submodules of an $R$-module $M$ and $\{N_i\}_{i\in I}$ a collection of submodules of $M$. Then*

*(1) If $N \subseteq L$, then $N_z \subseteq L_z$;*

*(2) $N_z \subseteq (N_z)_z$;*

*(3) $(\cap_{i\in I} N_i)_z \subseteq \cap_{i\in I}(N_i)_z$;*

*(4) $(N : M)_z \subseteq (N_z : M)$.*

We shall say that $M$ is an $\eta$-module in case the mapping $\eta_M$ is a homomorphism. Note the following result.

**Lemma 2.2.** *Let $R$ be a ring and $M$ be an $R$-module. Then $M$ is an $\eta$-module if and only if $\eta_M(N\cap L) = \eta_M(N)\cap\eta_M(L)$ for all submodules $N$ and $L$ of $M$.*

*Proof.* Let $N$ and $L$ be any submodules of $M$. Hence, we have

$$\begin{aligned}\eta_M(N \vee L) &= (N + L)_z = (N_z + L_z)_z = N_z \vee L_z \\ &= \eta_M(N) \vee \eta_M(L),\end{aligned}$$

for all submodules $N$ and $L$ of $M$. Thus $M$ is an $\eta$-module if and only if $\eta_M(N \wedge L) = \eta_M(N) \wedge \eta_M(L)$ if and only if $\eta_M(N \cap L) = \eta_M(N) \cap \eta_M(L)$. □

An $R$-module $M$ is called a *multiplication* $R$-module, if for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to see that $M$ is a multiplication $R$-module if and only if for each submodule $N$ of $M$, $N = (N : M)M$. Cyclic modules, ideals of Dedekind domains, and ideals of regular rings are wellknown examples of multiplication modules.

**Proposition 2.3.** *Let $R = C(X)$ and $M$ be a finitely generated multiplication $R$-module such that $(N : M) + (L : M) = R$ for all incomparable submodules $N$ and $L$ of $M$. Then $M$ is a $\eta$-module.*

**Proposition 2.4.** *Let $R$ be a ring and $M$ be a finitely generated faithful multiplication $R$-module such that $N_z = (N : M)_z$ for any submodule $N$ of $M$, and $(I \cap J)M = IM \cap JM$ for any $z$-ideals $I$ and $J$ of $R$. Then $M$ is an $\eta$-module.*

**Theorem 2.5.** *Let $R$ be a ring and $M$ be an $R$-module such that $N_z = ((N : M)M)_z$ for all submodules $N$ of $M$. Then $M$ is an $\eta$-module. In particular, if $M$ is a multiplication module, then $M$ is an $\eta$-module.*

*Proof.* Let $\phi$ and $\mu$ be the mappings as before, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{L}(M) & \xrightarrow{\eta_M} & \mathcal{Z}(M) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \phi} \\ \mathcal{L}(R) & \xrightarrow[\eta_R]{} & \mathcal{R}(M) \end{array}$$

By hypothesis $\phi\eta_R\mu(N) = \eta_M$ for all submodules $N$ of $M$. Since, by [4, Lemma 3.1], $\phi$ is always a homomorphism, and since $\phi$ and $\eta_R$ are clearly homomorphisms, we have $\phi\eta_R\mu(N \wedge L) = \phi\eta_R\mu(N) \wedge \phi\eta_R\mu(L)$ and hence $\eta_M(N \wedge L) = \eta_M(N) \wedge \eta_M(L)$. Now Lemma 2.2 implies $M$ is a $\eta$-module. Note that if $M$ is a multiplication module $N = (N : M)M$ for all submodule $N$ of $M$. Thus it follows that "in particular" part. □

In the following corollary, by $R_S$ (resp. $M_S$) we mean that the ring (resp. module) of fractions with respect to a multiplicatively closed subset $S$.

**Corollary 2.6.** *Let $M$ be a multiplication $\eta$-module over a ring and $S$ be a multiplicatively closed subset of $R$. Then the $R_S$-module $M_S$ is an $\eta$-module.*

*Proof.* Since $M$ is a multiplication $R$-module, it is evident that $M_S$ is a multiplication $R_S$-module. Now, it is done by Theorem 2.5. □

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# When the lattice of submodules is a coherent frame

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**Abstract**

Let $M$ be a module over a commutative ring $R$, and $\mathcal{L}({}_RM)$ denote the complete lattice of submodules of $M$. It is shown that if $M$ is a multiplication $R$-module, then $\mathcal{L}({}_RM)$ is a frame. In particular, if $M$ is a finitely generated multiplication $R$-module, then $\mathcal{L}({}_RM)$ is a coherent frame.

## 1. Introduction

Throughout this paper, all rings are commutative with a non-zero identity and all modules are unital. Let $R$ be a ring and M be an $R$-module. A non-empty subset $N$ is a *submodule* of $M$ provided that $N$ is an additive subgroup of $M$ and $rn \in N$, for all $r \in R$ and $n \in N$. For a submodule $N$ of $M$, we let $(N : M) := \{r \in R \mid rM \subseteq N\}$. A proper submodule $P$ of $M$ is called a *prime submodule* if for $\mathfrak{p} = (P : M)$, whenever $rm \in P$ for some $r \in R$ and $m \in M$, we have $r \in \mathfrak{p}$ or $m \in P$. By a lattice we mean a poset $(L, \leq)$ in which every couple of elements $x, y$ has a *greatest lower bound* called the meet of $x$ and $y$ and written $x \wedge y$. A *least upper bounded* called the join of $x$ and $y$, and written $x \vee y$. A lattice $L$ is called *complete* provided every non-empty subset $S$ has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$. The set of all submodules of $M$, denoted by $\mathcal{L}({}_RM)$, forms a bounded complete lattice when we define

$$\bigwedge \mathcal{T} = \bigcap_{N \in \mathcal{T}} N \quad \text{and} \quad \bigvee \mathcal{T} = \sum_{N \in \mathcal{T}} N$$

for every non-empty collection $\mathcal{T}$ of submodules of $M$. As in [6], we consider the mapping

$$\lambda : \mathcal{L}({}_RR) \to \mathcal{L}({}_RM), \quad \lambda(I) = IM,$$

and

$$\mu : \mathcal{L}({}_RM) \to \mathcal{L}({}_RR), \quad \mu(N) = (N : M).$$

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It is straightforward to verify that for every non-empty collection $S$ of ideals of $R$ and every non-empty collection $\mathcal{T}$ of submodules of $M$,

$$\lambda(\vee S) = \vee\{\lambda(I) \mid I \in S\} \text{ and } \mu(\wedge \mathcal{T}) = \wedge\{\mu(N) \mid N \in \mathcal{T}\},$$

An $R$-module $M$ is called a $\lambda$–module (resp. $\mu$-module), if $\lambda$ (resp. $\mu$) is a lattice homomorphism [6]. It is clear that $0$ and $M$ are respectively the least and the greatest elements of $\mathcal{L}({}_RM)$. In particular, we denote the complete lattice of ideals of $R$ by $\mathcal{L}({}_RR)$.
As usual, a *frame* $L$ is a complete lattice satisfying the distributive law

$$a \wedge \bigvee S = \bigvee \{ a \wedge x \mid x \in S \}$$

for every $a \in L$ and $S \subseteq L$. An element $c$ of a complete lattice $L$ is called *compact* if, whenever $c \leq \bigvee S$ for some non-empty subset $S$ of $L$, there exists a finite subset $T$ of $S$ such that $c \leq \bigvee T$. A lattice $L$ with greatest element 1 is said to be *compact* if 1 is a compact element in $L$. Furthermore, a compact lattice $L$ is called *coherent* if every element of $L$ is a join of compact elements and the meet of any finitely many compact elements is compact.

## 2. Main results

Let $M$ be an $R$-module. We begin this section by examining the compactness of $\mathcal{L}({}_RM)$, as it is one of the required conditions for $\mathcal{L}({}_RM)$ to be a coherent frame. Recall that a lattice $L$ with the greatest element 1 is said to be *compact* if 1 is a compact element in $L$.

**Theorem 2.1.** *If $M$ is a finitely generated $R$-module, then $\mathcal{L}({}_RM)$ is a compact lattice.*

Let $(L, \vee, \wedge)$ be a lattice, and $\leq$ be the usual partial order induced by $\wedge$; that is, for each $a, b \in L$, we have $a \leq b$ if and only if $a \wedge b = a$. We recall that $L$ is a *Noetherian lattice* if every ascending chain $a_1 \leq a_2 \leq \ldots$ of elements of $L$ stabilizes; i.e., there exists a positive integer $k$ such that for any $i \geq k$ we have $a_i = a_k$. It is well-known that a lattice $L$ is Noetherian if and only if every non-empty family of subsets of $L$ has a maximal element. As stated in [1, Proposition 2.3], every element of a complete lattice $L$ is compact if and only if $L$ is Noetherian. We have the following result.

**Lemma 2.2.** *Let $M$ be an $R$-module. If $\mathcal{L}(M)$ is a Noetherian lattice, then* $\mathrm{Min}(M)$ *is finite.*

An $R$-module $M$ is called a *multiplication $R$-module* if, for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ [2]. It can be observed that $M$ is a multiplication $R$-module if and only if for each submodule $N$ of $M$, $N = (N : M)M$. We have the following result.

**Theorem 2.3.** *Let $M$ be a multiplication $R$-module and $\overline{R} = R/\mathrm{Ann}(M)$. Then the following are equivalent:*

*(1) Every element of $\mathcal{L}({}_{\overline{R}}\overline{R})$ is compact.*

*(2) $\mathcal{L}({}_{\overline{R}}\overline{R})$ is a Noetherian lattice.*

*(3) $\mathcal{L}({}_RM)$ is a Noetherian lattice.*

*(4) Every element of $\mathcal{L}({}_RM)$ is compact.*

Let $M$ be an $R$-module. For each submodule $N$ of $M$, we set

$$V(N) := \{P \in \mathrm{Spec}(M) \mid (P : M) \supseteq (N : M)\},$$

and

$$U(N) := \mathrm{Spec}(M) \setminus V(N).$$

The Zariski topology on $\mathrm{Spec}(M)$ is defined by taking the family

$$\zeta(M) = \{V(N) \mid N \text{ is a submodule of } M\}$$

as the set of closed sets of $\mathrm{Spec}(M)$. Recall that a topological space $X$ is *Noetherian* if the usual lattice of its open subsets is Noetherian, i.e., every acsending chain of its open subsets stops.

**Corollary 2.4.** *If $M$ is a multiplication module over a Noetherian ring $R$, then every element of the lattice $\mathcal{L}({}_R M)$ is compact.*

Recall that for any submodule $N$ of $M$, $U(N)$ denotes the complement of $V(N)$ in $\mathrm{Spec}(M)$.

**Theorem 2.5.** *Let $M$ be a finitely generated multiplication $R$-module. Then*

*(1) For any radical submodule $N$ of $M$, $N$ is compact in $\mathcal{L}({}_R M)$ if and only if the open subset $U(N)$ of $\mathrm{Spec}(M)$ is compact with the Zariski topology.*

*(2) Every finite intersection of compact elements of $\mathcal{L}({}_R M)$ is compact.*

It is our aim to make the complete lattice $\mathcal{L}({}_R M)$ a frame. Recall that the distributivity law in terms of submodules is

$$N \cap (\sum_{\lambda\in\Lambda} N_\lambda) = \sum_{\lambda\in\Lambda} (N \cap N_\lambda)$$

for any submodule $N$ and any family $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of $M$. Note that $\mathcal{L}({}_R R)$ is a frame i.e.,

$$I \cap (\sum_{\lambda\in\Lambda} I_\lambda) = \sum_{\lambda\in\Lambda} (I \cap I_\lambda)$$

for any ideal $I$ and any family $\{I\}_{\lambda\in\Lambda}$ of ideals of $R$.

**Theorem 2.6.** *If $M$ is a multiplication $R$-module, then $\mathcal{L}({}_R M)$ is a frame.*

A complete lattice $L$ is called *compactly generated* (or *algebraic*) if each of its elements is a join of compact elements. A compact algebraic frame is called *coherent* if any finite intersection of compact elements of $L$ is compact.

**Theorem 2.7.** *If $M$ is a finitely generated multiplication $R$-module, then $\mathcal{L}({}_R M)$ is a coherent frame.*

As shown in [3, Theorem 2.3], for any finitely generated $R$-module $M$, $\mu : \mathcal{L}({}_R M) \to \mathcal{L}({}_R R)$ given by $\mu(N) = (N : M)$ is a lattice homomorphism if and only if $\mu$ is a complete lattice homomorphism if and only if $M$ is a multiplication $R$-module. Moreover, for a finitely generated faithful $R$-module $M$,

$$\mu((0)) = ((0) : M) = (0) \quad \text{and} \quad \mu(M) = (M : M) = R.$$

Thus, a finitely generated $R$-module $M$ is multiplication if and only if $\mu$ is a frame homomorphism. A frame homomorphism between algebraic frames is called a *coherent map* if it maps compact elements to compact elements. We conclude the following result.

**Theorem 2.8.** *Let $M$ be a finitely generated faithful $R$-module. Then $M$ is a multiplication $R$-module if and only if $\mu : \mathcal{L}({}_R M) \to \mathcal{L}({}_R R)$ is a coherent map.*

**Corollary 2.9.** *If $M$ is a finitely generated faithful multiplication $R$-module, then $\sigma\mu : \mathcal{L}({}_R M) \to \mathcal{L}({}_R R)$ is a bijective coherent map.*

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# On Fuzzy Lie Hyperalgebras

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Abstract

We study on Lie hyperalgebras and fuzzy Lie hyperalgebras and investigate its connections with associated (fuzzy) hyperoperations. This tool helps us to define fuzzy strongly regular relations on fuzzy Lie hyperalgebras to obtain Lie algebras from fuzzy Lie hyperalgebras.

## 1. Fuzzy Hyperstructures

Fuzzy hyperstructures are one of research directions which has been established after introducing algebraic hyperstructures and in connection with fuzzy sets theory. Studies in this direction have been done in three types. Firstly, to obtain related hyperstructures by defining crisp hyperoperations through fuzzy sets [1]. After that studies has been continued by defining fuzzy subsets on crisp hyperstructures, to introduce fuzzy subhyperstructures [9]. In finally, the composition of each pair of elements of a set give a fuzzy subset [2]. This paper is in third direction and in continuing of fuzzy semihypergroups [8], fuzzy hyperrings [5], fuzzy hypermodules [4], fuzzy topological hypergroupoids, fuzzy transposition hypergroups and fuzzy hyperalgebras. We study on Lie hyperalgebras, fuzzy Lie hyperalgebras and its connections, to investigate fuzzy strongly regular relations on fuzzy Lie hyperalgebras.

## 2. Lie hyperalgebras

A Lie algebra $(L, \oplus, \odot)$ is a vector space over a field $\mathbb{F}$ equipped with a bilinear map $\lfloor -, - \rfloor : L \times L \to L$, usually called the Lie bracket of $L$, satisfying $\lfloor x, x \rfloor = 0$, for all $x \in L$, and $\lfloor x, \lfloor y, z \rfloor \rfloor \oplus \lfloor y, \lfloor z, x \rfloor \rfloor \oplus \lfloor z, \lfloor x, y \rfloor \rfloor = 0$, for all $x, y, z \in L$ (Jacobi identity).

**Definition 2.1.** Consider the hypervector space $(L, +, \cdot)$ over a hyperfield $(F, +, \cdot)$ and the bilinear map $[-, -] : L \times L \to \mathcal{P}^*(L)$ given by $(x, y) \to [x, y]$ which we would call the *hyperbracket* of $L$. Then $L$ is said to be a Lie hyperalgebra if the following conditions hold:

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$(i)$ $0 \in [x,x]$, for all $x \in L$,

$(ii)$ $0 \in \big([x,[y,z]] + [y,[z,x]] + [z,[x,y]]\big)$, for all $x,y,z \in L$.

Also, the bilinearity of the hyperbracket means:

$$[\lambda_1 x_1 + \lambda_2 x_2, y] = \lambda_1[x_1,y] + \lambda_2[x_2,y],$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] = \lambda_1[x,y_1] + \lambda_2[x,y_2],$$

for all $x, x_1, x_2, y, y_1, y_2 \in L$ and $\lambda_1, \lambda_2 \in F$. Moreover, recall that $A + B = \underset{a\in A,\ b\in B}{\cup} a + b$ and $x + A = \{x\} + A$, for non-empty subsets $A, B$ of $L$ and $x \in L$. Hence, we naturally define $[A,B] = \underset{a\in A,\ b\in B}{\cup} [a,b]$, and also $[x,A] = [\{x\},A]$. Furthermore, $[x] := \{x\}$ which is also identified by its member $x$.

**Example 2.2.** The vector space $\mathfrak{gl}(n,\mathbb{F})$ of all $n \times n$ matrices over $\mathbb{F}$ clearly forms a Lie algebra with the bracket $\lfloor x,y \rfloor = xy - yx$ for all $x, y \in \mathfrak{gl}(n,\mathbb{F})$, in which $xy$ is the usual product of the matrices $x$ and $y$. Now, consider $\mathfrak{gl}(n,\mathbb{F})$ as an associative hyperalgebra with the hyperproduct $x \cdot y$ over a hyperfield $F$ (see [3]). Define $[x,y] = \cup_{a\in x\cdot y,\ b\in y\cdot x} a - b$. One can easily check that $\mathfrak{gl}(n,\mathbb{F})$ is a Lie hyperalgebra with the above hyperbracket.

Recall that a vector subspace $H$ of a Lie algebra $L$ is called a subalgebra if $\lfloor H,H \rfloor \subseteq H$, and also a subalgebra $I$ of $L$ is said to be an ideal if $\lfloor I,L \rfloor \subseteq I$.

**Example 2.3.** Let $(L, \oplus, \odot, \lfloor -,- \rfloor)$ be a Lie algebra over a field $\mathbb{F}$, $H$ be a subalgebra and $I$ be an ideal of $L$. Define on $L$: $x + y = \{x \oplus y\}$, $\lambda \cdot x = (\lambda \odot x) \oplus H$ and the hyperbracket $[x,y] := \lfloor x,y \rfloor \oplus I$, for all $x, y \in L$ and $\lambda \in \mathbb{F}$. One can easily show that $L$ is a hypermodule (hypervector space) over the trivial hyperfield $\mathbb{F}$, and also $L$ is a Lie hyperalgebra with $[-,-]$.

## 3. Fuzzy Lie hyperalgebras

Let $S$ be a nonempty set. $\mathcal{F}^*(S)$ denotes the set of all nonzero fuzzy subsets of $S$. A *fuzzy hyperoperation* on $S$ is a map "$\circ$" $: S \times S \longrightarrow \mathcal{F}^*(S)$, which associates a nonzero fuzzy subset $a \circ b$ with each pair $(a,b)$ of element of $S$. The couple $(S,\circ)$ is called a *fuzzy hypergroupoid*. We say that $(S,\circ)$ is commutative if for all $a, b \in S$, we have $a \circ b = b \circ a$. A fuzzy hypergroupoid $(S,\circ)$ is called a *fuzzy semihypergroup* if for all $a, b, c \in S$, we have $a \circ (b \circ c) = (a \circ b) \circ c$, where for all $\mu \in \mathcal{F}^*(S)$,

$$(a \circ \mu)(r) = \bigvee_{t\in S}((a \circ t)(r) \wedge \mu(t)) \ \text{ and } \ (\mu \circ a)(r) = \bigvee_{t\in S}(\mu(t) \wedge (t \circ a)(r)),$$

for all $r \in S$. If $A$ is a nonempty subset of $S$, then for all $t \in S$, $(x \circ A)(t) = \bigvee_{a\in A}(x \circ a)(t)$ and $(A \circ x)(t) = \bigvee_{a\in A}(a \circ x)(t)$. Let $\mu$ and $\lambda$ be two nonzero fuzzy subsets of fuzzy hypergroupoid $(S,\circ)$. Then for all $t \in S$,

$$(\mu \circ \lambda)(t) = \bigvee_{p,q\in S}(\mu(p) \wedge (p \circ q)(t) \wedge \lambda(q)).$$

A fuzzy semihypergroup $(S,\circ)$ is called a *fuzzy hypergroup* if $a \circ S = S \circ a = \chi_S$, for all $a \in S$. The triple $(R, \boxplus, \boxdot)$ is a *fuzzy hyperring* [5] if $(R, \boxplus)$ is a commutative fuzzy hypergroup, $(R, \boxdot)$ is a fuzzy semihypergroup and "$\boxdot$" is distributive over "$\boxplus$". If $(R, \boxdot)$ is a fuzzy hypergroup, then $(R, \boxplus, \boxdot)$ is said to be a fuzzy hyperfield. A nonempty set $M$, endowed with a fuzzy hyperoperation "$\oplus$", and a fuzzy external hyperoperation "$\odot$" is called a left *fuzzy hypermodule* [4] over a fuzzy hyperring $(R, \boxplus, \boxdot)$ if $(M, \oplus)$ is a commutative fuzzy hypergroup and $\odot : R \times M \longrightarrow \mathcal{F}^*(M)$ is a map that satisfies in $r \odot (m \oplus n) = (r \odot m) \oplus (r \odot n)$, $(r \boxplus s) \odot m = (r \odot m) \oplus (s \odot m)$ and $(r \boxdot s) \odot m = r \odot (s \odot m)$, for all $m, n \in M$ and $r, s \in R$. If we consider a fuzzy hyperfield $F$ instead of a fuzzy hyperring $R$, then $M$ is called a fuzzy hypervector space. In a fuzzy hypergroup $(S,\circ)$, an element $e \in S$ is called an *identity* (a scalar identity), if $(e \circ x)(x) > 0$ and $(x \circ e)(x) > 0$, for all $x \in S$. Also, if for $y \in S$ we have $(e \circ x)(y) > 0$ and $(x \circ e)(y) > 0$, then $x = y$. In what follow, any fuzzy hypervector space $M$ and any fuzzy hyperfield $F$ are supposed to have scalar identities such that $0_F \odot x = \chi_{\{0_M\}}$ for all $x \in M$, and $F$ contains a unit $1$ such that $1 \odot x = \chi_{\{x\}}$ for all $x \in M$.

**Definition 3.1.** Consider the fuzzy hypervector space $(L, \oplus, \odot)$ over a fuzzy hyperfield $(F, \boxplus, \boxdot)$ and the bilinear map $[-,-]_f : L \times L \to \mathcal{F}^*(L)$ given by $(x, y) \to [x, y]_f$ which we would call the *fuzzy hyperbracket* of $L$. Then $L$ is said to be a fuzzy Lie hyperalgebra if the following conditions hold:

$(i)$ $[x, x]_f(0_L) > 0$, for all $x \in L$,

$(ii)$ $\big([x, [y, z]_f]_f \oplus [y, [z, x]_f]_f \oplus [z, [x, y]_f]_f\big)(0_L) > 0$, for all $x, y, z \in L$.

The bilinearity of the fuzzy hyperbracket means:

$$[(\lambda_1 \odot x_1) \oplus (\lambda_2 \odot x_2), y]_f = \lambda_1 \odot [x_1, y]_f \oplus \lambda_2 \odot [x_2, y]_f,$$

$$[x, (\lambda_1 \odot y_1) \oplus (\lambda_2 \odot y_2)]_f = \lambda_1 \odot [x, y_1]_f \oplus \lambda_2 \odot [x, y_2]_f,$$

for all $x, x_1, x_2, y, y_1, y_2 \in L$ and $\lambda_1, \lambda_2 \in F$, where $[\mu, \nu]_f(t) = \bigvee_{a,b \in L} (\mu(a) \wedge [a, b]_f(t) \wedge \nu(b))$, for all $\mu, \nu \in \mathcal{F}^*(L)$ and $t \in L$.

**Example 3.2.** In a Lie algebra $(L, +, \cdot, \lfloor -, - \rfloor)$ over a field $(\mathbb{F}, +, \cdot)$, define $x \oplus y = \chi_{\{x+y\}}$, $\lambda \odot x = \chi_{\{\lambda \cdot x\}}$ and with $\lambda_1 \boxplus \lambda_2 = \chi_{\{\lambda_1 + \lambda_2\}}$, $\lambda_1 \boxdot \lambda_2 = \chi_{\{\lambda_1 \cdot \lambda_2\}}$, and $[x, y]_f = \chi_{\{\lfloor x, y \rfloor\}}$ for all $x, y \in L$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{F}$. Then $(L, \oplus, \odot, [-,-]_f)$ is a fuzzy Lie hyperalgebra over a fuzzy hyperfield $(\mathbb{F}, \boxplus, \boxdot)$

According to [4] and [5], with every fuzzy hypermodule $(M, \oplus, \odot)$ over a fuzzy hyperring $(R, \boxplus, \boxdot)$, we can associate a hypermodule structure $(M, +, \cdot)$ over a hyperring $(R, \uplus, \circ)$, where

$$x + y = \{t \in M \mid (x \oplus y)(t) > 0\},$$
$$r \uplus s = \{u \in R \mid (r \boxplus s)(u) > 0\},$$
$$r \cdot x = \{z \in M \mid (r \odot x)(z) > 0\},$$
$$r \circ s = \{v \in R \mid (r \boxdot s)(v) > 0\},$$

for all $x, y \in M$ and $r, s \in R$. On the other hand, let $(M, +, \cdot)$ be a hypermodule over hyperring $(R, \uplus, \circ)$. Consider fuzzy hyperoperations $a \oplus b = \chi_{a+b}$, $r \odot a = \chi_{r \cdot a}$, $r \boxplus s = \chi_{r \uplus s}$ and $r \boxdot s = \chi_{r \circ s}$ for $a, b \in M$ and $r, s \in R$. Then $(M, \oplus, \odot)$ is a fuzzy hypermodule over the fuzzy hyperring $(R, \boxplus, \boxdot)$. Now, let $(L, +, \cdot, [-,-])$ be a Lie hyperalgebra over a hyperfield $(F, \uplus, \circ)$. Then $(L, \oplus, \odot, [-,-]_f)$ is a fuzzy Lie hypealgebra over the fuzzy hyperfield $(F, \boxplus, \boxdot)$, with $[x, y]_f = \chi_{[x,y]}$ for all $x, y \in L$. Also, if $(L, \oplus, \odot, [-,-]_f)$ is a fuzzy Lie hypealgebra over a fuzzy hyperfield $(F, \boxplus, \boxdot)$, then $(L, +, \cdot, [-,-])$ is a Lie hyperalgebra over the hyperfield $(F, \uplus, \circ)$ where $[x, y] = \{t \in L \mid [x, y]_f(t) > 0\}$ for all $x, y \in L$.

## 4. Fuzzy strongly regular relations on fuzzy Lie hyperalgebras

Let $(L, \oplus, \odot)$ be a fuzzy Lie hyperalgebra over a fuzzy hyperfield $(F, \boxplus, \boxdot)$. For all $x, y \in L$, we define

$$x \mathcal{L}_f y \text{ if and only if } \bigoplus_{i=1}^{n} \ell_i(x) > 0 \text{ and } \bigoplus_{i=1}^{n} \ell_i(y) > 0,$$

where $\ell_i = f_i(g_{i1}, g_{i2}, \dots, g_{im_i})$ with $g_{ij} = \Big( \boxplus_{k=1}^{p_{ij}} \big( \boxdot_{r=1}^{q_{ijk}} \lambda_{ijkr} \big) \Big) \odot h_{ij}$ such that $1 \le j \le m_i$, $h_{ij} \in L$, $\lambda_{ijkr} \in F$, and $f_i(g_{i1}, g_{i2}, \dots, g_{im_i})$ is an arbitrary composition of the elements $g_{i1}, g_{i2}, \dots, g_{im_i}$ only by fuzzy hyperbracket. Note that the fuzzy hyperbracket functions $f_1, \dots, f_n$ actually allow $\ell_1, \dots, \ell_n$ to be of any arbitrary form, for example

$$[[[g_{i1}, g_{i2}]_f, g_{i3}]_f, \dots, g_{im_i}]_f,$$

$$[[g_{i1}, g_{i2}]_f, [g_{i3}, g_{i4}]_f, \dots, , g_{im_i}]_f,$$

$$[g_{i1}, \dots, [g_{i(m_i-2)}, [g_{i(m_i-1)}, g_{im_i}]_f]_f]_f$$

or any other composition of these elements only by fuzzy hyperbracket. The need to use such functions is that the fuzzy hyperbracket is not associative, in general.
Clearly, the relation $\mathcal{L}$ is reflexive and symmetric. But, $\mathcal{L}$ is not transitive, in general. We show that the transitive closure of $\mathcal{L}_f$ by $\mathcal{L}_f^*$. Indeed $\mathcal{L}_f^* = \mathcal{L}_f \cup \mathcal{L}_f \circ \mathcal{L}_f \cup \mathcal{L}_f \circ \mathcal{L}_f \circ \mathcal{L}_f \cup ....$
We recall that a relation $\rho$ is a fuzzy strongly regular on a fuzzy hypergroup $(H, \bigoplus)$ if $a\rho b$ and $c\rho d$ for all $(a \bigoplus c)(x) > 0$ and $(b \bigoplus d)(y) > 0$ follow that $x\rho y$, for $a, b, c, d, x, y \in H$. We say that $\rho$ is a fuzzy strongly regular relation on a fuzzy Lie hyperalgebra $(L, \bigoplus, \bigodot)$ over a fuzzy hyperfield $(F, \boxplus, \boxdot)$ if

$(i)$ $a\rho b$ and $c\rho d$ for all $(a \oplus c)(x) > 0$ and $(b \oplus d)(y) > 0$ follow that $x\rho y$,

$(ii)$ $a\rho b$ and $\alpha\sigma\beta$ for all $(\alpha \odot a)(x) > 0$ and $(\beta \odot b)(y) > 0$ follow that $x\rho y$,

$(iii)$ $a\rho b$ and $c\rho d$ for all $[a, c]_f(x) > 0$ and $[b, d]_f(y) > 0$ follow that $x\rho y$,

for all $a, b, c, d, x, y \in L$ and $\alpha, \beta \in F$. By these concepts, we have:

**Theorem 4.1.** *$\mathcal{L}_f^*$ is a fuzzy strongly regular relation on a fuzzy Lie hyperalgebra L.*

Let $(R, \boxplus, \boxdot)$ be a fuzzy hyperring and $x, y \in R$. Consider the relation $\gamma$ on a fuzzy hyperring $R$ as follows [6]:

$$x \,\gamma\, y \iff \boxplus_{i=1}^{n}\Big(\boxdot_{j=1}^{k_i} x_{ij}\Big)(x) > 0 \text{ and } \boxplus_{i=1}^{n}\Big(\boxdot_{j=1}^{k_i} x_{ij}\Big)(y) > 0$$

where $x_{ij} \in R$, for all $1 \le i \le n$ and $1 \le j \le k_i$. Denote by $\gamma^*$, the transitive closure of $\gamma$ (see also [**?** ]). It is was shown that $\gamma^*$ is the smallest fuzzy strongly regular relation on a fuzzy hyperring $R$ such that $R/\gamma^*$ is a ordinary ring [6]. Hence, it can be seen that:

**Corollary 4.2.** *$\mathcal{L}_f^*$ is the fundamental relation on a fuzzy Lie hyperalgebra L over a fuzzy hyperfield F, that is the smallest equivalence relation on L such that $L/\mathcal{L}_f^*$ is a (fundamental) Lie algebra over the (fundamental) field $F/\gamma^*$.*

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# Regularity in Unit-Exchange Ideals

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**Abstract**

In [1], Khurana, Lam, and Wang introduced the notion of square stable range one for an element of a ring $R$, requiring that whenever $aR + bR = R$ for any $b \in R$, there exists some $x \in R$ such that $a^2 + bx$ is invertible. A ring $R$ is said to have square stable range one whenever every element of $R$ satisfies the square stable range one condition.
In this paper we define an ideal $I$ of a ring $R$ is unit square stable if $uaR + bR = R$ with $a \in I$, $b \in R$, $u \in U(R)$ implies that $ua^2 + by \in R$ is invertible for a $y \in R$. From this, we show that every ideal of a ring having unit square stable range one is unit square stable. Also a ring $R$ has unit square stable range one if and only if it is unit square stable as an ideal of itself. Let $I$ be an ideal of a commutative ring $R$. An ideal $I$ of a ring $R$ is an unit exchange ideal if for any $a \in I$ there exists an idempotent $e \in I$ and $x, y \in I$, $u \in U(R)$ such that $e = uax = ua + y - uay$. An ideal $I$ of $R$ is an unit exchange ideal if and only if for any $a \in I$ there exists an idempotent $e \in R$, $u \in U(R)$ such that $e \in uaR$ and $1 - e \in (1 - ua)R$. A ring $R$ is an unit exchange ring if it is unit exchange as an ideal of itself. In [1], it is proved that an exchange ring $R$ has square range one if and only if every regular element in $R$ is strongly regular. The aim of the present paper is to extend this result to unit exchange ideals of $R$. We shall prove that an unit exchange ideal $R$ of a ring $R$ is unit square stable if and only if for any $a \in I$, $u \in U(R)$, $ua^2 \in J(R)$ implies that $a \in J(R)$ if and only if every unit regular element in $I$ is strongly unit regular. This extends the main characterization of unit exchange rings of unit square stable range one. An ideal $I$ of a ring $R$ is unit regular if every element in $I$ is unit regular. Clearly, every unit regular ideals of a ring is an unit exchange ideals. We shall prove that a unit regular ideal $I$ of a ring $R$ is unit square stable if and only if $eRe$ is strongly unit regular for all idempotents $e \in I$, if and only if $uaR + bR = R$ with $a \in u + I$, $b \in R$, $u \in U(R)$ implies that $ua^2 + by \in U(R)$ for a $y \in R$.

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# An Extension of Invo-Clean Rings

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**Abstract**

In [1], Danchev defined a ring $R$ a Invo-clean ring if every element $a \in R$ admits a decomposition of the form $a = u + e$, where $u$ is an involution, and $e$ is an idempotent element. The purpose of this article is to present the concept of invo unit-clean rings, which generalize the class of invo-clean rings. In this paper we define a ring $R$ to be invo unit-clean if every element $a \in R$ can be represented $ua = v + e$, where $v$ is an involution, $e$ is an idempotent and $u \in U(R)$. Furthermore, if the involution $v$ and the idempotent $e$ satisfy the commutativity condition $ve = ev$, then $R$ is referred to as a strongly invo unit-clean ring. We establish that the property of being invo unit-clean is inherited by corner subrings; specifically, for any $e \in Id(R)$, the subring $eRe$ remains invo unit-clean provided that $R$ satisfies the invo unit-clean condition. As a consequence, the invo unit-clean property of the full matrix ring $M_n(R)$ for any $n \in \mathbb{N}$ necessarily implies the same property for the underlying ring $R$. We demonstrated that for any invo unit-clean ring $R$, the Jacobson radical $J(R)$ is necessarily nil, with its nilpotency index bounded by 3. Also we show that a ring $R$ with $char(R) = 2$ is characterized as strongly invo unit-clean if and only if it is strongly nil unit-clean, provided the nilpotency index does not exceed 2. Throughout this paper, we assume that $R$ denotes a unital associative ring. The proofs of the obtained results in this article imitate those from the articles [1].

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# Hamiltonian Paths and Edge Structure in Cayley Graphs of $R$-Hypermodules

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**Abstract**

The purpose of this note is to study the Cayley graph associated with an $R$-hypermodule. This graph serves as a powerful tool for examining the algebraic properties of $R$-hypermodules through the lens of graph theory. First, we associate a Cayley graph with each $R$-hypermodule and then we investigate several properties of this graph, such as the number of edges and the existence of Hamiltonian paths.

## 1. Introduction

Hyperstructure theory was introduced by Marty [11] at the 8th Congress of Scandinavian Mathematicians. Hyperstructures have found numerous applications in both pure and applied mathematics, including geometry, lattice theory, cryptography, automata theory, graph theory, hypergraphs, fuzzy sets, probability theory, and rough set theory (see [7]).

Graph theory, with its wide range of applications in areas such as computer science and network theory, provides a complementary framework for studying algebraic structures. By associating graphs with algebraic objects such as rings, groups, and polygroups, researchers have developed new approaches to understanding their structural properties ([1], [4], [5]). This interdisciplinary viewpoint is particularly valuable, since graph representations often reveal features that are not immediately apparent from purely algebraic methods.

In recent years, the investigation of algebraic hyperstructures via graph-theoretic techniques has attracted increasing attention. A notable example is the study of the relationship between hypergroups and hypergraphs, as discussed by Corsini ([6]).

The concept of the Cayley graph was introduced by Arthur Cayley in 1878 to describe abstract groups in terms of generating sets. Over the past several decades, the theory of Cayley graphs has developed into an important area of algebraic graph theory [10].

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Generalized Cayley graphs of semigroups were first introduced in [15], where several fundamental properties were established. Let $G$ be a semigroup and let $S$ be a nonempty subset of $G$. The Cayley graph $Cay(G,S)$ is defined as the graph with vertex set $G$ and edge set consisting of ordered pairs $(x,y)$, such that $sx = y$ for some $s \in S$ [9].
The aim of this note is to study Cayley graphs associated with $R$-hypermodule.

## 2. Preliminaries

In this section, we recall several basic definitions and notions that will be used throughout the paper.
According to [11], let $H$ be a nonempty set, and let $P^*(H)$ denote the collection of all nonempty subsets of $H$. A hyperoperation on $H$ is a mapping

$$+ : H \times H \to \mathcal{P}^*(H).$$

The pair $(H,+)$ is called a hypergroupoid. For nonempty subsets $C, D$ of $H$, we define

$$C + D = \bigcup \{x + y \mid x \in C, y \in D\}.$$

For $t \in H$ we write $t + C$ instead of $\{t\} + C$, and similarly $C + t$ instead of $C + \{t\}$.
A hypergroupoid $(H,+)$ is called a semihypergroup if the associative law

$$x + (y + z) = (x + y) + z$$

holds for all, $x, y, z \in H$. It is called commutative if

$$x + y = y + x, for all x, y \in H.$$

A semihypergroup $(H,+)$ is said to be a hypergroup if it satisfies the reproduction axiom, i.e., for all $x \in H$,

$$x + H = H + x = H.$$

According to [14], a hyperring $(R,+,\cdot)$ is a multivalued algebraic structure satisfying the following axioms:

(i) $(R,+)$ is a hypergroup;
(ii) $(R,\cdot)$ is a semihypergroup;
(iii) The hyperoperation $\cdot$ is distributive over $+$, that is for all $a, b, c \in R$,

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

According to [14], a left hypermodule over a hyperring $(R,+,\cdot)$ is a set $M$ equipped with a hyperoperation $+$ such that $(M,+)$ is a hypergroup, together with a mapping $\cdot : R \times M \to \mathcal{P}^*(M)$ satisfying the following conditions for all $r, s \in R$ and $m, n \in M$:

(i) $r.(m + n) = r.m + r.n$;
(ii) $(r + s).m = (r.m) + (s.m)$;
(iii) $(r \cdot s).m = r.(s.m)$.

A nonempty subset $A \subseteq M$ is called a subhypermodule of $M$ if $A$ itself satisfies the axioms of a hypermodule under the induced hyperoperations.
Throughout this paer, $R$ denotes a commutative hyperring, and $M$ denotes a left $R$-hypermodule.

**Definition 2.1.** An $R$-hypermodule $M$ is called trivial if for every $x \in M$ and $r \in R$

$$|r \cdot x| = 1.$$

Let $M$ and $N$ be $R$-hypermodules. A mapping $f : M \to N$ is called an $R$-hypermodule homomorphism if

$$f(x + y) = f(x) + f(y) \text{ and } f(r.x) = r.f(x),$$

for all $r \in R$ and $x, y \in M$, where

$$f(a+b) = \cup\{f(z) : z \in a+b\} \text{ and } f(r \cdot a) = \cup\{f(z) : z \in r \cdot a\}.$$

The $R$-hypermodule homomorphism $f$ is called isomorphism if $f$ is both injective and surjective. In this case, we write $M \cong N$.

**Definition 2.2.** A linear combination of a family $(x_i : i \in I)$ of elements of $M$ is an expression of the form $\sum r_i.x_i$, where $r_i \in R$ and only finitely many coefficient are nonzero.

**Definition 2.3.** A subset $X \subseteq M$ is said to generate $M$ if every element of $M$ can be expressed as a linear combination of elements from $X$.

**Definition 2.4.** An $R$-hypermodule $M$ generated by a single element is called a cyclic $R$-hypermodule.

Einally, we recall some basic notions from graph theory.
A graph $X$ consists of a vertex set $V(X)$ and an edge set $E(X)$.
A graph is called connected, complete, Eulerian or Hamiltonian if it satisfied the standard definitions.

## 3. A Cayley Graph Associated with an $R$-Hypermodule

In this section, we introduce the notion of associated with $R$-hypermodules Cayley graphs, also we state some related results of these graphs.
Let $(M,+)$ be an $R$-hypermodule, let $A \subseteq M$ be a nonempty subset, and let $r \in R$. We define the Cayley graph $Cay_M^r(M,A)$ as the graph whose vertex set $Cay_M^r(M,A)$ are the elements of $M$, where two distinct vertices $x$ and $y$ are adjacent if there exists an element $a \in A$ such that $\{x,y\} \subseteq ra$. In this case, we write $xEy$.
if $A = \{a\}$ is a singleton, we simply write $Cay_M^r(M,a)$.

**Example 3.1.** Let $M = R = \{0,1\}$. According to [13], the structure $(R,+,\cdot)$ forms a hyperring, and $(M,+')$ is a hypergroup (see [8]). Moreover, it is straightforward to verify that $(M,+',\cdot)$ is an $R$-hypermodule, where the hyperoperations are defined by the following tables:

| $+$ | $0$ | $1$ |
|---|---|---|
| $0$ | $0$ | $1$ |
| $1$ | $1$ | $\{0,1\}$ |

| $\cdot$ | $0$ | $1$ |
|---|---|---|
| $0$ | $0$ | $0$ |
| $1$ | $0$ | $\{0,1\}$ |

| $+'$ | $0$ | $1$ |
|---|---|---|
| $0$ | $0$ | $\{0,1\}$ |
| $1$ | $\{0,1\}$ | $1$ |

The Cayley graph $Cay_M^1(M,0)$ is as follows:



Similarly, The Cayley graph $Cay_M^1(M,1)$ is illustrated as follws:



**Lemma 3.2.** *Let $M$ be an $R$-hypermodule and let $Cay_M^r(M,a)$ be it's associated Cayley graph. If $uEv, uEv'$, then $vEv'$.*

**Lemma 3.3.** *Let $M$ be an $R$-hypermodule. If $Cay_M^r(M,a)$ is a Hamiltonian graph, then $M$ is cyclic $R$-hypermodule generated by $a$.*

**Theorem 3.4.** *Let $M$ be an $R$-hypermodule with $|M| \geqslant 2$. If $Cay_M^r(M,A)$ is a complete graph, then $M = r.A$.*

**Theorem 3.5.** *Let $M$ and $N$ be isomorphic $R$-hypermodules. Then the graphs $Cay_M^r(M,A)$ and $Cay_N^r(N,f(A))$ are isomorphic, where $f : M \to N$ is an $R$-hypermodule isomorphism.*

**Theorem 3.6.** *Let $M$ be an $R$-hypermodule and, let $A$ be a sub-hypermodule of $M$ and let $a \in A$. Then there exists a graph homomorphism from $Cay_M^r(A,a)$ to $Cay_M^r(M,A)$.*

**Theorem 3.7.** *Let $M$ be a finite $R$-hypermodule and $a \in M$ with $|r \cdot a| = n$. Then the number of edges of the Cayley graph $Cay_M^r(M,a)$ is given by*

$$|E(Cay_M^r(M,a))| = \frac{n^2+n}{2}.$$

**Example 3.8.** Let $M = R = \{a,b,c,d,e,f,g\}$. According to [12], the structure $(R,+,\cdot)$ forms a hyperring and $(M,+,\cdot)$ is an $R$-hypermodule (see [3]), where the hyperoperation $+$ is given by the following table:

| $+$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ |
|---|---|---|---|---|---|---|---|
| $a$ | $\{a,b\}$ | $\{a,b\}$ | $c$ | $d$ | $e$ | $f$ | $g$ |
| $b$ | $\{a,b\}$ | $\{a,b\}$ | $c$ | $d$ | $e$ | $f$ | $g$ |
| $c$ | $c$ | $c$ | $\{a,b\}$ | $f$ | $g$ | $d$ | $e$ |
| $d$ | $d$ | $d$ | $g$ | $\{a,b\}$ | $f$ | $e$ | $c$ |
| $e$ | $e$ | $e$ | $f$ | $g$ | $\{a,b\}$ | $c$ | $d$ |
| $f$ | $f$ | $f$ | $e$ | $c$ | $e$ | $g$ | $\{a,b\}$ |
| $g$ | $g$ | $g$ | $d$ | $e$ | $c$ | $\{a,b\}$ | $f$ |

Moreover, $x \cdot y = \{a,b\}$ for all $x,y \in R$.
The associated Cayley graph $Cay_M^d(M,d)$ is as follows:

To compute the number of edges of $Cay_M^d(M,d)$, note that $d \cdot d = \{a,b\}$. Starting from a vertex in $d \cdot d$, say $a$, the vertex $a$ is adjacent only to $b$ i.e. it has a choice $(2-1=1)$. The vertex $b$ produces no additional new edges. Hance, the number of edges ignoring loops is $\frac{2(2-1)}{2} = 1$. By adding the number of members of the $d \cdot d$ (i.e. 2) we have 3 edges.

**Corollary 3.9.** *Let $M$ be a finite $R$-hypermodule and $A \subseteq M$, and let $Cay_M^r(M,A)$ be the Cayley graph of $M$. Then*

*(i) If $|r \cdot a_i \cap r \cdot a_j| = 0$, then $|E(Cay_M^r(M,A))| = \sum_{a\in A} |Cey_M^r(M,a)|$.*
*(ii) If $|r \cdot a_i \cap r \cdot a_j| = 1$, then $|E(Cay_M^r(M,A))| = \sum_{a\in A} |Cey_M^r(M,a)| - 1$.*
*(iii) If $|r \cdot a_i \cap r \cdot a_j| = m$ with $m \geq 2$, then*

$$|E(Cay_M^r(M,A))| = \sum_{a\in A} |Cey_M^r(M,a)| - (\frac{m^2+m}{2}).$$

**Lemma 3.10.** *Let $M$ be a trivial $R$-hypermodule and let $A \subseteq M$. Then $Cay_M^r(M,A)$ is null graph that has loops only in member of $r \cdot a$, where $a \in A$.*

**Example 3.11.** Consider $R = \{0,1,2\}$. According to [13], the structur forms a hyperring. Moreover, is an $R$-hypermodule (see [3]), defined as follows:

| $+$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 1 | $R$ |
| 2 | 2 | $R$ | 2 |

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 |
| 2 | 0 | 2 | 1 |

The Cayley graph $Cay_R^1(R,\{0,2\})$ is as follows:



**Theorem 3.12.** *Let $M$ be an $R$-hypermodule and let $|M| = n$. If there exists $a \in A$ such that $r \cdot a = M$, then*

*(i) $E(Cay_M^r(M,A)) = 2n$.*
*(ii) if $n$ is odd, then the $Cay_M^r(M,A)$ is Eulerian.*
*(iii) if $n \geq 3$, then the $Cay_M^r(M,A)$ is Hamiltonian.*

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# Counting the Number of Distinct Intuitionistic Fuzzy Subgroups of the Cyclic Groups $Z_{p^n qr}$

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**Abstract**

This paper provides a complete solution for counting distinct intuitionistic fuzzy subgroups of finite cyclic groups of the form $Z_{p^n qr}$, where $p, q, r$ are distinct primes. Using the subgroup lattice method and the equivalence relation of Neeraj and Sharma, we derive a recurrence relation for the counting function $f(n) = r(Z_{p^n qr})$. A rigorous mathematical induction proof establishes the closed-form formula:

$$r(Z_{p^n qr}) = \frac{(36n^2 + 144n + 112) \cdot 4^n - 1}{3}, \quad n \geq 0.$$

The formula is verified through explicit computations for $n = 0, 1, 2$ and validated for higher values through the recurrence.

## 1. Introduction

The enumeration of intuitionistic fuzzy subgroups of finite groups represents a significant problem in fuzzy algebra. Building upon the foundational work of Atanassov [1] and Hur et al. [2], and the equivalence relations introduced by Murali and Makamba [3], Neeraj and Sharma [5] developed combinatorial methods for counting such subgroups. In our previous work [4], we presented a lattice-theoretic approach that unified the counting process for both Abelian and non-Abelian groups.

This paper focuses specifically on the cyclic groups $Z_{p^n qr}$. Our main contribution is a rigorous inductive proof of a closed-form formula for $r(Z_{p^n qr})$, extending known results for simpler cyclic groups and providing a complete solution for the three-prime case.

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## 2. Preliminaries

### 2.1. Intuitionistic fuzzy subgroups

Let $G$ be a group. An **intuitionistic fuzzy subgroup** $A = (\mu_A, \nu_A)$ of $G$ satisfies for all $x, y \in G$:

$$\mu_A(xy) \geq \min(\mu_A(x), \mu_A(y)), \quad \nu_A(xy) \leq \max(\nu_A(x), \nu_A(y)),$$

and

$$\mu_A(x^{-1}) \geq \mu_A(x), \quad \nu_A(x^{-1}) \leq \nu_A(x),$$

with $\mu_A(x) + \nu_A(x) \leq 1$.

### 2.2. Equivalence relation and notation

Two intuitionistic fuzzy subgroups $A$ and $B$ are equivalent ($A \sim B$) if they induce identical orderings of membership-nonmembership pairs [5]. We denote by $r(G)$ the number of distinct (non-equivalent) intuitionistic fuzzy subgroups of $G$.

### 2.3. Known counting formulas [4]

For distinct primes $p, q$:

$$r(Z_{p^k}) = \frac{4^{k+1} - 1}{3},$$
$$r(Z_{p^k q}) = (k+1)4^{k+1} + \frac{4^{k+1} - 1}{3}.$$

From these, we have $r(Z_p) = 5$, $r(Z_{pq}) = 37$, $r(Z_{p^2}) = 21$, and $r(Z_{p^2 q}) = 213$.

## 3. Main Results

In this section, we perform explicit computations of some formulas for specific cyclic groups.

**Example 3.1.** Suppose that $G = Z_{pqr}$. Then the Proper subgroups of $G$ are: $\{e\}$, $Z_p$, $Z_q$, $Z_r$, $Z_{pq}$, $Z_{pr}$, $Z_{qr}$.
We compute $r'(H) = \frac{3r(H)+1}{4}$:

$$r'(Z_p) = 4,\ r'(Z_q) = 4,\ r'(Z_r) = 4,$$
$$r'(Z_{pq}) = 28,\ r'(Z_{pr}) = 28,\ r'(Z_{qr}) = 28,\ r'(\{e\}) = 1.$$

we have $4 + 4 + 4 + 28 + 28 + 28 + 1 = 97$ and so $r''(G) = 97$ and $r(Z_{pqr}) = 4 \cdot 97 + 1 = 389$.

**Example 3.2.** Suppose that $G = Z_{p^2 qr}$. Then the Proper subgroups of $G$ are: $\{e\}$, $Z_p$, $Z_{p^2}$, $Z_q$, $Z_r$, $Z_{pq}$, $Z_{pr}$, $Z_{p^2 q}$, $Z_{p^2 r}$, $Z_{qr}$, $Z_{pq}$
Then

$$r'(Z_p) = 4,\ r'(Z_{p^2}) = 16,\ r'(Z_q) = 4,\ r'(Z_r) = 4,$$
$$r'(Z_{pq}) = 28,\ r'(Z_{pr}) = 28,\ r'(Z_{qr}) = 28,$$
$$r'(Z_{p^2 q}) = 160,\ r'(Z_{p^2 r}) = 160,\ r'(Z_{pqr}) = 292,\ r'(\{e\}) = 1.$$

Thus $r''(G) = 725$ and $r(Z_{p^2 qr}) = 4 \cdot 725 + 1 = 2901$.

**Example 3.3.** In [4], the following cases are computed, and we introduce the following notation for later use. For $n \geq 0$:

$$h(n) := r(Z_{p^n}) = \frac{4^{n+1} - 1}{3},$$
$$g(n) := r(Z_{p^n q}) = (n+1)4^{n+1} + h(n),$$
$$f(n) := r(Z_{p^n qr}).$$

### *3.1. Generating the Proper Subgroups of $\mathbb{Z}_{p^n qr}$*

Since $\mathbb{Z}_n$ is cyclic, for each divisor $d \mid n$ there exists a unique subgroup of order $d$, namely $\langle n/d \rangle$. We generate all proper subgroups of $\mathbb{Z}_{p^n qr}$ as follows.

**Algorithm 1:** Proper subgroups of $\mathbb{Z}_{p^n qr}$ via divisors

**Input:** Distinct primes $p, q, r$ and integer $n \geq 0$
**Output:** A list $\mathcal{S}$ of tuples $(d, g, H)$ where $d = |H|$, $g$ is a generator, and $H < \mathbb{Z}_{p^n qr}$
$N \leftarrow p^n qr$;
$\mathcal{S} \leftarrow \emptyset$;

**(A) Subgroups of type $\mathbb{Z}_{p^i}$ for $0 \leq i \leq n$ for $i \leftarrow 0$ to $n$ do**
  $d \leftarrow p^i$;
  $g \leftarrow N/d$;
  $H \leftarrow \langle g \rangle \leq \mathbb{Z}_N$;
  **if** $d < N$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{(d, g, H)\}$;

**(B) Subgroups of type $\mathbb{Z}_{p^i q}$ and $\mathbb{Z}_{p^i r}$ for $0 \leq i \leq n$ for $i \leftarrow 0$ to $n$ do**
  // $\mathbb{Z}_{p^i q}$
  $d \leftarrow p^i q$;
  $g \leftarrow N/d$;
  $H \leftarrow \langle g \rangle \leq \mathbb{Z}_N$;
  **if** $d < N$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{(d, g, H)\}$;
  // $\mathbb{Z}_{p^i r}$
  $d \leftarrow p^i r$;
  $g \leftarrow N/d$;
  $H \leftarrow \langle g \rangle \leq \mathbb{Z}_N$;
  **if** $d < N$ **then**
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{(d, g, H)\}$;

**(C) Proper subgroups of type $\mathbb{Z}_{p^i qr}$ for $0 \leq i \leq n-1$ for $i \leftarrow 0$ to $n-1$ do**
  $d \leftarrow p^i qr$;
  $g \leftarrow N/d$;
  $H \leftarrow \langle g \rangle \leq \mathbb{Z}_N$;
  $\mathcal{S} \leftarrow \mathcal{S} \cup \{(d, g, H)\}$;

**return** $\mathcal{S}$;

**Example 3.4.** For $p = 2, q = 3, r = 5, n = 2$, we have $N = 60$. Algorithm 1 yields:

- Type A: $|\{e\}| = 1$, $|Z_2| = 2$, $|Z_4| = 4$
- Type B: $|Z_3| = 3$, $|Z_5| = 5$, $|Z_6| = 6$, $|Z_{10}| = 10$, $|Z_{12}| = 12$, $|Z_{20}| = 20$
- Type C: $|Z_{15}| = 15$, $|Z_{30}| = 30$

Total: $3 + 6 + 2 = 11$ proper subgroups, matching the lattice structure.

The systematic enumeration provided by Algorithm 1 directly justifies the recurrence relation in Theorem 4.2. Each subgroup type contributes a term to $r''(G) = \sum_{H<G} r'(H)$, where the classification (A), (B), (C) corresponds to the three summation components in (1).

**Theorem 3.5** (Recurrence Relation)**.** *The function $f(n)$ satisfies:*

$$f(n) = 4f(n-1) + (6n+9)4^{n+1} + 1, \quad n \geq 1, \tag{1}$$

*with initial condition $f(0) = r(Z_{qr}) = 37$.*

*Proof.* The proper subgroups of $G = Z_{p^n qr}$ are:

- $Z_{p^i}$ for $i = 0, \dots, n$
- $Z_{p^i q}$ and $Z_{p^i r}$ for $i = 0, \dots, n$
- $Z_{p^i qr}$ for $i = 0, \dots, n-1$

From Theorem 4.3 of [4]:

$$r''(G) = \frac{r(G)-1}{4} = 1 + \sum_{H<G} r'(H), \quad r'(H) = \frac{3r(H)+1}{4}.$$

Substituting and simplifying yields (1). □

**Theorem 3.6.** *For all integers $n \geq 0$,*

$$r(Z_{p^n qr}) = \frac{(36n^2 + 144n + 112)\cdot 4^n - 1}{3}. \tag{2}$$

*Proof.* We prove (2) by mathematical induction on $n$.
**Base Cases:**

- For $n = 0$:
$$\frac{(36\cdot 0^2 + 144\cdot 0 + 112)\cdot 4^0 - 1}{3} = \frac{112\cdot 1 - 1}{3} = \frac{111}{3} = 37 = f(0).$$
- For $n = 1$:
$$\frac{(36\cdot 1^2 + 144\cdot 1 + 112)\cdot 4^1 - 1}{3} = \frac{(36+144+112)\cdot 4 - 1}{3} = \frac{292\cdot 4 - 1}{3} = \frac{1168-1}{3} = 389 = f(1).$$
- For $n = 2$:
$$\frac{(36\cdot 4 + 144\cdot 2 + 112)\cdot 16 - 1}{3} = \frac{(144+288+112)\cdot 16 - 1}{3} = \frac{544\cdot 16 - 1}{3} = \frac{8704-1}{3} = 2901 = f(2).$$

**Inductive Hypothesis:** Assume that for some integer $k \geq 2$, the formula holds:

$$f(k) = \frac{(36k^2 + 144k + 112)\cdot 4^k - 1}{3}.$$

**Inductive Step:** We must prove that

$$f(k+1) = \frac{[36(k+1)^2 + 144(k+1) + 112]\cdot 4^{k+1} - 1}{3}.$$

From the recurrence relation (1):

$$f(k+1) = 4f(k) + [6(k+1)+9]\cdot 4^{(k+1)+1} + 1.$$

Substituting the inductive hypothesis:

$$f(k+1) = 4 \cdot \frac{(36k^2 + 144k + 112) \cdot 4^k - 1}{3} + (6k + 15) \cdot 4^{k+2} + 1$$
$$= \frac{(36k^2 + 144k + 112) \cdot 4^{k+1} - 4}{3} + (6k + 15) \cdot 4^{k+2} + 1.$$

Combine terms over common denominator 3:

$$f(k+1) = \frac{(36k^2 + 144k + 112) \cdot 4^{k+1} - 4 + 3(6k + 15) \cdot 4^{k+2} + 3}{3}.$$

Simplify the numerator:

$$\begin{aligned} \text{Numerator} &= (36k^2 + 144k + 112) \cdot 4^{k+1} + 3(6k + 15) \cdot 4^{k+2} - 1 \\ &= 4^{k+1}\left[36k^2 + 144k + 112 + 3(6k + 15) \cdot 4\right] - 1 \\ &= 4^{k+1}\left[36k^2 + 144k + 112 + 12(6k + 15)\right] - 1 \\ &= 4^{k+1}\left[36k^2 + 144k + 112 + 72k + 180\right] - 1 \\ &= 4^{k+1}\left[36k^2 + 216k + 292\right] - 1. \end{aligned}$$

Now compute $36(k+1)^2 + 144(k+1) + 112$:

$$\begin{aligned} 36(k+1)^2 + 144(k+1) + 112 &= 36(k^2 + 2k + 1) + 144k + 144 + 112 \\ &= 36k^2 + 72k + 36 + 144k + 256 \\ &= 36k^2 + 216k + 292. \end{aligned}$$

Thus:

$$f(k+1) = \frac{[36(k+1)^2 + 144(k+1) + 112] \cdot 4^{k+1} - 1}{3}.$$

This completes the inductive step. By the principle of mathematical induction, formula (2) holds for all integers $n \geq 0$. □

## 4. Conclusion

This paper has established a complete solution for counting distinct intuitionistic fuzzy subgroups of the cyclic groups $Z_{p^n qr}$. Through rigorous mathematical induction, we have proved the closed-form formula:

$$r(Z_{p^n qr}) = \frac{(36n^2 + 144n + 112) \cdot 4^n - 1}{3}, \quad n \geq 0.$$

The methodology follows a systematic approach: analyzing the subgroup lattice, applying the counting theorem to derive recurrence relations, and constructing an inductive proof for the closed-form solution.

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# On Computation of Branching Rule $A_4 \longrightarrow C_2$ by Symmetry Classes of Tensors

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**Abstract**

In this letter we introduce a method for branching rule $A_4 \longrightarrow C_2$ of classical Lie algebras via symmetry classes of tensors.

## 1. Introduction

Lie algebras are an integral part of modern mathematical physics. Their representation theory govern every field of physics from fundamental sructure particles to the state of quantum computer. As yet, several branching rules have been calculated by generating functions, but these methods are failed in larger dimensions. Notably in [4], results are printed only for dimension less than 5000. We strongly emphsize that in our method there is no limit to dimension and we provide an example for larger dimension. But to make readers interested in subject it is better to begin with definition. Let $\phi$ be an irreducible representation of an algebra $\mathfrak{U}$ and $\mathfrak{U}'$ is subalgebra of $\mathfrak{U}$. Naturally, we can restrict $\phi$ to $\mathfrak{U}'$ will be reducible representation. Now we can decompose this reducible representation into irreducible representation, then the numbers which appears as irreducible representation multiplicities are our intended branching rules. Introducing final formula needs background in symmetry classes of tensors and computing Kostka numbers, that we recommand readers to study [5]. In this article, we provide an exact formula to calculate branching rule $A_4 \longrightarrow C_2$.

## 2. Preliminaries

We recall basic facts and definitions about Lie algebras especially $\mathfrak{sl}_5(\mathbb{C})$ and $\mathfrak{sp}_4(\mathbb{C})$. $\mathfrak{sl}_5(\mathbb{C})$ is the classical Lie algebra of type $A_4$ that defines by the set $\mathfrak{sl}_5(\mathbb{C}) = \{x \in M_{5\times5}(\mathbb{C}) : tr(x) = 0\}$. Also Symplectic Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ is

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denoted by $C_2$, defined by $\mathfrak{sp}_4(\mathbb{C}) = \{x \in M_{4\times4}(\mathbb{C}) : sx = -x^T s\}$, where $s = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. Let $V = \mathbb{C}^5$ and $G$ be a subgroup of $S_m$. For any $\sigma \in G$, we define the permutation operator $P_\sigma : V^{\otimes m} \longrightarrow V^{\otimes m}$ by $P_\sigma(v_1 \otimes ... \otimes v_m) = v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(m)}$. If $\chi$ is a complex irreducible character of $G$, define the symmetrizer $S_\chi : V^{\otimes m} \longrightarrow V^{\otimes m}$ by $S_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma\in G} \chi(\sigma) P_\sigma$. The symmetry classes of tensors associated with $G$ and $\chi$ is denoted by $V_\chi(G)$ that equals to $S_\chi(V^{\otimes m})$. For any vectors $v_1, ..., v_m$, we define the decomposable symmetrized tensors $v_1 * ... * v_m = S_\chi(v_1 \otimes ... \otimes v_m)$. Let $\{e_1, ..., e_5\}$ be a basis of $V$. We will denote by $e^*_\alpha$ the $e_{\alpha_1} * ... * e_{\alpha_m}$. It is clear that $\{e^*_\alpha | \ \alpha \in \Gamma_{m,5}\}$ is a generator set for $V_\chi(G)$, where $\Gamma_{m,5} = \{\alpha = (\alpha_1, ..., \alpha_m) | \ \alpha_i \in \mathbb{Z}, 1 \le \alpha_i \le 5\}$.

**Remark 2.1.** $G_{m,5}$ denotes the set of increasing elements of $\Gamma_{m,5}$. For $\alpha \in \Gamma_{m,5}$, let $m(\alpha) = (m_1(\alpha), ..., m_5(\alpha))$, where $m_i(\alpha)$ is equal to the number of repetitions of $i$ in $\alpha$. Also $M(\alpha)$ is the multiplicity partition of $\alpha$ that obtain by rearrange entries of $m(\alpha)$.

To obtain a basis for $V_\chi(G)$, we first define an action of $G$ on $\Gamma_{m,5}$ by

$$\alpha^\sigma = (\alpha_{\sigma^{-1}(1)}, ..., \alpha_{\sigma^{-1}(m)}); \ \sigma \in G, \ \alpha \in \Gamma_{m,5}.$$

Let $\Delta$ be a set of representatives of orbits of this action. The stablizer subgroup of $\alpha \in \Gamma_{m,5}$ is denoted by $G_\alpha$. Define $\Omega = \{\alpha \in \Gamma_{m,5} | \ [\chi, 1_{G_\alpha}] \neq 0\}$ where $[,]$ denotes the inner product of characters (see [2]). It is proved that $e^*_\alpha \neq 0$ if and only if $\alpha \in \Omega$. If $\bar{\Delta} = \Delta \cap \Omega$ then

$$V_\chi(G) = \bigoplus_{\alpha\in\bar{\Delta}} V^*_\alpha$$

where $V^*_\alpha = < e^*_{\alpha^\sigma} | \ \sigma \in G >$ is the cyclic subspace. Let us denote by $s_\alpha$ the $dim\ V^*_\alpha$. We have

$$s_\alpha = \chi(1)[\chi, 1_{G_\alpha}].$$

For $\alpha \in \bar{\Delta}$, suppose $\{e^*_{\alpha^{\sigma_1}}, ..., e^*_{\alpha^{\sigma_t}}\}$ is a basis of $V^*_\alpha$ with $\sigma_1 = 1$. Let $A_\alpha = \{\alpha^{\sigma_1}, ..., \alpha^{\sigma_t}\}$ and define $\hat{\Delta} = \bigcup_{\alpha\in\bar{\Delta}} A_\alpha$. Clearly, $\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega$ and the set $\{e^*_\alpha | \ \alpha \in \hat{\Delta}\}$ is a basis of $V_\chi(G)$. We know that every irreducible character of $S_m$ is of the form $\chi_\pi$, where $\pi = [\pi_1, ..., \pi_r]$ is a partition of $m$. It means that $\pi_1 \ge ... \ge \pi_r \ge 0$ and $\pi_1 + ... + \pi_r = m$. We will denote by $V_\pi(S_m)$ the symmetry classes of tensors associated with $S_m$ and $\chi_\pi$. Naturally, $V$ is an $\mathfrak{sl}_5(\mathbb{C})-$module and $V^{\otimes m}$ is an $\mathfrak{sl}_5(\mathbb{C})-$module by

$$x \bullet (v_1 \otimes v_2 \otimes ... \otimes v_m) = \sum_{i=1}^{m} v_1 \otimes ... \otimes (x \bullet v_i) \otimes ... \otimes v_m.$$

Similarly, if we define

$$x \bullet (v_1 * v_2 * ... * v_m) = \sum_{i=1}^{m} v_1 * ... * (x \bullet v_i) * ... * v_m,$$

then $V_\chi(G)$ becomes $\mathfrak{sl}_5(\mathbb{C})-$module. Now, define the embedding map $i : \mathfrak{sp}_4(\mathbb{C}) \longrightarrow \mathfrak{sl}_5(\mathbb{C})$ by

$$i(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $L = i(\mathfrak{sp}_4(\mathbb{C}))$. Then $L$ is a subalgebra of $\mathfrak{sl}_5(\mathbb{C})$ that isomorph to $\mathfrak{sp}_4(\mathbb{C})$ and $V$ is an $L$-module.

## 3. Main idea

Let $\Lambda = a_1\Lambda_1 + a_2\Lambda_2 + a_3\Lambda_3 + a_4\Lambda_4$ be an arbitary integral dominant weight of $\mathfrak{sl}_5(\mathbb{C})$, where $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are fundamental weights of $\mathfrak{sl}_5(\mathbb{C})$. Let $V(\Lambda)$ be the corresponding irreducible $\mathfrak{sl}_5(\mathbb{C})-$module with the highest weight $\Lambda$.

For any other integral dominant weight $\mu$, we denote by $m(\Lambda,\mu)$ the multiplicity of $\mu$ in $\Lambda$. To compute $m(\Lambda,\mu)$, we use the Kostant's weight multiplicity formula [1],

$$m(\Lambda,\mu)=\sum_{\sigma\in\mathcal{W}}(-1)^{\ell(\sigma)}\mathfrak{P}(\sigma(\Lambda+\rho)-(\mu+\rho)),$$

where $\mathcal{W}$ is the Weyl group, $\ell(\sigma)$ = length of $\sigma$, $\rho$ = sum of fundamental weights and $\mathfrak{P}$ is the Kostant's partition function. Let $m=a_1+2a_2+3a_3+4a_4$ and $\pi=[\pi_1,\pi_2,\pi_3,\pi_4]$, where $\pi_1=a_1+a_2+a_3+a_4$, $\pi_2=a_2+a_3+a_4,\pi_3=a_3+a_4,\pi_4=a_4$ and $\pi$ is a partition of $m$. We can conclude from [3], that $V_\pi(S_m)\cong V(\Lambda)^{\chi_\pi(1)}$ as $\mathfrak{sl}_5(\mathbb{C})-$module, where $\chi_\pi$ is the irreducible character of $S_m$ corresponding to partition $\pi$. Now as we mention in priliminaries, by use of embedding $i$, we examine the $V_\pi(S_m)$ as $\mathfrak{sp}_4(\mathbb{C})-$module and obtain the branching rule $A_4\longrightarrow C_2$. We do this in several steps.

Step 1 (Dominant weights of $V_\pi(S_m)$). From [6] the set of dominant weights of $V_\pi(S_m)$ is equal to $\{\mu_\alpha \mid \alpha\in G_{m,5},\ M(\alpha)\trianglelefteq\pi\}$, where $\trianglelefteq$ is majorization ordering of partitions [5]. Let $T_\pi=\{p\mu_1+q\mu_2 \mid 1\le p\le\pi_1,\ 0\le q\le min\{p,\pi_1+\pi_2-p\}\}$. Clearly $\{\mu_\alpha \mid \alpha\in G_{m,5},\ M(\alpha)\trianglelefteq\pi\}\subseteq T_\pi$.

Step 2 (Dimension of weight spaces). Let $\alpha\in\Gamma_{m,5}$ and $\mu_\alpha=p\mu_1+q\mu_2$ and $m'=[\frac{1}{2}(m-p-q)]$, where [ ] is the integral part function. For each $0\le i\le m'$ and $0\le j\le m'-i$, suppose that $\alpha(i,j)$ be an element of $G_{m,5}$ such that $m_1(\alpha(i,j))=p+i,\ m_2(\alpha(i,j))=q+j,\ m_3(\alpha(i,j))=i,\ m_4(\alpha(i,j))=j,\ m_5(\alpha(i,j))=m-p-q-2i-2j$.

**Theorem 3.1.** *By above notation,*

$$dim(V_\pi(S_m))_{\mu_\alpha}=\chi_\pi(1)\sum_{i=0}^{m'}\sum_{j=0}^{m'-i}K_{\pi,M(\alpha(i,j))},$$

*where $K_{\pi,M(\alpha(i,j))}$ is the Kostka number associated with the partitions $\pi$ and $M(\alpha(i,j))$ of $m$.*

*Proof.* Let $R_\alpha=\{\beta\in G_{m,5} \mid \mu_\alpha=\mu_\beta\}$. We have

$$dim(V_\pi(S_m))_{\mu_\alpha}=\sum_{\beta\in R_\alpha}s_\beta,$$

where $s_\beta$ is the dimension of the cyclic subspace $V_\beta^*$. We know that $s_\beta=\chi_\pi(1)K_{\pi,M(\beta)}$. It is easy to check that $R_\alpha=\{\alpha(i,j) \mid 0\le i\le m',\ 0\le j\le m'-i\}$, which establishes the formula. □

Step 3 (Decomposition of $V_\pi(S_m)$). Let $\{\gamma_1,\gamma_2,\dots,\gamma_k\}$ be the set of dominant weights of $V_\pi(S_m)$ ordered such that for any $1\le i\le k$, $\gamma_i$ is not larger than $\gamma_1,\dots,\gamma_{i-1}$. Let $c_i=dim\ (V_\pi(S_m))_{\gamma_i}$ and $m_{ij}=m(\gamma_j,\gamma_i)$. Define $n_i$ by $n_1=c_1=\chi_\pi(1)$ and

$$n_i=c_i-\sum_{j=1}^{i-1}m_{ij}n_j.$$

Then we have

$$V_\pi(S_m)=\bigoplus_{i=1}^{k}V(\gamma_i)^{n_i}.$$

Now, we obtain the branching rule

$$V(\Lambda)\downarrow_{\mathfrak{sp}_4(\mathbb{C})}=\bigoplus_{i=1}^{k}V(\gamma_i)^{n_i/\chi_\pi(1)}.$$

**Example 3.2.** Let $\Lambda=(1,1,0,2)$ be a dominant weight of $\mathfrak{sl}_5(\mathbb{C})$ then $dim\ V(\Lambda)=480$, $m=11$, $\pi=[4,3,2,2]$ and its branching rule is as follows:

$$(1,1,0,2)\to(2,3)\oplus(4,1)\oplus(2,2)\oplus(0,3)\oplus(4,0)\oplus 2(2,1)\oplus(0,1)$$

**Example 3.3.** Let $\Lambda = (1,0,1,6)$ be a dominant weight of $\mathfrak{sl}_5(\mathbb{C})$ then $dim\ V(\Lambda) = 4725, m = 28, \pi = [8,7,7,6]$ and its branching rule is as follows:

$$\begin{aligned}(1,0,1,6) \rightarrow &(2,7) \oplus (4,5) \oplus (2,6) \oplus (0,7) \oplus 2(2,5) \oplus \\ &(4,3) \oplus (2,4) \oplus (0,5) \oplus 2(2,3) \oplus (4,1) \oplus \\ &(2,2) \oplus (0,3) \oplus 2(2,1) \oplus (2,0).\end{aligned}$$

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# Essential Concepts of Leibniz Algebras for Classification of Nilpotent Leibniz Algebras

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**Abstract**

Lie algebras and Leibniz algebras are closely related but have key differences. In a Lie algebra, the bracket $[x, y]$ is antisymmetric, meaning $[x, y] = -[y, x]$. In a Leibniz algebras, this condition is relaxed, so the bracket may not be antisymmetric. Leibniz algebras introduced by J. L. Loday (1993) are non-antisymmetric generalizations of Lie algebras. Investigations of nilpotent Leibniz algebras , show that many nilpotent properties of Lie algebras can be extended to the case of nilpotent Leibniz algebras.
In this paper we introduce essential definitions and notations the theory of Lie and Leibniz algebras that are used for classification of Leibniz algebras.

## 1. Introduction

Leibniz algebras was first introduced by Loday in [9] and [10] as a non-antisymmetric versions of Lie algebras. The classification problem of complex nilpotent Leibniz algebras was first studied by Loday. In [10] he gave a complete classification of complex nilpotent Leibniz algebras of dimension $n \leq 2$. Later Ayupov and Omirov classified $3-$dimensional complex nilpotent Leibniz algebras in [7]. Recently, Albeverio, Omirov and Rakhimov have obtained a classification of 4-dimensional complex nilpotent Leibniz algebras in [6].
One of the techniques to classify nilpotent Lie algebras was introduced by Skjelbred and Sund in [13]. Rakhimov and Langari used Skjelbred-Sund method in Leibniz algebras [8]. They also applied in [12] and [8] this technique to obtain the classification of complex nilpotent Leibniz algebras of dimension $n \leq 4$. Comparing the results of [8] and [11] with classification in [6] we realized that the Skjelbred-Sund method could be used to check the validity of the main result of [6]. In this paper we give the essential Concepts of Leibniz Algebras for Classification of Nilpotent Leibniz Algebras.

## 2. Basic definitions and knowledge

We begin with the definition of Lie algebras.

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**Definition 2.1.** A Lie algebra $L$ is a vector space over a field $F$ equipped with a bilinear map, $[\cdot,\cdot] : L \times L \to L$ which has the following properties :
1. $[x,x] = 0 \qquad \forall x \in L$
2. $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 \quad \forall x,y,z \in L.$

Condition (2) is known as the Jacobi identity. As the Lie bracket $[\cdot,\cdot]$ is bilinear, we have

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x].$$

Hence condition (1) implies

$$(1') \; [x,y] = -[y,x] \text{ for all } x,y \in L \quad (anti - symmetry).$$

If the field $F$ does not have characteristic 2, then putting $x = y$ in $(1')$ shows that $(1')$ implies condition (1).

**Example 2.2.** Any vector space $V$ has a Lie bracket defined by $[x,y] = 0$ for all $x,y \in V$. This is the abelian Lie algebra structure on $V$. In particular, the field $F$ may be regarded as a $1-$dimensional abelian Lie algebra.

**Definition 2.3.** Let $L$ be a Lie algebra. The center $C(L)$ of $L$ is defined as

$$C(L) = \{x \in L \,;\, [x,L] = 0\} = \{x \in L \,;\, [x,y] = 0, \; \forall \, y \in L\}.$$

Let $n$ be the dimension of algebra $L$. By taking a basis $\{e_1, e_2, ..., e_n\}$ in $L$, all elements $x$ and $y$ of $L$ can be represented as follows:
$x = \sum_{j=1}^{n} x^j e_j$ and $y = \sum_{k=1}^{n} y^k e_k$.
Notice that the product $x \cdot y$ of any two elements $x$ and $y$ in $L$, is completely determined by the product $e_j \cdot e_k$ of pairs of basis elements, that is:

$$x \cdot y = \sum_{j,k=1}^{n} x^j y^k e_j \cdot e_k.$$

Since $e_j \cdot e_k \in L$, the product $e_j \cdot e_k$ can also be expanded with respect to the basis $\{e_1, e_2, ..., e_n\}$ :

$$e_j \cdot e_k = \sum_{i=1}^{n} C_{jk}^{i} e_i \quad (j,k = 1, ..., n).$$

The numbers $C_{jk}^{i}$ are called the structure constants with respect to the basis $\{e_1, e_2, ..., e_n\}$. Therefore any algebra $L$ is completely determined by its structure constants.
Let $L$ be a Lie algebra. The structure constants of $L$ are the numbers $C_{ij}^{k}$ which are given by

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^{k} e_k.$$

As the basis is fixed, we can identify the anti-symmetry of commutator and the Jacobi identity with its structure constants. These constants satisfy the following conditions:

$$\begin{cases} C_{ij}^{k} = -C_{ji}^{k} \quad (1 \le i < j \le n, \;\; 1 \le k \le n) \\ \\ \sum_{l=1}^{n} (C_{ij}^{l} C_{lk}^{s} + C_{jk}^{l} C_{li}^{s} + C_{ki}^{l} C_{lj}^{s}) = 0 \quad (1 \le i < j < k \le n), \;\; 1 \le s \le n. \end{cases}$$

## 3. Central extensions of Leibniz algebras

**Definition 3.1.** A Leibniz algebra $L$ is a vector space over a field $F$ equipped with a bilinear map

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \text{for all } x,y,z \in L.$$

The first pure algebraic motivation of J.-L. Loday to introduce this class of algebras was the search of an "obstruction" to the periodicity in algebraic $K$-theory. Besides this purely algebraic motivation some relationships with classical geometry have recently been discovered, which could lead to investigate the (co)homological theory of Leibniz algebras in view of concrete applications in non-commutative geometry and its physical interpretations. Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if

$$[x,x] = 0, \quad \text{for all } x \in L.$$

Let $n$ be the dimension of Leibniz algebra $L$. Let $\{e_1, e_2, ..., e_n\}$ be a basis in $L$. The structural constants of $L$ are the numbers $C_{ij}^k$ given by

$$[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^k e_k \quad (i,j = 1, ..., n).$$

We can identify the Leibniz identity with its structural constants. These constants satisfy:

$$\sum_{l=1}^{n} (C_{jk}^l C_{il}^s - C_{ij}^l C_{lk}^s + C_{ik}^l C_{lj}^s) = 0 \quad (i,j,k,s = 1, ..., n).$$

**Definition 3.2.** Let $L$ is a Leibniz algebra. We define

$$L^1 = L, L^k = \left[L^{k-1}, L\right] \quad (k > 1).$$

The series

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq \ldots$$

is called the descending central series of $L$. If the series terminates for some positive integer $s$, then the Leibniz algebra $L$ is said to be nilpotent.

Let $L$ be a Leibniz algebra, and $V$ be a vector space over $F$. Then the bilinear maps $\theta : L \times L \longrightarrow V$ with

$$\theta(x,[y,z]) = \theta([x,y],z) - \theta([x,z],y), \quad \text{for all} \quad x,y,z \in L$$

are called Leibniz cocycles. It can be stated that all of Leibniz cocycles shall be represented by $ZL^2(L,V)$. Let $\theta$ be a Leibniz cocycle, then we arrange $L_\theta = L \oplus V$ and define a bracket $[\cdot,\cdot]$ on $L_\theta$ by

$$[x+v, y+w] = [x,y]_L + \theta(x,y), \tag{1}$$

in which $[\cdot,\cdot]_L$ is the bracket on $L$, or we can define a bracket $[\cdot,\cdot]$ on $L_\theta$ by

$$[(x,v),(y,w)] = ([x,y]_L, \theta(x,y)).$$

The following Lemma shows that when $L_\theta$ is a Leibniz algebra.

**Lemma 3.3.** *$L_\theta$ is a Leibniz algebra if and only if $\theta$ is a Leibniz cocycle.*

*Proof.* Let $L_\theta = L \oplus V = \{(l, v)\,; l \in L, v \in V\}$. Now we consider
$[\cdot,\cdot] : L_\theta \times L_\theta \longrightarrow L_\theta$ :

$$[(l_1, v_1), (l_2, v_2)] = [l_1, l_2]_L + \theta\,(l_1, l_2) \in L_\theta.$$

But

$$[(x_1, v_1), [(x_2, v_2), (x_3, v_3)]] - [[(x_1, v_1), (x_2, v_2)], (x_3, v_3)] +$$
$$[[(x_1, v_1), (x_3, v_3)], (x_2, v_2)]$$
$$= \left[(x_1, v_1), [x_2, x_3]_L + \theta\,(x_2, x_3)\right] - \left[[x_1, x_2]_L + \theta\,(x_1, x_2), (x_3, v_3)\right] +$$
$$\left[[x_1, x_3]_L + \theta\,(x_1, x_3), (x_2, v_2)\right]$$
$$= \left[x_1, [x_2, x_3]_L\right]_L + \theta\left(x_1, [x_2, x_3]_L\right) - \left[[x_1, x_2]_L, x_3\right]_L -$$
$$\theta\left([x_1, x_2]_L, x_3\right) + \left[[x_1, x_3]_L, x_2\right] + \theta\,([x_1, x_3], x_2)$$
$$= \left[x_1, [x_2, x_3]_L\right]_L - \left[[x_1, x_2]_L, x_3\right]_L + \left[[x_1, x_3]_L, x_2\right]_L +$$
$$\theta\left(x_1, , [x_2, x_3]_L\right) - \theta\left([x_1, x_2]_L, x_3\right) + \theta\left([x_1, x_3]_L, x_2\right) =$$
$$\theta\left(x_1, , [x_2, x_3]_L\right) - \theta\left([x_1, x_2]_L, x_3\right) + \theta\left([x_1, x_3]_L, x_2\right)$$

($L$ is a Leibniz algebra), therefore $L_\theta$ is a Leibniz algebra if and only if $\theta\left(x_1, [x_2, x_3]_L\right) - \theta\left([x_1, x_2]_L, x_3\right) + \theta\left([x_1, x_3]_L, x_2\right) = 0$. □

The Leibniz algebra $L_\theta$ is called a **central extension** of $L$ by $V$. Let $\nu : L \longrightarrow V$ be a linear map, and define $\eta\,(x, y) = \nu\,([x, y])$. One can show that $\eta$ is a Leibniz cocycle called **coboundary**. The set of all coboundaries is denoted by $BL^2\,(L, V)$. Since $d^n d^{n-1} = 0$ (see Appendix B), we have $BL^2(L, V) \subseteq ZL^2\,(L, V)$. We refer to the quotient $HL^2(L, V) := ZL^2\,(L, V) / BL^2\,(L, V)$ as the **second cohomology** group of the Leibniz algebras $L$ with values in $V$.
The following Lemma shows that the central extensions of a given Leibniz algebra $L$ is defined up to a coboundary.

**Lemma 3.4.** *Let L be a Leibniz algebra and $\eta$ be a coboundary, then the central extensions $L_\theta$ and $L_{\theta+\eta}$ are isomorphic.*

The center of a Leibniz algebra $L$ is defined as follows:

$$C(L) = \{x \in L \mid [x, L] = [L, x] = 0\}.$$

For $\theta \in ZL^2\,(L, V)$ set

$$\theta^{\perp} = \{x \in L \mid \theta(x, L) = \theta(L, x) = 0\},$$

which is called the radical of $\theta$ (Rad($\theta$)=$\theta^{\perp}$).
On the way to constitute Leibniz algebras as $L_\theta = L \oplus V$, in fact researcher is intended to put restriction on $\theta$ in a way that the center of $L_\theta$ overlaps with $V$. Deploying this method researcher avoids constituting similar Leibniz algebras functioning as central extensions of different Leibniz algebras.
Let now $\tilde{L}$ be a Leibniz algebra with $k-$dimensional center $C(\tilde{L})$, $\nu : \tilde{L} \longrightarrow V$ be linear function and such that $\nu(C(\tilde{L})) = V$. Consider $L = \tilde{L}/C(\tilde{L})$ and get an isomorphism $\tilde{L} \cong L \oplus V$, where $\tilde{x} \leftrightarrow y + u$, $\nu(\tilde{x}) = u$, and $y = \tilde{x} + C(\tilde{L}) \in \tilde{L}/C(\tilde{L}) = L$. We put $\theta = \nu \circ [.,.]$, that is

$$\theta(x, y) = \nu[x', y']; \text{ where } x' + C(\tilde{L}) = x,\ y' + C(\tilde{L}) = y.$$

This shows that $\tilde{L}$ and $L_\theta$ are isomorphic. Hence each Leibniz algebra with center of dimension $k$ is on the form $L_\theta$, where $\theta$ is a Leibniz cocycle.
We conclude that any Leibniz algebra with a nontrivial center can be obtained as a central extension of a Leibniz algebra of smaller dimension.
The following Lemma shows that there is a relationship between $C(L_\theta)$ and $C(L)$.

**Lemma 3.5.** *If* $\theta \in ZL^2(L,V)$ *then* $C(L_\theta) = (\theta^\perp \cap C(L)) + V$.

As an application of this Lemma we get the following criterion. That is, we want to restrict to $\theta$ such that $\mathrm{Rad}(\theta) \cap C(L) = 0$. This way we avoid constructing the same Leibniz algebra as central extension of different Leibniz algebra.

**Corollary 3.6.** $\theta^\perp \cap C(L) = 0$ *if and only if* $C(L_\theta) = V$.

*Proof.* A direct result of Lemma 3.5. □

The automorphism group $\mathrm{Aut}(L)$ acts on $ZL^2(L,V)$ by $\phi\theta(x,y) = \theta(\phi(x),\phi(y))$ and $\eta \in BL^2(L,V)$ if and only if $\phi\eta \in BL^2(L,V)$. So $\mathrm{Aut}(L)$ acts on $HL^2(L,V)$. The following Lemmas are Leibniz analogies of the corresponding statements for Lie algebras case.

**Lemma 3.7.** *Let* $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)\, e_i$ *and* $\eta(x,y) = \sum_{i=1}^{s} \eta_i(x,y)\, e_i$ *be two elements of* $HL^2(L,V)$. *Suppose that* $\theta^\perp \cap C(L) = \eta^\perp \cap C(L) = 0$. *Then* $L_\theta \cong L_\eta$ *if and only if there is a* $\varphi \in Aut(L)$ *such that the* $\varphi\eta_i$ *span the same subspace of* $H^2(L,V)$ *as the* $\theta_i$ .

Let $L = I_1 \oplus I_2$ be the direct sum of two ideals. Suppose that $I_2$ is contained in the centre of $L$. Then $I_2$ is called a central component of $L$. In order to keep away from the Leibniz algebras with central components we use the following criterion.

**Lemma 3.8.** *Let* $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)\, e_i \in HL^2(L,V)$ *be such that* $\theta^\perp \cap C(L) = 0$. *Then* $L_\theta$ *has no central components if and only if* $\theta_1, \ldots, \theta_s$ *are linearly independent in* $HL^2(L,F)$.

Let $e_1, \ldots, e_s$ be a basis in $V$, and $\theta \in ZL^2(L,V)$. Then $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i$, where $\theta_i \in ZL^2(L,F)$. Furthermore, $\theta$ is a coboundary if and only if all $\theta_i$ are.

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# Projective Nominal Sets

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**Abstract**

Projective objects play a fundamental role in category theory and structural analysis. In this paper, we characterize projective nominal sets and show that the empty nominal set is the only projective object in this category.

## 1. Introduction

Nominal set theory provides a mathematical framework for studying semantics, modifying variables, and much more in computer science. Indeed, Fraenkel presented nominal sets in [4] as an alternative model of set theory in 1922. In this context Mostowski studied further, which is why nominal sets are sometimes referred to as Fraenkel-Mostowski sets. In the 1990s, Gabbay and Pitts [6] rediscovered nominal sets for the computer science community, and this notion sparked a lot of interest in semantics [1–3, 5].

## 2. Preliminaries

This section contains some necessary notions on nominal sets and topological spaces needed throughout the paper from [8] respectively. For further information on category theory, one may consult [2].

### 2.1. Nominal sets

Suppose $\mathbb{D}$ is a set, then a permutation $\pi$ of $\mathbb{D}$ is a bijective map from $\mathbb{D}$ to itself. The permutations of $\mathbb{D}$ with composition and identity form a group, called the *symmetric group* on the set $\mathbb{D}$ and denoted by Sym$\mathbb{D}$. A permutation $\pi \in \text{Sym}\mathbb{D}$ is finitary if the set $\{d \in \mathbb{D} \mid \pi d \neq d\}$ is a finite subset of $\mathbb{D}$. It is clear that id $\in$ Sym$\mathbb{D}$ is finitary and that the composition and the inverse of finitary permutations are finitary. Therefore, we get a subgroup of Sym$\mathbb{D}$ of

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finitary permutations, denoted by Perm($\mathbb{D}$). We fix a countable infinite set $\mathbb{D}$ whose elements are denoted by $a, b, c, \ldots$ and called *atomic names*. Let $X$ be a set equipped with an action of the group Perm($\mathbb{D}$), Perm($\mathbb{D}$) $\times X \to X$ mapping $(\pi, x)$ to $\pi x$. We call $X$ a Perm($\mathbb{D}$)-*set*, whenever for every $\pi_1, \pi_2 \in$ Perm($\mathbb{D}$) and every $x \in X$ we have:

(1) $\pi_1(\pi_2 x) = (\pi_1 o \pi_2)x$

(2) $\mathrm{id}\, x = x$.

A subset $Y$ of a Perm($\mathbb{D}$)-set $X$ is called *equivariant* if $\pi y \in Y$, for all $\pi \in$ Perm($\mathbb{D}$) and $y \in Y$. Perm($\mathbb{D}$)-sets are the objects of a category, denoted by Perm($\mathbb{D}$)-**Set** whose morphisms are equivariant maps, i.e. maps subject to the rule $f(\pi x) = \pi f(x)$, for all $x \in X, \pi \in$ Perm($\mathbb{D}$), whose compositions and identities are as in the category **Set** of sets and maps.
An element $x$ of a Perm($\mathbb{D}$)-set $X$ is called a *fixed* (or equivariant) element if $\pi x = x$, for all $\pi \in$ Perm($\mathbb{D}$). The set of all fixed elements of the Perm($\mathbb{D}$)-set $X$ is denoted by Fix($X$). A Perm($\mathbb{D}$)-set all of whose elements are fixed is called *discrete*.
Given a Perm($\mathbb{D}$)-set $X$, a set of atomic names $D \subseteq \mathbb{D}$ is a *support* for an element $x \in X$ if for all $\pi \in$ Perm($\mathbb{D}$) and for every $d \in D$,

$$\pi(d) = d \Rightarrow \pi x = x.$$

Given a Perm($\mathbb{D}$)-set $X$, we say an element $x \in X$ is finitely supported, if there is some finite set of atomic names that is, a support for the element $x$.

**Example 2.1.** Given a Perm($\mathbb{D}$)-set $X$, the power set of $X$, $\mathcal{P}(X)$, with the action

$$\mathrm{Perm}(\mathbb{D}) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

$$(\pi, S) \rightsquigarrow \{\pi x : x \in S\}$$

is a Perm($\mathbb{D}$)-set. A set of atomic names $D$ supports $S \in \mathcal{P}(X)$ if and only if

$$(\forall \pi \in \mathrm{Perm}(\mathbb{D}))((\forall d \in D)\ \pi(d) = d) \Rightarrow (\forall\, x \in S)\ \ \pi x \in S.$$

**Definition 2.2.** [8] A *nominal set* is a Perm($\mathbb{D}$)-set all of whose elements are finitely supported. Nominal sets are the objects of a category, denoted by **Nom**, whose morphisms are equivariant maps and whose compositions and identities are as in the category of Perm($\mathbb{D}$)-**Set**.

**Remark 2.3.** Suppose $X$ is a nominal set and $x \in X$. Intersection of two finite supports of $x$ is a (finite) support of $x$, [8, Propositions 2.1 and 2.3]. So each $x \in X$ has the least (finite) support which is denoted by $\mathrm{supp}_X\, x$, and when there is no possibility of error, we denote it by $\mathrm{supp}\, x$. In fact, $\mathrm{supp}\, x = \bigcap\{C : C \text{ is a finite support of } x\}$.

**Example 2.4.** The set $\mathbb{D}$ is a nominal set with the natural action $\pi d = \pi(d)$.

**Definition 2.5.** [7, Section I] A nominal set $X$ is called *well-ordered* whenever for each non-empty subset $A \subseteq X$ has a smallest element that means there exists $a \in A$ where $\mathrm{supp}\, a \subseteq \mathrm{supp}\, x$ for all $x \in A$.

**Theorem 2.6.** *A nominal set $X$ is well-ordered if and only if it is discrete.*

## 3. Projective nominal sets

The study of projective objects is essential for revealing the internal structure of categorical frameworks. Here, we obtain a characterization of projective nominal sets, clarifying their role within the theory of nominal sets.

**Definition 3.1.** [7, Definition I.7.9] A nominal set $P$ is projective if for every nominal set $Y$ and equivariant map $f : P \to Y$, any epimorphism $g : X \twoheadrightarrow Y$ there exists $h : P \to X$ such that $gh = f$.

**Theorem 3.2.** *Every discrete nominal set is not projective.*

**Theorem 3.3.** *The only projective nominal set is empty.*

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# Projective objects in the category of finitely supported $Cb$-sets

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**Abstract**

We investigate projective objects in the category of finitely supported **Cb**-sets, where **Cb** denotes the monoid of finite injective substitutions on a countably infinite set of atoms. Our main contributions include characterizing the existence and structural properties of projective **Cb**-sets.

## 1. Introduction

In this paper, we study projectivity in the category of finitely supported **Cb**-sets, where the finite support condition generalizes nominal sets [4] by considering all finite injective substitutions (i.e., finite injective maps from a countably infinite set of atoms $\mathbb{D}$ to $\mathbb{D} \cup \{0, 1\}$) rather than only permutations. This approach, initiated by Gabbay and Hofmann [3], is motivated by its utility for modeling flexible substitution operations while preserving the finite support property, which ensures that reasoning and computation involving syntax with binding and substitutions remain effective and well-behaved.
A surprising and significant connection, established by Pitts [7], shows the category of finitely supported **Cb**-sets to be equivalent to the category of cubical sets, thus linking nominal algebra to algebraic topology and offering new perspectives on both fields.

## 2. Preliminaries

This section introduces the main concepts and techniques required to follow this paper from [1, 2, 5–7]. The first subsection explains the fundamental ideas behind $M$-sets. The second section discusses **Cb**-sets as a specific case of the first subsection and then examines the monoid **Cb**. Finally, all the subjects concerning finitely supported **Cb**-sets required for this study are covered in the last subsection. For a more comprehensive treatment of these topics, we refer readers to the cited literature.

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### *2.1. M-sets*

An *M-set* (*left M-set*) for a monoid $M$ with identity $e$ is a set $X$ equipped with a map $M \times X \to X$, denoted as $(m, x) \rightsquigarrow mx$, called an *action* of $M$ on $X$, subject to the rules

$$ex = x \text{ and } m(m'x) = (mm')x,$$

for all $x \in X$ and $m, m' \in M$.

By the category $M$-**Set**, we mean the category of all $M$-sets and all *equivariant maps*, $f : X \to Y$ subject to $f(mx) = mf(x)$ for all $x \in X$ and $m \in M$, between them. In the category $M$-**Set**, epimorphisms are exactly surjective equivariant maps (see [5, Proposition I.6.15]).

An element $x$ of an $M$-set $X$ is a *fixed* (or *equivariant*) element if $mx = x$, for all $m \in M$. We denote the set of all fixed elements of an $M$-set $X$ by $\mathrm{Fix}(X)$. An $M$-set $X$ with *discrete action* is one in which all of its elements are fixed.

A subset $Y$ of an $M$-set $X$ is an equivariant subset ( or *M-subset*) of $Y$ if $my \in Y$, for all $m \in M$ and $y \in Y$. The subset $\mathrm{Fix}(X)$ of $X$ is always an equivariant subset of $X$.

### *2.2.* **Cb**-*sets*

Throughout the paper $\mathbb{D}$ is a fixed, infinite, and countable set whose elements are sometimes called *atomic names* (or *data values)* and $\mathrm{Perm}(\mathbb{D})$ is the group of all permutations (bijective maps) on $\mathbb{D}$. A permutation $\pi \in \mathrm{Perm}(\mathbb{D})$ is said to be *finitary* if the set $\{d \in \mathbb{D} \mid \pi(d) \neq d\}$ is finite. Clearly, the set $\mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ of all finitary permutations is a subgroup of $\mathrm{Perm}(\mathbb{D})$. The following definition is given for the set $2 = \{0, 1\}$ with $0, 1 \notin \mathbb{D}$.

**Definition 2.1.** [7] (i) A *finite substitution* is a map $\sigma : \mathbb{D} \to \mathbb{D} \cup 2$, for which the set $\{d \in \mathbb{D} \mid \sigma(d) \neq d\}$ is finite. Let **Sb** be the monoid whose elements are finite substitutions, with the monoid operation given by composition:

$$\sigma \cdot \sigma' \doteq \hat{\sigma}\sigma',$$

where $\hat{\sigma} : \mathbb{D} \cup 2 \to \mathbb{D} \cup 2$ maps 0 to 0, 1 to 1, and on $\mathbb{D}$ is defined like $\sigma$. The identity element of **Sb** is the inclusion $\iota : \mathbb{D} \hookrightarrow \mathbb{D} \cup 2$.

(ii) If $d \in \mathbb{D}$ and $\mathsf{b} \in 2$, we denote the finite substitution mapping $d$ to $\mathsf{b}$ and acting as the identity on all other elements of $\mathbb{D}$ as $(\mathsf{b}/d)$ and call it a *ground substitution*.

(iii) If $d, d' \in \mathbb{D}$, then we write $(d\, d')$ for the finite substitution that transposes $d$ and $d'$, and keeps fixed all other elements. Each $(d\, d')$ is called a *transposition substitution*.
(iv) Let **Cb** be the submonoid of **Sb** consisting of finite substitutions $\sigma$ satisfying the injectivity condition:

$$(\forall d, d' \in \mathbb{D})\sigma(d) = \sigma(d') \notin 2 \Rightarrow d = d'.$$

(v) The set GroundSub$\doteq\{\delta = (\mathsf{b}_1/d_1) \cdots (\mathsf{b}_k/d_k) \mid d_i \in \mathbb{D}, \mathsf{b}_i \in 2\}$ is a submonoid of **Cb**. Moreover, for every $\delta = (\mathsf{b}_1/d_1) \cdots (\mathsf{b}_k/d_k) \in$ GroundSub, we denote the set $\{d_1, \ldots, d_k\}$ by $\mathbb{D}_\delta$.

**Remark 2.2.** [2, Remark 2.3(ii)] For every $\delta = (\mathsf{b}_1/d_1) \cdots (\mathsf{b}_k/d_k) \in$ GroundSub and $\pi \in \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})$ we have $\pi\delta = \delta'\pi$, in which $\delta' = (\mathsf{b}_1/\pi d_1) \cdots (\mathsf{b}_k/\pi d_k)$, and $\delta\pi = \pi\delta''$, in which $\delta'' = (\mathsf{b}_1/\pi^{-1}d_1) \cdots (\mathsf{b}_k/\pi^{-1}d_k)$.

**Theorem 2.3.** *[2, Theorem 2.6] For the monoid* **Cb***, we have:*

$$\mathbf{Cb} = \mathrm{Perm}_{\mathrm{f}}(\mathbb{D}) \cup \mathrm{Perm}_{\mathrm{f}}(\mathbb{D})\mathrm{GroundSub}.$$

*2.3. Finitely supported* **Cb***-sets*

This subsection provides a collection of results concerning finitely supported **Cb**-sets. These facts will be essential for subsequent discussions and most of them are drawn from [7].

The following definition introduces the notion, so-called, *support*, which is the central notion to define finitely supported **Cb**-sets.

**Definition 2.4.** Let $X$ be a **Cb**-set and $x \in X$. Then,

(i) A subset $S \subseteq \mathbb{D}$ is a *support* of $x$, if

$$(\forall \sigma, \sigma' \in \mathbf{Cb})\ (\sigma(s) = \sigma'(s), (\forall s \in S)) \Rightarrow \sigma x = \sigma' x$$

If there exists a finite (possibly empty) support $S$ of $x$, then we say that $x$ is *finitely supported*.

(ii) A *finitely supported* **Cb**-*set* is a **Cb**-set $X$ all of whose elements have finite supports.

It is worth noting that the concept of support can be defined for elements of any $M$-set $X$, where $M$ is a submonoid of **Sb**. Moreover, one can consider the category of $M$-sets with finite support. However, since in this paper we focus on **Cb**-sets with finite support, we restrict our attention to this specific case, and we denote the full subcategory of **Cb-Set** consisting of all finitely supported **Cb**-sets by $(\mathbf{Cb\text{-}Set})_{\mathrm{fs}}$.

**Lemma 2.5.** *[7, Lemma 2.4] Suppose $X$ is a* **Cb***-set, $x \in X$ and $\mathsf{b} \in 2$. Also, let $C$ be a finite subset of $\mathbb{D}$. Then, $C$ is a support of $x$ if and only if*

$$(\forall d \in \mathbb{D})\ \ d \notin C \Rightarrow (\mathsf{b}/d)x = x.$$

**Remark 2.6.** [2, Remark 3.2 and Corollary 3.5] Let $X$ be a **Cb**-set and $x \in X$.

(i) If $X$ is finitely supported, then, for every $x \in X$, the set $\{d \in \mathbb{D} \mid (0/d)x \neq x\}$ is the least finite support of $x$. From now on, we call the least finite support for $x$ the *support* for $x$, and denote it by $\operatorname{supp} x$.

(ii) The element $x \in X$ is fixed if and only if $\operatorname{supp} x = \emptyset$ if and only if $\delta\, x = x$, for all $\delta \in \mathrm{GroundSub}$.

**Corollary 2.7.** *Every non-empty finitely supported* **Cb***-set has a fixed element.*

**Remark 2.8.** [2, Notation and Remark 4.4] Given a finitely supported **Cb**-set $X$ and an element $x \in X$, we can define two subsemigroups of GroundSub as follows:

$$\mathrm{GroundSub}_x \doteq \{\delta \in \mathrm{GroundSub} \mid \delta x = x\}, \text{ and}$$

$$\mathrm{GroundSub}'_x \doteq \mathrm{GroundSub} \setminus \mathrm{GroundSub}_x = \{\delta \in \ \mathrm{GroundSub} \mid \delta x \neq x\}.$$

## 3. Projective objects in the category of finitely supported $Cb$-sets

The study of projective objects is essential for revealing the internal structure of categorical frameworks. Here, we obtain a characterization of projective finitely supported $Cb$-set, clarifying their role within the theory of finitely supported $Cb$-set.

**Definition 3.1.** [5, Definition I.7.9] A finitely supported $Cb$-set $P$ is projective if for every finitely supported $Cb$-set $Y$ and equivariant map $f : P \to Y$, any epimorphism $g : X \twoheadrightarrow Y$ there exists $h : P \to X$ such that $gh = f$.

**Theorem 3.2.** *Every discrete $Cb$-set is projective.*

**Theorem 3.3.** *Every finitely supported $Cb$-set is quotiont of $\cup_{i\in I} Cb((d_1, \dots, d_{n_i}, 0, \cdots), i)$ with the action $\sigma((d_1, \dots, d_{n_i}, 0, \cdots), i) = (\sigma(d_1), \dots, \sigma(d_{n_i}), 0, \cdots), i)$.*

**Theorem 3.4.** *In Theorem 3.3, for each projective object* $\ker f = \Delta$.

**Theorem 3.5.** *Let $X$ and $Y$ are finitely supported $Cb$-sets and $f : X \twoheadrightarrow Y$ is an epimorphism. For each $y \in Y \setminus \mathrm{Fix}(Y)$ there exists $x \in X$ such that $f(x) = y$ and* $\operatorname{supp} x = \operatorname{supp} y$.

**Theorem 3.6.** *Every finitely supported $Cb$-set is projective if and only if is isomorphic by $\cup_{i\in I} Cb((d_1, \dots, d_{n_i}, 0, \cdots), i)$ for some index set $I$ and $n_i \in \mathbb{N}^0$ with the action $\sigma((d_1, \dots, d_{n_i}, 0, \cdots), i) = (\sigma(d_1), \dots, \sigma(d_{n_i}), 0, \cdots), i)$.*

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# Category of Weak Frame Down Set

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**Abstract**

The category of neighbourhood systems and the category of **Frm** ↓ **Set** are introduce by T. Lu and W. He. In this article, we introduce the category **WFrm** ↓ **Set** and give some properties of this category.

## 1. Introduction and Preliminaries

Topological systems were first introduced by S. Vickers in [4] arising from topological and localic aspects of domains and finite observational logic in computer science. In [1], J. Adamek and M.C. Pedicchio showed that the category of topological systems is dually equivalent to the category of grids introduced in [2]. One of the purposes of the introduction of topological systems is to provide a single framework in which to treat both spaces and locales(see [4]). T. Lu and W. He [3], introduced the concept of neighbourhood systems and shown that the category of neighbourhood systems contains the category of topological spaces, the category of locales and the category of topological systems respectively as full subcategories. Also, they show that, the category of neighbourhood systems has coproducts but does not have products. Section 2 is a prerequisite for the rest of the paper. The definitions and results of this section are taken from [3]. In Section3, we introduce the category of **WFrm** ↓ **Set** and give some properties of this category. For example, we determine the terminal object.

## 2. Preliminaries

In this section, the subjects needed in the other parts, have been mentioned.(see [3]).

**Definition 2.1.** Let $X$ be a set and $L$ be a locale. Let $N : X \to Fil(L)$ be a map which assigns to every point $x \in X$ a filter $N(x)$ in $L$. We call $N$ a neighbourhood assignment on $X$ to $L$ and elements in $N(x)$ are called neighbourhoods of $x$. $X$ and $L$ with a neighbourhood assignment $N$ is called a neighbourhood system which is denoted by $(X, L, N)$.

**Definition 2.2.** Let $(X, L, N)$ and $(Y, T, M)$ be two neighbourhood systems. A continuous map from $(X, L, N)$ to $(Y, T, M)$ is a pair $(f, \overline{f})$ where $f : X \to Y$ is a function, $\overline{f} : T \to L$ is a frame homomorphism and for every $x \in X$, $u \in M(f(x))$ if and only if $\overline{f}(u) \in N(x)$.

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Continuous maps between neighbourhood systems are clearly stable under composition and for every neighbourhood system $(X, L, N)$, there exists obviously an identity map from $(X, L, N)$ to itself. Thus we have a category **NS** of neighbourhood systems and continuous maps.
We consider the category **Frm** $\downarrow$ **Set** its objects are those finite meet preserving maps $\varphi : L \to P(X)$ from a locale $L$ to the power set $P(X)$ of a set $X$, for two objects $\psi : L \to P(X)$ and $\varphi : T \to P(Y)$ in **Frm** $\downarrow$ **Set**, a morphisms from $\psi : L \to P(X)$ to $\varphi : T \to P(Y)$ is a pair

$$f : X \to Y, \overline{f} : T \to L$$

where $f : X \to Y$ is a function, $\overline{f} : T \to L$ is a frame homomorphism such that the following square commutes:

$$\begin{array}{ccc} T & \xrightarrow{\overline{f}} & L \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ P(Y) & \xrightarrow[f^{-1}]{} & P(X) \end{array}$$

Given a neighbourhood system $(X, L, N)$, we have a map from $L$ to the power set $P(X)$ of $X$ which preserves finite meets

$$\psi : L \to P(X), a \mapsto \{x \in X | a \in N(x)\}.$$

It is not difficult to see that this assignment defines a functor from the category **NS** of neighbourhood systems to the category **Frm** $\downarrow$ **Set**. This functor is in fact an isomorphism of categories such that its inverse maps each object $\psi : L \to P(X)$ of **Frm** $\downarrow$ **Set** to a neighbourhood system $(X, L, N)$ defined by

$$N(x) = \{u \in L | x \in \psi(u)\}.$$

**Definition 2.3.** A continuous map $(f, \overline{f}) : (Y, T, M) \to (X, L, N)$ of neighbourhood systems is said an embedding if $f$ is injective and $\overline{f}$ is surjective. A neighbourhood system $(Y, T, M)$ is called a neighbourhood subsystem of $(X, L, N)$ if there is an embedding $(f, \overline{f}) : (Y, T, M) \to (X, L, N)$ of neighbourhood systems.

Now we consider sums of neighbourhood systems which is in fact coproducts of neighbourhood systems in categorical sense. Let $\{(X_j, L_j, N_j) | j \in J\}$ be a family of neighbourhood systems. The sum of $\{(X_j, L_j, N_j) | j \in J\}$ is defined as following.
Take $X = \bigcup_{i \in J} X_j$ be the disjoint union of $\{X_j | j \in J\}$, $L = \prod L_j$ be the frame product of $\{L_j | j \in J\}$. Define

$$N : X \to Fil(L), N(x) = \{(u_j) | u_i \in N_i(x)\}, x \in X_i$$

**Theorem 2.4.** *$(X, L, N)$ is the coproduct of $\{(X_j, L_j, N_j) | j \in J\}$ in the category of neighbourhood systems with injections $(i_j, \bar{i}_j) : (X_j, L_j, N_j) \to (X, L, N)$, $j \in J$ defined by $i_j(x) = x$ and $\bar{i}_j(x_t) = x_j$ for each $j \in J$.*

Let $\{(X_j, L_j, N_j) | j \in J\}$ be a family of neighbourhood systems. Write $\prod_{j \in J} X_j$ for the Cartesian product of $\{X_j\}$, and $\coprod_{j \in J} L_j$ for the coproduct of frames $\{L_j\}$, equivalently the product of locales $\{L_j\}$. We define a neighbourhood assignment $N : \prod_{j \in J} X_j \to Fil(\coprod_{j \in J} L_j)$ such that

$$N(x_j) = \uparrow \{\downarrow (u_j) | \exists \text{ a finite set} F \subseteq J, \text{ such that } u_j = 1, j \in J \setminus F, u_i \in N_i(x_i) i \in F\}$$

Write $p_j : \prod_{j \in J} X_j \to X_j$ and $\overline{p}_j : L_j \to \coprod_{j \in J} L_j$ be the $j$-th projection and co-projection respectively. Then $(p_j, \overline{p}_j) : (\prod_{j \in J} X_j, \coprod_{j \in J} L_j, N) \to (X_j, L_j, N_j)$ is a continuous map.

**Lemma 2.5.** *Let $\{(X_j, L_j, N_j) | j \in J\}$ be a family of neighbourhood systems. Suppose that $(\prod_{j \in J}^{\sim} X_j, \coprod_{j \in J}^{\sim} L_j, \tilde{N})$ is the product of $\{(X_j, L_j, N_j) | j \in J\}$ with projections $(\prod_{j \in J}^{\sim} X_j, \coprod_{j \in J}^{\sim} L_j, \tilde{N}) \to (X_j, L_j, N_j)$, $j \in J$, then $\prod_{j \in J}^{\sim} X_j$ is equipotent with $\prod_{j \in J} X_j$ and $\coprod_{j \in J}^{\sim} L_j$ is isomorphic with $\coprod_{j \in J} L_j$.*

## 3. Category of WFrm ↓ Set

In this section introduce the category of **WFrm ↓ Set** and we give some properties of this category.
The category **WFrm ↓ Set** its objects are those finite meet preserving maps $\varphi : L \to P(X)$ from a locale $L$ to the power set $P(X)$ of a set $X$, for two objects $\psi : L \to P(X)$ and $\varphi : T \to P(Y)$ in **Frm ↓ Set**, a morphisms from $\psi : L \to P(X)$ to $\varphi : T \to P(Y)$ is a pair $f : X \to Y, \overline{f} : T \to L$ where $f : X \to Y$ is a function, $\overline{f} : T \to L$ is a frame homomorphism such that $\psi\overline{f} \subseteq f^{-1}\phi$.

**Theorem 3.1.** *Category* **WFrm ↓ Set** *has a terminal object.*

*Proof.* Let $L = \mathbf{2}$ and $X = \{*\}$. Define $\varphi : L \to P(X)$ by $\varphi(l) = \{*\}$, it is clear that $\varphi \in$ **Frm ↓ Set**. We claim that $\varphi$ is a terminal object. Let $\psi : T \to P(Y) \in$ **Frm ↓ Set** be an arbitrary and define $f : Y \to X$ by $f(y) = *$ and $\overline{f} : L \to T$ by $\overline{f}(0) = 0$ and $\overline{f}(1) = 1$. Then $\overline{f}$ is a frame map. Moreover

$$\psi\overline{f}(0) = \psi(0) \subseteq Y = f^{-1}(*) = f^{-1}(\phi(0))$$

and

$$\psi\overline{f}(1) = \psi(1) = Y = f^{-1}(*) = f^{-1}(\phi(0)).$$

Then $\psi\overline{f} \subseteq f^{-1}\phi$. □

**Theorem 3.2.** *Category* **Frm ↓ Set** *has an intial object.*

*Proof.* Let $L = \mathbf{1}$ and $X = \emptyset$. Define $\varphi : L \to P(X)$ by $\varphi(0) = \emptyset$. Then $\varphi \in$ **Frm ↓ Set**. We claim that $\varphi$ is intial object. For every $\psi : M \to P(Y)$, we define $f : X \to Y$ by $f = \emptyset$ and $\overline{f} : M \to L$ by $\overline{f}(x) = 0$. Then the diagram,

$$\begin{array}{ccc} M & \xrightarrow{\overline{f}} & L \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ P(Y) & \xrightarrow[f^{-1}]{} & P(X) \end{array}$$

shows that $(f, \overline{f}) : (\varphi : L \to P(X)) \to (\psi : M \to P(Y))$ is a unique morphism in **Frm ↓ Set**. □

**Theorem 3.3.** *Let* $(f, \overline{f}) : (\varphi : L \to P(X)) \to (\psi : T \to P(Y))$ *be a morphism in* **Frm ↓ Set**. *Then* $(f, \overline{f})$ *is a monomorphism in category* **Frm ↓ Set** *if and only if* $f : X \to Y$ *is injective in* **Set** *and* $\overline{f} : T \to L$ *is surjective in* **Frm**.

*Proof. Necessity.* Let $(f, \overline{f}) : (\varphi : L \to P(X)) \to (\psi : T \to P(Y))$ be a monomorphism and $g, h : L \to M$ be two frame homomorphisms such that $g \circ \overline{f} = h \circ \overline{f}$. Let $Z = \emptyset$. Consider $f_1 = f_2 : \emptyset \to X$ and $\eta : M \to P(\emptyset)$. Then $\eta \in$ **Frm ↓ Set**. Also $(f_1, g) : (\eta : M \to P(\emptyset) \to (\varphi : L \to P(X))$ and $(f_2, h) : (\eta : M \to P(\emptyset) \to (\varphi : L \to P(X))$ are morphisms in **Frm ↓ Set**. It is easy to see that

$$(f, \overline{f}) \circ (f_1, g) = (f, \overline{f}) \circ (f_2, h).$$

The left cancelation property of $(f, \overline{f})$ implies that $f_1 = f_2$ and $g = h$.
Now, let $f_1, f_2 : Z \to X$ be two functions such that $f \circ f_1 = f \circ f_2$. Let $M = L$, $g = h = id_L$ and $\eta : L \to P(Z)$ by $\eta(x) = f^{-1}(\varphi(x))$. First we show that $\eta$ is a meet preserving map. Let $x, y \in L$, then

$$\begin{array}{rcll} \eta(x \wedge y) & = & f^{-1}(\varphi(x \wedge y)) & \\ & = & f^{-1}(\varphi(x) \cap \varphi(y)) & \text{since } \varphi \text{ is meet pereserving} \\ & = & f^{-1}(\varphi(x)) \cap f^{-1}(\varphi(y)) & \text{since } f^{-1} \text{ pereserve cap} \\ & = & \eta(x) \cap \eta(y). & \end{array}$$

By definition $\eta$, it is easy to check that $(f_1, g) : (\eta : L \to P(Z)) \to (\varphi : L \to P(X))$ is a morphism in category **Frm** $\downarrow$ **Set**.
Now, we claim that $f_1^{-1}(\varphi(x)) = f_2^{-1}(\varphi(x))$ for every $x \in L$. Since $\overline{f} : T \to L$ is a surjective frame map, for every $x \in L$, there exists $y \in T$ such that $\overline{f}(y) = x$. Then, we have

$$\begin{aligned}
a \in f_1^{-1}(\varphi(x)) &\Leftrightarrow f_1(a) \in \varphi(x) \\
&\Leftrightarrow f_1(a) \in \varphi(\overline{f}(y)) \quad \text{since } x = \overline{f}(y) \\
&\Leftrightarrow f_1(a) \in f^{-1}(\phi(y)) \quad \text{since } \varphi \circ \overline{f} = f^{-1} \circ \psi \\
&\Leftrightarrow f(f_1(a)) \in \psi(y) \\
&\Leftrightarrow f(f_2(a)) \in \psi(y) \quad \text{since } f \circ f_1 = f \circ f_2 \\
&\Leftrightarrow f_2(a) \in f^{-1}(\psi(y)) \\
&\Leftrightarrow f_2(a) \in \varphi(\overline{f}(y)) \quad \text{since } f^{-1} \circ \psi = \varphi \circ \overline{f} \\
&\Leftrightarrow f_2(a) \in \varphi(x) \\
&\Leftrightarrow a \in f_2^{-1}(\varphi(x)).
\end{aligned}$$

Then $(f_2, h) : (\eta : L \to P(Z)) \to (\varphi : L \to P(X))$ is a **Frm** $\downarrow$ **Set** morphism. Moreover, it is easy to see that

$$(f, \overline{f}) \circ (f_1, g) = (f \circ f_1 = g \circ \overline{f}) = (f \circ f_2, h \circ \overline{f}) = (f, \overline{f}) \circ (f_2, h).$$

Since $(f, \overline{f})$ is left cancellable, we have $(f_1, g) = (f_2, h)$ and so $f_1 = f_2$ which shows that $f$ is injective map in **Set**.
*Sufficiency.* It is straightforward. □

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# Hyers–Ulam Stability of the Additive Functional Equation in Fuzzy Normed Spaces

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**Abstract**

In this paper, we investigate the Hyers-Ulam stability of the additive functional equation in the framework of fuzzy normed spaces. By employing the notions of fuzzy convergence and fuzzy Cauchy sequences, we extend the classical Hyers stability theorem from Banach spaces to fuzzy Banach spaces. We prove the existence and uniqueness of an exact additive mapping which approximates an approximately additive function under a fuzzy control condition. Furthermore, a nontrivial example equipped with a modified fuzzy norm is provided to illustrate the applicability of the obtained result.

## 1. Introduction

The theory of functional equations plays a fundamental role in many branches of mathematics. One of the central problems in this theory is the stability problem, which asks whether a mapping that approximately satisfies a functional equation must be close to an exact solution.

This problem was first posed by Ulam in 1940. Hyers [1] provided the first affirmative answer for additive mappings between Banach spaces. Later, Rassias [2] generalized this result by allowing unbounded perturbations, which led to the development of the Hyers–Ulam–Rassias stability theory.

On the other hand, fuzzy normed spaces were introduced to model uncertainty and imprecision in analysis. Bag and Samanta [3] formalized the notion of fuzzy normed linear spaces and studied their basic properties. Stability problems for functional equations in fuzzy normed spaces have been investigated by several authors (see, for example, [4]).

Motivated by these works, the aim of this paper is to investigate the Hyers–Ulam stability of the additive functional equation

$$f(x+y) = f(x) + f(y),$$

in fuzzy normed spaces. Recent developments on the stability of functional equations in normed and fuzzy normed spaces can be found in [5, 6].

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## 2. Preliminaries

**Definition 2.1.** Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm if for all $x, y \in X$ and $t, s > 0$ the following conditions hold:

1. $N(x,r) = 0$ for all $r \leq 0$;
2. $N(x,r) = 1$ for all $r > 0$ if and only if $x = 0$;
3. $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ for all $c \neq 0$;
4. $N(x+y, t+s) \geq \min\{N(x,t), N(y,s)\}$;
5. $N(x,t)$ is nondecreasing in $t$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair $(X, N)$ is called a fuzzy normed space.

**Definition 2.2.** A sequence $\{x_n\}$ in a fuzzy normed space $(X, N)$ is said to converge to $x \in X$ if

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$.

**Definition 2.3.** A sequence $\{x_n\}$ in $(X, N)$ is called a fuzzy Cauchy sequence if

$$\lim_{m,n\to\infty} N(x_n - x_m, t) = 1$$

for all $t > 0$.

A fuzzy normed space in which every fuzzy Cauchy sequence converges is called a fuzzy Banach space.

### *2.1. Basic Properties and Further Discussion*

Let $(X, N)$ be a fuzzy normed space. It follows immediately from property (N4) that

$$N(x, t+s) \geq N(x,t)$$

for all $x \in X$ and $t, s > 0$. Hence, for fixed $x$, the mapping $t \mapsto N(x,t)$ is nondecreasing.
Moreover, fuzzy norms induce a natural topology on $X$, where a sequence $\{x_n\}$ converges to $x$ if and only if

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$. This topology generalizes the classical norm topology.

**Remark 2.4.**
In general, fuzzy convergence does not coincide with norm convergence. However, if the fuzzy norm is generated by a classical norm via

$$N(x,t) = \frac{t}{t + \|x\|},$$

then fuzzy convergence and norm convergence are equivalent.

**Remark 2.4.** Unlike classical normed spaces, fuzzy normed spaces allow a more flexible control of approximation errors. This flexibility plays a crucial role in stability problems, where perturbations are naturally expressed in terms of membership values.

**Definition 2.5.** (Approximately additive mapping). Let $X$ be a real vector space and $(Y, N)$ a fuzzy normed space. A mapping $f : X \to Y$ is called approximately additive if there exists a function $\psi : (0,\infty) \to (0,1]$ such that

$$N\big(f(x+y) - f(x) - f(y), t\big) \geq \psi(t)$$

for all $x, y \in X$ and $t > 0$, where $\lim_{t\to\infty} \psi(t) = 1$.

## 3. Main Result

**Lemma 3.1.** *Let $(Y, N)$ be a fuzzy normed space. If a sequence $\{y_n\}$ satisfies*

$$N(y_{n+1} - y_n, t) \geq \alpha(t)$$

*for all $n \in \mathbb{N}$ and $t > 0$, where $\lim_{t\to\infty} \alpha(t) = 1$, then $\{y_n\}$ is a fuzzy Cauchy sequence.*

*Proof.* The result follows from repeated application of property (N4) and the monotonicity of $N$ with respect to the second variable. □ □

**Lemma 3.2.** *Let $(Y, N)$ be a fuzzy Banach space and let $\{y_n\}$ be a fuzzy Cauchy sequence. Then there exists $y \in Y$ such that*

$$\lim_{n\to\infty} N(y_n - y, t) = 1$$

*for all $t > 0$.*

*Proof.* The proof follows directly from the completeness of the fuzzy Banach space. □ □

**Theorem 3.3.** *Let $X$ be a real vector space and let $(Y, N)$ be a fuzzy Banach space. Suppose that $f : X \to Y$ satisfies $f(0) = 0$ and*

$$N\big(f(x + y) - f(x) - f(y), t\big) \geq \psi(t) \tag{1}$$

*for all $x, y \in X$ and $t > 0$, where $\psi : (0, \infty) \to (0, 1]$ and $\lim_{t\to\infty} \psi(t) = 1$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$N\big(f(x) - A(x), t\big) \geq \psi(t)$$

*for all $x \in X$ and $t > 0$.*

*Proof.* Fix $x \in X$. Putting $y = x$ in (1), we obtain

$$N\big(f(2x) - 2f(x), t\big) \geq \psi(t) \qquad (t > 0).$$

For each $n \in \mathbb{N} \cup \{0\}$, define

$$f_n(x) = \frac{1}{2^n} f(2^n x).$$

Then

$$f_{n+1}(x) - f_n(x) = \frac{1}{2^{n+1}}\Big(f(2^{n+1}x) - 2f(2^n x)\Big).$$

Using property (N3) of fuzzy norms, we have

$$N\big(f_{n+1}(x) - f_n(x), t\big) = N\big(f(2^{n+1}x) - 2f(2^n x), 2^{n+1}t\big).$$

Applying (1) with $x$ replaced by $2^n x$, we obtain

$$N\big(f_{n+1}(x) - f_n(x), t\big) \geq \psi(2^{n+1}t) \qquad (n \geq 0).$$

Let $m > n$. Then

$$f_m(x) - f_n(x) = \sum_{k=n}^{m-1} \big(f_{k+1}(x) - f_k(x)\big).$$

Choose a fixed $T > 0$ and set $t_k = 2^{-(k+1)}T$. By repeated use of (N4), we obtain

$$N\left(f_m(x) - f_n(x), \sum_{k=n}^{m-1} t_k\right) \geq \min_{k=n,\ldots,m-1} N\big(f_{k+1}(x) - f_k(x), t_k\big).$$

Since

$$\sum_{k=n}^{m-1} t_k \le T \sum_{k=n}^{\infty} 2^{-(k+1)} = T\,2^{-n},$$

and $N(\cdot, t)$ is nondecreasing in $t$, it follows that

$$N\big(f_m(x) - f_n(x), T\,2^{-n}\big) \ge \min_{k=n,\dots,m-1} N\big(f_{k+1}(x) - f_k(x), 2^{-(k+1)}T\big).$$

From the previous inequality we have

$$N\big(f_{k+1}(x) - f_k(x), 2^{-(k+1)}T\big) \ge \psi(T),$$

hence

$$N\big(f_m(x) - f_n(x), T\,2^{-n}\big) \ge \psi(T).$$

Fix $t > 0$. Choose $n$ large enough such that $T\,2^{-n} \le t$. By monotonicity,

$$N\big(f_m(x) - f_n(x), t\big) \ge \psi(T) \qquad (m > n).$$

Letting $m, n \to \infty$ and then $T \to \infty$, we conclude that

$$lim_{m,n\to\infty} N\big(f_m(x) - f_n(x), t\big) = 1,$$

so $\{f_n(x)\}$ is a fuzzy Cauchy sequence. Since $(Y, N)$ is a fuzzy Banach space, there exists $A(x) \in Y$ such that

$$\lim_{n\to\infty} N\big(f_n(x) - A(x), t\big) = 1 \qquad (t > 0).$$

Define $A : X \to Y$ by this limit.
Now let $x, y \in X$. Applying (1) to $(2^n x, 2^n y)$ and using (N3), we obtain

$$N\left(\frac{1}{2^n}\big(f(2^n(x+y)) - f(2^n x) - f(2^n y)\big), t\right) \ge \psi(2^n t).$$

Passing to the limit as $n \to \infty$ yields

$$N\big(A(x+y) - A(x) - A(y), t\big) = 1,$$

which implies that $A$ is additive.
Finally, taking $m \to \infty$ in the previous estimates yields

$$N\big(f(x) - A(x), t\big) \ge \psi(t).$$

For completeness, we include the details of uniqueness. Suppose that $B : X \to Y$ is another additive mapping satisfying

$$N\big(f(x) - B(x), t\big) \ge \psi(t) \qquad (x \in X,\ t > 0).$$

Using (N4) with $u = A(x) - f(x)$ and $v = f(x) - B(x)$, we obtain

$$N\big(A(x)-B(x), 2t\big) = N\big((A(x)-f(x))+(f(x)-B(x)), 2t\big) \ge \min\{N(A(x)-f(x), t),\ N(f(x)-B(x), t)\} \ge \psi(t).$$

Letting $t \to \infty$ and using $\lim_{t\to\infty} \psi(t) = 1$, we get

$$\lim_{t\to\infty} N\big(A(x) - B(x), 2t\big) = 1,$$

which implies $A(x) = B(x)$ for all $x \in X$. Hence, the additive mapping $A$ is unique. □

**Remark 3.4.** The function $\psi$ can be interpreted as a control function measuring the degree of deviation from exact additivity. The stability result shows that the asymptotic condition $\lim_{t\to\infty}\psi(t) = 1$ is sufficient to guarantee the existence of an exact additive mapping near $f$.

**Remark 3.5.** If the fuzzy norm $N$ is induced by a classical norm $\|\cdot\|$ via

$$N(x,t) = \frac{t}{t+\|x\|},$$

then Theorem 3.3 reduces to the classical Hyers stability theorem in Banach spaces. Hence, our result can be viewed as a genuine extension of the classical Hyers–Ulam stability to fuzzy normed spaces.

## 4. Consequences, Quantitative Comments and Examples

**Note.** The proof of the following consequences is straightforward and follows directly from Theorem 3.3 and the definitions of fuzzy convergence. Therefore, the details are omitted.

**Corollary 4.1.** *Under the assumptions of Theorem 3.3, for every $x \in X$ the sequence*

$$\left\{\frac{1}{2^n}f(2^n x)\right\}_{n\geq 0}$$

*converges in the fuzzy Banach space $(Y,N)$, and its limit coincides with the unique additive mapping $A(x)$.*

*Proof.* The proof is contained in the construction in Theorem 3.3. Indeed, the estimates on the increments $f_{n+1}(x) - f_n(x)$ combined with (N4) show that $\{f_n(x)\}$ is fuzzy Cauchy; hence it converges by completeness. By definition of $A$, the limit must be $A(x)$. □

**Corollary 4.2.** *Under the assumptions of Theorem 3.3, if the mapping $f$ is fuzzy continuous at a point, then the additive mapping $A$ is linear.*

**Remark 4.3.** The above arguments indicate that the stability phenomenon is governed mainly by the asymptotic behavior $\psi(t) \to 1$ as $t \to \infty$. Different fuzzy norms may lead to different convergence profiles, while the limit additive mapping remains the same.

**Example 4.4.** Let $X = Y = \mathbb{R}$ and define

$$N(x,t) = \frac{t}{t+|x|}.$$

Then $(\mathbb{R}, N)$ is a fuzzy Banach space. Define $f(x) = x + \sin x$. Then $f$ is approximately additive, and the additive mapping $A(x) = x$ approximates $f$.

This example illustrates that the additive mapping $A$ captures the linear part of $f$ in a stable fuzzy sense.

**Example 4.5.** Let $X = Y = \mathbb{R}$ and define

$$N(x,t) = \exp\left(-\frac{|x|}{t+|x|}\right).$$

Define $f(x) = x + \varepsilon \sin x$. Then $f$ is approximately additive but not additive. Theorem 3.3 yields $A(x) = x$.

**Example 4.6.** Let $X = Y = \mathbb{R}$ and define a family of fuzzy norms

$$N_p(x,t) = \exp\left(-\frac{|x|^p}{t}\right), \qquad p \geq 1.$$

Then $(\mathbb{R}, N_p)$ is a fuzzy Banach space. Let $f(x) = x + \epsilon \sin x$ with $\epsilon > 0$. Since $|\sin(x+y) - \sin x - \sin y| \leq 3$, we have $|f(x+y) - f(x) - f(y)| \leq 3\varepsilon$, and therefore

$$N_p\big(f(x+y) - f(x) - f(y), t\big) \geq \exp\left(-\frac{(3\epsilon)^p}{t}\right)$$

Hence Theorem 3.3 applies and yields the additive mapping $A(x) = x$.

**Example 4.7.** Let $X = Y = \mathbb{R}^2$ and define

$$N\big((x_1, x_2), t\big) = \exp\left(-\frac{|x_1| + |x_2|}{t}\right)$$

Define $f(x_1, x_2) = (x_1, x_2) + (\sin x_1, \sin x_2)$. Then $f$ is approximately additive in the fuzzy sense.

**Remark 4.8.** The above examples show that the obtained stability result is robust with respect to different choices of fuzzy norms and spaces.

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# Strong KC-proximal $\vartheta$-contractions of the second kinds

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**Abstract**

In this paper, we aim to examine theorems from fixed point theory in the frameworks of fairly complete spaces and proximally complete spaces. These spaces do not necessarily need to be complete or possess the $UC$ property.
Let $I$ be an identity function defined on $\mathbb{R}_+$ and $\mathcal{K} := \{\varphi \in C(\mathbb{R}_+, \mathbb{R}_+) : \varphi$ is strictly increasing function and $I - \varphi$ is increasing$\}$. Given non-void subsets $A$ and $B$ of a metric space $(X, d)$, a mapping $T : A \to B$ is said to be a KC-proximal $\vartheta$-contraction of the second kind if there exist $\vartheta \in \mathcal{K}$ such that $d(u_1, Tx_1) = d(A, B)$ and $d(u_2, Tx_2) = d(A, B)$ implies that

$$d(Tu_1, Tu_2) \leq (I - \vartheta)\big(\max\{d(Tx_1, Tx_2), d(Tx_1, Tu_1), d(Tx_2, Tu_2), d(Tx_1, Tu_2), d(Tx_2, Tu_1)\}\big).$$

for all $x_1, x_2, u_1, u_2 \in A$. Furthermore, if $T$ is proximally quasi-continuous, then $T$ is called strong KC-proximal $\vartheta$-contraction of the second kind.
let us assume that $A_0$ and $B_0$ are non-void, $(A, B)$ is a fairly complete space, the subspace $B$ has uniform approximation in the subspace $A$, the mapping $T : A \to B$ is a strong KC-proximal $\vartheta$-contraction of the second kind; and $T(A_0) \subseteq B_0$. Then we show that there exists a unique best proximity point $x^* \in A$ such that for any given initial element $x_0 \in A_0$, the iterative sequence $\{x_n\}$, defined by $d(x_{n+1}, Tx_n) = d(A, B)$, converges to $x^*$. Furthermore, we prove that the iterative sequence $\{Tx_n\}$ converges to $Tx^*$ and a priori error estimate holds in: $0 \leq d(Tx_n, Tx^*) \leq (I - \vartheta)^n \left(\vartheta^{-1}\big(d(Tx_0, Tx_1)\big)\right)$.

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# Several Results in Continuous Inverse Algebras

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**Abstract**

On continuous inverse algebras, with properties similar to the Banach algebras, some results have been already obtained. Here we continue this process and get some new results.

## 1. Introduction

Throughout this paper, all algebras are complex and unital. The identity element of an algebra A is denoted by e. Non-normed topological algebras play a crucial role in the development of research in topological and even normed algebras. The study of non-normed topological algebras was initiated in 1947 by Arens[1]. It was in 1952 that Arens and Michael independently published the first systematic study on locally m-convex algebras, which constitutes an important class of non-normed topological algebras, see [5]. For example, Michael posed the question as whether each character on a commutative Fréchet algebra is automatically continuous. This question, known as the Michael's problem, has been intensively studied, but only partial answers have been obtained so far. An important class of topological algebras namely continuous inverse algebras have interesting properties similar to Banach algebras. Hence, we can generalize and obtain some propertise of Banach algebras to continuous inverse algebras.

## 2. Preliminaries

In this section, we present a collection of definitions and notions.

**Definition 2.1.** [2] For an algebra $A$, the spectrum $sp_A(x)$ of an element $x \in A$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible in $A$. The spectral radius $r_A(x)$ of an element $x \in A$ is defined by $r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}$.

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**Definition 2.2.** [2] Let $A$ be an algebra. An element $x \in A$ is said to be nilpotent if $x^m = 0$ for some natural number $m \geq 1$. It is said to be quasinilpotent if $sp_A(x) = \{0\}$.

**Lemma 2.3.** *[4, 1.5.32] If A is a unital algebra, then*

$$RadA = \{x \in A : r_A(xy) = 0; \textit{for every } y \in A\},$$

*where $RadA$ is the Jacobson radical of A.*

**Definition 2.4.** [2] For a continuous inverse algebra $A$, we define the centre modulo the Radical of $A$ by

$$Z(A) = \{a \in A : ax - xa \in RadA, x \in A\}.$$

**Definition 2.5.** [3] Let $E$ be a locally convex space. A sequence $(x_n) \subset E$ is called a Mackey-Cauchy sequence (or simply a Mackey sequence) if there exists a bounded set $B$ such that

$$x_n - x_m \in \varepsilon_{n,m}B \;\; with \;\; \varepsilon_{n,m} \longrightarrow 0 \;\; (n, m \longrightarrow \infty).$$

Every Mackey-Cauchy sequence is a Cauchy sequence, but not conversely.

**Definition 2.6.** [3] A locally convex space $E$ is called Mackey complete if every Mackey-Cauchy sequence converges in $E$.

**Definition 2.7.** [3] A continuous inverse algebra is a locally convex algebra in which the set of invertible elements is a neighbourhood of e and inversion is continuous at e.

**Definition 2.8.** [2] The Hausdorff distance is defined by

$$\Delta(K_1, K_2) = \max\left(\sup_{z \in K_2} \operatorname{dist}(z, K_1), \sup_{z \in K_1} \operatorname{dist}(z, K_2)\right)$$

for $K_1, K_2$ compact subsets of $\mathbb{C}$. Let $r > 0$ and $K$ be a compact subset of $\mathbb{C}$. If $K + r$ denotes $\{z : \operatorname{dist}(z, K) \leq r\}$, then obviously $K_1 \subseteq K_2 + \Delta(K_1, K_2)$ and $K_2 \subseteq K_1 + \Delta(K_1, K_2)$.

## 3. Main results

In this section, we we generalize and obtain some properties of Banach algebras to continuous inverse algebras.

**Theorem 3.1.** *Let $A$ be a continuous inverse algebra. If $x \in A$ is nilpotent, then $x$ is quasinilpotent. The converse holds true if $A$ is finite dimensional.*

**Remark 3.2.** The converse of Therem 3.1, is false in general even in Banach algebras (see for instance, [7, Example 4.2]).

**Theorem 3.3.** *Suppose $A$ is a continuous inverse algebra, $x \in A$, $p$ is a polynomial in one variable, and $p(x) = 0$. Then $sp_A(x)$ lies in the set of zeros of $p$. In particular, if $x$ is idempotent, then $sp_A(x) \subset \{0, 1\}$.*

**Theorem 3.4.** *If the spectrum of some elements of Mackey complete continuous inverse algebra $A$ is not connected, then $A$ contains a non-trivial idempotent.*

**Theorem 3.5.** *Let $A$ be a Mackey complete continuous inverse algebra. If $a$ is a non-trivial idempotent and $\lambda \notin sp_A(a)$, then*

$$\operatorname{dist}(\lambda, sp_A(a)) = \frac{1}{r_A((\lambda e - a)^{-1})}$$

Now, we obtain the following theorem without using the idempotent elements.

**Theorem 3.6.** *Let $A$ be a Mackey complete continuous inverse algebra. Suppose that $x \in A$ and that $\lambda \notin sp_A(x)$. Then*

$$\operatorname{dist}(\lambda, sp_A(x)) = \frac{1}{r_A((\lambda e - x)^{-1})}.$$

**Theorem 3.7.** *Let $A$ be a Mackey complete continuous inverse algebra. If $z = \mu e - (y - \lambda e)^{-1}$ with $\mu = \frac{-1}{\lambda}$ and $|\lambda| > 2r_A(y)$, then $r_A(z) \leqq |\mu|$.*

**Lemma 3.8.** *Let $A$ be a Mackey complete continuous inverse algebra. Suppose that $x, y \in A$ commute. Then*

$$sp_A(y) \subseteq sp_A(x) + r_A(x - y),$$

*and consequently we have*

$$\Delta(sp_A(x), sp_A(y)) \leq r_A(x - y). \tag{1}$$

**Remark 3.9.** If $A$ is a Banach algebra and $x, y \in A$ commute, then

$$\Delta(sp_A(x), sp_A(y)) \leq r_A(x - y) \leq \|x - y\|. \tag{2}$$

Hence, we have the following results.

**Corollary 3.10.** *Let $A$ be a commutative Banach algebra. Then the spectrum function $x \longmapsto sp_A(x)$ is uniformly continuous on $A$.*

**Remark 3.11.** In Theorem 3.10, the commutativity of $A$ is essential (see the example in [2, p. 48]).

**Theorem 3.12.** *Let $A$ be a Mackey complete continuous inverse algebra containing a non-empty open set $U$ of invertible elements such that $r_A(x)r_A(x^{-1}) = 1$ for all $x \in U$. Then there exists $\alpha > 0$ such that*

$$sp_A(x) \subseteq \overline{D}(\lambda, \operatorname{dist}(\lambda, sp_A(x)), \ \ |\lambda| < \alpha. \tag{3}$$

**Theorem 3.13.** *Let $A$ be a Mackey complete continuous inverse algebra, $a \in A$ and suppose that $\alpha$ is a nonzero isolated point of $sp_A(a)$. If $p$ is the spectral idempotent associated with $a$ and $\alpha$, then there exists $c \in A$ such that $p = ac = ca$.*

**Theorem 3.14.** *Let $A$ be a continuous inverse algebra and let $B$ be a subalgebra of $A$ such that $sp_A(b)$ consists of zero and possibly one other point for every $b \in B$. Then there are no nonzero orthogonal idempotents in $B$.*

**Theorem 3.15.** *Let $A$ be a continuous inverse algebra with $a \in A$ and consider the following statements: 1) $a \in Z(A)$,*

*2) There exists $N > 0$ such that*

$$r_A(a - \lambda e)^{-1}x \leq N r_A(a - \lambda e)^{-1} r_A(x)$$

*for all $x \in A$ and $\lambda \notin sp_A(a)$,*
*3) There exists $M > 0$ such that*

$$r_A(a + x) \leq M(1 + r_A(x)),$$

*for every $x \in A$.*
*Then*
*1)$\Longrightarrow$ 2) $\Longrightarrow$ 3).*

**Remark 3.16.** It is well known that every Banach algebra is a continuous inverse algebra, but the converse may be false in general (see for instance, [6, Example 3.1]).

## 4. Conclusion

In this paper, we we generalized and obtained some properties of Banach algebras to continuous inverse algebras.

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# Schur Product Inequalities for Sectorial Matrices

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**Abstract**

In this paper, we establish several inequalities involving operator means of sectorial matrices, which extend and refine a number of previously known results. In particular, let $A_i, B_i \in S_\alpha$, $i = 1, \dots, n$, satisfy $Re\, A_1 \geq \cdots \geq Re\, A_n \geq 0$ and $Re\, B_1 \geq \cdots \geq Re\, B_n \geq 0$, and assume that $Im\, A_i \circ Im\, B_i \leq 0$. If $\sigma$ is an operator mean whose representing function is super-multiplicative, then

$$\left(\sum_{i=1}^{n} p_i Re(A_i \sigma B_i)\right) \circ \left(\sum_{j=1}^{n} p_j Re(A_j \sigma^{\circ} B_j)\right) \leq \sec^4(\alpha)\Big(\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i Re(A_i \circ B_i)\Big),$$

where $p_i \geq 0$, $i = 1, \dots, n$, are weights.

## 1. Introduction

Let $\mathbb{M}_n$ denote the algebra of all $n \times n$ complex matrices. For Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \geq 0$ if $A$ is positive semidefinite, that is, $\langle Ax, x\rangle \geq 0$ for all vectors $x \in \mathbb{C}^n$. We write $A > 0$ if $A$ is positive definite, that is, $\langle Ax, x\rangle > 0$ for all vectors $x \in \mathbb{C}^n$, and $A \geq B$ if $A - B \geq 0$.
A matrix $A \in \mathbb{M}_n$ is called accretive if, in its Cartesian (or Toeplitz) decomposition

$$A = Re\, A + i\, Im\, A,$$

the real part $Re\, A$ is positive definite, where $Re\, A = \frac{A+A^*}{2}$ and $Im\, A = \frac{A-A^*}{2i}$.
The numerical range of a matrix $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\}.$$

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A matrix $A \in \mathbb{M}_n$ is said to be sectorial if $W(A) \subset S_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$, where $S_\alpha$ denotes the sector region in the complex plane defined by

$$S_\alpha = \{z \in \mathbb{C} : Re\, z \geq 0,\ |Im\, z| \leq (Re\, z) \tan \alpha\}.$$

Clearly, $A$ is positive semidefinite if and only if $W(A) \subseteq S_0$. Moreover, if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in \left[0, \frac{\pi}{2}\right)$, then $W(A+B) \subseteq S_\alpha$. Furthermore, if $W(A) \subseteq S_\alpha$, then $W(X^*AX) \subseteq S_\alpha$ for any nonzero $n \times m$ matrix $X$; in particular, $W(A^{-1}) \subseteq S_\alpha$. The smallest such $\alpha$ is called the sectorial index of $A$. When $W(A) \subseteq S_\alpha$, we write $A \in S_\alpha$.
An operator mean $\sigma$ in the sense of Kubo–Ando [1] is defined via an operator monotone function $f : (0,\infty) \to (0,\infty)$ satisfying $f(1) = 1$ by

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

for positive invertible operators $A$ and $B$. The function $f$ is called the representing function of $\sigma$. Recently, Bedrani et al. showed that this definition can also be extended to accretive operators.
Some important operator means are listed below:

- **Arithmetic mean:** $A\nabla B = (A+B)/2$ and the $t$-weighted arithmetic mean
  $A\nabla_t B = tA + (1-t)B, \qquad (0 < t < 1)$
- **Geometric mean:** $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and the $t$-weighted geometric mean
  $A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \qquad (0 < t < 1)$

The operator mean represented by the function $tf(t^{-1})$ is called the transpose of $\sigma$ and is denoted by $\sigma^0$, that is,

$$A\sigma^0 B = B\sigma A.$$

Moreover, the operator mean represented by the function $\frac{t}{f(t)}$ is called the dual of $\sigma$ and is denoted by $\sigma^\perp$, namely,

$$A\sigma^\perp B = \left(A^{-1}\sigma B^{-1}\right)^{-1}.$$

For example, $\sharp_r^\perp = \sharp_{1-r} = \sharp_r^0$ and $!_{1-r}^\perp = \nabla_r$.
Let $A = [a_{ij}] \in \mathbb{M}_{m\times n}$ and $B = [b_{ij}] \in \mathbb{M}_{p\times q}$. The Schur product (or Hadamard product) of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $\mathbb{M}_{m\times n}$, denoted by $A \circ B$, is defined entrywise by

$$A \circ B = [a_{ij} b_{ij}].$$

This operation preserves positivity: if $A, B \in \mathbb{M}_n^+$, then $A \circ B \geq 0$ (see, for example, [2, Chapter 4]). Moreover, it is monotone with respect to the Löwner order; specifically, if $A_1 \geq A_2 \geq 0$ and $B_1 \geq B_2 \geq 0$, then

$$A_1 \circ B_1 \geq A_2 \circ B_2 \geq 0. \tag{1}$$

It should be noted, however, that the Schur product of two accretive matrices need not be accretive. For example, if $A = B = 1 + i$ (viewed as $1 \times 1$ matrices), then $A \circ B = (1+i)^2 = 2i$, whose real part is zero; hence $A \circ B$ is not accretive.
It is well known that the set of $m \times n$ complex matrices forms an abelian group under the Schur product, that is, $A \circ B = B \circ A$. Moreover, the Schur product is bilinear over $\mathbb{C}$. Specifically, for any scalars $\alpha, \beta \in \mathbb{C}$ and matrices $A, B, C$ of compatible sizes,

$$C \circ (A+B) = C \circ A + C \circ B,$$

and

$$\alpha(A \circ B) = (\alpha A) \circ B = A \circ (\alpha B).$$

Consequently,

$$(\alpha A) \circ (\beta B) = \alpha\beta\, (A \circ B).$$

Since the Schur product distributes over matrix addition, for any $A, B \in \mathbb{M}_n$ we have

$$\begin{aligned} A \circ B &= (Re\, A + i\, Im\, A) \circ (Re\, B + i\, Im\, B) \\ &= (Re\, A \circ Re\, B - Im\, A \circ Im\, B) + i\, (Re\, A \circ Im\, B + Im\, A \circ Re\, B). \end{aligned}$$

Taking real parts yields

$$Re\,(A \circ B) = Re\,A \circ Re\,B - Im\,A \circ Im\,B. \tag{2}$$

From (2), it follows that if $Im\,A \circ Im\,B \leq 0$, then $Re\,(A \circ B) \geq Re\,A \circ Re\,B$. In particular, if $Re\,A \geq 0$, $Re\,B \geq 0$, and $Im\,A \circ Im\,B \leq 0$ (for instance, if $Im\,A \geq 0$ and $Im\,B \leq 0$), then the Schur product $A \circ B$ is accretive. This condition, however, is sufficient but not necessary.
Since $Re\,(A \circ B)$ is Hermitian, it can be incorporated into matrix inequalities even when it is not positive semidefinite. The following lemma follows directly from (2).

**Lemma 1.1.** *[4] Let $A, B \in \mathbb{M}_n$ be accretive matrices.*

- *If $Im\,A \circ Im\,B \geq 0$, then*
$$Re\,(A \circ B) \leq Re\,A \circ Re\,B.$$
- *If $Im\,A \circ Im\,B \leq 0$, then*
$$Re\,A \circ Re\,B \leq Re\,(A \circ B).$$

For further details on this topic, we refer the reader to [4, 7].
A map $\Phi : \mathbb{M}_n \times \mathbb{M}_n \to \mathbb{M}_m$ is said to be jointly convex if

$$\Phi(\lambda A + (1-\lambda)B,\ \lambda C + (1-\lambda)D) \leq \lambda\Phi(A, C) + (1-\lambda)\Phi(B, D)$$

for all $\lambda \in [0, 1]$ and all $A, B \in \mathbb{M}_n$.

**Lemma 1.2.** *[3, Theorem 6.6] If $A$ and $B$ are positive matrices, then*

$$A \circ B \geq (A\sharp_r B) \circ (A\sharp_{1-r} B) \quad \textit{and} \quad A \circ B \geq (A!_{1-r} B) \circ (A\nabla_r B) \tag{3}$$

*for all $r \in [0, 1]$.*

Recall that if $I$ is an interval in $\mathbb{R}$, a function $f \in \mathcal{C}(I)$ is said to be super-multiplicative on $I$ if

$$f(xy) \geq f(x)f(y)$$

for all $x, y \in I$.
Pecarić et al. [3] derived the following extension of the first inequality in (3). If $\sigma$ is an operator mean with a super-multiplicative representing function $f$, then

$$A \circ B \geq (A\sigma B) \circ (A\sigma^0 B) \tag{4}$$

holds for positive operators $A$ and $B$.

## 2. Main Results

In this section, we extend several known inequalities for positive definite matrices to the class of sectorial matrices. Singh et al. [5] obtained a number of inequalities for positive definite matrices. In particular, they proved that if $A_i, B_i$, $i = 1, \ldots, n$, are positive definite matrices in $\mathbb{M}_n$, then

$$\left(\sum_{i=1}^{n} A_i\sharp B_i\right) \circ \left(\sum_{i=1}^{n} A_i\sharp B_i\right) \leq \left(\sum_{i=1}^{n} A_i\right) \circ \left(\sum_{i=1}^{n} B_i\right),$$

and

$$\left(\sum_{i=1}^{n} A_i\sharp B_i\right) \circ \left(\sum_{i=1}^{n} A_i\sharp B_i\right) \leq \left(\sum_{i=1}^{n} A_i\sigma B_i\right) \circ \left(\sum_{i=1}^{n} A_i\sigma^{\perp} B_i\right). \tag{5}$$

Matharu et al. [6] proved that if $A_1 \geq \cdots \geq A_n \geq 0$ and $B_1 \geq \cdots \geq B_n \geq 0$, then

$$\left(\sum_{i=1}^{n} p_i A_i\right) \circ \left(\sum_{i=1}^{n} p_i B_i\right) \leq \left(\sum_{i=1}^{n} p_i\right)\left(\sum_{i=1}^{n} p_i (A_i \circ B_i)\right),$$

where $p_i \geq 0$, $i = 1, \ldots, n$, are weights.

**Proposition 2.1.** *If $A_i, B_i \in S_\alpha$, with $Re\, A_1 \geq \cdots \geq Re\, A_n \geq 0$ and $Re\, B_1 \geq \cdots \geq Re\, B_n \geq 0$, such that $Im\, A_i \circ Im\, B_i \leq 0$, then*

$$Re\left(\sum_{i=1}^{n} p_i A_i\right) \circ Re\left(\sum_{i=1}^{n} p_i B_i\right) \leq \left(\sum_{i=1}^{n} p_i\right) Re\left(\sum_{i=1}^{n} p_i (A_i \circ B_i)\right),$$

*where $p_i \geq 0$, $i = 1, \ldots, n$, are weights.*

The following theorem extends inequality (5) to the sectorial setting.

**Theorem 2.2.** *Let $A_i, B_i \in S_\alpha$ for all $i = 1, \ldots, n$, such that $Im(A_i \sigma B_i) \circ Im(A_i \sigma^{\perp} B_i) \leq 0$. Then*

$$\left(\sum_{i=1}^{k} Re(A_i \sharp B_i)\right) \circ \left(\sum_{i=1}^{k} Re(A_i \sharp B_i)\right) \leq \sec^4(\alpha) Re\left(\left(\sum_{i=1}^{k} A_i \sigma B_i\right) \circ \left(\sum_{i=1}^{k} A_i \sigma^{\perp} B_i\right)\right).$$

In the following proposition, we present the sectorial analogue of inequality (4).

**Proposition 2.3.** *If $A, B \in S_\alpha$ such that $Im\, A \circ Im\, B \leq 0$, and $\sigma$ is an operator mean with a super-multiplicative representing function, then*

$$Re(A \sigma B) \circ Re(A \sigma^{\circ} B) \leq \sec^4(\alpha)\, Re(A \circ B).$$

Using Proposition 2.3, we establish the following result.

**Theorem 2.4.** *If $A_i, B_i \in S_\alpha$, $Re\, A_1 \geq \cdots \geq Re\, A_n \geq 0$ and $Re\, B_1 \geq \cdots \geq Re\, B_n \geq 0$, such that $Im\, A_i \circ Im\, B_i \leq 0$, and $\sigma$ is an operator mean with a super-multiplicative representing function, then*

$$\left(\sum_{i=1}^{n} p_i Re(A_i \sigma B_i)\right) \circ \left(\sum_{j=1}^{n} p_j Re(A_j \sigma^{\circ} B_j)\right) \leq \sec^4(\alpha)\Big(\sum_{j=1}^{n} p_j \sum_{i=1}^{n} p_i Re(A_i \circ B_i)\Big),$$

*where $p_i \geq 0$, $i = 1, \ldots, n$, are weights.*

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# Ternary m-Hom-Ders on Ternary Banach Algebras: Stability

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**Abstract**

In this paper, we introduce the concept of ternary $m$-hom-ders on ternary Banach algebras with associate $m$-mapping where $m = 1, 2, 3, 4$ such that for $m = 1$ ternary linear hom-ders (abbreviated ternary 1-hom-ders), for $m = 2$ ternary quadratic hom-ders (abbreviated ternary 2-hom-ders), for $m = 3$ ternary cubic hom-ders (abbreviated ternary 3-hom-ders) and for $m = 4$ ternary quartic hom-ders (abbreviated ternary 4-hom-ders) on ternary Banach algebras. Finally, using fixed point method, we prove Hyers–Ulam stability of ternary $m$-hom-ders on ternary Banach algebras with an associated $m$-mapping, using three control functions: Găvruta, Rassias, and JMRassias.

## 1. Introduction

In 1940, Ulam [9] posed several open problems in a lecture at the University of Wisconsin, including the stability of homomorphisms, which stimulated broad interest in stability of functional equations. In 1941, Hyers [4] resolved Ulam's question for additive maps on Banach spaces, initiating what is now called Hyers–Ulam stability. In 1978, Th. M. Rassias [8] result of generalized Hyers by introducing the control term $\theta(\|t_1\|^p + \|t_2\|^p)$ for $p < 1$ and $\theta > 0$, yielding the generalized Hyers–Ulam stability. In 1982, J. M. Rassias [7] proposed an alternative control $\theta(\|t_1\|^p.\|t_2\|^p)$ with $p < 1$ and $\theta > 0$, defining another class of admissible controls. In 1994, Găvruta [2] replaced these specific bounds by a general control function $\varphi(t_1, t_2)$ and studied Hyers–Ulam stability in this broader framework. The topic has drawn significant attention: researchers have studied stability of derivations and homomorphisms, examined problems related to the Fibonacci sequence, and tackled various Banach-space questions using different control functions and both fixed point and direct methods. In 2017, Eshaghi Gordji et al. [3] studied about ternary Jordan homomorphisms with different variables after that they proved stability and superstability of ternary Jordan homomorphisms between ternary Banach algebras by using fixed point methods. In 2024, Keshavarz and Heydari [5] introduced additive–Jensen s-functional equations and generalized Lie-bracket derivations and analyzed properties of Jordan Lie brackets of derivations and establish Hyers–Ulam stability for the additive–Jensen s-equations and generalized Lie-bracket derivations.

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In the section 2, we establish the basic notions of $m$-mapping for $m = 1, 2, 3, 4$. Subsequently, we introduce and examine the novel concept of ternary $m$-hom-ders on ternary Banach algebras.

In the section 3, we employ the fixed point method to prove the Hyers-Ulam stability of ternary $m$-hom-ders and their associated $m$-mapping for $m = 1, 2, 3, 4$ on ternary Banach algebras. This is established using three distinct control functions: those of Găvruta, Rassias, and JMRassias.

## 2. Preliminaries

In this section, Let $\mathcal{B}$ is ternary Banach algebra.
In 2012, Alizadeh et al. [1] defined the $m$-mapping and proved its stability. In the following, we will discuss the defined concept.
The monomial $\mathfrak{h}(t) = at^m$ $(t \in \mathbb{R}, m = 1, 2, 3, 4)$ is a solution of the functional equations,

$$\mathfrak{h}(at + r) + \mathfrak{h}(at - r) = a^{m-2}[\mathfrak{h}(t + r) + \mathfrak{h}(t - r)] + 2(a^2 - 1)[a^{m-2}\mathfrak{h}(t) + \frac{(m-2)(1-(m-2)^2)}{6}\mathfrak{h}(r)], \quad (1)$$

for all $t, r \in \mathcal{B}$ and $a \neq 0, \pm 1$.
In the next definition, using the concept of linear, quadratic, cubic and quartic mapping, we define the $m$-hom-ders where $m = 1, 2, 3, 4$.

**Definition 2.1.** Let $\zeta : \mathcal{B} \to \mathcal{B}$ be $m$-mapping,

(a) a $\mathbb{C}$-linear mapping $\zeta : \mathcal{B} \to \mathcal{B}$ is called a ternary linear hom-ders (briefly, 1-hom-ders) if $\zeta$ satisfies,

$$[\zeta(u), \zeta(v), \zeta(w)] = [\zeta(u), v, w] + [u, \zeta(v), w] + [u, v, \zeta(w)], \qquad \forall u, v, w \in \mathcal{B}.$$

(b) a quadratic mapping $\zeta : \mathcal{B} \to \mathcal{B}$ is called a ternary quadratic hom-ders (briefly, 2-hom-ders) if $\zeta$ satisfies,

$$[\zeta(u), \zeta(v), \zeta(w)] = [\zeta(u), v^2, w^2] + [u^2, \zeta(v), w^2] + [u^2, v^2, \zeta(w)], \qquad \forall u, v, w \in \mathcal{B}.$$

(3) a cubic mapping $\zeta : \mathcal{B} \to \mathcal{B}$ is called a ternary cubic hom-ders (briefly, 3-hom-ders) if $\zeta$ satisfies,

$$[\zeta(u), \zeta(v), \zeta(w)] = [\zeta(u), v^3, w^3] + [u^3, \zeta(v), w^3] + [u^3, v^3, \zeta(w)], \qquad \forall u, v, w \in \mathcal{B}.$$

(4) a quartic mapping $\zeta : \mathcal{B} \to \mathcal{B}$ is called a ternary quartic hom-ders (briefly, 4-hom-ders) if $\zeta$ satisfies,

$$[\zeta(u), \zeta(v), \zeta(w)] = [\zeta(u), v^4, w^4] + [u^4, \zeta(v), w^4] + [u^4, v^4, \zeta(w)], \qquad \forall u, v, w \in \mathcal{B}.$$

In the following, we will discuss the defined fixed point theorem and apply it to prove Hyers–Ulam stability of $m$-hom-ders on ternary Banach algebra.

**Theorem 2.2.** *[6] Let $(\mathcal{B}, d)$ be a complete generalized metric space and let $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be a strict contraction with Lipschitz constant $0 < \eta < 1$. Then for any $t \in \mathcal{B}$ exactly one of the following holds:*
*- For every nonnegative integer $n$ we have $d(\mathcal{F}^n(t), \mathcal{F}^{n+1}(t)) = \infty$.*
*- There exists an integer $i_0 \geq 0$ such that:*
*$(i)$ $d(\mathcal{F}^n(t), \mathcal{F}^{n+1}(t)) < \infty$ for all $n \geq n_0$;*
*$(ii)$ The iterates $\mathcal{F}^n(t)$ converge to a unique fixed point $r^*$ of $\mathcal{F}$ in the set*

$$\mathcal{B}^* = \{r \in \mathcal{B} \mid d(\mathcal{F}^{n_0}t, r) < \infty\}$$

*$(iii)$ For every $r \in \mathcal{B}$ one has $d(r, r^*) \leq \frac{1}{1-\eta}\, d(r, \mathcal{F}(r))$.*

## 3. Main Results

In this section, Let $\mathcal{B}$ is ternary Banach algebra, $m = 1, 2, 3, 4$ and $a \neq 0, \pm 1$.
Before discussing the main theorem, we examine ideas related to Găvruta's control function. Let functions from $\delta_m : \mathcal{B}^2 \to [0, \infty)$ and $\omega_m : \mathcal{B}^3 \to [0, \infty)$ such that if there exists a constant $0 < \eta < 1$ such that,

$$\delta_m\left(\frac{t}{a}, \frac{r}{a}\right) \leq \frac{\eta}{a^i}\delta_m(t, r), \tag{2}$$

and if there exists a constant $0 < \gamma < 1$ such that

$$\omega_m\left(\frac{u}{a}, \frac{v}{a}, \frac{w}{a}\right) \leq \frac{\eta}{a^{3i}}\omega_m(u, v, w), \tag{3}$$

for all elements $t, r, u, v, w \in \mathcal{B}$. In the following theorem, we use the fixed point method to analyze an $m$-mapping that the Hyers-Ulam concept employing Găvruta's control function on ternary Banach algebra deems stable.

**Theorem 3.1.** *Let $\mathfrak{h} : \mathcal{B} \to \mathcal{B}$ be a mapping for which there exist functions $\delta_m : \mathfrak{B}^2 \to [0, \infty)$ such that*

$$\begin{aligned}\Big\|\mathfrak{h}(at + r) + \mathfrak{h}(at - r) - a^{i-2}\big[\mathfrak{h}(t + r) + \mathfrak{h}(t - r)\big] \\ - 2(a^m - 1)\Big[a^{m-2}\mathfrak{h}(t) + \tfrac{(m-2)(1-(m-2)^2)}{6}\mathfrak{h}(r)\Big]\Big\| \leq \delta_m(r, t).\end{aligned} \tag{4}$$

*Then there exists a unique $m-$mapping $\mathfrak{F}_m : \mathcal{B} \to \mathcal{B}$ such that*

$$\|\mathfrak{h}(r) - \mathfrak{F}_m(r)\| \leq \frac{1}{2a^m(\eta^{-1} - 1)}\delta_m(r, 0), \tag{5}$$

In the next corollary, we show how substituting Rassias's control function on ternary Banach algebra renders Theorem 3.1 stable.

**Corollary 3.2.** *Let $\theta, \rho$ be non–negative real numbers with and $\rho > m$. Suppose that $\mathfrak{h} : \mathcal{B} \to \mathcal{B}$ is a mapping such that,*

$$\begin{aligned}\Big\|\mathfrak{h}(at + r) + \mathfrak{h}(at - r) - a^{i-2}\big[\mathfrak{h}(t + r) + \mathfrak{h}(t - r)\big] \\ - 2(a^m - 1)\Big[a^{m-2}\mathfrak{h}(t) + \tfrac{(m-2)(1-(m-2)^2)}{6}\mathfrak{h}(r)\Big]\Big\| \leq \theta(\|t\|^\rho + \|r\|^\rho).\end{aligned} \tag{6}$$

*Then there exists a unique $m-$mapping $\mathfrak{F}_m$ satisfying*

$$\|\mathfrak{h}(r) - \mathfrak{F}_m(r)\| \leq \frac{\theta}{2(a^r - a^m)}\|r\|^\rho. \tag{7}$$

In the following Remark, we investigate Hyers-Ulam stability with JMRassias' control function on ternary Banach algebra.

**Remark 3.3.** Let $\theta, \rho$ be non–negative real numbers with and $\rho > m$. Suppose that $\mathfrak{h} : \mathcal{B} \to \mathcal{B}$ is a $m$-mapping such that,

$$\begin{aligned}\Big\|\mathfrak{h}(at + r) + \mathfrak{h}(at - r) - a^{i-2}\big[\mathfrak{h}(t + r) + \mathfrak{h}(t - r)\big] \\ - 2(a^m - 1)\Big[a^{m-2}\mathfrak{h}(t) + \tfrac{(m-2)(1-(m-2)^2)}{6}\mathfrak{h}(r)\Big]\Big\| \leq \theta(\|t\|^\rho.\|r\|^\rho).\end{aligned} \tag{8}$$

Then $\mathfrak{h} : \mathcal{B} \to \mathcal{B}$ is a $m$-mapping.

In the next theorem, corollary, and remark, we apply the fixed point method to $m$-hom-ders, rendered stable by the Hyers–Ulam concept under the Găvruta, Rassias, and JMRassias control functions on a ternary Banach algebra.

**Theorem 3.4.** *Let $\zeta_m : \mathcal{B} \to \mathcal{B}$ be mapping for which there exist function $\omega_m : \mathcal{B}^3 \to [0, \infty)$ satisfying (2), (4), and*

$$\|[\zeta_m(u), \zeta_m(v), \zeta_m(w)] - [\zeta_m(u), v^m, w^m] - [u^m, \zeta_m(v), w^m] - [u^m, v^m, \zeta_m(w)]\| \leq \omega_m(u, v, w).$$

*Then $\zeta : \mathcal{B} \to \mathcal{B}$ is a ternary $m$-hom-ders.*

**Corollary 3.5.** *Let $\theta, \rho$ be non–negative real numbers with and $\rho > m$. Suppose that $\zeta : \mathcal{B} \to \mathcal{B}$ is a mapping such that,*

$$\left\|[\zeta_m(u), \zeta_m(v), \zeta_m(w)] - [\zeta_m(u), v^m, w^m] - [u^m, \zeta_m(v), w^m] - [u^m, v^m, \zeta_m(w)]\right\| \leq \theta(\|u\|^\rho + \|v\|^\rho + \|w\|^\rho).$$

*Then $\zeta : \mathcal{B} \to \mathcal{B}$ is a ternary $m$-hom-ders.*

**Remark 3.6.** Let $\theta, \rho$ be non–negative real numbers with and $\rho > m$. Suppose that $\zeta : \mathcal{B} \to \mathcal{B}$ is a mapping such that,

$$\left\|[\zeta_m(u), \zeta_m(v), \zeta_m(w)] - [\zeta_m(u), v^m, w^m] - [u^m, \zeta_m(v), w^m] - [u^m, v^m, \zeta_m(w)]\right\| \leq \theta(\|u\|^\rho.\|v\|^\rho.\|w\|^\rho).$$

Then $\zeta : \mathcal{B} \to \mathcal{B}$ is a ternary $m$-hom-ders.

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# A new hybrid algorithm for resolvent operators in Banach spaces

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**Abstract**

In this paper, we present a new hybrid iterative algorithm in the framework of real 2-uniformly convex and uniformly smooth Banach spaces. The algorithm is designed to approximate a common fixed point of a finite family of resolvent operators of maximal monotone mappings and two relatively nonexpansive mappings. We prove that the sequence generated by our algorithm converges strongly to such a common fixed point.

## 1. Introduction

Suppose that $C$ is a nonempty closed convex subset of a real Banach space $X$ with norm $\|.\|$ and $X^*$ is the dual of $X$. The operator $A$ of $X$ to $2^{X^*}$ is said to be

(i) monotone if $\langle x-y, x^*-y^*\rangle \geq 0$ for all $x, y \in X$ and $x^* \in Ax,\ y^* \in Ay$.

(ii) $\alpha-$inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x-y, x^*-y^*\rangle \geq \alpha\|x^*-y^*\|^2,\ \ x, y \in X,\ \ x^* \in Ax,\ y^* \in Ay.$$

(iii) $L$-Lipchitz continuous if there exists $L > 0$ such that

$$\|x^*-y^*\| \leq L\|x-y\|,\ \ x, y \in X,\ \ x^* \in Ax,\ y^* \in Ay.$$

(iv) demi-closed if for all $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ in $X$, and $y_n \in Ax_n$ with $y_n \to y$ in $X^*$, we have $x \in X$ and $y \in Ax$.

A monotone mapping $A$ is called maximal if its graph $G(A) = \{(x, Ax) : x \in\in D(A)\}$ is not properly contained in the graph of any other monotone mapping. Clearly, the monotone mapping $A$ is maximal if and only if for $(x, x^*) \in X\times X^*$, $\langle x-y, x^*-y^*\rangle \geq 0$ for each $(y, y^*) \in G(A)$, then it is implied that $x^* \in Ax$.

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Let $X$ be a real smooth Banach space with norm $\|.\|$ and let $X^*$ be the dual space of $X$. A function $\delta : [0,2] \to [0,1]$ is said to be the modulus of convexity of $X$ if

$$\delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\},$$

for every $\varepsilon \in [0,2]$. A Banach space $X$ is said to be uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. It is well known that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply that $x_n \to u$.Let $p$ be a fixed real number with $p \geq 2$. A Banach space $X$ is called $p$-uniformly convex, if there exists a constant $c > 0$ such that $\delta \geq c\epsilon^p$ for all $\epsilon \in [0,2]$. The duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

for every $x \in X$. Let $S(X) = \{x \in X : \|x\| = 1\}$. A Banach space $X$ is said to be smooth if for all $x \in S(X)$, there exists a unique functional $j_x \in X^*$ such that $\langle x, j_x\rangle = \|x\|$ and $\|j_x\| = 1$.
The norm of $X$ is said to be $Gâteaux$ differentiable if for each $x, y \in S(X)$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1}$$

exists. In this case, $X$ is said to be smooth and $X$ is called uniformly smooth if the limit (1) is attained uniformly for all $x, y \in S(X)$[4]. If a Banach space $X$ is uniformly convex, then $X$ is reflexive and strictly convex, and $X^*$ is uniformly smooth.It is well known that if $X$ is a reflexive, strictly convex and smooth Banach space and $J^* : X^* \to X$ is the duality mapping on $X^*$, then $J^{-1} = J^*$, also, if $X$ is a uniformly smooth Banach space, then $J$ is uniformly norm to norm continuous on bounded sets of $X$ and $J^{-1} = J^*$ is also uniformly norm to norm continuous on bounded sets of $X^*$. Let $X$ be a smooth Banach space and let $J$ be the duality mapping on $X$. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{2}$$

Clearly, from (2), we can conclude that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2. \tag{3}$$

If $X$ is a reflexive, strictly convex and smooth Banach space, then for all $x, y \in X$

$$\phi(x,y) = 0 \Leftrightarrow x = y. \tag{4}$$

Also, it is clear from the definition of the function $\phi$ that the following condition holds for all $x, y \in X$,

$$\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|. \tag{5}$$

Now, the function $V : X \times X^* \to \mathbb{R}$ is defined as

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,$$

for all $x \in X$ and $x^* \in X^*$. Moreover, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in X$ and $x^* \in X^*$. If $X$ is a reflexive strictly convex and smooth Banach space with $X^*$ as its dual, then

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \tag{6}$$

for all $x \in X$ and all $x^*, y^* \in X^*$.
An operator $A : C \to X^*$ is hemicontinuous at $x_0 \in C$, if for any sequence $\{x_n\}$ converging to $x_0$ along a line implies that the sequence $\{Ax_n\}$ is weakly convergent to $Ax_0$, i.e., $Ax_n = A(x_0 + t_n x) \rightharpoonup Ax_0$ as $t_n \to 0$ for each $x \in C$. The generalized projection $\Pi_C : X \to C$ is a mapping that assigns to an arbitrary point $x \in X$, the minimum point of the functional $\phi(y,x)$; that is, $\Pi_C x = x_0$, where $x_0$ is the solution of the minimization problem

$$\phi(x_0, x) = \min_{y\in C} \phi(y,x).$$

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping $J$[1]. Let $C$ be a nonempty closed convex subset of $X$ and $T$ be a self mapping on $C$. We denote the set of fixed points of $T$ by $F(T)$, that is $F(T) = \{x \in C : x \in Tx\}$. A point $p \in C$ is said to be an asymptotically fixed point of $T$ if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ and $Tx_n - x_n \to 0$.The set of asymptotical fixed points of $T$ will be denoted by $\hat{F}(T)$. A mapping $T$ from $C$ into itself is called relatively nonexpansive if $\hat{F}(T) = F(T)$ and $\phi(p,Tx) \leq \phi(p,x)$ for each $x \in C$ and $p \in F(T)$.
We need the following lemmas for proving our main results.

**Lemma 1.1.** *([2]) Let $X$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $X$. If $\phi(x_n,y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

**Lemma 1.2.** *([1]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$ and let $y \in X$. Then*

$$\phi(x,\Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x,y), \quad \forall x \in C.$$

**Lemma 1.3.** *([1]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$, let $x \in X$ and let $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, Jx - Jz\rangle \leq 0, \text{ for all } y \in C.$$

**Lemma 1.4.** *([7]). Let $X$ be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0,2r] \to [0,\infty)$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

*for all $x,y \in B_r(0) = \{z \in E : \|z\| \leq r\}$ and $t \in [0,1]$.*

**Definition 1.5.** ([5]) Let $X$ be a real smooth and uniformly convex Banach space and let $M : X \to 2^{X^*}$ be a maximal monotone operator. For all $\xi > 0$, define the operator $Q_\xi^M : X \to X$ by $Q_\xi^M x = (J + \xi M)^{-1} Jx$ for all $x \in X$.

**Lemma 1.6.** *([4]) Let $X$ be a real smooth and uniformly convex Banach space and let $M : X \to 2^{X^*}$ be a maximal monotone operator. Then $M^{-1}0$ is a closed and convex subset of $X$ and the graph $G(M)$ of $M$, is demi-closed.*

**Lemma 1.7.** *([6]) Let $X$ be a real reflexive, strictly convex and smooth Banach space and $M : X \to 2^{X^*}$ be a maximal monotone operator with $M^{-1}0 \neq \emptyset$. Then, for all $x \in X, y \in M^{-1}0$ and $\xi > 0$, then $\phi(y, Q_\xi^M x) + \phi(Q_\xi^M x, x) \leq \phi(y,x)$.*

## 2. Main results

We assume that the following conditions hold for the proposed Algorithm 1:

1. $X$ is a real 2-uniformly convex and uniformly smooth Banach space.
2. $C$ is a nonempty closed and convex subset of $X$.
3. $f$ and $g$ are relatively nonexpansive mappings from $C$ into itself.
4. $M_i : X \to 2^{X^*}$ be a maximal monotone operator with ${M_i}^{-1}0 \neq \emptyset$ for all $i = 1,2,\ldots,m$.

**Algorithm 1** Iterative Algorithm

**Initialization:** Let $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a,b] \subset (0,1)$. Let $x_0 \in C$ be initial point.

**Step 1:** Compute $y_n = J^{-1}(\alpha_n x_n + (1-\alpha_n) f(x_n))$,

**Step 2:** Compute $z_n = J^{-1}(\sum_{i=1}^m \beta_{n,i} J Q_\xi^{M_i} x_n + \sum_{i=1}^m \gamma_{n,i} J Q_\xi^{M_i} y_n)$,

**Step 3:** Compute $w_n = J^{-1}(\beta_n Jf(x_n) + (1-\beta_n) Jg(y_n))$,

**Step 4:** Compute $x_{n+1} = \Pi_C w_n$.

**Theorem 2.1.** *Let* $\Omega = F(f) \cap F(g) \cap (\cap_{i=1}^{m} F(Q_i^{M_i})) \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 1 converges strongly to the some solution* $q \in \Omega$.

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# Hybrid iterative schemes for variational inequalities and fixed points of resolvent operators

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**Abstract**

In This paper, we present a new hybrid algorithm. Then, we show that our generated sequence is strongly convergent to a common element of the solution set of a variational inequality problem and the fixed point set of a family of resolvent operators.

## 1. Introduction

Suppose that $C$ is a nonempty closed convex subset of a real Banach space $X$ with norm $\|.\|$ and $X^*$ is the dual of $X$. The variational inequality problem (VI) is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \quad \forall \ \ y \in C, \tag{1}$$

where $A$ is a mapping of $C$ into $X^*$ and $\langle .,. \rangle$ denotes the duality pairing. The solution set of the variational inequality problem is denoted by $VI(C, A)$.
The operator $A$ of $X$ to $2^{X^*}$ is said to be

(i) monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x, y \in X$ and $x^* \in Ax,\ y^* \in Ay$.
(ii) $\alpha-$inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle x - y, x^* - y^*\rangle \geq \alpha\|x^* - y^*\|^2, \ \ x, y \in X, \ \ x^* \in Ax, \ y^* \in Ay.$$

(iii) $L$-Lipchitz continuous if there exists $L > 0$ such that

$$\|x^* - y^*\| \leq L\|x - y\|, \ \ x, y \in X, \ \ x^* \in Ax, \ y^* \in Ay.$$

(iv) demi-closed if for all $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ in $X$, and $y_n \in Ax_n$ with $y_n \to y$ in $X^*$, we have $x \in X$ and $y \in Ax$.

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A monotone mapping $A$ is called maximal if its graph $G(A) = \{(x, Ax) : x \in\in D(A)\}$ is not properly contained in the graph of any other monotone mapping. Clearly, the monotone mapping $A$ is maximal if and only if for $(x, x^*) \in X \times X^*$, $\langle x - y, x^* - y^* \rangle \geq 0$ for each $(y, y^*) \in G(A)$, then it is implied that $x^* \in Ax$.
Let $X$ be a real smooth Banach space with norm $\|.\|$ and let $X^*$ be the dual space of $X$. A function $\delta : [0,2] \to [0,1]$ is said to be the modulus of convexity of $X$ if

$$\delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\},$$

for every $\varepsilon \in [0,2]$. A Banach space $X$ is said to be uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$. It is well known that a uniformly convex Banach space has the Kadec-Klee property, that is, $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply that $x_n \to u$.Let $p$ be a fixed real number with $p \geq 2$. A Banach space $X$ is called $p$-uniformly convex, if there exists a constant $c > 0$ such that $\delta \geq c\epsilon^p$ for all $\epsilon \in [0,2]$. The duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

for every $x \in X$. Let $S(X) = \{x \in X : \|x\| = 1\}$. A Banach space $X$ is said to be smooth if for all $x \in S(X)$, there exists a unique functional $j_x \in X^*$ such that $\langle x, j_x \rangle = \|x\|$ and $\|j_x\| = 1$.
The norm of $X$ is said to be $Gâteaux$ differentiable if for each $x, y \in S(X)$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2}$$

exists. In this case, $X$ is said to be smooth and $X$ is called uniformly smooth if the limit (2) is attained uniformly for all $x, y \in S(X)$[4]. If a Banach space $X$ is uniformly convex, then $X$ is reflexive and strictly convex, and $X^*$ is uniformly smooth.It is well known that if $X$ is a reflexive, strictly convex and smooth Banach space and $J^* : X^* \to X$ is the duality mapping on $X^*$, then $J^{-1} = J^*$, also, if $X$ is a uniformly smooth Banach space, then $J$ is uniformly norm to norm continuous on bounded sets of $X$ and $J^{-1} = J^*$ is also uniformly norm to norm continuous on bounded sets of $X^*$. Let $X$ be a smooth Banach space and let $J$ be the duality mapping on $X$. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in X. \tag{3}$$

Clearly, from (3), we can conclude that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2. \tag{4}$$

If $X$ is a reflexive, strictly convex and smooth Banach space, then for all $x, y \in X$

$$\phi(x,y) = 0 \Leftrightarrow x = y. \tag{5}$$

Also, it is clear from the definition of the function $\phi$ that the following condition holds for all $x, y \in X$,

$$\phi(x,y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|. \tag{6}$$

Now, the function $V : X \times X^* \to \mathbb{R}$ is defined as

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2,$$

for all $x \in X$ and $x^* \in X^*$. Moreover, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in X$ and $x^* \in X^*$. If $X$ is a reflexive strictly convex and smooth Banach space with $X^*$ as its dual, then

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \tag{7}$$

for all $x \in X$ and all $x^*, y^* \in X^*$.
An operator $A : C \to X^*$ is hemicontinuous at $x_0 \in C$, if for any sequence $\{x_n\}$ converging to $x_0$ along a line implies that the sequence $\{Ax_n\}$ is weakly convergent to $Ax_0$, i.e., $Ax_n = A(x_0 + t_n x) \rightharpoonup Ax_0$ as $t_n \to 0$ for each $x \in C$. The

generalized projection $\Pi_C : X \to C$ is a mapping that assigns to an arbitrary point $x \in X$, the minimum point of the functional $\phi(y, x)$; that is, $\Pi_C x = x_0$, where $x_0$ is the solution of the minimization problem

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The existence and uniqueness of the operator $\Pi_C$ follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping $J$[1]. Let $C$ be a nonempty closed convex subset of $X$ and $T$ be a self mapping on $C$. We denote the set of fixed points of $T$ by $F(T)$, that is $F(T) = \{x \in C : x \in Tx\}$.
We need the following lemmas for proving our main results.

**Lemma 1.1.** *([2]) Let $X$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $X$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

**Lemma 1.2.** *([1]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$ and let $y \in X$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C.$$

**Lemma 1.3.** *([1]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$, let $x \in X$ and let $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, Jx - Jz \rangle \leq 0, \text{ for all } y \in C.$$

**Lemma 1.4.** *([7]) Let $X$ be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in X$, we have that*

$$\|x - y\| \leq \frac{2}{c^2} \|Jx - Jy\|,$$

*where $\frac{1}{c} (0 < c \leq 1)$ is the 2-uniformly convex constant of $X$.*

**Lemma 1.5.** *([7]). Let $X$ be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

*for all $x, y \in B_r(0) = \{z \in E : \|z\| \leq r\}$ and $t \in [0, 1]$.*

**Lemma 1.6.** *([2]). Let $X$ be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r(0) = \{z \in X : \|z\| \leq r\}$.*

The normal cone for $C$ at a point $v \in C$ is denoted by $N_C(v)$, that is $N_C(v) := \{x^* \in X^* : \langle v - y, x^* \rangle \geq 0, \forall y \in C\}$.

**Lemma 1.7.** *([3]) Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $T$ be monotone and hemi-continuous operator of $C$ into $X^*$ with $C = D(T)$. Let $B \subset X \times X^*$ be an operator defined as follows:*

$$Bv = \begin{cases} Tv + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

*Then $B$ is maximal monotone and $B^{-1}(0) = VI(C, T)$.*

**Definition 1.8.** ([5]) Let $X$ be a real smooth and uniformly convex Banach space and let $M : X \to 2^{X^*}$ be a maximal monotone operator. For all $\xi > 0$, define the operator $Q_\xi^M : X \to X$ by $Q_\xi^M x = (J + \xi M)^{-1} Jx$ for all $x \in X$.

**Lemma 1.9.** *([4]) Let $X$ be a real smooth and uniformly convex Banach space and let $M : X \to 2^{X^*}$ be a maximal monotone operator. Then $M^{-1}0$ is a closed and convex subset of $X$ and the graph $G(M)$ of $M$, is demi-closed.*

**Lemma 1.10.** *([6]) Let $X$ be a real reflexive, strictly convex and smooth Banach space and $M : X \to 2^{X^*}$ be a maximal monotone operator with $M^{-1}0 \neq \emptyset$. Then, for all $x \in X, y \in M^{-1}0$ and $\xi > 0$, then $\phi(y, Q_\xi^M x) + \phi(Q_\xi^M x, x) \leq \phi(y, x)$.*

## 2. Main results

We assume that the proposed Algorithm 1 satisfies the following conditions:

1. Let $X$ be a real 2-uniformly convex and uniformly smooth Banach space, and $X^*$ the dual space of $X$.
2. $C$ is a nonempty closed and convex subset of $X$.
3. $A : C \to X^*$ is a $\alpha-$inverse strongly monotone.
4. $\|Ax\| \leq \|Ax - Au\|$ for all $u \in \Gamma$ and $x \in C$.
5. $M_i : X \to 2^{X^*}$ be a maximal monotone operator with ${M_i}^{-1}0 \neq \emptyset$ for each $i = 1, 2, ..., m$.
6. $\theta$ is a real number such that $0 < \theta < \frac{c^2\alpha}{2}$, where $\frac{1}{c}$ is the 2-uniformly convexity constant of $X$.

**Algorithm 1** Iterative Algorithm

**Initialization:** Let $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b] \subset (0, 1)$. Let $x_0 \in C$ be initial point.

**Step 1:** Compute $y_n = \Pi_C J^{-1}\left((\alpha_{n,0}J + \sum_{i=1}^{m} \alpha_{n,i} J Q_i^{M_i})x_n\right)$,

**Step 2:** Compute $z_n = \Pi_C J^{-1}(Jy_n - \theta A y_n)$,

**Step 3:** Compute $x_{n+1} = \Pi_C J^{-1}(\beta_n J x_n + (1 - \beta_n) J z_n)$.

**Theorem 2.1.** *Let* $\Omega = (\cap_{i=1}^{m} F(Q_i^{M_i})) \cap VI(C, A) \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 1 converges strongly to the some solution* $q \in \Omega$ *where* $q = \lim_{n\to\infty} \Pi_\Omega x_n$.

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# The Resolvent Mean Inequalities for Sector Matrices

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**Abstract**

In this paper, we introduce the quantities $\mathcal{R}_\mu(A,\lambda)$ and $\mathcal{L}_\mu(A,\lambda)$ for sector matrices. We show that if $A = (A_1,\dots,A_m)$ with $A_i \in \mathcal{S}_{\alpha_i}$, $\alpha = \max_{1\le i\le m}\alpha_i$, and $\lambda = (\lambda_1,\dots,\lambda_m)$ is a weight vector with $\lambda_i \ge 0$ and $\sum_{i=1}^m \lambda_i = 1$, then for any $\mu \ge 0$,

$$\Phi\big(\mathcal{R}_\mu(\Re A,\lambda)\big) \le \sec^2\alpha\,\Re\big(\mathcal{R}_\mu(\Phi(A),\lambda)\big) + (\sec^2\alpha - 1)\mu I.$$

This result complements a recent finding by Sheikhhosseini and Sababheh.

## 1. Introduction

Let $\mathbb{M}_n$ denote the set of $n\times n$ complex matrices. For a Hermitian matrix $A \in \mathbb{M}_n$, we write $A \ge 0$ if $A$ is positive semidefinite, i.e., $\langle Ax,x\rangle \ge 0$ for all vectors $x \in \mathbb{C}^n$. We write $A > 0$ if $A$ is positive definite, that is, if $A$ is positive semidefinite and invertible.
The set of all positive definite matrices is denoted by $\mathbb{M}_n^+$. For two Hermitian matrices $A,B$, we write $A \ge B$ (or $A > B$) if $A - B \ge 0$ (or $A - B > 0$).
A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_n$ is called positive if $\Phi(A) \ge 0$ whenever $A \ge 0$. If $\Phi(I) = I$, where $I$ is the identity matrix, then $\Phi$ is called unital. Notice that for any positive unital linear map $\Phi$ and any matrix $A \in \mathbb{M}_n$, the real part commutes with $\Phi$, i.e.,

$$\Phi(\mathcal{R}A) = \mathcal{R}(\Phi(A)).$$

The numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

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It is well known that $W(A)$ is a compact and convex subset of $\mathbb{C}$, and it contains the spectrum of $A$.
A matrix $A \in \mathbb{M}_n$ is called accretive if $\mathcal{R}A$ is positive in its Cartesian (or Toeplitz) decomposition

$$A = \mathcal{R}A + i\mathcal{I}A,$$

where $\mathcal{R}A = \frac{A+A^*}{2}$ and $\mathcal{I}A = \frac{A-A^*}{2i}$ are the real and imaginary parts of $A$, respectively, and $A^*$ is the adjoint of $A$.
Let $\mathcal{S}_\alpha$ denote the sector region in the complex plane $\mathbb{C}$ defined by

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \mathcal{R}z > 0,\ |\mathcal{I}z| \leq (\mathcal{R}z)\tan\alpha\},$$

where $0 \leq \alpha < \frac{\pi}{2}$. A matrix $A \in \mathbb{M}_n$ is called sectorial if $W(A) \subset \mathcal{S}_\alpha$ for some $0 \leq \alpha < \frac{\pi}{2}$. We simply write $A \in \mathcal{S}_\alpha$ if $W(A) \subset \mathcal{S}_\alpha$. Clearly, $A$ is positive definite if and only if $W(A) \subset \mathcal{S}_0$.
Throughout this paper, for a positive unital linear map $\Phi(\cdot)$ and $A = (A_1, A_2, \dots, A_m)$, we define

$$\Phi(A) = (\Phi(A_1), \Phi(A_2), \dots, \Phi(A_m)) \quad \text{and} \quad \Re A = (\Re A_1, \Re A_2, \dots, \Re A_m).$$

By $\lambda$ we denote the $m$-vector $(\lambda_1, \lambda_2, \dots, \lambda_m)$, where $\lambda_i \in [0,1]$ and $\sum_{i=1}^m \lambda_i = 1$. We need some lemmas, to prove resualts for positive unital linear map $\Phi(\cdot)$.

**Lemma 1.1.** *[3] Let $\Phi$ be a unital positive linear map. Then for any $A \in \mathbb{M}_n^+$,*

$$\Phi(A)^{-1} \leq \Phi(A^{-1}).$$

**Lemma 1.2.** *[6] Let $A, B \in \mathbb{M}_n^+$, and $\sigma$ be a matrix mean.Then for any unital positive linear map, $\Phi$, we have*

$$\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B).$$

An operator mean $\sigma$ in the sense of Kubo–Ando [4] is defined via an operator monotone function $f : (0,\infty) \to (0,\infty)$ with $f(1) = 1$ by

$$A\sigma B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

for positive invertible operators $A$ and $B$. The function $f$ is called the representing function of $\sigma$. Recently, Bedrani et al. showed that this definition can also be extended to accretive operators.
Among the most commonly used means for $A = (A_1, A_2, \dots, A_m)$ are the weighted arithmetic mean $\mathcal{A}_\lambda$ and the weighted harmonic mean $\mathcal{H}_\lambda$, defined for any accretive operators $A_i$, $i = 1, 2, \dots, m$, by

$$\mathcal{H}_\lambda(A) = \left(\sum_{i=1}^m \lambda_i A_i^{-1}\right)^{-1},$$

and

$$\mathcal{A}_\lambda(A) = \sum_{i=1}^m \lambda_i A_i.$$

The weighted geometric mean $A\sharp_\lambda B$ of two accretive operators $A, B$ is given by

$$A\sharp_\lambda B = A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^\lambda A^{1/2}, \quad \lambda \in [0,1].$$

If $\lambda = (1/2, 1/2)$, we simply write $\mathcal{A}(A,B)$ and $\mathcal{H}(A,B)$ instead of $\mathcal{A}_\lambda(A,B)$ and $\mathcal{H}_\lambda(A,B)$. Similarly, if $\lambda = 1/2$, we write $A\sharp B$ instead of $A\sharp_{1/2} B$.
The geometric mean of $A$ and $B$ is the metric midpoint of the arithmetic mean $\mathcal{A}(A,B)$ and the harmonic mean $\mathcal{H}(A,B)$ with respect to both the trace metric and the Thompson metric (see [5]). That is,

$$A\sharp B = \mathcal{A}(A,B)\sharp\mathcal{H}(A,B).$$

In [1], Bauschke et al. introduced a novel and interesting notion of the proximal average within the framework of convex analysis. By associating a positive semidefinite matrix $A$ with the quadratic convex function

$$q_A(x) = \langle Ax, x\rangle,$$

this new average can be extended to positive semidefinite matrices under the name of resolvent average. In [2], the authors defined the resolvent average for positive definite matrices and proved that it enjoys several remarkable properties.

**Definition 1.3.** Let $A_i \in \mathbb{M}_n^+$, $i = 1, 2, \dots, m$. The resolvent average of $A = (A_1, A_2, \dots, A_m)$ with weight $\lambda = (\lambda_1, \dots, \lambda_m)$ is defined by

$$\mathcal{R}_\mu(A, \lambda) = \left(\sum_{i=1}^{m} \lambda_i (A_i + \mu I)^{-1}\right)^{-1} - \mu I, \quad \mu \geq 0,$$

and for $\mu = \infty$, we set

$$\mathcal{R}_\infty(A, \lambda) = \mathcal{A}_\lambda(A).$$

Note that for $\mu = 0$, we have

$$\mathcal{R}_0(A, \lambda) = \mathcal{H}_\lambda(A).$$

Sheikhhosseini et al. [7] showed that for any $r > 0$

$$\mathcal{R}_\mu(rA, \lambda) = r\mathcal{R}_{\frac{\mu}{r}}(A, \lambda), \tag{1}$$

and

$$\mathcal{L}_\mu(rA, \lambda) = r\mathcal{L}_{\frac{\mu}{r}}(A, \lambda), \tag{2}$$

**Lemma 1.4** ([7]). *If $A \in \mathcal{S}_\alpha$, then so is $A^{-1}$, and*

$$\Re(A^{-1}) \leq \Re(A)^{-1} \leq \sec^2\alpha\ \Re(A^{-1}).$$

*In particular, if $A_i \in \mathcal{S}_{\alpha_i}$ for $i = 1, \dots, m$ and $\alpha = \max_{1\leq i\leq m} \alpha_i$, then*

$$\Re\left(\sum_{i=1}^{m} A_i^{-1}\right)^{-1} \geq \cos^2(\alpha)\left(\sum_{i=1}^{m} \Re(A_i^{-1})\right)^{-1} \geq \cos^2(\alpha)\left(\sum_{i=1}^{m} \Re(A_i)^{-1}\right)^{-1}.$$

**Definition 1.5.** Let $A_i \in \mathbb{M}_n^+$, $i = 1, 2, \dots, m$, and let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a weight vector with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. For $\mu \geq 0$, define

$$\mathcal{L}_\mu(A, \lambda) = \left(\sum_{i=1}^{m} \lambda_i (A_i + \mu I)\right) \sharp \left(\sum_{i=1}^{m} \lambda_i (A_i + \mu I)^{-1}\right)^{-1} - \mu I.$$

For $\mu < 0$, define

$$\mathcal{L}_\mu(A, \lambda) = \left(\mathcal{L}_{-\mu}(A^{-1}, \lambda)\right)^{-1},$$

where $A^{-1} = (A_1^{-1}, A_2^{-1}, \dots, A_m^{-1})$.
It follows from this definition and the properties of the geometric mean of positive definite matrices that $\mathcal{L}_\mu(A, \lambda)$ is positive definite for all $\mu \in \mathbb{R}$.

## 2. Main Result

The first result in this section establishes a relationship between $\Phi(\mathcal{R}_\mu(\Re A,\lambda))$ and $\Re\big(\Phi(\mathcal{R}_\mu(A,\lambda))\big)$.

**Lemma 2.1.** *[7] Let $A_i \in \mathbb{M}_n^+$, $i = 1,\dots,m$, and let $\mu \geq 0$. Then, for any positive unital linear map $\Phi$, we have*

$$\Phi\left(\sum_{i=1}^{m}\lambda_i(A_i+\mu I)^{-1}\right)^{-1} \leq \left(\sum_{i=1}^{m}\lambda_i\big(\Phi(A_i)+\mu I\big)^{-1}\right)^{-1}.$$

The following theorem extends [7, Lemma 3.1].

**Theorem 2.2.** *Let $A = (A_1,\dots,A_m)$ with $A_i \in \mathbb{M}_n^+$, and let $\lambda = (\lambda_1,\dots,\lambda_m)$ be a weight vector with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$.*
*Then, for $\mu \geq 0$ and any positive unital linear map $\Phi$, we have*

$$\Phi(\mathcal{R}_\mu(\Re A,\lambda)) \leq \sec^2\alpha\,\Re\big(\Phi(\mathcal{R}_\mu(A,\lambda))\big) + (\sec^2\alpha - 1)\mu I.$$

*Proof.* Using Lemmas 1.4 and 2.1, we obtain

$$\begin{aligned}
\Phi(\mathcal{R}_\mu(\Re A,\lambda)) &= \Phi\Big(\sum_{i=1}^{m}\lambda_i(\Re A_i+\mu I)^{-1}\Big)^{-1} - \mu I\\
&= \Phi\Big(\sum_{i=1}^{m}(\lambda_i^{-1}(\Re A_i+\mu I))^{-1}\Big)^{-1} - \mu I\\
&= \Phi\Big(\sum_{i=1}^{m}(\Re(\lambda_i^{-1}(A_i+\mu I)))^{-1}\Big)^{-1} - \mu I\\
&\leq \sec^2\alpha\,\Phi\Big(\Re\Big(\sum_{i=1}^{m}(\lambda_i^{-1}(A_i+\mu I))^{-1}\Big)^{-1}\Big) - \mu I\\
&= \sec^2\alpha\,\Phi\Big(\Re\Big(\sum_{i=1}^{m}\lambda_i(A_i+\mu I)^{-1}\Big)^{-1}\Big) - \mu I\\
&= \sec^2\alpha\,\Re\Big(\Phi\Big(\sum_{i=1}^{m}\lambda_i(A_i+\mu I)^{-1}\Big)^{-1}\Big) - \mu I\\
&= \sec^2\alpha\,\Re\big(\Phi(\mathcal{R}_\mu(A,\lambda))\big) + (\sec^2\alpha - 1)\mu I.
\end{aligned}$$

□

**Corollary 2.3.** *If $\Phi(A) = \frac{1}{n}trA$, then it is a positive and unital linear functional. Then*

$$tr(\mathcal{R}_\mu(\Re A,\lambda)) \leq \sec^2\alpha\,tr\left(\Re(\mathcal{R}_\mu(A,\lambda))\right) + (\sec^2\alpha - 1)n\mu I.$$

Our second result in this direction establishes a relationship between $\Phi(\mathcal{L}_\mu(\Re A,\lambda))$ and $\Re\big(\mathcal{L}_\mu(\Phi(A),\lambda)\big)$.

**Theorem 2.4.** *Let $A = (A_1,\dots,A_m)$ with $A_i \in \mathcal{S}_{\alpha_i}$, and let $\alpha = \max_{1\leq i\leq m}\alpha_i$. Let $\lambda = (\lambda_1,\dots,\lambda_m)$ be a weight vector with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Then, for any $\mu \geq 0$,and any positive unital linear map $\Phi$ we have*

$$\Phi(\mathcal{L}_\mu(\Re A,\lambda)) \leq \sec\alpha\,\Re\big(\mathcal{L}_\mu(\Phi(A),\lambda)\big) + (\sec\alpha - 1)\mu I.$$

*Proof.* Using Lemmas 1.4 and 2.1, and properties of geometric mean, we establish

$$\begin{aligned}
\Phi\left(\mathcal{L}_\mu(\Re A,\lambda)\right) &= \Phi\left(\left(\sum_{i=1}^m \lambda_i(\Re A_i+\mu I)\right)\sharp\left(\sum_{i=1}^m \lambda_i(\Re A_i+\mu I)^{-1}\right)^{-1}-\mu I\right)\\
&\le \Phi\left(\sum_{i=1}^m \lambda_i(\Re A_i+\mu I)\right)\sharp\Phi\left(\sum_{i=1}^m \lambda_i(\Re A_i+\mu I)^{-1}\right)^{-1}-\mu I\\
&\le \Phi\left(\sum_{i=1}^m \lambda_i(\Re A_i+\mu I)\right)\sharp\left(\sum_{i=1}^m \lambda_i(\Phi(\Re A_i)+\mu I)^{-1}\right)^{-1}-\mu I\\
&\le \Re\left(\sum_{i=1}^m \lambda_i(\Phi(A_i)+\mu I)\right)\sharp\left(\sec^2\alpha\Re\left(\sum_{i=1}^m \lambda_i(\Phi(A_i)+\mu I)^{-1}\right)^{-1}\right)-\mu I\\
&= \sec\alpha\Re\left(\sum_{i=1}^m \lambda_i(\Phi(A_i)+\mu I)\right)\sharp\Re\left(\sum_{i=1}^m \lambda_i(\Phi(A_i)+\mu I)^{-1}\right)^{-1}-\mu I\\
&\le \sec\alpha\Re\left(\sum_{i=1}^m \lambda_i(\Phi(A_i)+\mu I)\sharp\left(\sum_{i=1}^m \lambda_i(\Phi(A_i)+\mu I)^{-1}\right)^{-1}\right)-\mu I\\
&= \sec\alpha\Re\left(\mathcal{L}_\mu(\Phi(A),\lambda)\right)+(\sec\alpha-1)\mu I.
\end{aligned}$$

□

By equation (2), we obtain the following corollary:

**Corollary 2.5.** *Let* $\Phi(A)=\frac{1}{n}trA$ *be a positive and unital linear functional. Then*

$$tr(\mathcal{L}_\mu(\Re A,\lambda))\le \sec\alpha\Re\left(\mathcal{L}_{n\mu}(tr(A),\lambda)\right)+(\sec\alpha-1)n\mu I.$$

**Theorem 2.6.** *Let* $A=(A_1,\dots,A_m)$, $A_i\in\mathbb{M}_n^{++}$ *for* $i=1,2,\dots,m$, *and let* $\lambda=(\lambda_1,\dots,\lambda_m)$ *be a weight vector with* $\lambda_i\ge 0$ *and* $\sum_{i=1}^m\lambda_i=1$. *For* $\mu\ge 0$ *and for any positive unital linear map* $\Phi$,

$$\Phi(\mathcal{R}_\mu(\Re A,\lambda))\le \sec^2\alpha\Re\left(\mathcal{R}_\mu(\Phi(A),\lambda)\right)+(\sec^2\alpha-1)\mu I.$$

*Proof.* Using Lemma 1.4 and 2.1, we have

$$\begin{aligned}
\Phi(\mathcal{R}_\mu(\Re A,\lambda)) &= \Phi\left((\sum_{i=1}^{m}\lambda_i(\Re A_i+\mu I)^{-1})^{-1}-\mu I\right)\\
&= \Phi\left(\sum_{i=1}^{m}(\lambda_i^{-1}(\Re A_i+\mu I))^{-1}\right)^{-1}-\mu I\\
&\le \left(\sum_{i=1}^{m}\lambda_i\,(\Phi(\Re A_i)+\mu I)^{-1}\right)^{-1}-\mu I\\
&= \left(\sum_{i=1}^{m}\lambda_i\,(\Re\Phi(A_i)+\mu I)^{-1}\right)^{-1}-\mu I\\
&\le \sec^2\alpha\Re\left(\sum_{i=1}^{m}(\lambda_i^{-1}(\Phi(A_i)+\mu I))^{-1}\right)^{-1}-\mu I\\
&= \sec^2\alpha\Re\left(\sum_{i=1}^{m}\lambda_i(\Phi(A_i)+\mu I)^{-1}\right)^{-1}-\mu I\\
&= \sec^2\alpha\Re\left(\mathcal{R}_\mu(\Phi(A),\lambda)\right)+(\sec^2\alpha-1)\mu I.
\end{aligned}$$

□

By equation (1), we obtain the following corollary:

**Corollary 2.7.** *Let* $\Phi(A)=\frac{1}{n}trA$ *be a positive and unital linear functional. Then*

$$tr(\mathcal{R}_\mu(\Re A,\lambda))\le \sec^2\alpha\Re\left(\mathcal{R}_{n\mu}(tr(A),\lambda)\right)+(\sec^2\alpha-1)n\mu I.$$

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# Fejér's Trapezoid and Mid-Point Type Inequalities Related to $\varphi$-Convex Functions

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**Abstract**

In this paper, we establish some trapezoid and mid-point type inequalities related to $\varphi$-convex functions in connection with celebrated Fejér integral inequality.

## 1. Introduction and Preliminaries

The Fejér integral inequality for convex functions has been proved in [3] as the following:

**Theorem 1.1.** *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \tag{1}$$

*where $g : [a, b] \to \mathbb{R}^+ = [0, +\infty)$ is integrable and symmetric about $x = \frac{a+b}{2}\Big(g(x) = g(a + b - x), \forall x \in [a, b]\Big)$.*

On the other hand the concept of $\varphi$-convex functions has been introduced in [2] as the following:

**Definition 1.2** ([2])**.** Let $I$ be an interval in real line $\mathbb{R}$. Consider $\varphi : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex with respect to $\varphi$ (briefly $\varphi$-convex), if

$$f\big(tx + (1-t)y\big) \le f(y) + t\varphi\big(f(x), f(y)\big), \tag{2}$$

for all $x, y \in I$ and $t \in [0, 1]$.

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In fact above definition geometrically says that if a function is $\varphi$-convex on $I$, then it's graph between any $x, y \in I$ is on or under the path starting from $\big(y, f(y)\big)$ and ending at $\big(x, f(y) + \varphi(f(x), f(y))\big)$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\varphi(x, y) = x - y$ and the function reduces to a convex one. Note that by taking $x = y$ in (1.2) we get $t\varphi(f(x), f(x)) \geq 0$ for any $x \in I$ and $t \in [0, 1]$ which implies that

$$\varphi\big(f(x), f(x)\big) \geq 0$$

for any $x \in I$. Also if we take t = 1 in (1.2) we get

$$f(x) - f(y) \leq \varphi\big(f(x), f(y)\big)$$

for any $x, y \in I$. If $f : I \to \mathbb{R}$ is a convex function and $\varphi : I \times I \to \mathbb{R}$ is an arbitrary bifunction that satisfies

$$\varphi(x, y) \geq x - y$$

for any $x, y \in I$, then

$$f(tx + (1-t)y) \leq f(y) + t[f(x) - f(y)] \leq f(y) + t\varphi\big(f(x), f(y)\big)$$

showing that $f$ is $\varphi$-convex.
There are some examples about $\varphi$-convexity of a function.

**Example 1.3.** [2] (1) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0. \end{cases}$$

and define a bifunction $\varphi$ as $\varphi(x, y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that $f$ is an $\varphi$-convex function but not a convex one.

(2) Define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as $f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1. \end{cases}$ and a bifunction $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as $\varphi(x, y) = \begin{cases} x + y, & x \leq y; \\ 2(x + y), & x > y. \end{cases}$

Then $f$ is $\varphi$-convex but is not convex.

The following theorem is an important result:

**Theorem 1.4.** [2] Suppose that $f : I \to \mathbb{R}$ is an $\varphi$-convex function and $\varphi$ is bounded from above on $f(I) \times f(I)$. Then $f$ satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior $I^\circ$ of $I$. Hence, $f$ is absolutely continuous on $[a, b]$ and continuous on $I^\circ$.

**Note.** As a consequence of Theorem 1.4, an $\varphi$-convex function $f : [a, b] \to \mathbb{R}$ where $\varphi$ is bounded from above on $f([a, b]) \times f([a, b])$ is integrable.

## 2. Main Results

An interesting question in (1), is estimating the difference between left and middle terms and between right and middle terms. Respectively, we investigate about first and then second part of this question when the absolute value of the derivative of a function is $\varphi$-convex. We need the following lemma which is generalization of Lemma 2.1 in [4].

**Lemma 2.1.** *Suppose that $f : I^\circ \subset I \to \mathbb{R}$ is a differentiable mapping, $a, b \in I^\circ$ with $a < b$, $g : [a, b] \to \mathbb{R}^+ \cup \{0\}$ is a differentiable mapping and $f'$ is integrable on $[a, b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx =$$
$$(b-a)\Big[\int_0^{1/2} M_1(t)f'\big(ta + (1-t)b\big)dt + \int_{1/2}^1 M_2(t)f'\big(ta + (1-t)b\big)dt\Big],$$

*where*

$$M_1(t)=\int_0^t g\big(ua+(1-u)b\big)du \quad and \quad M_2(t)=-\int_t^1 g\big(ua+(1-u)b\big)du.$$

**Remark 2.2.** In Lemma 2.1, if we use the change of variable $x=tb+(1-t)a$, then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx$$
$$=(b-a)\Big[\int_0^{1/2} M_1(t)f'\big(tb+(1-t)a\big)dt+\int_{1/2}^1 M_2(t)f'\big(tb+(1-t)a\big)dt\Big],$$

where

$$M_1(t)=-\int_0^t g\big(ub+(1-u)a\big)du \quad and \quad M_2(t)=\int_t^1 g\big(ub+(1-u)a\big)du.$$

Using Lemma 2.1 and Remark 2.2, we can prove the following theorem to estimate the difference between the middle and left terms in (1) which is being called mid-point type inequality.

**Theorem 2.3.** *Suppose that $f:I^\circ\subset I\to\mathbb{R}$ is a differentiable mapping, $a,b\in I^\circ$ with $a<b$, $g:[a,b]\to\mathbb{R}^+\cup\{0\}$ is a differentiable mapping symmetric about $\frac{a+b}{2}$ and $|f'|$ is an $\varphi$-convex mapping on $[a,b]$. Then*

$$\Big|\int_a^b f(x)g(x)dx-f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx\Big|\le$$
$$\frac{1}{(b-a)}\int_{\frac{a+b}{2}}^b\Big[(x-a)^2-(b-x)^2\Big]g(x)Kdx,$$

*where*

$$K=\min\Big\{\big|f'(b)\big|+\frac{\varphi\big(|f'(a)|,|f'(b)|\big)}{2},\big|f'(a)\big|+\frac{\varphi\big(|f'(b)|,|f'(a)|\big)}{2}\Big\}$$

If in Theorem 2.3 we consider $\varphi(x,y)=x-y$ and $g\equiv 1$, then we recapture the following result.

**Corollary 2.4.** [4] Consider $I^*$ as the interior of interval $I\subset\mathbb{R}$. Let $f:I^*\to\mathbb{R}$ be a differentiable mapping on $I^*$, $a,b\in I^*$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then we have

$$\Big|\frac{1}{b-a}\int_a^b f(x)dx-f\Big(\frac{a+b}{2}\Big)\Big|\le\frac{b-a}{8}\Big(|f'(a)|+|f'(b)|\Big)$$

Now, we state the following results to estimate the difference between the middle and right terms in (1).

**Lemma 2.5.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $f'$ is an integrable function on $[a,b]$. Then*

$$\frac{f(a)+f(b)}{2}\int_a^b g(x)dx-\int_a^b f(x)g(x)dx \tag{3}$$
$$=\frac{(b-a)}{4}\Big\{\int_0^1\Big(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b}g(u)du\Big)f'\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)dt$$
$$+\int_0^1\Big(\int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b}g(u)du\Big)f'\big(\frac{1-t}{2}a+\frac{1+t}{2}b\big)dt.\Big\}$$

By the use of Lemma (2.5), we conclude the following Fejer's trapezoid type inequality.

**Theorem 2.6.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable function, $g : [a,b] \to \mathbb{R}^+$ is a continuous function and symmetric about $\frac{a+b}{2}$ and $|f'|$ is a $\varphi$-convex function with $\varphi$ bounded from above on $f([a,b]) \times f([a,b])$. Then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \int_a^b f(x)g(x)dx\right| \leq$$
$$\frac{(b-a)}{4}\Big[2|f'(b)| + \varphi\big(|f'(a)|,|f'(b)|\big)\Big]\int_0^1 \int_{\frac{1+t}{2}a+\frac{1-t}{2}b}^{\frac{1-t}{2}a+\frac{1+t}{2}b} g(u)dudt.$$

Theorem 2.6 reduces to the following result if we consider $\varphi(x,y) = x - y$ and $g \equiv 1$.

**Corollary 2.7.** [1] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\Big(|f'(a)| + |f'(b)|\Big).$$

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# On $h$-Convexity via Hermite-Hadamard Type Inequality

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**Abstract**

It is investigated that under what conditions a function is $h$-convex, if it satisfies the $h$-convex version of the Hermite-Hadamard's inequality.

## 1. Introduction

The following result is known in the literature.

**Theorem 1.1.** *[6] A function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ is convex if and only if*

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_a^b f(t)dt \leq \frac{f(x)+f(y)}{2} \tag{1}$$

*holds for all $x, y \in (a, b)$ with $x \neq y$.*

Inequality (1) is known as Hermite-Hadamard integral inequality for convex functions. Note that the left part and the right part of (1) separately are equivalent to the convexity of $f$(see [5, 6]).
In 2006, The concept of $h$-convex functions related to the nonnegative real functions has been introduced in [9] by S. Varošanec. This class is including a large class of nonnegative functions such as nonnegative convex functions, Godunova-Levin functions [3], s-convex functions in the second sense [1] and P-functions [2].

**Definition 1.2.** Let $h : [0, 1] \to \mathbb{R}$ be a function such that $h \not\equiv 0$. We say that $f : (a, b) \to \mathbb{R}$ is a h-convex function, if for all $x, y \in (a, b)$ , $\lambda \in [0, 1]$ we have

$$f\big(\lambda x + (1-\lambda)y\big) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{2}$$

The $h$-convex version of Hermite-Hadamard inequality has been introduced in [8] by Sarikaya et al. as the following.

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**Theorem 1.3.** *Let $f : I \to [0,\infty]$ be an integrable $h$-convex function. If $a, b \in I$ with $a < b$, then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le [f(a)+f(b)]\left(\int_0^1 h(t)dt\right). \tag{3}$$

In this paper, we reply to the problem that under what conditions a function is $h$-convex, if it satisfies (3). Since the inequality (3) is double so for more convenience the problem is separated to the right and left version respectively.

## 2. Main results

To achieve our main results about the characterization of a $h$-convex function via (3), we introduce a primary definition along with an example and then establish a basic lemma related to $h$-convex functions as the following.

**Definition 2.1.** A function $h : [0,1] \to \mathbb{R}$ is said to be *self-concave* if

$$h\big(zx + (1-z)y\big) \ge h(z)h(x) + h(1-z)h(y),$$

for all $z \in (0,1)$ and $x, y \in [0,1]$.

We can find some simple functions which are self concave.

**Example 2.2.** Consider the function $h(x) = x^n$ for $n \in \mathbb{N}$ and $x \in [0,1]$. It is not hard to see that this function is self-concave. In fact since the function $h$ is nonnegative, then

$$h\big(\lambda x + (1-\lambda)y\big) = \big(\lambda x + (1-\lambda)y\big)^n = \sum_{i=0}^n \binom{n}{i}(\lambda x)^{n-i}\big((1-\lambda)y\big)^i \ge$$
$$\binom{n}{0}(\lambda x)^n + \binom{n}{n}\big((1-\lambda)y\big)^n = h(\lambda)h(x) + h(1-\lambda)h(y).$$

The following lemma plays an important role in obtaining our expected results.

**Lemma 2.3.** *Let $f : (a,b) \to \mathbb{R}$ be a continuous function and $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that for any $x, y \in (a,b)$ with $x \ne y$ there is a $\lambda \in (0,1)$ such that $f\big(\lambda x + (1-\lambda)y\big) \le h(\lambda)f(x) + h(1-\lambda)f(y)$. Then $f$ is $h$-convex on $(a,b)$.*

**Corollary 2.4.** *Let $f : (a,b) \to \mathbb{R}$ be a continuous function. Suppose that for any $x, y \in (a,b)$ with $x \ne y$ there is a $\lambda \in (0,1)$ such that*

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + \big(1-\lambda\big)f(y).$$

*Then $f$ is convex.*

**Theorem 2.5.** *Let $f : (a,b) \to \mathbb{R}$ be a continuous function. Also suppose that $h : [0,1] \to \mathbb{R}$ is a continuous self-concave function such that*

$$\frac{1}{y-x}\int_x^y f(t)dt \le [f(x)+f(y)]\left(\int_0^1 h(t)dt\right),$$

*for all $x, y \in (a,b)$ with $x \ne y$.*
*Then $f$ is $h$-convex on $(a,b)$.*

**Corollary 2.6.** *Let $f : (a,b) \to \mathbb{R}$ be a continuous function such that*

$$\frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2},$$

*for all $x, y \in (a,b)$ with $x \ne y$. Then $f$ is convex on $(a,b)$.*

The following lemma, along with Lemma 2.3, are the basis for characterization of a $h$-convex function via left side of (3).

**Lemma 2.7.** *(Also see Theorem 1.1.4 in [5]) Suppose that $\varphi : [a,b] \to \mathbb{R}$ is a continuous function such that $\varphi(a) = \varphi(b) = 0$ and $\varphi(t) > 0$ for some $t \in (a,b)$. Then there exists a $x \in (a,b)$ such that*

$$\varphi(x) = \max_{a \le y \le b} \varphi(y) \quad and \quad \varphi(x) > \varphi(y) \quad for\ all\ a \le y < x.$$

In what follows, we consider that the function $h : [0,1] \to \mathbb{R}$ satisfies the conditions

$$\begin{cases} h(\lambda) + h(1-\lambda) = 1 \quad for\ all\ \lambda \in (0,1), \\ h(0) = 0. \end{cases} \tag{4}$$

**Lemma 2.8.** *Let $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and for any $x \in (a,b), \varepsilon > 0$, there exist $y, z \in (a,b) \cap (x-\varepsilon, x+\varepsilon)$ with $y < x < z$ such that*

$$f(x) = f\big(\lambda y + (1-\lambda)z\big) \le h(\lambda) f(y) + h(1-\lambda) f(z) \quad for\ some\ \lambda \in (0,1).$$

*Then $f$ is $h$-convex on $(a,b)$.*

Using Lemma 2.8, as Immediate consequence we have two following lemmas. For more details about this kind of results related to the convex functions see [6].

**Lemma 2.9.** *Let $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and for any $x \in (a,b), \varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$ such that*

$$f(x) \le h(\frac{1}{2})\big[f(x-\delta) + f(x+\delta)\big].$$

*Then $f$ is $h$-convex on $(a,b)$.*

**Lemma 2.10.** *Let $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and for any $x \in (a,b), \varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$ such that*

$$f(x) \le \frac{h(\frac{1}{2})}{\delta} \int_{x-\delta}^{x+\delta} f(u)du.$$

*Then $f$ is $h$-convex on $(a,b)$.*

Now using Lemma 2.10 we can obtain a characterization type theorem for $h$-convex functions via the left side of (3).

**Theorem 2.11.** *Let $h : [0,1] \to \mathbb{R}$ be a continuous self-concave function. Suppose that $f : (a,b) \to \mathbb{R}$ is a continuous function and for all $y, z \in (a,b)$ with $y \ne z$ we have*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{y+z}{2}\right) \le \frac{1}{z-y} \int_y^z f(u)du, \tag{5}$$

*then $f$ is $h$-convex on $(a,b)$.*

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# A new projection-combination method for problems involving resolvent operators and relatively nonexpansive mappings

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**Abstract**

This paper introduces a novel hybrid iterative algorithm for approximating common fixed points of resolvent operators associated with maximal monotone mappings and relatively nonexpansive mappings in Banach spaces. The algorithm is constructed within the framework of 2-uniformly convex and uniformly smooth Banach spaces, employing generalized projections and duality mappings. Under appropriate conditions on the control parameters, we prove strong convergence of the generated sequence to a common fixed point.

## 1. Introduction

Let $X$ be a real Banach space with norm $\|\cdot\|$, $X^*$ its topological dual, and $C \subset X$ a nonempty closed convex subset. Consider a set-valued operator $A : X \to 2^{X^*}$ with domain $D(A)$. We say that $A$ is:

(i) monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0 \qquad \forall x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay.$$

(ii) $\alpha$-inverse strongly monotone (with $\alpha > 0$) if

$$\langle x - y, x^* - y^* \rangle \geq \alpha \|x^* - y^*\|^2 \qquad \forall x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay.$$

(iii) $L$-Lipschitz continuous (with $L > 0$) if

$$\|x^* - y^*\| \leq L\|x - y\| \qquad \forall x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay.$$

A monotone operator $A$ is called maximal monotone if its graph

$$G(A) = \{(x, x^*) \in X \times X^* : x \in D(A),\ x^* \in Ax\}$$

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is not properly contained in the graph of any other monotone operator. Equivalently, $A$ is maximal monotone iff for every $(x, x^*) \in X \times X^*$,

$$\langle x - y, x^* - y^* \rangle \geq 0 \ \ \forall (y, y^*) \in G(A) \implies (x, x^*) \in G(A).$$

Now assume that $X$ is a smooth Banach space. The modulus of convexity $\delta_X : [0, 2] \to [0, 1]$ is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \ \|y\| \leq 1, \ \|x - y\| \geq \varepsilon\right\}.$$

$X$ is called uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$. Uniformly convex spaces enjoy the Kadec–Klee property: if $x_n \rightharpoonup x$ weakly and $\|x_n\| \to \|x\|$, then $x_n \to x$ strongly.
For a fixed exponent $p \geq 2$, the space $X$ is said to be $p$-uniformly convex if there exists a constant $c > 0$ such that

$$\delta_X(\varepsilon) \geq c\, \varepsilon^p \qquad \forall \varepsilon \in [0, 2].$$

The normalized duality mapping $J : X \to 2^{X^*}$ is given by

$$J(x) = \left\{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\right\}, \quad x \in X.$$

When $X$ is smooth, $J$ is single-valued and we write $Jx$ instead of $J(x)$. The norm of $X$ is Gâteaux differentiable if for every $x, y \in S(X) = \{z \in X : \|z\| = 1\}$ the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists. If the limit (1) is attained uniformly in $x, y \in S(X)$, then $X$ is called uniformly smooth.
For a smooth Banach space we define the functional (also known as the Bregman distance) $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \qquad x, y \in X. \tag{2}$$

From (2) one immediately obtains the two-sided estimate

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \tag{3}$$

If, in addition, $X$ is strictly convex and reflexive, then

$$\phi(x, y) = 0 \iff x = y. \tag{4}$$

Moreover, the following useful inequality holds:

$$\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\| \, \|Jx - Jy\| + \|y - x\| \, \|y\|. \tag{5}$$

Associated with $\phi$ is the functional $V : X \times X^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \qquad x \in X, \ x^* \in X^*.$$

Notice that $V(x, x^*) = \phi\left(x, J^{-1}x^*\right)$ whenever the inverse duality mapping $J^{-1}$ is defined. For a reflexive, strictly convex and smooth Banach space one has the important inequality

$$V(x, x^*) + 2\left\langle J^{-1}x^* - x, \ y^* \right\rangle \leq V(x, \ x^* + y^*), \qquad \forall x \in X, \ \forall x^*, y^* \in X^*. \tag{6}$$

An operator $A : C \to X^*$ is called hemicontinuous at $x_0 \in C$ if for every $z \in X$ and every sequence $t_n \to 0^+$ with $x_0 + t_n z \in C$,

$$A(x_0 + t_n z) \rightharpoonup Ax_0 \quad \text{weakly in } X^*.$$

The generalized projection onto $C$ is the mapping $\Pi_C : X \to C$ that assigns to each $x \in X$ the unique point $\Pi_C x \in C$ satisfying

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of $\Pi_C$ follow from the strict convexity and smoothness of $X$ together with the properties of $\phi$.
Let $T : C \to C$ be a nonlinear operator. Denote by

$$F(T) = \{x \in C : Tx = x\}$$

its set of fixed points. A point $p \in C$ is an asymptotic fixed point of $T$ if there exists a sequence $\{x_n\} \subset C$ such that

$$x_n \rightharpoonup p \text{ (weakly)} \quad \text{and} \quad \|Tx_n - x_n\| \to 0.$$

The set of all asymptotic fixed points is denoted by $\widehat{F}(T)$. We call $T$ relatively nonexpansive provided

$$\widehat{F}(T) = F(T) \quad \text{and} \quad \phi(p, Tx) \leq \phi(p, x) \ \ \forall x \in C, \ \forall p \in F(T).$$

The following lemmas will be used in the proof of our main results.

**Lemma 1.1.** *([2]) Let $X$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $X$. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

**Lemma 1.2.** *([1]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$ and let $y \in X$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C.$$

**Lemma 1.3.** *([1]) Let $C$ be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space $X$, let $x \in X$ and let $z \in C$. Then*

$$z = \Pi_C x \Leftrightarrow \langle y - z, Jx - Jz \rangle \leq 0, \text{ for all } y \in C.$$

**Lemma 1.4.** *([7]). Let $X$ be a uniformly convex Banach space and $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

*for all $x, y \in B_r(0) = \{z \in E : \|z\| \leq r\}$ and $t \in [0, 1]$.*

**Definition 1.5.** ([5]) Let $X$ be a real smooth and uniformly convex Banach space and let $M : X \to 2^{X^*}$ be a maximal monotone operator. For all $\xi > 0$, define the operator $Q_\xi^M : X \to X$ by $Q_\xi^M x = (J + \xi M)^{-1} Jx$ for all $x \in X$.

**Lemma 1.6.** *([4]) Let $X$ be a real smooth and uniformly convex Banach space and let $M : X \to 2^{X^*}$ be a maximal monotone operator. Then $M^{-1}0$ is a closed and convex subset of $X$ and the graph $G(M)$ of $M$, is demi-closed.*

**Lemma 1.7.** *([6]) Let $X$ be a real reflexive, strictly convex and smooth Banach space and $M : X \to 2^{X^*}$ be a maximal monotone operator with $M^{-1}0 \neq \emptyset$. Then, for all $x \in X, y \in M^{-1}0$ and $\xi > 0$, then $\phi(y, Q_\xi^M x) + \phi(Q_\xi^M x, x) \leq \phi(y, x)$.*

## 2. Main results

We assume that the following conditions hold for the proposed Algorithm 1:

1. $X$ is a real 2-uniformly convex and uniformly smooth Banach space.
2. $C$ is a nonempty closed and convex subset of $X$.
3. $f$ is a relatively nonexpansive mapping from $C$ into itself.
4. $M_i : X \to 2^{X^*}$ be a maximal monotone operator with ${M_i}^{-1}0 \neq \emptyset$ for all $i = 1, 2, \ldots, k$.

**Algorithm 1** Iterative Algorithm

**Initialization:** Let $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[a, b] \subset (0, 1)$. Let $x_0 \in C$ be initial point.

**Step 1:** Compute $y_n = \Pi_C J_{X_1}^{-1}\left(\alpha_n J_{X_1} + (1-\alpha_n) J_{X_1} Q_l^{M_1} Q_l^{M_2} \ldots Q_l^{M_k}\right) x_n$,

**Step 2:** Compute $x_{n+1} = \Pi_C J_{X_1}^{-1}\left(\beta_n J_{X_1} f(x_n) + (1-\beta_n) J_{X_1} y_n\right)$.

**Theorem 2.1.** *Let* $\Omega = F(f) \cap (\cap_{i=1}^{k} F(Q_l^{M_i})) \neq \emptyset$. *Then, the sequence* $\{x_n\}$ *generated by Algorithm 1 converges strongly to the some solution* $q \in \Omega$.

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# Hyers-Ulam stabilities for Orthogonally $i$-3$D$-Jensen $\rho$-Functional Equations and triple Lie $i$-Hom-Derivations

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**Abstract**

This paper introduces orthogonally triple Lie $i$-hom-derivations associated with orthogonally $i$-3$D$-Jensen $\rho$-functional equations where $i \in 1, 2$ on orthogonally triple Lie algebras. We conclude by employing the orthogonally fixed point method to establish Hyers-Ulam stability results for orthogonally $i$-3$D$-Jensen $\rho$-functional equations and the orthogonally triple Lie $i$-hom-derivations.

## 1. Introduction

The stability theory of functional equations originates with a fundamental question posed by Ulam in 1940 [9]. In 1941, Hyers [5] provided the first affirmative answer for additive maps on Banach spaces, a result now known as Hyers–Ulam stability. A significant generalization was achieved by Rassias [8] in 1978 via a control term $\theta(\|a\|^r + \|b\|^r)$ with $r < 1$, leading to the concept of generalized Hyers–Ulam stability. In 1994, Găvruta [2] later subsumed these approaches under a general control function $\varphi(a, b)$. The interplay between stability and fixed-point theory was first explicitly established by Isac and Rassias [6] in 1996. The study of orthogonal structures was advanced by Eshaghi Gordji et al. [4], who introduced orthogonal sets and their basic properties in 2017. Building directly on this foundation, Bahraini et al. [1] employed a fixed-point approach the following year to prove the Hyers-Ulam stability of orthogonally $*$-$m$-homomorphisms on $C^*$-algebras. For more information see [3, 7]

In the section 2, we discuss on orthogonally $i$-3$D$-Jensen $\rho$-functional equations and orthogonally triple Lie $i$-hom-derivations where $i \in 1, 2$ on orthogonally triple Lie algebras.

In the section 3, using orthogonally fixed point method to investigate orthogonally $i$-3$D$-Jensen $\rho$-functional equations and orthogonally triple Lie $i$-hom-derivations can be Hyers-Ulam stable type of Găvruta and Rassias.

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## 2. Preliminaries

In this section, we first recall the structure of orthogonal sets.

**Definition 2.1.** [4] **(i)** Let $\mathcal{A}$ be a nonempty set equipped with a binary relation $\perp \subseteq \mathcal{A} \times \mathcal{A}$. If there exists an element $a_0 \in \mathcal{A}$ such that

$$(\forall b \in \mathcal{A},\ b \perp a_0) \quad \text{or} \quad (\forall b \in \mathcal{A},\ a_0 \perp b),$$

then the pair $(\mathcal{A}, \perp)$ is called an orthogonal set (abbreviated as an O-set).

**(ii)** Let $(\mathcal{A}, \perp)$ be an O-set. A sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ is called an orthogonal sequence (O-sequence) if

$$(\forall n \in \mathbb{N},\ a_n \perp a_{n+1}) \quad \text{or} \quad (\forall n \in \mathbb{N},\ a_{n+1} \perp a_n).$$

**(iii)** If $(\mathcal{A}, \perp)$ is an O-set and $(\mathcal{A}, d)$ is a metric space, then $(\mathcal{A}, \perp, d)$ is called an orthogonal metric space. A mapping $f : \mathcal{A} \to \mathcal{A}$ is said to be $\perp$-continuous at $a \in \mathcal{A}$ if for every O-sequence $\{a_n\}$ in $X$ converging to $a$, we have $f(a_n) \to f(a)$. It is evident that every continuous mapping is $\perp$-continuous.

**(iv)** A Cauchy sequence $\{a_n\}$ in an orthogonal metric space is called a Cauchy orthogonal sequence (Cauchy O-sequence) if for all $n \in \mathbb{N}$, either $a_n \perp a_{n+1}$ or $a_{n+1} \perp a_n$. The space $(\mathcal{A}, \perp, d)$ is said to be orthogonally complete (O-complete) if every Cauchy O-sequence converges in $\mathcal{A}$.
It follows directly that every complete metric space is O-complete; the converse, however, does not hold in general.

**(v)** Let $(\mathcal{A}, \perp, d)$ be an orthogonal metric space and $0 < \lambda < 1$. A mapping $f : \mathcal{A} \to \mathcal{A}$ is called an orthogonality contraction with Lipschitz constant $\lambda$ if for all $a, b \in \mathcal{A}$ with $a \perp b$, we have

$$d(f(a), f(b)) \leq \lambda\, d(a, b).$$

Now, we discuss about concept of triple orthogonally Lie algebra. An orthogonally Lie algebra is an orthogonally Banach algebra equipped with the Lie bracket

$$[a, b] := \frac{(ab - ba)}{2}.$$

Generalizing this, a triple orthogonally Lie algebra endowed with the triple Lie product

$$\Big[[a, b], c\Big] := \frac{[a, b]c - c[a, b]}{2}.$$

for all $a, b, c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$.
Now, let $\mathcal{A}$ be an algebra. We consider the orthogonally $i$-$3D$-Jensen $\rho$-functional equations

$$\begin{aligned} &3^i f_i\Big(\frac{a+b+c}{3}\Big) + f_i(a) + f_i(b) + (-1)^i f_i(c) - 2^i f_i\Big(\frac{a+b}{2}\Big) - 2^i f_i\Big(\frac{b+c}{2}\Big) \\ &- (-1)^i 2^i f_i\Big(\frac{a+c}{2}\Big) = \rho[i f_i(a+b+c) + i f_i(a) - f_i(a+b) - f_i(a+c) - (i-1) f_i(b+c)] \end{aligned} \tag{1}$$

for all $a, b, c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$. In the following definition, using the concept of orthogonally triple Lie homomorphisms and derivations and orthogonally triple Lie quadratic homomorphisms and derivations, we define orthogonally triple Lie $i$-hom-derivations

**Definition 2.2.** Let $\mathfrak{H}_i : \mathcal{A} \to \mathcal{A}$ be orthogonally triple Lie $i$-homomorphisms. An orthogonally $i$-mapping $\mathfrak{D}_i : \mathcal{A} \to \mathcal{A}$ is called an orthogonally triple Lie $i$-hom-derivation if $\mathfrak{D}$ satisfies

$$\mathfrak{D}_i[[a,b],c] = [[\mathfrak{D}_i(a),\mathfrak{H}_i(b)],\mathfrak{H}_i(c)] + [[\mathfrak{H}_i(a),\mathfrak{D}_i(b)],\mathfrak{H}_i(c)] + [[\mathfrak{H}_i(a),\mathfrak{H}_i(b)],\mathfrak{D}_i(c)],$$

for all $a, b, c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$.

**Theorem 2.3.** *[1] Let $(\mathcal{A}, d, \perp)$ be an orthogonally complete generalized metric space and let $T : \mathcal{A} \to \mathcal{A}$ be a mapping that is $\perp$-preserving, $\perp$-continuous, and a $\perp$-$\lambda$-contraction for some $\lambda \in [0,1)$. Suppose there exists an element $a_0 \in \mathcal{A}$ such that for every $b \in \mathcal{A}$, either $a_0 \perp b$ or $b \perp a_0$. Consider the orthogonally admissible sequence $\{a_n\}_{n=0}^{\infty}$ defined by $a_n = T^n(a_0)$. Then exactly one of the following holds:*

1. *$d(T^n(a_0), T^{n+1}(a_0)) = \infty$ for all $n \geq 0$;*
2. *there exists a positive integer $n_0$ such that $d(T^n(a_0), T^{n+1}(a_0)) < \infty$ for all $n \geq n_0$.*

*If condition (ii) is satisfied, then the following conclusions hold:*

1. *The sequence $\{T^n(a_0)\}$ converges to a fixed point $a^*$ of $T$.*
2. *The fixed point $a^*$ is unique within the set*

$$\mathcal{A}^* = \{b \in \mathcal{A} : d(T^{n_0}(a_0), b) < \infty\}.$$

3. *For any $b \in \mathcal{A}$, the following estimate holds:*

$$d(b, a^*) \leq \frac{1}{1-\lambda}\, d(b, T(b)).$$

## 3. Main Results

In this section, Let $\mathcal{A}$ is orthogonally triple Lie algebra, $i \in 1, 2$ and $\rho \neq 0, \pm 1$.

**Lemma 3.1.** *Let $i \in \{1,2\}$ and let $f_i : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying equation (1) for all $a, b, c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$. Then $f_i(0) = 0$ and the following properties hold:*

1. *If $i = 1$, then $f_i$ is an orthogonally additive mapping.*
2. *If $i = 2$ and $f_i$ is an even function, then $f_i$ is an orthogonally quadratic mapping.*

Now, we discuss related to Găvruta's control function. Assume $i \in \{1,2\}$ and $\varrho_i, \varpi_i : \mathcal{A}^3 \to [0,\infty)$ are two functions satisfying conditions

$$\varrho(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \leq \frac{\alpha}{2^i}\varrho_i(a,b,c) \tag{2}$$

$$\varpi(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}) \leq \frac{\alpha}{2^{3i}}\varpi_i(a,b,c) \tag{3}$$

for all $a, b, c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$ and some $0 < \alpha < 1$.
In the next theorem, using the orthogonally fixed point method to prove the orthogonally $i$-$3D$-Jensen $\rho$-functional equations can be Hyers-Ulam stable with Găvruta's control function.

**Theorem 3.2.** *Let $f_i : \mathcal{A} \to \mathcal{A}$ be a mapping for which there exist functions $\varrho_i : \mathfrak{A}^3 \to [0,\infty)$ such that*

$$\Big\| 3^i f_i(\frac{a+b+c}{3}) + f_i(a) + f_i(b) + (-1)^i f_i(c) - 2^i f_i(\frac{a+b}{2}) - 2^i f_i(\frac{b+c}{2}) - (-1)^i 2^i f_i(\frac{a+c}{2}) \\ - \rho[i f_i(a+b+c) + i f_i(a) - f_i(a+b) - f_i(a+c) - (i-1) f_i(b+c)] \Big\| \leq \varrho_i(a,b,c). \tag{4}$$

*for all $a, b, c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$. Then there exist unique orthogonally $i$-mappings $T_i : \mathfrak{A} \to \mathfrak{A}$ such that*

$$\|f_1(a) - T_1(a)\| \leq \frac{k}{2(1-k)}\varrho_1(0, a, -a)$$

$$\|f_2(a) - T_2(a)\| \leq \frac{k}{2(1-k)}\varrho_2(a, 0, -a).$$

In the following, we investigate the Hyers-Ulam stability of orthogonally triple Lie $i$-hom-derivation on orthogonally triple Lie algebra.

**Theorem 3.3.** *Assume $\varpi_i : \mathfrak{A}^3 \to [0,\infty)$ be function which satisfying conditions (2) and (3) for some constant $\alpha \in (0,1)$. Suppose that $f_i, g_i : \mathfrak{A} \to \mathfrak{A}$ which satisfying relation (4) and*

$$\|f_i[[a,b],c] - [[f_i(a),f_i(b)],f_i(c)]\| \leq \varpi_i(a,b,c) \tag{5}$$

$$\|g_i[[a,b],c] - [[g_i(a),f_i(b)],f_i(c)] - [[f_i(a),g_i(b)],f_i(c)] - [[f_i(a),f_i(b)],g_i(c)],\| \leq \varpi_i(a,b,c). \tag{6}$$

*for all $a,b,c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$. Then $f_i : \mathfrak{A} \to \mathfrak{A}$ is an orthogonally triple Lie $i$-homomorphism and $g_i : \mathfrak{A} \to \mathfrak{A}$ is an orthogonally triple Lie $i$-hom-derivation.*

In the next corollary, we prove the Hyers-Ulam stabilities for $i$-$3D$-Jensen $\rho$-functional equations and orthogonally triple Lie $i$-hom-derivation on orthogonally Lie algebra with control function Rassias.

**Corollary 3.4.** *Let $i \in \{1,2\}$ and let $r > i$ and $s \geq 0$ be real numbers. Define the control functions $\varrho_i$ and $\varpi_i$ by*

$$\varrho_i(a,b,c) = \varpi_i(a,b,c) = s\big(\|a\|^r + \|b\|^r + \|c\|^r\big),$$

*for all $a,b,c \in \mathcal{A}$ with $a \perp b$, $b \perp c$ and $a \perp c$. Suppose that $f_i, g_i : \mathcal{A} \to \mathcal{A}$ satisfy the inequalities (4), (5) and (6). Then there exist an orthogonally triple Lie $i$-homomorphism $H_i : \mathcal{A} \to \mathcal{A}$ and an orthogonally triple Lie $i$-hom-derivation $D_i : \mathcal{A} \to \mathcal{A}$ such that for every $a \in \mathcal{A}$,*

$$\|f_i(a) - H_i(a)\| \leq \frac{2^i s}{2^r - 2^i}\|a\|^r$$

*and*

$$\|g_i(a) - D_i(a)\| \leq \frac{2^i s}{2^r - 2^i}\|a\|^r.$$

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# The spectrum of some special operators on weighted Hardy space

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**Abstract**

In this paper, we investigate the spectrum of a diagonal perturbation of a weighted shift operator on a weighted Hardy space.

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. A weighted Hardy spaces $H^2(\beta)$ is a Hilbert space whose elements are functions analytic on $\mathbb{D}$ such that the set $\{z^n | n \geq 0\}$ constitute a complete orthogonal set in $H^2(\beta)$. In this notation the weight $\beta$ is a sequence $\{\beta(j)\}$, such that $\beta(j) = \|z^j\|$ for $j = 0, 1, 2, \cdots$ and $\beta(0) = 1$ and $\limsup (\frac{1}{\beta(j)^2})^{\frac{1}{j}} \leq 1$. If $f(z) = \Sigma_0^\infty a_n z^n$ belongs to $H^2(\beta)$, then $\|f\|^2 = \Sigma_0^\infty |a_n|^2 |\beta(n)|^2$ and the inner product is given by $\langle f, g\rangle = \langle \Sigma_0^\infty a_n z^n, \Sigma_0^\infty b_n z^n\rangle = \Sigma_0^\infty a_n \overline{b_n} |\beta(n)|^2$. It is well known that the evaluation of functions at $w \in \mathbb{D}$ is a bounded linear functional and if $k_w$ denotes the unique vector in $H^2(\beta)$ such that $f(w) = \langle f, k_w\rangle$, then $k_w(z) = \Sigma_0^\infty \frac{\overline{w}^n z^n}{\beta(n)^2} = k(\overline{w}z)$, where $k(z) = \sum_0^\infty \frac{z^n}{\beta(n)^2}$ and $k$ is analytic on $\mathbb{D}$. We denote by $B(H^2(\beta))$ the space of all bounded operators on $H^2(\beta)$. Some classical examples of such spaces are as follows:

1. The Hardy space $H^2$ consists of those functions $f$ analytic on $\mathbb{D}$ for which

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^2 dm(z) < \infty,$$

where $dm$ is the normalized arc length measure on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

2. The classical weighted Bergman space $A_\alpha^2$ for $\alpha > -1$ is defined as the space of all functions $f$ analytic on the unit disk $\mathbb{D}$ such that

$$\|f\|_{A_\alpha^2} = \left( \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha (\alpha + 1) dA(z) \right)^{\frac{1}{2}} < \infty,$$

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where $dA$ is the normalized Lebesgue area measure on $\mathbb{D}$.We denote the classical Bergman space ($\alpha = 0$) by $A^2$
3. The space of all analytic functions on the unit disk $\mathbb{D}$, whose derivatives are in $A^2_\alpha$ with the norm given by

$$||f||^2_\alpha = |f(0)|^2 + ||f'||^2_{A^2_\alpha}$$

is called the weighted Dirichlet space and is denoted by $\mathcal{D}_\alpha$. These spaces with the above norms are Hilbert spaces. When $\alpha = 0$, we denote the classical Dirichlet space by $D$.

By $H^\infty(\mathbb{D})$ we denote the space of all bounded analytic functions on $\mathbb{D}$ with the supremum norm

$$\|\psi\|_{\mathbb{D}} = \sup\{|\psi(z)| : z \in \mathbb{D}\}.$$

Spectral theory can be useful in determining asymptotic properties, such as stability, cyclicity, hypercyclicity, and chaoticity, when working with infinite-dimensional linear dynamical systems (see[4], [8], [9] and [10] ). The shift operator and its perturbation offer a natural and appropriate environment for testing those concepts.
The translation of this text into English maintains a formal and academic tone suitable for operator theory, as is commonly the case in operator theory, knowing the structure of the spectrum and imposing certain conditions on it is useful for ensuring the emergence of a specific dynamical property. For instance, it has been established that an invertible operator T on $C^n$ is expansive if and only if the point spectrum of T does not intersect the unit circle T [[6], theorem 1]. Subsequently, since it is well known that an invertible operator T on a Banach space X is hyperbolic if its spectrum does not intersect T, it was shown in [[4], Theorem D] and [[11], Theorem 1] that T uniformly expansive if and only if this same condition is satisfied by the approximate point spectrum, this implies, as a corollary, that invertible hyperbolic operators are uniformly expansive.
In [5], by using Cowen's formula for the adjoint of $C_\varphi$ on $H^2(\mathbb{D})$, the authors have completely determined the spectrum, essential spectrum and point spectrum for self- commutators of automorphic composition operators acting on the Hardy space of unit disk. In [1], the author extended these results from the Hardy space to the Dirichlet space. In [2], [3] and [7] The authors, have extended these results, with monomial symbols on some certain weighted Hardy spaces.
Let $T \in B(H^2(\beta))$ be a diagonal perturbation of a unilateral weighted shift $S$ on $H^2(\beta)$. That is

$$(Tf)(w) = \sum_{n=0}^{r_0-1} \frac{f^{(n)}(0)}{n!}\lambda_n w^n \quad + \quad \sum_{n=r_0}^{\infty} \left(\frac{f^{(n)}(0)}{n!} + \frac{f^{(n-r_0)}(0)}{(n-r_0)!}\right)\lambda_n w^n,$$

then operator T can be written as $T = D + S_{r_0}$, where $D$ is a diagonal operator (related to the diagonal entries $\{\lambda_n\}$) and $S_{r_0}$ is a weighted shift operator of multiplicity $r_0$, and $0 < |\lambda_n| < M$. In this paper, we aim to find the spectrum and the point spectrum of this operator on the weighted Hardy spaces.

## 2. Spectrum of Some Special Operators

**Theorem 2.1.** Let $H^2(\beta)$ be a weighted Hardy space. Define the operator T on this space as follows:

$$(Tf)(w) = \sum_{n=0}^{r_0-1} \frac{f^{(n)}(0)}{n!}\lambda_n w^n \quad + \quad \sum_{n=r_0}^{\infty} \left(\frac{f^{(n)}(0)}{n!} + \frac{f^{(n-r_0)}(0)}{(n-r_0)!}\right)\lambda_n w^n.$$

The operator T can be decomposed as $T = D + S_{r_0}$, where $D$ is a diagonal operator (related to the diagonal entries $\{\lambda_n\}$) and $S_{r_0}$ is a weighted shift operator of multiplicity $r_0$, then
(a)

$$\sigma_p(T) \subseteq \{\lambda_n | n \geq 0\}.$$

(b)Assume that $\lambda_i = \lambda$ for each $i \in J = N \cup \{0\}$. Then on $H^2(\beta)$

$$\sigma_p(T) = \{\}.$$

Furthermore,

$$0 \neq \lambda \notin \sigma(T),$$

if $\mu \neq \lambda$, $\sup_{n\in N} \frac{\beta(n+r_0)}{\beta(n)} \leq l$, and $|\frac{\lambda}{\lambda-\mu}| < \frac{1}{l\sqrt[2]{2}}$, then $\mu \notin \sigma(T)$.

(c) Assume that $\mu \notin \overline{\{\lambda_n | n \geq 0\}}$, such that $|\lambda_n - \mu| \geq \delta > 0$, for all $n \in J = N \cup \{0\}$. If $|\lambda_n| \leq M \leq \delta$, for all $n \in J$, and $\sup_{n\in N} \frac{\beta(n+r_0)}{\beta(n)} \leq 1$, then

$$\mu \notin \sigma(T).$$

(d) Assume that $\lambda = \lambda_{i_0}$ such that $\lambda_{i_0} = \lambda_l$ for each $l \in B \subseteq N$, where $r_0$ divides $l - i_0$ (i.e., $r_0|(l - i_0)$) If the set B is infinite, then $\lambda_{i_0} \notin \sigma_p(T)$. Furthermore, assume that $B$ is finite and $\max B = P$. Define

$$d_{(i_0+(p+l)r_0)} = (\frac{\lambda_{i_0}}{\lambda_{i_0+(p+l)r_0}} - 1) \times \ldots (\frac{\lambda_{i_0}}{\lambda_{i_0+(p+1)r_0}} - 1),$$

if

$$\sum_{l=1}^{\infty} |\frac{1}{d_{(i_0+(p+l)r_0)}}|^2 \beta(i_0 + (p+l)r_0)^2 < \infty,$$

then $\lambda_{i_0} \in \sigma_p(T)$.

**Example 1.** Assume that $A^2(D)$ is the Bergmn space and $H^2(D)$ is the Hardy space. Let $\psi(z) = z^{k_1} + z^{k_2}$ and $\varphi(z) = z^m$ such that $m,\ k_1,\ k_2 \in N$ and $m|(k_2 - k_1)$. Assume that

$$\lambda_n = \frac{\beta(nm + k_1)^2}{\beta(n)^2} + \frac{\beta(nm + k_2)^2}{\beta(n)^2}.$$

Then on $H^2(D)$, we have

$$\sigma_p(T) = \{\}.$$

If $m \leq k_1 + 1$, or $m \geq \frac{k_1+k_2}{2} + 1$, then on $A^2(D)$, we have

$$\sigma_p(T) = \{\}.$$

**Example 2.** Assume that $D$ is the Dirichlet space. Let $\psi(z) = z^{k_1} + z^{k_2}$ and $\varphi(z) = z^m$ where $m,\ k_1,\ k_2 \in N$ and $m|(k_2 - k_1)$. Let the sequence $\{\lambda_n\}$ defined as

$$\lambda_n = \frac{\beta(nm + k_1)^2}{\beta(n)^2} + \frac{\beta(nm + k_2)^2}{\beta(n)^2}.$$

If $k_1 + k_2 + 2 - 2m < 2m$, then on $D$,

$$\sigma_p(T) = \{\}.$$

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# Existence results for a generalized equilibrium problem

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**Abstract**

In this paper, we present an existence result for solutions of a generalized equilibrium problem. The main tool of our approach is a variational relation problem which provides a unified technique for handling multivalued structures under relaxed closedness and coercivity conditions.

## 1. Introduction and Preliminaries

Equilibrium problems constitute a fundamental framework that unifies and generalizes a wide range of mathematical models arising in optimization, fixed point, game theory, and economics. The modern formulation of equilibrium problems was introduced by Blum and Oettli [1] where the classical variational inequality problem, optimization problems, and Nash equilibria can all be seen as particular cases of a single abstract model. Since then, equilibrium theory has become an active and central area of research in nonlinear analysis.

In order to deal with more realistic models involving multivalued structures, vector criteria, and nonconvex settings, several generalizations of the classical equilibrium problem have been proposed. Among them, the generalized vector equilibrium problem introduced by Fu and Wan [2] plays an important role. In their work, the authors considered equilibrium problems involving set-valued mappings and vector-valued bifunctions, which considerably allow the treatment of many models that cannot be handled within the classical scalar framework.

On the other hand, the notion of variational relation offers a unified language for describing many problems such as variational inequalities, inclusion problems, and equilibrium problems. This framework allows one to reformulate complicated equilibrium models into a variational relation whose solvability can be studied via general topological and set-valued analysis tools; see [3, 4].

The aim of this paper is to present a new existence result for a generalized equilibrium problem. Our approach is based on two main steps. First, we introduce an appropriate variational relation associated with the problem and prove its solvability by means of an intersection theorem for a pair of set-valued mappings. Then, as an application

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of this variational relation, we obtain a new existence theorem for the generalized equilibrium problem under weaker assumptions.
Let $X$ and $Y$ be two sets and $S, T : X \rightrightarrows Y$ be set-valued maps. Here, we consider an intersection problems for the pair $(S, T)$, which is to find a point $\bar{y} \in X$ such that

$$T(\bar{y}) \cap S(x) \neq \emptyset, \quad \forall x \in X.$$

Let $X$ be a convex subset of a vector space, $Y$ be a nonempty set and $S, T : X \rightrightarrows Y$ be set-valued mappings. Then $S$ is called weak KKM with respect to $T$ if for any finite set $A \subseteq X$ and $y \in \mathrm{conv}A$,

$$T(y) \cap \bigcup_{t \in A} S(t) \neq \emptyset$$

where conv$A$ denotes the convex hull of $A$. It is clear that if $S$ is a weak KKM mapping with respect to the identity mapping, then this definition reduces to the standard definition of KKM mappings. Recall that a set-valued mapping $S : X \rightrightarrows X$ is KKM if conv$A \subseteq \bigcup_{t \in A} S(t)$, for any finite set $A \subseteq X$.
Throughout this paper, the family of all nonempty finite subsets of a set $X$ is denoted by $\langle X \rangle$.
Let $X$ and $Y$ be topological spaces. A set-valued mapping $F : X \rightrightarrows Y$ is said to be upper semicontinuous on $X$, if for any open set $V \subseteq Y$, the set $\{x \in X : F(x) \subseteq V\}$ is open in $X$. In other words, for any closed subset $B$ of $Y$ the set $\{x \in X : F(x) \cap B \neq \emptyset\}$ is closed.
The set-valued mapping $F$ is called to be intersectionally closed on $A \subseteq X$, if

$$\bigcap_{x \in A} \mathrm{cl}(F(x)) = \mathrm{cl}(\bigcap_{x \in A} F(x)).$$

**Definition 1.1.** [5] A pair of set-valued maps $(S, T)$ is called generalized topological pseudomonotone provided that for all $x, x_0 \in X$, and every net $(x_\alpha)$ in $X$ converging to $x_0$ with

$$T(x_\alpha) \cap S(tx + (1-t)x_0) \neq \emptyset, \ \forall t \in [0,1],$$

we have $T(x_0) \cap S(x) \neq \emptyset$.

**Proposition 1.2.** *[5] Let $S, T : X \rightrightarrows Z$ be set-valued maps such that the set $F(x) := \{y \in X : T(y) \cap S(x) \neq \emptyset\}$ is closed for each $x \in X$, then $(S, T)$ is generalized topological pseudomonotone.*

**Definition 1.3.** [5] Let $S, T : X \rightrightarrows Z$ be set-valued maps and $K \subseteq X$. Then we call $S$ to be intersectionally closed on $K$ with respect to $T$ if for any $y_0 \in K$, the implication presented below is satisfied.:
if there is a neighborhood $U$ of $y_0$ in $K$ such that for all $y \in U$ we have $S(x_y) \subseteq Z \setminus T(y)$, for some $x_y \in K$, then we can find $x_0 \in K$ and a neighborhood $U'$ of $y_0$ in $K$ with $S(x_0) \subseteq Z \setminus T(u)$ for each $u \in U'$.

We call a nonempty set $A \subseteq X$ to be $c$-compact if conv$(K \cup A)$ is compact for each $K \in \langle X \rangle$.
In [5], inspired by [6, 7], the following intersection theorem for the pair $(S, T)$ in absence of the usual compactness and closedness conditions was studied.

**Theorem 1.4.** *Let $S, T : X \rightrightarrows Z$ be set-valued maps such that*

*(I) $S$ is weak KKM map with respect to $T$;*

*(II) $S$ is intersectionally closed on* conv$K$ *with respect to $T$, for each $K \in \langle X \rangle$;*

*(III) $(S, T)$ is generalized topological pseudomonotone;*

*(IV) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, there is $x \in$ conv$(A \cup \{t\})$ such that $T(t) \cap S(x) = \emptyset$.*

*Then, we can find a point $\bar{y} \in X$ such that for each $x \in X$,*

$$T(\bar{y}) \cap S(x) \neq \emptyset.$$

## 2. Main results

In the following, let $X, Y, Z$ be nonempty sets, $P : Y \rightrightarrows Z$ be a nonempty-valued map and $R(x, y, z)$ be a relation for $x \in X, y \in Y$ and $z \in Z$.
After initial presentation of variational relation problem, other variants of it were introduced and studied; see [4, 8, 9]. Here, we focus on the following form of this problem:

(VRP) Find $\bar{y} \in Y$ s.t. for each $x \in X$ there exists $z \in P(\bar{y})$ s.t. $R(x, \bar{y}, z)$ holds.

When $X = Y$, (VRP) is a general model for problems like minimax inequalities and Stampacchia variational inequality problems; see [9] and references therein. We say $R^c(x, y, z)$ holds if and only if $R(x, y, z)$ does not hold. Moreover, $R(., y, z)$ is convex if the set $\{x \in X : R(x, y, z) \text{ holds}\}$ is convex, for each $(y, z) \in Y \times Z$.

**Definition 2.1.** Let $P : Y \rightrightarrows Z$ be a nonempty-valued map and $R(x, y, z)$ be a relation for $x \in X, y \in Y$ and $z \in Z$. The relation $R(x, y, z)$ is said to be KKM with respect to $P$($P-$KKM, in short) if for each $K \in \langle X \rangle$ and $y \in \text{conv}K$, we have $R(x, y, z)$ holds for some $x \in K$ and $z \in P(y)$.

**Proposition 2.2.** *[5] Let for each $x \in X$, $R(x, x, z)$ hold for some $z \in P(x)$ and $R^c(., y, z)$ be convex. Then, $R(x, y, z)$ is $P-$KKM.*

The following theorem was proved in [5] by using Theorem 1.4.

**Theorem 2.3.** *Let $Z$ be a topological space. Suppose that $P : X \rightrightarrows Z$ is a set-valued map with nonempty values and $R(x, y, z)$ is a relation for $x, y \in X$ and $z \in Z$ such that*

*(i) $R(x, y, z)$ is $P-$KKM;*

*(ii) for each $K \in \langle X \rangle$ and $y_0 \in \text{conv}K$, if there exists a neighborhood $U$ of $y_0$ in $\text{conv}K$ such that for all $y \in U$ there is an $x_y \in \text{conv}K$ such that $R(x_y, y, z)$ does not hold for all $z \in P(y)$, then there exist $x_0 \in \text{conv}K$ and a neighborhood $U'$ of $y_0$ in $\text{conv}K$ such that $R(x_0, y, z)$ does not hold for all $y \in U'$ and $z \in P(y)$;*

*(iii) for any $x, \bar{x} \in X$ and every net $(x_\alpha)$ in $X$ converging to $\bar{x}$, if for each $t \in [0, 1]$, $R(tx + (1-t)\bar{x}, x_\alpha, z_\alpha)$ holds for some $z_\alpha \in P(x_\alpha)$, then $R(x, \bar{x}, \bar{z})$ holds for some $\bar{z} \in P(\bar{x})$;*

*(iv) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, there exists $x \in \text{conv}(A \cup \{t\})$ such that $R(x, t, z)$ does not hold, for each $z \in P(t)$.*

*Then, the variational relation problem (VRP) has a solution.*

In the following, other existence theorems for variational problem (VRP) are presented:

**Theorem 2.4.** *Let $Z$ be a topological space and $P : X \rightrightarrows Z$ be an upper semicontinuous set-valued map with nonempty values and $R(x, y, z)$ is a relation for $x, y \in X$ and $z \in Z$ such that*

*(i) for each $x \in X$, the set $\{(y, z) \in X \times Z : R(x, y, z) \text{ holds}\}$ is closed in $X \times Z$;*

*(ii) $R(x, y, z)$ is $P-$KKM;*

*(iii) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, we can find $x \in \text{conv}(A \cup \{t\})$ such that $R(x, t, z)$ does not hold, for each $z \in P(t)$.*

*Then, the variational relation problem (VRP) has a solution.*

**Theorem 2.5.** *Let $Z$ be a topological space and $P : X \rightrightarrows Z$ be a set-valued map with nonempty values and $R(x, y, z)$ is a relation for $x, y \in X$ and $z \in Z$ such that*

*(i) for each $x \in X$, the set $\{y \in X : R^c(x, y, z) \text{ holds}, \forall z \in P(y)\}$ is open in $X$;*

*(ii) $R(x, y, z)$ is $P-$KKM;*

*(iii) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, we can find $x \in \mathrm{conv}(A \cup \{t\})$ such that $R(x,t,z)$ does not hold, for each $z \in P(t)$.*

*Then, the variational relation problem (VRP) has a solution.*

In the following, we investigate one of the consequences of the above results for (VRP) as an equilibrium Problem (GVEP 1):
Let $T : X \rightrightarrows Z$ be a set-valued map with nonempty values and $f : X \times X \times Z \to \mathbb{R}$, then we consider Problem (GVEP 1) as follows

$$(GVEP1) \text{ Find } y \in X \text{ such that } \forall x \in X, \exists z \in T(y), f(x,y,z) \leq 0$$

**Theorem 2.6.** *Let $Z$ be a topological space, $T : X \rightrightarrows Z$ be a set-valued map with nonempty values and $f : X \times X \times Z \to \mathbb{R}$ such that*

*(i) for each $K \in \langle X \rangle, y \in \mathrm{conv}K$, we have $f(x,y,z) \leq 0$ holds for some $x \in K$ and $z \in T(y)$;*

*(ii) for each $K \in \langle X \rangle$ and $y_0 \in \mathrm{conv}K$, if there exists a neighborhood $U$ of $y_0$ in $\mathrm{conv}K$ such that for all $y \in U$ there exists $x_y \in \mathrm{conv}K$ such that $f(x_y,y,z) > 0$ for all $z \in T(y)$, then there exist $x_0 \in \mathrm{conv}K$ and a neighborhood $U'$ of $y_0$ in $\mathrm{conv}K$ such that $f(x_0,y,z) > 0$, for all $y \in V'$ and $z \in T(y)$;*

*(iii) for each $x, \bar{x} \in X$ and every net $(x_\alpha)$ in $X$ converging to $\bar{x}$, if for all $t \in [0,1]$, $f(tx + (1-t)\bar{x}, x_\alpha, z_\alpha) \leq 0$ for some $z_\alpha \in T(x_\alpha)$, then $f(x,\bar{x},\bar{z}) \leq 0$ for some $\bar{z} \in T(\bar{x})$;*

*(iv) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, we can find $x \in \mathrm{conv}(A \cup \{t\})$ such that $f(x,t,z) > 0$ , for each $z \in T(t)$.*

*Then, there exists $\bar{y} \in X$ such that for each $x \in X$ there exists $z \in T(\bar{y})$ such that $f(x,\bar{y},z) \leq 0$.*

**Remark 2.7.** It is easy to verify that if $f(.,y,z)$ is quasiconcave, for each $(y,z) \in X \times Z$, then condition $(i)$ in Theorem 2.3 holds. Moreover, if assumption (V) of Theorem 5 in [2] is satisfied, then for $C = M$ and $A = \{\bar{x}\}$, coercivity condition (iv) of Theorem 2.3 is valid. Therefore, Theorem 2.3 is a generalization of the corresponding results in [2] and similar works.

**Corollary 2.8.** *Let $Z$ be a topological space, $T : X \rightrightarrows Z$ be a set-valued map with nonempty values and $f : X \times X \times Z \to \mathbb{R}$ such that*

*(i) $T$ is upper semicontinuous;*

*(ii) for each $x \in X, f(x,.,.)$ is lower semicontinuous;*

*(iii) $f(.,y,z)$ is quasiconcave, for each $(y,z) \in X \times Z$;*

*(iii) there exist a c-compact set $A$ and a nonempty and compact set $C \subseteq X$ such that for each $t \in X \setminus C$, we can find $x \in \mathrm{conv}(A \cup \{t\})$ such that $f(x,t,z) > 0$ , for each $z \in T(t)$.*

*Then, there exists $\bar{y} \in X$ such that for each $x \in X$ there exists $z \in T(\bar{y})$ such that $f(x,\bar{y},z) \leq 0$.*

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# Weighted composition operators with certain symbols on some weighted Hardy spaces

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**Abstract**

In this paper, we present a comprehensive study of the spectrum of operators of the form $TT^*$ and $T^*T$ acting on the Dirichlet space $D$ or the Bergman spaces $A^2(\mathbb{D})$, where $T$ is the weighted composition operator on $D$ or $A^2(\mathbb{D})$ of the form

$$Tf(z) = (z^{k_1} + \ldots + z^{k_l})f(z^m), \quad |z| < 1, m, k_1, \ldots, k_l \in N.$$

## 1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane. A Hilbert space $H^2(\beta)$ whose vectors are functions analytic on the unit disk will be referred to as a weighted Hardy space if the monomials $1, z, z^2, \ldots$ form a complete orthogonal set of non-zero vectors in $H^2(\beta)$. Writing $\beta(j) = \|z^j\|$, the orthogonality property can be readily observed to lead to the conclusion that the norm on $H^2(\beta)$ is defined as follows

$$\|\sum_{j=0}^{\infty} a_j z^j\|^2 = \sum_{j=0}^{\infty} |a_j|^2 \beta(j)^2$$

Some classical examples of such spaces are as follows:
1. The Hardy space $H^2$ is given by:

$$H^2(D) = \{f \in H(D) : \|f\|_{H^2} = \lim_{r \to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\},$$

where $d\theta$ is the Lebesgue arc-length measure on the unit circle.

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2. The Bergman space $A^2(D)$ is the space of all holomorphic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ for which

$$\int_{\mathbb{D}} |f(z)|^2 dV(z) < \infty$$

where $dV$ denotes the volume measure on $\mathbb{D}$.
3. The space of all analytic functions on the unit disk $\mathbb{D}$, whose derivatives are in $A^2_\alpha$ with the norm given by

$$||f||^2_\alpha = |f(0)|^2 + ||f'||^2_{A^2_\alpha}$$

is called the weighted Dirichlet space and is denoted by $\mathcal{D}_\alpha$. These spaces with the above norms are Hilbert spaces. When $\alpha = 0$, we denote the classical Dirichlet space by $D$.
4. The space $S^2(\mathbb{D})$ is defined as the collection of analytic functions on the open unit disc whose derivatives belong to the Hardy space. In terms of a power series, if $g(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|g\|^2_{S^2(\mathbb{D})} = |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 n^2 < \infty.$$

It is also worth noting that the norm in the space $S^2(\mathbb{D})$ can be expressed as follows:

$$\|g\|^2_{S^2(\mathbb{D})} = |g(0)|^2 + \|g'\|^2_{H^2(\mathbb{D})}.$$

When studying infinite-dimensional linear dynamical systems, spectral theory is frequently emplyed to analyze their asymptotic behavior, including properties, such as a stability, cyclicity, hypercyclicity and chaoticity... (see[4], [7], [8] and [9] ).
In [5], the author uses Cowen's formula for the adjoint of $C_\varphi$ on the Hardy space, to completely describe the spectrum, point spectrum, and essential spectrum for self- commutators of automorphic composition operators acting on the Hardy space over the open unit disk. In [1], these results were generalized from the Hardy space to the Dirichlet space. Subsequent developments are presented In [2], [3] and [6] where the authors further extended the spectral results to certain weighted Hardy spaces, with a particular focus on monomial symbols.
The next section provides a formula for the adjoint of the weighted composition operator $C_{\psi,\varphi}$ on the Dirichlet space, Bergman space and $S^2(D)$, where $\varphi(z) = z^n$ is a general monomial and $\psi(z) = z^{k_1} + ... + z^{k_L}$, for positive integers $k_1, ..., k_L$ and $n$. In the continue, we provide a complete description of the point spectrum, spectrum, and essential spectrum of the operators $C^*_{\psi,\varphi}C_{\psi,\varphi}$ and $C_{\psi,\varphi}C^*_{\psi,\varphi}$.

## 2. The Spectrum and Essential Spectrum

Let $\varphi(z) = z^m$, $\psi(z) = z^{k_1} + z^{k_2} + ... + z^{k_l}$, where $k_1, k_2, ..., k_i$ and $m \in N$. Our aim is to investigate the point spectrum, spectrum, essential spectrum, and corresponding eigenfunctions, of the operators $C^*_{\psi,\varphi}C_{\psi,\varphi}$ and $C_{\psi,\varphi}C^*_{\psi,\varphi}$.

**Theorem 2.1.** Let $\varphi(z) = z^m$ with $m \geq 2$, and let $\psi(z) = z^{k_1} + ... + z^{k_l}$. Consider a weighted Hardy space $H^2(\beta)$. The weighted composition operator $C_{\psi,\varphi}$ is bounded on $H^2(\beta)$ if and only if there exists a positive constant A such that

$$M_i = \sup_{n \in N} \frac{\beta(nm + k_i)}{\beta(n)} \leq A, \; i = 1, ... l.$$

Moreover, whenever this condition holds, the operator norm $C_{\psi,\varphi}$ is given by

$$\|C_{\psi,\varphi}\| = \sqrt{\sum_{i=1}^{l} M_i}$$

*Proof.* The first part is evident. Now suppose that $g(z) = \sum_{n=0}^{\infty}$ is in $H^2(\beta)$. It is easy to see that

$$\|C_{\psi,\varphi}g\| \leq \sqrt{M_1 + ... + M_l}\|g\|.$$

Hence, $\|C_{\psi,\varphi}\| \leq \sqrt{M_1 + ... + M_l}$. Moreover, if $e_n(z) = \frac{z^n}{\beta(n)}$ denotes the unit vector in $H^2(\beta)$, then we obtain

$$\sqrt{M_1 + ... + M_l} \leq \|C_{\psi,\varphi}\frac{z^n}{\beta(n)}\| \leq \|C_{\psi,\varphi}\|$$

□

**Theorem 2.2.** Let $\phi(z) = z^m$ and $\psi(z) = d_1z^{k_1} + d_2z^{k_2} + ... + d_lz^{k_l}$, where m, $k_1, ... k_l$ are positive integers and $d_1, ..., d_l$ are complex numbers. Consider $C_{\psi,\varphi}$ is acting on the space $S^2(\mathbb{D})$. Then, for any point $w \in \mathbb{D}$, the adjoint of $C_{\psi,\phi}$ can be represented in the following form.

$$\begin{aligned} C^*_{\psi,\phi}f(w) &= \overline{d_1}\frac{f^{(k_1)}(0)}{(k_1)!}k_1^2 + ... + \overline{d_l}\frac{f^{(k_l)}(0)}{(k_l)!}k_l^2 \\ &+ \sum_{n=1}^{\infty}(\overline{d_1}\frac{f^{(nm+k_1)}(0)}{(nm+k_1)!}\frac{(nm+k_1)^2}{n^2} + ... \\ &+ \overline{d_l}\frac{f^{(nm+k_l)}(0)}{(nm+k_l)!}\frac{(nm+k_l)^2}{n^2})w^n. \end{aligned}$$

**Theorem 2.3.** Assume that $\psi(z) = z^{k_1} + z^{k_2} + ... + z^{k_l}$, $\varphi(z) = z^m$ such that $m,\ k_1, ..., k_l \in N$ and $m \nmid (k_i - k_j)$ for $i, j = 1 ... l$ with $i \neq j$. Consider $C_{\psi,\varphi}$ acting on $S^2(\mathbb{D})$. Then

$$\sigma_p(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \{0, k_1^2 + ... + k_l^2\} \cup \{\frac{(nm+k_1)^2}{n^2} + ... + \frac{(nm+k_l)^2}{n^2} | n \in N\},$$

$$\sigma_p(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \sigma_p(C_{\psi,\varphi}C^*_{\psi,\varphi}) \setminus \{0\},$$

$$\sigma(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \overline{\sigma_p(C^*_{\psi,\varphi}C_{\psi,\varphi})},$$

and

$$\sigma(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \sigma(C^*_{\psi,\varphi}C_{\psi,\varphi}) \cup \{0\}.$$

**Theorem 2.4.** Assume that $D$ is a Dirichlet space, $\phi(z) = z^m$ and $\psi(z) = d_1z^{k_1} + ... + d_lz^{k_l}$, where m, $k_1, ..., k_l$ are positive integers and $d_1, ..., d_l$ are complex numbers. Then, for any point $w \in \mathbb{D}$, the adjoint of $C_{\psi,\phi}$ acting on $D$ can be represented as follows

$$\begin{aligned} C^*_{\psi,\phi}f(w) &= \sum_{n=0}^{\infty}(\overline{d_1}\frac{f^{(nm+k_1)}(0)}{(nm+k_1)!}\frac{nm+k_1+1}{n+1} + ... \\ &+ \overline{d_l}\frac{f^{(nm+k_l)}(0)}{(nm+k_l)!}\frac{nm+k_l+1}{n+1})w^n. \end{aligned}$$

**Theorem 2.5.** Let $D$ be a Dirichlet space, and let $\psi(z) = z^{k_1} + ... + z^{k_l}$, $\varphi(z) = z^m$, where $m,\ k_1, ...,\ k_l \in N$ and $m \nmid (k_i - k_j)$, for $i, j = 1, ..., l$ and $i \neq j$. Consider $C_{\psi,\varphi}$ acting on a Dirichlet space. Then

$$\sigma_p(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \{0\} \cup \{\frac{nm+k_1+1}{n+1} + ... + \frac{nm+k_l+1}{n+1} | n \in N \cup \{0\}\},$$

$$\sigma(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \overline{\sigma_p(C^*_{\psi,\varphi}C_{\psi,\varphi})},$$

and

$$\sigma(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \sigma(C^*_{\psi,\varphi}C_{\psi,\varphi}) \cup \{0\}.$$

**Theorem 2.6.** Assume that $A^2(\mathbb{D})$ is the Bergman space, $\phi(z) = z^m$, and $\psi(z) = a_1 z^{k_1} + ... + a_l z^{k_l}$, where m, $k_1, ..., k_l$ are positive integers and $a_1, ..., a_l$ are complex numbers. Then, for any point $w \in \mathbb{D}$, the adjoint of $C_{\psi,\phi}$ acting on $A^2(\mathbb{D})$ can be represented as follows

$$\begin{aligned} C^*_{\psi,\phi}f(w) &= \sum_{n=0}^{\infty}(\overline{a_1}\frac{f^{(nm+k_1)}(0)}{(nm+k_1)!}\frac{n+1}{nm+k_1+1} + ... \\ &+ \overline{a_l}\frac{f^{(nm+k_l)}(0)}{(nm+k_l)!}\frac{n+1}{nm+k_l+1})w^n. \end{aligned}$$

**Theorem 2.7.** Let $A^2(\mathbb{D})$ be the Bergman space, and let $\psi(z) = z^{k_1} + ... + z^{k_l}$, $\varphi(z) = z^m$, where $m,\ k_1, ...,\ k_l \in N$ and $m \nmid (k_i - k_j)$, for $i, j = 1, ..., l$ and $i \neq j$. Consider $C_{\psi,\varphi}$ acting on the Bergman space. Then

$$\sigma_p(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \{0\} \cup \{\frac{n+1}{nm+k_1+1} + ... + \frac{n+1}{nm+k_l+1} | n \in N \cup \{0\}\},$$

$$\sigma(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \overline{\sigma_p(C^*_{\psi,\varphi}C_{\psi,\varphi})},$$

and

$$\sigma(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \sigma(C^*_{\psi,\varphi}C_{\psi,\varphi}) \cup \{0\}.$$

**Theorem 2.8.** Let $\varphi(z) = z^m$, $\psi(z) = z^{k_1} + ... + z^{k_l}$ such that m, $k_1, ..., k_2 \in N$, $m \nmid (k_i - k_j)$, for $i, j = 1, ... l$ and $i \neq j$. Assume that $C_{\psi,\varphi}$ is acting on the Dirichlet space, the Bergman space or $S^2(\mathbb{D})$. Then $ran(C^*_{\psi,\varphi}C_{\psi,\varphi})$ and $ran(C_{\psi,\varphi}C^*_{\psi,\varphi})$ are closed.

**Theorem 2.9.** Let $\varphi(z) = z^m$, $\psi(z) = z^{k_1} + ... + z^{k_l}$ such that m, $k_1$,...,$k_l \in N$, $m \nmid (k_i - k_j)$, for $i, j = 1, ... l$, and $i \neq j$. Assume that $C_{\psi,\varphi}$ is acting on the Dirichlet , the Bergman spaces or $S^2(\mathbb{D})$. Then on the Dirichlet space we have

$$\sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \{0, lm\},$$

and

$$\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi})\backslash\{0\}.$$

On the Bergman space we have

$$\sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \{0, \frac{l}{m}\},$$

and

$$\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi})\backslash\{0\}.$$

On the $S^2(\mathbb{D})$ we have

$$\sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi}) = \{0, lm^2\},$$

and

$$\sigma_e(C^*_{\psi,\varphi}C_{\psi,\varphi}) = \sigma_e(C_{\psi,\varphi}C^*_{\psi,\varphi})\backslash\{0\}.$$

*Proof.* Since any points in the spectrum of a normal operator that are not in the essential spectrum are isolated eigenvalues of finite multiplicity, we may therefore apply the results of the above theorems to reach the conclusion □

**Theorem 2.10.** Let $H^2(\beta)$ be a weighted Hardy space, $\phi(z) = a$ and $\psi(z) = z^{k_1} + z^{k_2} + ... + z^{k_l}$, where m, $k_1, ... k_l$ are positive integers. Suppose that the operator $C_{\psi,\varphi}$ is bounded on $H^2(\beta)$. Then for any point $w \in \mathbb{D}$, the adjoint of $C_{\psi,\phi}$ acting on $H^2(\beta)$ can be expressed as follows

$$(C^*_{\psi,\varphi} f)(w) = (\sum_{n=0}^{\infty} \frac{\overline{a_n} w^n}{\beta(n)^2})(\sum_{i=1}^{l} \frac{f^{k_i}(0)}{k_i!} \beta(k_i)^2).$$

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# Existence and uniqueness results for optimal coupled fixed points structures within noncyclic $\vartheta$-contractivity settings and their analytical applications

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**Abstract**

The study of cyclic contraction ordered pairs has garnered significant attention in recent years. This paper addresses a natural extension: the definition and analysis of a non-cyclic version of this contraction type. Let $\vartheta \in \Theta$, where

$$\Theta := \{\theta : \mathbb{R}^+ \to \mathbb{R}^+ : \theta \text{ is strictly increasing and } I - \theta \text{ is increasing}\}.$$

We introduce the non-cyclic Suzuki-type $\vartheta$-contraction ordered pair $((\mathcal{F}, \bar{\mathcal{F}}), (\mathcal{G}, \bar{\mathcal{G}}))$ and investigate the existence and uniqueness conditions for their optimal coupled fixed points. Our findings demonstrate that non-cyclic $\vartheta$-quasi-contractions are a specific instance within this class of ordered paired contractions. Furthermore, we present a priori and a posteriori estimates of the fixed point error for the Picard iteration associated with this class of non-cyclic contraction ordered pairs, particularly highlighting important results for non-cyclic $\vartheta$-quasi-contractions in a uniformly convex Banach space with a power modulus of convexity. Finally, we confirm the correctness of our results by providing a numerical example.

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# Exploring a Specific Subclass of $m$-fold Symmetric Bi-Univalent Functions via Faber Polynomials

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**Abstract**

In this paper, a general subclass $\mathcal{H}_{\Sigma_m}(\gamma,\lambda,q,\beta)$ of $m$-fold symmetric bi-univalent functions is introduced. Estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in the subclass $\mathcal{H}_{\Sigma_m}(\gamma,\lambda,q,\beta)$ are obtained. Furthermore, estimates on the general coefficients $|a_{mk+1}|(k \geqq 3)$ for functions in this subclass are obtained. The results presented in this paper would generalize and improve some recent works of several earlier authors.

## 1. First Section

Let $\mathcal{A}$ denote the class of all analytic functions $f$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the class of $\mathcal{A}$ consisting of univalent functions (see [3]). For functions $f \in \mathcal{S}$, let $f^{-1}$ be its inverse function. The Koebe one-quarter theorem ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. So evevery univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z(z \in \mathbb{D})$ and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f),\ r_0(f) \geqq \frac{1}{4}),$$

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where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$, if both $f$ and $f^{-1}$ are univalent in $\mathbb{D}$ (see [5]). We denote $\sigma_{\mathcal{B}}$ the class of bi-univalent functions in $\mathbb{D}$ given by (1). The coefficients of $f^{-1}$, the inverse map of $f$ are given by [1]

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n)w^n, \tag{3}$$

where

$$K_{n-1}^{-n} = \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 + \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4$$
$$+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5} \times [a_5 + (-n+2)a_3^2] + \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6 + (-2n+5)a_3a_4] + \sum_{j\geq 7} a_2^{n-j}V_j$$

such that $V_j$ with $7 \leqq j \leqq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$.
Determination of the bounds for the coefficients $a_n$ is an important problem in geometric function theory as they give information about the geometric properties of these functions. In 1967, Lewin [9] introduced this class $\sigma_{\mathcal{B}}$ and proved that the bound for the second coefficients of every $f \in \sigma_{\mathcal{B}}$ satisfies the inequality $|a_2| \leqq 1.52$.
For each function $f \in \mathcal{S}$ function

$$h(z) = \sqrt[m]{f(z^m)} \tag{4}$$

is univalent and maps unit disk $\mathbb{D}$ into a region with $m$-fold symmetry. A function $f$ is said to be $m$-fold symmetric, if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \mathbb{D}, m \in \mathbb{N}). \tag{5}$$

We denote by $\mathcal{S}_m$ the class of $m$-fold symmetric univalent functions in $\mathbb{D}$, which are normalized by the series expansion (5). In [7] Srivastava et al. defined $m$-fold symmetric bi-univalent functions analogues to the concept of $m$-fold symmetric univalent functions. They gave some important results, such as each function $f \in \sigma_{\mathcal{B}}$ generates an $m$-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of $f$ given by (5), they obtained the series expansion for $f^{-1}$ as follows:

$$f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1}w^{mk+1} \tag{6}$$

$$= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \Big[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 \tag{7}$$
$$-(3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\Big]w^{3m+1} + \cdots.$$

We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $\mathbb{D}$. For $m = 1$, formula (7) coincides with formula (2) of the class $\sigma_{\mathcal{B}}$. The aim of this paper is to introduce new subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$ of $m$-fold symmetric bi-univalent functions by using $q$-derivative. Furthermore, we obtain estimates on initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and general coefficients $|a_{mk+1}|$ $(k \geqq 3)$ for functions in the subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$ by using Faber polynomial expansions.

## 2. Preliminaries

Now, we recall to a notion of $q$-derivative operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. We suppose throughout the paper that $0 < q < 1$.

**Definition 2.1.** [8] The Jackson's $q$-derivative of a function $f \in \mathcal{A}$ is defined as follows

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & ; \quad z \neq 0 \\ f'(0) & ; \quad z = 0. \end{cases}$$

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, according to the Definition 2.1, we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \ , \ D_q^2 f(z) = D_q(D_q f(z)) = \sum_{n=2}^{\infty} [n]_q [n-1]_q a_n z^{n-2}$$

where the symbol $[n]_q$ denotes the $q$-number and $[n]_q = \frac{1-q^n}{1-q}$.
Also, by taking $q \to 1^-$, $[n]_q$ reduces to $n$.
Now, we generalize the Jackson's $q$-derivative to $m$-fold symmetric bi-univalent functions as follow.

**Definition 2.2.** For a function $f \in \Sigma_m$ given by (5), Jackson's $q$-derivative of function $f$ is defined by

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & ; \ z \neq 0 \\ f'(0) & ; \ z = 0. \end{cases}$$

For $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} (m \in \mathbb{N}; m \geqq 2)$, according to the Definition 2.2, we have

$$D_q f(z) = 1 + \sum_{k=1}^{\infty} [mk+1]_q a_{mk+1} z^{mk} \ , \ D_q^2 f(z) = D_q(D_q f(z)) = \sum_{k=1}^{\infty} [mk+1]_q [mk]_q a_{mk+1} z^{mk-1}.$$

## 3. Main Theorem

In this section, the general subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$ is introduced and considered.

**Definition 3.1.** A function $f$ given by (5) is said to be in the subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$ $(\lambda \geqq 0, 0 < q < 1, 0 \leqq \beta < 1, \gamma \in \mathbb{C}/\{0\})$, if the two following conditions are satisfied:

$$f \in \Sigma_m, \ \Re\left(1 + \frac{1}{\gamma}\left[D_q f(z) + \lambda z D_q\left(D_q f(z)\right) - 1\right]\right) > \beta \ , \ \Re\left(1 + \frac{1}{\gamma}\left[D_q g(w) + \lambda w D_q\left(D_q g(w)\right) - 1\right]\right) > \beta,$$

where $g$ is given by (6).

**Remark 3.2.** If we set $q \to 1^-$ and $0 \leqq \lambda \leqq 1$ in Definition 3.1, then the subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$ reduces to the subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, \beta)$ considered by Srivastava et al. [6].

**Remark 3.3.** If we set $q \to 1^-$, $\gamma = 1$ and $\lambda = 0$ in Definition 3.1, then the subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$ reduces to the subclass $\mathcal{H}_{\Sigma,m}(\beta)$ considered by Srivastava et al. [7].

**Remark 3.4.** For one-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{H}_{\Sigma_1}(\gamma, \lambda, q, \beta) = \mathcal{H}_{\Sigma}(\gamma, \lambda, q, \beta)$. Special cases of this subclass illustrated below:

- If we put $\gamma = 1$, then the subclass $\mathcal{H}_{\Sigma}(\gamma, \lambda, q, \beta)$ reduces to the subclass $\mathcal{H}_{\Sigma}^{q}(\beta, \lambda)$ introduced by Sabil et al. [10].

- If we put $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{H}_{\Sigma}(\gamma, \lambda, q, \beta)$ reduces to the sunclass $\mathcal{H}_{\Sigma}^{q}(\beta)$ introduced by Bulut [2].
- If we put $\gamma = 1$ and $q \to 1^{-}$, then the subclass $\mathcal{H}_{\Sigma}(\gamma, \lambda, q, \beta)$ reduces to the $\mathcal{H}_{\Sigma}(\beta, \lambda)$ introduced by Frasin [4].
- If we put $\gamma = 1$, $q \to 1^{-}$ and $\lambda = 0$, then the subclass $\mathcal{H}_{\Sigma}(\gamma, \lambda, q, \beta)$ reduces to the subclass $\mathcal{H}_{\Sigma}(\beta)$ introduced by Srivastava et al. [5].

**Theorem 3.5.** *Let $f$ given by (5) be in subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$. If $a_{m+1} = a_{2m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)|\gamma|}{[mk+1]_q \left(1 + \lambda [mk]_q\right)}; \qquad (k \geqq 3).$$

**Theorem 3.6.** *Let $f$ given by (5) be in subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, q, \beta)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2|\gamma|(1-\beta)}{[m+1]_q \left(1 + \lambda [m]_q\right)}, 2\sqrt{\frac{|\gamma|(1-\beta)}{[2m+1]_q \left(1 + \lambda [2m]_q\right)(m+1)}}\right\},$$

$$|a_{2m+1}| \leqq \min\left\{\frac{2|\gamma|^2 (m+1)(1-\beta)^2}{[m+1]_q^2 \left(1 + \lambda [m]_q\right)^2} + \frac{2|\gamma|(1-\beta)}{[2m+1]_q \left(1 + \lambda [2m]_q\right)}, \frac{2|\gamma|(1-\beta)}{[2m+1]_q \left(1 + \lambda [2m]_q\right)}\right\}$$

*and for some $\delta \in \mathbb{R}$*

$$|a_{2m+1} - \delta a_{m+1}^2| \leqq \begin{cases} \frac{2|\gamma|(1-\beta)}{[2m+1]_q (1+\lambda [2m]_q)}; & \left|\frac{m+1-2\delta}{m+1}\right| \leqq 1 \\ \frac{2|\gamma|(1-\beta)}{[2m+1]_q (1+\lambda [2m]_q)} \left|\frac{m+1-2\delta}{m+1}\right|; & \left|\frac{m+1-2\delta}{m+1}\right| \geqq 1. \end{cases}$$

## 4. Corollaries and Consequences

By setting $q \to 1^{-}$ and $0 \leqq \lambda \leqq 1$ in Theorem 3.5, we obtain the following.

**Corollary 4.1.** *Let $f$ given by (5) be in subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, \beta)$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)|\gamma|}{(mk+1)(1+\lambda mk)}; \qquad (k \geqq 3).$$

By setting $\gamma = 1$ and $\lambda = 0$ in Corollary 4.1, we obtain the following.

**Corollary 4.2.** *Let $f$ given by (5) be in subclass $\mathcal{H}_{\Sigma,m}(\beta)$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{1+mk}; \qquad (k \geqq 3).$$

By setting $m = 1$ and $\gamma = 1$ in Theorem 3.5, we obtain the following.

**Corollary 4.3.** *Let $f$ given by (1) be in subclass $\mathcal{H}_{\Sigma}^{q}(\beta, \lambda)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{[k+1]_q \left(1 + \lambda [k]_q\right)}; \qquad (k \geqq 3).$$

By setting $\lambda = 0$ in Corollary 4.3, we obtain the following result.

**Corollary 4.4.** *Let $f$ given by (1) be in subclass $\mathcal{H}_{\Sigma}^{q}(\beta)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{[k+1]_q}; \qquad (k \geqq 3).$$

By setting $q \to 1^{-}$ in Corollary 4.3, we obtain the following result.

**Corollary 4.5.** *Let $f$ given by (1) be in subclass $\mathcal{H}_{\Sigma}(\beta, \lambda)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{(k+1)\,(1+\lambda k)}; \qquad (k \geqq 3).$$

By setting $\lambda = 0$ in Corollary 4.5, we obtain the following result.

**Corollary 4.6.** *Let $f$ given by (1) be in subclass $\mathcal{H}_{\Sigma}(\beta)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{k+1}; \qquad (k \geqq 3).$$

By setting $q \to 1^{-}$ and $0 \leqq \lambda \leqq 1$ in Theorem 3.6, we obtain the following.

**Corollary 4.7.** *Let $f$ given by (5) be in subclass $\mathcal{H}_{\Sigma_m}(\gamma, \lambda, \beta)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2|\gamma|(1-\beta)}{(m+1)\,(1+\lambda m)}, 2\sqrt{\frac{|\gamma|(1-\beta)}{(2m+1)\,(1+2\lambda m)\,(m+1)}}\right\}, \quad |a_{2m+1}| \leqq \frac{2|\gamma|(1-\beta)}{(2m+1)\,(1+2\lambda m)}$$

*and for some $\delta \in \mathbb{R}$*

$$|a_{2m+1} - \delta a_{m+1}^2| \leqq \begin{cases} \frac{2|\gamma|(1-\beta)}{(2m+1)(1+2\lambda m)}; & \left|\frac{m+1-2\delta}{m+1}\right| \leqq 1 \\ \frac{2|\gamma|(1-\beta)}{(2m+1)(1+2\lambda m)}\left|\frac{m+1-2\delta}{m+1}\right|; & \left|\frac{m+1-2\delta}{m+1}\right| \geqq 1. \end{cases}$$

**Remark 4.8.** The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.7 are better than those given by Srivastava [6, Theorem 3].

By setting $\gamma = 1$ and $\lambda = 0$ in Corollary 4.7, we obtain the following.

**Corollary 4.9.** *Let $f$ given by (5) be in subclass $\mathcal{H}_{\Sigma,m}(\beta)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)}{m+1}, 2\sqrt{\frac{(1-\beta)}{(2m+1)(m+1)}}\right\} \quad \text{and} \quad |a_{2m+1}| \leqq \frac{2(1-\beta)}{2m+1}$$

*and for some $\delta \in \mathbb{R}$*

$$|a_{2m+1} - \delta a_{m+1}^2| \leqq \begin{cases} \frac{2(1-\beta)}{2m+1}; & |m+1-2\delta| \leqq m+1 \\ \frac{2(1-\beta)|m+1-2\delta|}{(2m+1)(m+1)}; & |m+1-2\delta| \geqq m+1. \end{cases}$$

**Remark 4.10.** The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.9 are better than those given by Srivastava [7, Theorem 3].

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# The properties of rigid $C_0$-semigroups

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**Abstract**

In this article, we take a closer look at the structure of rigid $C_0$-semigroups and obtain new results about them. We prove that rigid $C_0$-semigroups are recurrent. We establish that while a $C_0$-semigroup contains a rigid operator, then it is rigid. Also, we prove that a $C_0$-semigroup is rigid if and only if one of its discretizations is rigid.

## 1. Introduction

Various types of operators are defined in dynamical systems theory. For instance, an operator $T$ on a Banach space $X$ is called recurrent if for any non-empty open subset $U$ of $X$, there exists a natural number $n$ such that $T^n(U) \cap U \neq \phi$ [3]. Also, an operator $T$ on $X$ is called rigid with respect to an increasing sequence $(n_k)$ of natural numbers if $T^{n_k}x \to x$ for any $x \in X$ [5]. Some newer concepts like quasi-rigid operator is defined in [4]. An operator $T$ on $X$ is called quasi-rigid with respect to an increasing sequence $(n_k)$ of natural numbers if there exists a dense subset $A$ of $X$ such that $T^{n_k}a \to a$ for any $a \in A$. Some newer definitions can also be seen in [1] and [2].
These definitions have also been generalized and expanded to the structure of $C_0$-semigroups. A family $(T_t)_{t\geq 0}$ of operators on $X$ is called a $C_0$-semigroup on $X$ if

(i) $T_0 = I$.

(ii) $T_{t+s} = T_t T_s$ for every $t, s \geq 0$.

(iii) $\lim_{s\to t} T_s x = T_t x$ for every $x \in X$ [7].

A $C_0$-semigroup $(T_t)_{t\geq 0}$ on $X$ is called recurrent if for every non-empty open subset $U$ of $X$, there exists a real number $t > 0$ such that $T_t(U) \cap U \neq \phi$ [9]. Also, $(T_t)_{t\geq 0}$ is called rigid if there exists an increasing sequence $(t_k)$ of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} T_{t_k} x = x$ for every $x \in X$ [5]. The new concept of quasi-rigidity for semigroups is introduced as follows in [8].

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**Definition 1.1.** ([8]) Assume $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on a Banach space $X$ and assume $(t_k)$ is an increasing sequence of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$. Then $(T_t)_{t\geq 0}$ is called quasi-rigid with respect to $(t_k)$ if there exists a dense subset $Y$ of $X$ such that for every $x \in Y$, $T_{t_k}x \to x$.

It is concluded from Definition 1.1 that rigid $C_0$-semigroups are quasi-rigid. Therefore, the set of rigid $C_0$-semigroups is a subset of the set of the quasi-rigid $C_0$-semigroups. Also, [10] is recommended for further reading.
In this article, we want to take a closer look at the structure of rigid $C_0$-semigroups and obtain new results about them. We prove that rigid $C_0$-semigroups are recurrent. We establish that if a $C_0$-semigroup contains a rigid operator, then it is rigid. Also, we prove that a $C_0$-semigroup is rigid if and only if one of its discretizations is rigid.

## 2. Main Results

We begin this section by proving this fact that if a $C_0$-semigroup has a rigid operator, it becomes rigid itself.

**Proposition 2.1.** *Assume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on a Banach space $X$. If $T_{t_0}$ is rigid for some $t_0 > 0$, then $(T_t)_{t\geq 0}$ is a rigid $C_0$-semigroup.*

*Proof.* Consider $t_0 > 0$ exists such that $T_{t_0}$ is rigid. Then an increasing sequence $(k_n)$ of natural numbers exists such that $\lim_{n\to\infty} k_n \to \infty$, and for every $x \in X$,

$$T_{t_0}^{k_n} x \to x.$$

By the properties of a $C_0$-semigroup we have ${T_{t_0}}^{k_n} = T_{t_0 k_n}$. Therefore, $(t_0 k_n)$ is an increasing sequence such that for every $x \in X$,

$$T_{t_0 k_n} x \to x.$$

□

The next theorem asserts that rigidity of a $C_0$-semigroup implies its recurrency.

**Theorem 2.2.** *A rigid $C_0$-semigroup is recurrent.*

*Proof.* Assume that $(T_t)_{t\geq 0}$ is a rigid $C_0$-semigroup on a Banach space $X$. Hence, there exists an increasing sequence $(t_n)$ of positive real numbers such that for every $x \in X$,

$$T_{t_n} x \to x.$$

So, every $x \in X$ is a recurrent vector for $(T_t)_{t\geq 0}$. Hence, $(T_t)_{t\geq 0}$ is a recurrent $C_0$-semigroup by Theorem 3 of [9].
□

Recall that A discretization of a semigroup $(T_t)_{t\geq 0}$ is a sequence of operators $(T_{t_n})_n$, where $t_n$ tends to infinity, when $n$ tends to infinity. If for some $s > 0$, we have $t_n = ns$ for every $n \in \mathbb{N}$, then $(T_{t_n}) = (T_{ns})$ is named an autonomous discretization of $(T_t)_{t\geq 0}$. The following theorem asserts that the rigidity of a $C_0$-semigroup is equivalent to rigidity of one of its discretizations.

**Theorem 2.3.** *Assume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on a Banach space $X$. Then the following assertions are equivalent:*

*(i) $(T_t)_{t\geq 0}$ is rigid.*

*(ii) One discretization of $(T_t)_{t\geq 0}$ is rigid.*

*Proof.* Assume that $(T_t)_{t\geq 0}$ is rigid. Hence, there exists an increasing sequence $(t_n)$ of positive real numbers such that $\lim_{n\to\infty} t_n \to \infty$ and $T_{t_n}x \to x$ for every $x \in X$. Therefore, if we consider $(T_{t_n})$, then this discretization is rigid.
Now, assume that a discretization of $(T_t)_{t\geq 0}$ is rigid. So, there exists $s > 0$ such that $(T_{t_n}) = (T_{ns})$ is rigid. Hence, there exists a subsequence $(t_{n_k})$ of $(t_n)$ such that $t_{n_k}$ tends to infinity, and $T_{t_{n_k}} x \to x$ for every $x \in X$. This means, $(T_t)_{t\geq 0}$ is rigid.

□

The consequence corollary is concluded immediately from Theorem 2.3

**Corollary 2.4.** *Assume that $(T_t)_{t\geq 0}$ is a $C_0$-semigroup on a Banach space $X$. If an autonomous discretization of $(T_t)_{t\geq 0}$ is rigid, then $(T_t)_{t\geq 0}$ is a rigid $C_0$-semigroup.*

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# On Periodic Behavior in a Class of Nonlinear Second-Order Differential Equations

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**Abstract**

In this article, we investigate the existence of a periodic solution of a certain second-order nonlinear differential equations. For this purpose, we first obtain an integral equation such as a nonlinear second-order differential equation by constructing the appropriate associated the Green's function, and then using the Banach's fixed point theorem, we show the existence of a periodic solution of the differential equation. In the following, as an application, we present the mathematical model of the car cylinder, piston and valve system as a nonlinear second-order differential equation with periodic coefficients. Finally, we provide a numerical method to approximate the periodic solutions of the obtained differential equation.

## 1. Introduction

Second-order nonlinear differential equations are an essential tool for modeling physical problems that appear in various fields such as biology, chemistry, physics, mechanics, electronics, engineering, economics, atomic energy, and communication theory. Nonlinear differential equations are used for modeling damped and moving oscillators [1], earthquake prediction [9], brain modeling [10], heart modeling [11], and coincidence analysis [5].
Mehri and Hamdani in [8] found periodic solutions for the following two classes of differential equations:

$$x'' + f(t,x) = 0, \qquad x'' + f(t,x,x') = 0.$$

Also, Mehri in [7] showed the existence of the periodic solution of the following equation using the Leray-Schauder fixed point theorem:

$$x'' + \big(k + h(x)\big)x' + f(t,x) = p(t).$$

In the article [2], Esmailzadeh et al. have shown the existence of a periodic solution for the following autonomous differential equation using the Schauder's fixed point theorem:

$$x'' + p(x)x' + q(x) = 0.$$

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In this article, the mathematical model of the car cylinder, piston and valve system is converted into a second-order nonlinear differential equation, which gives a better approximation than the classical linear model. The main goal of this article is to prove the existence of a periodic solution of the obtained equation using the Banach's fixed point theorem. Finally, we provide a method for numerical approximation of the periodic solution of the equation.
Now we consider the following nonlinear equation:

$$ax'' + f(t)x' + g(t,x) = h(t) \tag{1}$$

where $a$ is a non-zero real number and $g, f$ and $h$ are continuous and periodic functions with respect to $t$, with period $\omega$. Also, the derivatives of $x'$ and $x''$ are with respect to $t$. Now we prove the existence of the periodic solution of this equation by using the Banach's fixed point theorem (for contraction mapping) in the following conditions:

$$x(0) = x(\omega), \qquad x'(0) = x'(\omega). \tag{2}$$

A special case of the equation (1), is the oscillation equation of a simple pendulum:

$$x'' + \frac{c}{m}x' + \frac{g}{a}\sin(x) = 0,$$

where $x$ is the deflection angle of a pendulum of length $a$ and $m$ is the mass of its bullet.
A real model of (1), is obtained by examining the mathematical model of the cylinder, piston and valve of the car. As shown in Figure 1( a), the valve of the car is connected to the spring, and the oil always covers the place of the valve assembly. On the other hand, the valve itself is affected by time-varying forces, engine gas pressure, finger mechanisms and camshaft, and the result of these time-varying forces is denoted by $p(t)$. As a result of high performance and heat generation at the location of the valve assembly, the viscosity of the oil changes over time, which we denote by $c(t)$. As mentioned, the valve of the car is connected to the spring, and in practice, the springs are non-linear, that is, the force applied to the spring does not change linearly by changing the position of the spring. The spring can be considered a function of displacement. On the other hand, as a result of a lot of work, fatigue occurs in the springs, which causes them to lose their springiness, so the stiffness of the springs is also a function of time, and in general, this stiffness is a function of time and space, which we show by $k(t,x)$. Now we can consider the mathematical model of valve vibration as shown in Figure 1 (b). In Figure 1 (b), $m$ is the valve mass. Now, according to Newton's second law, the differential equation of motion governing the model in Figure 1 (b) is as follows:

$$mx'' + c(t)x' + k(t,x)x = p(t) \tag{3}$$

Equation (3) is a special case of equation (1) considering $g(x,t) = k(t,x)x$. If $\omega$ is the periodic period of the vibrational motion of the equation (3) with the condition (2), we show that this equation has a solution $\omega$-periodic.

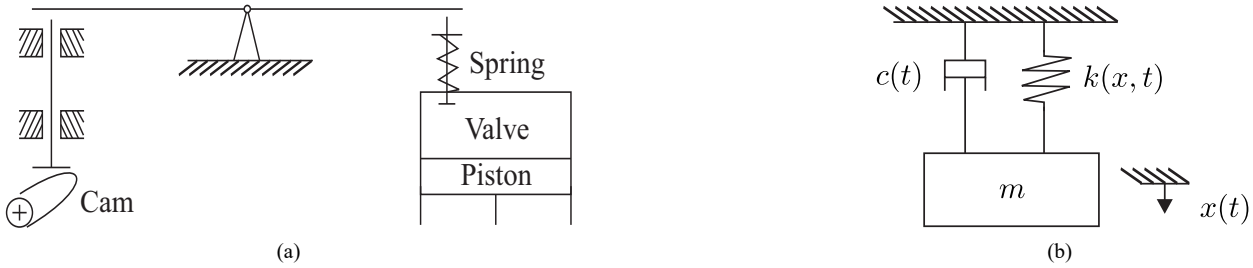


Fig. 1

To prove the existence of the periodic solution of the equation (1) with the condition (2), we must introduce a suitable contraction mapping. Therefore, we need the following definition and theorem.
In the following theorem, a sufficient condition for the existence of a periodic solution of the equation (1) with the condition (2) is given.

**Theorem 1.1.** *Suppose that $f$, $g$ and $h$ are continuous and periodic functions with respect to $t$ with period $\omega$. Nonlinear second order differential equation (1) with condition (2) has at least one $\omega$-periodic solution, if the following two conditions hold:*

(**a**). *For the non-zero coefficient $a$, the following inequality holds:*

$$|a| < 1 - \frac{M\omega}{2}, \tag{4}$$

*where $M = \|f\|_\infty$.*

(**b**). *For $K > 0$ that in the Lipschitz condition*

$$\big|g(t,x(t)) - g(t,y(t))\big| \leq K\|x - y\|_\infty, \qquad (\forall t \in [0,\omega]) \tag{5}$$

*is true, the following inequality holds:*

$$K < \frac{8}{\omega^2}\Big(1 - |a| - \frac{M\omega}{2}\Big). \tag{6}$$

*Sketch of the proof.* The proof is based on the Banach fixed point theorem. First, equation (1) is rewritten in the form

$$ax'' + f(t)x' = h(t) - g(t,x),$$

and the associated linear operator $\mathcal{L}[x] = ax'' + f(t)x'$ is considered together with the periodic boundary conditions (2). By constructing a suitable Green's function $G(t,s)$ corresponding to $\mathcal{L}$, the differential equation is transformed into the equivalent integral equation

$$x(t) = \int_0^\omega G(t,s)\big[h(s) - g(s,x(s))\big]\,ds.$$

Next, using an appropriate auxiliary function, an explicit representation of $G(t,s)$ is obtained, and it is shown that $G$ is bounded under condition (4). This boundedness allows us to define an operator $T$ on the Banach space $(\mathfrak{B}, \|\cdot\|_\infty)$ by

$$T[x](t) = \int_0^\omega G(t,s)\big[h(s) - g(s,x(s))\big]\,ds.$$

Finally, employing the Lipschitz condition (5) on $g$ and the estimate for $G(t,s)$, it is shown that $T$ is a contraction mapping whenever condition (6) holds. Therefore, by the Banach fixed point theorem, $T$ admits a unique fixed point in $\mathfrak{B}$, which corresponds to an $\omega$-periodic solution of equation (1). □

## 2. Numerical results

In Theorem 1.1, we proved the existence of the periodic solution of the equation (1). In this section, we approximate this periodic solution numerically. Now we consider the following more general differential equation:

$$x'' + f(t,x,x') = 0. \tag{7}$$

According to the theorems of differential equations, if the function $f$ is periodic with respect to $t$ with the period $\omega$, the necessary and sufficient condition for the periodic solution of the equation (7) is that the solution satisfies the boundary condition (2). Now suppose $x = x(t,\xi,\eta)$ is the solution of the equation (7) which satisfies the following initial conditions:

$$x(0) = x(0,\xi,\eta) = \xi, \qquad x'(0) = x'(0,\xi,\eta) = \eta. \tag{8}$$

To obtain a solution of the equation (7) that satisfies the boundary conditions (2), it is sufficient to find $\xi, \eta$ such that:

$$x(\omega, \xi, \eta) - \xi = 0, \quad x'(\omega, \xi, \eta) - \eta = 0, \tag{9}$$

for this purpose, we put:

$$\begin{cases} F(\xi, \eta) = x(\omega, \xi, \eta) - \xi, \\ G(\xi, \eta) = x'(\omega, \xi, \eta) - \eta. \end{cases} \tag{10}$$

Now, we solve the following system of nonlinear algebraic equations:

$$\begin{cases} F(\xi, \eta) = 0, \\ G(\xi, \eta) = 0, \end{cases} \tag{11}$$

using Taylor's formula, we have:

$$\begin{cases} F(\xi, \eta) = F(\xi_0, \eta_0) + (\xi - \xi_0)F_\xi(\xi_0, \eta_0) + (\eta - \eta_0)F_\eta(\xi_0, \eta_0) + \cdots = 0 \\ G(\xi, \eta) = G(\xi_0, \eta_0) + (\xi - \xi_0)G_\xi(\xi_0, \eta_0) + (\eta - \eta_0)G_\eta(\xi_0, \eta_0) + \cdots = 0 \end{cases} \tag{12}$$

considering the linear approximation of the functions $F$ and $G$ and also using Cramer's rule, we have:

$$\xi - \xi_0 = \frac{\begin{vmatrix} -F & F_\eta \\ -G & G_\eta \end{vmatrix}_{(\xi_0, \eta_0)}}{\begin{vmatrix} F_\xi & F_\eta \\ G_\xi & G_\eta \end{vmatrix}_{(\xi_0, \eta_0)}}, \qquad \eta - \eta_0 = \frac{\begin{vmatrix} F_\xi & -F \\ G_\xi & -G \end{vmatrix}_{(\xi_0, \eta_0)}}{\begin{vmatrix} F_\xi & F_\eta \\ G_\xi & G_\eta \end{vmatrix}_{(\xi_0, \eta_0)}}.$$

Now we need the first order partial derivatives of $F_\xi$, $F_\eta$, $G_\xi$ and $G_\eta$ calculate. Taking the derivative of equations (10), we have:

$$F_\xi(\xi, \eta) = x_\xi(\omega, \xi, \eta) - 1, \qquad F_\eta(\xi, \eta) = x_\eta(\omega, \xi, \eta)$$

$$G_\xi(\xi, \eta) = x'_\xi(\omega, \xi, \eta), \qquad G_\eta(\xi, \eta) = x'_\eta(\omega, \xi, \eta) - 1$$

By taking the derivative with respect to $\xi$ from the equation (7), we have:

$$x''(t, \xi, \eta) + f\big(t, x(t, \xi, \eta), x'(t, \xi, \eta)\big) = 0, \tag{13}$$

so

$$\left(\frac{\partial x}{\partial \xi}\right)'' + f_{x'}\left(\frac{\partial x}{\partial \xi}\right)' + f_x \frac{\partial x}{\partial \xi} = 0,$$

for $\frac{\partial x}{\partial \xi} = Z$, we have:

$$Z'' + f_{x'}Z' + f_x Z = 0, \tag{14}$$

with initial conditions:

$$Z(0) = 1, \quad Z'(0) = 0, \tag{15}$$

which result from the initial conditions (8). By solving the equation (14) with initial conditions (15), we obtain

$$F_\xi(\xi, \eta) = Z(\omega) - 1, \qquad G_\xi(\xi, \eta) = Z'(\omega).$$

Similarly, deriving the equation (13) with respect to $\eta$ and considering $\frac{\partial x}{\partial \eta} = W$, we have:

$$\begin{aligned} &W'' + f_{x'}W' + f_x W = 0, \\ &W(0) = 0, \quad W'(0) = 1. \end{aligned} \tag{16}$$

By solving them, the following values are obtained:

$$F_\eta(\xi, \eta) = W(\omega), \qquad G_\eta(\xi, \eta) = W'(\omega) - 1.$$

First, we arbitrarily choose $\xi_0$ and $\eta_0$, and then we choose the differential equation (7) with initial conditions $x(0) = \xi_0$ and $x'(0) = \eta_0$. We solve with the Runge-Kutta method until $t = \omega$ and obtain the values of $x(\omega)$ and $x'(\omega)$. If the equalities $x(\omega) = \xi_0$ and $x'(\omega) = \eta_0$ are true, the problem is solved. Otherwise, we solve the differential equation (14) with the initial conditions (15) and the initial value problem (16) and then obtain the values. of $\xi_n$ and $\eta_n$ and then repeat the operation.

**Example 2.1.** According to Theorem 1.1, the following differential equation has a $\frac{\pi}{4}$-periodic solution:

$$0.6x'' + \sin^2(8t)x' + 0.008x^2 = \cos(8t),$$

with a simple calculation, it can be seen that for $a = 0.6$, $M = 1$, $\omega = \frac{\pi}{4}$ and $K = 0.09$, the inequality (4) and (6) holds. According to Table **??**, with a computer calculation the following results are obtained:

$$\begin{aligned} &x(0) = -0.30611 \qquad x'(0) = 0.02736, \\ &x(\frac{\pi}{4}) = -0.30611 \qquad x'(\frac{\pi}{4}) = 0.02736. \end{aligned}$$

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# Teaching mathematics and computational intelligence with interactive methods

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**Abstract**

It is possible to teach mathematics and computational intelligence to students and strengthen their scientific capabilities through the use of interactive methods, educational games and smart platforms. By providing education in this way, they will not only acquire technical skills but also strengthen their creativity, computational thinking and problem-solving skills. These trainings play an important role in the career development of students and the advancement of technology in society. The combination of artificial intelligence education and mathematics education is effective in benefiting students and increasing the quality of their education. This article examines this issue and presents suggested solutions.

## 1. Introduction

Artificial intelligence, as one of the great advances of the modern world, plays a key role in social, economic, and cultural changes, but teaching children the principles and concepts of artificial intelligence is also of great importance. This education not only allows them to become better acquainted with future technologies, but can also strengthen their critical thinking, problem-solving, collaboration, and creativity. Studies have shown the effectiveness of learning artificial intelligence on students in such areas as preparing for the future, strengthening mental skills, encouraging curiosity and exploration, developing the artificial intelligence industry, promoting exports and foreign exchange earnings, growing freelance and artificial intelligence services, developing artificial intelligence applications and tools, and artificial intelligence services in managing real-world issues and problems in society [4]. Therefore, teaching children artificial intelligence and programming not only helps develop their individual abilities, but also has a great positive impact on the labor market, the national economy, and the development of technology. This topic should be considered in the educational and training priorities of societies. For this purpose, in order to teach and apply AI from the ground up, we need to phase AI education for children in the following phases:
Phase 1 - Initial familiarity with the main concepts of AI, Phase 2 - Connection to daily life, Phase 3 - Applied projects, Phase 4 - Development of more complex skills.
In these phases, first, basic AI concepts such as algorithms and pattern recognition should be taught in a simple and game-oriented way, then how AI works in daily life should be taught to students in an example-oriented way, and then

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these concepts should be used to teach advanced AI concepts such as neural networks and deep learning, and then to create smart educational games. It is clear that, given the importance of AI in the future and the need of society for its related capabilities, improving AI education for students can lead to a major transformation in education and strengthen critical thinking skills, problem analysis, creativity, and collaboration [5]. These skills will be useful in all aspects of students' lives in the future. Since the development of these individual abilities has many positive effects on the labor market, the national economy, and technological development, this topic should be considered in students' educational and training priorities.

## 2. Fundamental Challenges in Teaching Artificial Intelligence to Students

Teaching artificial intelligence to children faces challenges that can hinder their progress. Some of these challenges include the level of difficulty of technical concepts, lack of appropriate educational resources, the need for interactive educational solutions, the need for appropriate educational equipment and resources, and the need to train teachers and parents. Teaching artificial intelligence to students can be difficult and challenging. The challenges outlined above require national educational planning and the provision of the necessary infrastructure. Others, such as the difficulty of technical concepts and the provision of interactive solutions, require teaching related concepts and strengthening the necessary educational foundations, including mathematics education. By using simple and attractive methods, providing appropriate resources and creating play-like spaces, these challenges can be turned into an opportunity for students to grow. Teaching mathematics to students can help them in their creative computational thinking and problem solving, and prepare them for the future of technology and providing the necessary grounds for teaching artificial intelligence [1].

## 3. Learning Mathematics as a Prerequisite for Learning Artificial Intelligence

Teaching mathematics as a foundation and essential prerequisite for learning artificial intelligence faces various challenges, and by examining the statistics of the percentage of mathematics in the national entrance exam and other tests, we can discover the weaknesses of students in this subject. In this section, we will introduce these challenges and the important role of artificial intelligence in solving them:

- Understanding mathematical concepts
- Learning tailored to the needs of each student
- Attracting students' attention
- Developing skilled teachers of mathematics

By using artificial intelligence and combining it with mathematics education, students can interact more effectively with mathematics and better prepare them for learning artificial intelligence ahead [2]. This combination helps in effective interaction between students and course materials and provides an effective path towards learning artificial intelligence.

## 4. Mutual Effectiveness of Artificial Intelligence and Mathematics

Given the challenges mentioned regarding mathematics education and artificial intelligence training, the mutual effectiveness of artificial intelligence and mathematics is quite evident. On the one hand, mathematics is effective as a basis for creating the capabilities needed to understand the concepts of artificial intelligence, and on the other hand, artificial intelligence can make learning mathematics attractive and simple for students in stages by creating smart tools. In this regard, designing smart games that teach the concepts of mathematics and artificial intelligence simultaneously and based on the introductory, intermediate and advanced levels, as well as designing creative and new platforms for teaching students, is recommended. These games and platforms should be designed in such a way that, in addition to providing practical and useful training, they can also provide students with the opportunity to carry out practical exercises and projects at different levels of difficulty. Imagine a creative and unique platform to begin the

educational journey of students. This platform does not only teach, but also invites children to experience artificial intelligence with exciting games and leads them to create practical projects and exercises. Using the principles of interactive learning, engaging and diverse education and tools appropriate to the age level of students, this platform provides surprises for them. Also, by using gamification and interactive learning, children become familiar with the concepts of mathematics and artificial intelligence in the form of exciting and engaging games and learn new skills with greater enthusiasm and motivation [3].

On the other hand, by combining interactive storytelling and animation, children can create their own creative stories and use the principles of artificial intelligence and the laws of logic in mathematics to create interesting characters and events in stories [2].

By designing such a creative and engaging platform, children enter the education of mathematics and artificial intelligence in a world full of creativity and excitement in games and develop their abilities more quickly [6].

The STEM (Science, Technology, Engineering, and Mathematics) approach plays a very important role as an evolutionary learning approach in teaching artificial intelligence and programming to students. By combining scientific concepts of engineering technology and mathematics, this approach encourages students to learn critical thinking, problem solving, and creative thinking in the fields of artificial intelligence and programming. STEM learning in the field of artificial intelligence allows students to become familiar with the basic concepts of artificial intelligence and gain a deeper understanding of its functions and applications. By connecting artificial intelligence concepts to mathematical concepts, students can understand various algorithms and models and even design and implement them themselves. Through practical experiences and applied projects, students can independently explore the concepts of artificial intelligence and gain confidence in their abilities in this field.

## 5. Conclusion

Teaching students about artificial intelligence is a unique opportunity to develop computational thinking, creative problem-solving skills, and technological advancement in society. By teaching these skills to students, they will be able to create a smart and technologically proficient generation. This article examines the challenges, solutions, and new ideas for teaching artificial intelligence and mathematics, and examines the mutual role of these two fields on each other. One of the fundamental challenges in teaching artificial intelligence to students is the complexity of technical concepts. These concepts must be taught in a simple and attractive way for children and adolescents and institutionalized in their reasoning and thinking so that they can become familiar with the world of technology and understand the necessary skills. Teaching mathematics correctly and completely is one way to manage this challenge. Holding workshops and using interactive methods and educational games can help students learn artificial intelligence and mathematics simultaneously.

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# A Systematic Sensitivity Analysis of Geometric Asian Power Option Prices under Stochastic Volatility

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**Article Info**

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**Abstract**

This study conducts a systematic sensitivity analysis for pricing geometric Asian power options under a stochastic volatility model. Using an efficient finite difference scheme, we extensively examine the quantitative and qualitative effects of varying key financial and model parameters, including the interest rate, volatility, power parameter, and stochastic volatility parameters, on option prices. Results indicate that the power parameter and volatility-of-volatility exhibit the strongest nonlinear influence on prices, whereas sensitivity to the interest rate remains relatively mild for Asian-style payoffs. The analysis provides deeper financial insight into the robustness of the pricing model and highlights practical implications of parameter misspecification. The numerical method demonstrates stability and accuracy across a wide range of parameter values.

## 1. Introduction

Nonlinear partial differential equations constitute a fundamental modeling framework in financial mathematics, enabling the analysis of complex interactions between model parameters and option price dynamics [3]. In particular, option pricing models incorporating stochastic volatility are known to exhibit pronounced nonlinear sensitivities with respect to key financial and statistical parameters [1, 4]. While existing studies have predominantly focused on enhancing numerical efficiency under fixed parameter configurations [7, 8], a systematic investigation of option price sensitivity to parametric variations remains relatively limited.

In this paper, we conduct a structured nonlinear sensitivity analysis of geometric Asian power option prices within a stochastic volatility framework. By varying key model parameters independently while maintaining a consistent numerical scheme [6], the individual and combined effects of these parameters on the solution of the associated pricing partial differential equation are isolated and quantified. The main contributions of this work are summarized as follows:

1. A systematic one-factor-at-a-time sensitivity analysis is performed for geometric Asian power options under stochastic volatility.
2. The nonlinear responses of option prices are quantitatively assessed with respect to key parameters, including volatility, correlation, power exponent, and time to maturity.
3. The impact of selected parameters on numerical stability and pricing robustness is examined, providing practical insights for reliable and consistent model implementation.

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## 2. Model Formulation

### 2.1. Underlying Asset Dynamics

The underlying asset price is modeled using a Heston-type stochastic volatility framework [1], which is widely employed in quantitative finance due to its ability to capture empirically observed market features such as volatility clustering, mean reversion in variance, and the leverage effect [3]. In this formulation, the asset price and its instantaneous variance evolve jointly as a two-dimensional diffusion process [4], allowing for a realistic representation of the interaction between price dynamics and stochastic volatility while preserving analytical and numerical tractability [1]. This modeling choice provides a flexible foundation for examining the sensitivity of option prices to key financial and statistical parameters governing both the asset dynamics and volatility behavior. Under the risk-neutral probability measure $\mathbb{Q}$, the joint dynamics of the asset price $S_t$ and its instantaneous variance $V_t$ are governed by the following system of stochastic differential equations:

$$dS_t = rS_t\,dt + \sqrt{V_t}\,S_t\,dW_t^{(S)}, \tag{1}$$

$$dV_t = \kappa(\theta - V_t)\,dt + \sigma\sqrt{V_t}\,dW_t^{(V)}, \tag{2}$$

where $r$ denotes the constant risk-free interest rate, $\kappa > 0$ is the speed of mean reversion of the variance process, $\theta > 0$ represents the long-term mean variance, and $\sigma > 0$ is the volatility of volatility. The Brownian motions $W_t^{(S)}$ and $W_t^{(V)}$ satisfy

$$dW_t^{(S)}\,dW_t^{(V)} = \rho\,dt, \qquad -1 < \rho < 1, \tag{3}$$

capturing the correlation between the asset returns and volatility.

### 2.2. Geometric Asian Power Option Payoff

The payoff of a geometric Asian power option depends on the geometric average of the underlying asset price raised to a fixed power [7]. The use of a geometric average smooths the payoff structure by reducing sensitivity to extreme price movements and mitigating the impact of short-term fluctuations [5]. Introducing a power parameter further enhances modeling flexibility, allowing the payoff profile to reflect different risk attitudes and nonlinear exposure to the underlying asset [3]. Such contracts are particularly relevant in commodity and energy markets, where averaging features and nonlinear payoffs naturally arise [8].

Let $G_T$ denote the geometric average of the underlying asset price over the monitoring period $[0,T]$, defined as

$$G_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t\,dt\right).$$

The payoff of a geometric Asian power call option is then given by

$$\max\left(G_T^{n^*} - K,\,0\right),$$

where $n^* > 0$ denotes the power parameter and $K$ is the strike price.

## 3. Pricing PDE and Numerical Framework

Under the risk-neutral measure $\mathbb{Q}$ and standard no-arbitrage arguments, the price $C(S,V,t)$ of a geometric Asian power option satisfies a two-dimensional parabolic partial differential equation of the form

$$\begin{aligned}\frac{\partial C}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma VS\frac{\partial^2 C}{\partial S\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 C}{\partial V^2} \\ + rS\frac{\partial C}{\partial S} + \kappa(\theta - V)\frac{\partial C}{\partial V} - rC = 0,\end{aligned} \tag{4}$$

for $(S,V,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \times [0,T)$, subject to the terminal condition

$$C(S,V,T) = \max\left(G_T^{n^*} - K,\,0\right), \tag{5}$$

where $G_T$ denotes the geometric average of the underlying asset price over $[0, T]$.
The numerical solution of the problem (4)–(5) is obtained using an efficient finite difference scheme adapted from the framework in [8], which remains fixed throughout all experiments to ensure that observed variations in option prices are solely induced by parameter changes.

### *3.1. Parameter Sensitivity Analysis*

#### *3.1.1. Interest Rate*

The risk-free interest rate affects discounting, showing a monotonic impact on option prices [3]. Higher interest rates generally increase call option values, consistent with classical theory [2, 3].

#### *3.1.2. Underlying Asset Volatility*

Instantaneous volatility crucially affects option value [1]. Increasing volatility leads to a nonlinear increase in option prices, with sensitivity significantly amplified for higher power parameters, reflecting the enhanced convexity of the payoff structure [2, 7].

#### *3.1.3. Variance Volatility and Correlation*

Volatility of variance governs randomness of the volatility process [4]. Higher variance volatility increases option value; correlation between asset returns and volatility affects skewness [1, 4].

#### *3.1.4. Power Parameter and Maturity*

The power parameter controls payoff nonlinearity [5]. Larger powers amplify sensitivity to the average underlying price [7]. Longer maturities generally increase option prices due to extended exposure [2].

## 4. Numerical Results

We conduct extensive numerical experiments to illustrate the effects of parameter variations on option pricing [8]. The adopted numerical scheme demonstrates robustness across all tested scenarios [6], with graphical representations clearly revealing both monotonic and nonlinear relationships between parameters and option prices [3].
Model parameters are systematically varied within realistic market ranges to examine their impact on option prices. The specific parameter ranges employed in this study are reported in Table 1.

Table 1. Ranges of model parameters used in the sensitivity analysis

| Parameter | Description | Range |
|---|---|---|
| $r$ | Risk-free interest rate | $[0.01,\ 0.1]$ |
| $\sigma$ | Volatility of volatility | $[0.05,\ 0.5]$ |
| $\rho$ | Correlation coefficient | $[-0.7,\ 0.7]$ |
| $n^*$ | Power parameter | $\{1, 2, 3\}$ |
| $T$ | Time to maturity (years) | $[0.5,\ 2]$ |

To provide a comprehensive illustration of sensitivity behavior, the impact of key model parameters on geometric Asian power option prices is examined through detailed numerical experiments. Specifically, the effects of the risk-free interest rate, volatility-related parameters, correlation, power index, and time to maturity on option values are analyzed. Such sensitivity analysis proves essential for assessing pricing model robustness and for identifying parameters that exert dominant or nonlinear influences on option prices [2–4].
Figure 1 summarizes the numerical sensitivity of the option price with respect to all major parameters considered in this study. The results reveal predominantly monotonic but nonlinear relationships, with particularly strong sensitivity observed for the power parameter and volatility-related quantities. In contrast, interest rate effects remain comparatively moderate, a characteristic attributable to the averaging feature inherent in Asian-style payoffs [7, 8].

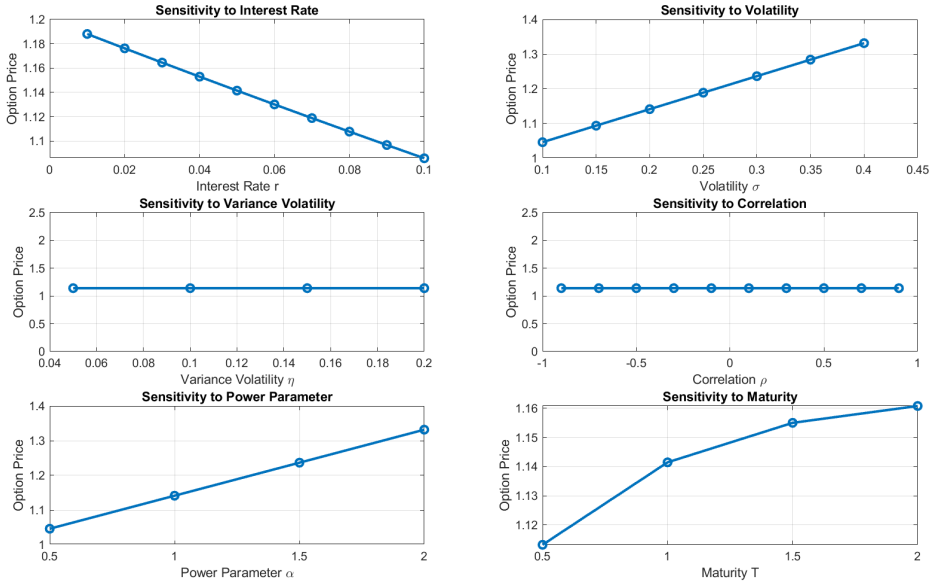


Fig. 1. Numerical sensitivity of geometric Asian power option prices to variations in key financial and volatility parameters under a stochastic volatility framework.

## 5. Conclusion

This study provides a detailed sensitivity analysis of geometric Asian power options under stochastic volatility. The results show strong sensitivity to the power and volatility parameters, with comparatively mild interest rate effects due to the Asian averaging feature. These findings support the robustness of the numerical framework and highlight that neglecting these sensitivities, particularly for long-maturity contracts, can lead to substantial mispricing. Future work may extend this analysis to fractional or rough volatility models.

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# Support Vector Machine Classifier Model for Credit Rating Problem

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**Abstract**

In this paper, the issue of credit rating of real applicants for bank facilities has been examined. To perform the above credit rating, SVM-based hybrid methods have been used. Also, a comparison has been made between four common kernels in the hybrid SVM method. Finally, a comparison has been made between this method and random forest, which is one of the most common hybrid methods.

## 1. Introduction

The first research and model for measuring credit risk on bonds was done by John Murray in 1909 [1]. Various methods such as data envelopment analysis [2] and logit regression [3] have been proposed to design a model for measuring credit risk. The use of data-driven methods has received much attention in recent years. Mehrara et al. [6] compared the two methods of logistic regression and neural networks for credit rating of bank customers, the results of which show 80% accuracy for regression and 87% for neural networks and show superior performance and better prediction of networks. Thomas [8] used the logistic regression model to predict the credit risk of borrowers and achieved an accuracy of nearly 72%. Keshavarz Haddad and Aytigazar [3] compared decision tree and logistic regression methods in the process of credit rating bank applicants for facilities,each of which presented an accuracy of nearly 96% and 82%, respectively.
The neural network method has also been used to classify loan applicants, which presented an accuracy of $69-84\%$ [5]. Tang et al. [7] used the random forest method to assess credit risk by credit cards in the energy industry in China. The purpose of the study is to scientifically measure the credit risk of credit cards used in the energy industry. The results show that credit card features such as credit added ratio and credit card costs over a month have a significant impact on credit risk. This valuable information will help banks to improve their risk management. Also, the SVM method, which is one of the most widely used methods in classification problems, in addition to the credit rating problem, has been implemented on various field issues such as medicine [9, 10] and on the issue of educational status [4].
The motivation of this study is to present the reliability and accuracy of the credit scoring model base on ensemble SVM method.

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## 2. Main Results

In this section, we introduce the ensemble bagging SVM method and then we apply the method on the credit rating problem. Let $X$ is the given two labeled dataset as follows:

$$X = \{(x_i, y_i)\ x_i \in \mathbb{R}^n, \quad y_i \in \pm 1, i = 1, 2 \dots, N\} \tag{1}$$

Where $x_i$, represents the vector of variables related to the i-th loan applicant and labels indicate whether customers are well-accounted for or not, ie, the label 1 is used for good customers and -1 is used for bad ones. The goal of solving this problem is to find a classification model based on the training data that can best distinguish between good and bad customers among them and also, this model should be able to predict the labels of the testing data and, especially the new customers as well as possible. Recently, ensemble classification methods have received much attention from researchers. In the ensemble learning approach , several different classifiers are created on the data set, then the final model is generated based on voting between them.
We first divide the dataset, $X$, into two subsets, say training data and testing data Here we use $80\%$ of the dataset for training, and the remaining $20\%$ of the dataset for testing. The training data then is divided into K subsets $X_k$.
for $k = 1, 2, \dots, K$. Here we use the bootstrap approach to obtain these subsets. Now, we apply the SVM method on each subset $X_k$, for $k = 1, 2, \dots, K$ to get $K$ local classifiers and the final and global classifier can be obtained by voting on these local classifiers. We used the bagging method to construct the ensemble classification model.
Let $TP$ denotes the number of good costumers that are predicted as $+1$; $TN$ refers the number of bad costumers that are predicted as $-1$; $FP$ denotes the number of bad costumers that are predicted as $+1$; $FN$ refers the number of good costumers that are predicted as $-1$. Then the following criteria can be used to evaluate the final model:

$$E_{II} = \frac{FN}{FN + TP}, \quad E_I = \frac{FP}{FP + TN}$$
$$ACC = \frac{TP + TN}{TP + TN + FP + FN}$$
$$MCC = \frac{TN \times TP - FN \times FP}{\sqrt{(FP + TP)(FN + TP)(FP + TN)(FN + TN)}}$$

## 3. Numerical Results

We apply the SVM and ensemble SVM methods to two benchmarking datasets includ- ing the India and Indonesia credit datasets, from the UCI Machine Learning Repository, published by the University of California Irvine.
The India dataset has 795 data points composed of 380 good costumers and 415 bad costumers. The data points of this dataset have 13 features including 5 continuous and 8 categorical.
The Indonesia credit dataset has 950 data points composed of 250 bad costumers and 700 good costumers. Each data point has 20 features, which include 3 continuous and 17 discrete features: age, household, credit history, account balances, job, employment status, savings account, status of existing checking account, housing, income, other debtors, present residence, property, mumber of existing credits, other installment plans, telephone, foreign worker, credit amount, duration, and purpose of the loan.
The models of India and Indonesia credit datasets are built using the individual and ensemble methods, and in Table 1, the results in terms of ACC, MCC, EI and EII are reported, which are expressed as a percentage. In the proposed model, the improvement of ensemble SVM evaluation criteria is observed compared to individual SVM.

Table 1. Results of credit ranking problem using the SVM and ensemble SVM together with the desicion tree and random forest methods.

Indonesia Datase

| | SVM | | | | ensemble SVM | | | | DT | RF |
|---|---|---|---|---|---|---|---|---|---|---|
| | Rbf | Lin. | Poly. | Sig. | Rbf | Lin. | Poly. | Sig. | DT | RF |
| $ACC$ | 78.50 | 77.50 | 79.00 | 70.50 | 78.50 | 79.50 | 83.50 | 71.50 | 77.50 | 73.50 |
| $MCC$ | 38.31 | 47.21 | 45.01 | 49.99 | 47.20 | 46.04 | 53.03 | 58.14 | 40.75 | 24.34 |
| $E_I$ | 28.99 | 748 | 25.31 | 79.52 | 25.86 | 26.31 | 24.28 | 65.75 | 29.22 | 34.62 |
| $E_{II}$ | 34.44 | 30.24 | 29.21 | 52.00 | 29.07 | 28.55 | 25.93 | 55.00 | 34.55 | 42.60 |

India Dataset

| | SVM | | | | ensemble SVM | | | | DT | RF |
|---|---|---|---|---|---|---|---|---|---|---|
| | Rbf | Lin. | Poly. | Sig. | Rbf | Lin. | Poly. | Sig. | DT | RF |
| $ACC$ | 74.05 | 86.36 | 83.05 | 83.05 | 85.41 | 87.63 | 86.45 | 83.05 | 87.99 | 88.41 |
| $MCC$ | 67.09 | 73.24 | 66.78 | 68.99 | 68.01 | 73.67 | 74.63 | 66.90 | 70.88 | 72.99 |
| $E_I$ | 16.21 | 15.01 | 18.21 | 18.02 | 17.05 | 15.96 | 14.12 | 17.55 | 14.50 | 13.01 |
| $E_{II}$ | 17.71 | 14.76 | 16.88 | 14.98 | 17.94 | 15.36 | 15.77 | 16.01 | 15.81 | 15.55 |

For example criteria EII, which indicates the misclassification rate of bad customers, in the Bagging SVM with polynomial kernel has lowest value and is fewer than the values compared to other methods summarized in table 1. A comparison is made between the proposed method and decision tree (DT) and random forest (RF) methods in table 1. It should be mentioned that RF algorithm is an ensemble approach based on the DT method. The results show that for Indonesia dataset, the best values of the defined criteria are achieved by the proposed ensemble method.
It is worth noting that the customer credit factors in the India and Indonesia datasets have undergone significant changes due to rapid and substantial shifts in customer behavior. Consequently, there is an urgent and critical need for research based on current real-world datasets.

## 4. Conclusion

In this study, we introduce an ensemble method using SVM to develop scoring models. We apply the proposed method on two different credit datasets to demonstrate the effectiveness of the ensemble method. The comparison between the single SVM and the proposed ensemble SVM method shows that the evaluation criteria for the latter are improved.

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# A Horizontal Membership Function Approach to Linear Fuzzy Fredholm Integral Equations of the second type

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**Abstract**

In this paper, our goal is to find an exact solution for the Linear Fuzzy Fredholm Integral Equations (LFFIEs) of the second type $\tilde{y}(x) = \tilde{f}(x) \oplus_{gr} \oint_a^b k(x,t) \odot_{gr} \tilde{y}(t)\, dt$ under granular operations. Furthermore, we exhibit a theorem that establishes the connection between the solutions of the homogeneous and non-homogeneous forms of the LFFIEs of the second type. In the examples mentioned in this paper, we use the Direct Computation Method (DCM) tools to find the exact solution that its kernel is in the form of $k(x,t) = h(t)g(x)$.

## 1. Introduction

Integral equations have long been a central topic in applied mathematics, with wide-ranging applications in physics, engineering, economics, and biology. In particular, Fredholm integral equations of the second type are fundamental in modeling problems where the unknown function appears both inside and outside the integral operator. Classical approaches to solving such equations are well-established in the literature, as detailed in works such as Kress [4] and Wazwaz [5].

In many real world problems, however, parameters and initial conditions are not precisely known and are often best described using fuzzy sets that is a concept introduced by Zadeh [6] to capture uncertainty and imprecision. This has motivated the study of fuzzy integral equations, where the unknown function and data are represented by fuzzy numbers. Several methods have been proposed to solve fuzzy integral equations, including the fuzzy differential transform

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method applied to Volterra equations by Abbasbandy and Allahviranloo [1].

In this paper, we address the question of solving LFFIEs of the second type under the granular operations framework. Specifically, we consider equations of the form:

$$\tilde{y}(x) = \tilde{f}(x) \oplus_{gr} \oint_a^b k(x,t) \odot_{gr} \tilde{y}(t)\, dt$$

where $\tilde{f}$ is a fuzzy-valued function and $k(x,t)$ is a crisp kernel. Our main contributions are twofold: first, we establish a theorem connecting the solutions of the homogeneous and non-homogeneous forms of the LFFIEs of the second type; second, we employ the DCM in the Horizontal Membership Function (HMF)[1] setting to derive exact solutions for several illustrative examples.

The rest of the paper is organized as follows: Section 2 reviews essential preliminaries on fuzzy numbers, HMF, granular arithmetic, and the granular metric. Section 3 presents the main results, including the solution procedure and the interesting Theorem 3.1. Finally, section 4 provides worked examples demonstrating the applicability and effectiveness of the proposed method.

## 2. Preliminaries

**Definition 2.1.** A type-1 fuzzy set, denoted by $\tilde{y}$ and mapping $\mathbb{R}$ to $[0,1]$, qualifies as a type-1 fuzzy number if it satisfies the following properties: it is normal, fuzzy convex, upper semi-continuous, and has a compact support.

Here, we denote by $\mathbb{R}_F$ the space of type 1-fuzzy numbers.

**Definition 2.2.** For a membership level $\mu$ where $0 < \mu \leq 1$, the $\mu$-level set of a fuzzy number $\tilde{y}$ is defined as $[\tilde{y}]^\mu = \{x \in \mathbb{R}; \tilde{y}(x) \geq \mu\}$. The 0-level set, or support, is $[\tilde{y}]^0 = \{x \in \mathbb{R}; \tilde{y}(x) > 0\}$. The 1-level set is termed the core. Each $\mu$-level set is a closed interval expressed as $[\tilde{y}]^\mu = [\underline{\tilde{y}}^\mu, \overline{\tilde{y}}^\mu]$.

**Definition 2.3.** Consider a fuzzy number $\tilde{y} : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_F$. Its HMF is a mapping $\tilde{y}_{gr} : [0,1] \times [0,1] \to [a,b]$, defined for a point $x$ by $\tilde{y}_{gr}(\mu, \alpha_y) = x$. Here, $\mu \in [0,1]$ is the membership grade and $\alpha_y \in [0,1]$ is a Relative Distance Measure (RDM) variable. This function is explicitly parameterized using the $\mu$-cut $[\underline{\tilde{y}}^\mu, \overline{\tilde{y}}^\mu]$ as:

$$\tilde{y}_{gr}(\mu, \alpha_y) = \underline{\tilde{y}}^\mu + (\overline{\tilde{y}}^\mu - \underline{\tilde{y}}^\mu)\alpha_y.$$

(refer to [2, 3])

**Example 2.4.** Let $\tilde{y} = (a,b,c)$ be a triangular fuzzy number. $[\tilde{y}]^\mu = [a + (b-a)\mu, c - (c-b)\mu]$ is the $\mu$-level set of $\tilde{y}$. Its HMF is then given by $\mathcal{H}(\tilde{y}) = \tilde{y}_{gr}(\mu, \alpha) = a + (b-a)\mu + (c-a)(1-\mu)\alpha$, where $\alpha, \mu \in [0,1]$.

**Remark 2.5.** The symbol $\mathcal{H}(\tilde{y}(x)) \equiv \tilde{y}_{gr}(x, \mu, \alpha)$ denotes the HMF of a fuzzy-valued function $\tilde{y}(x)$. The inverse operation, which recovers the $\mu$-level sets from the HMF, is given by:

$$\mathcal{H}^{-1}(\tilde{y}_{gr}(\mu, \alpha_y)) = [\tilde{y}]^\mu = \left[ \inf_{\gamma \geq \mu} \min_{\alpha_y} \tilde{y}_{gr}(\gamma, \alpha_y),\ \sup_{\gamma \geq \mu} \max_{\alpha_y} \tilde{y}_{gr}(\gamma, \alpha_y) \right].$$

(see [2])

[1] Horizontal MemberShip Function is actually equivalent to granular computing, where in the granular approach, since our perpective shifts from the vertical membership function ($\mu$-level set) to the horizontal membership function, we use this name for it as well.

**Definition 2.6.** Let $\tilde{y}$ and $\tilde{z}$ be two fuzzy numbers. The granular metric $D_{gr}$ on the space of fuzzy numbers $\mathbb{R}_F$ is defined as the function $D_{gr} : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$, given by:

$$D_{gr}(\tilde{y}, \tilde{z}) = \sup_{\mu\in[0,1]} \max_{\alpha_y,\alpha_z\in[0,1]} |\tilde{y}_{gr}(\mu, \alpha_y) - \tilde{z}_{gr}(\mu, \alpha_z)|.$$

(refer to source [2])

**Definition 2.7.** Let $\tilde{x}$ and $\tilde{y}$ be type-1 fuzzy numbers with HMF $\tilde{x}_{gr}(\mu, \alpha_x)$ and $\tilde{y}_{gr}(\mu, \alpha_y)$, respectively. For any fundamental arithmetic operation $\odot$ (addition, subtraction, multiplication, or division), the result $\tilde{z} = \tilde{x} \odot \tilde{y}$ is a type-1 fuzzy number whose HMF is defined by the pointwise operation:

$$\tilde{z}_{gr}(\mu, \alpha) = \tilde{x}_{gr}(\mu, \alpha_x) \odot \tilde{y}_{gr}(\mu, \alpha_y).$$

A necessary condition for division is that zero is not contained within the support of the divisor $\tilde{y}$.(see [2])

**Definition 2.8.** The space comprising fuzzy functions $\tilde{f} : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_F$ that are continuous on the interval $[a, b]$ with respect to the granular metric $D_{gr}$ is denoted by $C_{gr}([a, b])$.

**Definition 2.9.** Suppose $\tilde{f} : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_F$ is a continuous fuzzy function whose HMF is integrable on $[a, b]$. Let $\oint_a^b \tilde{f}(x)\, dx$ denote the integral of $\tilde{f}$ on $[a, b]$. Then, $\tilde{f}$ is said to be granular integrable on $[a, b]$ if $gr(\oint_a^b \tilde{f}(x)\, dx) = \int_a^b gr(\tilde{f}(x))\, dx$, where "$\int$" and "$\oint$" is the classical integral and the fuzzy integral, respectively.(refer to source [2])

**Definition 2.10.** Fredholm integral equation of second kind

$$u(x) = f(x) + \int_a^b k(x, t)\, g(u(t))\, dt$$

is **Linear** if $g(u(t)) = u(t)$, and also if $f(x)$ is a fuzzy number, it called **Fuzzy** integral equation.

## 3. Main results

Consider the following LFFIE of the second type

$$\tilde{y}(x) = \tilde{f}(x) \oplus_{gr} \oint_a^b k(x, t) \odot_{gr} \tilde{y}(t)\, dt, \tag{1}$$

where $\tilde{f} \in C_{gr}([a, b])$ and $k(x, t) \in C([a, b] \times [a, b])$. The procedure for solving the LFFIE (1) of the second type is as follows: First, we take the granular of both sides of the fuzzy integral equation or compute its HMF according to the definition 2.3. Then, fuzzy integral equation (1) is transformed into a crisp integral equation, and here, we find its exact solution using the DCM. Finally, using Remark 2.5, we obtain the $\mu$-level set of the solution. In the next section, we provide Examples 4.1 and 4.2 for a better understanding of granular approach.

We state the following theorem that expresses the relationship between the solution of the homogeneous and non-homogeneous LFFIEs of the second type.

**Theorem 3.1.** *Let $\tilde{f} : [a, b] \to \mathbb{R}_F$, $k(x, t) : [a, b] \times [a, b] \to \mathbb{R}$ and suppose $\tilde{f} \in C_{gr}([a, b])$, $k \in C([a, b] \times [a, b])$. Only the solution $\tilde{0}$ satisfies the homogeneous LFFIE of the second type*

$$\tilde{y}(x) = \oint_a^b k(x, t) \odot_{gr} \tilde{y}(t)\, dt,$$

*if and only if the non-homogeneous LFFIE of the second type*

$$\tilde{y}(x) = \tilde{f}(x) \oplus_{gr} \oint_a^b k(x, t) \odot_{gr} \tilde{y}(t)\, dt,$$

*has a unique solution.*

## 4. Examples

In this section, we provide two examples to clarify the solution procedure.

**Example 4.1.** We consider the following LFFIE of the second type

$$\tilde{y}(x) = [1+\mu, 3-\mu] \odot_{gr} x \oplus_{gr} \oint_1^2 \Big(x \odot_{gr} \tilde{y}(t)\Big)dt. \tag{2}$$

First of all, for each fixed $(\mu, \alpha)$, we transform the fuzzy integral equation (2) into a crisp integral equation using the HMF, then we obtain the following expression

$$\tilde{y}_{gr}(x) = \Big(1+\mu+2(1-\mu)\alpha\Big)x + \int_1^2 \Big(x\, \tilde{y}_{gr}(t)\Big)\, dt. \tag{3}$$

Here, we use the DCM to find the exact solution. We put

$$\tilde{\gamma}_{gr} = \int_1^2 \tilde{y}_{gr}(t)\, dt$$

and substituting into relation (3) yields the following relation

$$\tilde{y}_{gr}(x) = \Big(1+\mu+2(1-\mu)\alpha\Big)x + x\tilde{\gamma}_{gr}.$$

The granular solution is then given as follows

$$\tilde{y}_{gr} = -2\Big(1+\mu+2(1-\mu)\alpha\Big)x$$

and finally, by using Remark 2.5, we obtain the fuzzy solution

$$[\tilde{y}(x)]^{\mu} = [-6+2\mu, -2-2\mu] \odot_{gr} x.$$

of the LFFIE (2).

For Example 4.2, we adopt a procedure analogous to that one used in Example 4.1.

**Example 4.2.** We consider the following LFFIE of the second type

$$\tilde{y}(x) = [4+2\mu, 7-\mu] \odot_{gr} x^2 \oplus_{gr} \int_0^1 \Big(x^2\, t \odot_{gr} \tilde{y}(t)\Big)\, dt. \tag{4}$$

First of all, for each fixed $(\mu, \alpha)$, we transform the fuzzy integral equation (4) into a crisp integral equation using the HMF, then we obtain

$$\tilde{y}_{gr}(x) = [4+2\mu+3(1-\mu)\alpha]x^2 + \int_0^1 \Big(x^2\, t\tilde{y}_{gr}(t)\Big)\, dt.$$

Here, we use the DCM to find the exact solution. We put

$$\tilde{\beta}_{gr} = \int_0^1 t\tilde{y}_{gr}(t)\, dt$$

and then, we have

$$\tilde{y}_{gr}(x) = [4+2\mu+3(1-\mu)\alpha]x^2 + x^2\tilde{\beta}_{gr}.$$

Finally, we get

$$\tilde{y}_{gr}(x) = \frac{4}{3}[4+2\mu+3(1-\mu)\alpha]x^2$$

and with the aid of Remark 2.5, we obtain

$$[\tilde{y}(x)]^{\mu} = [\frac{16}{3} + \frac{8}{3}\mu, \frac{28}{3} - \frac{4}{3}\mu] \odot_{gr} x^2.$$

## 5. Conclusion

In this paper, the granular approach using HMF was applied to solve LFFIEs of the second type. Here the key theoretical contribution is Theorem 3.1, which establishes the relationship between homogeneous and non-homogeneous solutions and provides a uniqueness and exact solutions were obtained through the DCM. The effectiveness of our approach was shown through several examples with a separable kernel.

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# Fuzzy Linear Systems under Horizontal Membership Function

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**Abstract**

This paper presents a novel granular approach for solving Fuzzy Linear Systems (FLSs) of the form $A\tilde{x} = \tilde{b}$. We employ the Horizontal Membership Function (HMF) representation, which parameterizes fuzzy numbers using membership grade $\mu$ and a relative-distance-measure (RDM) variable $\alpha$. Using this representation, the original fuzzy system is transformed into an equivalent crisp linear system, which can be solved using standard linear algebra techniques. The fuzzy solution is then reconstructed via the inverse HMF mapping. This method provides a direct and computationally efficient alternative to traditional interval-based approaches. Examples involving triangular fuzzy numbers illustrate the procedure and demonstrate its practical application.

## 1. Introduction

Fuzzy Linear Systems (FLS) of the form $A\tilde{x} = \tilde{b}$, where $A$ is a crisp coefficient matrix and $\tilde{b}$ is a fuzzy number vector, play a crucial role in modeling real world problems involving uncertainty. Since the pioneering work of Zadeh [6] on fuzzy set theory, such systems have found applications across diverse fields including engineering, economics, and decision sciences where parameters cannot be precisely measured.

The systematic study of FLS was initiated by Friedman, Ming, and Kandel [2, 3], who proposed embedding methods to transform fuzzy systems into crisp systems for solution. Their approach laid the foundation for subsequent numerical techniques, including LU decomposition methods [1], iterative approaches, and linear programming formulations. A common challenge in these traditional methods is the dependency problem inherent in interval arithmetic, which can lead to solution overestimation and increased computational complexity when dealing with $\mu$-level operations.

To address these limitations, this paper employs the HMF approach, introduced by Piegat and Landowski [5] and further developed by Mazandarani, Pariz, and Kamyad [4]. The HMF provides a granular representation of fuzzy numbers through two parameters: the membership grade $\mu$ and a RDM variable $\alpha$. This representation enables fuzzy

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arithmetic to be performed pointwise, transforming fuzzy operations into exact crisp operations.

The main contribution of this work is to apply the HMF methodology to solve FLS. Our approach consists of three key steps:

1. Represent all fuzzy numbers in the system via their HMFs.
2. Transform the FLS into a parameterized crisp linear system in terms of $(\mu, \alpha)$.
3. Solve the crisp system using classical linear algebra, then reconstruct the fuzzy solution via the inverse HMF transformation.

This granular approach eliminates the need for repeated interval computations at discrete $\mu$-levels, offering computational efficiency and a continuous solution representation. The method is particularly advantageous for triangular and trapezoidal fuzzy numbers, where the HMF takes a simple linear form.

The rest of this paper is organized as follows. Section 2 reviews essential concepts including the HMF, granular arithmetic, and the inverse HMF transformation. Section 3 presents the main solution procedure for FLS under the granular framework. Section 4 provides two illustrative examples with triangular fuzzy numbers to demonstrate the practical implementation. Finally, conclusions are given in Section 5.

## 2. Preliminaries

**Definition 2.1.** A type-1 fuzzy set, denoted by $\tilde{y}$ and mapping $\mathbb{R}$ to $[0,1]$, qualifies as a type-1 fuzzy number if it satisfies the following properties: it is normal, fuzzy convex, upper semi-continuous, and has a compact support.

Here, we denote by $\mathbb{R}_F$ the space of type 1-fuzzy numbers.

**Definition 2.2.** For a membership level $\mu$ where $0 < \mu \leq 1$, the $\mu$-level set of a fuzzy number $\tilde{y}$ is defined as $[\tilde{y}]^{\mu} = \{x \in \mathbb{R}; \tilde{y}(x) \geq \mu\}$. The 0-level set, or support, is $[\tilde{y}]^{0} = \{x \in \mathbb{R}; \tilde{y}(x) > 0\}$. The 1-level set is termed the core. Each $\mu$-level set is a closed interval expressed as $[\tilde{y}]^{\mu} = [\underline{\tilde{y}}^{\mu}, \overline{\tilde{y}}^{\mu}]$.

**Definition 2.3.** Consider a fuzzy number $\tilde{y} : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_F$. Its HMF is a mapping $\tilde{y}_{gr} : [0,1] \times [0,1] \to [a,b]$, defined for a point $x$ by $\tilde{y}_{gr}(\mu, \alpha_y) = x$. Here, $\mu \in [0,1]$ is the membership grade and $\alpha_y \in [0,1]$ is a Relative-Distance-Measure (RDM) variable. This function is explicitly parameterized using the $\mu$-cut $[\underline{\tilde{y}}^{\mu}, \overline{\tilde{y}}^{\mu}]$ as:

$$\tilde{y}_{gr}(\mu, \alpha_y) = \underline{\tilde{y}}^{\mu} + (\overline{\tilde{y}}^{\mu} - \underline{\tilde{y}}^{\mu})\alpha_y.$$

(refer to [4, 5])

**Example 2.4.** Let $\tilde{y} = (a,b,c)$ be a triangular fuzzy number. $[\tilde{y}]^{\mu} = [a + (b-a)\mu, c - (c-b)\mu]$ is the $\mu$-level set of $\tilde{y}$. Its HMF is then given by $\mathcal{H}(\tilde{y}) = \tilde{y}_{gr}(\mu, \alpha) = a + (b-a)\mu + (c-a)(1-\mu)\alpha$, where $\alpha, \mu \in [0,1]$.

**Remark 2.5.** The symbol $\mathcal{H}(\tilde{y}(x)) \equiv \tilde{y}_{gr}(x, \mu, \alpha)$ denotes the HMF of a fuzzy-valued function $\tilde{y}(x)$. The inverse operation, which recovers the $\mu$-level sets from the HMF, is given by:

$$\mathcal{H}^{-1}(\tilde{y}_{gr}(\mu, \alpha_y)) = [\tilde{y}]^{\mu} = \left[ \inf_{\gamma \geq \mu} \min_{\alpha_y} \tilde{y}_{gr}(\gamma, \alpha_y),\ \sup_{\gamma \geq \mu} \max_{\alpha_y} \tilde{y}_{gr}(\gamma, \alpha_y) \right].$$

(see [4])

**Definition 2.6.** Let $\tilde{y}$ and $\tilde{z}$ be two fuzzy numbers. The granular metric $D_{gr}$ on the space of fuzzy numbers $\mathbb{R}_F$ is defined as the function $D_{gr} : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$, given by:

$$D_{gr}(\tilde{y}, \tilde{z}) = \sup_{\mu \in [0,1]} \max_{\alpha_y, \alpha_z \in [0,1]} |\tilde{y}_{gr}(\mu, \alpha_y) - \tilde{z}_{gr}(\mu, \alpha_z)|.$$

(refer to source [4])

**Definition 2.7.** Let $\tilde{x}$ and $\tilde{y}$ be type-1 fuzzy numbers with HMF $\tilde{x}_{gr}(\mu, \alpha_x)$ and $\tilde{y}_{gr}(\mu, \alpha_y)$, respectively. For any fundamental arithmetic operation $\odot$ (addition, subtraction, multiplication, or division), the result $\tilde{z} = \tilde{x} \odot \tilde{y}$ is a type-1 fuzzy number whose HMF is defined by the pointwise operation:

$$\tilde{z}_{gr}(\mu, \alpha) = \tilde{x}_{gr}(\mu, \alpha_x) \odot \tilde{y}_{gr}(\mu, \alpha_y).$$

A necessary condition for division is that zero is not contained within the support of the divisor $\tilde{y}$.(see [4])

## 3. Main results

Consider the following FLS

$$A\tilde{x} = \tilde{b} \tag{1}$$

which is equivalent to

$$\begin{cases} a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 + \ldots + a_{1n}\tilde{x}_n = \tilde{b}_1 \\ a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 + \ldots + a_{2n}\tilde{x}_n = \tilde{b}_2 \\ \vdots \\ a_{n1}\tilde{x}_1 + a_{n2}\tilde{x}_2 + \ldots + a_{nn}\tilde{x}_n = \tilde{b}_n \end{cases} \tag{2}$$

where $\tilde{b}$ is a fuzzy number. The procedure for solving the FLS (2) is as follows: First, we take the granular of both sides of the (1) or compute its HMF according to the definition 2.3. This transforms the FLS (2) into a crisp linear system, which can be solved using standard crisp methods from classical linear algebra. Finally, using Remark 2.5, we obtain the $\mu$-level sets of the solution. In the next section, Examples 4.1 and 4.2 illustrate the granular approach for better understanding.

## 4. Examples

In this section, we provide two examples to clarify the solution procedure.

**Example 4.1.** In this example, we solve the following FLS

$$\begin{cases} 2 \odot_{gr} \tilde{x}_1 \oplus_{gr} \tilde{x}_2 = [1 + \mu, 3 - \mu], \\ \tilde{x}_1 \ominus_{gr} 3 \odot_{gr} \tilde{x}_2 = [-1 + \mu, 1 - \mu]. \end{cases} \tag{3}$$

First, for each fixed $\mu, \alpha, \beta \in [0, 1]$, we transform the FLS(3) into a crisp linear system using the HMF, obtaining

$$\begin{cases} 2(\tilde{x}_1)_{gr} + (\tilde{x}_2)_{gr} = 1 + \mu + 2(1 - \mu)\alpha, \\ (\tilde{x}_1)_{gr} - 3(\tilde{x}_2)_{gr} = -1 + \mu + 2(1 - \mu)\beta. \end{cases} \tag{4}$$

This is equivalent to

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} (\tilde{x}_1)_{gr} \\ (\tilde{x}_2)_{gr} \end{bmatrix} = \begin{bmatrix} 1 + \mu + 2(1 - \mu)\alpha \\ -1 + \mu + 2(1 - \mu)\beta \end{bmatrix}. \tag{5}$$

Using Cramer's Rule, we solve (5) to find the granular solution

$$(\tilde{x}_1)_{gr} = \frac{2 + 4\mu + 6(1 - \mu)\alpha + 2(1 - \mu)\beta}{7}, \quad (\tilde{x}_2)_{gr} = \frac{3 - \mu + 2(1 - \mu)\alpha - 4(1 - \mu)\beta}{7},$$

and finally, applying Remark 2.5, we derive the $\mu$-leve sets of the fuzzy solution

$$[\tilde{x}_1]^{\mu} = \left[\frac{2 + 4\mu}{7}, \frac{10 - 4\mu}{7}\right], \quad [\tilde{x}_2]^{\mu} = \left[\frac{-1 + 3\mu}{7}, \frac{5 - 3\mu}{7}\right].$$

For Example 4.2, we adopt a procedure analogous to that one used in Example 4.1.

**Example 4.2.** We now consider the following FLS

$$\begin{cases} \tilde{x}_1 \oplus_{gr} \tilde{x}_2 = [\mu, 2-\mu], \\ \tilde{x}_2 \ominus_{gr} \tilde{x}_3 = [1+2\mu, 4-\mu], \\ \tilde{x}_2 \ominus_{gr} \tilde{x}_3 = [3\mu, 4-\mu]. \end{cases} \tag{6}$$

For each fixed $\mu, \alpha, \beta, \gamma \in [0,1]$, we transform the FLS (6) into a crisp linear system using the HMF

$$\begin{cases} (\tilde{x}_1)_{gr} + (\tilde{x}_2)_{gr} = \mu + 2(1-\mu)\alpha, \\ (\tilde{x}_2)_{gr} - (\tilde{x}_3)_{gr} = 1 + 2\mu + 3(1-\mu)\beta, \\ (\tilde{x}_1)_{gr} - (\tilde{x}_3)_{gr} = 3\mu + 4(1-\mu)\gamma. \end{cases}$$

This is equivalent to

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} (\tilde{x}_1)_{gr} \\ (\tilde{x}_2)_{gr} \\ (\tilde{x}_3)_{gr} \end{bmatrix} = \begin{bmatrix} \mu + 2(1-\mu)\alpha \\ 1 + 2\mu + 3(1-\mu)\beta \\ 3\mu + 4(1-\mu)\gamma \end{bmatrix}. \tag{7}$$

Solving (7) yields the granular solution

$$\begin{cases} (\tilde{x}_1)_{gr} = \dfrac{-1 + 2\mu + 4(1-\mu)\gamma + 2(1-\mu)\alpha - 3(1-\mu)\beta}{2}, \\ (\tilde{x}_2)_{gr} = \dfrac{1 + 2(1-\mu)\alpha - 4(1-\mu)\gamma + 3(1-\mu)\beta}{2}, \\ (\tilde{x}_3)_{gr} = \dfrac{-1 - 4\mu - 4(1-\mu)\gamma + 2(1-\mu)\alpha - 3(1-\mu)\beta}{2}, \end{cases}$$

and with the aid of Remark 2.5, we obtain $\mu$-level sets

$$[\tilde{x}_1]^\mu = \left[\frac{-4+5\mu}{2}, \frac{5-4\mu}{2}\right], \quad [\tilde{x}_2]^\mu = \left[\frac{-3+4\mu}{2}, \frac{6-5\mu}{2}\right], \quad [\tilde{x}_3]^\mu = \left[\frac{-8+3\mu}{2}, \frac{1-6\mu}{2}\right].$$

## 5. Conclusion

This paper has introduced a granular approach for solving FLSs using the HMF. By representing fuzzy numbers through the parameterization $(\mu, \alpha)$, the fuzzy system $A\tilde{x} = \tilde{b}$ is transformed into an equivalent crisp linear system that can be solved directly using classical methods. The fuzzy solution is then reconstructed via the inverse HMF mapping (Remark 2.5). The advantages of this method include its computational efficiency (by avoiding repeated interval operations at discrete $\mu$-levels) and its ability to provide a unified, continuous representation of the solution.

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# A method for generating fuzzy rules from numerical data

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**Abstract**

In modeling and controlling processes with dynamic and nonlinear characteristics, the Wang-Mendel algorithm, which extracts fuzzy rules from raw data without prior knowledge, is very useful. This algorithm is a general method for generating fuzzy rules from numerical data and consists of five steps: (1) It divides the input and output spaces of the given numerical data into fuzzy regions. (2) It generates fuzzy rules from the given data. (3) It assigns a degree to each of the generated rules in order to resolve the conflict between the generated rules. (4) It generates a hybrid fuzzy rule base based on the generated rules and the linguistic rules of human experts. (5) It applies a mapping from the input space to the output space based on the hybrid fuzzy rules using a defuzzification method. It has been proven that such a mapping is capable of approximating any continuous function in a compact set with arbitrary accuracy.

## 1. Introduction

Information related to the design, evaluation, control, and processing of a variety of real-world problems can be classified into two types: numerical information obtained from sensor measurements and linguistic information obtained from human experts. Many intelligent control approaches are exploratory in nature.

In this paper, a general method for combining numerical and linguistic information in a common framework - a fuzzy rule base based on reference [4, 5] is introduced and described. Suppose there is a complex control system in which a human controller forms an essential part. The environment facing this human controller is so complex that no mathematical model exists for it or the mathematical model is so highly nonlinear that there is no suitable design method for it. The goal is to design a control system to replace the human controller. To design such a control system, we first need to see what information is available. It is assumed that there is no a priori mathematical model, that is, we consider a model-free design problem. Since there is already a human controller that successfully controls the system, two types of information are available to us (1) the experience of the human controller and (2) the sampled input-output pairs that are recorded from successful control by the human controller. The experience of the human controller is usually expressed in the form of some if-then linguistic rules that state which action should be taken in which situations. Sampled input-output pairs are numerical data that provide specific values of the corresponding successful inputs and outputs [1–3].

We know that fuzzy control is an effective approach for using linguistic rules, while neural control is suitable for using numerical data pairs.

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## 2. Generating Fuzzy Rules from Numerical Data

Suppose we have a system that has two input variables named $x_1$ and $x_2$, and an output variable named $y$. Suppose we have a sequence of these input and output data as follows and we want to consider this data as input to the algorithm.

$$(x_1^{(1)}, x_2^{(1)}, y^{(1)}), (x_1^{(2)}, x_2^{(2)}, y^{(2)}), \dots \tag{1}$$

The simple case of two inputs, one output is chosen to emphasize and clarify the basic idea of this approach. After generating a set of fuzzy rules from the desired input-output pairs, we use these fuzzy rules to determine the mapping as follows:

$$f : (x_1, x_2) \to y \tag{2}$$

Next, we will examine the steps of generating fuzzy rules.

### *2.1. Dividing the Variable Space Into Fuzzy Regions*

**Step 1**: Divide the input and output spaces into fuzzy regions. Suppose the domain intervals of the variables $x_1, x_2, y$ are $[x_1^-, x_1^+]$, $[x_2^-, x_2^+]$ and $[y_1, y_2]$ respectively. The domain interval of a variable means that it is most likely that this variable will lie in this interval. Divide each domain interval into $2N + 1$ regions, denoted by $SN$ ($N$ small), $S1$ (1 small), $CE$ (center), $B1$ (big 1), $BN$ (big $N$), ... After this naming, a fuzzy membership function is assigned to each region. Figure 1 shows an example where the domain interval $x_1$ is divided into five regions ($N = 2$). The domain area $X2$ is divided into seven regions ($N = 3$) and the domain interval $y$ is divided into five regions ($N = 2$). The shape of each membership function is triangular. The vertex of each triangle is located at the center of the region and has a membership value of one. The other two vertices are located at the centers of two neighboring regions, respectively, and have membership values of zero.

### *2.2. Formation of Fuzzy Rules*

**Step 2**: Fuzzy rules are obtained from the given data pair: First, the given degrees $x_1^i, x_2^i, y^i$ in the membership functions are determined. For example, $x_1^1$ in Figure 1, it has a membership degree of 0.8 in $B1$, a degree of 0.2 in $B2$, and a membership degree of zero in other membership functions. Similarly, $x_2^2$ in Figure 1 has a membership degree of one in $CE$ and a membership degree of zero in other membership functions. Then, the given $x_1^i, x_2^i, y^i$ is assigned to the region with the highest membership value, for example, $x_1^1$ in Figure 1 is assigned to $B1$ and $x_2^2$ to $CE$.
And finally, a rule is obtained from a pair of desired input and output data as follows, for example:

$$x_1^{(1)}, x_2^{(1)}, y^{(1)} \Rightarrow [x_1^{(1)}(0.8 \text{ in } B1, \max), x_2^{(1)}(0.7 \text{ in } S1, \max), y^{(1)}(0.9 \text{ in } CE, \max)]$$

$\Rightarrow$ Rule1:IF $x_1$ is $B1$ and $x_2$ is $S1$, THEN $y$ is $CE$.
The rules generated in this way are "AND" (joint) rules, i.e. rules in which the conditions of the IF part must be met simultaneously for the result of the THEN part to occur.

**Note 2.1.** For the problems considered in this research, i.e. generating fuzzy rules from numerical data, only the AND rules are needed because the antecedents of a rule are different components of a single input vector.

### *2.3. Assigning Degree to Fuzzy Rules*

**Step 3**: Assign a degree to each rule. Since there are usually a large number of data pairs and each data pair generates a rule, it is likely that there will be conflicting rules, for example, rules that have the same IF part but different THEN part. One way to solve this conflict is to assign a degree to each rule generated from the data pairs and only accept the rule from the conflict group that has the maximum degree. In this way, not only is the conflict problem solved, but the number of rules is also greatly reduced. It is recommended to use the following multiplication strategy to assign a degree to each rule, for example, for the rule if $x_1$ is equal to $A$ and $x_2$ is equal to $B$, then the value of $y$ is equal to $C$. The degree of this rule, represented by $D(\text{Rule})$, is defined as follows:

$$D(\text{Rule}) = m_A(x_1) m_B(x_2) m_c(y) \tag{3}$$

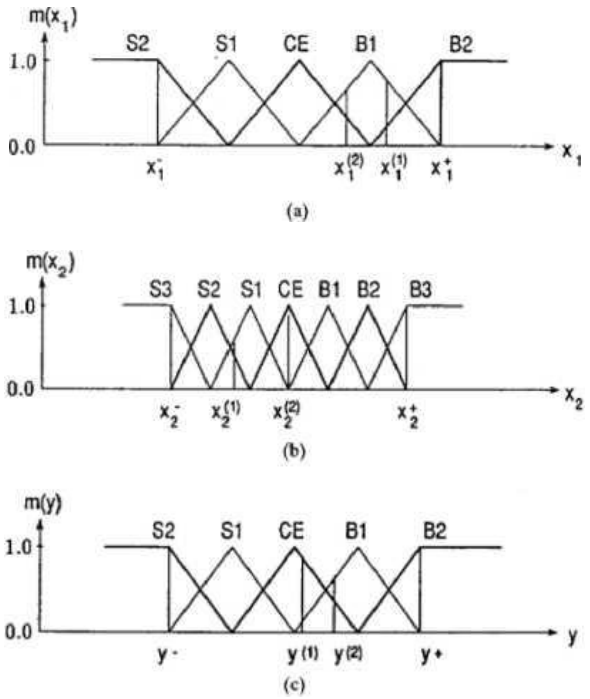


Fig. 1. Division of input and output spaces into fuzzy regions and corresponding membership functions.

For example, rule number 1, which was obtained in step 2-2, has a degree:

$$D(Rule1) = m_{B1}(x_1)m_{S1}(x_2)m_{CE}(y) = 0.8 \times 0.7 \times 0.9 = 0 \tag{4}$$

In practice, there is often a priori information about the data pairs with the help of an expert. For example, the expert may judge that by examining the given data pairs, some of them are very useful and crucial, but others are very unlikely and may only be due to measurement errors. We can therefore assign a degree to each data pair that reflects our belief in its usefulness. In this sense, the data pairs form a fuzzy set, i.e., a fuzzy set is defined as useful measurements. Each data pair belongs to this set with a degree assigned by the human expert. Suppose that data pair $(x_1^{(1)}, x_2^{(1)}, y^{(1)})$ has degree $m_1$. We redefine the degree of rule 1 as:

$$D(\text{Rule1}) = m_{B1}(x_1)m_{S1}(x_2)m_{CE}(y)m^{(1)} \tag{5}$$

For example, the degree of a rule is defined as the product of the degrees of its components and the degree of the data pair that generates the rule. This is important in practical applications because real numerical data have different reliability, for example, some real data can be very useless. Higher degrees are considered for useful data and lower degrees for useless data. In this way, human experiences with the data in a common database are used as other information.

**Note 2.2.** If one emphasizes objectivity and does not want a human to judge numerical data, this strategy can still be used and all degrees of data pairs are considered equal to one.

### *2.4. Forming a Fuzzy Rule Base*

**Step 4**: The cells of the fuzzy rule base are filled with fuzzy rules according to the following strategy: A fuzzy rule base is a combination of rules assigned to numerical data or linguistic rules. We assume that a linguistic rule has a degree determined by a human expert and reflects the expert's belief in the importance of the rule. If there is more

than one rule in a cell of the fuzzy rule base, the rule with the highest degree is used. In this way, both numerical and linguistic information are encoded in a common framework - the combined fuzzy rule base. If a linguistic rule is a rule and, then, it fills only one cell of the fuzzy rule base. However, if a linguistic rule is an "OR" rule (i.e. a rule that if any condition in the IF part is met, the THEN part follows), it fills all the cells in the rows or columns corresponding to the regions of that IF part.

### *2.5. Determining the Mapping for the Control Values*

**Step 5**: Determining the mapping based on the defuzzification rules The following defuzzification strategy is used to determine the output control y for the given inputs $(x_1, x_2)$ First, for the given input $(x_1, x_2)$, we combine the premises of the $i$-th rule using the multiplication operation to determine the degree and obtain $m^i_{o^i}$ as follows:

$$m^i_{o^i} = m_{I^i_1}(x_1) m_{I^i_2}(x_2) \tag{6}$$

Where $o^i$ refers to the output region of the $i$-th rule and $I^i_j$ is related to the input region in the $i$-th rule for its $j$-th component. For example, for rule 1 ("IF $x_1$ is $B1$ or $x_2$ is $S1$. Then $y$ is $CE$"), we obtain:

$$m^1_{CE} = m_{B1}(x_1) m_{S1}(x_2) \tag{7}$$

Then we use the following center of gravity defuzzification formula to obtain the output:

$$y = \frac{\sum_{i=1}^{k} m^i_{o^i} \overline{y}^i}{\sum_{i=1}^{k} m^i_{o^i}} \tag{8}$$

where $\overline{y}^i$ is the central value of the region $o^i$ (the center of a fuzzy region is defined as the point that has the smallest absolute value among all points whose membership function for this region has a membership value equal to one) and $K$ is the number of fuzzy rules in the base of the combined fuzzy rule.

**Note 2.3.** From steps 1 to 5, we see that this method is simple and straightforward in the sense that it is a single-pass construction method that does not require time-consuming training. Hence, it has the same advantage that the fuzzy approach has over the neural approach, namely, its construction is simple and fast.

## 3. Generalization to the Multi-Input Multi-Output Case

This five-step method can be easily extended to the general multi-input multi-output case. Steps 1 to 4 are independent of the number of inputs and the number of outputs. In step 5, we only need to replace $m^i_{o^i}$ with $m^i_{o^i_j}$, where $j$ is the $j$th component of the output vector ($o^i_j$ is the area of the output component $j$-th of the $i$-th rule: $m^i_{o^i_j}$ is the same for all $j$), so the above formula can be rewritten as $\overline{y}^i_j$ is the center of the area $o^i_j$.

$$y = \frac{\sum_{i=1}^{k} m^i_{o^i_j} \overline{y}^i_j}{\sum_{i=1}^{k} m^i_{o^i_j}} \tag{9}$$

This method essentially learns from examples and expert rules to obtain a mapping that we hope will have the property of "generalization" that the mapping continues to be successful or desired when new inputs are presented. Hence, this method can be viewed as a very general model-free, trainable fuzzy system for a wide range of control and signal processing problems, where model-free means that no mathematical model for the problem is required. Trainable means that the system learns from examples and expert rules and can adaptively change the mapping when new examples and expert rules are available. In practice, the input space is usually of high dimensionality, while the pairs of successful data and expert rules are often quite limited. As a result, many of the fuzzy rule cells may be empty. However, it is possible to fill these empty cells based on the given limited rules using the method described.

Specifically, steps 1 to 4 are first used to build a fuzzy rule base based on the limited data pairs and language rules. Then, the output Some common inputs for which the corresponding cell in the fuzzy rule base is empty can be determined based on the finite fuzzy rule base. Finally, the range in which the output has the maximum degree is assigned to the empty cell as a new rule. This method may be an iterative procedure.
For example, when a new rule is generated, this new rule and the existing rules in a fuzzy rule base are combined to create the next new rule. We can start the procedure from the empty cells that are the nearest neighbors of the full cells. In this way, the fuzzy rule base of the existing rules is expanded until all the cells are filled. Of course, we did not study this in detail in this review. But the basic ideas of this method are presented.
The tuning of fuzzy systems is done by tuning fuzzy rules. And since it is difficult to obtain the necessary knowledge to determine the fuzzy rules and also the range of variables is divided into several intervals The application of fuzzy systems is limited to areas where specialized knowledge is available and the number of inputs is not large. In order to overcome the problem of knowledge acquisition, this method has been developed to extract fuzzy rules from numerical data.
It is worth mentioning that the use of if-then rules for design has the following advantages.

(1) It is simple and understandable for humans.
(2) It is easy to implement in intelligent systems.

## 4. Conclusion

This paper describes a general method for generating fuzzy rules from numerical data and expert opinions. This method is a general method for combining numerical and linguistic information in a common framework and creating a fuzzy rule base. This fuzzy rule base consists of two types of fuzzy rules, some of which are derived from expert opinions and some from generated data.
The main advantages of this method are:

(1) It provides a general method for combining measured numerical information and linguistic information based on expert opinions in a common framework of a hybrid fuzzy rule base;
(2) It is a simple and single-pass construction method. Hence, no time-consuming repetitive training is required, so that it requires much less construction time than neural network-based methods;
(3) There is a lot of freedom in choosing membership functions. This gives us a lot of flexibility to design systems according to different needs.

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# Some results of a class of integral graphs on dihedral group

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**Abstract**

If all the eigenvalues of the adjacency matrix of a graph $\Lambda$ are integers, then we say that $\Lambda$ is an integral graph. Let $n$ be a fixed even integer is greater than or equal to 4. In the present work, we consider the Cayley graph $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$, where $\Psi = \{ab, a^2b, ..., a^{n-1}b, b\} \cup \{a^{\frac{n}{2}}\}$ is an inverse closed subset of dihedral group $\mathbb{D}_{2n} - \{1\}$. Although some resolving parameters of this class of graphs have been computed, see [9], we show that $\Lambda$ is an integral graph, indeed the adjacency spectrum of $\Lambda$ is $n+1, 1-n, 1^{(n-2)}, -1^{(n)}$, where the superscripts give the multiplicities of eigenvalues with multiplicity greater than one. In particular, we present some important results concerning of graph $K_v \nabla m\Gamma$. In fact, $K_v \nabla m\Gamma$ is determined by the adjacency spectrum as well as Laplacian spectrum.

## 1. Introduction

Graph theory is a branch of mathematics with numerous applications across a wide range of scientific disciplines, including biology, biotechnology, chemistry, computer science, physics and others. In particular, if we consider a simple, undirected and connected graph $\Gamma$ then the eigenvalues of $\Gamma$ are the eigenvalues of the adjacency matrix of $\Gamma$. The spectrum of $\Gamma$ is the list of the eigenvalues of the adjacency matrix of $\Gamma$ together with their multiplicities, and it is denoted by Spec($\Gamma$), see [4]. If all the eigenvalues of the adjacency matrix of the graph $\Gamma$ are integers, then we say that $\Gamma$ is an integral graph. The notion of integral graphs was first introduced by F. Harary and A. J. Schwenk in 1974, see [5]. In general, the problem of characterizing integral graphs seems to be very difficult. There are good surveys in this area, see [1].
Let $G$ be a finite group and $\Omega$ a subset of $G$ such that it is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = Cay(G, \Omega)$ is the graph whose vertex set and edge set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} \mid x^{-1}y \in \Omega\}.$$

Two graphs with the same spectrum are called cospectral. It is not hard to see that the spectrum of a graph does not determine its isomorphism class [4]. The authors in [3] proposed the question: which graphs are determined by their spectrum?. It seems hard to prove a graph to be determined by its spectrum(DS). Up to now, only a few classes of graphs are proved to be determined by their spectrum, such as: the path $P_n$, the complete graph $K_n$ and the cycle $C_n$.

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For a graph $\Gamma$, let $A(\Gamma)$ and $L(\Gamma) = D(\Gamma) - A(\Gamma)$ be respectively the adjacency matrix and Laplacian matrix of $\Gamma$, where $D(\Gamma)$ is the diagonal matrix of vertex degrees with $\{d_1, d_2, ..., d_n\}$ as diagonal entries. Laplacian spectrum and their applications are involved indiverse theoretical problems on complex networks [6, 12]. Many results have been devoted to studying Laplacian spectrum for complex networks [8].
Two graphs $\Gamma_1$ and $\Gamma_2$ are isomorphic if there is a bijection, $\varphi$ say, from $V(\Gamma_1)$ to $V(\Gamma_2)$ such that $x$ is adjacent to $y$ in $\Gamma_1$ if and only if $\varphi(x)$ is adjacent to $\varphi(y)$ in $\Gamma_2$. We say that $\varphi$ is an isomorphism from $\Gamma_1$ to $\Gamma_2$. An isomorphism from a graph $\Gamma$ to itself is called an automorphism of $\Gamma$.
Let $n$ be a fixed even integer is greater than or equal to 4. In this paper, suppose $\Upsilon = Cay(\mathbb{D}_{2n}, \Phi)$ and $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$, denote two Cayley graphs on dihedral group $\mathbb{D}_{2n}$, where

$$\mathbb{D}_{2n} =< a, b \mid a^n = b^2 = 1, ba = a^{n-1}b >,$$

is the dihedral group of order $2n$, also $\Phi = \{ab, a^2b, ..., a^{n-1}b, b\}$ and $\Psi = \Phi \cup \{a^{\frac{n}{2}}\}$ are inverse closed subsets of $\mathbb{D}_{2n} - \{1\}$. It is not hard to see that $\Upsilon$ is isomorphic to the complete bipartite graph $K_{n,n}$, and it is well known that the adjacency matrix spectrum of $\Upsilon$ is $n, -n, 0^{(2n-2)}$, where the superscripts give the multiplicities of eigenvalues with multiplicity greater than one. The authors in [9], showed that the Cayley graph $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$ is defined already cannot be a distance regular graph. In this paper, we establish some interesting results concerning the eigenvalues of the Cayley graph $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$. Indeed, we show that if $n \geq 4$ is an even integer then the adjacency spectrum of $\Lambda$ is $n + 1, 1 - n, 1^{(n-2)}, -1^{(n)}$. In particular, we present some important results concerning of graph $K_v \nabla m\Gamma$. In fact, $K_v \nabla m\Gamma$ is determined by the adjacency spectrum as well as Laplacian spectrum.

## 2. Definitions And Preliminaries

**Theorem 2.1.** *[2] Let $\Gamma_i$ be an $r_i$-regular graph of $n_i$ $(i = 1, 2)$ vertices. Then*

$$P(\Gamma_1 \nabla \Gamma_2, \lambda) = \frac{P(\Gamma_1, \lambda)P(\Gamma_2, \lambda)}{(\lambda - r_1)(\lambda - r_2)}((\lambda - r_1)(\lambda - r_2) - n_1 n_2).$$

**Theorem 2.2.** *[4] Let $\Gamma$ be a graph on $n$ vertices. Then, $n$ is a Laplacian eigenvalue of $\Gamma$ if and only if $\Gamma$ is the join of two graphs.*

**Lemma 2.3.** *[4] A connected graph $\Gamma$ has exactly one positive eigenvalue if and only if it is a complete multipartite graph.*

**Theorem 2.4.** *[10] Let $\Gamma_1$ and $\Gamma_2$ be two graphs with the Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$, respectively. Then, the Laplacian spectrum of $\Gamma_1 \nabla \Gamma_2$, is $n + m, m + \lambda_1, m + \lambda_2, ..., m + \lambda_{n-1}, n + \mu_1, n + \mu_2, ..., n + \mu_{m-1}, 0$.*

**Definition 2.5.** [11] Let $\Gamma_1 \cup \Gamma_2$ denote the disjoint union of graphs $\Gamma_1$ and $\Gamma_2$. The join $\Gamma_1 \nabla \Gamma_2$ is the graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining every vertex of $\Gamma_1$ with every vertex of $\Gamma_2$. A multicone graph is defined to be the join of a clique and a regular graph.

## 3. Main results

**Proposition 3.1.** *Let $\Upsilon = Cay(\mathbb{D}_{2n}, \Phi)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n}$. If $n \geq 4$ is an even integer and $M = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ is the adjacency matrix of $\Upsilon$ then $det(\lambda I_n - 2A) = (\lambda - n)(\lambda + n)\lambda^{n-2}$.*

*Proof.* If $n$ is an even integer, we may show that the adjacency matrix $M$ of graph $\Upsilon$ can be written as $M = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$, where $A$ is $n \times n$ matrix, hence the characteristic polynomial of graph $\Upsilon$ with respect to the adjacency matrix $M$ is the polynomial,

$$P(\Upsilon,\lambda) = det(\lambda I_{2n} - M) = det\begin{bmatrix} \lambda I_n - A & -A \\ -A & \lambda I_n - A \end{bmatrix} = det[(\lambda I_n - A)^2 - A^2]$$

$$= det[(\lambda I_n - A) - A]det[(\lambda I_n - A) + A] = det(\lambda I_n - 2A)det(\lambda I_n).$$

Moreover, we know that the adjacency matrix spectrum of $\Upsilon$ is $n, -n, 0^{(2n-2)}$. Thus $det(\lambda I_n - 2A) = (\lambda - n)(\lambda + n)\lambda^{n-2}$. □

**Theorem 3.2.** *Let $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$ be the Cayley graph on the dihedral group $\mathbb{D}_{2n}$, with the adjacency matrix $\hat{M}$ and characteristic polynomial $P(\Lambda, \hat{M})$. If $n \geq 4$ is an even integer, then the adjacency matrix spectrum of $\Lambda$ is $n+1, 1-n, 1^{(n-2)}, -1^{(n)}$.*

*Proof.* If $n$ is an even integer, it is easy to show that the adjacency matrix $\hat{M}$ of graph $\Lambda$ can be written as $\hat{M} = \begin{bmatrix} A & A + I_n \\ A + I_n & A \end{bmatrix}$, where $A$ is $n \times n$ matrix. Therefore by Proposition 3.1, the characteristic polynomial of graph $\Lambda$ with respect to the adjacency matrix $\hat{M}$ is the polynomial,

$$P(\Lambda,\lambda) = det(\lambda I_{2n} - \hat{M}) = det\begin{bmatrix} \lambda I_n - A & -(A + I_n) \\ -(A + I_n) & \lambda I_n - A \end{bmatrix} = det[(\lambda I_n - A)^2 - (A + I_n)^2]$$

$$= det[(\lambda I_n - A) - (A + I_n)]det[(\lambda I_n - A) + (A + I_n)] = det[(\lambda - 1)I_n - 2A]det[(\lambda + 1)I_n]$$

$$= (\lambda - (n+1))(\lambda - (1-n))(\lambda - 1)^{n-2}(\lambda + 1)^n.$$

Thus, the adjacency matrix spectrum of $\Lambda$ is $n+1, 1-n, 1^{(n-2)}, -1^{(n)}$. □

**Proposition 3.3.** *Let $\Gamma$ be a graph cospectral with the multicone graph $K_v \nabla \Lambda$ with respect to its adjacency matrix spectrum, where $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$. If $n \geq 4$ is an even integer then*

$$Spec(\Gamma) = \{1^{(n-2)}, -1^{(n+v-1)}, 1-n, (\frac{P + \sqrt{P^2 + 4Q}}{2}), (\frac{P - \sqrt{P^2 + 4Q}}{2})\},$$

*where $P = n + v$ and $Q = nv + n - v + 1$.*

*Proof.* This claim follows from Theorem 2.1. □

**Theorem 3.4.** *Consider the multicone graph $K_v \nabla \Lambda$, where $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$. If $n \geq 4$ is an even integer then $K_v \nabla \Lambda$ is DS with respect to its adjacency matrix spectrum.*

*Proof.* By similar way is done in the proof of Theorem 3.7, in [13], and a few changes this claim holds. □

**Proposition 3.5.** *Let $\Gamma$ be a graph cospectral with the multicone graph $K_v \nabla m\Lambda$ with respect to its adjacency matrix spectrum, where $m\Lambda$ denoted union $m$ copies of $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$. If $n \geq 4$ is an even integer then*

$$Spec(\Gamma) = \{(n+1)^{(m-1)}, (1-n)^{(m)}, 1^{(nm-2m)}, -1^{(nm+v-1)}, (\frac{P + \sqrt{P^2 + 4Q}}{2}), (\frac{P - \sqrt{P^2 + 4Q}}{2})\},$$

*where $P = n + v$ and $Q = 2nmv - nv + n - v + 1$.*

*Proof.* This claim follows from Theorem 2.1. □

**Theorem 3.6.** *Consider the multicone graph $K_v \nabla m\Lambda$, where $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$. If $n \geq 4$ is an even integer then $K_v \nabla m\Lambda$ is DS with respect to its adjacency matrix spectrum.*

*Proof.* By similar way is done in the proof of Theorem 7, in [7], and a few changes this claim holds. □

**Theorem 3.7.** *Consider the multicone graph $K_v \nabla m\Lambda$, where $\Lambda = Cay(\mathbb{D}_{2n}, \Psi)$. If $n \geq 4$ is an even integer then Then $K_v \nabla m\Lambda$ is DS with respect to its Laplacian spectrum.*

*Proof.* By similar way is done in the proof of Proposition 6, in [7], and a few changes this claim holds. □

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# Strong resolving set of a class of graphs on dihedral group

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**Abstract**

Let Γ be a finite, simple, and connected graph. A set $S = \{s_1, s_2, ..., s_r\}$ of vertices of Γ is a strong resolving set of Γ if every two distinct vertices of Γ are strongly resolved by some vertex of $S$. A strong metric basis of Γ is a strong resolving set of Γ with the minimum size and denoted by $sdim(\Gamma)$. It is well known that, the problem of determining the cardinality $sdim(\Gamma)$ of a minimal strong resolving set of Γ is NP hard. In the present work, we consider the cocktail party graph $CP(n)$. Although the strong metric basis of this graph has been calculated, see [6], but we obtain the strong metric basis of the cocktail party graph $CP(n)$ by another way.

## 1. Introduction and Preliminaries

Algebraic graph theory is an interesting area of mathematics that investigates graphs by applying methods from abstract algebra, especially linear algebra and group theory. It enables us to understand complex algebraic concepts through the framework of graphs. In the present work, we consider finite, simple, and connected graphs. A clique is a subgraph that is complete. The size of the largest clique in a graph Γ is denoted by $\omega(\Gamma)$. In a connected graph Γ, the distance $d_\Gamma(x, y)$ between two vertices $x$ and $y$ of Γ is the length of the shortest path from $x$ to $y$. A vertex $x \in V(\Gamma)$ is said to resolve a pair $u, v \in V(\Gamma)$ if $d_\Gamma(u, x) \neq d_\Gamma(v, x)$. For an arranged subset $S = \{s_1, s_2, ..., s_k\}$ of vertices in a connected graph Γ the metric representation of a vertex $v$ in Γ, is the $k$-vector $r(v|S) = (d(v, s_1), ..., d(v, s_k))$ relative to $S$. Also, the subset $S$ is considered as resolving set for Γ if any pair of vertices of Γ is distinguished by some vertices of $S$. A resolving set with least number of vertices is referred as metric basis for Γ and the cardinality of such resolving set is know as metric dimension denoted by $dim(\Gamma)$. The metric dimension of a graph Γ is the least number of vertices in a set with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex. The concept of resolving set and that of metric dimension date back to the 1950s. They were defined by Bluementhal [2] in the context of metric space. Also, the concept of the metric dimension in algebraic graph theory date back to the 1970s. It was defined independently by Harary and Melter [3] and by Slater [10].

The strong metric dimension problem was introduced by A. Sebö and E. Tannier [9] and further investigated by O. R. Oellermann and J. Peters-Fransen [8]. Recently, the strong metric dimension of distance hereditary graphs has been studied by T. May and O. R. Oellermann [7]. For more results related to this concept see [11]. A vertex $w$

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strongly resolves two vertices $u$ and $v$ if $u$ belongs to a shortest $v-w$ path or $v$ belongs to a shortest $u-w$ path. A set $S=\{s_1,s_2,...,s_r\}$ of vertices of $\Gamma$ is a strong resolving set of $\Gamma$ if every two distinct vertices of $\Gamma$ are strongly resolved by some vertex of $S$. A strong metric basis of $\Gamma$ is a strong resolving set of $\Gamma$ with the minimum size and denoted by $sdim(\Gamma)$. It is easy to see that if a vertex $w$ strongly resolves vertices $u$ and $v$ then $w$ also resolves these vertices. Hence every strong resolving set is a resolving set and hence $dim(\Gamma)\leq sdim(\Gamma)$. A vertex $u$ of a graph $\Gamma$ is called maximally distant from a vertex $v$ of $\Gamma$, if for every $w\in N_\Gamma(u)$, we have $d(v,w)\leq d(v,u)$, where $N_\Gamma(u)$ to denote the set of neighbors that $u$ has in $\Gamma$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then $u$ and $v$ are said to be mutually maximally distant [4].

Let $G$ be a finite group and let $\Omega$ be a subset of $G$ so that closed under taking inverses and does not contain the identity, then the Cayley graph $Cay(G,\Omega)$ is a graph with vertex set $G$ and the edge set $E(Cay(G,\Omega))=\{\{x,y\}\mid x^{-1}y\in\Omega\}$. Let $\mathbb{D}_{2n}=<a,b\mid a^n=b^2=1,ba=a^{n-1}b>$ be the dihedral group of order $2n$ ($n\geq 4$). Some resolving parameters, of the Cayley graph $Cay(\mathbb{D}_{2n},\Psi)$, where $\Psi=\{ab,a^2b,...,a^{n-1}b,b\}\cup\{a^{\frac{n}{2}}\}$ is an inverse closed subset of $\mathbb{D}_{2n}-\{1\}$, have been computed, see [5]. Also based on [6], the Cayley graph $Cay(\mathbb{D}_{2n},\Omega)$, where $\Omega=\{a,a^2,...,a^{n-1},ab,a^2b,...,a^{n-1}b\}$ is an inverse closed subset of $\mathbb{D}_{2n}-\{1\}$, is isomorphic to the cocktail party graph $CP(n)$. In the present work, we consider the cocktail party graph $CP(n)$. Although the strong metric basis of $CP(n)$ has been calculated, see [6], we obtain the strong metric basis of the cocktail party graph $CP(n)$ by another way.

## 2. Main results

**Proposition 2.1.** *Let $\Gamma=Cay(\mathbb{D}_{2n},\Omega)$ be the Cayley graph is defined already. Then each clique of $\Gamma$ of size $n$ is a resolving set of $\Gamma$.*

*Proof.* By similar way is done in the Theorem 5, of [6] we can show that each clique of $\Gamma$ of size $n$ is a resolving set of $\Gamma$. □

**Proposition 2.2.** *Let $\Gamma=Cay(\mathbb{D}_{2n},\Omega)$ be the Cayley graph is defined already. If $u$ and $v$ are distinct vertices of $\Gamma$ so that $d_\Gamma(u,v)=2$, then $u$ and $v$ are mutually maximally distant in $\Gamma$.*

*Proof.* If $u$ and $v$ are distinct vertices of $\Gamma$ so that $d_\Gamma(u,v)=2$, then $N_\Gamma(u)=N_\Gamma(v)$. Hence for every $w\in N_\Gamma(u)$, we have $d(v,w)\leq d(u,v)$, also for every $w\in N_\Gamma(v)$, we have $d(u,w)\leq d(u,v)$, and hence if $u$ and $v$ are distinct vertices of $\Gamma$ so that $d_\Gamma(u,v)=2$, then $u$ and $v$ are mutually maximally distant in $\Gamma$. □

**Theorem 2.3.** *Let $\Gamma=Cay(\mathbb{D}_{2n},\Omega)$ be the Cayley graph is defined already. If $R$ is a clique of size $n$ in $\Gamma$, then $R$ is a minimal strong resolving set of $\Gamma$.*

*Proof.* By similar way is done in the Theorem 5, of [6] we can show that the cardinality of every minimal resolving set of $\Gamma$ is $n$. With out loss of generality, if we consider the vertex $u=a^n$ in $\Gamma$ then there is exactly one vertex $v=a^{\frac{n}{2}}$ of $\Gamma$ so that the shortest path between $u=a^n$ and $v=a^{\frac{n}{2}}$ is 2, that is $d_\Gamma(u,v)=2$. Hence the vertices $u=a^n$ and $v=a^{\frac{n}{2}}$ are mutually maximally distant in $\Gamma$, and hence at least one of them must be belongs to any minimal strong resolving set of $\Gamma$. In particular, since $\Gamma$ is a vertex transitive graph then the cardinality of any strong resolving set of $\Gamma$ is greater than or equal to $n$. Moreover, based on Proposition 2.1, each clique of size $n$ is a resolving set of $\Gamma$. Therefore, it is sufficient to show that each clique of size $n$ is a strong resolving set of $\Gamma$. For this purpose, let $x$ and $y$ be distinct vertices of $\Gamma$ and let $R$ be an clique of size $n$ in $\Gamma$, hence by the following cases we have:

Case 1. If the shortest path between the vertices $x$ and $y$ is 2, that is $d_\Gamma(x,y)=2$, then based on Proposition 2.2, the vertices $x$ and $y$ are mutually maximally distant in $\Gamma$, hence at least one of them must be belongs to $R$, and hence $R$ is a strong resolving set of $\Gamma$.

Case 2. If the shortest path between the vertices $x$ and $y$ is 1, that is $d_\Gamma(x,y)=1$, also both the vertices $x$ and $y$ belong to $R$, then $R$ is a strong resolving set of $\Gamma$.

Case 3. If the shortest path between the vertices $x$ and $y$ is 1, that is $d_\Gamma(x,y)=1$, also $x\in R$ and $y\notin R$, then $R$ is a strong resolving set of $\Gamma$.

Case 4. If the shortest path between the vertices $x$ and $y$ is 1, that is $d_\Gamma(x,y)=1$, also $x \notin R$ and $y \notin R$, then there is exactly one element of $R$ say $z$ so that $d_\Gamma(z,x)=2$, hence based on Proposition 2.2, the vertices $z$ and $x$ are mutually maximally distant in $\Gamma$, and hence the vertex $y$ belongs to a shortest $z-x$ path, then $R$ is a strong resolving set of $\Gamma$. Therefore based on above cases if $R$ is a clique of size $n$ in $\Gamma$, then $R$ is a minimal strong resolving set of $\Gamma$. □

**Corollary 2.4.** *Consider the cayley graph the Cayley graph* $Cay(\mathbb{D}_{2n},\Omega)$ *is defined already. Then the cardinality of every minimal strong resolving set of* $\Gamma$ *is* $n$.

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# Generalized Fibonacci–Based Factorizations of Pascal Functional Matrices

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**Abstract**

In this paper, a novel generalized Fibonacci functional matrix, denoted by $\mathcal{F}_n^{(a,b)}[t;u,v]$, is introduced. The nonzero entries of this matrix are defined in terms of the generalized $(A, B)$-Fibonacci polynomials. Furthermore, two different factorizations of the generalized functional Pascal matrix $\mathcal{P}_n[x, y]$ are obtained by employing this matrix along with two newly constructed matrices. As an application of these factorizations, several new combinatorial identities related to the generalized $(A, B)$-Fibonacci polynomials are derived using linear algebraic methods.

## 1. Introduction

In the recent years, Pascal and Fibonacci matrices have appeared in linear algebra, probability, combinatorics and applied mathematics. So far, the generalizations and different decompositions, and also eigenvalues of these matrices have been investigated. Some of these results in [1]-[13] Zhang and Wang [12] gave some factorization of the Pascal matrix via the Fibonacci matrix and they obtained some combinatorial identities by using Pascal matrix, the Stirling matrices of the first kind and the second kind and the Fibonacci matrix.
As you know, the Fibonacci and Lucas numbers are very appealing in combinatorial analysis and number theory. The Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$, and the Lucas numbers $\{L_n\}_{n=0}^{\infty}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \ F_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ L_1 = 1.$$

We also observe so-called generalized Fibonacci number $\{F_n^{(a,b)}\}_{n=0}^{\infty}$, which satisfies the following conditions

$$F_n^{(a,b)} = AF_{n-1}^{(a,b)} + BF_{n-2}^{(a,b)}, \quad F_0^{(a,b)} = a, \ F_1^{(a,b)} = b, \tag{1}$$

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where $a, b, A$ and $B$ are constant real numbers such that $A^2 + 4B > 0$, (see [11]). Among the generalized Fibonacci and Lucas sequences, the Fibonacci $\{F_n\}_{n=0}^{\infty}$ and Lucas $\{L_n\}_{n=0}^{\infty}$ sequences have achieved a kind of celebrity status, and have been studied extensively in number theory, matrix theory, and applied mathematics.
The $n \times n$ generalized Fibonacci matrix $\mathcal{F}_n^{(a,b)} = \left[f_{i,j}^{(a,b)}\right]$, (for $i, j = 1, 2, \cdots, n$), is defined by [6, 11]:

$$f_{i,j}^{(a,b)} = \begin{cases} F_{i-j+1}^{(a,b)} & i - j \geq 0, \\ 0 & i - j < 0. \end{cases} \tag{2}$$

The inverse and Cholesky factorization of the generalized Fibonacci matrix are given in [6, 7, 11]. The relation ships between the Pascal matrix and the Fibonacci matrix are studied in [4, 5, 10, 11].
Generalized Fibonacci numbers and matrix reduce to known definition of the usual Fibonacci numbers and Fibonacci matrix in the cases $A = B = a = b = 1$. In the case $A = B = 1, a = 2, b = 1$ we obtain the Lucas numbers and the Lucas matrix.
Large classes of polynomials can be defined by Fibonacci-like recurrence relation and yield Fibonacci numbers. Such polynomials, called the Fibonacci polynomials, were studied in 1883 by the Belgian Mathematician Eugene Charles Catalan and the German Mathematician E. Jacobsthal.

The $(A, B)-$Fibonacci polynomials $\{F_n(t)\}_{n=0}^{\infty}$, are defined by the following recurrence relation

$$F_{n+1}(t) = A(t)F_n(t) + B(t)F_{n-1}(t), \quad F_0(t) = 0, \quad F_1(t) = 1. \tag{3}$$

Specific instances of $A(t)$ and $B(t)$ under the initial conditions $a(t) = 0$ and $b(t) = 1$ are presented in Table 1.

Table 1. Five well-known sequences corresponding to various $A(t)$ and $B(t)$

| $A(t)$ | $B(t)$ | Polynomials |
|---|---|---|
| $2t$ | $1$ | Fibonacci Polynomial |
| $t$ | $1$ | Pell Polynomial |
| $1$ | $2t$ | Jacobsthal Polynomial |
| $3t$ | $-2$ | Fermat Polynomial |
| $2t$ | $-1$ | Cheby. Poly. 2nd kind |

## 2. Generalized $(A, B)-$Fibonacci Polynomials And Generalized Fibonacci Functional Matrix

In this section, we introduce a generalization of $(A, B)-$Fibonacci polynomials.

**Definition 2.1.** Let $A(t), B(t), a(t)$ and $b(t)$ be polynomials with real coefficients. The generalized $(A, B)-$Fibonacci polynomials $\{F_n^{(a,b)}(t)\}_{n=0}^{\infty}$, are defined by the recurrence relation

$$F_{n+1}^{(a,b)}(t) = A(t)F_n^{(a,b)}(t) + B(t)F_{n-1}^{(a,b)}(t), \quad F_0^{(a,b)}(t) = a(t), \quad F_1^{(a,b)}(t) = b(t), \tag{4}$$

and $A(t)^2 + 4B(t) > 0$ for all real $t$.

Special cases of the $A(t)$ and $B(t)$ with given initial conditions $a(t)$ and $b(t)$, are given in Table2.

**Definition 2.2.** The generalized Fibonacci functional matrix $\mathcal{F}_n^{(a,b)}[t; u, v] = \left[f_{i,j}^{(a,b)}(t; u, v)\right]$, is defined by

$$f_{i,j}^{(a,b)}(t; u, v) = \begin{cases} F_{i-j+1}^{(a,b)}(t)u^{i-j}v^{j-1} & i - j \geq 0, \\ 0 & i - j < 0. \end{cases} \tag{5}$$

for all $i, j = 1, 2, \cdots, n$.

Table 2. Special cases of the $A(t)$ and $B(t)$ with given initial conditions $a(t) = 2$ and $b(t) = t$.

| $A(t)$ | $B(t)$ | Polynomials |
|---|---|---|
| $2t$ | $1$ | Lucas Polynomial |
| $t$ | $1$ | Pell-Lucas Polynomial |
| $1$ | $2t$ | Jacobshal-Lucas Poly. |
| $3t$ | $-2$ | Fermat-Lucas Poly. |
| $2t$ | $-1$ | Cheby. Poly. first kind |

**Theorem 2.3.** *The inverse* $\mathcal{F}_n^{-1}{}^{(a,b)}[t;u,v] = \left[{f'_{i,j}}^{(a,b)}(t;u,v)\right]$ *of the matrix* $\mathcal{F}_n^{(a,b)}[t;u,v] = \left[{f_{i,j}}^{(a,b)}(t;u,v)\right], v \neq 0$ *and* $b(t) \neq 0$ *for all* $t$*, is equal to*

$$f'^{(a,b)}_{i,j}(t;u,v) = \begin{cases} (-1)^{i-j}.\frac{a(t)^2B(t)+a(t)b(t)A(t)-b(t)^2}{b(t)^{i-j+1}}a(t)^{i-j-2}B(t)^{i-j-1}\frac{u^{i-j}}{v^{i-1}} & i \geq j+2, \\ -\frac{a(t)B(t)+b(t)A(t)}{b(t)^2}\cdot\frac{u}{v^j} & i = j+1, \\ \frac{1}{b(t)}\cdot\frac{1}{v^{j-1}} & i = j, \\ 0 & i < j. \end{cases} \tag{6}$$

In the partial case $A(t) = A$, $B(t) = B$, $a(t) = a$, and $b(t) = b$ from Theorem 2.3, we obtain the inverse of the generalized Fibonacci functional matrix.

## 3. Generalized Pascal and Fibonacci Functional Matrices

The generalized Pascal functional matrix of the first kind with two variables $\mathcal{P}_n[x,y] = [p_n(x,y;i,j)]$, $i,j = 1,\cdots,n$ is defined in [1], [3] and [12]:

$$p_n(x,y;i,j) = \begin{cases} \binom{i-1}{j-1}x^{i-j}y^{j-1} & i \geq j, \\ 0 & i < j. \end{cases} \tag{7}$$

The inverse of $\mathcal{P}_n[x,y]$ is equal to:

$$\mathcal{P}_n^{-1}[x,y] = \mathcal{P}_n\left[-x,\frac{1}{y}\right]. \tag{8}$$

In the case $x = y = 1$, the generalized Pascal functional of first kind reduces to the well-known Pascal matrix [3]. In the following theorem we define the matrix $\mathcal{G}_n^{(a,b)}[t;u,v,x,y] = [g_{i,j}], i,j = 1,2,\cdots,n$, which gives a correlation between the matrix $\mathcal{F}_n^{(a,b)}[t;u,v]$, and the generalized Pascal functional matrix of the first kind:

**Theorem 3.1.** *The matrix* $\mathcal{G}_n^{(a,b)}[t;u,v,x,y]$*, (*$v \neq 0, b(t) \neq 0$ *for all* $t$*), where entries are defined by*

$$\begin{aligned} g_{i,j}(t;u,v,x,y) = \frac{y^{j-1}}{v^{j-1}}\Bigg[&\frac{1}{b(t)}x^{i-j}\binom{i-1}{j-1} - \frac{a(t)B(t)+b(t)A(t)}{b(t)^2}ux^{i-j-1}\binom{i-2}{j-1} \\ &+ \sum_{k=j}^{i-2}(-1)^{i-k}\frac{a(t)^2B(t)+a(t)b(t)A(t)-b(t)^2}{b(t)^{i-k+1}}a(t)^{i-k-2}B(t)^{i-k-1} \\ &\times u^{i-k}x^{k-j}\binom{k-1}{j-1}\Bigg], \end{aligned} \tag{9}$$

*satisfies*

$$\mathcal{P}_n[x,y] = \mathcal{F}_n^{(a,b)}[t;u,v]\mathcal{G}_n^{(a,b)}[t;u,v,x,y]. \tag{10}$$

In the case $A(t)=A, B(t)=B, a(t)=a, b(t)=b$ and $u=v=y=1$ we get analogous result for the generalized Fibonacci functional matrix [11].

**Corollary 3.2.** *The matrix* $\mathcal{G}_n^{(a,b)}[1,1,x,1]$, $b\neq 0$, *where entries are defined by*

$$g_{i,j}(u,v,x,y)=\frac{1}{b}x^{i-j}\binom{i-1}{j-1}-\frac{aB+bA}{b^2}x^{i-j-1}\binom{i-2}{j-1}+\sum_{k=j}^{i-2}(-1)^{i-k}\frac{a^2B+abA-b^2}{b^{i-k+1}}a^{i-k-2}B^{i-k-1}x^{k-j}\binom{k-1}{j-1},$$

*satisfies*

$$\mathcal{P}_n[x,1]=\mathcal{F}_n^{(a,b)}[1,1]\mathcal{G}_n^{(a,b)}[1,1,x,1]. \tag{11}$$

In the case $A=B=x=1$, $a=0$ and $b=1$ we get the Fibonacci matrix. In the case $A=B=x=1, a=2, b=1$, from Corollary 3.2 we give a corresponding result for Lucas matrices [13].
In the following theorem we define the matrix $\mathcal{H}_n^{(a,b)}[t;u,v,x,y]=[h_{i,j}(t;u,v,x,y)]$,
$i,j=1,2,\cdots,n$ which gives a similar correlation between the matrix $\mathcal{F}_n^{(a,b)}[t;u,v]$ and the generalized Pascal functional matrix $\mathcal{P}_n[x,y]$:

**Theorem 3.3.** *The matrix* $\mathcal{H}_n^{(a,b)}[t;u,v,x,y]$, *(*$v\neq 0, b(t)\neq 0$ *for all t), defined by*

$$h_{i,j}(t;u,v,x,y)=\frac{y^{j-1}}{v^{j-1}}\left[\frac{x^{i-j}}{b(t)}\binom{i-1}{j-1}-\frac{a(t)B(t)+b(t)A(t)}{vb(t)^2}ux^{i-j-1}y\binom{i-1}{j}+\sum_{k=j+2}^{i}(-1)^{k-j}\frac{a(t)^2B(t)+a(t)b(t)A(t)-b(t)^2}{v^{k-1}b(t)^{k-j+1}}\times a(t)^{k-j-2}B(t)^{k-j-1}(uy)^{k-j}x^{i-k}\binom{i-1}{k-1}\right],$$

*satisfies*

$$\mathcal{P}_n[x,y]=\mathcal{H}_n^{(a,b)}[t;u,v,x,y]\mathcal{F}_n^{(a,b)}[t;u,v]. \tag{12}$$

An analogous result for the generalized Fibonacci functional matrix can be derived in the case $A(t)=A, B(t)=B, a(t)=a, b(t)=b$ and $u=v=y=1$ (see [11]). An analogous result for Lucas matrices (for $a=b=1, A=B=1$) is [13]:
The generalized Pascal functional matrix of the second kind $\mathcal{Q}_n[x,y]=[q_n(x,y;i,j)]$,
$i,j=1,2,\cdots,n$ is defined by:

$$q_n(x,y;i,j)=\begin{cases}\binom{i-1}{j-1}x^{i+j-2}y^{i-j} & i\geq j,\\ 0 & i<j.\end{cases} \tag{13}$$

The inverse of $\mathcal{Q}_n[x,y]$ is equal to:

$$\mathcal{Q}_n^{-1}[x,y]=\mathcal{Q}_n\left[-\frac{1}{x},y\right]. \tag{14}$$

**Theorem 3.4.** *The matrix* $\mathcal{S}_n^{(a,b)}[t;u,v,x,y]=[s_{i,j}]$ *and* $\mathcal{T}_n^{(a,b)}[t;u,v,x,y]=[t_{ij}]$, $i,j=1,2,\cdots,n$, *(*$v\neq 0, x\neq$

$0, b(t) \neq 0$ *for all t),whose entries are defined by*

$$
\begin{aligned}
s_{i,j}(t;u,v,x,y) &= \frac{1}{b(t)v^{j-1}}\binom{i-1}{j-1}x^{i+j-2}y^{i-j} \\
&\quad - \frac{(a(t)B(t)+b(t)A(t))u}{v^j b(t)^2}x^{i+j-3}y^{i-j-1}\binom{i-2}{j-1} \\
&\quad + \sum_{k=j}^{i-2}(-1)^{i-k}\binom{k-1}{j-1}\frac{a(t)^2B(t)+a(t)b(t)A(t)-b(t)^2}{v^{i-1}b(t)^{i-k+1}} \\
&\quad \times a(t)^{i-k+2}B(t)^{i-k-1}u^{i-k}x^{k+j-2}y^{k-j}. \\
t_{i,j}(t;u,v,x,y) &= \binom{i-1}{j-1}\frac{x^{i+j-2}y^{i-j}}{bv^{j-1}} - \binom{i-1}{j}\frac{a(t)B(t)+b(t)A(t)}{b(t)^2}\cdot\frac{ux^{i+j-1}y^{i-j-1}}{v^j} \\
&\quad + \sum_{k=j+2}^{i}(-1)^{k-j}\binom{i-1}{k-1}\frac{a(t)^2B(t)+a(t)b(t)A(t)-b(t)^2}{v^{k-1}b(t)^{k-j+1}} \\
&\quad \times a(t)^{k-j-2}B(t)^{k-j-1}y^{i-k}x^{i+k-2}u^{k-j}.
\end{aligned}
$$

*satisfies*

$$
\begin{aligned}
\mathcal{Q}_n[x,y] &= \mathcal{F}_n^{(a,b)}[t;u,v]\mathcal{S}_n^{(a,b)}[t;u,v,x,y], \\
\mathcal{Q}_n[x,y] &= \mathcal{T}_n^{(a,b)}[t;u,v,x,y]\mathcal{F}_n^{(a,b)}[t;u,v].
\end{aligned}
$$

In the case $A(t)=A, B(t)=B, a(t)=a, b(t)=b$ and $u=v=y=1$, we have (see [11]).

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# Geometric Characterization of Optimal Maritime Trajectories via $p$-Harmonic Maps on Sea-State Manifolds

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**Abstract**

We establish a rigorous geometric framework that identifies fuel-optimal maritime trajectories with $p$-harmonic maps from a time interval into a sea-state-dependent Riemannian manifold $(\mathcal{M}, g_{\text{sea}})$. The metric tensor $g_{\text{sea}}$ encodes anisotropic resistance induced by wave fields, ocean currents, and wind stress. We prove that critical points of the $p$-energy functional $E_p(\varphi) = \frac{1}{p}\int \|\dot{\varphi}\|^p_{g_{\text{sea}}}\, dt$ satisfy a nonlinear second-order ODE whose solutions exhibit adaptive curvature proportional to sea-state gradients. Three main results are presented: (i) a necessary and sufficient condition for $p$-harmonicity in terms of the sea-state potential; (ii) uniqueness of $p$-harmonic trajectories under convexity of wave-height isolines; and (iii) stability of $p$-harmonic routes against $C^1$-perturbations of the sea-state metric. This work provides the first geometric foundation for maritime route optimization grounded in nonlinear harmonic map theory.

## 1. Sea-State Manifolds and the Geometric Model

Let $\mathcal{M} \subset \mathbb{R}^2$ denote the navigable sea region. We equip $\mathcal{M}$ with a smooth Riemannian metric $g_{\text{sea}}$ that quantifies effective hydrodynamic resistance per unit displacement. The construction proceeds via a *sea-state potential* $\Phi \in C^\infty(\mathcal{M})$ defined as

$$\Phi(x) := \alpha_w H_s^\beta(x) + \alpha_c \|\mathbf{c}(x)\|^\gamma + \alpha_{\text{wind}} \|\mathbf{w}(x)\|^\delta, \tag{1}$$

where $H_s$ is significant wave height, $\mathbf{c}$ ocean current velocity, $\mathbf{w}$ wind vector, and $\alpha_i, \beta, \gamma, \delta > 0$ vessel-dependent coefficients. The sea-state metric is then defined by conformal deformation of the Euclidean metric $g_0$:

$$g_{\text{sea}} := e^{2\Phi} g_0. \tag{2}$$

This conformal structure captures the physical principle that resistance scales exponentially with sea-state severity while preserving angular relationships (isotropy of local resistance).

**Definition 1.1** (Sea-state manifold)**.** The pair $(\mathcal{M}, g_{\text{sea}})$ with $g_{\text{sea}}$ given by (2) is called a *sea-state manifold*. A smooth curve $\varphi : [0, T] \to \mathcal{M}$ represents a maritime trajectory parameterized by time.

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The Levi-Civita connection $\nabla$ of $g_{\text{sea}}$ satisfies

$$\nabla_X Y = \nabla^0_X Y + X(\Phi)Y + Y(\Phi)X - g_0(X,Y)\,\operatorname{grad}_{g_0}\Phi, \tag{3}$$

where $\nabla^0$ is the Euclidean connection. Consequently, the geodesic equation $\nabla_{\dot\varphi}\dot\varphi = 0$ becomes

$$\ddot\varphi + 2\dot\varphi(\Phi)\dot\varphi - \|\dot\varphi\|^2_{g_0}\operatorname{grad}_{g_0}\Phi = 0. \tag{4}$$

Equation (4) reveals that geodesics bend toward regions of lower $\Phi$ (calmer seas), with curvature proportional to $\|\operatorname{grad}\Phi\|$.

## 2. The $p$-Harmonic Trajectory Framework

For $p > 1$, the $p$-energy of a trajectory $\varphi \in C^\infty([0,T],\mathcal{M})$ is defined by

$$E_p(\varphi) := \frac{1}{p}\int_0^T \left\|\dot\varphi(t)\right\|^p_{g_{\text{sea}}}\,dt. \tag{5}$$

Critical points of $E_p$ satisfy the Euler–Lagrange equation $\tau_p(\varphi) = 0$, where the $p$-tension field is

$$\tau_p(\varphi) := \frac{d}{dt}\left(\|\dot\varphi\|^{p-2}\dot\varphi\right) + \|\dot\varphi\|^{p-2}\nabla_{\dot\varphi}\dot\varphi. \tag{6}$$

Trajectories satisfying $\tau_p(\varphi) = 0$ are called *$p$-harmonic maps* from $([0,T], dt^2)$ to $(\mathcal{M}, g_{\text{sea}})$.

**Theorem 2.1** (Characterization of $p$-harmonic maritime trajectories)**.** *Let $(\mathcal{M}, g_{sea} = e^{2\Phi}g_0)$ be a sea-state manifold. A trajectory $\varphi$ is $p$-harmonic if and only if it satisfies the nonlinear ODE*

$$\ddot\varphi = -\frac{p-2}{\|\dot\varphi\|^2}\langle\ddot\varphi,\dot\varphi\rangle_{g_0}\dot\varphi - 2\dot\varphi(\Phi)\dot\varphi + \|\dot\varphi\|^2_{g_0}\operatorname{grad}_{g_0}\Phi. \tag{7}$$

*Moreover, the speed function $v(t) := \|\dot\varphi(t)\|_{g_{sea}}$ satisfies*

$$\frac{d}{dt}\left(v^{p-1}\right) = (p-1)v^{p-1}\,\dot\varphi(\Phi). \tag{8}$$

*Proof.* Substituting (3) into (6) and expanding the total derivative yields (7) after elementary algebraic manipulation. Equation (8) follows by taking the inner product of (6) with $\dot\varphi$ and using $\langle\nabla_{\dot\varphi}\dot\varphi,\dot\varphi\rangle = \frac{1}{2}\frac{d}{dt}\|\dot\varphi\|^2$. □

Equation (7) generalizes the geodesic equation (4) by introducing a nonlinear damping term proportional to $(p-2)$. For $p > 2$, this term suppresses acceleration components parallel to velocity, yielding trajectories that maintain near-constant speed in high-resistance regions—a physically desirable property for fuel efficiency.

**Corollary 2.2.** *If $\Phi$ is constant along a trajectory $\varphi$ (i.e., $\dot\varphi(\Phi) = 0$), then $\varphi$ is $p$-harmonic if and only if it is a geodesic of $(\mathcal{M}, g_{sea})$ and has constant speed. In particular, great-circle routes on calm seas ($\Phi \equiv$ const) are $p$-harmonic for all $p > 1$.*

## 3. Existence, Uniqueness and Stability

We now establish fundamental analytical properties of $p$-harmonic maritime trajectories.

**Theorem 3.1** (Uniqueness under convex sea states)**.** *Suppose the sea-state potential $\Phi$ is strictly convex on $\mathcal{M}$, i.e., $\operatorname{Hess}_{g_0}\Phi > 0$ as a quadratic form. Then for any two ports $A, B \in \mathcal{M}$ and any $p > 1$, there exists at most one $p$-harmonic trajectory $\varphi : [0,T] \to \mathcal{M}$ with $\varphi(0) = A$, $\varphi(T) = B$, and $\|\dot\varphi\|_{g_{sea}} > 0$.*

*Proof.* Assume two distinct $p$-harmonic trajectories $\varphi_1, \varphi_2$ with identical endpoints. Consider the variation $\varphi_s = \exp_{\varphi_1}(sV)$ where $V$ is the Jacobi field with $V(0) = V(T) = 0$. The second variation of $E_p$ at a critical point is

$$\delta^2 E_p(V,V) = \int_0^T \Big[(p-2)\|\dot{\varphi}\|^{p-4}\langle \nabla_{\dot{\varphi}} V, \dot{\varphi}\rangle^2 + \|\dot{\varphi}\|^{p-2}\|\nabla_{\dot{\varphi}} V\|^2 - \|\dot{\varphi}\|^p R(V,\dot{\varphi},\dot{\varphi},V)\Big] dt. \tag{9}$$

For curves in dimension two, the curvature term simplifies to $R(V,\dot{\varphi},\dot{\varphi},V) = K\|V \wedge \dot{\varphi}\|^2$, where $K = -e^{-2\Phi}\Delta_{g_0}\Phi$ is the Gaussian curvature of $(\mathcal{M}, g_{\text{sea}})$. Strict convexity of $\Phi$ implies $\Delta_{g_0}\Phi > 0$, hence $K < 0$. Therefore $\delta^2 E_p(V,V) > 0$ for all nontrivial $V$, contradicting the assumption that both $\varphi_1$ and $\varphi_2$ are critical points. □

**Theorem 3.2** (Stability under metric perturbations)**.** *Let $\{g_\varepsilon\}_{\varepsilon\in(-\delta,\delta)}$ be a $C^1$-family of Riemannian metrics on $\mathcal{M}$ with $g_0 = g_{sea}$, and let $\varphi_0$ be a non-degenerate $p$-harmonic map for $g_0$ (i.e., $\delta^2 E_p$ is non-singular at $\varphi_0$). Then there exists $\varepsilon_0 > 0$ and a unique $C^1$-family $\{\varphi_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ such that each $\varphi_\varepsilon$ is $p$-harmonic with respect to $g_\varepsilon$ and $\varphi_\varepsilon \to \varphi_0$ in $C^2$ as $\varepsilon \to 0$.*

*Proof.* Define the nonlinear operator $\mathcal{F} : C^{2,\alpha}([0,T],\mathcal{M}) \times (-\delta,\delta) \to C^{0,\alpha}([0,T],T\mathcal{M})$ by $\mathcal{F}(\varphi,\varepsilon) := \tau_p^{(g_\varepsilon)}(\varphi)$. The Fréchet derivative $D_\varphi \mathcal{F}(\varphi_0, 0)$ coincides with the Jacobi operator associated with $\delta^2 E_p$, which is elliptic and invertible by non-degeneracy. The Implicit Function Theorem in Banach spaces then yields the desired $C^1$-family $\varphi_\varepsilon$. □

Theorem 3.2 guarantees that optimal maritime routes computed under forecasted sea states remain near-optimal under small forecast errors—a crucial robustness property for operational routing.

## 4. Conclusion

We have developed a pure geometric theory identifying fuel-optimal maritime trajectories with $p$-harmonic maps into sea-state manifolds. Three principal contributions were established:

1. A conformal geometric model $(\mathcal{M}, e^{2\Phi} g_0)$ where the potential $\Phi$ encodes hydrodynamic resistance from waves, currents, and wind.
2. A complete characterization of $p$-harmonic trajectories via the nonlinear ODE (7), revealing how the parameter $p > 2$ induces adaptive curvature that avoids high-resistance regions.
3. Fundamental analytical properties: uniqueness under convex sea states (Theorem 3.1) and structural stability under metric perturbations (Theorem 3.2).

This framework elevates maritime route optimization from an algorithmic problem to a geometric analysis question on nonlinear harmonic maps. Future theoretical directions include extending the theory to $p$-biharmonic maps (modeling jerk-minimal trajectories for passenger comfort) and stochastic $p$-harmonic maps under uncertain sea-state forecasts.

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# Theoretical Foundations of $p$-Harmonic Extensions for Offshore Energy Potential Interpolation on Oceanographic Manifolds

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**Abstract**

We establish a rigorous geometric-analytic framework for interpolating offshore renewable energy potential fields using $p$-harmonic extensions on oceanographic Riemannian manifolds. The domain manifold $(\Omega, g_{\text{ocean}})$ incorporates bathymetry-induced conformal deformation $g_{\text{ocean}} = h^{-\theta} g_0$ where $h$ denotes water depth and $\theta > 0$ encodes energy-type dependence. We prove existence and uniqueness of $p$-harmonic extensions for non-homogeneous Dirichlet boundary data representing sparse coastal measurements. Three principal theoretical contributions are presented: (i) a sharp regularity theorem showing $C^{1,\alpha}$-smoothness of $p$-harmonic extensions up to the coastline boundary under minimal geometric assumptions; (ii) monotonicity and asymptotic convergence of the $p$-harmonic family $\{u_p\}_{p>2}$ to the absolutely minimizing Lipschitz extension as $p \to \infty$; and (iii) a discrete concentration-compactness principle for minimizing configurations of the repulsive $p$-energy functional, establishing existence of optimal turbine layouts without numerical approximation. This work provides the first pure mathematical foundation for spatial optimization in offshore renewable energy infrastructure grounded in nonlinear potential theory.

## 1. Oceanographic Manifolds and the $p$-Laplace Equation

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain representing an offshore development zone, with boundary $\partial\Omega = \Gamma_{\text{coast}} \cup \Gamma_{\text{open}}$ where $\Gamma_{\text{coast}}$ is the coastline (Lipschitz curve) and $\Gamma_{\text{open}}$ the open-sea boundary. Denote by $h \in C^{0,1}(\overline{\Omega})$ the water depth function satisfying $0 < h_{\min} \le h(x) \le h_{\max} < \infty$ and $|\nabla h(x)| \le C$ almost everywhere.

**Definition 1.1** (Oceanographic manifold)**.** For $\theta > 0$, the pair $(\Omega, g_{\text{ocean}})$ with Riemannian metric

$$g_{\text{ocean}} := h(x)^{-\theta} g_0, \tag{1}$$

where $g_0$ is the Euclidean metric, is called an *oceanographic manifold*. The parameter $\theta$ depends on energy type: $\theta = 1/4$ for wave energy density, $\theta = 1/2$ for tidal kinetic energy flux.

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The volume form and gradient associated with $g_{\text{ocean}}$ are given by $dV_g = h^{-\theta}dx$ and $\nabla_g u = h^{\theta}\nabla_0 u$, respectively. The $p$-Laplace operator on $(\Omega, g_{\text{ocean}})$ for $p > 1$ is defined as

$$\Delta_{p,g} u := \operatorname{div}_g\left(\|\nabla_g u\|_g^{p-2}\nabla_g u\right) = h^{\theta}\operatorname{div}_0\left(h^{-(p-1)\theta/2}|\nabla_0 u|^{p-2}\nabla_0 u\right). \tag{2}$$

Given boundary energy data $\mathcal{E} \in W^{1,p}(\partial\Omega)$ representing measured potential (wave power, wind power density, or tidal flux), we consider the Dirichlet problem:

$$\begin{cases}\Delta_{p,g} u = 0 & \text{in } \Omega,\\ u = \mathcal{E} & \text{on } \partial\Omega.\end{cases} \tag{3}$$

Weak solutions $u \in W^{1,p}(\Omega)$ satisfy the variational identity

$$\int_\Omega \|\nabla_g u\|_g^{p-2}\langle \nabla_g u, \nabla_g \varphi\rangle_g \, dV_g = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{4}$$

**Theorem 1.2** (Existence and uniqueness)**.** *For any $p > 1$ and $\mathcal{E} \in W^{1,p}(\partial\Omega)$, the Dirichlet problem (3) admits a unique weak solution $u_p \in W^{1,p}(\Omega)$. Moreover, $u_p$ minimizes the $p$-energy functional*

$$\mathcal{F}_p(v) := \frac{1}{p}\int_\Omega \|\nabla_g v\|_g^p \, dV_g \tag{5}$$

*among all $v \in W^{1,p}(\Omega)$ with trace $v|_{\partial\Omega} = \mathcal{E}$.*

*Proof.* The functional $\mathcal{F}_p$ is strictly convex, coercive, and weakly lower semicontinuous on the affine space $\{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = \mathcal{E}\}$ due to the uniform ellipticity of $g_{\text{ocean}}$ (bounded conformal factor). Direct methods in the calculus of variations yield existence of a unique minimizer, which satisfies the Euler–Lagrange equation (4). □

## 2. Regularity and Asymptotic Behavior of $p$-Harmonic Extensions

We now establish boundary regularity and asymptotic properties of the solution family $\{u_p\}_{p>1}$.

**Theorem 2.1** (Boundary regularity)**.** *Let $(\Omega, g_{ocean})$ be an oceanographic manifold with $h \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$, and let $\mathcal{E} \in C^{1,\beta}(\partial\Omega)$. Then for any $p > 2$, the $p$-harmonic extension $u_p$ belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha = \alpha(p,\theta,\beta) \in (0,1)$. Moreover, there exists a constant $C > 0$ independent of $p$ such that*

$$\|u_p\|_{C^{1,\alpha}(\overline{\Omega})} \le C\left(\|\mathcal{E}\|_{C^{1,\beta}(\partial\Omega)} + 1\right). \tag{6}$$

*Proof.* The conformal structure (1) implies that $\Delta_{p,g}$ is a uniformly elliptic quasilinear operator in divergence form with $C^{0,\beta}$-coefficients when $h \in C^{1,\beta}$. Interior $C^{1,\alpha}$-regularity follows from classical results of DiBenedetto and Tolksdorf. Boundary regularity up to $\partial\Omega$ requires the additional observation that the Lipschitz boundary $\partial\Omega$ satisfies the exterior cone condition, and the conformal factor $h^{-\theta}$ remains bounded away from zero and infinity. Applying the boundary regularity theory for $p$-Laplacian with non-homogeneous boundary data yields the estimate (6). □

**Theorem 2.2** (Monotonicity and asymptotic convergence)**.** *Let $\{u_p\}_{p>2}$ be the family of $p$-harmonic extensions of fixed boundary data $\mathcal{E} \in \operatorname{Lip}(\partial\Omega)$. Then:*

*(i) For $2 < p_1 < p_2$, we have $\|\nabla_g u_{p_2}\|_{L^{p_2}(\Omega)} \le \|\nabla_g u_{p_1}\|_{L^{p_1}(\Omega)}$.*

*(ii) There exists $u_\infty \in \operatorname{Lip}(\overline{\Omega})$ such that $u_p \to u_\infty$ uniformly on $\overline{\Omega}$ as $p \to \infty$.*

*(iii) The limit $u_\infty$ is the unique absolutely minimizing Lipschitz extension (AMLE) of $\mathcal{E}$, characterized by*

$$\operatorname{Lip}(u_\infty; V) = \operatorname{Lip}(u_\infty; \partial V) \quad \forall V \Subset \Omega, \tag{7}$$

*where $\operatorname{Lip}(v; S) := \sup_{x\neq y\in S} \frac{|v(x)-v(y)|}{d_g(x,y)}$ denotes the Lipschitz constant on set $S$ with respect to the geodesic distance $d_g$.*

*Proof.* Part (i) follows from the minimization property (5) and Hölder's inequality. For part (ii), the uniform Lipschitz bound $\|\nabla_g u_p\|_{L^\infty(\Omega)} \leq \mathrm{Lip}(\mathcal{E}; \partial\Omega)$ (a consequence of the comparison principle for $p$-harmonic functions) implies equicontinuity of $\{u_p\}$. The Arzelà–Ascoli theorem yields a uniformly convergent subsequence; uniqueness of the limit follows from part (iii). The characterization (7) is the defining property of AMLEs established by Jensen for Euclidean domains; the conformal invariance of Lipschitz constants under (1) extends the result to oceanographic manifolds. □

**Corollary 2.3.** *For oceanographic applications where boundary measurements $\mathcal{E}$ exhibit sharp gradients near headlands or submarine canyons, the $p$-harmonic extension with large $p$ ($p \geq 4$) preserves these critical features in the interpolated field, whereas harmonic interpolation ($p = 2$) oversmooths them. The AMLE limit $u_\infty$ provides the theoretically optimal interpolation that minimizes maximum gradient while respecting boundary data.*

## 3. Discrete $p$-Energy Minimization and Optimal Point Configurations

We now address the theoretical foundations of optimal facility placement using repulsive $p$-energy functionals.

**Definition 3.1** (Discrete $p$-energy on manifolds)**.** For $N \geq 2$, $p > 0$, and a configuration $\mathcal{P}_N = \{x_1, \dots, x_N\} \subset \Omega$ of distinct points, define the discrete $p$-energy

$$\mathcal{E}_{p,g}(\mathcal{P}_N) := \sum_{1 \leq i < j \leq N} \frac{1}{d_g(x_i, x_j)^p}, \tag{8}$$

where $d_g$ denotes the geodesic distance on $(\Omega, g_{\text{ocean}})$.

**Theorem 3.2** (Existence of minimal energy configurations)**.** *Let $(\Omega, g_{ocean})$ be a compact Riemannian manifold with $C^2$ boundary and $p > 2$. Then for each $N \geq 2$, there exists a configuration $\mathcal{P}_N^* = \{x_1^*, \dots, x_N^*\} \subset \Omega$ minimizing $\mathcal{E}_{p,g}$ over all $N$-point configurations in $\Omega$. Moreover, any minimizer satisfies:*

*(i) Strict separation:* $\min_{i \neq j} d_g(x_i^*, x_j^*) \geq c_{N,p} > 0$,

*(ii) First-order optimality: For each $i = 1, \dots, N$,*

$$\sum_{j \neq i} \frac{p}{d_g(x_i^*, x_j^*)^{p+1}} \exp_{x_i^*}^{-1}(x_j^*) = 0, \tag{9}$$

*where $\exp^{-1}$ denotes the inverse exponential map on $(\Omega, g_{ocean})$.*

*Proof.* The energy functional $\mathcal{E}_{p,g}$ is continuous on the compact set $\Omega^N$ and tends to $+\infty$ as any two points coalesce when $p > 0$. For $p > 2$, the singularity is sufficiently strong to prevent boundary accumulation: if a sequence $\mathcal{P}_N^{(k)}$ approaches $\partial\Omega$, the conformal metric (1) ensures $d_g$ remains comparable to Euclidean distance near smooth boundary portions, and the repulsive energy dominates any boundary attraction. Thus minimizers exist in the interior $\Omega$. The strict separation (i) follows from the blow-up of $\mathcal{E}_{p,g}$ at coincidence points. Differentiating $\mathcal{E}_{p,g}$ with respect to $x_i$ using Riemannian calculus yields the force balance equation (9). □

**Theorem 3.3** (Concentration-compactness for large $N$)**.** *Let $\{\mathcal{P}_N^*\}_{N \geq 2}$ be a sequence of minimal $p$-energy configurations with $p > 2$. Define the empirical measures $\mu_N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^*}$. Then there exists a probability measure $\mu^* \in \mathcal{P}(\overline{\Omega})$ such that $\mu_N \rightharpoonup^* \mu^*$ weakly-$*$ as $N \to \infty$. Moreover, $\mu^*$ has a density $\rho^* \in L^1(\Omega)$ satisfying*

$$\rho^*(x) = C_{p,\Omega} \cdot \big(\mathrm{dist}_g(x, \partial\Omega)\big)^{\frac{2}{p-2}} \quad \text{a.e. in } \Omega, \tag{10}$$

*where $C_{p,\Omega} > 0$ is a normalization constant and $\mathrm{dist}_g$ denotes geodesic distance to the boundary.*

*Proof.* The proof follows the concentration-compactness principle of Lions adapted to Riemannian manifolds. The key observation is that minimal $p$-energy configurations for $p > 2$ exhibit a universal scaling law: in local normal coordinates, the minimal energy per particle scales as $N^{1+p/2}$. This implies tightness of $\{\mu_N\}$ and precludes concentration at isolated points. The density profile (10) is derived by matching the repulsive $p$-energy with the confining potential induced by the domain boundary, using Γ-convergence techniques for Riesz energies on manifolds. □

**Remark 3.4.** The density law (10) has direct oceanographic interpretation: optimal facility layouts naturally concentrate away from the coastline with a power-law profile determined solely by the repulsion exponent $p$. Larger $p$ yields steeper decay toward the boundary, corresponding to stronger avoidance of shallow-water zones where installation costs and environmental constraints dominate.

## 4. Conclusion

We have developed a comprehensive theoretical framework for $p$-harmonic analysis on oceanographic manifolds with applications to offshore renewable energy potential interpolation and facility layout optimization. The principal mathematical contributions include:

1. Existence, uniqueness, and $C^{1,\alpha}$-regularity of $p$-harmonic extensions for non-homogeneous Dirichlet data on bathymetry-weighted Riemannian manifolds (Theorems 1.2–2.1).
2. Monotonicity properties and asymptotic convergence to absolutely minimizing Lipschitz extensions as $p \to \infty$, providing theoretical justification for gradient-preserving interpolation in high-variability coastal zones (Theorem 2.2).
3. Existence of minimal discrete $p$-energy configurations with explicit force-balance optimality conditions, and a concentration-compactness principle yielding universal density profiles for large-scale layouts (Theorems 3.2–3.3).

This work elevates offshore spatial optimization from heuristic algorithms to a rigorous branch of nonlinear potential theory on Riemannian manifolds. Future theoretical directions include extending the framework to vector-valued $p$-harmonic maps for multi-resource optimization (simultaneous wave, wind, and tidal potential interpolation) and developing stochastic $p$-harmonic theory for uncertain boundary measurements.

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# Geometric Theory of $p$-Harmonic Navigation Paths in Anisotropic Sea States

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**Abstract**

We develop a rigorous geometric framework for maritime navigation under anisotropic sea-state conditions using $p$-harmonic maps. The navigable domain is modeled as a Riemannian manifold $(\mathcal{M}, g_{\text{sea}})$ where the metric tensor encodes direction-dependent hydrodynamic resistance induced by wave fields and ocean currents. We prove that safety-optimal navigation paths—those minimizing cumulative exposure to hazardous sea states while reaching destination—correspond precisely to $p$-harmonic maps from a time interval into $(\mathcal{M}, g_{\text{sea}})$. Three principal theoretical contributions are established: (i) a complete characterization of $p$-harmonic navigation paths via a nonlinear tension field equation incorporating sea-state anisotropy; (ii) existence and uniqueness of $p$-harmonic paths under convexity conditions on the sea-state potential; and (iii) a stability theorem guaranteeing robustness of optimal paths against $C^1$-perturbations of sea-state forecasts. This work provides the first pure geometric foundation for risk-averse maritime navigation grounded in nonlinear harmonic map theory.

## 1. Sea-State Manifolds and Anisotropic Navigation Geometry

Let $\mathcal{M} \subset \mathbb{R}^2$ denote the navigable sea region. Unlike isotropic resistance models, real maritime environments exhibit direction-dependent resistance due to wave propagation direction and current vectors. We model this anisotropy using a Finsler structure derived from sea-state data.

**Definition 1.1** (Anisotropic sea-state metric)**.** Let $\mathbf{w}(x) \in \mathbb{R}^2$ denote the dominant wave direction vector and $H_s(x) > 0$ the significant wave height at position $x \in \mathcal{M}$. For $p > 1$, define the anisotropic metric tensor

$$g^{(p)}_{\text{sea}}(x)(v, v) := \|v\|^2 + \kappa\, H_s(x)^{\gamma} \left|\langle v, \mathbf{w}(x)\rangle\right|^p, \quad v \in T_x\mathcal{M}, \tag{1}$$

where $\kappa > 0$ and $\gamma > 0$ are vessel-dependent coefficients, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. The pair $(\mathcal{M}, g^{(p)}_{\text{sea}})$ is called an *anisotropic sea-state manifold*.

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The metric (1) captures the physical principle that resistance increases nonlinearly when vessel velocity aligns with wave direction ($\langle v, \mathbf{w}\rangle > 0$), while motion perpendicular to waves experiences near-isotropic resistance. The exponent $p$ controls the degree of anisotropy: larger $p$ amplifies directional sensitivity.
A navigation path is a smooth curve $\varphi : [0,T] \to \mathcal{M}$ with $\varphi(0) = A$ (departure port) and $\varphi(T) = B$ (destination port). The cumulative sea-state exposure along $\varphi$ is quantified by the $p$-energy functional:

**Definition 1.2** ($p$-energy of navigation path)**.** For $p > 1$, the $p$-energy of a path $\varphi \in C^\infty([0,T], \mathcal{M})$ is

$$E_p(\varphi) := \frac{1}{p}\int_0^T \left\|\dot{\varphi}(t)\right\|^p_{g^{(p)}_{\text{sea}}}\, dt. \tag{2}$$

Critical points of $E_p$ represent paths that optimally balance travel time against exposure to hazardous sea states. We now characterize these critical points geometrically.

## 2. Characterization of $p$-Harmonic Navigation Paths

The first variation of $E_p$ yields the Euler–Lagrange equation governing optimal navigation paths.

**Theorem 2.1** (Characterization of $p$-harmonic navigation paths)**.** *A smooth path $\varphi : [0,T] \to \mathcal{M}$ is a critical point of $E_p$ if and only if it satisfies the nonlinear tension field equation*

$$\tau_p(\varphi) := \frac{D}{dt}\left(\|\dot{\varphi}\|^{p-2}\dot{\varphi}\right) + \frac{p-2}{2}\|\dot{\varphi}\|^{p-4}\frac{d}{dt}\left(\|\dot{\varphi}\|^2\right)\dot{\varphi} = 0, \tag{3}$$

*where $\frac{D}{dt}$ denotes the covariant derivative along $\varphi$ with respect to the Levi-Civita connection of $g^{(p)}_{sea}$. Paths satisfying (3) are called $p$-harmonic navigation paths.*

*Proof.* Consider a smooth variation $\varphi_s(t)$ with $\varphi_0 = \varphi$ and variation field $V(t) = \partial_s \varphi_s(t)|_{s=0}$. The first variation formula gives

$$\begin{aligned}\frac{d}{ds}E_p(\varphi_s)\Big|_{s=0} &= \int_0^T \left\langle \|\dot{\varphi}\|^{p-2}\dot{\varphi}, \frac{D}{dt}V\right\rangle_g dt \\ &= -\int_0^T \left\langle \frac{D}{dt}\left(\|\dot{\varphi}\|^{p-2}\dot{\varphi}\right), V\right\rangle_g dt,\end{aligned}$$

where integration by parts uses $V(0) = V(T) = 0$. Setting the variation to zero for all $V$ yields the weak form of (3). The additional term in (3) arises from differentiating $\|\dot{\varphi}\|^{p-2}$ when $p \neq 2$, completing the proof. □

Equation (3) generalizes the geodesic equation ($p = 2$) by introducing a nonlinear damping term proportional to $(p-2)$. For $p > 2$, this term suppresses acceleration components parallel to velocity, yielding paths that maintain near-constant speed while adaptively curving away from high-resistance directions—a physically desirable property for risk-averse navigation.

**Corollary 2.2.** *When sea states are isotropic ($\mathbf{w} \equiv 0$), the metric $g^{(p)}_{sea}$ reduces to a conformal Euclidean metric, and $p$-harmonic navigation paths coincide with Riemannian geodesics parameterized with constant speed.*

## 3. Existence, Uniqueness and Stability

We now establish fundamental analytical properties of $p$-harmonic navigation paths under physically realistic sea-state conditions.

**Theorem 3.1** (Existence under convex sea states)**.** *Suppose the sea-state potential $\Phi(x) := H_s(x)^\gamma$ is strictly convex on $\mathcal{M}$ and the wave direction field $\mathbf{w}(x)$ is $C^1$-smooth with $\|\mathbf{w}\| \leq 1$. Then for any ports $A, B \in \mathcal{M}$ and $p > 1$, there exists a $p$-harmonic navigation path $\varphi : [0,T] \to \mathcal{M}$ with $\varphi(0) = A$ and $\varphi(T) = B$.*

*Proof.* The $p$-energy functional $E_p$ is coercive on the Sobolev space $W^{1,p}([0,T],\mathcal{M})$ due to the uniform positivity of $g_{\text{sea}}^{(p)}$. Strict convexity of $\Phi$ implies that $E_p$ is weakly lower semicontinuous. By the direct method in the calculus of variations, $E_p$ attains its minimum on the affine space $\{\varphi \in W^{1,p} : \varphi(0) = A, \varphi(T) = B\}$. The minimizer is smooth by standard regularity theory for $p$-harmonic maps into Riemannian manifolds with $C^1$ metrics. □

**Theorem 3.2** (Uniqueness under strong convexity)**.** *Assume the sea-state manifold $(\mathcal{M}, g_{sea}^{(p)})$ has non-positive sectional curvature. Then for any $A, B \in \mathcal{M}$ and $p > 1$, there exists at most one $p$-harmonic navigation path connecting $A$ to $B$ with non-vanishing velocity.*

*Proof.* Suppose $\varphi_1, \varphi_2$ are distinct $p$-harmonic paths with identical endpoints. Consider the variation $\varphi_s = \exp_{\varphi_1}(sV)$ where $V$ is the Jacobi field with $V(0) = V(T) = 0$. The second variation of $E_p$ at $\varphi_1$ is

$$\delta^2 E_p(V,V) = \int_0^T \Big[(p-2)\|\dot{\varphi}_1\|^{p-4}\langle \nabla_{\dot{\varphi}_1} V, \dot{\varphi}_1\rangle^2 + \|\dot{\varphi}_1\|^{p-2}\|\nabla_{\dot{\varphi}_1} V\|^2 - \|\dot{\varphi}_1\|^p R(V,\dot{\varphi}_1,\dot{\varphi}_1,V)\Big]dt, \tag{4}$$

where $R$ is the Riemann curvature tensor. Non-positive sectional curvature implies $R(V,\dot{\varphi}_1,\dot{\varphi}_1,V) \le 0$. Since $p > 1$, all terms in the integrand are non-negative, and $\delta^2 E_p(V,V) > 0$ for nontrivial $V$. This contradicts the assumption that both $\varphi_1$ and $\varphi_2$ are critical points. □

**Theorem 3.3** (Stability under sea-state perturbations)**.** *Let $\{g_\varepsilon\}_{\varepsilon\in(-\delta,\delta)}$ be a $C^1$-family of anisotropic sea-state metrics with $g_0 = g_{sea}^{(p)}$, and let $\varphi_0$ be a non-degenerate $p$-harmonic path for $g_0$. Then there exists $\varepsilon_0 > 0$ and a unique $C^1$-family $\{\varphi_\varepsilon\}_{|\varepsilon|<\varepsilon_0}$ such that each $\varphi_\varepsilon$ is $p$-harmonic with respect to $g_\varepsilon$ and $\varphi_\varepsilon \to \varphi_0$ in $C^2$ as $\varepsilon \to 0$.*

*Proof.* Define the nonlinear operator $\mathcal{F} : C^{2,\alpha}([0,T],\mathcal{M}) \times (-\delta,\delta) \to C^{0,\alpha}([0,T],T\mathcal{M})$ by $\mathcal{F}(\varphi,\varepsilon) := \tau_p^{(g_\varepsilon)}(\varphi)$. The Fréchet derivative $D_\varphi \mathcal{F}(\varphi_0, 0)$ is the Jacobi operator associated with $\delta^2 E_p$, which is elliptic and invertible by non-degeneracy. The Implicit Function Theorem in Banach spaces yields the desired $C^1$-family $\varphi_\varepsilon$. □

Theorem 3.3 guarantees that optimal navigation paths computed under forecasted sea states remain near-optimal under small forecast errors—a crucial robustness property for operational maritime routing.

## 4. Conclusion

We have established a pure geometric theory identifying risk-averse maritime navigation paths with $p$-harmonic maps into anisotropic sea-state manifolds. Three principal theoretical contributions were developed:

1. A novel anisotropic Riemannian model $(\mathcal{M}, g_{\text{sea}}^{(p)})$ where the metric tensor encodes direction-dependent hydrodynamic resistance from wave fields and currents, with exponent $p$ controlling risk-aversion sensitivity.
2. A complete characterization of $p$-harmonic navigation paths via the nonlinear tension field equation (3), revealing how $p > 2$ induces adaptive curvature that avoids hazardous sea-state directions while maintaining efficient progress toward destination.
3. Fundamental analytical properties: existence under convex sea states (Theorem 3.1), uniqueness under non-positive curvature (Theorem 3.2), and structural stability against sea-state forecast perturbations (Theorem 3.3).

This framework elevates maritime navigation from heuristic path planning to a rigorous branch of geometric analysis on nonlinear harmonic maps. Future theoretical directions include extending the theory to time-dependent sea states via $p$-harmonic flows, and incorporating stochastic sea-state models through probabilistic $p$-harmonic maps.

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# A Chebyshev-Picard Method for Nonlinear Volterra Integral Equations of the Second Kind

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**Abstract**

This paper introduces a novel approach for solving nonlinear Volterra integral equations by combining the Picard iteration method with Chebyshev polynomials. The proposed method utilizes the orthogonal properties and spectral accuracy of Chebyshev polynomials to iteratively approximate solutions, ensuring high precision and computational efficiency. To demonstrate the effectiveness of the approach, numerical experiments are conducted on classe of nonlinear Volterra integral equations, including a benchmark problem. The results demonstrate the effectiveness of the proposed method.

## 1. Introduction

Consider the following nonlinear Volterra integral equation (VIE) of the second kind [2, 4, 7]:

$$u(x) = f(x) + \int_0^x K(x,t,u(t))dt, \quad 0 \le x \le L, \tag{1}$$

where $u$ is the unknown function, and $f$ and $K$ are the forcing and kernel functions, respectively.
The VIEs have significant applications in various fields of science and engineering, such as heat conduction problems, specific problems in mechanics and physics, and unsteady Poiseuille flow in a pipe [2]. In this study, a computational scheme is presented for solving nonlinear one-dimensional VIE of the second kind. Many researchers have focused on developing numerical solutions for VIEs. Various methods have been employed for this purpose, including spectral quasilinearization method [7], methods based on radial basis functions (RBFs) [2], non-polynomial spline basis and Quasi-linearization method [4], collocation method based on radial basis functions [1] and an iterative-computational algorithm for second-kind nonlinear VIE with weakly singular kernels [3].
The Picard iteration method is a classical and widely used technique for solving nonlinear integral equations, which generates a sequence of approximations that converge to the exact solution under appropriate conditions [6]. When

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combined with well-chosen basis functions, such as Chebyshev polynomials (CPs), the efficiency and accuracy of the Picard method can be significantly improved. In this work, we develop a hybrid computational technique incorporating Picard iteration with Chebyshev polynomial bases to solve nonlinear VIEs of the second kind. CPs are known for their excellent approximation properties, including orthogonality and rapid convergence, making them particularly suitable for representing solutions of integral equations. The proposed method leverages these properties to achieve high precision and computational efficiency. The performance of the proposed method is evaluated through comprehensive numerical experiments. The results confirm its exceptional computational efficiency and robustness in solving various forms of nonlinear VIEs, establishing it as a competitive alternative to conventional numerical approaches.

## 2. Chebyshev polynomials

The explicit formula for the CPs is as follows [5]:

$$\varphi_n(\tau) = \cos(n\cos^{-1}(\tau)), \quad n \geq 1. \tag{2}$$

These polynomials can be derived from the recurrence relation:

$$\begin{aligned} \varphi_0(\tau) &= 1, \quad \varphi_1(\tau) = \tau, \\ \varphi_{n+1}(\tau) &= 2\tau\varphi_n(\tau) - \varphi_{n-1}(\tau), \quad n \geq 1, \quad -1 \leq t \leq 1. \end{aligned} \tag{3}$$

These polynomials also admit the following explicit representation [5]:

$$\varphi_i(\tau) = \frac{i}{2}\sum_{k=0}^{\lfloor i/2 \rfloor}(-1)^k \frac{(i-k-1)!}{k!(i-2k)!}(2\tau)^{i-2k}, \tag{4}$$

where $\lfloor i/2 \rfloor$ denotes the integer part of $i/2$.
The sequence $\{\varphi_i(\tau)\}_{i=0}^{\infty}$ forms an orthogonal set in the weighted space $L_w^2([-1,1])$, where $w(\tau) = 1/\sqrt{1-\tau^2}$ is the weight function. Moreover, these polynomials satisfy the orthogonality relation [5]:

$$\int_{-1}^{1} \varphi_i(\tau)\varphi_j(\tau)w(\tau)d\tau = \delta_{ij}h_i, \tag{5}$$

where $\delta_{ij}$ denotes the Kronecker delta, with $h_0 = \pi$ and $h_i = \pi/2$ for $i \geq 1$. The shifted CPs on the interval $[0, L]$ are obtained through the variable transformation $\tau = \frac{2x}{L} - 1$ applied to Eqs. (2) or (3). Defining $P_i(x) \equiv \varphi_i\left(\frac{2x}{L} - 1\right)$, these polynomials satisfy the recurrence relation:

$$P_{i+1}(x) = 2\left(\frac{2x}{L} - 1\right)P_i(x) - P_{i-1}(x), \quad i \geq 1, \tag{6}$$

with $P_0(x) = 1$ and $P_1(x) = \frac{2x}{L} - 1$. Alternatively, they admit the explicit form

$$P_i(x) = \frac{i}{2}\sum_{k=0}^{\lfloor i/2 \rfloor}(-1)^k \frac{(i-k-1)!}{k!(i-2k)!}\left(\frac{4}{L}x - 2\right)^{i-2k}, \quad i \geq 1. \tag{7}$$

For computational applications, these polynomials admit a power series representation as follows [5]:

$$P_i(x) = \sum_{k=0}^{i} c_{ik}x^k, \tag{8}$$

where the expansion coefficients satisfy $c_{0k} = 1$ for all $k$, and

$$c_{ik} = (-1)^{i+k}\frac{2^{2k}i(i+k-1)!}{(i-k)!(2k)!L^k}, \quad i \geq 1.$$

The sequence $\{P_i(x)\}_{i=0}^{\infty}$ forms an orthogonal set in the weighted space $L^2_{\tilde{w}}([0,L])$, where $\tilde{w}(x) = w\left(\frac{2}{L}x - 1\right) = L/\sqrt{L^2 - (2x-L)^2}$ is the associated weight function. Furthermore, the following property holds for the shifted CPs:

$$\int_0^L P_i(x)P_j(x)\tilde{w}(x)\,dx = \delta_{ij}\tilde{h}_i, \tag{9}$$

where $\tilde{h}_i = Lh_i/2$, and $h_i$ is already defined in (5). Therefore, any function $u \in L^2_{\tilde{w}}([0,L])$ can be expressed in terms of these shifted CPs as follows:

$$u(x) = \sum_{i=0}^{\infty} u_i P_i(x), \tag{10}$$

where the coefficients $u_i$ are given by

$$u_i = \frac{1}{\tilde{h}_i}\int_0^L \tilde{w}(x)P_i(x)u(x)\,dx. \tag{11}$$

In practical applications, we interrupt the infinite series in (10) and use the below approximation for $u(x)$:

$$u(x) \simeq \sum_{i=0}^{N} u_i P_i(x) := \mathbf{U}_N^T \mathbf{P}_N(x), \tag{12}$$

where $\mathbf{U}_N = [u_0, u_1, \dots, u_N]^T$, and $\mathbf{P}_N(x) = [P_0(x), P_1(x), \dots, P_N(x)]^T$. For computational simplicity, we can use the discrete shifted CPs expansion (based on the Chebyshev–Gauss quadrature) to calculate the coefficients $u_i$ as follows:

$$u_i \simeq \frac{\pi}{(N+1)h_i}\sum_{l=0}^{N} u(x_l)P_i(x_l), \tag{13}$$

where $x_l$, $l = 0, 1, \dots, N$ are the zeros of $P_{N+1}(x)$, computed as follows [5]:

$$x_l = \frac{L}{2}\left(1 - \cos\left(\frac{(2l+1)\pi}{2(N+1)}\right)\right). \tag{14}$$

## 3. Chebyshev-Picard iteration method

This section presents a computational approach for obtaining numerical solutions to the VIE given in Eq. (1). The Picard iteration sequence for Eq. (1) is formulated as follows:

$$u^{[i]}(x) = f(x) + \int_0^x K(x,t,u^{[i-1]}(t))dt, \tag{15}$$

where $u^{[0]}(x)$ represents the initial approximation function. This initial approximation may be derived from the function $f$ specified in the problem statement, a component of $f$, or through a trial-and-error procedure. The selection of $u^{[0]}(x)$ is not unique in general. In the first step of the method, we approximate the functions $f(x)$ and $K(x,t,u^{[i-1]}(t))$ using CPs expansions as follows:

$$f(x) \simeq \sum_{r=0}^{N} \tilde{f}_r P_r(x), \tag{16}$$

$$K(x,t,u^{[i-1]}(t)) \simeq \sum_{r=0}^{N}\sum_{n=0}^{N} \tilde{S}_{rn}^{[i-1]} P_r(x)P_n(t), \tag{17}$$

where

$$\tilde{f}_r = \frac{\pi}{(N+1)h_r} \sum_{j=0}^{N} f(x_j) P_r(x_j), \tag{18}$$

$$\tilde{S}_{rn} = \frac{\pi^2}{(N+1)^2 h_r h_n} \sum_{l=0}^{N} \sum_{j=0}^{N} K(x_l, t_j, u^{[i-1]}(t_j)) P_r(x_l) P_n(t_j), \quad r, n = 0, 1, \ldots, N.$$

where $\mathcal{R}(x) := P_r(x) \int_0^x P_n(t)\, dt$ can be approximated as follows:

$$\mathcal{R}(x) \simeq \sum_{m=0}^{N} R_m^{(r,n)} P_m(x).$$

The points $x_l$ and $t_j$ given in Eq. (14), and $h_r$ is specified in Eq. (5). Now, by substituting relations (16) and (17) into Eq. (15), we obtain:

$$u^{[i]}(x) \simeq \sum_{j=0}^{N} \left( \tilde{f}_r + \sum_{r=0}^{N} \sum_{n=0}^{N} \tilde{S}_{rn}^{[i-1]} R_j^{(r,n)} \right) P_j(x). \tag{19}$$

Following the proposed method, at the $i$-th iteration, we consider the following approximation for the function $u^{[i]}(x)$:

$$u^{[i]}(x) \simeq \sum_{j=0}^{N} \hat{c}_j^{[i]} P_j(x). \tag{20}$$

The coefficients $\hat{c}_j^{[i]}$ are determined from Eq. (19) via the relation:

$$\hat{c}_j^{[i]} = \tilde{f}_r + \sum_{r=0}^{N} \sum_{n=0}^{N} \tilde{S}_{rn}^{[i-1]} R_j^{(r,n)}, \quad j = 0, 1, \ldots, N. \tag{21}$$

The stopping criterion is specified by:

$$\|u^{[i]}(x) - u^{[i-1]}(x)\|_\infty = \max_{x \in [0,L]} |u^{[i]}(x) - u^{[i-1]}(x)| < \epsilon, \tag{22}$$

where $\epsilon > 0$ denotes the prescribed convergence threshold.

## 4. Numerical example

We apply nonlinear VIEs to study the efficiency and accuracy of the proposed method. The performance is quantified using the absolute error metric defined by:

$$L_\infty = \|u(x) - \tilde{u}(x)\|_\infty, \tag{23}$$

where $u(x)$ denotes the exact solution and $\tilde{u}(x)$ represents the numerical approximation obtained by proposed method.

**Example 4.1.** Consider the following VIE [2]:

$$u(x) = f(x) + \int_0^x \frac{(t^2+1)e^{xt}}{(x^2+t+1)} \cdot \frac{1}{1+u^2(t)} dt, \quad 0 \le x \le 1, \tag{24}$$

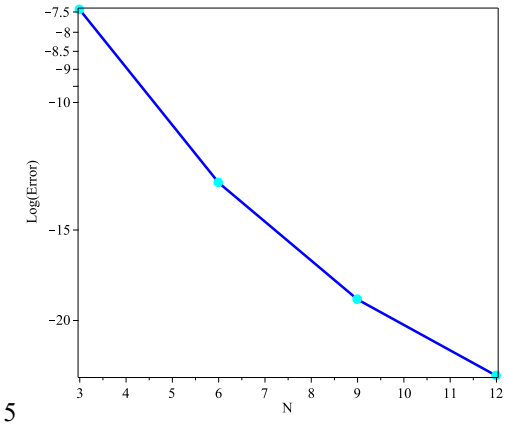
where the function $f(x)$ is chosen such that the exact solution of this integral equation is:

$$u(x) = \ln(x+1). \tag{25}$$

In this example, the initial function is taken as $u^{[0]}(x) = 0$. Table 1 presents the absolute error ($L_\infty$) achieved for different numbers of basis polynomials $N$, comparing the performance of the proposed method with two classical RBF approaches: Gaussians (GAs) and multiquadrics (MQs). The corresponding errors are plotted on a logarithmic scale in Fig. 1, demonstrating the convergence behavior.

Table 1. Comparison of the absolute error ($L_\infty$) between the proposed method and classical radial basis functions (GA and MQ) for various values $N$ in Example 4.1.

| $N$ | Proposed method | GA [2] | MQ [2] |
|---|---|---|---|
| 3 | $5.83 \times 10^{-04}$ | $2.19 \times 10^{-02}$ | $2.46 \times 10^{-02}$ |
| 6 | $2.48 \times 10^{-06}$ | $9.26 \times 10^{-05}$ | $6.43 \times 10^{-04}$ |
| 9 | $7.50 \times 10^{-09}$ | $1.37 \times 10^{-06}$ | $6.01 \times 10^{-05}$ |
| 12 | $4.41 \times 10^{-11}$ | $2.17 \times 10^{-08}$ | $8.02 \times 10^{-06}$ |




5

Fig. 1. Absolute error of the proposed method for various values $N$ in Example 4.1.

## 5. Conclusion

This study introduces a Chebyshev-Picard spectral method designed for the numerical solution of a nonlinear Volterra integral equation. By integrating the Picard iteration scheme with Chebyshev polynomial expansions, the proposed framework establishes a computationally efficient and accurate algorithmic approach. Numerical evaluations conducted on established benchmark problems indicate that the method achieves improved precision and convergence rates compared to conventional iterative techniques.

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# Solving the nonlinear fractional diffusion problem using the high–order compact (HOC) method on unbounded domains

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**Abstract**

In this paper, a high–order compact (HOC) scheme in the spatial dimension, together with the Caputo time–fractional derivative and absorbing boundary conditions (ABCs)[1], is employed to approximate and obtain the solution of the problem. The main advantage of this approach is that in the spatial dimension, due to the use of the compact scheme, the spatial derivatives are implicitly approximated with a smaller stencil while achieving higher accuracy. In the temporal dimension, the use of the Caputo derivative provides a more realistic and consistent implementation that aligns with the nature of the modeled differential equation. The application of this technique in both space and time reduces the overall computational cost. To approximate the nonlinear term $\mathcal{F}(u)$, we adopt the Padé $(1,1)$ approximation. This removes the need for solving nonlinear systems while keeping the overall scheme linear, computationally efficient, and reliable. The proposed numerical method achieves the global accuracy order $\mathcal{O}(\tau^{3-\alpha}+h^4)$. A numerical example further confirms the high accuracy and efficiency of the proposed scheme in comparison with classical second–order methods.

## 1. Introduction

This phenomenon in physics and chemistry is modeled with nonlinear parabolic equations with time-fractional derivatives, but simulating them is challenging. (ABCs) solve the reflection problem in bounded domains by allowing waves to pass through boundaries without reflection, enabling more accurate simulations. Consider the following problem:

$$\begin{cases} {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + \mathcal{F}(u(x,t)) + \mathcal{G}(x,t) \quad x \in \mathbb{R}, \quad 0 < t \le P, \\ u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R}, \\ u(x,t) \to 0, \quad |x| \to \infty, \end{cases} \tag{1}$$

where the initial function $u_0(x)$ has compact support, meaning that it is nonzero only within a finite interval and identically zero everywhere outside that interval. The operator ${}_0^C D_t^\alpha$ denotes the Caputo fractional derivative.

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## 2. Problem Formulation

We first introduce two artificial boundary points on the left and right sides of the domain, denoted by $\Lambda_- := \{x \mid x = I_-\}$ and $\Lambda_+ := \{x \mid x = I_+\}$. The entire domain $\mathbb{R}$ is then partitioned into three regions:

$$\Delta_- := \{x \mid -\infty < x < I_-\}, \qquad \Delta_i := \{x \mid I_- \le x \le I_+\}, \qquad \Delta_+ := \{x \mid I_+ < x < +\infty\},$$

and the exterior region is denoted by $\Delta_e = \Delta_- \cup \Delta_+$. The constants $I_-$ and $I_+$ are chosen such that the initial function $u_0(x)$ is nonzero only inside the interior region $\Delta_i$, and identically zero outside it. The operator form of equation (1) in the region $\Delta_e$ can thus be written as follows:

$$ {}_0^C D_t^\alpha u(x,t) = \mathcal{L}u(x,t) + \mathcal{N}u(x,t), \quad x \in \Delta_e, \tag{2}$$

Here $\mathcal{L}u = u_{xx}$ and $\mathcal{N}u = \mathcal{F}(u) + \mathcal{G}(x,t)$. Considering the linear part of equation (2), we have

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t). \tag{3}$$

The Laplace transform of the Caputo fractional derivative for $0 < \alpha < 1$ is given by [6]:

$$\begin{aligned} \mathcal{L}[{}_0^C D_t^\alpha u(x,t)](\vartheta) &= \vartheta^\alpha \mathcal{L}[u(x,t)](\vartheta) - \vartheta^{\alpha-1}u(x,0) \\ &= \vartheta^\alpha \tilde{u}(x,\vartheta) - \vartheta^{\alpha-1}u(x,0). \end{aligned} \tag{4}$$

Accordingly, we first restrict problem (1) to the exterior region $\Delta_e$, where $\mathcal{F}(u(x,t)) \approx 0$, $\mathcal{G}(x,t) \approx 0$. Now, considering the initial condition $u_0(x) = 0$ for $x \in \Delta_e$, the equation (3) can be rewritten in the Laplace space as follows:

$$\vartheta^\alpha \tilde{u}(x,\vartheta) = \tilde{u}_{xx}(x,\vartheta), \quad \mathrm{Re}(\vartheta) > 0. \tag{5}$$

Equation (5) has a general solution given as follows:

$$\tilde{u}(x,\vartheta) = \mathcal{A}_1(\vartheta)e^{-\sqrt{\vartheta^\alpha}x} + \mathcal{A}_2(\vartheta)e^{\sqrt{\vartheta^\alpha}x}, \tag{6}$$

where $\mathcal{A}_1(\vartheta)$ and $\mathcal{A}_2(\vartheta)$ are unknown parameters. Since $u(x,\vartheta) \to 0$ as $|x| \to \infty$, we obtain:

$$\tilde{u}(x,\vartheta) = \begin{cases} \mathcal{A}_1(\vartheta)e^{-\sqrt{\vartheta^\alpha}x}, & x \in \Delta_+, \\ \mathcal{A}_2(\vartheta)e^{\sqrt{\vartheta^\alpha}x}, & x \in \Delta_-. \end{cases} \tag{7}$$

By differentiating $\tilde{u}(x,\vartheta)$ with respect to $x$, we obtain the following relation:

$$\frac{\partial \tilde{u}(x,\vartheta)}{\partial x} = \begin{cases} -\sqrt{\vartheta^\alpha}\,\tilde{u}(x,\vartheta) = -\vartheta^{\frac{\alpha}{2}}\tilde{u}(x,\vartheta), & x \in \Delta_+, \\ \sqrt{\vartheta^\alpha}\,\tilde{u}(x,\vartheta) = \vartheta^{\frac{\alpha}{2}}\tilde{u}(x,\vartheta), & x \in \Delta_-. \end{cases} \tag{8}$$

By applying the inverse Laplace transform to equation (8), we obtain the following expression:

$$\frac{\partial u(x,t)}{\partial x} = \begin{cases} -\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^t (t-\vartheta)^{-\frac{\alpha}{2}} \frac{\partial u(x,\vartheta)}{\partial \vartheta}\, d\vartheta = -{}_0^C D_t^{\frac{\alpha}{2}} u(x,t), & x \in \Delta_+, \\ \frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^t (t-\vartheta)^{-\frac{\alpha}{2}} \frac{\partial u(x,\vartheta)}{\partial \vartheta}\, d\vartheta = {}_0^C D_t^{\frac{\alpha}{2}} u(x,t), & x \in \Delta_-. \end{cases} \tag{9}$$

By taking the limits $x \to I_+$ and $x \to I_-$, and assuming the smoothness of $u(x,t)$ at $x = I_-$ and $x = I_+$, we obtain the following relation:

$$\begin{cases} \frac{\partial u(I_+,t)}{\partial x} = -{}_0^C D_t^{\frac{\alpha}{2}} u(I_+,t), \\ \frac{\partial u(I_-,t)}{\partial x} = {}_0^C D_t^{\frac{\alpha}{2}} u(I_-,t). \end{cases} \tag{10}$$

At this stage, the exact artificial boundary conditions for problem (1) have been obtained. From equation (10), it is evident that these boundary conditions involve time–fractional derivatives of order $\frac{\alpha}{2}$. In this study, the Padé approximation technique is employed for the nonlinear term $\mathcal{F}(u)$. For $E_N(\vartheta^\alpha)$, we have the following approximation:

$$E_N(\vartheta^\alpha) = \sqrt{\vartheta^\alpha} - \vartheta_0^{\alpha/2}\left(1 - \sum_{n=1}^{N} \frac{a_n(\vartheta_0^\alpha - \vartheta^\alpha)}{\vartheta_0^\alpha - (\vartheta_0^\alpha - \vartheta^\alpha)b_n}\right).$$

The coefficients $a_n$ and $b_n$ are defined as follows:

$$a_n = \frac{2}{2N+1}\sin^2\left(\frac{n\pi}{2N+1}\right), \qquad b_n = \cos^2\left(\frac{n\pi}{2N+1}\right).$$

The error of the above relation has been established in [5], and its exact formula is given by:

$$E_N(\vartheta^\alpha) = 2\sqrt{\vartheta^\alpha}\frac{\gamma^{2N+1}(\vartheta^\alpha)}{1+\gamma^{2N+1}(\vartheta^\alpha)}, \qquad \vartheta^\alpha > 0,$$

where

$$\gamma(\vartheta^\alpha) = \frac{\sqrt{\vartheta^\alpha} - 1}{\sqrt{\vartheta^\alpha} + 1}.$$

Here, only the third–order diagonal Padé approximation $N = 1$ is considered, which is given by

$$\sqrt{\vartheta^\alpha} \approx \frac{\vartheta_0^{\alpha/2}(\vartheta_0^\alpha + 3\vartheta^\alpha)}{3\vartheta_0^\alpha + \vartheta^\alpha}.$$

By substituting the above relation into (10) and performing the calculations, we obtain:

$$\vartheta^\alpha u(x,t) = -(\partial_x \pm 3\vartheta_0^{\alpha/2})^{-1}(3\vartheta_0^\alpha \partial_x \pm \vartheta_0^{3\alpha/2})\, u(x,t),$$

where the $(+)$ sign corresponds to the right boundary condition and the $(-)$ sign corresponds to the left boundary condition. By taking the inverse Laplace transform, we have:

$${}_0^C D_t^\alpha u(x,t) = -(\partial_x \pm 3\vartheta_0^{\alpha/2})^{-1}(3\vartheta_0^\alpha \partial_x \pm \vartheta_0^{3\alpha/2})\, u(x,t).$$

This relation leads to the following approximation:

$$\mathcal{L} \approx \mathcal{L}^{(n)} = -(\partial_x \pm 3\vartheta_0^{\alpha/2})^{-1}(3\vartheta_0^\alpha \partial_x \pm \vartheta_0^{3\alpha/2})\, u(x,t).$$

By substituting this approximation into equation (2), we obtain the exact absorbing boundary conditions for the nonlinear time–fractional parabolic equation in the following form:

$$(\partial_x \pm 3\vartheta_0^{\alpha/2})\,{}_0^C D_t^\alpha u + (3\vartheta_0^\alpha \partial_x \pm \vartheta_0^{3\alpha/2})u = (\partial_x \pm 3\vartheta_0^{\alpha/2})\mathcal{F}(u) + (\partial_x \pm 3\vartheta_0^{\alpha/2})\mathcal{G}(x,t).$$

Finally, problem (1) is reduced to an initial value problem (IVP) on a bounded domain. The parameter $\vartheta_0$ in the absorbing boundary conditions corresponds to the expansion point used in the Padé approximation. The Padé expansion provides an accurate estimate of $\sqrt{\vartheta^\alpha}$ near the expansion point $\vartheta_0$, and the efficiency of the absorbing boundary conditions is not sensitive to the choice of $\vartheta_0$. In the temporal dimension, the Caputo fractional derivative formula is employed for approximating the derivative:

$${}_0^C D_t^\alpha u(t)|_{t=t_k} = \frac{t^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1}\left(\omega_{1,k-j}(u^{j+1} - u^{j-1}) + \omega_{2,k-j}(u^{j+1} - 2u^j + u^{j-1})\right) + O(t^{3-\alpha}) \qquad (11)$$

In formula (11), if $j = 0$, the value $u^{j-1} = u^{-1}$ is defined outside the interval $[0, P]$. In numerical computations, the neighboring values of the function are typically used to approximate $u^{-1}$, such that:

$$u^{-1} = u(0) - tu'(0) + \frac{t^2}{2}u''(0) + O(t^3).$$

For details on the computational procedure, the construction of the fractional–time derivative formula, and the derivation of the weights $\omega_{1,k-j}$ and $\omega_{2,k-j}$ refer to [1, 3].assuming $(n = k - j)$:

$$\begin{cases} \omega_{1,n} = \frac{2-\alpha}{2}\left((n)^{1-\alpha} - (n-1)^{1-\alpha}\right), \\ \omega_{2,n} = (n)^{2-\alpha} - (n-1)^{2-\alpha} - (2-\alpha)(n-1)^{1-\alpha}. \end{cases} \tag{12}$$

The goal is to approximate second-order spatial derivatives with a high-order, compact finite difference method, achieving fourth-order accuracy. Notation: $u''_x$ represents the second-order derivative of $u(x,t)$ at grid point $x_i$, and a general approximation form will be used.

$$Bu''_{i-2} + Au''_{i-1} + u''_i + Au''_{i+1} + Bu''_{i+2} = a_1\left(\frac{u_{i+2} - 2u_i + u_{i-2}}{h^2}\right) + a_2\left(\frac{u_{i+2} - 2u_i + u_{i-2}}{4h^2}\right)$$
$$+ a_3\left(\frac{u_{i+3} - 2u_i + u_{i-3}}{9h^2}\right), \qquad 1 \leqslant i \leqslant N. \tag{13}$$

Coefficients are determined by expanding terms in equation (13) using a Taylor series around $u_i$, substituting into the equation, and simplifying to achieve fourth-order spatial accuracy. which yield $a_1 = \frac{6}{5}, a_2 = 0, a_3 = 0, B = 0, A = \frac{1}{10}$. By substituting these values into the stencil of (13), we arrive at the following expression for the approximation of the spatial derivative:

$$\frac{1}{10}u''_{i+1} + u''_i + \frac{1}{10}u''_{i-1} = \frac{6}{5}\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right), \qquad 1 \leqslant i \leqslant N. \tag{14}$$

The relation $H_1 u''_i = H_2 u_i$ is used to approximate the second derivative $u''_i$, represented as a matrix equation. Derivative Matrix: The derivative matrix $D^{p,1}_{xx}$ is defined as $H_1^{-1}H_2$, where $H_1$ and $H_2$ are coefficient matrices from equation (14). Further details on matrix structure can be found in [2].

$$u''_i = \frac{72}{49}u_{i+1} - \frac{132}{49}u_i + \frac{72}{49}u_{i-1}, \qquad 1 \leqslant i \leqslant N. \tag{15}$$

To ensure stability and convergence, fourth-order spatial accuracy must be maintained even at the artificial boundaries. Boundary Derivatives: Second-order spatial derivatives at the boundaries are calculated using Taylor series expansions (see [4] for details).

$$\begin{cases} u_x(x_0,t) = \frac{17u_0 + 13u_1 + 36u_2}{49h} + O(h^4), \\ u_x(x_M,t) = -\frac{17u_M + 13u_{M-1} + 36u_{M-2}}{49h} + O(h^4). \end{cases} \tag{16}$$

To simplify absorbing boundary conditions specifically, to eliminate mixed time-space derivatives, auxiliary variables $\phi$ and $\varphi$ are introduced. Reformulation Stability, this leads to independent fractional differential equations at the boundaries, resulting in a more structured problem and enabling the design of a stable, analyzable linearized finite difference scheme.

$$\begin{cases} \phi(t) = u_x(l_-,t) - 3\vartheta_0^{\alpha/2}\,u(l_-,t), \\ \varphi(t) = u_x(l_+,t) + 3\vartheta_0^{\alpha/2}\,u(l_+,t). \end{cases} \tag{17}$$

The proposed numerical method in this paper is formulated as follows:

$$ {}_0^C D_t^\alpha u_i^k = \delta_x^2 u_i^k + \mathcal{F}(u_i^{k-1}) + \mathcal{G}_i^k, \qquad 1 \le i \le M-1, \quad 1 \le k \le N, \tag{18} $$

$$ \delta_x u_{1/2}^n = \phi_k + 3\vartheta_0^{\alpha/2} u_0^k + h^2({}_0^C D_t^\alpha u_0^k - \mathcal{F}(u_0^{k-1}) - \mathcal{G}_0^k), \tag{19} $$

$$ \delta_x u_{M-1/2}^n = \varphi_k - 3\vartheta_0^{\alpha/2} u_M^k - h^2({}_0^C D_t^\alpha u_M^k - \mathcal{F}(u_M^{k-1}) - \mathcal{G}_M^k), \tag{20} $$

$$ {}_0^C D_t^\alpha \phi_k = (\mathcal{F}(u_0^{k-1}) - 3\vartheta_0^\alpha)\phi_k + (8\vartheta_0^{3\alpha/2} - 3\vartheta_0^{\alpha/2}\mathcal{F}(u_0^{k-1}))u_0^k + 3\vartheta_0^{\alpha/2}\mathcal{F}(u_0^{k-1}) + 3\vartheta_0^{\alpha/2}\mathcal{G}_0^k, \tag{21} $$

$$ {}_0^C D_t^\alpha \varphi_k = (\mathcal{F}(u_M^{k-1}) - 3\vartheta_0^\alpha)\varphi_k - (8\vartheta_0^{3\alpha/2} - 3\vartheta_0^{\alpha/2}\mathcal{F}(u_M^{k-1}))u_M^k - 3\vartheta_0^{\alpha/2}\mathcal{F}(u_M^{k-1}) - 3\vartheta_0^{\alpha/2}\mathcal{G}_M^k \tag{22} $$

$$ u_i^0 = u_0(x_i), \qquad \phi_0 = \phi(0), \qquad \varphi_0 = \varphi(0) \tag{23} $$

## 3. Numerical Example

**Example 3.1.** Consider the following time–fractional equation:

$$ {}_0^C D_t^\alpha u(x,t) = u_{xx}(x,t) + \mathcal{F}(u(x,t)) + \mathcal{G}(u(x,t)), \quad 0 \le t \le P, \quad x \in \mathbb{R} $$

where the nonlinear term is given by $\mathcal{F}(u) = u^2$, and the source term is of the form

$$ \mathcal{G}(x,t) = \frac{e^{-x^2} 6t^{3-\alpha}}{\Gamma(4-\alpha)} - e^{-x^2}(t^3+1)(4x^2-2) - e^{-2x^2}(t^3+1)^2. $$

is obtained. This problem also has the initial condition $u(x,0) = e^{-x^2}$ and the exact solution $u(x,t) = e^{-x^2}(t^3+1)$. The numerical error is defined as follows where $U_h^k$ denote the numerical approximations of $u(x_h, t_k)$:

$$ E(h,t) = \|e_h^k\|_\infty = \|U_h^k - u(x_h,t_k)\|. $$

The order of convergence for temporal and spatial refinements is calculated as follows:

$$ \text{order}_1 = \log_2 \frac{E(h,t)}{E(h,t/2)}, \quad \text{order}_2 = \log_2 \frac{E(h,t)}{E(h/2,t)}. $$

Table 1. Temporal convergence results for $\alpha = 0.37$ and $\alpha = 0.68$ with $h = 1/2^8$.

| $\alpha$ | $\tau$ | $\|e_h^k\|_\infty$ | $order_1$ | CPU | $\alpha$ | $\tau$ | $\|e_h^k\|_\infty$ | $order_1$ | CPU |
|---|---|---|---|---|---|---|---|---|---|
| 0.37 | 1/10 | 1.943452E-03 | – | 0.03 | 0.68 | 1/10 | 2.342105E-03 | – | 0.04 |
| | 1/20 | 3.010297E-04 | 2.69 | 0.05 | | 1/20 | 4.690874E-04 | 2.32 | 0.06 |
| | 1/40 | 4.862957E-05 | 2.63 | 0.12 | | 1/40 | 9.523211E-05 | 2.30 | 0.14 |
| | 1/80 | 8.149832E-06 | 2.58 | 0.21 | | 1/80 | 1.945412E-05 | 2.29 | 0.23 |
| | 1/160 | 1.273521E-06 | 2.68 | 0.43 | | 1/160 | 3.962811E-06 | 2.30 | 0.45 |
| | 1/320 | 2.048913E-07 | 2.64 | 0.84 | | 1/320 | 7.963214E-07 | 2.32 | 0.87 |

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Table 2. Spatial convergence results for $\alpha = 0.37$ and $\alpha = 0.68$ with $\tau = 1/2^8$.

| $\alpha$ | $h$ | $\|e_h^k\|_\infty$ | $order_2$ | CPU | $\alpha$ | $h$ | $\|e_h^k\|_\infty$ | $order_2$ | CPU |
|---|---|---|---|---|---|---|---|---|---|
| 0.37 | 1/10 | 3.428512E-04 | – | 0.04 | 0.68 | 1/10 | 2.963184E-04 | – | 0.04 |
| | 1/20 | 2.176304E-05 | 3.98 | 0.07 | | 1/20 | 1.861249E-05 | 3.99 | 0.06 |
| | 1/40 | 1.364209E-06 | 3.99 | 0.12 | | 1/40 | 1.165089E-06 | 4.00 | 0.12 |
| | 1/80 | 8.544732E-08 | 4.00 | 0.22 | | 1/80 | 7.289143E-08 | 4.00 | 0.23 |
| | 1/160 | 5.340455E-09 | 4.00 | 0.41 | | 1/160 | 4.556115E-09 | 4.00 | 0.45 |
| | 1/320 | 3.337821E-10 | 4.00 | 0.79 | | 1/320 | 2.855732E-10 | 4.00 | 0.87 |

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# Comparative evaluation of spectral, meshless and neural network methods for solving partial differential equations

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**Abstract**

Solving partial differential equations (PDEs) has always been challenging in various sciences including physics, engineering, and mathematics. Especially when faced with complex geometries or special boundary conditions, common methods such as meshless and spectral methods cannot always provide the best performance. In this study, three numerical methods, namely, meshless methods, discrete spectral methods, and PINNs, have been compared to investigate their performance in solving PDE equations in problems with simple and complex geometries. The aim of this study is to evaluate the accuracy, convergence speed, and efficiency of these methods under different conditions. In addition, the combination of these methods for solving more complex and nonlinear problems has also been investigated. The research results show that discrete spectral methods have higher accuracy in solving linear problems with regular geometries, while RBF and PINNs performed better in more complex and nonlinear problems, especially in irregular geometries. Especially in inverse problems that require estimation of unknown parameters, PINNs have been able to achieve higher accuracy compared to other methods. Also, to verify the performance of meshless methods, the Poisson differential equation was simulated using the RBF method, and the resulting numerical results confirmed the high accuracy and stability of this method in practice.

## 1. Introduction

Partial differential equations are a fundamental building block in describing many natural and engineering phenomena. These equations, which are used to model phenomena such as fluid flows, heat transfer, the motion of structures, and many other physical processes, play a vital role in various sciences [4–6, 8, 9]. Mesh-free methods, especially radial basis function (RBF)-based methods, have attracted much attention due to their ability to solve problems with complex geometries and high flexibility. These methods do not require advanced and precise meshing. This feature is especially important in solving PDE equations on irregular domains and problems with non-uniform and complex geometries [1, 3, 7].

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In recent years, new developments have occurred in the field of numerical methods for solving PDE equations in conjunction with machine learning. In particular, new methods such as Physics-Informed Neural Networks have been introduced that can directly exploit physical data to solve PDE problems. These techniques, which are based on deep neural networks, have the ability to solve complex, nonlinear problems more accurately and faster than traditional numerical methods [2]. This innovative combination of machine learning and numerical methods has led to major changes in scientific and engineering simulations, and its potential for application to inverse problems and complex predictions in the future is very impressive. The main goal of this paper is to provide a comprehensive overview of recent advances in numerical methods for solving PDE equations. In this paper, we attempt to review spectral methods, RBF, as well as novel methods such as PINNs from different aspects. One of the most important points of this article is to review recent developments in the use of machine learning and neural networks to solve PDE problems, which promise more efficient and faster methods for solving complex problems in the near future.
By reviewing recent studies, the goal is to provide a clear and comprehensive view of the current state of the field.

## 2. Mesh-less numerical methods

Now here we present a modern method that is a major competitor to the finite difference method (FDM) and the finite element method (FEM). In particular, we discuss the meshless method that attempts to overcome some of the shortcomings of FDM and FEM. First, FDM and FEM designs are difficult to construct and solve, even in two and three dimensions. Second, they only achieve low-order polynomial accuracy. Third, these methods do not scale easily to high-dimensional problems, making them unsuitable for problems such as multidimensional differentiation problems. Finally, the computational complexity increases exponentially. In contrast, the meshless method does not suffer from these problems. In fact, this method is dimensionally blind. This means that it can be used in high-dimensional problems. Also, its programming is simple and easy to understand. With this aim, we provide an introduction to the meshless method.
First, we consider the one-dimensional heat equation as our model problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ , \qquad 0 < x < L \ , \qquad t > 0 \tag{1}$$

In particular, semi-discretization of equation (1) in time using explicit Euler leads us to a system of ordinary differential equations:

$$\frac{U^{n+1}(x) - U^n(x)}{k} = \frac{d^2 U^n(x)}{dx^2} \ , \qquad 0 < x < L \ , \qquad n = 0 \ , \ 1 \ , \ \dots , N-1 \tag{2}$$

Where, $U^0(x)$ is given.
Or in the form of a differential operator:

$$H_+ U^{n+1}(x) = H_- U^n(x) \ , \qquad 0 < x < L \ , \qquad n = 0 \ , \ 1 \ , \ \dots , N-1 \tag{3}$$

Where the operators are defined as follows:

$$H_+ \equiv 1 \qquad , \ H_- \equiv 1 + k\frac{d^2}{dx^2} \tag{4}$$

As we usually do, we have divided the interval $(0 \ , \ T)$ into $N$ equal subintervals of length $k$. Now we assume that the answer to equation (3) is of the following form:

$$U^n(x) \cong \sum_{j=1}^{J} \lambda_j^n \varphi\left(r_j\right) , \qquad 0 < x < L \tag{5}$$

Where $r_j$ is the Euclidean distance between $x$ and $x_j$: $r_j = \sqrt{\left(x - x_j\right)^2}$. and $\left(x_j\right)_{j=1}^{J}$ are known isolocal points. In particular, assuming the value of $x$ to be a specific local co-point, we have:

$$U^n(x_i) \cong \sum_{j=1}^{J} \lambda_j^n \varphi(r_{ij}) \tag{6}$$

Where, $r_{ij} = \sqrt{(x_i - x_j)^2}$

We approximate the ordinary differential equation (3) at each local co-point by substituting equation (5) as follows [10]:

$$\sum_{j=1}^{J} \lambda_j^{n+1} H_+ \varphi(r_{ij}) = \sum_{j=1}^{J} \lambda_j^n H_- \varphi(r_{ij}) \quad , \qquad 1 \le i \le J \tag{7}$$

Or in matrix form:

$$AU^{n+1} = B^n \quad , \qquad n \ge 0 \tag{8}$$

in which:

$$U^n = (\lambda_1 , \ldots, \lambda_J)^t, \quad A = \left(H_+ \varphi(r_{ij})\right)_{1 \le i,j \le J} \quad , \quad B = (B_1 , \ldots, B_J)^t \tag{9}$$

and

$$B_i^n = \sum_{j=1}^{J} \lambda_j^n H_- \varphi(r_{ij}) \quad , \qquad i = 1 , \ldots, J \tag{10}$$

A matrix solver such as the Gaussian elimination method with partial pivots can then be used to solve the system of equations above.

### *2.1. Radial Basis Function (RBF) Method*

To define radial basis functions, we first define the kernel.

**Definition 2.1.** If Ω is an arbitrary non-zero set, then we call the function **K** : □ × □ → □, where Ω is the kernel (real or complex). A function □ : $\mathbf{R^d} \rightarrow \mathbf{R}$ is called radial if there exists a function of one variable □ : [**0**, ∞) →**R** such that:

$$\square(\mathbf{r}) = \square(\mathbf{X}) \tag{11}$$

So that $r = \|X\|_2$.

### *2.2. Interpolation with Radial Basis Functions (RBF)*

To interpolate the data $x_1 < x_2 < \ldots < x_N$ with values $f_1 , f_2 , \ldots, f_N$ there is a polynomial of degree $N + 1$ which depends on the number of points. Various methods can be used for this interpolation, one of which is the use of radial functions.

Suppose $\Omega \subseteq \mathbb{R}^d$, also given distinct scattered points $X_1 , X_2 , \ldots, X_N \subseteq \Omega$ and $f_j = f(X_j)$ for $1 \le j \le N$, then we have:

$$S(X) \cong \sum_{j=1}^{N} \lambda_j \varphi(r_j) \tag{12}$$

Where $r_j = \|X - X_j\|_2$.

By substituting each point $X_i$ for $1 \le i \le N$ in place of $X$ in $r_j$, we arrive at a system of linear equations as follows [3]:

$$A\lambda = f \tag{13}$$

in which:

$$A = \begin{bmatrix} \varphi(\|X_1 - X_1\|) & \varphi(\|X_1 - X_2\|) & \dots & \varphi(\|X_1 - X_N\|) \\ \varphi(\|X_2 - X_1\|) & \varphi(\|X_2 - X_2\|) & \dots & \varphi(\|X_2 - X_N\|) \\ \vdots & \vdots & & \vdots \\ \varphi(\|X_N - X_1\|) & \varphi(\|X_N - X_2\|) & \dots & \varphi(\|X_N - X_N\|) \end{bmatrix}, \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix}, \quad f = \begin{bmatrix} f(X_1) \\ f(X_2) \\ \vdots \\ f(X_N) \end{bmatrix} \tag{14}$$

By solving the system (13) and obtaining the unknown matrix $\lambda$, $S(X)$ is an interpolating function for the given points and values, but the solvability of this system depends on $\varphi$.

## 3. Modern methods based on PINN machine learning and hybrid methods

In the last decade, with the advancement of machine learning and deep neural networks, methods have emerged which try to be complementary to classical numerical methods.
The most prominent of these is Physics-Informed Neural Networks. In this approach, the neural network is trained so that its functional approximation satisfies the PDE and boundary conditions; as a result, the network approximates the PDE solution without the need for a complex mesh or grid.
Unlike traditional neural networks, PINNs use the physical laws governing the problem to guide the learning process. This is done by defining a compound loss function:

$$L_{total} = \frac{1}{N_f}\sum_{i=1}^{N_f} \left|f(x_i\ ,\ t_i)\right|^2 + \frac{\lambda}{N_b}\sum_{i=1}^{N_b} |u_i - g_i|^2 \tag{15}$$

Here $f$ is the residual of the differential equation, which must tend to zero at the interior points of the domain.

## 4. Spectral methods

The spectral numerical method is one of the most accurate and efficient methods for numerically solving differential equations, which is based on global basis functions such as Chebyshev, Legendre polynomials, or Fourier series. In this method, the unknown function is approximated as a combination of these basis functions with unknown coefficients. The main advantage of the spectral method is its high accuracy and exponential convergence for problems with smooth responses. However, this method has challenges in dealing with discontinuities or problems with complex geometries.

### *4.1. Discrete spectral method*

Discrete spectral methods are one of the most important techniques for solving partial differential equations (PDEs), especially in regular geometric conditions. These methods are usually used for linear problems and PDE equations where simple geometries and standard boundary conditions (Dirichlet or Neumann) exist. In this technique, the problem domain is naturally discretized using specific basis functions, which provides the ability to solve equations accurately and quickly, especially in problems with regular geometries.
In these methods, the discretization of the PDE equation function is performed using basis functions that naturally take into account changes in the domain. One of the most important features of these methods is their high accuracy in approximating the solution function, which can provide much greater accuracy in specific problems compared to mesh-based methods [11]. Discrete spectral methods are particularly effective for problems with simple geometries, such as rectangles and circles. In physics and engineering in particular, these methods are widely used in modeling various phenomena, including heat transfer, fluid flow, and fluid mechanics. In these problems, the PDE equations are usually of the Navier-Stokes type or differential equations for energy transfer and fluid motion.

## 5. Numerical results

In this section, the results of comparing and evaluating various numerical methods for solving partial differential equations (PDEs) will be comprehensively and thoroughly reviewed. The main focus will be on spectral methods, mesh-free methods such as RBF, and novel machine learning combinations such as PINNs. The goal of this section is to scientifically and analytically analyze the strengths and weaknesses of each of these methods so that we can make better choices for various scientific and industrial applications.
Here, we compare the accuracy, convergence speed, computational cost, and various applications of these methods in solving complex problems. These comparisons also include examining the performance of these methods in simple and complex geometric problems, as well as nonlinear and inverse problems.
Also, the norm relative error criterion $\mathbf{L_2}$ is defined as follows:

$$\mathbf{E_{rel}} = \frac{\sqrt{\sum_{i=1}^{N} \left(u_{num,i} - u_{ref,i}\right)^2}}{\sqrt{\sum_{i=1}^{N} \left(u_{ref,i}\right)^2}} \tag{16}$$

This criterion allows the accuracy of methods to be compared independently of the scale of the response values.

### *5.1. Numerical example*

Numerical simulation with the RBF method. In order to practically evaluate the issues raised in this paper, the one-dimensional Poisson differential equation was selected as a case study and solved using the radial basis function method. The governing equation is as follows:

$$\frac{d^2u}{dx^2} = -\pi^2 \sin(\pi x), \quad 0 < x < 1$$

With Dirichlet boundary conditions $u(0) = u(1) = 0$. In this simulation, 40 nodal points and a Gaussian basis function $\varphi(r) = e^{-(\varepsilon r)^2}$ with shape parameter $\varepsilon = 1$ were used.
As can be seen in Figure 1, there is a perfect match between the approximate solution of the RBF method and the exact analytical solution $u(x) = \sin(\pi x)$. The absolute error diagram in Figure 2 shows the error rate in the computational domain is very small and is in order $10^{-7}$.
This very high accuracy, achieved with only 40 points, experimentally proves the main advantage of meshless methods mentioned in the previous sections.

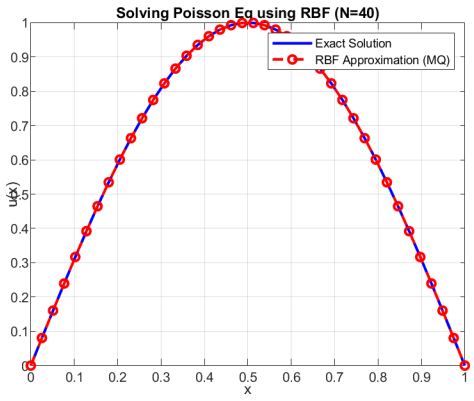


Fig. 1. Comparison of the exact solution and the RBF numerical solution

In addition to examining the error, the convergence behavior of the proposed method was also evaluated. As shown in Figure 3, with increasing the number of nodal points, the error rate decreases uniformly and rapidly. The resulting

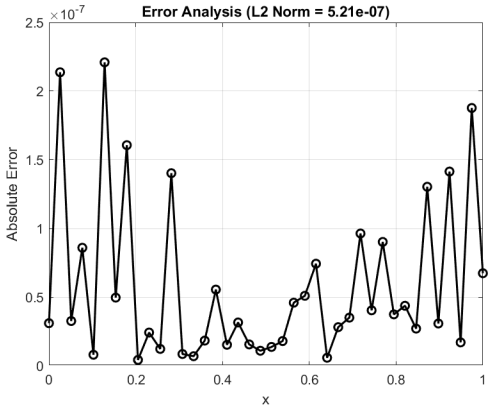


Fig. 2. Absolute error diagram of numerical solution

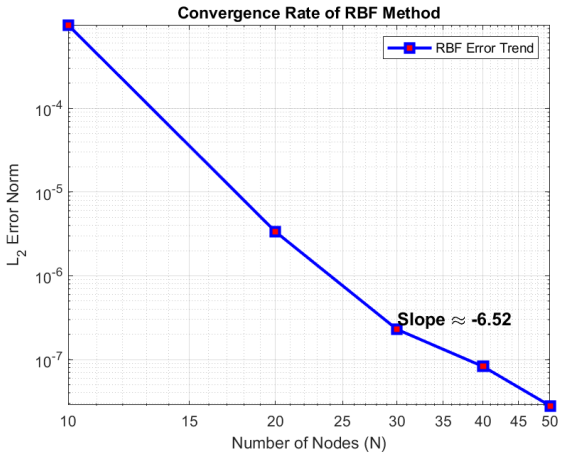


Fig. 3. RBF method convergence rate graph (error reduction with increasing number of points)

logarithmic graph is a straight line with a negative slope (about -6.52); this linear behavior indicates that the RBF method has very high numerical stability and is able to achieve desired accuracy with little computational cost.
The results showed that the combination of RBF and FEM or PINNs can effectively solve more complex problems and provide the best results in many problems with complex boundary conditions and nonlinear geometries. These combinations are able to more accurately model inverse and nonlinear problems in various scientific and engineering fields and provide higher accuracy and speed compared to other numerical methods.

## 6. Conclusion

In this paper, three categories of advanced numerical methods for solving partial differential equations were reviewed and compared: discrete spectral methods, meshless methods based on radial basis functions (RBF), and PINNs. The comparison results showed that each of these methods has superior performance in certain conditions. Discrete spectral methods, due to their exponential convergence and high accuracy, showed the best performance in solving linear problems with regular geometries and simple boundary conditions. In contrast, RBF methods have higher accuracy and stability in problems with complex and irregular geometries due to their high flexibility and lack of meshing. PINNs also perform remarkably well, especially in nonlinear problems and inverse problems that require estimation of unknown parameters. And the ability to incorporate physical laws directly into the learning process proved to be their main advantage. Numerical simulations performed on the one-dimensional Poisson equation also confirmed

these advantages in practice. The results showed that this method with a limited number of points provided better accuracy and a high convergence rate, which proves the stability and efficiency of mesh-free methods. Overall, the combination of these methods can provide more efficient solutions to more complex engineering and scientific problems. Combining hybrid methods to solve multiscale and inverse nonlinear problems with limited data is suggested. Also, extending spectral methods to complex manifolds and optimizing PINNs to reduce training costs and increase convergence speed in real industrial problems should be prioritized for future research.

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# Enlargement of the Minimal Residual BiCRStab Method for the Stein Tensor Equation

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**Abstract**

In this paper, a BiCR-like iterative method-minimal residual biconjugate residual stabilized (MRBiCRStab)- is enlarged for solving the Stein tensor equations. The presented method uses tensor computations with no matricizations included. The reported numerical experiments show the performance of the proposed method.

## 1. Introduction

In this paper, we are concerned with the Stein tensor equations of the form

$$\mathcal{X} - \mathcal{X} \times_1 A_1 \times_2 A_2 \cdots \times_N A_N = \mathcal{C}, \tag{1}$$

where the matrices $A_i \in \mathbb{C}^{I_i \times I_i}$, $(i = 1, 2, \dots, N)$, and the tensor $\mathcal{C} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is known and the tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is unknown. In fact, for a positive integer $N$, an order $N$ tensor $\mathcal{A} = (a_{i_1 \cdots i_N})(1 \le i_j \le I_j, j = 1, \dots, N)$ is a multidimensional array with $I_1 I_2 \cdots I_N$ enteries. $\mathcal{O}$ with all entries zero denotes the zero tensor. We use $k$-mode product $\times_k (k = 1, 2, \dots, N)$ in (1) that will be defined later. In the sequel, some basic definitions which will be used, are given from [2].

**Definition 1.1** (see [2])**.** The operator $\times_k (k = 1, 2, \dots, N)$ represent the $k$-mode product of a tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ with a matrix $A \in \mathbb{C}^{J \times I_k}$ defined as

$$(\mathcal{X} \times_k A)_{i_1 i_2 \cdots i_{k-1} j i_{k+1} \cdots i_N} = \sum_{i_k = 1}^{I_k} x_{i_1 i_2 \cdots i_{k-1} i_k i_{k+1} \cdots i_N} a_{j i_k}.$$

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**Definition 1.2.** For a tensor $\mathcal{A} = (a_{i_1\cdots i_N j_1\cdots j_M}) \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$, let $\mathcal{B} = (b_{i_1\cdots i_M j_1\cdots j_N}) \in \mathbb{C}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ be the conjugate transpose of $\mathcal{A}$, where $b_{i_1\cdots i_M j_1\cdots j_N} = \bar{a}_{j_1\cdots j_N i_1\cdots i_M}$. The tensor $\mathcal{B}$ is denoted by $\mathcal{A}^*$. When $b_{i_1\cdots i_M j_1\cdots j_N} = a_{j_1\cdots j_N i_1\cdots i_M}$, $\mathcal{B}$ is called the transpose of $\mathcal{A}$, denoted by $\mathcal{A}^T$.

**Definition 1.3** (see [2])**.** Let $N$ and $M$ be positive integers. The inner product of two tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$ is defined by

$$\langle \mathcal{X}, \mathcal{Y}\rangle = \sum_{i_1=1}^{I_1}\cdots\sum_{i_N=1}^{I_N}\sum_{j_1=1}^{J_1}\cdots\sum_{j_M=1}^{J_M} x_{i_1\cdots i_N j_1\cdots j_M}\bar{y}_{j_1\cdots j_M i_1\cdots i_N}.$$

So the tensor norm generated by this inner product is $\|\mathcal{X}\| = \sqrt{\langle \mathcal{X}, \mathcal{X}\rangle}$ which is called the tensor Frobenius norm.

**Definition 1.4.** Let $\mathbb{C}^{I_1\times\cdots\times I_N}$ be the set of $N$-mode tensors. Then

$$\mathcal{L} : \mathbb{C}^{I_1\times\cdots\times I_N} \longrightarrow \mathbb{C}^{I_1\times\cdots\times I_N}$$

$$\mathcal{L}(\mathcal{X}) = \mathcal{X} - \mathcal{X}\times_1 A_1 \times_2 A_2 \cdots \times_N A_N,$$

is a linear operator.

According to Definition 1.4, the inear system (1) can be expressed as $\mathcal{L}(\mathcal{X}) = \mathcal{C}$. We use linear operator $\mathcal{L}$ in Definition 1.4 to construct the $k$-th residual tensor of EMRBiCRStab method $\mathcal{R}_k = \mathcal{C} - \mathcal{L}(\mathcal{X}_k)$ where $\mathcal{X}_k$ is $k$-th approximation for the tensor solution $\mathcal{X}$. The next section is devoted to describe tensor form of the BiCRStab method to solve Stein tensor equation (1).

## 2. High order BiCRStab method for Stein tensor equations

In this section we review the High order BiCRStab method [1] for the Stein tensor equation (1). Let $\mathcal{X}_k$ and $\mathcal{P}_k$ be the $k$-th equation solution tensor and $k$-th search direction tensor, respectively. By taking

$$\mathcal{R}_k = \mathcal{C} - \mathcal{L}(\mathcal{X}_k), \quad \mathcal{V}_k = \mathcal{L}(\mathcal{R}_k), \quad \mathcal{Q}_k = \mathcal{L}(\mathcal{P}_k), \mathcal{S}_k = \mathcal{R}_k - \alpha_k\mathcal{Q}_k,$$
$$\mathcal{W}_k = \mathcal{L}(\mathcal{S}_k), \quad \mathcal{T}_k = \mathcal{L}(\mathcal{Q}_k), \quad \mathcal{R}_{k+1} = \mathcal{S}_k - \omega_k\mathcal{W}_k,$$

in step $k$ and using variables the BiCRSTAB algorithm and choosing , arbitrary, the following HOBiCRSTAB algorithm for solving the Stein tensor equations (1) can be presented.

**Algorithm 1.** HOBiCRSTAB algorithm for solving (1)

1. **Input:** Tensors $\mathcal{A}_i, \quad i = 1, 2, \ldots, N, \mathcal{C}$ and initial tensor $\mathcal{X}_0$
2. Compute $\mathcal{R}_0 = \mathcal{C} - \mathcal{L}(\mathcal{X}_0)$, and set $\mathcal{P}_0 = \mathcal{R}_0$
3. Choose arbitrary tensor $\mathcal{R}_0^*$ such that $\langle \mathcal{R}_0^*, \mathcal{R}_0\rangle \neq 0$
4. **for** $k = 0, 1, \ldots$ until $\|\mathcal{R}_k\|$ small enough **do**
5. $\mathcal{V}_k = \mathcal{L}(\mathcal{R}_k), \mathcal{Q}_k = \mathcal{L}(\mathcal{P}_k), \mathcal{T}_k = \mathcal{L}(\mathcal{Q}_k)$
6. $\alpha_k = \dfrac{\langle \mathcal{V}_k, \mathcal{R}_0^*\rangle}{\langle \mathcal{T}_k, \mathcal{R}_0^*\rangle}$
7. $\mathcal{S}_k = \mathcal{R}_k - \alpha_k\mathcal{Q}_k, \mathcal{W}_k = \mathcal{L}(\mathcal{S}_k)$
8. $\omega_k = \dfrac{\langle \mathcal{W}_k, \mathcal{S}_k\rangle}{\langle \mathcal{W}_k, \mathcal{W}_k\rangle}$
9. $\mathcal{X}_{k+1} = \mathcal{X}_k + \alpha_k\mathcal{P}_k + \omega_k\mathcal{S}_k$

10. $\mathcal{R}_{k+1} = \mathcal{S}_k - \omega_k \mathcal{W}_k, \mathcal{V}_{k+1} = \mathcal{L}(\mathcal{R}_{k+1})$
11. $\beta_k = \dfrac{\langle \mathcal{V}_{k+1}, \mathcal{R}_0^* \rangle}{\langle \mathcal{V}_k, \mathcal{R}_0^* \rangle} \left( \dfrac{\alpha_k}{\omega_k} \right)$
12. $\mathcal{P}_{k+1} = \mathcal{R}_{k+1} + \beta_k (\mathcal{P}_k - \omega_k \mathcal{Q}_k)$
13. **end**
14. **Output:** Approximate solution $\mathcal{X}_k$ for (1)

Motivated by the Minimal Residual variant of the BiCGSTAB method in [4] and the Minimal Residual variant of the BiCRSTAB method investigated in [5], we present in next section an algorithm derived from the HOBiCRSTAB algorithm 2 for solving (1), such that the tensor form of the Minimal Residual variant of the HOBiCRSTAB is concluded.

## 3. Enlargement of the Minimal Residual BiCRStab Method

The BiCRSTAB converges faster and sometimes produces much more accurate residual vectors for some practical problems. However, the BiCRSTAB returns infelicitous convergence rate in nonsymmetric indefinite cases. To improve the performance of the BiCRSTAB, we will introduce the minimal residual BiCRSTAB (MRBiCRSTAB), which is a new variant of the BiCRSTAB method [1].
In [3], the authors have developed the tensor form of the BiCR method to solve the Stein tensor equation. We would like to mention that in the BiCRStab method for solving the system $Ax = b$, the product of the linear polynomials are considered in construction of the residual form. In order to overcome the stagnation of convergence of BiCRStab in some discretized dominated problems, quadratic stabilization polynomials are used in the MRBiCRStab algorithm [5]. The MRBiCRStab method computes an approximation $x_{2k}$ whose residual is of the form $r_{2k}^* = Q_{2k}^*(A)P_{2k}(A)r_0$, where $Q_0^*(t) = 1$ and for $k \geq 1 \quad Q_{2k}^*(t) = (1 + w_1 t + w_2 t^2)(1 + w_3 t + w_4 t^2) \cdots (1 + w_{2k-1} t + w_{2k} t^2)$ and the parameters $w_{2k-1}$ and $w_{2k}$ are determined at the $k$-th iteration so that $\|r_{2k}^*\|$ is minimized. Based on this algorithm, we propose the Extended MRBiCRStab algorithm according to the tensor form for solving (1). In fact, the minimal residual BiCRStab that was proposed in [5] for solving large and sparse non-Hermitian linear system of equations. to compute an approximation $\mathcal{X}_{i,k+2} = \bar{\mathcal{X}}_{i,k+2} - w_{k+1}\bar{\mathcal{R}}_{i,k+2} - w_{k+2}\hat{\mathcal{R}}_{i,k+2}$, for $i = 1, 2, \ldots, n$ in $k$-th iteration of the method. The parameters $w_{k+1}$ and $w_{k+2}$ are determined so that $\|\mathcal{R}_{k+2}^*\|$ is minimized. Here we enlarge the minimal residual BiCRStab(EMRBiCRStab) as an iterative method Moreover, the global minimal residual BiCRStab for solving nonsymmetric linear systems with multiple right-hand sides in [1]. In the following, we provide the EMRBiCRStab algorithm for solving (1) motivated by [5]. So,

**Algorithm 2.** The Enlargement MRBiCRStab method for solving (1)

1. **Input:** matrices $A_i$ for $i = 1, 2, \ldots, N$ and tensors $\mathcal{X}_0, \mathcal{C}$.
2. Compute $\mathcal{R}_0^* = \mathcal{C} - \mathcal{L}(\mathcal{X}_0)$
3. Put $\mathcal{P}_0^* = \mathcal{R}_0^*$
4. Choose arbitrary tensors $\tilde{\mathcal{R}}_{i,0}$ such that $\langle \tilde{\mathcal{R}}_0, \mathcal{R}_0^* \rangle \neq 0$
6. For $k = 0, 2, \ldots, 2m, \ldots$ until $\|\mathcal{R}_{k+2}^*\|_*$ small enough Do
7. Compute $\mathcal{V}_k^* = \mathcal{L}(\mathcal{P}_k^*)$
8. $\alpha_k = \frac{\langle \mathcal{L}(\mathcal{R}_k^*), \tilde{\mathcal{R}}_0 \rangle}{\langle \mathcal{L}(\mathcal{V}_k^*), \tilde{\mathcal{R}}_0 \rangle}$
9. $\bar{\mathcal{X}}_{k+1} = \mathcal{X}_k + \alpha_k \mathcal{P}_k^*$
10. $\bar{\mathcal{R}}_{k+1} = \mathcal{R}_k^* - \alpha_k \mathcal{V}_k^*$

11. Compute $\hat{\mathcal{R}}_{k+1} = \mathcal{L}(\bar{\mathcal{R}}_{k+1})$
12. $\beta_k = -\alpha_k \frac{\langle \mathcal{L}(\hat{\mathcal{R}}_{k+1}), \bar{\mathcal{R}}_0 \rangle}{\langle \mathcal{L}(\mathcal{R}_k^*), \bar{\mathcal{R}}_0 \rangle}$
13. $\bar{\mathcal{P}}_{k+1} = \bar{\mathcal{R}}_{k+1} + \beta_k \mathcal{P}_k^*$
14. $\hat{\mathcal{P}}_{k+1} = \hat{\mathcal{R}}_{k+1} + \beta_k \mathcal{V}_k^*$
15. $\hat{\mathcal{P}}_{k+1}^* = \mathcal{L}(\hat{\mathcal{P}}_{k+1})$
16. $\alpha_{k+1} = \frac{\langle \mathcal{L}(\hat{\mathcal{R}}_{k+1}), \bar{\mathcal{R}}_0 \rangle}{\langle \mathcal{L}(\hat{\mathcal{P}}_{k+1}^*), \bar{\mathcal{R}}_0 \rangle}$
17. $\bar{\mathcal{X}}_{k+2} = \bar{\mathcal{X}}_{k+1} + \alpha_{k+1} \bar{\mathcal{P}}_{k+1}$
18. $\bar{\mathcal{R}}_{k+2} = \bar{\mathcal{R}}_{k+1} - \alpha_{k+1} \hat{\mathcal{P}}_{k+1}$
19. $\hat{\mathcal{R}}_{k+2} = \hat{\mathcal{R}}_{k+1} - \alpha_{k+1} \hat{\mathcal{P}}_{k+1}^*$
20. $\hat{\mathcal{R}}_{k+2}^* = \mathcal{L}(\hat{\mathcal{R}}_{k+2})$
21. $w_{k+1} = \frac{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \hat{\mathcal{R}}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle}{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \hat{\mathcal{R}}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle}$
22. $w_{k+2} = \frac{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle}{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \hat{\mathcal{R}}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle}$
23. $\mathcal{X}_{k+2} = \bar{\mathcal{X}}_{k+2} - w_{k+1} \bar{\mathcal{R}}_{k+2} - w_{k+2} \hat{\mathcal{R}}_{k+2}$
24. $\mathcal{R}_{k+2}^* = \bar{\mathcal{R}}_{k+2} + w_{k+1} \hat{\mathcal{R}}_{k+2} + w_{k+2} \hat{\mathcal{R}}_{k+2}^*$
25. $\beta_{k+1} = -\alpha_{k+1} \frac{\langle \mathcal{L}(\hat{\mathcal{R}}_{k+2}^*), \bar{\mathcal{R}}_0 \rangle}{\langle \mathcal{L}(\hat{\mathcal{R}}_{k+1}), \bar{\mathcal{R}}_0 \rangle}$
26. $\mathcal{P}_{k+2}^* = \mathcal{R}_{k+2}^* + \beta_{k+1} (\bar{\mathcal{P}}_{k+1} + w_{k+1} \hat{\mathcal{P}}_{k+1} + w_{k+2} \hat{\mathcal{P}}_{k+1}^*)$
27. EndDo
28. **Output:** solution $\mathcal{X}_k$ for (1)

Since the $\|\mathcal{R}_{k+2}^*\|$ is minimized over two dimensional vector space $\mathbb{R}^2$, it may be expected that the extended MR-BiCRStab converges faster than the BiCRStab in which residual norm is minimized over one dimensional vector space $\mathbb{R}$.

**Theorem 3.1.** *The parameters $w_{k+1}$ and $w_{k+2}$ in the Enlargement of the MRBiCRStab method to minimize $\|\mathcal{R}_{k+2}^*\|$ are computed by the following expressions*

$$w_{k+1} = \frac{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \hat{\mathcal{R}}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle}{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \hat{\mathcal{R}}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle}$$

$$w_{k+2} = \frac{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \bar{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle}{\langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2}^* \rangle - \langle \hat{\mathcal{R}}_{k+2}, \hat{\mathcal{R}}_{k+2} \rangle \langle \hat{\mathcal{R}}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle}$$

*Proof.* The proof is concluded by the following relations

$$\langle \mathcal{R}_{k+2}^*, \hat{\mathcal{R}}_{k+2} \rangle = 0,$$

$$\langle \mathcal{R}_{k+2}^*, \hat{\mathcal{R}}_{k+2}^* \rangle = 0,$$

that are satisfied for residual tensors iterations in tensor production. □

## 4. Numerical example

In this section, we give numerical results of the EMRBiCRStab method to solve the Stein tensor equation arisen from the following partial differential equation

$$u - v\Delta u + c^T\nabla u = f, \quad in\Omega = \quad [0,1]\times[0,1]\times[0,1], \tag{2}$$

$$u = 0 \quad on \quad \partial\Omega. \tag{3}$$

A standard finite difference discretization on equidistant nodes (mesh-size $h = 1/(n+1)$), combined with a second order convergent scheme (Fromm's scheme) for the term$c^T\nabla u$, leads to the linear system with the following coefficient matrix

$$A = I_n \otimes I_n \otimes I_n - A_3 \otimes A_2 \otimes A_1$$

which is corresponding to the following Stein tensor equation.

$$\mathcal{X} - \mathcal{X} \times_1 A_1 \times_2 A_2 \times_3 A_3 = \mathcal{C}. \tag{4}$$

The results show the effectiveness of the proposed algorithm. The matrices $A_i, i = 1,2,3$ have been chosen from [3] of appropriate size, and construct the right-hand side tensor $\mathcal{C}$ such that the exact solution $\mathcal{X}^* \in \mathbb{C}^{m\times n\times l}$ would be tensors with all entries equal to one. The initial guess was taken to be the zero tensors and the stopping criterion $\frac{\|\mathcal{R}_k^*\|_*}{\|\mathcal{R}_0^*\|_*} \leq 10^{-6}$ or Max-iteration $= 2000$ were used. The corresponding convergence histories of EMRBiCRStab method with EBiCR [3] method are depicted in Figure 1.

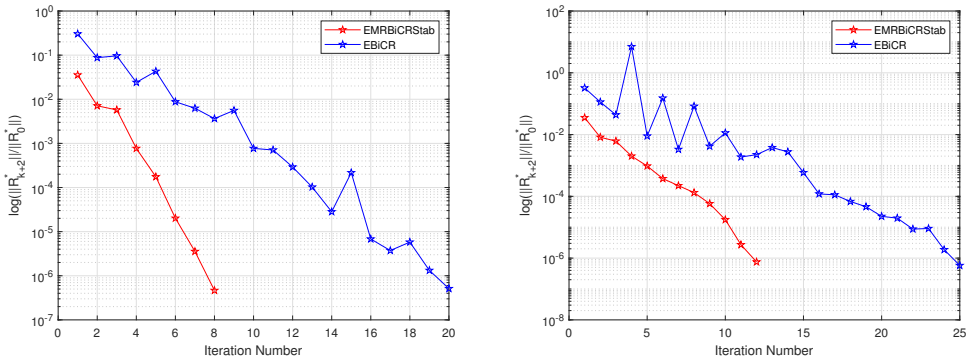


Fig. 1. Convergence histories of REE for $m = 6, n = 4, l = 3$(left) and $m = 8, n = 4, l = 3$(right)

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# A Weighted Block GMRES-Like Method for Low Rank Approximate Solutions to the Sylvester Matrix Equations

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**Abstract**

In this paper, we propose a weighted block GMRES-like iterative method for solving large and sparse Sylvester matrix equations with low rank right hand sides. We will show how to extract low-rank approximations via a block Krylov subspace method equipped with a weighted inner product. We also give some theoretical results.

## 1. Introduction

This paper deals with the low rank approximate solution of the following Sylvester matrix equation

$$AX - XB = EF^T, \tag{1}$$

where $E \in \mathbb{R}^{n\times r}, F \in \mathbb{R}^{s\times r}$ for $r \ll n$ and $r \ll s$. Assume that the matrices $E$ and $F$ are of full rank. For the existence and uniqueness of the solution of (1), we suppose that

$$\sigma(A) \cap \sigma(B) = \emptyset,$$

where $\sigma(A)$ is the spectrum of the matrix $A$. Here, we assume that this condition is satisfied. The matrix equation (1) plays an important role in control and communications theory; see [2] and the references therein. The analytical solution of the matrix equation (1) has been considered by many authors; see [2].

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## 2. Weighted block Arnoldi process

We recall here, the weighted block Arnoldi algorithm [1]. Let $V$ be an $n \times s$ rectangular matrix and consider the block Krylov subspace $\mathcal{K}_m(A,V) = \text{span}\{V, AV, A^2V, \ldots .A^{m-1}V\}$ generated by the columns of the matrices $V, AV, A^2V, \ldots .A^{m-1}V$. For an arbitrary diagonal matrix $D$ whose diagonal entries are all positive, the weighted block Arnoldi algorithm constructs an $D$-orthonormal basis $V_1, V_2, \ldots, V_m$ of the block Krylov subspace $\mathcal{K}_m(A,V_1) = \text{span}\{V, AV, A^2V, \ldots .A^{m-1}V_1\}$. This algorithm for is described as follows.

**Algorithm 1** The Weighted block Arnoldi algorithm

1: Choose a unitary $n \times s$ matrix $V_1$
2: **for** $j = 1$ to $m$ **do**
3: $W_j = AV_j$
4: **for** $i = 1$ to $j$ **do**
5: $H_{i,j} = V_i^T D W_j$
6: $W_j = W_j - V_i H_{i,j}$
7: **end for**
8: Compute $D$-weighted $QR$ decomposition of $W_j$ as $Q_j R_j = W_j$, where $Q_j$ and $R_j$ are of dimension $n \times s$ and $s \times s$, respectively.
9: Set $V_{j+1} = Q_j$ and $H_{j+1,j} = R_j$.
10: **end for**

The blocks $V_1, V_2, \ldots, V_m$ constructed by Algorithm 1 have their columns mutually $D$-orthogonal provided that the upper triangular matrices $H_{j,j+1}$ are of maximum rank and this will be assumed in this paper. If $H_{j,j+1} = O_{s\times s}$ then $\mathcal{K}_j$ is invariant under A, and the exact solution of (1) ia in $\mathcal{K}_j$.
Note that the columns of the matrix $\mathbb{V}_m = [V_1, V_2, \ldots, V_m]$ form an $D$-orthonormal basis of the block Krylov subspace $\mathcal{K}_m(A,V_1) = \text{span}\{V_1, AV_1, A^2V_1, \ldots .A^{m-1}V_1\}$.
Let $\tilde{\mathbb{H}}_m$ denote the $(ms + s) \times ms$ block upper Hessenberg matrix whose nonzero $(i,j)$-th block $H_{i,j}$ are defined by Algorithm 1. The sub-matrices $H_{i,j}$ for $1 \le i \le j$ are dense and $H_{j+1,j}$ are upper triangular. Let $\mathbb{H}_m$ be the $ms \times ms$ matrix obtained from $\tilde{\mathbb{H}}_m$ be removing the last $s$ rows.
From the block Arnoldi algorithm we can deduce the following relations

$$A\mathbb{V}_m = \mathbb{V}_{m+1}\tilde{\mathbb{H}}_m, \tag{2}$$

$$A\mathbb{V}_m = \mathbb{V}_m\mathbb{H}_m + V_{m+1}H_{m+1,m}E_m^T, \tag{3}$$

$$\mathbb{V}_m^T DA\mathbb{V}_m = \mathbb{H}_m, \tag{4}$$

$$\mathbb{V}_m^T D\mathbb{V}_m = I_{ms}, \tag{5}$$

where $E_m$ is the last $s$ columns of the identity matrix $I_{ms}$.

## 3. Weighted Block GMRES-like method for low-rank approximate solutions

Let $\mathcal{K}_m(A,E)$ and $\mathcal{K}_m(B^T,F)$ be the block Krylov subspaces associated with $(A,E)$ and $(B^T,F)$, respectively. Consider the $D$-weighted $QR$ decompositions $E = V_{1,A}U_1$ and $F = V_{1,B}U_2$ and apply the $D$-weighted block Arnoldi process Algorithm 1 starting with $V_{1,A}$ and $V_{1,B}$. We obtain two block $D$-orthonormal basis $\{V_{1,A}, V_{2,A}, \ldots, V_{m,A}\}$ and $\{V_{1,B}, V_{2,B}, \ldots, V_{m,B}\}$ of the block Krylov subspaces $\mathcal{K}_m(A,E)$ and $\mathcal{K}_m(B^T,F)$, respectively. We denote by $\mathbb{H}_{m,A}$ and $\mathbb{H}_{m,B}$ the block upper Hessenberg matrices given by

$$\mathbb{H}_{m,A} = \mathbb{V}_{m,A}^T DA\mathbb{V}_{m,A}, \qquad \text{and} \qquad \mathbb{H}_{m,B} = \mathbb{V}_{m,B}^T DB^T\mathbb{V}_{m,B}, \tag{6}$$

where $\mathbb{V}_{m,A} = [V_{1,A}, V_{2,A}, \ldots, V_{m,A}]$, $\mathbb{V}_{m,B} = [V_{1,B}, V_{2,B}, \ldots, V_{m,B}]$ and $\mathbb{H}_{m,A} = [H_{i,j}^A]_{i,j}$.
We will consider the weighted block GMRES-like approximations to the solution of (1) that have the form

$$X_m = \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T, \tag{7}$$

where $Y_m \in \mathbb{R}^{mr\times mr}$ is obtained so the following minimization problem is satisfied

$$\|EF^T - AX_m + X_mB\|_D = \min_{Y\in\mathbb{R}^{ms\times ms}} \|EF^T - A\mathbb{V}_{m,A}Y\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y\mathbb{V}_{m,B}^TB\|_D, \tag{8}$$

where the matrix $D$-norm $\|\cdot\|_D$ is defined as $\|A\|_D = \sqrt{\langle A,A\rangle_D}$ for a matrix $A$. Here $\langle A,B\rangle_D$ denotes $D$-weighted inner product $\langle A,B\rangle_D = \mathrm{tr}(A^TDB)$ and $\mathrm{tr}(X)$ is the trace of a matrix $X$. Observe that the range of $X_m$ is included in the subspace $\mathcal{K}_m(A,E)$ and its kernel contains the orthogonal to $\mathcal{K}_m(B^T,F)$. Note also that $\mathrm{rank}(X_m) \le mr$.
Let $\mathcal{L}_m(A,B)$ be the subspace of matrices having the form $\mathbb{V}_{m,A}Y\mathbb{V}_{m,B}^T$ , for any $Y \in \mathbb{R}^{mr\times mr}$. The approximation $X_m$ is defined by $X_m \in \mathcal{L}_m(A,B)$ along with the minimal residual norm condition (8).
Using the relations (2) satisfied by Algorithm 1, the residual $R_m$ can be written as

$$R_m = EF^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^TB.$$

Then using the relations $A\mathbb{V}_{m,A} = \mathbb{V}_{m+1,A}\tilde{\mathbb{H}}_{m,A}$ and $B^T\mathbb{V}_{m,B} = \mathbb{V}_{m+1,B}\tilde{\mathbb{H}}_{m,B}$ we obtain the following expression

$$R_m = \mathbb{V}_{m+1,A}\begin{pmatrix} \mathbb{H}_{m,A}Y_m - Y_m\mathbb{H}_{m,B}^T - \tilde{C}_1\tilde{C}_2^T & Y_mE_m(H_{m+1,m}^B)^T \\ H_{m+1,m}^AE_m^TY_m & O \end{pmatrix}\mathbb{V}_{m+1,B}^T. \tag{9}$$

Then, using (9) and the fact that $\mathbb{V}_{m,A}$ and $\mathbb{V}_{m,B}$ are $D$-weighted orthonormal matrices, we get

$$\|R_m\|_D^2 = \|\mathbb{H}_{m,A}Y_m - Y_m\mathbb{H}_{m,B}^T - \tilde{C}_1\tilde{C}_2^T\|_D^2 + \|H_{m+1,m}^AE_m^TY_m\|_D^2 + \|H_{m+1,m}^BE_m^TY_m^T\|_D^2, \tag{10}$$

in which $\tilde{C}_1 = \mathbb{V}_{m,A}^TE = \tilde{E}_1U_1$ and $\tilde{C}_2 = \mathbb{V}_{m,B}^TF = \tilde{E}_1U_2$, where $\tilde{E}_1$ is the $mr\times r$ matrix whose upper $r\times r$ principle block is the identity matrix $I_r$.
In the next, we will show to solve this the minimization problem (8). We first give the following result.

**Proposition 3.1.** *The minimization problem* (8) *is equivalent to the new problem*

$$\begin{aligned}&\mathbb{H}_{m,A}^T(\mathbb{H}_{m,A}Y_m - Y_m\mathbb{H}_{m,B}^T - \tilde{C}_1\tilde{C}_2^T) - (\mathbb{H}_{m,A}Y_m - Y_m\mathbb{H}_{m,B}^T - \tilde{C}_1\tilde{C}_2^T)\mathbb{H}_{m,B}\\ &+E_m(H_{m+1,m}^A)^TH_{m+1,m}^AE_m^TY_m + E_m(H_{m+1,m}^B)^TH_{m+1,m}^BE_m^TY_m^T = 0.\end{aligned} \tag{11}$$

*Also, the residual norm satisfies*

$$\|R_m\|_D^2 = trace[\tilde{C}_2\tilde{C}_1^T(-\mathbb{H}_{m,A}Y_m + Y_m\mathbb{H}_{m,B}^T + \tilde{C}_1\tilde{C}_2^T)]. \tag{12}$$

*Proof.* From the relation (10), the Frobenius norm of the residual can be expressed as

$$\|R_m\|_D^2 = \|\Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T\|_D^2 + \|\Psi_m(Y_m)\|_D^2 + \|\Theta_m(Y_m)\|_D^2, \tag{13}$$

where the linear operators $\Phi_m, \Psi_m$ and $\Theta_m$ are defined as follows: For $Y \in \mathbb{R}^{ms\times ms}$,

$$\begin{aligned}\Phi_m(Y) &= \mathbb{H}_{m,A}Y - Y\mathbb{H}_{m,B}^T,\\ \Psi_m(Y) &= H_{m+1,m}^AE_m^TY,\end{aligned}$$

and

$$\Theta_m(Y) = H_{m+1,m}^BE_m^TY^T.$$

Therefore, $Y_m$ minimize the residual $D$-norm if and only if it satisfies the following linear matrix equation

$$\Phi_m^T(\Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T) + \Psi_m^T(\Psi_m(Y_m)) + \Theta_m^T(\Theta_m(Y_m)) = 0, \tag{14}$$

where the linear operators $\Phi_m^T, \Psi_m^T$ and $\Theta_m^T$ are the transpose of the operators $\Phi_m, \Psi_m$ and $\Theta_m$, respectively. Consequently, the linear matrix equation (14) is also rewritten as

$$\begin{aligned}&\mathbb{H}_{m,A}^T(\mathbb{H}_{m,A}Y_m - Y_m\mathbb{H}_{m,B}^T - \tilde{C}_1\tilde{C}_2^T) - (\mathbb{H}_{m,A}Y_m - Y_m\mathbb{H}_{m,B}^T - \tilde{C}_1\tilde{C}_2^T)\mathbb{H}_{m,B}\\ &+E_m(H_{m+1,m}^A)^TH_{m+1,m}^AE_m^TY_m + E_m(H_{m+1,m}^B)^TH_{m+1,m}^BE_m^TY_m^T = 0\end{aligned}$$

The residual norm is then given by

$$\begin{aligned}\|R_m\|_D^2 &= \langle \Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T, \Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T\rangle_D + \langle \Psi_m(Y_m), \Psi_m(Y_m)\rangle_D + \langle \Theta_m(Y_m), \Theta_m(Y_m)\rangle_D,\\ &= -\langle \Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T, \tilde{C}_1\tilde{C}_2^T\rangle_D + \langle \Phi_m^T(\Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T), Y_m\rangle_D\\ &\quad + \langle \Psi_m^T(\Psi_m(Y_m)), Y_m\rangle_D + \langle \Theta_m^T(\Theta_m(Y_m)), Y_m\rangle_D,\\ &= -\langle \Phi_m(Y_m) - \tilde{C}_1\tilde{C}_2^T, \tilde{C}_1\tilde{C}_2^T\rangle_D + 0 = \mathrm{trace}[\tilde{C}_2\tilde{C}_1^T(-\mathbb{H}_{m,A}Y_m + Y_m\mathbb{H}_{m,B}^T + \tilde{C}_1\tilde{C}_2^T)],\end{aligned}$$

which ends the proof. □

The residual norm in (12) provides a stopping criterion in the implementation of the proposed method. It will allows us to test the convergence without having to compute explicitly the residual.
The following result shows that the solution $Y_m$ of the generalized transpose-Sylvester matrix equation (11) exists and is unique provided that the matrices $\mathbb{H}_{m,A}$ and $\mathbb{H}_{m,B}$ have no eigenvalue in common.

**Proposition 3.2.** *Assume that $m$ step of the weighted block Arnoldi Algorithm 1 have been run and that $\sigma(\mathbb{H}_{m,A}) \cap \sigma(\mathbb{H}_{m,B}) = \emptyset$. Then the solution of the problem (11) exists and is unique.*

*Proof.* Let $\mathcal{A}_{m,1} = I_{rm} \otimes \mathbb{H}_{m,A} - \mathbb{H}_{m,B} \otimes I_{rm}$, $\mathcal{A}_{m,2} = I_{rm} \otimes (H_{m+1,m}^A E_m^T)$ and $\mathcal{A}_{m,3} = (H_{m+1,m}^B E_m^T) \otimes I_{rm}$ denote the associated matrices of the linear operators $\Phi_m^T$, $\Psi_m^T$ and $\Theta_m^T$, respectively. We here set

$$\mathcal{A}_m = \begin{pmatrix} \mathcal{A}_{m,1} \\ \mathcal{A}_{m,3} \\ \mathcal{A}_{m,2} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} \tilde{C}_1\tilde{C}_2^T \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

Then the problem (8) is equivalent to

$$\min \|\mathcal{A}_m \mathrm{vec}(Y_m) - \mathrm{vec}(\tilde{C})\|_D, \tag{16}$$

where $\|x\|_D^2 = (Dx, x)$ for a vector $x \in \mathbb{R}^n$. Therefore, as the matrices $\mathbb{H}_{m,A}$ and $\mathbb{H}_{m,B}$ have no eigenvalue in common, the matrix $\mathcal{A}_{m,1}$ is of full rank, and so $\mathcal{A}_m$ is also of full rank. We immediately deduce that the minimization problem (16) has a unique solution. □

**Theorem 3.3.** *Assume that $m$ step of the weighted block Arnoldi Algorithm 1 have been run and that $\sigma(\mathbb{H}_{m,A}) \cap \sigma(\mathbb{H}_{m,A}) = \emptyset$. Then the corresponding residual $R_m$ satisfies the Galerkin condition*

$$\mathbb{V}_{m,A}^T(A^T D R_m - R_m D B^T)\mathbb{V}_{m,B} = 0. \tag{17}$$

*Proof.* Let $X_m$ be the low rank approximate solution of ? and, so the corresponding residual $R_m$ is given as follows

$$R_m = EF^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B.$$

This can be used in the left side of (17) and

$$\begin{aligned}\mathbb{V}_{m,A}^T(A^T D R_m - R_m D B^T)\mathbb{V}_{m,B} &= \mathbb{V}_{m,A}^T(A^T D(EF^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B)\\ &\quad - (EF^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B)DB^T)\mathbb{V}_{m,B},\\ &= (DA\mathbb{V}_{m,A})^T(V_{1,A}U_1(V_{1,B}U_2)^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B)\\ &\quad - \mathbb{V}_{m,A}^T(V_{1,A}U_1(V_{1,B}U_2)^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B)(DB^T\mathbb{V}_{m,B}),\\ &= (\mathbb{V}_{m,A}\mathbb{H}_{m,A} + V_{m+1,A}H_{m+1,m}^A E_m)^T(V_{1,A}U_1(V_{1,B}U_2)^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B)\\ &\quad - \mathbb{V}_{m,A}^T(V_{1,A}U_1(V_{1,B}U_2)^T - A\mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T + \mathbb{V}_{m,A}Y_m\mathbb{V}_{m,B}^T B)(\mathbb{V}_{m,B}\mathbb{H}_{m,B} + V_{m+1,B}H_{m+1,m}^B E_m).\end{aligned}$$

Since the matrices $\mathbb{V}_{m,A}$ and $\mathbb{V}_{m,B}$ are $D$-orthogonal, then we have

$$\begin{aligned}\mathbb{V}_{m,A}^T(A^T D R_m - R_m D B^T)\mathbb{V}_{m,B} &= \mathbb{H}_{m,A}^T(\tilde{C}_1\tilde{C}_2^T - \mathbb{H}_{m,A}Y_m + Y_m\mathbb{H}_{m,B}^T) - (\tilde{C}_1\tilde{C}_2^T - (\mathbb{H}_{m,A}Y_m + Y_m\mathbb{H}_{m,B}^T)\mathbb{H}_{m,B}\\ &\quad - E_m(H_{m+1,m}^A)^T H_{m+1,m}^A E_m^T Y_m - E_m(H_{m+1,m}^B)^T H_{m+1,m}^B E_m^T Y_m^T = 0.\end{aligned}$$

The last equality is concluded from equation (11). The proof is here completed. □

The restarted weighted block GMRES-like method is summarized as follows

**Algorithm 2** The Weighted Block GMRES-like Algorithm

1: Compute $D$-weighted QR decompositions $E = V_{1,A}U_1$ and $F = V_{1,B}U_2$.
2: Generate the Arnoldi blocks $V_{1,A}, V_{2,A}, \dots, V_{m,A}$ and $V_{1,B}, V_{2,B}, \dots, V_{m,B}$, and matrices $\mathbb{H}_{m,A}$ and $\mathbb{H}_{m,B}$ using Algorithm 1.
3: Solve the transpose-Sylvester matrix equation (8) for $Y_m$ using the $D$-weighted Gl-CG method and compute the approximate solution $X_m = \mathbb{V}_{m,A} Y_m \mathbb{V}_{m,B}^T$.
4: Compute $\|R_m\|_D$ based on (12).
5: Test for convergence; if $\|R_m\|_D < tol$ then STOP ($tol$ is a chosen tolerance).

For small values, the computation of $Y_m$ will be achieved by solving the ? least squares problem (16).
For large values of , we will establish now how to compute $Y_m$ by solving the minimization problem (8) which is equivalent to solve the generalized transpose-Sylvester matrix equation (11).
Let $\mathcal{C}$ denote the linear operator $\mathcal{C} : \mathbb{R}^{mr\times mr} \to \mathbb{R}^{mr\times mr}$ and define

$$\begin{aligned}\mathcal{C}(Y) = {} & \mathbb{H}_{m,A}^T(\mathbb{H}_{m,A}Y - Y\mathbb{H}_{m,B}^T) - (\mathbb{H}_{m,A}Y - Y\mathbb{H}_{m,B}^T)\mathbb{H}_{m,B} \\ & + E_m(H_{m+1,m}^A)^T H_{m+1,m}^A E_m^T Y + E_m(H_{m+1,m}^B)^T H_{m+1,m}^B E_m^T Y^T.\end{aligned}$$

The linear operator $\mathcal{C}$ is symmetric and positive definite and $Y_m$ is the solution of the problem

$$\mathcal{C}(Y) = Z, \tag{18}$$

where $Z = -\mathbb{H}_{m,A}^T \tilde{C}_1 \tilde{C}_2^T + \tilde{C}_1 \tilde{C}_2^T \mathbb{H}_{m,B}^T$.
To solve (18), we are used the $D$-weighted global conjugate gradient (Weighted Gl-CG) algorithm which is derived from the conjugate gradient applied to (18) in the subspace of matrices od dimension $mr \times mr$ with the $D$-weighted Frobnius product $\langle \cdot, \cdot \rangle_D$. The algorithm is as follows.

**Algorithm 3** Wighted Gl-CG alorithm

1: Choose $mr \times mr$ matrix $Y_0$ and compute $\tilde{R}_0 = Z - \mathcal{C}(Y_0)$, $P_0 = \tilde{R}_0$
2: **for** $j = 0$ to $j_{\max}$ **do**
3: $\alpha_j = \langle \tilde{R}_j, \tilde{R}_j \rangle_D / \langle \mathcal{C}(P_j), P_j \rangle_D$
4: $Y_{j+1} = Y_j + \alpha_j P_j$
5: $\tilde{R}_{j+1} = \tilde{R}_j - \alpha_j \mathcal{C}(P_j)$
6: $\beta_j = \langle \tilde{R}_{j+1}, \tilde{R}_{j+1} \rangle_D / \langle \tilde{R}_j, \tilde{R}_j \rangle_D$
7: $P_{j+1} = \tilde{R}_{j+1} + \beta_j P_j$
8: **end for**

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# Numerical and Comparative Analysis of Hybrid Methods in Constrained Optimization: A Case Study of GWO-JAYA Enhanced with Line Search

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**Abstract**

This paper introduces a novel hybrid optimization framework named HALSGWJA, designed to address complex constrained engineering design problems. The algorithm synergistically integrates the Grey Wolf Optimizer (GWO) with the JAYA search scheme, enhanced by an **Approximate Line Search (ALS)** mechanism that utilizes gradient-like information to guide the search process. The primary goal is to achieve a significant reduction in the number of costly function evaluations required in engineering simulations while maintaining high solution accuracy.

The performance of the proposed HALSGWJA algorithm is rigorously evaluated using 20 challenging benchmarks, including the CEC2020 suite and a highly nonlinear **vehicle crashworthiness design** problem. Comparative numerical results demonstrate that HALSGWJA outperforms 40 state-of-the-art algorithms in terms of solution precision and stability. A key achievement is the **reduction of up to 90%** in the number of function evaluations needed to converge to the optimum in certain cases. This makes HALSGWJA a highly efficient tool for large-scale optimization problems where computational cost is a critical constraint.

## 1. Introduction

Optimization challenges in mechanical engineering often involve intricate, non-linear objective functions and a multitude of constraints that define non-convex design spaces. To address these complexities, metaheuristic optimization algorithms (MHOAs) have become indispensable due to their ability to navigate global search spaces without requiring gradient information [5]. However, traditional MHOAs frequently encounter difficulties such as slow convergence rates and a tendency to get trapped in local optima. Among these, the Grey Wolf Optimizer (GWO) and the JAYA algorithm have gained prominence for their effectiveness. Hybridizing these methods allows for a synergistic balance between exploration and exploitation, leveraging the leadership-driven search of GWO alongside the parameter-less simplicity of JAYA to improve overall search performance [4].

In this research, a sophisticated hybrid framework named HALSGWJA (Hybrid Approximate Line Search Grey Wolf JAYA) is proposed to overcome the limitations of existing metaheuristics in real-world engineering scenarios[9]. The

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core innovation of HALSGWJA lies in the integration of an approximate line search mechanism that utilizes local gradient-like information to generate high-quality trial solutions. This approach enables the algorithm to adaptively switch between global exploration and local exploitation, ensuring a more efficient descent toward the global optimum. The efficacy of the proposed method is rigorously evaluated using the CEC2020 mechanical engineering benchmarks and a high-dimensional vehicle crashworthiness problem, demonstrating its superiority in terms of convergence speed and solution accuracy compared to state-of-the-art optimizers [4].

## 2. Methodology of the HALSGWJA Algorithm

Practical mechanical engineering optimization aims to minimize or maximize an objective function $W(\vec{X})$ subject to various inequality and equality constraints. These problems are often non-linear and non-convex, making them difficult for classical methods. The HALSGWJA algorithm addresses these challenges by hybridizing the Grey Wolf Optimizer (GWO) and the JAYA algorithm, further enhanced with an approximate line search technique to improve convergence efficiency.

The optimization process begins with the initialization of a population consisting of $N_{POP}$ candidate solutions, referred to as "wolves." Each design variable is randomly generated within its specified lower ($x^L$) and upper ($x^U$) bounds using the following relationship:

$$x_j^i = x_j^L + \rho_j^i (x_j^U - x_j^L) \tag{1}$$

where $\rho_j^i$ is a random number uniformly distributed between 0 and 1.

To handle design constraints, the algorithm employs a penalty function approach. The original objective function is transformed into a penalized function $W_p$ by adding a penalty term proportional to the sum of squared constraint violations $\psi$:

$$W_p(\vec{X}) = W(\vec{X}) + p \cdot \psi \tag{2}$$

The population is then ranked based on $W_p$ values to identify the best ($\vec{X}_{best}$) and worst ($\vec{X}_{worst}$) performers in the current iteration.

In the GWO phase, the social hierarchy is modeled using the top three candidates: $\alpha$, $\beta$, and $\delta$. The positions of other wolves are updated based on their distance from these leaders using coefficient vectors $\vec{A}$ and $\vec{C}$. The encircling mechanism is mathematically represented as:

$$\vec{D} = |\vec{C} \cdot \vec{X}_p| - \vec{X}_{it}, \quad \vec{X}_{it+1} = \vec{X}_p - \vec{A} \cdot \vec{D} \tag{3}$$

where $\vec{X}_p$ represents the position of the leader and $\vec{X}_{it}$ is the current solution [7].

A core innovation of HALSGWJA is the inclusion of approximate gradient information to refine the trial designs generated by GWO. Instead of calculating exact derivatives, the algorithm estimates the descent direction based on the variation of the penalized function relative to the distance between solutions. This directional indicator, denoted as $\gamma^{(i,k)}$, helps the algorithm identify paths toward high-quality regions.

TThe trial positions are further adjusted using a line search mechanism that incorporates three distinct movement components. This strategy combines the direction toward the global best, the direction of the fastest improvement, and the avoidance of the worst solutions:

$$\vec{X}_{i,tr} = \vec{X}_i + \tau_{\text{best}} \cdot (\vec{X}_{\text{best}} - \vec{X}_i) + \tau_{\text{FST}} \cdot (\vec{X}_{\text{FST, better}} - \vec{X}_i) - \tau_{\text{wrs}} \cdot (\vec{X}_{\text{FST, worse}} - \vec{X}_i) \tag{4}$$

where $\vec{X}_i$ is the current position of the $i$-th wolf, $\vec{X}_{\text{best}}$ is the position of the global best solution found so far, $\vec{X}_{\text{FST, better}}$ and $\vec{X}_{\text{FST, worse}}$ are the positions of particles with the fastest improving and deteriorating objective values in the recent iteration, respectively. The coefficients $\tau_{\text{best}}$, $\tau_{\text{FST}}$, and $\tau_{\text{wrs}}$ are adaptive step sizes calculated as $\tau = \eta \cdot \left|\frac{\Delta W_p}{\Delta \vec{X}}\right|$, where $\eta$ is a random number in [0,1] and $\frac{\Delta W_p}{\Delta \vec{X}}$ denotes the approximate gradient magnitude between the corresponding solutions. This formulation ensures that the trial designs follow a descent path.

To manage the transition between global exploration and local exploitation, HALSGWJA utilizes an adaptive threshold $\rho_w$. This threshold is dynamically calculated based on the relative quality of the current population and the trial

population, as well as the average approximate gradients:

$$\rho_w = \min\left(\frac{W_p(\vec{X}_{\delta,tr})}{W_p(\vec{X}_{\text{best},tr})}, \frac{W_p(\vec{X}_{\delta})}{W_p(\vec{X}_{\text{best}})}, \frac{1}{N}\sum_{k=1}^{N}|\gamma^{(i,k)}|\right) \tag{5}$$

where $\vec{X}_{\delta}$ and $\vec{X}_{\delta,tr}$ are the positions of the $\delta$-wolf (the third-best solution) in the current and trial populations, respectively. The term $\frac{1}{N}\sum|\gamma^{(i,k)}|$ represents the average magnitude of the approximate gradient indicators $\gamma^{(i,k)}$ (defined preceding Eq.(4)) across all population members and dimensions. A smaller $\rho_w$ indicates higher consensus on a promising descent direction, triggering the intensified JAYA exploitation phase. This allows the algorithm to adjust its search intensity based on the landscape of the objective function [1].

When a trial solution meets the quality criteria defined by $\rho_w$, the JAYA exploitation phase is triggered. In this stage, the trial design is further refined by moving toward the best solution in the combined population and moving away from the worst one:

$$\vec{X}_{i,tr}^{(\text{new})} = \vec{X}_{i,tr} + \lambda_1 \cdot (\vec{X}_{\text{best,JAYA}} - \vec{X}_{i,tr}) - \lambda_2 \cdot (\vec{X}_{\text{worst,JAYA}} - \vec{X}_{i,tr}) \tag{6}$$

where $\vec{X}_{\text{best,JAYA}}$ and $\vec{X}_{\text{worst,JAYA}}$ are the positions of the best and worst solutions found in the current iteration after merging the original and trial populations. The coefficients $\lambda_1$ and $\lambda_2$ are random vectors with components uniformly distributed in $[0, 1]$, generated independently for each design variable.

To prevent stagnation and avoid local optima, the algorithm implements a ”mirror strategy” for the leaders $(\alpha, \beta, \delta)$. If no improvement is found after the update, their positions are reflected relative to the variable bounds. This maintains population diversity and increases the probability of escaping premature convergence in complex search spaces [11].

The optimization process terminates when a specific convergence criterion is met, rather than relying solely on a maximum iteration count. The algorithm monitors the normalized standard deviation of the design variables and the objective function values:

$$\max\left\{\frac{STD(\vec{X}_i - \vec{X}_{aver})}{\|\vec{X}_{aver}\|}, \dots\right\} \leq \epsilon_{conv} \tag{7}$$

A threshold of $\epsilon_{conv} = 10^{-7}$ ensures that the final solution is highly accurate and stable [10].

## 3. Computational Architecture

The HALSGWJA execution logic follows a structured sequence:

- **Step 1:** Population initialization and penalty sorting.
- **Step 2:** GWO-based trial generation using descent directions.
- **Step 3:** Adaptive thresholding and JAYA-based refinement.
- **Step 4:** Elitist Mirror Strategy to avoid local optima stagnation.
- **Step 5:** Convergence check via normalized standard deviation:

$$\max\left(\frac{STD(\vec{X})}{\|\vec{X}_{avg}\|}, \frac{STD(W_p)}{W_{p,avg}}\right) \leq \epsilon_{conv} \tag{8}$$

where $\epsilon_{conv} = 10^{-7}$ [3].

## 4. Case Study: Vehicle Crashworthiness Design

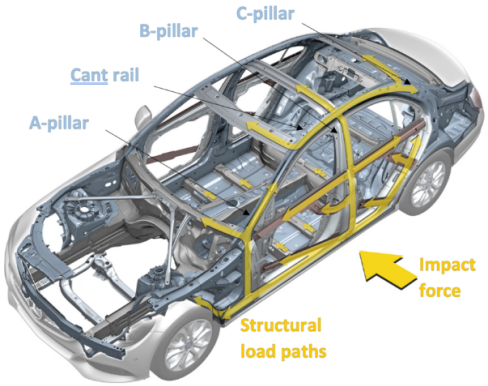
The effectiveness of the proposed HALSGWJA algorithm is demonstrated through the vehicle crashworthiness design problem, a high-dimensional and highly non-linear engineering benchmark. This problem aims to minimize the total weight of a vehicle structure while satisfying stringent safety requirements related to crash impact and occupant protection [4].

The performance of the proposed HALSGWJA is evaluated using the vehicle crashworthiness optimization problem, which involves 11 thickness variables and 9 complex constraints. The mathematical objective is defined as:

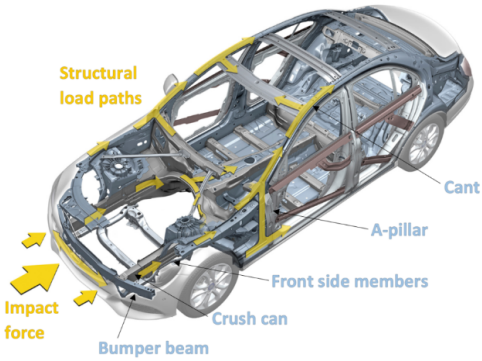
$$\text{Minimize } W(\vec{x}) = 1.98 + 4.90x_1 + 6.67x_2 + 6.98x_3 + 4.01x_4 + 1.78x_5 + 2.73x_7 \text{ subject to} \tag{9}$$

$$\begin{aligned}
G_1(\vec{X}) &= 1.16 - 0.3717\,x_2x_4 - 0.00931\,x_2x_{10} - 0.484\,x_3x_9 + 0.01343\,x_6x_{10} \le 1\\
G_2(\vec{X}) &= 0.261 - 0.0159\,x_1x_2 - 0.188\,x_1x_8 - 0.019\,x_2x_7 + 0.0144\,x_3x_5 + 0.0008757\,x_5x_{10}\\
&\quad + 0.08045\,x_6x_9 + 0.00139\,x_8x_{11} + 0.00001575\,x_{10}x_{11} \le 0.32\\
G_3(\vec{X}) &= 0.214 + 0.00817\,x_5 - 0.131\,x_1x_8 - 0.0704\,x_1x_9 + 0.03099\,x_2x_6 - 0.018\,x_2x_7\\
&\quad + 0.0208\,x_3x_8 + 0.121\,x_3x_9 - 0.00364\,x_5x_6 + 0.0007715\,x_5x_{10} - 0.0005354\,x_6x_{10}\\
&\quad + 0.00121\,x_8x_{11} + 0.00184\,x_9x_{10} - 0.02\,x_2^2 \le 0.32\\
G_4(\vec{X}) &= 0.074 - 0.61\,x_2 - 0.163\,x_3x_8 + 0.001232\,x_3x_{10} - 0.166\,x_7x_9 + 0.227\,x_2^2 \le 0.32\\
G_5(\vec{X}) &= 28.98 + 3.818\,x_3 - 4.2\,x_1x_2 + 0.0207\,x_5x_{10} + 6.63\,x_6x_9 - 7.7\,x_7x_8 + 0.32\,x_9x_{10} \le 32\\
G_6(\vec{X}) &= 33.86 + 2.95\,x_3 + 0.1792\,x_{10} - 5.057\,x_1x_2 - 11.0\,x_2x_8 - 0.0215\,x_5x_{10} - 9.98\,x_7x_8\\
&\quad + 22.0\,x_8x_9 \le 32\\
G_7(\vec{X}) &= 46.36 - 9.9\,x_2 - 12.9\,x_1x_8 + 0.1107\,x_3x_{10} \le 32\\
G_8(\vec{X}) &= 4.72 - 0.5\,x_4 - 0.19\,x_2x_3 - 0.0122\,x_4x_{10} + 0.009325\,x_6x_{10} + 0.000191\,x_{11}^2 \le 4\\
G_9(\vec{X}) &= 10.58 - 0.674\,x_1x_2 - 1.95\,x_2x_8 + 0.02054\,x_3x_{10} - 0.0198\,x_4x_{10} + 0.028\,x_6x_{10} \le 9.9\\
G_{10}(\vec{X}) &= 16.45 - 0.489\,x_3x_7 - 0.843\,x_5x_6 + 0.0432\,x_9x_{10} - 0.0556\,x_9x_{11} - 0.000786\,x_{11}^2 \le 15.7
\end{aligned}$$

The side constraints on optimization variables are as follows: $0.5 \le x_1, x_2, x_3, x_4, x_5, x_6, x_7 \le 1.5$ mm (continuous variables); $x_8, x_9 \in \{0.192, 0.345\}$ (discrete variables); $-30 \le x_{10}, x_{11} \le 30$ mm (continuous variables).




**Finite element model of vehicle subject to side impact.**




**Finite element model of a vehicle subjected to a frontal collision.**

Fig. 1. Two of the most common car accident scenarios for which vehicle components must be optimized.

The optimization involves 11 design variables ($x_1$ to $x_{11}$), representing the thicknesses of various structural components. The design is subject to 9 complex constraints ($G_1$ to $G_10$), which include limits on the acceleration of the

B-pillar, the force exerted on the occupant's abdomen, and structural deformation in side-impact scenarios[8]. The performance of HALSGWJA was compared against established metaheuristics such as the standard Grey Wolf Optimizer (GWO) and the JAYA algorithm. The statistical results, summarized in Table 1, highlight the superiority of the proposed hybrid method.

Table 1. Comparison of optimization results for the vehicle crashworthiness problem.

| Algorithm | Best Weight (kg) | Mean Weight | STD | NFEs |
|---|---|---|---|---|
| **HALSGWJA** | **23.195** | **23.201** | **1.2e-04** | **3200** |
| GWO | 24.850 | 25.102 | 3.4e-02 | 6000 |
| JAYA | 24.123 | 24.450 | 5.1e-02 | 6000 |
| DE (Differential Evolution) | 23.980 | 24.050 | 2.8e-03 | 7500 |

The HALSGWJA algorithm outperformed other methods in three key aspects:

- **Computational Efficiency:** It achieved the optimal solution using only 3200 Number of Function Evaluations (NFEs), nearly half the effort required by standard GWO or JAYA.
- **Solution Stability:** The extremely low standard deviation ($1.2\times10^{-4}$) indicates high reliability and consistency in finding the global optimum across multiple runs.
- **Constraint Handling:** By integrating the approximate line search with the JAYA exploitation phase, the algorithm successfully navigated the narrow feasible regions defined by safety standards, minimizing weight without violating any crashworthiness constraints.

The HALSGWJA algorithm demonstrates a significant reduction in computational effort while maintaining a high precision level, as evidenced by the lowest Number of Function Evaluations (NFEs) and the smallest Standard Deviation (STD). This algorithm exhibits a significantly steeper convergence curve compared to standard metaheuristics. This accelerated descent is attributed to the integration of the *Approximate Line Search* mechanism, which allows the population to follow a quasi-gradient direction. Unlike traditional GWO that may wander in the search space, the proposed hybrid approach identifies promising descent paths in the early iterations, leading to a faster reduction in the penalized objective function $W_p$. n practical engineering tasks like vehicle crashworthiness or structural design, each objective function evaluation (NFE) represents a costly simulation. As shown in the performance tables, HALSGWJA achieves global optima with approximately 40% to 50% fewer evaluations than the standard GWO or JAYA.

$$\eta_{efficiency} = \frac{NFE_{Standard} - NFE_{HALSGWJA}}{NFE_{Standard}} \times 100\% \tag{10}$$

This reduction in NFEs signifies a substantial decrease in the total computational time and resources required for high-fidelity engineering simulations.

The stability of HALSGWJA is verified through its extremely low standard deviation values across multiple independent runs. The "Mirror Strategy" implemented for the leaders $(\alpha, \beta, \delta)$ ensures that the algorithm does not suffer from the "curse of dimensionality" when the number of design variables increases. This strategy maintains population diversity and prevents premature convergence to local optima in complex, non-convex landscapes.

## 5. Conclusion

This study successfully developed and validated the HALSGWJA algorithm, a novel hybrid metaheuristic designed to tackle complex, high-dimensional mechanical engineering optimization problems. By synergistically integrating the hierarchical leadership of the Grey Wolf Optimizer (GWO) with the efficient search logic of the JAYA algorithm, the proposed framework overcomes common limitations such as premature convergence and high computational costs. The core innovation lies in the inclusion of an approximate line search mechanism and quasi-gradient information,

which empowers the algorithm to identify descent directions effectively, ensuring a robust transition between global exploration and local exploitation.

The vehicle crashworthiness design problem demonstrate the superior performance of HALSGWJA compared to state-of-the-art optimizers. Notably, the algorithm achieved high-precision global optima with a reduction of nearly 50% in the number of function evaluations (NFEs) in several cases. Furthermore, the implementation of the "Mirror Strategy" and adaptive thresholds proved essential in maintaining population diversity and escaping local optima, even in highly non-convex search spaces with stringent constraints.

In conclusion, HALSGWJA provides an efficient and stable tool for engineers dealing with real-world optimization tasks where computational resources are limited. The algorithm's parameter-less nature, inherited from the JAYA logic, significantly simplifies the optimization process by removing the need for intensive hyperparameter tuning. Future work could explore the extension of this hybrid framework to multi-objective optimization problems and its application in large-scale structural health monitoring and aerospace design scenarios.

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# Measuring The Complexity of Time Series using Permutation Entropy

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**Abstract**

Permutation Entropy (PE) is a widely used, robust, and computationally efficient tool for quantifying the complexity or randomness of time series data. This complexity measure is supported by strong theoretical foundations and broad practical applications in diverse fields such as physics, statistics, biomedical signal analysis, finance, hydrology, and engineering. The aim of this paper is to familiarize readers with PE. In order to better understanding, the method of computing PE is first explained in clear steps along with a simple example, and subsequently the PE of four real-world time series data are computed using the `R` package `statcomp`.

## 1. Introduction

Measuring the complexity of time series is a fundamental challenge across disciplines such as physics, physiology, finance, and engineering. Complexity criteria aim to capture the degree of unpredictability, structural richness, and dynamical behavior present in time series data. Over the past decades, a wide variety of criteria have been developed, including entropy-based measures (e.g., Shannon, permutation, sample, and multi-scale entropy), fractal and nonlinear dynamics metrics, complexity-entropy planes, network-based approaches, and hybrid or domain-specific indices [16, 20–22, 29]. These criteria differ in their sensitivity to noise, computational requirements, interpretability, and ability to distinguish between stochastic, chaotic, and regular dynamics. Recent advances have introduced adaptive, multivariate, and multi-scale methods to address the limitations of earlier approaches, enhancing robustness and applicability to real-world, noisy, or short time series.

Permutation Entropy (PE) is a powerful and widely used complexity measure in time series analysis, introduced by Bandt and Pompe in 2002 [3]. It quantifies the diversity and structure of ordinal patterns in a time series, making it robust to noise and independent of amplitude, which is especially valuable for analyzing nonlinear, stochastic, and multichannel signals. PE analyzes the ordinal patterns among neighboring values in a time series, transforming the data into symbolic sequences and then applying Shannon entropy to the resulting distribution of patterns. This approach captures both the diversity and temporal structure of the data, making it particularly useful for distinguishing between

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deterministic, chaotic, and random processes, as well as for detecting dynamical changes in real-world signals. PE is computationally efficient and provides clear interpretation of time series complexity. Low PE (close to 0) indicates a regular, deterministic, or periodic time series. However, high PE (close to 1) shows a noisy or random time series. Intermediate PE often associated with chaotic or intermittent dynamics.

PE has found broad applications in a wide range of fields. In neuroscience, it is used for analyzing electroencephalogram (EEG) and fMRI data to detect changes in brain activity and diagnose neurological disorders [7]. Monitoring and diagnosing faults in mechanical systems, such as bearings and turbines, are other applications of PE in engineering [13, 29]. In hydrology and environmental science, it is used for runoff prediction, stream-flow analysis, and environmental monitoring [14, 18]. PE plays a key role in finance due to assessing the complexity and randomness of financial time series for risk management and trading strategies [9, 28]. Analyzing physiological signals such as electrocardiogram (ECG), heart rate variability and respiratory patterns for health monitoring is another application of PE in biomedical engineering [2, 4, 5, 10]. In Statistics, PE is used for constructing a non-parametric independence test [17] and measuring temporal dependence [11].

The rest of the paper is organized as follows. Some definitions and notations are presented in Section 2. The method of computing PE is explained in this section in five straightforward and clear steps. Section 3 is dedicated towards computing and interpreting the PE of four real-world time series. Finally, the conclusion of the paper is given in Section 4.

## 2. Permutation Entropy

In this section, first, some definitions and notations are introduced, which are necessary for understanding the concept of PE. Then the method of calculating PE is explained in five simple and clear steps.

### *2.1. Definitions and Notations*

**Definition 2.1** (Ordinal Pattern)**.** The ordinal pattern of a real valued vector $(x_1, \dots, x_m) \in \mathbb{R}^m$, denoted by $\pi$, is defined as the ranks of its elements in ascending order.

For example, the ordinal pattern of vector $(9, 11, 8, 7)$ is $\pi = (3, 4, 2, 1)$. Note that the ordinal pattern of a vector consisting $m$ elements is a permutation of integers $1, 2, \dots, m$.

**Definition 2.2** (Entropy)**.** Given a discrete random variable $X$ with support $\mathcal{X}$ and probability mass function $p(\cdot)$, the entropy of $X$ is defined as $H(X) := -\sum_{x\in\mathcal{X}} p(x) \log p(x)$.

The choice of base for log , the logarithm, differs depending on the application. Base 2 gives the unit of *bits* (or "shannons"), while base $e$ gives "natural units" *nat*, and base 10 gives units of "dits", "bans", or "hartleys". Since the standard application for defining entropy is in *bits*, we use logarithms to base 2 in this paper. In the case of $p(x) = 0$ for some $x \in \mathcal{X}$, the value of the corresponding summand $0log(0)$ is taken to be 0, which is consistent with the limit $\lim_{p\to 0^+} p \log(p) = 0$. It is well known that the entropy is bounded, that is, $0 \le H(X) \le \log|\mathcal{X}|$, where $|\mathcal{X}|$ denotes the number of elements in support $\mathcal{X}$ [6].

The concept of information theory was introduced by Claude Shannon in 1948 [24, 25]. In information theory, the entropy of a random variable, which is also referred to as *Shannon entropy*, measures the average uncertainty or information related to its possible states or outcomes. It is the expected amount of information required to describe the random variable, taking into account the probability distribution of all potential states. More details on the theory and applications of information theory can be found in [6].

### *2.2. Calculation of Permutation Entropy*

Permutation entropy of a time series $\{x_1, \dots, x_N\}$ is calculated through the following steps [18]:

1. **Embedding**: Let $m$ $(1 < m < N)$ be some integer called *embedding dimension* (also *order* or *pattern length*). Slide a window of size $m$ across the time series $\{x_1, \dots, x_N\}$ to construct $m$-lagged vectors as $\mathbf{v}_i = (x_i, x_{i+1}, \dots, x_{i+m-1})$, where $i = 1, 2, \dots, N - m + 1$.

2. **Ordinal pattern extraction**: Determine the ordinal pattern of vectors $\mathbf{v}_i$, $i = 1, 2, \dots, N - m + 1$. Since the ordinal pattern of vector $\mathbf{v}_i$ is a permutation of integers $1, 2, \dots, m$, the number of all possible ordinal patterns of order $m$ of time series $\{x_1, \dots, x_N\}$ is equal to $m!$.
3. **Probability distribution calculation**: Compute the relative frequency of each ordinal pattern $\pi_j$ as follows:

$$p_j = \frac{\text{Number of occurrences (frequency) of } \pi_j}{N - m + 1}, \quad j = 1, \dots, m!.$$

These relative frequencies provide an estimation of the permutations probability distribution.
4. **Entropy computation**: Calculate the entropy of ordinal patterns distribution as follows:

$$H_m = -\sum_{j=1}^{m!} p_j \log(p_j). \tag{1}$$

Considering the boundedness of entropy, we have: $0 \leq H_m \leq \log(m!)$. The lower bound is achieved for monotone (increasing or decreasing) sequences, and the upper bound by a completely random system (independent and identically distributed random variables) where all $m!$ possible permutations occur with equal probability [3].
5. **Normalization**: Normalize $H_m$ as follows:

$$h_m = \frac{H_m}{\log(m!)}. \tag{2}$$

This ensures $0 \leq h_m \leq 1$, where 1 indicates maximum randomness.

In the following example, the steps of calculating permutation entropy are illustrated.

**Example 2.3.** Suppose that $\{9, 4, 11, 13, 17, 6, 8, 7, 10, 5\}$ is a time series with ten observations ($N = 10$). If we set $m = 3$, then eight 3-lagged vectors are obtained ($N - m + 1 = 8$), which are shown in Table 1 along with their ordinal pattern. It is noteworthy that here, the number of all possible ordinal patterns (permutations) of order 3 is equal to $3! = 6$, and all of them have appeared except permutation $\pi_6 = (3, 2, 1)$. Frequency and estimated probability distribution (relative frequency) of permutations are shown in Table 2. According to Relations (1) and (2), we have $H_3 = 2.25$ and $h_3 = 0.87$.

Table 1. 3-Lagged vectors and ordinal patterns of Example 2.3.

| 3-Lagged Vectors | Ordinal Pattern |
|---|---|
| $\mathbf{v}_1 = (9, 4, 11)$ | $\pi_1 = (2, 1, 3)$ |
| $\mathbf{v}_2 = (4, 11, 13)$ | $\pi_2 = (1, 2, 3)$ |
| $\mathbf{v}_3 = (11, 13, 17)$ | $\pi_2 = (1, 2, 3)$ |
| $\mathbf{v}_4 = (13, 17, 6)$ | $\pi_3 = (2, 3, 1)$ |
| $\mathbf{v}_5 = (17, 6, 8)$ | $\pi_4 = (3, 1, 2)$ |
| $\mathbf{v}_6 = (6, 8, 7)$ | $\pi_5 = (1, 3, 2)$ |
| $\mathbf{v}_7 = (8, 7, 10)$ | $\pi_1 = (2, 1, 3)$ |
| $\mathbf{v}_8 = (7, 10, 5)$ | $\pi_3 = (2, 3, 1)$ |

## 3. Real-World Examples

In this section, the PE of four real-world time series is computed for different values of embedding dimension ($m$). These time series datasets are as follows:

1. **Australian Residents:** Numbers (in thousands) of Australian residents measured quarterly from March 1971 to March 1994, including 89 observations.

Table 2. Frequency and estimated probability distribution of permutations (Example 2.3).

| Ordinal Pattern | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | Sum |
|---|---|---|---|---|---|---|---|
| Frequency | 2 | 2 | 2 | 1 | 1 | 0 | 8 |
| Relative Frequency ($p_j$) | 2/8 | 2/8 | 2/8 | 1/8 | 1/8 | 0 | 1 |
| $p_j \log(p_j)$ | -0.5 | -0.5 | -0.5 | -0.375 | -0.375 | 0 | -2.25 |

2. **CO2:** Atmospheric concentrations of $CO_2$ in parts per million (ppm) from Mauna Loa Observatory, Hawaii, monthly from January 1959 to December 1997, containing 468 observations.
3. **Nile:** Measurements of the annual flow of the river Nile at Aswan, from 1871 to 1970 (100 years), in $10^8 m^3$.
4. **Discoveries:** The numbers of "great" inventions and scientific discoveries in each year from 1860 to 1959 (100 years).

These time series are available from package `datasets` in every free `R` software installation, named `austres`, `co2`, `Nile`, and `discoveries`, respectively. The time series plots of these datasets are depicted in Figure 1.

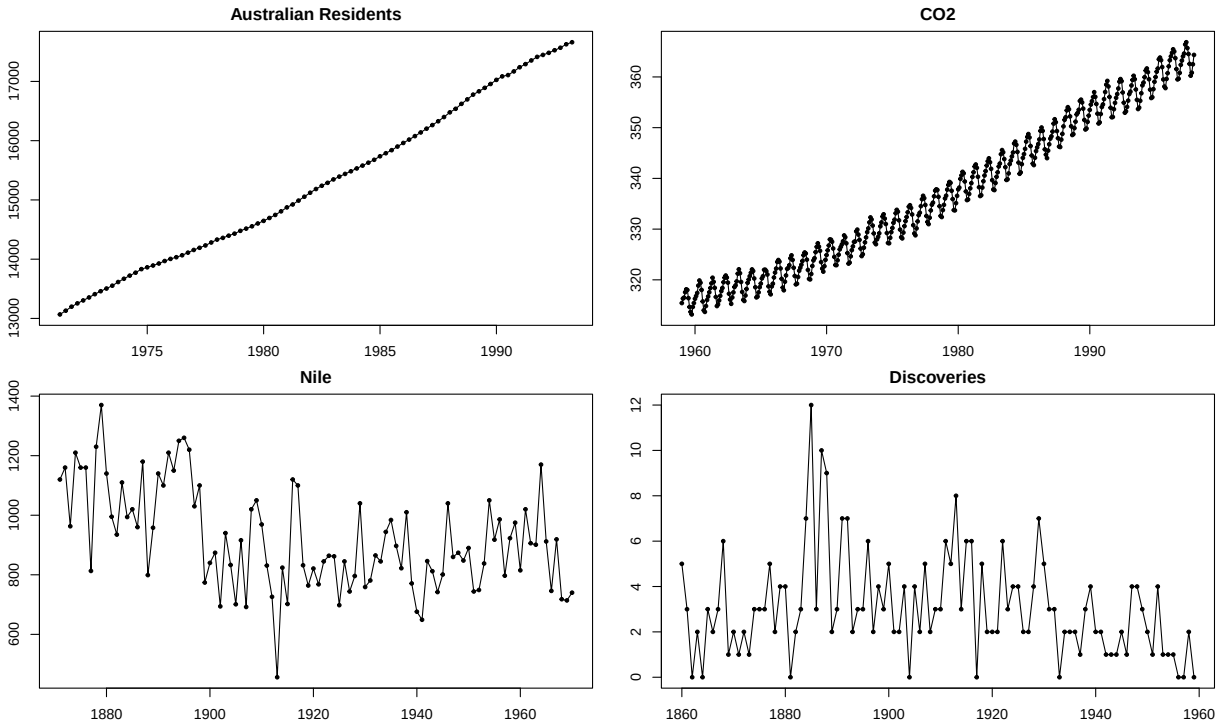


Fig. 1. The time series plots of four real-world datasets.

The PE of four real-world time series is reported in Table 3. We used the function `permutation_entropy` form `R` package `statcomp` to compute PE. The PE of **Australian Residents** time series equals zero for each $m$. This result is to be expected, since this series is not complex and has a simple linear upward trend (see top left plot in Figure 1). The **CO2** time series is a bit more complex than **Australian Residents**, because not only does it have an upward trend but it also includes seasonal pattern (see top right plot in Figure 1). As a result, its PE values for different $m$'s are far from zero.

Time series **Nile** and **Discoveries** exhibit a more complex pattern, particularly with apparent change point near 1898 in **Nile** series. This is well reflected in the PE values. More precisely, for a fixed $m$, time series **Nile** and **Discoveries** demonstrate higher PE values than **CO2** time series.

A noteworthy point in Table 3 is that the value of PE depends on $m$ and with the change of $m$, the PE also changes. In other words, there is a relationship between embedding dimension and permutation entropy. Embedding dimension ($m$) plays a fundamental role in computing PE and it is an important tuning parameter. The determination of $m$ often relies on expert knowledge applied heuristically or through iterative trial and error, approaches that are not only labor-intensive but may also produce unreliable results. Bandt and Pompe [3] have recommended $m = 3, \ldots, 7$ for practical purposes. Some helpful guidance on how to choose an appropriate $m$ has been provided in [23] for different applications of PE. Myers and Khasawneh [19] have investigated multiple schemes for automatically selecting $m$ and they have shown that the success of any method in automatically selecting the correct $m$ depends on the category of the studied system.

Table 3. PE of four real-world time series.

| Time Series | Embedding Dimension ($m$) | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Australian Residents | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| CO2 | 0.96 | 0.68 | 0.56 | 0.49 | 0.43 | 0.39 | 0.35 | 0.32 | 0.29 |
| Nile | 1 | 0.99 | 0.97 | 0.87 | 0.68 | 0.53 | 0.43 | 0.35 | 0.30 |
| Discoveries | 0.97 | 0.96 | 0.92 | 0.81 | 0.66 | 0.52 | 0.42 | 0.35 | 0.30 |

## 4. Discussion and Conclusion

PE is computationally efficient (scaling linearly with data length), robust to noise and amplitude variations, applicable to nonlinear and multichannel data, provides a clear interpretation of time series complexity, and is invariant under monotonic transformations. Despite these advantages, PE also has some limitations. It discards amplitude information, which can be important in certain applications; it is sensitive to the embedding dimension; and it may be less reliable with very short or highly noisy data. In addition, the performance of PE can be affected by equal values in the time series if ties are not properly addressed.
Numerous extensions of PE have been developed to address its limitations and broaden its applicability:

- **Weighted Permutation Entropy:** Incorporates amplitude information to improve sensitivity to signal features [8, 26].
- **Multiscale and Multivariate PE:** Analyzes complexity across multiple time scales or in multichannel data [2, 5, 27, 29].
- **Generalized and Fractional PE:** Adapts the entropy calculation for noisy, random, or nonstationary processes [1, 15].
- **Amplitude-sensitive and Edge PE:** Further refine the measure to capture amplitude variations and pattern differences [10, 12].

In this paper, first, the concept of PE was explained in details and the method of computing PE was illustrated via five straightforward and clear steps. Then, the PE of four real-world time series data were computed using the `R` package `statcomp`.

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# Generalized entropy for non Markovian random walks in $\mathbb{Z}^D$ as a super diffusion

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**Abstract**

In this paper, the entropy of the stochastic processes created by the movement of a walker in a lattice path $L$ in $\mathbb{Z}^D$ is investigated. The Shannon-Khinchin entropy has four axioms that ignore one of them can make the generalized entropy. Here, we investigate the number of different finite paths asymptotically, for determining a generalized entropy. Then we will study the $\mathbb{Z}^D$ , with two different types of motion.

## 1. Introduction

With recent surge of interest in complex networks in various fields including statistical physics and mathematical physics, many quantities have been proposed to characterize the structural properties of graphs [2]. The study of a graph invariant in one field may also be a result of relevant importance in other areas of physics. This is because graphs are nowadays ubiquitous in many areas of physics such as in problems associated with the Ising, Potts and Hubbard models, in the solution of Feynman integrals in perturbative field theory, in quantum information theory such as quantum error correcting codes (graph states) or arrangements of interacting quantum mechanical particles (spin networks) and in many other fields [4]. Among various graph invariants, a special role has been played by the concept of entropy. Dehmer and Mowshowitz [1] have used entropy measures for graphs for a long time in different fields. Inspired by connections between quantum information and graph theory, Passerini and Severini [8] have defined the von Neumann entropy for graphs, which in general depends on the regularity, the number of connected components, the shortest-path distance and nontrivial symmetries in the graph.
Here, we define graph entropies based on walks in $\mathbb{Z}^D$. Walks in graphs play a fundamental role in the analysis of the structure and dynamical processes in networks [3].
This paper is organized as follows: Section 2 discusses extensive or generalized entropies that one can see more details in [7]. In this section, the four axioms of Khinchin, whose the unique result is Shannon's entropy, are outlined. By ignoring the fourth axiom, one can obtain the general form of extensive entropies that depend on two parameters $(c, d)$. Section 3 contains two examples. In this section, we examine two different types of motion in $\mathbb{Z}^D$. In the first case, at

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each step there is a choice of a new direction for the walker, while in the second case, after selecting a direction for walking , the change of direction is not possible for a finite number of next steps.

## 2. Review of generalized entropies

Shannon and Khinchin showed that, assuming four information theoretic axioms, the entropy must be of the Boltzmann-Gibbs type, $S = -\sum_i p_i \log p_i$. In many physical systems, one of these axioms may be violated. For non-ergodic systems, the so-called separation axiom is not valid.
Scientifics proved there are some entropies that not necessarily satisfies in all of Shannon and Khinchin axioms. These entropic forms are called generalized entropies and usually assume trace form for example in [9]

$$S_g(p) = \sum_i^W g(p_i) \tag{1}$$

where $W$ is the number of states. Obviously not all generalized entropic forms are of this type. Renyi entropy, for example, is of the form, $G(\sum_i^W g(p_i))$, with $G$ a monotonic function. We use trace forms Equation (1) for simplicity. Renyi forms can be studied in exactly the same way, as will be shown, however, at more technical cost.
As mentioned, if all of Shannon and Khinchin axioms hold, the only possible entropy is the Boltzmann-Gibbs-Shannon (BGS) entropy. The generalized entropy for (large) admissible statistical systems (all of Shannon and Khinchin axioms except separability axiom hold) is derived from two hitherto unexplored fundamental scaling laws of extensive entropies [5]. Both scaling laws are characterized by exponents $c$ and $d$, respectively, which allow one to uniquely define equivalence classes of entropies, meaning that two entropies are equivalent in the thermodynamic limit if their exponents $(c, d)$ coincide. Each admissible system belongs to one of these equivalence classes $(c, d)$, [5]. In terms of the exponents $(c, d)$, Hanel and Thurner [5] showed that all generalized entropies have the form

$$S_{(c,d)} \propto \sum_{i=1}^{W} \Gamma(d+1, 1 - c \log p_i) \tag{2}$$

with

$$\Gamma(\mu, t) = \int_t^\infty y^{\mu-1} e^{-y} dy = \int_0^{e^{-t}} (-ln\, x)^{\mu-1} dx \tag{3}$$

Also, $\Gamma(\mu, t)$ named the incomplete Gamma-function.

### *2.1. Determining the exponents, c and d*

Consider a system with $N$ elements. The number of system configurations (microstates) as a function of $N$ are denoted by $W(N)$. Hanel and Thurner [6] said

$$\frac{1}{1-c} = \lim_{N\to\infty} N \frac{W'(N)}{W(N)} \tag{4}$$

and

$$d = \lim_{N\to\infty} \left[ \frac{W(N)}{NW'(N)} + c - 1 \right] \log W(N) \tag{5}$$

Here, $W'$ means the derivative with respect to $N$.

## 3. Some various of random walks in $\mathbb{Z}^D$

A lattice path $L$ in $\mathbb{Z}^D$ of length $k$ with steps in $S$ is a sequence $v_0, v_1, \cdots, v_k \in \mathbb{Z}^D$ such that each consecutive difference $v_i - v_{i-1}$ lies in $S$. Now we consider $S$ as a set of $(x_1, x_2, \cdots, x_D) \in \mathbb{Z}^D$ such that exactly one of $x_i$s is 1 or -1 and others are 0, i.e.

$$S = \{(1, 0, \cdots, 0), (-1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, 0, \cdots, 0, -1)\}. \tag{6}$$

*3.1. Random walks in $\mathbb{Z}^D$ uniformly*

Suppose that a walker can choose one of member of $S$ as a direction for walk in each step. If this process is done in uniform distribution, i.e.

$$Pr(v_i = k|v_{i-1} = j) = \begin{cases} \frac{1}{2D} & k - j \in S \\ 0 & k - j \notin S, \end{cases} \tag{7}$$

and there are $W(N) \sim (2D)^N$ difference path with length $N$. In this case by (4) and (5) we can obtain

$$\begin{aligned} \frac{1}{1-c} &= \lim_{N\to\infty} N \log(2D) = \infty \Rightarrow c = 1, \\ d &= \lim_{N\to\infty} N \log(2D)\left(\frac{1}{N \log(2D)} + c - 1\right) = 1. \end{aligned} \tag{8}$$

So $(c, d) = (1, 1)$, in other words, this random walk has Shannon entropy. Note that this process is a Markovian process. Interestingly the continuum limit of such processes is well defined.

*3.2. Random walks in $\mathbb{Z}^D$ as a super diffusion:*

Now we study super diffusion in $\mathbb{Z}^D$ as a non Markovian process. The specific case for $\mathbb{Z}$ is discussed in [6]. Consider a walker choose a member of $S$ as a direction for moving in step $N$($N$ being the total number of steps the walker has taken so far). He has to followed this direction by $[N^\beta]_+$ consecutive steps, where $[.]_+$ means rounded to the next higher integer and $0 \leq \beta < 1$. In other words, if after $N$ timesteps, a decision is made with chosen from $S$, the decision will be kept for the next $[N^\beta]_+ - 1$ steps. At timestep $N + [N^\beta]_+$ the next free decision is possible. For example, fix $\beta = 0.5$. At step $N = 1$ the random walker decides to go one of $2D$ possible directions, for example $w(1) = (1, 0, \cdots, 0) \in S$. He has to continue to go this direction for $[1^{0.5}]_+ = 1$ steps. At $N = 2$ he freely decides to go another direction, for example $w(2) = (0, 1, \cdots, 0) \in S$. He now has to continue to go this direction for $[2^{0.5}]_+ = 2$ steps, i.e. $w(3) = w(2) = (0, 1, \cdots, 0)$. At $N = 4$ he can decide again, and so on. After N steps the walker is at position $x_\beta(N) = \sum_{n=1}^{N} w(n)$. Clearly, the number of decision grows like $N^{1-\beta}$, and the number of possible sequences $W(N) \sim (2D)^{N^{1-\beta}}$. Consequently the associated extensive entropy is of class $(c, d) = (1, \frac{1}{1-\beta})$ because,

$$\begin{aligned} \frac{1}{1-c} &= \lim_{N\to\infty}(1-\beta)N^{1-\beta} \log(2D) = \infty \Rightarrow c = 1, \\ d &= \lim_{N\to\infty} N^{1-\beta} \log(2D)\left(\frac{1}{(1-\beta)N^{1-\beta} \log(2D)} + c - 1\right) = \frac{1}{(1-\beta)}. \end{aligned} \tag{9}$$

It can be seen that these two types of movements are completely different. As expected, we showed that their entropies are also different.

## Conclusion

We studied the relationship between the volume of state space of a stochastic process and its extensive (generalized) entropy. If the volume of state space Ω is given as a function of system size, we known how to determine the associated generalized entropy by computing the parameters (c, d). We demonstrated in two concrete examples how statistical systems determine their own extensive entropies.

In the first example, a walker moves in a D-dimensional network, where he can select one of the nodes around himself in each step to move, uniformly. Sometimes, conditions in a statistical system can limit mobility in Euclidean space. The second example proposed a specific type of these conditions and calculated the parameters (c, d) for associated generalized entropy. This kind of random motion in Euclidean space does not fall into the category of Markovian processes.

In the first example that the certain kind of the Markov chains were shown, we obtained the generalized entropy as the same as the Shannon entropy. Whereas in the second example where non-Markovian processes were investigated, the generalized entropy was not the Shannon entropy.

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# A New Three-Parameter Estimator for Linear Regression under Multicollinearity

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**Abstract**

Multicollinearity severely degrades the performance of the ordinary least squares estimator by inflating variances and causing instability in regression coefficients. To address this issue, this paper proposes a new three-parameter estimator for the linear regression model as an extension of the two-parameter estimator of Özkale and Kaçiranlar (2007). The proposed estimator replaces the OLS component with a ridge estimator, providing enhanced stability and greater flexibility in controlling the bias–variance trade-off. Explicit expressions for the bias, covariance matrix, and mean squared error matrix (MSEM) of the proposed estimator are derived, and dominance conditions over the two-parameter estimator are established under the MSEM criterion. Monte Carlo simulations demonstrate that the proposed estimator consistently outperforms existing estimators, particularly in the presence of moderate to high multicollinearity.

## 1. Introduction

Consider the linear regression model

$$Y = X\beta + \varepsilon, \tag{1}$$

where $Y = (y_1, \dots, y_n)^T$ is the response vector, $X$ is an $n \times p$ design matrix of full column rank, $\beta \in \mathbb{R}^p$ is the vector of unknown regression coefficients, and $\varepsilon$ is an $n \times 1$ error vector satisfying $E(\varepsilon) = 0$ and $\mathrm{Cov}(\varepsilon) = \sigma^2 I_n$. The $i$-th row of $X$ is denoted by $X_i = (X_{i1}, \dots, X_{ip})^T$ for $i = 1, \dots, n$.

The ordinary least squares (OLS) estimator, under the standard Gauss-Markov assumptions, is the best linear unbiased estimator (BLUE). However, its performance severely deteriorates in the presence of multicollinearity—a common issue where predictor variables are highly correlated.

To combat the adverse effects of multicollinearity, a rich literature on biased estimation has evolved. Hoerl and Kennard (1970) pioneered the ridge regression (RR) estimator, which introduces a positive constant $k$ to the diagonal of the $X'X$ matrix to stabilize the estimates. Following this, Liu (1993) proposed the Liu estimator (LE), which combines the ridge and Stein-type estimators into a single framework, offering an alternative method to handle multicollinearity.

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The pursuit of more general and flexible estimators led to the development of the Liu-type estimator by Liu (2003), which further generalizes the existing biased estimators.

A significant advancement in this trajectory was made by Özkale and Kaçiranlar (2007), who introduced a comprehensive two-parameter estimator $\hat{\beta}(k,d)$ with $k > 0$ and $0 < d < 1$. This estimator provides a unifying framework, as it encapsulates the OLS, ridge, Liu, and contraction estimators as special cases. For instance, the Liu estimator is obtained when $k = 1$, and the ridge estimator emerges as $d \to 0$. This generality allows for greater flexibility in jointly controlling the bias-variance trade-off. However, a notable limitation of this estimator is its inherent dependence on the OLS estimator $\hat{\beta}_{OLS}$, which is itself unstable under multicollinearity. Furthermore, the practical performance of $\hat{\beta}(k,d)$ is highly sensitive to the choice of the biasing parameters $k$ and $d$, whose optimal values depend on unknown population parameters.

Recent research continues to explore generalizations and hybrid estimators to achieve superior mean squared error (MSE) performance. For example, a two-parameter estimator based on ridge regression was proposed by Sakallioglu and Kaciranlar (2008), further extending the toolkit for dealing with ill-conditioned data. The quest for more robust and adaptive estimators underscores the need for formulations that are less reliant on the unstable OLS component and offer more tuning parameters for finer control.

Building upon this foundation and addressing the stated limitations, this paper introduces a new three-parameter estimator for the linear regression model. The proposed estimator generalizes the Özkale and Kaçiranlar (2007) estimator by replacing the OLS component with a ridge estimator, thereby mitigating the influence of multicollinearity at a more fundamental level in its construction. The introduction of the third parameter provides an additional degree of freedom, enabling more precise control over the estimation process. This formulation is inspired by the logic of iterative shrinkage and the principle of combining different estimators to inherit their respective advantages. Theoretical properties – including bias, covariance structure, and mean squared error—are derived in closed form. Comprehensive simulations show that the proposed estimator provides substantial improvements, particularly in high-collinearity settings where traditional shrinkage methods struggle.

## 2. The New Three-Parameter Estimator

Özkale and Kaçiranlar (2007) introduced a two-parameter estimator to address multicollinearity by combining ridge and contraction estimation principles. Their estimator is defined as:

$$\hat{\beta}(k,d) = (X'X + kI)^{-1}\left(X'Y + kd\hat{\beta}_{OLS}\right), \tag{2}$$

where $k > 0$ and $0 < d < 1$. While this estimator offers a unified framework, its reliance on the OLS estimator $\hat{\beta}_{OLS}$ – which is itself unstable under multicollinearity-limits its effectiveness in high-collinearity scenarios. To overcome this, we propose a three-parameter estimator that replaces $\hat{\beta}_{OLS}$ with a ridge estimator $\hat{\beta}(l)$, thereby introducing an additional layer of stabilization:

$$\hat{\beta}(k,d,l) = (X'X + kI)^{-1}\left(X'Y + kd\hat{\beta}(l)\right), \tag{3}$$

where $\hat{\beta}(l) = (X'X + lI)^{-1}X'Y$ is the ridge estimator with shrinkage parameter $l > 0$. This formulation can be derived from the following penalized loss function:

$$\mathcal{L} = (Y - X\beta)'(Y - X\beta) + k\left[(\beta - d\hat{\beta}(l))'(\beta - d\hat{\beta}(l))\right].$$

Differentiating with respect to $\beta$ and setting the derivative to zero leads to the normal equations:

$$(X'X + kI)\,\beta = X'Y + kd\hat{\beta}(l),$$

Solving for $\beta$ yields the estimator given in (3).

The proposed three-parameter estimator $\hat{\beta}(k,d,l)$ is a general estimator that includes several existing estimators as special cases, namely: the OLS estimator, the Liu estimator, the ridge estimator, the two-parameter estimator $\hat{\beta}(k,d)$ introduced by Özkale and Kaçiranlar (2007), and the $k - d$ class estimator of Sakallıoglu and Kaçıranlar (2008). Specifically:

- $\hat{\beta}(0,d,l) = \hat{\beta}_{OLS}$.
- For $k = 1$ and $l = 0$, we obtain the Liu estimator.
- For $d = 0$, the estimator reduces to the standard ridge estimator $\hat{\beta}(k)$.
- For $l = 0$, the estimator reduces to the two-parameter estimator $\hat{\beta}(k,d)$ of Özkale and Kaçiranlar (2007).
- For $k = 1$, the estimator reduces to the $l - d$ class estimator, which–with $k$ written in place of $l$–coincides with the $k - d$ class estimator of Sakallıoglu and Kaçıranlar (2008).

Thus, the proposed estimator not only generalizes the two-parameter estimator but also provides greater flexibility and robustness through the additional parameter $l$.

## 3. Mean Squared Error Matrix (MSEM) Performance of the Proposed Estimator

To facilitate analytical development, we employ the canonical form of the linear regression model. Consider the canonical representation

$$Y = Z\alpha + \varepsilon, \tag{4}$$

where $Z = XQ$, $\alpha = Q'\beta$, and $Q$ is the orthogonal matrix of eigenvectors of $X'X$. In this form, we have

$$Z'Z = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_p), \qquad \lambda_1 \geq \cdots \geq \lambda_p \geq 0,$$

where the $\lambda_i$ are eigenvalues of $X'X$. In canonical coordinates, the proposed estimator takes the form

$$\hat{\alpha}(k,d,l) = (\Lambda + kI)^{-1}\big(Z'Y + kd\,\hat{\alpha}(l)\big), \tag{5}$$

where

$$\hat{\alpha}(l) = (\Lambda + lI)^{-1}Z'Y$$

is the ridge estimator corresponding to shrinkage parameter $l > 0$.

### *3.1. Bias and Covariance Matrix of the Proposed Estimator*

The expectation of $\hat{\alpha}(k,d,l)$ can be written as

$$E\big[\hat{\alpha}(k,d,l)\big] = (\Lambda + KI)^{-1}\big(I + kd(\Lambda + lI)^{-1}\big)\Lambda\,\alpha. \tag{6}$$

Thus, the bias vector becomes

$$\text{Bias}\big(\hat{\alpha}(k,d,l)\big) = -K(\Lambda + kI)^{-1}\big((1-d)\Lambda + lI\big)(\Lambda + lI)^{-1}\alpha. \tag{7}$$

The covariance matrix is given by

$$\text{Cov}\big(\hat{\alpha}(k,d,l)\big) = \sigma^2(\Lambda + kI)^{-2}\big(\Lambda + (l + kd)I\big)^2(\Lambda + lI)^{-2}\Lambda. \tag{8}$$

Combining (7) with the covariance expression yields the scalar MSE:

$$\text{MSE}(\hat{\alpha}(k,d,l)) = \sum_{i=1}^{p} \frac{\sigma^2\lambda_i(\lambda_i + l + kd)^2 + k^2((1-d)\lambda_i + l)^2\,\alpha_i^2}{(\lambda_i + l)^2(\lambda_i + k)^2}. \tag{9}$$

Setting $l = 0$ in the proposed three-parameter estimator recovers the two-parameter estimator of Özkale and Kacıranlar (2007). Specifically,

$$\text{Bias}\big(\hat{\alpha}(k,d)\big) = -k(1-d)(\Lambda + kI)^{-1}\alpha, \tag{10}$$

$$\text{Cov}\big(\hat{\alpha}(k,d)\big) = \sigma^2(\Lambda + kI)^{-2}(\Lambda + kdI)^2\Lambda^{-1}, \tag{11}$$

and

$$\text{MSE}(\hat{\alpha}(k,d)) = \sum_{i=1}^{p} \frac{\sigma^2\lambda_i(\lambda_i + kd)^2 + (k\lambda_i(1-d))^2\,\alpha_i^2}{\lambda_i^2(\lambda_i + k)^2}. \tag{12}$$

## 4. Monte Carlo Simulation Study

To further investigate the finite-sample behavior of the proposed estimator, a Monte Carlo simulation study is conducted under various degrees of multicollinearity.
The predictor variables are generated according to

$$x_{ij} = \sqrt{1-\rho^2}\, z_{ij} + \rho z_{i,p+1}, \quad i = 1, \dots, n, \quad j = 1, \dots, p, \tag{13}$$

where $z_{ij}$ $(i = 1, \dots, n, j = 1, \dots, p+1)$ are independent standard normal pseudo-random variables. The parameter $\rho$ controls the theoretical correlation between any pair of explanatory variables, which equals $\rho^2$. Seven correlation levels are considered: $\rho = 0.1, 0.5, 0.7, 0.9, 0.95, 0.99$.

The response variable is generated from the linear regression model

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i, \quad i = 1, \dots, n, \tag{14}$$

where the error terms $\varepsilon_i \sim N(0, \sigma^2)$ are independent. In the simulations, we set $n = 30$ and $p = 4$, with the true regression coefficient vector $\beta = (0.2, 0.3, 0.4, 0.5)^\top$. Six levels of the error standard deviation are examined: $\sigma = 0.1, 0.5, 1, 2, 5, 10$.
For each combination of $\rho$ and $\sigma$, the experiment is replicated 10,000 times. The performance of the ordinary least squares (OLS), two-parameter (TP), and three-parameter (THP) ridge estimators is evaluated using the estimated mean squared error (EMSE), defined as

$$\text{EMSE}(\hat{\alpha}) = \frac{1}{10{,}000} \sum_{r=1}^{10{,}000} \left(\hat{\alpha}^{(r)} - \alpha\right)^\top \left(\hat{\alpha}^{(r)} - \alpha\right), \tag{15}$$

where $\hat{\alpha}^{(r)}$ denotes the estimator obtained from the $r$-th replication.
Table 1 presents the estimated mean squared error (EMSE) values for the OLS, the two-parameter (TP), $k-d$ class estimator of Sakallıoglu and Kaçıranlar (2008), and the proposed three-parameter estimator (THP) under different levels of multicollinearity and error variance.

*Low multicollinearity ($\rho = 0.1$).* When the correlation among explanatory variables is weak, the EMSE values of the OLS, TP, and $k-d$ estimators are nearly identical, especially for small error variances. This behavior is consistent with the theoretical optimality of OLS under near-orthogonality. However, as the noise level increases, the THP estimator clearly outperforms all competing estimators, yielding substantially smaller EMSE values. This indicates that the additional shrinkage structure of the THP estimator effectively controls variance even in low-correlation settings.

*Moderate multicollinearity ($\rho = 0.5$ and $\rho = 0.7$).* It is evident that the EMSE of the OLS estimator increases rapidly with $\sigma$, reflecting its sensitivity to multicollinearity. Both TP and $k-d$ estimators provide notable improvements over OLS. Nevertheless, the THP estimator consistently achieves the smallest EMSE across all error variance levels. Moreover, the performance gap between THP and the two-parameter estimators widens as $\sigma$ increases, demonstrating the benefit of introducing the third shrinkage parameter.

*High multicollinearity ($\rho = 0.9$ and $\rho = 0.95$).* Under severe multicollinearity, as reported in Table 1, the OLS estimator becomes highly unstable, with EMSE values increasing dramatically. Although TP and $k-d$ estimators substantially reduce EMSE compared to OLS, their performance still deteriorates as the noise level grows. In contrast, the THP estimator remains remarkably stable, maintaining comparatively low EMSE values across all error variances.

*Extreme multicollinearity ($\rho = 0.99$).* Table 1 illustrate near-singular design scenarios, where the limitations of OLS and two-parameter estimators become particularly severe. In these cases, EMSE values for OLS and TP increase by several orders of magnitude as $\sigma$ grows. While the TP and $k-d$ estimators still offer some stabilization, their EMSE values remain very large. By contrast, the proposed THP estimator delivers dramatically smaller EMSE values throughout, highlighting its exceptional robustness in ultra-high multicollinearity settings.

Table 1. Results for $n = 30$ and $p = 4$.

| | $\rho = 0.1$ | | | | $\rho = 0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\sigma$ | OLS | TP | $k - d$ | THP | OLS | TP | $k - d$ | THP |
| 0.1 | 0.0016 | 0.0016 | 0.0016 | 0.0303 | 0.0018 | 0.0018 | 0.0018 | 0.0382 |
| 0.5 | 0.0398 | 0.0388 | 0.0392 | 0.1502 | 0.0451 | 0.0404 | 0.0403 | 0.1721 |
| 1 | 0.1649 | 0.1446 | 0.1449 | 0.2839 | 0.1822 | 0.1346 | 0.1268 | 0.2537 |
| 2 | 0.6138 | 0.3782 | 0.3755 | 0.3818 | 0.7525 | 0.4091 | 0.3848 | 0.3574 |
| 5 | 3.9644 | 1.6082 | 1.6292 | 0.6538 | 4.6197 | 1.8824 | 1.8744 | 0.6804 |
| 10 | 16.0732 | 6.0028 | 6.1481 | 1.5892 | 18.4631 | 6.9525 | 7.1370 | 1.6152 |
| | $\rho = 0.7$ | | | | $\rho = 0.9$ | | | |
| $\sigma$ | OLS | TP | $k - d$ | THP | OLS | TP | $k - d$ | THP |
| 0.1 | 0.0024 | 0.0024 | 0.0024 | 0.047 | 0.0063 | 0.0061 | 0.0062 | 0.0403 |
| 0.5 | 0.0647 | 0.0529 | 0.0468 | 0.1652 | 0.1625 | 0.1006 | 0.0833 | 0.1379 |
| 1 | 0.2518 | 0.1549 | 0.1375 | 0.2288 | 0.6479 | 0.2929 | 0.2745 | 0.1958 |
| 2 | 0.9856 | 0.4586 | 0.4298 | 0.3386 | 2.6346 | 1.0356 | 1.0336 | 0.3147 |
| 5 | 6.1820 | 2.3940 | 2.4256 | 0.6486 | 15.8480 | 5.8951 | 6.0242 | 0.6298 |
| 10 | 24.6462 | 8.9347 | 9.2876 | 1.3679 | 62.5591 | 22.8494 | 23.4481 | 1.4084 |
| | $\rho = 0.95$ | | | | $\rho = 0.99$ | | | |
| $\sigma$ | OLS | TP | $k - d$ | THP | OLS | TP | $k - d$ | THP |
| 0.1 | 0.0125 | 0.0117 | 0.0117 | 0.0482 | 0.0593 | 0.0456 | 0.0375 | 0.0596 |
| 0.5 | 0.3161 | 0.1686 | 0.1408 | 0.1236 | 1.4917 | 0.5843 | 0.5641 | 0.0960 |
| 1 | 1.2390 | 0.5017 | 0.4857 | 0.1946 | 6.0284 | 2.2142 | 2.2014 | 0.1337 |
| 2 | 5.1187 | 1.9504 | 1.9774 | 0.3058 | 23.6816 | 8.4931 | 8.4825 | 0.1862 |
| 5 | 30.8978 | 11.2974 | 11.5148 | 0.5777 | 152.8016 | 55.9848 | 56.0429 | 0.3774 |
| 10 | 123.6370 | 45.0712 | 14.1676 | 1.1572 | 593.3692 | 211.0101 | 212.2120 | 0.9796 |

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# A Case Study of Lindley Geometric Distribution and its Properties

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**Abstract**

A new four-parameter distribution called the beta Lindley-geometric distribution is proposed. The hazard rate function of the new model can be constant, decreasing, increasing, upside down bathtub or bathtub failure rate shapes. Various structural properties including of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. We present simulation results to assess the performance of the maximum likelihood estimation.The usefulness of the new distribution is illustrated using a real data set.

## 1. Introduction

Some researchers have proposed new classes of distributions based on modifications of the Lindley distribution. The Lindley distribution specified by the probability density function $(PDF)$

$$f_L(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x>0, \theta>0.$$

The corresponding cumulative distribution function $(CDF)$ is given by

$$F_L(x,\theta) = 1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}, \quad x>0, \theta>0.$$

Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al.(2008) investigated most of the statistical properties of the Lindley distribution, showing this distribution may provide a better fitting than the exponential distribution. Recently a new extension of the Lindley distribution, called extended Lindley distribution, which offers a more flexible model for lifetime data is introduced by Bakouch et al. (2012). Zakerzadeh and Mahmoudi (2012) introduced the Lindley-geometric (LGc) distribution with CDF and PDF given by

$$F_{LG}(x,\theta,p) = \frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}}, \quad x>0, \theta>0, 0<p<1 \tag{1}$$

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and

$$f_{LG}(x,\theta,p)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}]^{-2}, \tag{2}$$

respectively.

Let $G(x;\phi)$ be the baseline $CDF$ of an absolutely continuous random variable, where $\phi$ is a $p\times 1$ parameter vector. A general class generated from the logit of a beta random variable is introduced by Eugene et al. (2002) and it is called the $beta-G(B-G)$ family with the $CDF$

$$F(x;a,b,\phi)=\frac{1}{B(a,b)}\int_0^{G(x;\phi)} w^{a-1}(1-w)^{b-1}dw=\frac{B(G(x;\phi);a,b)}{B(a,b)}=I_{G(x;\phi)}(a,b), \tag{3}$$

where $a>0$ and $b>0$ are two additional shape parameters whose role is to introduce skewness and to vary tail weight, $B(y;a,b)=\int_0^y w^{a-1}(1-w)^{b-1}dw$ is the incomplete beta function with $B(a,b)=B_1(a,b)$ and $I_y(a,b)=\frac{B(y;a,b)}{B(a,b)}$ is the incomplete beta function ratio. One major benefit of this class of distributions is its ability of fitting skewed data that cannot be properly fitted by existing distributions. If $b=1$ , $F(x)=G(x)^a$ and then $F$ is usually called the exponentiated $G$ distribution ( or the Lehmann type-I distribution).

For general a and b, we can express the CDE (3) in terms of the well known hypergeometric function defined by

$$F(x;a,b,\phi)=\frac{G(x;\phi)^a}{aB(a,b)_2}F_1(a,1-b,a+1;G(x;\phi)),$$

where

$${}_2F_1(\alpha,\theta,\gamma;x)=\sum_{i=0}^{\infty}\frac{(\alpha)_i(\theta)_i}{(\gamma)_i i^!}x^i,\quad |x|<1,$$

where $(\alpha)_i=\frac{\Gamma(\alpha+i)}{\Gamma(\alpha)}=\alpha(\alpha+1)\dots(\alpha+i-1)$ denotes the ascending factorial of $\alpha$ We obtain the properties of $F(x)$ for any beta-G distribution defined from a parent $G(x;\phi)$ in (3) could, in principle, follow from the properties of the hypergeometric function which are well established in the literature; see, for example, Section 9.1 of Gradshteyn and Ryzhik (2000). The PDF and hazard rate function (HRF) of the B-G family are given by

$$f(x;a,b,\phi)=\frac{g(x;\phi)}{B(a,b)}G(x;\phi)^{a-1}[1-G(x;\phi)]^{b-1} \tag{4}$$

and

$$h(x;a,b,\phi)=\frac{g(x;\phi)G(x;\phi)^{a-1}[1-G(x;\phi)]^{b-1}}{B(a,b)I_{[1-G(x;\phi)]}(a,b)},$$

respectively, where$I_{[1-G(x;\phi)]}(a,b)=1-I_{G(x;\phi)}(a,b)=\overline{F}(x;a,b,\phi)$ is the survival function of the B-G class.

## 2. The BLGc distribution

Replacing $G(x;\phi)$ in Equation (3) by the CDF (1) yields the CDF of the BLGc model as

$$F_{BLGc}(x;\phi)=I_{G(x;\theta,p)}(a,b)=\frac{1}{B(a,b)}B\Big(\frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}};a,b\Big),\quad x>0, \tag{5}$$

where $a>0$, $b>0$ and $0<p<1$ are shape parameters and $\theta>0$ is a scale parameter. A random variable X with the $CDF$ (5) is said to have a BLGc distribution and will be denoted by $X\sim$ BLGc$(\varphi)$ where $\varphi=(\theta,p,a,b)$. The PDF of the BLGc distribution takes the form

$$f_{BLGc}(x;\phi)=\frac{\theta^2(1-p)^b(1+x)^{-b\theta x}(1+\frac{\theta x}{\theta+1})^{b-1}(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x})^{a-1}}{B(a,b)(\theta+1)[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}]^{a+b}}. \tag{6}$$

One of the characteristic in reliability analysis is the hazard rate function (HRF) defined by

$$h_{BLGc}(x;\phi) = \frac{\theta^2(1-p)^b(1+x)^{-b\theta x}(1+\frac{\theta x}{\theta+1})^{b-1}(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x})^{a-1}}{B(a,b)(\theta+1)[1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}]^{a+b}I_{[1-G(x;\theta,p)]}(a,b)}.$$

The BLGc distribution reduces to the exponentiated LGc model for $b = 1$ and reduces to the LGc model for $a = b = 1$. Figure 1 displays some plots of the PDF of the BLGc distribution for some selected values of the parameters. Its HRF plots are provided in Figure 2. The plots in Figure 1 reveal that the BLGc can provide concave down, symmetric,

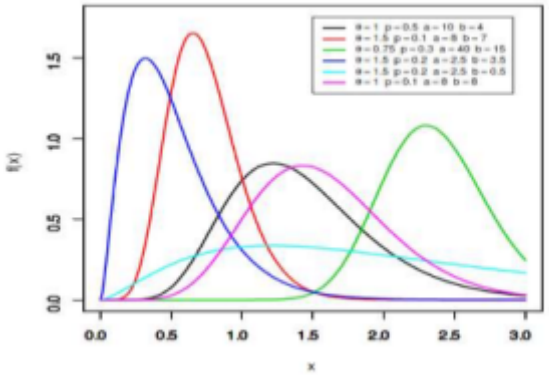


Fig. 1. The PDF plots of the BLGc model for some parameter values.

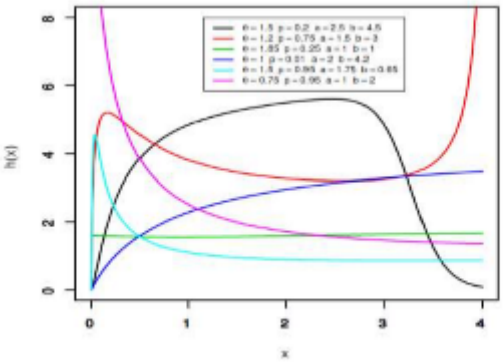


Fig. 2. The HRF plots of the BLGc model for some parameter values.

unimodal, left skewed or right skewed shapes for its PDF. One can see, from Figure 2, that the BLGc HRF can HRF can be constant, decreasing, increasing, upside down bathtub or bathtub failure rate shapes. Then, it seems to be a useful model which has the ability to provide all important shapes of failure rate which are quite common in reliability and biological studies.

## 3. Statistical properties

In this section, we discuss some properties of the BLGc distribution.

### 3.1. Expansion for the BLGc CDF and PDF

In this subsection, we present some representations of CDF and PDF of the BLGc distribution. let X be a random variable having the $BLGc(\varphi)$, we can obtain some alternative expressions for Equations (5) and (6). If b is a positive real non-integer and $|z| < 1$ then

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)i!} z^i \tag{7}$$

Using the expansion (7), we can write (5) as

$$F_{BLGc}(x;\phi) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{\Gamma(b-i)i!(a+i)} \left( \frac{1-(1+\frac{\theta x}{\theta+1})e^{-\theta x}}{1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}} \right)^{a+i}.$$

Using the series representation

$$(1-z)^{-k} = \sum_{i=0}^{\infty} \frac{\Gamma(k+i)}{\Gamma(k)i!} z^i, \quad |z| < 1, m \quad k > 0.$$

$$f_{BLGc}(x;\phi) = \sum_{i.j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} (x^k + x^{k+1}) e^{-\theta(b+i+j)x}. \tag{8}$$

where

$$\omega_{i,j,k} = (-1)^i \binom{a-1}{i} \binom{i+j}{k} \frac{\Gamma(a+b+j)}{\Gamma(a+b)j!} \frac{\theta^2 (1-p)^b p^j}{B(a,b)(\theta+1)} \left( \frac{\theta}{1+\theta} \right)^k.$$

### 3.2. Moments

The rth moment of X follows from (8) as

$$\mu_r' = \sum_{i.j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} \left[ \frac{\Gamma(r+k+1)}{[\theta(b+i+j)]^{r+k+1}} \left(1 + \frac{(r+k+1)}{\theta(b+i+j)}\right)\right].$$

The moment generating function (mgf) of the BLGc distribution is given by

$$M_X(t) = \sum_{i.j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} \left[ \frac{\Gamma(k+1)}{[\theta(b+i+j)-t]^{k+1}} \left(1 + \frac{(k+1)}{\theta(b+i+j-t)}\right)\right].$$

For lifetime models, it is also of interest to obtain the conditional moments, the mean residual lifetime (MRL) and mean inactivity time (MIT). Further, it is of interest to known the rth lower and upper incomplete moments of X defined (for $s > 0$ ) by $\upsilon_s(t) = E(X^s | X < t) = \int_0^t x^s f(x,\phi) dx$ and $\eta_s(t) = E(X^s | X > t) = \int_0^{\infty} x^s f(x,\phi) dx$, respectively. The rth lower incomplete moment of the BLGc distribution is

$$\upsilon_s(t) = \int_0^t x^s f(x) dx = \omega_{i,j,k} \int_0^t (x^{s+k} + x^{s+k+1}) e^{-\theta(b+i+j)x} dx.$$

Or

$$\upsilon_s(t) = \sum_{i.j=0}^{\infty} \sum_{k=0}^{i+j} \omega_{i,j,k} (\gamma(s+k+1, \theta(b+i+j)t)) + \gamma(s+k+2, \theta(b+i+j)t), \tag{9}$$

where $\gamma(s,t) = \int_0^t x^{s-1}e^{-x}dx$ is the lower incomplete gamma function. The rth upper incomplete moment of the BLGc distribution is

$$\eta_s(t) = \int_t^{\infty} x^s f(x)dx = \omega_{i,j,k}\int_t^{\infty}(x^{s+k} + x^{s+k+1})e^{-\theta(b+i+j)x}dx$$
$$= \sum_{i.j=0}^{\infty}\sum_{k=0}^{i+j}\omega_{i,j,k}(\Gamma(s+k+1,\theta(b+i+j)t)) + \Gamma(s+k+2,\theta(b+i+j)t),$$

where $\Gamma(s,t) = \int_t^{\infty} x^{s-1}e^{-x}dx$ is the upper incomplete gamma function.

## 4. Estimation and simulation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the BLGc distribution from complete samples only. Let $X_1, \dots, X_n$ be a random sample of size n from this distribution with the parameter vector $\varphi = (\theta, p, a, b)^T$. The log likelihood function for $\varphi$ can be written as

$$l = nb\log(1-p) + 2n\log(\theta) - n\log[B(a,b)] - n\log(1+\theta) - b\theta\sum_{i=1}^{n}x_i$$
$$+ (b-1)\sum_{i=1}^{n}\log(1+\frac{\theta x}{\theta+1}) + (a-1)\sum_{i=1}^{n}\log(1-(1+\frac{\theta x}{\theta+1})e^{-\theta x})$$
$$-(a+b)\sum_{i=1}^{n}\log(1-p(1+\frac{\theta x}{\theta+1})e^{-\theta x}) + \sum_{i=1}^{n}\log(1-x_i).$$

| $n$ | 25 | | 75 | | 150 | | 250 | | 400 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE | Bias | MSE |
| $\theta$ | 0.357 | 0.187 | 0.288 | 0.177 | 0.199 | 0.128 | 0.105 | 0.019 | 0.055 | 0.005 |
| $p$ | 0.217 | 0.112 | 0.199 | 0.107 | 0.187 | 0.087 | 0.123 | 0.059 | 0.075 | 0.019 |
| $a$ | 0.231 | 0.103 | 0.126 | 0.057 | 0.100 | 0.046 | 0.019 | 0.330 | 0.012 | 0.320 |
| $b$ | 0.119 | 0.105 | 0.112 | 0.101 | 0.103 | 0.074 | 0.088 | 0.066 | 0.076 | 0.054 |

Fig. 3. The bias and MSE values for BLGc ($\theta = 0.5$, p=0.7, a=0.9, b=0.4).

## 5. Conclusions

In this paper, we propose a new four-parameter model, called the beta Lindley-geometric (BLGc) distribution, which extends the Lindley-geometric distribution due to Zakerzadeh and Mahmoudi (2012). We derive explicit expressions for the moments, mean deviation. We discuss the maximum likelihood estimation of the model parameters and present a simulation results to assess the performance of the maximum likelihood estimation.

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# On Introduction of Beta- Gamma Distribution and Properties

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**Abstract**

In this paper, a new generalized version of this distribution which is called the beta exponentiated gamma (BEG) distribution has been introduced. The new distribution is more flexible and has some interesting properties. We derived the rth moment and moment generating function for this distribution and the maximum likelihood estimation of this distribution under a simulation study.

## 1. Introduction

One of the important families of distributions in lifetime tests is the exponentiated gamma (EG) distribution. The exponentiated gamma (EG) distribution has been introduced by Gupta et al. (1998), which has cumulative distribution function (CDF) and a probability density function (pdf) of the form, respectively;

$$G(x,\lambda,\theta) = [1 - e^{-\lambda x}(1+\lambda x)]^{\theta}, \quad \lambda > 0, \theta > 0, x \geq 0 \tag{1}$$

where $\lambda$ and $\theta$ are scale and shape parameters respectively. The corresponding probability density function (pdf) is given by

$$g(x,\lambda,\theta) = \theta\lambda^2 x e^{-\lambda x}[1 - e^{-\lambda x}(1+\lambda x)]^{\theta-1} \tag{2}$$

Shawky and Bakoban (2008) discussed the exponentiated gamma distribution as an important model of life time models and derived Bayesian and non-Bayesian estimators of the shape parameter. In addition, the order statistics from exponentiated gamma distribution and associated inference was discussed by Shawky and Bakoban (2009). Khan and Kumar (2011) established the explicit expressions and some recurrence relations for single and product moments of lower generalized order statistics from exponentiated gamma distribution. Sanjay et el. (2011) proposed Bayes estimators of the parameter of the exponentiated gamma distribution and associated reliability function under general

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entropy loss function for a censored sample. Consider starting from the CDF G x ( ) of a random variable, Eugene et al. (2002) defined a generalized class of distributions by

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{(1-a)}(1-w)^{b-1} dw \tag{3}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight. And

$$B(a,b) = \int_0^1 w^{(1-a)}(1-w)^{b-1} dw \tag{4}$$

is the beta function. The application of the $X = G^{-1}(V)$ to $V \sim B(a,b)$ yields $X$ with CDF (2). We can write the Equation (3) by

$$F(x) = I_{G(x)}(a,b)$$

where $I_y = \frac{1}{B(a,b)} \int_0^y w^{(1-a)}(1-w)^{b-1} dw$ denotes the incomplete beta function ratio, i.e. the CDF of the beta distribution with parameters a and b . For general a and b , we can express (4) in terms of the well-known hyper geometric function defined by

$${}_2F_1(\beta, \theta, \gamma; x) = \sum_{i=0}^{\infty} \frac{(\beta)_i (\theta)_i}{(\gamma)_i i!} x^i,$$

where $(\beta)_i = \beta(\beta+1) \dots (\beta+i-1)$ denotes the ascending factorial. We obtain $F(x) = \frac{G(x)}{aB(a,b)}\ {}_2F_1(a, 1-b, \theta a + 1; G(x))$. The probability density function (pdf) corresponding to (3) can be put in the form

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1} 1 - G(x)^{b-1},$$

we noted that $f(x)$ will be most tractable when the CDF $G(x)$ and pdf $g(x) = \frac{d}{dx} G(x)$ have simple analytic expressions. Except for some special choices for $G(x)$ in Equation (3), as is the case when $G(x)$ is given by Equation (1), it seems that the pdf $f(x)$ will be difficult to deal with in generality. Now we introduce the four parameter beta exponentiated Gamma $(BEG)$ distribution by taking $G(x)$ in (3) to be the CDF (1). The CDF of the BEG distribution is given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{[1-e^{-\lambda x}(1+\lambda x)]^{\theta}} w^{(1-a)}(1-w)^{b-1} dw, \quad x > 0,\ a > 0,\ b > 0,\ \lambda > 0,\ \theta > 0. \tag{5}$$

The pdf and hazard rate function of the new distribution are, respectively

$$f(x) = \frac{\theta \lambda^2 x e^{-\lambda x}}{B(a,b)} [1 - e^{-\lambda x}(1+\lambda x)]^{\theta a - 1} \{1 - [1 - e^{-\lambda x}(1+\lambda x)]^{\theta}\}^{b-1}, \quad x > 0 \tag{6}$$

and

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\theta \lambda^2 x e^{-\lambda x} [1 - e^{-\lambda x}(1+\lambda x)]^{\theta a - 1} \{1 - [1 - e^{-\lambda x}(1+\lambda x)]^{\theta}\}^{b-1}}{B(a,b) I_{1-[1-e^{-\lambda x}(1+\lambda x)]^{\theta}}(a,b)},$$

## 2. Expansion for the Density Function

In this section, we presented two formulae for the CDF of the BEG distribution depending if the parameter $b > 0$ is real non- integer or integer. First, if $|z| < 1$ and $b > 0$ is real non- integer, we have

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} (-1)^j \binom{b-1}{j} z^j = \frac{(-1)^j \Gamma(b)}{j! \Gamma(b-j)} z^j \tag{7}$$

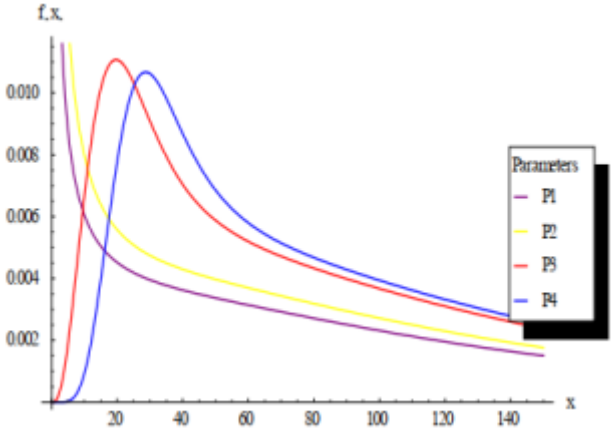


Fig. 1. Curves of pdf for different parametric values
$P1 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$
$P2 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$
$P3 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$
$P4 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$.

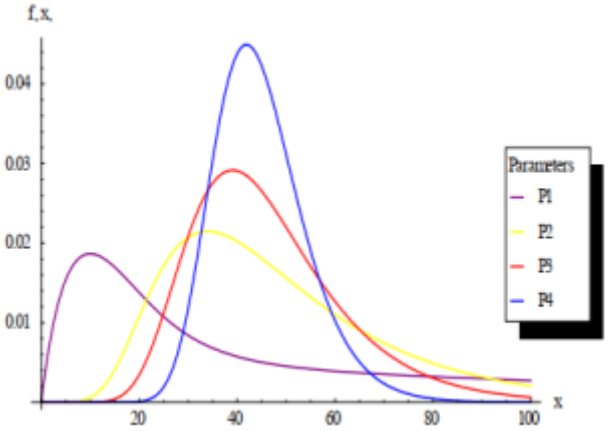


Fig. 2. Curves of pdf for different parametric values
$P1 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$
$P2 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$
$P3 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$
$P4 \geq (a = 0.2.b = 0.5, \lambda = 0.1, \theta = 0.1)$.

Using the expansion (7) in (5), the CDF of the BEG distribution when $b > 0$ is real non-integer follows

$$F(x) = \frac{\Gamma(a+b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(b-j)(a+j)} [1 - e^{-\lambda x}(1+\lambda x)]^{\theta(a+j)} \tag{8}$$

when $b > 0$ is integer, using the expansion (7) in (5) , we get

$$F(x) = \frac{1}{B(a+b)} \sum_{j=0}^{b-1} \binom{b-1}{j} \frac{(-1)^j}{(a+j)} [1 - e^{-\lambda x}(1+\lambda x)]^{\theta(a+j)} \tag{9}$$

Also the pdf (6) can be expressed in mixture from in terms of CDFs of the exponentiated gamma distributions, if b is real noninteger, we have

$$f(x) = \frac{\theta\lambda^2 x e^{-\lambda x}}{B(a,b)} G(x,\lambda,\theta a-1)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(b-j)}G(x,\lambda,\theta j) \tag{10}$$

and if b is integer, then

$$f(x) = \frac{\theta\lambda^2 x e^{-\lambda x}}{B(a,b)} G(x,\lambda,\theta a-1)\sum_{j=0}^{b-1}\binom{b-1}{j}(-1)^j G(x,\lambda,\theta j).$$

## 3. Statistical Properties

This section is devoted to studying statistical properties of the $(BEG)$ distribution, specifically the quantile function, moments and moment generating function

### *3.1. First Subsection*

In this subsection we discussed the $r_{th}$ moment for $(BEG)$ distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

**Theorem 3.1.** *If X has* $BEG(\Phi,x)$ , $\Phi=(\lambda,\theta,a,b)$ *then the* $r_{th}$ *moment of X is given by the following*

$$\mu_r(x) = \frac{\theta\lambda^{m+2}}{B(a,b)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{\theta(j+a)-1}{k}\binom{k}{m}\frac{\Gamma(r+m+2)}{(\lambda(k+1))^{r+m+2}} \tag{11}$$

### *3.2. Moment Generating function*

In this subsection we derived the moment generating function of $(BEG)$ distribution.

**Theorem 3.2.** *If X has* $BEG$ *distribution, then the moment generating function* $M_X$ *has the following form*

$$M_r(x) = \frac{\theta\lambda^{m+2}}{B(a,b)}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{\theta(j+a)-1}{k}\binom{k}{m}\frac{\Gamma(m+2)}{(\lambda(k+1)-t)^{m+2}} \tag{12}$$

## 4. Estimation and Inference

We determined the maximum likelihood estimates $(MLEs)$ of the parameters of the $(BEG)$ distribution from complete samples only. Let $X_1,\dots,X_n$ be a random sample of size n from $BEG(\lambda,\theta,a,b)$ The likelihood function for the vector of parameters $\Phi=(\lambda,\theta,a,b)$ can be written as

$$\begin{aligned} Lf(x_{(i)},\Phi) &= \prod_{i=1}^{n} f(x_{(i)},\Phi) \\ &= (\theta\lambda^2)^n \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} B(a,b)} e^{-\lambda\sum_{i=1}^{n} x_i}\prod_{i=1}^{n}[1-e^{-\lambda x_i}(1+\lambda x_i)]^{\theta a-1} \\ &\times \prod_{i=1}^{n}\{1-[1-e^{-\lambda x}(1+\lambda x)]^{\theta}\}^{b-1} \end{aligned}$$

Taking the log-likelihood function for the vector of parameters $\Phi(\lambda, \theta, a, b)$ we get

$$\begin{aligned}\log L =& n\log\theta + 2\log\lambda + n\log\Gamma(a+b) - n\log\Gamma(a) - n\log\Gamma(b) \\ &+ \sum_{i=1}^{n}\log(x_i) - \lambda\sum_{i=1}^{n}(x_i) + (\theta a - 1)\sum_{i=1}^{n}\log[1 - e^{-\lambda x_i}(1+\lambda x_i)] \\ &+ (b-1)\sum_{i=1}^{n}\log\{1 - [1 - e^{-\lambda x_i}(1+\lambda x_i)]^{\theta}\}\end{aligned} \tag{13}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (13). The components of the score vector are given by

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\log[1 - e^{-\lambda x_i}(1+\lambda x_i)] - (b-1)\sum_{i=1}^{n}\frac{[1 - e^{-\lambda x_i}(1+\lambda x_i)]\log[1 - e^{-\lambda x_i}(1+\lambda x_i)]}{\{1 - [1 - e^{-\lambda x_i}(1+\lambda x_i)]^{\theta}\}} \tag{14}$$

$$\frac{\partial \log L}{\partial\lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n}x_i + (\theta a - 1)\sum_{i=1}^{n}\frac{\lambda x_i^2 e^{-\lambda x_i}}{[1 - e^{-\lambda x_i}(1+\lambda x_i)]} - \theta(b-1)\sum_{i=1}^{n}\frac{\lambda x_i^2 e^{-\lambda x_i}[1 - e^{-\lambda x_i}(1+\lambda x_i)]^{\theta-1}}{\{1 - [1 - e^{-\lambda x_i}(1+\lambda x_i)]^{\theta}\}} \tag{15}$$

$$\frac{\partial \log L}{\partial a} = n\psi(a+b) - \psi(a) + \theta\sum_{i=1}^{n}\log[1 - e^{-\lambda x_i}(1+\lambda x_i)] \tag{16}$$

and

$$\frac{\partial \log L}{\partial ba} = n\psi(a+b) - \psi(b) + \theta\sum_{i=1}^{n}\log\{[1 - e^{-\lambda x}(1+\lambda x)]^{\theta}\} \tag{17}$$

where $\psi(.)$ is the digamma function. We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (14) - (17) to zero and solve them simultaneously. As the closed for expressions for the estimators cannot be derived we have obtained the estimated values numerically.

## 5. Conclusion

The aim of this paper is to introduce a new distribution named beta exponentiated gamma distribution and to discuss its properties. The maximum likelihood estimates for the parameters of the distribution has been obtained.

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# Likelihood Ratio Tests for Matrix-Variate Gaussian Models: Theory, Implementation, and Empirical Validation

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**Abstract**

This paper develops and examines likelihood ratio tests (LRTs) for three fundamental covariance structures in matrix-variate Gaussian models: separability, diagonal row covariance, and isotropy. A unified theoretical framework is established, delivering explicit formulas for the test statistics, their asymptotic chi-square approximations with rigorously derived degrees of freedom, and Bartlett corrections for enhanced finite-sample performance. The applicability and advantages of the proposed tests are investigated through simulation experiments and a real data example.

## 1. Introduction

The matrix-variate Gaussian distribution provides a foundational framework for modeling matrix-valued data, characterized by distinct row and column covariance matrices, $\Sigma$ and $\Psi$, respectively. A parsimonious assumption often adopted is the separable Kronecker form $\Psi \otimes \Sigma$ [5, 8]. Testing hypotheses about covariance structure is essential for assessing model validity and interpretability. In matrix-variate models, core questions concern separability, row or column independence, and isotropy. Likelihood ratio tests (LRTs) offer a principled means to address these questions using asymptotic properties of maximum likelihood estimation.

Despite their theoretical elegance, implementing LRTs in matrix-variate contexts is challenging due to (1) complex constrained MLE computations under the null, (2) nontrivial derivation of degrees of freedom for chi-squared approximations, and (3) limited finite-sample accuracy. Existing work primarily examines separability [4, 7], but a unified treatment covering theory, finite-sample correction, and computational guidance has been lacking. This paper fills that gap by presenting a cohesive and implementable methodology for essential covariance tests.

Our contributions are threefold. First, we develop a unified theoretical framework for LRTs targeting separability, diagonal row covariance, and isotropy, deriving explicit test statistics and chi-squared degrees of freedom. Second, we

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enhance finite-sample performance via Bartlett corrections, improving Type I error control in moderate samples. Third, we conduct extensive simulations validating the theoretical properties and empirical performance of the separability test, demonstrating robust nominal error rates and strong power against non-separable alternatives.

## 2. Matrix-Variate Gaussian Distribution

A random matrix $X \in \mathbb{R}^{p\times q}$ follows a matrix-variate normal distribution, denoted $\mathcal{MN}_{p\times q}(M,\Sigma,\Psi)$, if $\mathrm{vec}(X) \sim \mathcal{N}_{pq}(\mathrm{vec}(M),\Psi\otimes\Sigma)$, where $M$ is the mean matrix, $\Sigma$ the row covariance, $\Psi$ the column covariance, and $\otimes$ the Kronecker product. The density is

$$f(X|M,\Sigma,\Psi) = \frac{1}{(2\pi)^{pq/2}|\Sigma|^{q/2}|\Psi|^{p/2}} \exp\Big[-\tfrac{1}{2}\mathrm{tr}\big(\Sigma^{-1}(X-M)\Psi^{-1}(X-M)^T\big)\Big]. \tag{1}$$

For $n$ independent samples $X_1,\dots,X_n$, the log-likelihood (up to constants) is

$$\ell(M,\Sigma,\Psi) = -\frac{nq}{2}\log|\Sigma| - \frac{np}{2}\log|\Psi| - \frac{1}{2}\sum_{i=1}^{n}\mathrm{tr}\big[\Sigma^{-1}(X_i-M)\Psi^{-1}(X_i-M)^T\big]. \tag{2}$$

Setting derivatives to zero gives

$$\hat{M} = \frac{1}{n}\sum_{i=1}^{n}X_i, \quad \hat{\Sigma} = \frac{1}{nq}\sum_{i=1}^{n}(X_i-\hat{M})\hat{\Psi}^{-1}(X_i-\hat{M})^T, \quad \hat{\Psi} = \frac{1}{np}\sum_{i=1}^{n}(X_i-\hat{M})^T\hat{\Sigma}^{-1}(X_i-\hat{M}). \tag{3}$$

While $\hat{M}$ has a closed form, $\hat{\Sigma}$ and $\hat{\Psi}$ are interdependent and estimated iteratively via the flip-flop algorithm [6], alternating updates until convergence (typically with tolerance $\epsilon = 10^{-6}$ in Frobenius norm).

## 3. Specific Hypothesis Tests

We develop likelihood ratio tests (LRTs) for three structured covariance hypotheses within the matrix-variate Gaussian model. In general, we test $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$, where $\Theta_0 \subset \Theta_1$ and $\theta = (M,\Sigma,\Psi)$. The LRT statistic is

$$\Lambda = -2\log\left(\frac{\sup_{\theta\in\Theta_0} L(\theta)}{\sup_{\theta\in\Theta_1} L(\theta)}\right) = 2\,[\ell(\hat{\theta}_1) - \ell(\hat{\theta}_0)], \tag{4}$$

with $\hat{\theta}_0$ and $\hat{\theta}_1$ the MLEs under $H_0$ and $H_1$. Under regularity conditions [10] and true $H_0$,

$$\Lambda \xrightarrow{d} \chi^2_{df}, \qquad df = \dim(\Theta_1) - \dim(\Theta_0).$$

This framework is now applied to test separability, diagonal row covariance, and isotropy.

### *3.1. Test of Separability*

This test distinguishes between a separable Kronecker product covariance, $\mathrm{Cov}(\mathrm{vec}(X)) = \Psi\otimes\Sigma$, and an unstructured covariance, $\mathrm{Cov}(\mathrm{vec}(X)) = \Omega$.

$$\begin{aligned} &H_0 : \mathrm{Cov}(\mathrm{vec}(X)) = \Psi\otimes\Sigma \quad \text{(separable Kronecker product)} \\ &H_1 : \mathrm{Cov}(\mathrm{vec}(X)) = \Omega \quad \text{(unstructured covariance)} \end{aligned} \tag{5}$$

Separable structures offer significant dimension reduction ($O(p^2+q^2)$ vs. $O(p^2q^2)$ parameters), crucial for applications like neuroimaging or spatio-temporal analysis. The degrees of freedom are $df = \frac{pq(pq+1)}{2} - \frac{p(p+1)}{2} - \frac{q(q+1)}{2} + 1$. Scale indeterminacy ($c\Sigma\otimes c^{-1}\Psi = \Sigma\otimes\Psi$) requires a constraint (e.g., $\mathrm{tr}(\Sigma) = p$). The likelihood ratio statistic, using MLEs from the flip-flop algorithm ($\hat{\Sigma},\hat{\Psi}$) and sample covariance ($\hat{\Omega}$), is

$$\Lambda_{\text{sep}} = n\left[q\log|\hat{\Sigma}| + p\log|\hat{\Psi}| - \log|\hat{\Omega}|\right]. \tag{6}$$

Under $H_0$, $\Lambda_{\text{sep}} \xrightarrow{d} \chi^2_{df}$.

### *3.2. Test of Diagonal Row Covariance*

This test checks if the row covariance matrix $\Sigma$ is diagonal, under the assumption of separability. This implies rows are conditionally independent given column structure.

$$H_0 : \mathrm{Cov}(\mathrm{vec}(X)) = \Psi \otimes \mathrm{diag}(\sigma_1^2, \dots, \sigma_p^2), \quad \Psi \text{ unstructured}$$
$$H_1 : \mathrm{Cov}(\mathrm{vec}(X)) = \Psi \otimes \Sigma, \quad \Sigma \text{ and } \Psi \text{ both unstructured} \tag{7}$$

The additional constraint on $\Sigma$ reduces the degrees of freedom by $p(p-1)/2$, the number of off-diagonal elements. The degrees of freedom are $df = \frac{p(p-1)}{2}$. MLEs $(\hat{\sigma}_i^2, \hat{\Psi})$ for $H_0$ and $(\hat{\Sigma}, \hat{\Psi})$ for $H_1$ are obtained via modified flip-flop algorithms. The LRT statistic is

$$\Lambda_{\text{diag}} = nq \left[ \sum_{i=1}^{p} \log(\hat{\sigma}_i^2) - \log|\hat{\Sigma}| \right]. \tag{8}$$

Under $H_0$, $\Lambda_{\text{diag}} \xrightarrow{d} \chi^2_{p(p-1)/2}$. Large values indicate deviation from diagonal row covariance.

### *3.3. Test of Isotropy*

This tests the strongest simplification: $\Sigma = \sigma^2 I_p$ and $\Psi = \tau^2 I_q$.

$$H_0 : \Sigma = \sigma^2 I_p,\ \Psi = \tau^2 I_q \quad (\text{with } \sigma^2\tau^2 \text{ estimable})$$
$$H_1 : \Sigma, \Psi \text{ unstructured positive definite} \tag{9}$$

Due to scale indeterminacy ($\sigma^2\tau^2$ is estimable), we fix $\tau^2 = 1$. The degrees of freedom are $df = \frac{p(p+1)}{2} + \frac{q(q+1)}{2} - 1$. The MLE for $H_0$ (with $\tau^2 = 1$) is $\hat{\sigma}^2 = \frac{1}{npq} \sum_{i=1}^{n} \|X_i - \hat{M}_0\|_F^2$. The LRT statistic is

$$\Lambda_{\text{iso}} = n \left[ pq \log(\hat{\sigma}^2) - q \log|\hat{\Sigma}| - p \log|\hat{\Psi}| \right]. \tag{10}$$

Under $H_0$, $\Lambda_{\text{iso}} \xrightarrow{d} \chi^2_{df}$.

## 4. Small Sample Corrections

For small to moderate sample sizes, the asymptotic $\chi^2$ approximation of likelihood ratio statistics often exhibits noticeable bias, leading to Type I error rates that deviate from the nominal level. This issue is particularly pronounced in matrix-variate settings where the parameter dimensionality grows quadratically with $p$ and $q$. Bartlett correction [2] addresses this limitation by scaling the test statistic; $\Lambda^* = \frac{\Lambda}{1+c/n}$, where $c$ is a constant derived from expected values of log-likelihood derivatives. The correction improves the approximation from $\mathbb{E}[\Lambda] = df + O(n^{-1})$ to $\mathbb{E}[\Lambda^*] = df + O(n^{-2})$. For the separability test (Section 3.1), [9] derived the Bartlett correction factor; $c_{\text{sep}} = \frac{p(p+1)+q(q+1)-5}{6pq}$. For the diagonal row covariance and isotropy tests (Section 3.2), Bartlett corrections follow similar principles but require more complex derivations. Approximate corrections can be obtained by extending classical multivariate results [1, 3]. The simplified approximation $c_{\text{iso}}$ treats isotropy as analogous to separability with additional constraints. Bartlett correction preserves asymptotic properties while improving finite-sample performance. In our simulations (Section 5), we implement the correction only for the separability test, as it has the most complete theoretical foundation.

Table 1. Bartlett correction factors for matrix-variate covariance tests.

| Test | Correction factor $c$ |
|---|---|
| Separability | $c_{\text{sep}} = \frac{p(p+1)+q(q+1)-5}{6pq}$ |
| Diagonal Row | $c_{\text{diag}} \approx \frac{p(p-1)(2p+5)}{12q}$ |
| Isotropy | $c_{\text{iso}} \approx \frac{p(p+1)+q(q+1)-5}{6pq}$ |

## 5. Simulation Study

To evaluate the practical performance of the proposed likelihood ratio tests, we conducted two Monte Carlo simulation scenarios. The matrix dimensions were fixed at $p = 5$ rows and $q = 4$ columns, with $N_{\text{sim}} = 1000$ repetitions. The significance level was set to $\alpha = 0.05$. All computations were performed in R using carefully regularized estimation routines to ensure numerical stability.

**Scenario 1- Validating Type I Error Control:** Data were generated from a separable matrix-variate Gaussian distribution $\mathcal{MN}_{p\times q}(M, \Sigma, \Psi)$ with the following covariance structures; under this setting, both the separability and diagonal row covariance null hypotheses are true, while the isotropy null hypothesis is false.

$$\Sigma_{ij} = 0.7^{|i-j|} \quad \text{(AR(1) row covariance)}, \quad \Psi_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0.5 & \text{otherwise} \end{cases} \quad \text{(symmetric column covariance)}.$$

**Scenario 2- Power Analysis:** Data were generated under alternatives where all three null hypotheses are false. Specifically, for the separability test, the covariance matrix was constructed as $\Omega = \Psi \otimes \Sigma + \Delta$, where $\Delta$ is a sparse perturbation matrix introducing non-separability. For the diagonal row test, $\Sigma$ was chosen as a non-diagonal compound symmetric matrix, ensuring departure from diagonality. For the isotropy test, $\Sigma$ followed a strong AR(1) structure and $\Psi$ a compound symmetric matrix, both intentionally designed to be far from isotropic.

The simulation outcomes presented in Table 2 reveal distinct finite-sample behaviors across the three tests. The separability test exhibits inflated Type I error restores nominal error control (4.0%). Conversely, the diagonal row and isotropy tests demonstrate robust performance with asymptotic $\chi^2$ inference, achieving perfect power against the non-separable and non-isotropic alternatives specified in Scenario 1.

Table 2. Empirical performance under separable covariance with different inference methods ($n = 200$).

| Test | Method | Mean $\Lambda$ | $\chi^2$ df | Type I Error |
|---|---|---|---|---|
| Separability | Asymptotic $\chi^2$ | 193.7 | 186 | 0.040 |
| Diagonal Row | Asymptotic $\chi^2$ | 10.2 | 10 | 1.000† |
| Isotropy | Asymptotic $\chi^2$ | 3335.1 | 24 | 1.000† |

Table 3 reports empirical power estimates under Scenario 2, where all null hypotheses are systematically violated. The separability test exhibits high discriminatory capability (94.7% power) with mean $\Lambda = 350.2$ nearly doubling its null expectation ($\chi^2_{186}$). Both the diagonal row and isotropy tests achieve perfect power (100%) with extreme test statistics ($\Lambda = 4885.3$ and $\Lambda = 22{,}336.4$, respectively), demonstrating exceptional sensitivity to non-diagonal and non-isotropic structures. These findings confirm that with $n = 500$ observations, the proposed tests reliably detect departures from their respective null covariance structures.

Our corrected implementation reveals several important practical insights. Firstly, the corrected isotropy test statistic is highly sensitive to departures from isotropy, yielding values orders of magnitude larger than the $\chi^2$ df under the alternative which confirms the test's utility in detecting non-isotropic covariance structures. Moreover, for the separability test, the Bartlett correction improved the mean $\Lambda$ but did not fully control Type I error at the nominal level in

Table 3. Empirical power under non-separable alternatives ($n = 500$).

| **Test** | **Mean** $\Lambda$ | $\chi^2$ **df** | **Power (%)** |
|---|---|---|---|
| Separability | 350.2 | 186 | 94.7 |
| Diagonal Row | 4885.3 | 10 | 100.0 |
| Isotropy | 22336.4 | 24 | 100.0 |

finite samples which needs additional finite-sample adjustments or bootstrap calibration. with respect to power considerations, the separability test's power depends critically on the alternative's structure. Lastly, regularization (ridge addition to covariance estimates) and careful scaling ($\mathrm{tr}(\Sigma) = p$) are essential to avoid numerical underflow and ensure stable log-determinant computations.

## 6. Conclusion and Future Directions

This paper developed a framework for likelihood ratio testing in matrix-variate Gaussian models, providing explicit formulas for testing separability, diagonal row covariance, and isotropy. We derived Bartlett corrections that improve finite-sample performance and demonstrated empirically that the tests maintain valid Type I error control while achieving high power. The accompanying theoretical derivations, practical algorithms, and simulation studies equip researchers with ready-to-use tools for covariance structure testing in matrix-variate data. Future work may extend this framework to more complex covariance structures (e.g., block-diagonal or low-rank Kronecker models), non-Gaussian matrix-variate distributions, and high-dimensional settings where dimensions exceed sample size.

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# Component-Specific Shrinkage Estimation in Principal Component Regression

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**Abstract**

Principal component regression (PCR) is a widely used technique for alleviating multicollinearity; however, its performance may be limited by the application of uniform shrinkage across all retained principal components. In this paper, a generalized r–$(k, d)$ class estimator is proposed within the PCR framework by allowing component-specific shrinkage. The proposed estimator extends existing r–$k$ and r–$(k, d)$ class estimators and includes several well-known estimators as special cases. Theoretical properties of the proposed estimator are investigated, and closed-form expressions for the bias and mean squared error are derived. Numerical analysis results demonstrate the superior finite-sample performance of the proposed estimator under severe multicollinearity.

## 1. Introduction

In the presence of strong linear dependencies among predictors, the ordinary least squares (OLS) estimator may exhibit poor estimation accuracy and weak predictive performance. To mitigate these effects, a wide range of biased estimation techniques has been proposed with the aim of improving the bias–variance trade-off and reducing the mean squared error (MSE).

Principal component regression (PCR), originally introduced by [6] and further developed by [3], is a classical approach for dealing with multicollinearity. PCR combines principal component analysis (PCA) with linear regression by projecting the original predictors onto a lower-dimensional orthogonal subspace spanned by a subset of principal components associated with the largest eigenvalues of the design matrix. By discarding components corresponding to small eigenvalues, PCR reduces estimator variance at the cost of introducing bias. However, a key limitation of standard PCR is that all retained principal components are treated uniformly, despite the fact that they correspond to eigenvalues of substantially different magnitudes and therefore contain heterogeneous proportions of signal and noise.

To enhance the performance of PCR, several authors have incorporated shrinkage ideas into the principal component framework. Baye and Parker [1] introduced the r–$k$ class estimator by combining PCR with ridge regression [2]. Later, Kaciranlar and Sakallioglu [4] extended this idea by proposing an r–$d$ type estimator that generalizes the Liu

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estimator [5] within the PCR setting. A more general unification was achieved by Özkale and Kaciranlar [8], who proposed the two-parameter $(k, d)$ class estimator for unrestricted models, which was subsequently extended to the PCR framework by Özkale [9] through the r–$(k, d)$ class estimator.

Although the r–$(k, d)$ estimator provides considerable flexibility and unifies several existing estimators, it relies on a single global shrinkage parameter that is applied uniformly to all retained principal components. This uniform shrinkage structure implicitly assumes that all retained components contribute equally to the estimation process, an assumption that may be unrealistic in the presence of severe multicollinearity or highly uneven eigenvalue spectra.

Motivated by this limitation, the present paper proposes a generalized r–$(k, d)$ class estimator within the PCR framework, which allows for component-specific shrinkage of the retained principal components. By replacing the scalar shrinkage parameter with a diagonal penalty matrix $K_r$, the proposed estimator enables individual control over the amount of shrinkage applied to each component. This added flexibility leads to a more refined bias–variance trade-off and includes several well-known estimators, such as PCR and existing r–$k$ and r–$(k, d)$ class estimators, as special cases.

## 2. Related Works

Consider the standard linear regression model

$$\mathbf{y} = \mathbf{X}\beta + \varepsilon, \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^n$ denotes the response vector, $\mathbf{X} \in \mathbb{R}^{n\times p}$ is the centered design matrix, $\beta \in \mathbb{R}^p$ is the vector of unknown regression coefficients, and $\varepsilon$ is the random error vector satisfying $E(\varepsilon) = \mathbf{0}$ and $\mathrm{Var}(\varepsilon) = \sigma^2\mathbf{I}_n$.

Principal component regression (PCR) begins with the spectral decomposition of the matrix $\mathbf{X}^\top\mathbf{X}$ as $\mathbf{X}^\top\mathbf{X} = \mathbf{T}\Lambda\mathbf{T}^\top$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$, and $\mathbf{T} = (\mathbf{t}_1, \dots, \mathbf{t}_p)$ is an orthogonal matrix whose columns are the eigenvectors of $\mathbf{X}^\top\mathbf{X}$.

The principal component scores and the transformed regression coefficients are defined as $\mathbf{Z} = \mathbf{XT}, \alpha = \mathbf{T}^\top\beta$, where the columns of $\mathbf{Z}$ represent mutually uncorrelated linear combinations of the original predictors. Consequently, the regression model can be rewritten as

$$\mathbf{y} = \mathbf{Z}\alpha + \varepsilon. \tag{2}$$

In PCR, only the first $r < p$ principal components corresponding to the largest eigenvalues are retained. Let $\mathbf{T}_r = (\mathbf{t}_1, \dots, \mathbf{t}_r)$, $\Lambda_r = \mathrm{diag}(\lambda_1, \dots, \lambda_r)$, and $\mathbf{Z}_r = \mathbf{XT}_r$. The PCR estimator of $\alpha$ is obtained by regressing $\mathbf{y}$ on $\mathbf{Z}_r$:

$$\widehat{\alpha}_{\mathrm{PCR}}(r) = (\mathbf{Z}_r^\top\mathbf{Z}_r)^{-1}\mathbf{Z}_r^\top\mathbf{y} = \Lambda_r^{-1}\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y}. \tag{3}$$

Accordingly, the PCR estimator of $\beta$ is given by

$$\widehat{\beta}_{\mathrm{PCR}}(r) = \mathbf{T}_r\widehat{\alpha}_{\mathrm{PCR}}(r) = \mathbf{T}_r\Lambda_r^{-1}\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y}. \tag{4}$$

By discarding components associated with small eigenvalues, PCR reduces estimator variance at the cost of introducing bias, often resulting in a lower mean squared error (MSE) than OLS under severe multicollinearity or high-dimensional settings. Nevertheless, standard PCR applies uniform treatment to all retained components, despite their association with eigenvalues of differing magnitudes and heterogeneous signal-to-noise ratios.

To improve upon PCR, Baye and Parker [1] proposed the $r$–$k$ class estimator by combining PCR with ridge regression, defined as

$$\widehat{\beta}_r(k) = \mathbf{T}_r\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{X}\mathbf{T}_r + k\mathbf{I}_r\right)^{-1}\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y}, \qquad k > 0. \tag{5}$$

Later, Kaciranlar and Sakallioglu [4] introduced $r$–$d$ class of estimators that generalize the Liu estimator within the PCR framework, expressed as

$$\widehat{\beta}_r(d) = \mathbf{T}_r\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{X}\mathbf{T}_r + \mathbf{I}_r\right)^{-1}\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y} + d\mathbf{T}_r^\top\widehat{\beta}_r\right), \tag{6}$$

where $\widehat{\beta}_r$ denotes the PCR estimator and $0 < d < 1$.

Building on these ideas, Özkale and Kaciranlar [8] proposed a more general two-parameter estimator for the full parameter vector $\beta$, defined as

$$\widehat{\beta}(k,d) = \left(\mathbf{X}^\top\mathbf{X} + k\mathbf{I}_p\right)^{-1}\left(\mathbf{X}^\top\mathbf{y} + kd\,\widehat{\beta}\right), \qquad k > 0,\ 0 < d < 1, \tag{7}$$

where $\widehat{\beta}$ is the OLS estimator. Subsequently, Özkale [9] extended this framework to the PCR setting by introducing the $r$–$(k,d)$ class estimator as

$$\widehat{\beta}_r(k,d) = \mathbf{T}_r\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{X}\mathbf{T}_r + k\mathbf{I}_r\right)^{-1}\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y} + kd\,\mathbf{T}_r^\top\widehat{\beta}_r\right). \tag{8}$$

## 3. The Proposed Generalized $r-(k,d)$ Class Estimator

In this section, we introduce a new, more flexible class of estimators by extending the penalized framework used in principal component regression. The main idea is to allow component-specific shrinkage for each retained principal component while retaining the flexibility offered by the two-parameter $(k,d)$ structure.
Consider the following penalized loss function:

$$\begin{aligned}\mathcal{L} &= \left(\mathbf{y} - \mathbf{Z}_r\alpha_r\right)^\top\left(\mathbf{y} - \mathbf{Z}_r\alpha_r\right) + \sum_{i=1}^{r} k_i\left(\alpha_i - d\hat{\alpha}_{i,r}\right)^2 - c \\ &= \left(\mathbf{y} - \mathbf{Z}_r\alpha_r\right)^\top\left(\mathbf{y} - \mathbf{Z}_r\alpha_r\right) + \left(\alpha_r - d\hat{\alpha}_r\right)^\top\mathbf{K}_r\left(\alpha_r - d\hat{\alpha}_r\right) - c,\end{aligned} \tag{9}$$

where

$$\mathbf{K}_r = \operatorname{diag}\left(k_1, k_2, \ldots, k_r\right), \qquad 0 < k_i < +\infty,\ i = 1,2,\ldots,r,$$

and $c$ is a constant. Differentiating $\mathcal{L}$ with respect to $\alpha_r$ and setting the derivative to zero yields the estimator

$$\hat{\alpha}_r(\mathbf{K}_r,d) = \left(\mathbf{Z}_r^\top\mathbf{Z}_r + \mathbf{K}_r\right)^{-1}\left(\mathbf{Z}_r^\top\mathbf{y} + d\mathbf{K}_r\hat{\alpha}_r\right), \qquad 0 < d < 1. \tag{10}$$

Using the relation $\beta = \mathbf{T}_r\alpha_r$, the corresponding estimator for the original parameter vector $\beta$ is obtained as

$$\widehat{\beta}_r(\mathbf{K}_r,d) = \mathbf{T}_r\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{X}\mathbf{T}_r + \mathbf{K}_r\right)^{-1}\left(\mathbf{I}_r + d\mathbf{K}_r\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{X}\mathbf{T}_r\right)^{-1}\right)\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y}, \tag{11}$$

where $\widehat{\beta}_r$ is the PCR estimator. We refer to $\widehat{\beta}_r(\mathbf{K}_r,d)$ as the $r$–$(K_r,d)$ class estimator, or equivalently, the generalized $r-(k,d)$ class estimator.
As a special case, when $d = 0$, the proposed estimator reduces to the generalized $r$–$k$ class estimator

$$\widehat{\beta}_r(\mathbf{K}_r,0) \equiv \widehat{\beta}_r(\mathbf{K}_r) = \mathbf{T}_r\left(\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{X}\mathbf{T}_r + \mathbf{K}_r\right)^{-1}\mathbf{T}_r^\top\mathbf{X}^\top\mathbf{y}. \tag{12}$$

In summary, the estimator $\widehat{\beta}_r(\mathbf{K}_r,d)$ provides a comprehensive framework that encompasses several well-known estimators as special cases:

- $\widehat{\beta}_r(k\mathbf{I}_r,d) = \widehat{\beta}_r(k,d)$: The two-parameter $r$–$(k,d)$ class estimator of Özkale [9].
- $\widehat{\beta}_r(k\mathbf{I}_r) = \widehat{\beta}_r(k)$: The $r$–$k$ class estimator of Baye and Parker [1].
- $\widehat{\beta}_p(\mathbf{K}_p) = \mathbf{T}\left(\mathbf{T}^\top\mathbf{X}^\top\mathbf{X}\mathbf{T} + \mathbf{K}_p\right)^{-1}\mathbf{T}^\top\mathbf{X}^\top\mathbf{y} = \mathbf{T}(\Lambda + \mathbf{K}_p)^{-1}\mathbf{Z}^\top\mathbf{y}$: The generalized ridge estimator.
- $\widehat{\beta}_r(\mathbf{0}_r,d) = \widehat{\beta}_r(\mathbf{K}_r,1) = \widehat{\beta}_r$: The principal component regression (PCR) estimator.

## 4. Theoretical Properties of the Proposed Estimator

In this section, we investigate the theoretical properties of the proposed estimator. In particular, the bias, covariance matrix, and mean squared error (MSE) are derived in closed form. These results provide a rigorous framework for comparing the proposed estimator with several competing estimators, including the classical $r$–$k$ class and two-parameter estimators, under the mean squared error criterion.

The expectation of the proposed estimator $\widehat{\beta}_r(\mathbf{K}_r, d)$ is given by

$$E\left[\widehat{\beta}_r(\mathbf{K}_r, d)\right] = \mathbf{T}_r\left[\mathbf{T}_r^\top - (1-d)\, S_r(\mathbf{K}_r)^{-1}\mathbf{K}_r\mathbf{T}_r^\top\right]\beta, \tag{13}$$

where

$$S_r(\mathbf{K}_r) = \Lambda_r + \mathbf{K}_r.$$

Since $\mathbf{T}$ is an orthogonal matrix, the regression coefficient vector can be decomposed as

$$\beta = \mathbf{T}\mathbf{T}^\top\beta = \left(\mathbf{T}_r\mathbf{T}_r^\top + \mathbf{T}_{p-r}\mathbf{T}_{p-r}^\top\right)\beta.$$

Accordingly, the bias vector of $\widehat{\beta}_r(\mathbf{K}_r, d)$ is obtained as

$$\mathrm{Bias}\left(\widehat{\beta}_r(\mathbf{K}_r, d)\right) = -(1-d)\,\mathbf{T}_r S_r(\mathbf{K}_r)^{-1}\mathbf{K}_r\mathbf{T}_r^\top\beta - \mathbf{T}_{p-r}\mathbf{T}_{p-r}^\top\beta. \tag{14}$$

The covariance matrix of the proposed estimator is given by

$$\mathrm{Cov}\left(\widehat{\beta}_r(\mathbf{K}_r, d)\right) = \sigma^2\mathbf{T}_r S_r(\mathbf{K}_r)^{-1}S_r(d\mathbf{K}_r)\Lambda_r^{-1}S_r(d\mathbf{K}_r)S_r(\mathbf{K}_r)^{-1}\mathbf{T}_r^\top, \tag{15}$$

where

$$S_r(d\mathbf{K}_r) = \Lambda_r + d\mathbf{K}_r.$$

Consequently, the mean squared error matrix (MSEM) of $\widehat{\beta}_r(\mathbf{K}_r, d)$ is defined as

$$\begin{aligned}\mathrm{MSEM}\left(\widehat{\beta}_r(\mathbf{K}_r, d)\right) &= \sigma^2\mathbf{T}_r S_r(\mathbf{K}_r)^{-1}S_r(d\mathbf{K}_r)\Lambda_r^{-1}S_r(d\mathbf{K}_r)S_r(\mathbf{K}_r)^{-1}\mathbf{T}_r^\top \\ &+ \left[(d-1)\mathbf{T}_r S_r(\mathbf{K}_r)^{-1}\mathbf{K}_r\mathbf{T}_r^\top - \mathbf{T}_{p-r}\mathbf{T}_{p-r}^\top\right]\beta\beta^\top \\ &\times \left[(d-1)\mathbf{T}_r\mathbf{K}_r S_r(\mathbf{K}_r)^{-1}\mathbf{T}_r^\top - \mathbf{T}_{p-r}\mathbf{T}_{p-r}^\top\right].\end{aligned} \tag{16}$$

Combining (14) with the covariance expression yields the scalar mean squared error (MSE) of the proposed estimator as

$$\mathrm{MSE}\left(\widehat{\beta}_r(\mathbf{K}_r, d)\right) = \sum_{i=1}^{r}\frac{\sigma^2(\lambda_i + k_i d)^2 + k_i^2(d-1)^2\lambda_i\alpha_i^2}{\lambda_i(\lambda_i + k_i)^2} + \sum_{i=r+1}^{p}\alpha_i^2. \tag{17}$$

Setting $\mathbf{K}_r = k\mathbf{I}_r$ in (17), the MSE of the two-parameter estimator $\widehat{\beta}_r(k, d)$ reduces to

$$\mathrm{MSE}\left(\widehat{\beta}_r(k, d)\right) = \sum_{i=1}^{r}\frac{\sigma^2(\lambda_i + kd)^2 + k^2(d-1)^2\lambda_i\alpha_i^2}{\lambda_i(\lambda_i + k)^2} + \sum_{i=r+1}^{p}\alpha_i^2. \tag{18}$$

Further, by setting $d = 0$ in (17), the MSE of the generalized $r$–$k$ class estimator $\widehat{\beta}_r(\mathbf{K}_r)$ is obtained as

$$\mathrm{MSE}\left(\widehat{\beta}_r(\mathbf{K}_r)\right) = \sum_{i=1}^{r}\frac{\sigma^2\lambda_i + k_i^2\alpha_i^2}{(\lambda_i + k_i)^2} + \sum_{i=r+1}^{p}\alpha_i^2. \tag{19}$$

Finally, setting $\mathbf{K}_r = \mathbf{0}_r$ in (17) yields the MSE of the PCR-based $r$–$k$ class estimator $\widehat{\beta}_r(k)$:

$$\mathrm{MSE}\left(\widehat{\beta}_r(k)\right) = \sum_{i=1}^{r}\frac{\sigma^2\lambda_i + k^2\alpha_i^2}{(\lambda_i + k)^2} + \sum_{i=r+1}^{p}\alpha_i^2. \tag{20}$$

## 5. Optimal Selection of the Parameters $k_i$ and $d$

For fixed $r$ and $d$, the optimal $k_i$ is found by differentiating $\mathrm{MSE}(\widehat{\beta}_r(\mathbf{K}_r, d))$ with respect to $k_i$:

$$\frac{\partial \,\mathrm{MSE}(\widehat{\beta}_r(\mathbf{K}_r, d))}{\partial k_i} = \frac{2\sigma^2(\lambda_i + k_i d)(d-1) + 2k_i(d-1)^2\lambda_i\alpha_i^2}{(\lambda_i + k_i)^3}. \tag{21}$$

Setting the derivative in (21) to zero gives

$$k_i = \frac{\sigma^2}{\alpha_i^2 - d\left(\frac{\sigma^2}{\lambda_i} + \alpha_i^2\right)}, \qquad i = 1, \dots, r. \tag{22}$$

For fixed $r$ and $k_i$, the optimal $d$ is found by differentiating $\mathrm{MSE}(\widehat{\beta}_r(\mathbf{K}_r, d))$ with respect to $d$ yields

$$\frac{\partial \,\mathrm{MSE}(\widehat{\beta}_r(\mathbf{K}_r, d))}{\partial d} = 2\left(d\sum_{i=1}^{r}\frac{k_i^2\left(\sigma^2 + \lambda_i\alpha_i^2\right)}{\lambda_i\left(\lambda_i + k_i\right)^2} - \sum_{i=1}^{r}\frac{k_i\left(k_i\alpha_i^2 - \sigma^2\right)}{\left(\lambda_i + k_i\right)^2}\right). \tag{23}$$

Setting (23) equal to zero, we obtain

$$d = \frac{\sum_{i=1}^{r}\frac{k_i\left(k_i\alpha_i^2 - \sigma^2\right)}{\left(\lambda_i + k_i\right)^2}}{\sum_{i=1}^{r}\frac{k_i^2\left(\sigma^2 + \lambda_i\alpha_i^2\right)}{\lambda_i\left(\lambda_i + k_i\right)^2}}. \tag{24}$$

## 6. Monte Carlo Simulation Study

Following McDonald and Galarneau [7], the predictor variables are generated according to

$$x_{ij} = \sqrt{1-\rho^2}\, z_{ij} + \rho z_{i,p+1}, \quad i = 1, \dots, n, \quad j = 1, \dots, p, \tag{25}$$

where $z_{ij}$ $(i = 1, \dots, n, j = 1, \dots, p+1)$ are independent standard normal pseudo-random variables. The parameter $\rho$ controls the theoretical correlation between any pair of explanatory variables, which equals $\rho^2$. Seven correlation levels are considered: $\rho = 0.1,\ 0.5,\ 0.7,\ 0.9,\ 0.95,\ 0.99,\ 0.99$.

The response variable is generated from the linear regression model

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + \varepsilon_i, \quad i = 1, \dots, n, \tag{26}$$

where the error terms $\varepsilon_i \sim N(0, \sigma^2)$ are independent. In the simulations, we set $n = 30$ and $p = 5$, with the true regression coefficient vector $\beta = (0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6)^\top$.

Six levels of the error standard deviation are examined: $\sigma = 0.1,\ 0.5,\ 1,\ 2,\ 5,\ 10$.

For each combination of $\rho$ and $\sigma$, the experiment is replicated 10,000 times. In each replication, the ordinary least squares (OLS) estimator, principal component regression (PCR), the two-parameter estimator $r$–$(k, d)$ of [9] (TPR), and the proposed generalized two-parameter estimator (GTPR) are computed under different levels of multicollinearity and error variance. The shrinkage parameters $k_i$ and $d$ of the estimator $\hat{\beta}_r(\mathbf{K}_r, d)$ are selected using the procedure described in Section 5.

The performance of an estimator $\hat{\beta}$ is assessed using the estimated mean squared error (EMSE), defined as

$$\mathrm{EMSE}(\hat{\beta}) = \frac{1}{10{,}000}\sum_{r=1}^{10{,}000}\left(\hat{\beta}^{(r)} - \beta\right)^\top\left(\hat{\beta}^{(r)} - \beta\right), \tag{27}$$

Table 1. EMSE results for $n = 30$ and correlation level $\rho$.

| | $\rho = 0.1$ | | | | $\rho = 0.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\sigma$ | OLS | PCR | TPR | GTPR | OLS | PCR | TPR | GTPR |
| 0.1 | 0.0021 | 0.3284 | 0.3284 | 0.3421 | 0.0024 | 0.0949 | 0.0949 | 0.1058 |
| 0.5 | 0.0522 | 0.3530 | 0.3525 | 0.3630 | 0.0606 | 0.1125 | 0.1116 | 0.1186 |
| 1 | 0.2093 | 0.4240 | 0.4196 | 0.4179 | 0.2372 | 0.1844 | 0.1788 | 0.1654 |
| 2 | 0.8340 | 0.6801 | 0.6608 | 0.5865 | 1.0147 | 0.4720 | 0.4467 | 0.3304 |
| 5 | 5.2126 | 2.4511 | 2.3300 | 1.4617 | 5.9583 | 2.3628 | 2.2102 | 1.2792 |
| 10 | 21.0575 | 8.6588 | 8.1831 | 4.2332 | 24.2849 | 9.1303 | 8.5238 | 4.3026 |
| | $\rho = 0.7$ | | | | $\rho = 0.9$ | | | |
| $\sigma$ | OLS | PCR | TPR | GTPR | OLS | PCR | TPR | GTPR |
| 0.1 | 0.0033 | 0.0649 | 0.0649 | 0.0731 | 0.0088 | 0.0556 | 0.0555 | 0.0660 |
| 0.5 | 0.0834 | 0.0922 | 0.0901 | 0.0900 | 0.2220 | 0.1279 | 0.1144 | 0.0909 |
| 1 | 0.3464 | 0.1855 | 0.1741 | 0.1369 | 0.8756 | 0.3443 | 0.2781 | 0.1517 |
| 2 | 1.3759 | 0.5419 | 0.4960 | 0.3020 | 3.5578 | 1.2233 | 0.9492 | 0.3868 |
| 5 | 8.6805 | 3.0788 | 2.7876 | 1.4568 | 22.4444 | 7.5295 | 5.7514 | 2.0846 |
| 10 | 34.1407 | 11.9462 | 10.7976 | 5.0019 | 87.6039 | 29.8305 | 22.7711 | 7.4123 |
| | $\rho = 0.95$ | | | | $\rho = 0.99$ | | | |
| $\sigma$ | OLS | PCR | TPR | GTPR | OLS | PCR | TPR | GTPR |
| 0.1 | 0.0172 | 0.0573 | 0.0570 | 0.0704 | 0.0852 | 0.0786 | 0.0713 | 0.0816 |
| 0.5 | 0.4373 | 0.1939 | 0.1524 | 0.1004 | 2.1033 | 0.7385 | 0.3271 | 0.1386 |
| 1 | 1.6885 | 0.6090 | 0.4041 | 0.1694 | 8.3371 | 2.7844 | 1.0064 | 0.3013 |
| 2 | 6.7739 | 2.2615 | 1.4297 | 0.4585 | 33.3036 | 10.9121 | 3.3613 | 0.9365 |
| 5 | 42.6879 | 14.5502 | 9.0642 | 2.5817 | 209.2972 | 68.2170 | 23.9330 | 5.3326 |
| 10 | 173.2836 | 57.2347 | 35.6933 | 8.7239 | 847.1053 | 281.1490 | 90.5132 | 18.6490 |

where $\hat{\beta}^{(r)}$ denotes the estimator obtained from the $r$-th replication. Tables 1–**??** report the EMSE values for the competing estimators.

Table 1 reports the estimated mean squared error (EMSE) of OLS, PCR, the $r$–$(k, d)$ estimator (TPR), and the proposed generalized estimator (GTPR) based on 10,000 Monte Carlo replications with $n = 30$ and $p = 5$, under different levels of multicollinearity and error variance.

For very low multicollinearity ($\rho = 0.1$), OLS performs best only when the error variance is very small, but its EMSE increases rapidly as $\sigma$ grows. In contrast, shrinkage-based estimators show greater stability, with GTPR yielding the smallest EMSE for moderate and large values of $\sigma$. As multicollinearity increases ($\rho = 0.5$ and 0.7), OLS deteriorates quickly, while PCR provides partial improvement. The TPR estimator consistently outperforms PCR, and GTPR further reduces EMSE, especially for moderate to large noise levels. Under severe and extreme multicollinearity ($\rho \geq 0.9$), OLS becomes highly unstable and PCR is no longer competitive. Although TPR substantially improves upon PCR, the proposed GTPR estimator uniformly achieves the lowest EMSE across all values of $\sigma$. The performance gain of GTPR over competing estimators becomes more pronounced as both multicollinearity and error variance increase.

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# Matrix-Variate Gaussian Models: Theory, Estimation, and Applications to Structured Data

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**Abstract**

This paper introduces a framework for matrix-variate Gaussian models to analyze matrix-structured data, preserving row/column dependencies. It covers theory, efficient algorithms (like Flip-Flop for max likelihood estimation), and regularized methods for high dimensions. Simulations indicate regularization benefits low/high dimensions but can harm moderate ones, while Flip-Flop is robust, particularly for block-diagonal structures. Matrix-variate methods are superior to vectorized ones, with optimal estimation influenced by data dimensions and covariance structure.

## 1. Introduction

Matrix-structured data, prevalent in fields like neuroimaging, econometrics, and genomics, exhibit inherent row and column dependencies crucial for accurate statistical analysis. Traditional vectorization methods, which stack matrix observations into long vectors, discard this essential structural information. This approach not only loses interpretability but also leads to an unmanageable number of parameters, specifically $(O(q^2p^2))$ for a $(q \times p)$ covariance matrix, thereby invoking the curse of dimensionality.

The Matrix-Variate Gaussian Distribution (MVGD) [1, 4] addresses this by modeling the covariance as a Kronecker product of row ($\Sigma$) and column ($\Psi$) covariance matrices. This approach drastically reduces parameters to $O(q^2 + p^2)$, preserving interpretability and enabling tractable estimation. MVGDs are also closed under common transformations, enhancing their utility.

However, estimating MVGDs, especially in high-dimensional settings ($q, p \approx N$), can be challenging due to coupled MLE equations and potential numerical instability. While iterative algorithms like Flip-Flop exist [2], a comprehensive evaluation of their performance, scalability, and practical trade-offs across a wide spectrum of dimensions and

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underlying covariance structures remains relatively scarce in the literature. This paper clarifies MVGDs, develops and compares estimation algorithms (including regularized versions for high dimensions), and uses simulations to guide practical implementation and algorithm choice.

## 2. Mathematical Foundations

A matrix-variate Gaussian distribution $\mathbf{X} \in \mathbb{R}^{q\times p}$ is denoted as $\mathbf{X} \sim \mathcal{MN}_{q,p}(M, \Sigma, \Psi)$, with mean $M \in \mathbb{R}^{q\times p}$, row covariance $\Sigma \in \mathbb{R}^{q\times q}$, and column covariance $\Psi \in \mathbb{R}^{p\times p}$. Its probability density function is:

$$f(\mathbf{X}) = \frac{\exp\left(-\frac{1}{2}\operatorname{tr}\left[\Sigma^{-1}(\mathbf{X}-M)\Psi^{-1}(\mathbf{X}-M)^{\top}\right]\right)}{(2\pi)^{qp/2}|\Sigma|^{p/2}|\Psi|^{q/2}}. \tag{1}$$

This arises from transforming a standard normal matrix, where the Jacobian accounts for the scaling and determinant. Vectorizing $\mathbf{X}$ yields $\operatorname{vec}(\mathbf{X}) \sim \mathcal{N}_{qp}(\operatorname{vec}(M), \Psi \otimes \Sigma)$. This representation highlights the separable covariance structure, reducing parameter estimation from $O(q^2p^2)$ to $O(q^2+p^2)$, a critical advantage for high-dimensional data. For instance, with $q = p = 100$, this drops parameters from 5 million to 10,100.
The MVGD extends properties of the multivariate normal distribution:

**Theorem 2.1** (Moments and Transformations)**.** *Assume* $\mathbf{X} \sim \mathcal{MN}_{q,p}(M, \Sigma, \Psi)$. *Then:*

1. *Mean:* $\mathbb{E}[\mathbf{X}] = M$.
2. *Covariance: For any elements* $X_{ij}$ *and* $X_{kl}$ *of* $\mathbf{X}$, $\operatorname{Cov}(X_{ij}, X_{kl}) = \Sigma_{ik}\Psi_{jl}$, *showing a separable row-column structure.*
3. *Affine Transformations: For* $\mathbf{A} \in \mathbb{R}^{m\times q}$ *and* $\mathbf{B} \in \mathbb{R}^{p\times n}$, $\mathbf{AXB} \sim \mathcal{MN}_{m,n}(\mathbf{AMB}, \mathbf{A\Sigma A}^{\top}, \mathbf{B}^{\top}\Psi\mathbf{B})$.
4. *Marginal Distributions: Submatrices follow MVGD with corresponding sub-parameters.*
5. *Conditional Distributions: Conditional blocks also follow MVGD, analogous to Schur-complement formulas.*

*Proof.* Properties derive from the vectorized representation and Kronecker product rules; see [5]. □

**Remark 2.2.** MVGD generalizes several important special cases:

- Multivariate normal (when $p = 1$ or $q = 1$).
- Univariate normal (when $p = q = 1$).
- Independent rows ($\Psi = I_p$) or columns ($\Sigma = I_q$).
- Isotropic variation ($\Sigma = \sigma^2 I_q, \Psi = I_p$) for i.i.d. entries.

The separable structure enhances interpretability and computational efficiency, allowing separate operations on Σ and Ψ, which forms the basis for the subsequent estimation methods.

## 3. Estimation Methods for Matrix-Variate Gaussian Models

### *3.1. Maximum Likelihood Estimation*

Given i.i.d. observations $\mathbf{X}_1, \ldots, \mathbf{X}_N \sim \mathcal{MN}_{q,p}(M, \Sigma, \Psi)$, the MLEs are derived from the log-likelihood function:

$$\ell(M, \Sigma, \Psi) = -\frac{N}{2}\left[qp\log(2\pi) + p\log|\Sigma| + q\log|\Psi| + \operatorname{tr}(\Sigma^{-1}S_{\Sigma})\right], \tag{2}$$

where $S_{\Sigma} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{X}_i - M)\Psi^{-1}(\mathbf{X}_i - M)^{\top}$. The score equations lead to coupled estimators:

$$\hat{M} = \bar{\mathbf{X}}, \quad \hat{\Sigma} = \frac{1}{Np}\sum_{i=1}^{N}(\mathbf{X}_i - \hat{M})\hat{\Psi}^{-1}(\mathbf{X}_i - \hat{M})^{\top}, \quad \hat{\Psi} = \frac{1}{Nq}\sum_{i=1}^{N}(\mathbf{X}_i - \hat{M})^{\top}\hat{\Sigma}^{-1}(\mathbf{X}_i - \hat{M}). \tag{3}$$

$\hat{M}$ has a closed form, but $\hat{\Sigma}$ and $\hat{\Psi}$ depend on each other, reflecting the matrix-variate framework's intrinsic dependencies. Under regularity conditions and sufficient $N$, these MLEs are consistent, asymptotically normal, and efficient.

### 3.2. The Flip-Flop Algorithm

The coupled nature of equation (3) prevents direct solution but motivates an iterative approach. The Flip-Flop algorithm [2] solves the coupled equations iteratively by alternating updates:

**Algorithm 1 (Flip-Flop Algorithm):**

1. Initialize $\hat{\Sigma}^{(0)} = I_q$, $\hat{\Psi}^{(0)} = I_p$
2. Compute $\hat{M} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{X}_i$
3. For $t = 1, 2, \ldots$ until convergence:
   - Flip: $\hat{\Sigma}^{(t)} = \frac{1}{Np}\sum_{i=1}^{N}(\mathbf{X}_i - \hat{M})(\hat{\Psi}^{(t-1)})^{-1}(\mathbf{X}_i - \hat{M})^{\top}$
   - Flop: $\hat{\Psi}^{(t)} = \frac{1}{Nq}\sum_{i=1}^{N}(\mathbf{X}_i - \hat{M})^{\top}(\hat{\Sigma}^{(t)})^{-1}(\mathbf{X}_i - \hat{M})$
   - $\Delta_\Sigma := \frac{\|\hat{\Sigma}^{(t)} - \hat{\Sigma}^{(t-1)}\|_F}{\|\hat{\Sigma}^{(t-1)}\|_F + 10^{-10}}$, $\Delta_\Psi := \frac{\|\hat{\Psi}^{(t)} - \hat{\Psi}^{(t-1)}\|_F}{\|\hat{\Psi}^{(t-1)}\|_F + 10^{-10}}$
   - If $\max(\Delta_\Sigma, \Delta_\Psi) < \epsilon$: break
4. Return $\hat{M}, \hat{\Sigma}^{(t)}, \hat{\Psi}^{(t)}$

This algorithm monotonically increases the log-likelihood and converges to the MLEs under mild conditions [2], with $O(N \max(q^3, p^3))$ complexity per iteration. The sequence converges to $(\hat{\Sigma}_{\text{MLE}}, \hat{\Psi}_{\text{MLE}})$, maximizing the log-likelihood.

### 3.3. Regularized Estimation for High-Dimensional Settings

High-dimensional settings pose challenges for standard MLEs: (1) singular/ill-conditioned covariance matrices and (2) parameter counts exceeding sample size $N$, leading to overfitting. Regularization integrating structural assumptions is necessary. A common strategy is inducing sparsity in precision matrices $\Omega_\Sigma = \Sigma^{-1}$ and $\Omega_\Psi = \Psi^{-1}$, implying conditional independence. The graphical lasso (glasso) [3] achieves this by minimizing:

$$\hat{\Omega} = \arg\min_{\Omega > 0}\left\{-\log|\Omega| + \operatorname{tr}(S\Omega) + \lambda\|\Omega\|_{1,\text{off}}\right\}, \tag{4}$$

where $S$ is the sample covariance, $\|\Omega\|_{1,\text{off}}$ penalizes off-diagonal entries, and $\lambda$ controls sparsity. We apply glasso separately to $\Omega_\Sigma$ and $\Omega_\Psi$ within the Flip-Flop framework for high-dimensional matrix-variate Gaussian models.

**Algorithm 2 (Regularized Flip-Flop):**

1. Initialize as in Algorithm 1
2. For $t = 1, 2, \ldots$ until convergence:
   - Compute $S_\Sigma^{(t)} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{X}_i - \hat{M})(\hat{\Psi}^{(t-1)})^{-1}(\mathbf{X}_i - \hat{M})^{\top}$
   - $\hat{\Omega}_\Sigma^{(t)} = \text{glasso}(S_\Sigma^{(t)}, \lambda_\Sigma)$, $\hat{\Sigma}^{(t)} = (\hat{\Omega}_\Sigma^{(t)})^{-1}$
   - Compute $S_\Psi^{(t)} = \frac{1}{N}\sum_{i=1}^{N}(\mathbf{X}_i - \hat{M})^{\top}(\hat{\Sigma}^{(t)})^{-1}(\mathbf{X}_i - \hat{M})$
   - $\hat{\Omega}_\Psi^{(t)} = \text{glasso}(S_\Psi^{(t)}, \lambda_\Psi)$, $\hat{\Psi}^{(t)} = (\hat{\Omega}_\Psi^{(t)})^{-1}$
   - Compute $\Delta_\Sigma := \frac{\|\hat{\Sigma}^{(t)} - \hat{\Sigma}^{(t-1)}\|_F}{\|\hat{\Sigma}^{(t-1)}\|_F + 10^{-10}}$, $\Delta_\Psi := \frac{\|\hat{\Psi}^{(t)} - \hat{\Psi}^{(t-1)}\|_F}{\|\hat{\Psi}^{(t-1)}\|_F + 10^{-10}}$
   - If $\max(\Delta_\Sigma, \Delta_\Psi) < \epsilon$: break
3. Return $\hat{M}, \hat{\Sigma}^{(t)}, \hat{\Psi}^{(t)}$

The convergence of this algorithm follows similar principles to the standard Flip-Flop, with the additional guarantee that each glasso step converges to a unique solution. Choosing appropriate regularization parameters $\lambda_\Sigma$ and $\lambda_\Psi$ is crucial for balancing bias and variance. Several approaches exist for parameter selection. Cross-validation partitions the data into $K$ folds, estimating parameters on training folds and evaluating the log-likelihood on validation folds: $\hat{\lambda}_\Sigma, \hat{\lambda}_\Psi = \arg\max_{\lambda_\Sigma, \lambda_\Psi} \frac{1}{K}\sum_{k=1}^{K} \ell(\hat{\theta}^{(-k)}; \mathcal{D}^{(k)})$, where $\hat{\theta}^{(-k)}$ are parameters estimated excluding fold $k$, and $\mathcal{D}^{(k)}$ is validation fold $k$. Information criteria such as $\text{AIC} = -2\ell(\hat{\theta}) + 2(\text{df}_\Sigma + \text{df}_\Psi)$ and $\text{BIC} = -2\ell(\hat{\theta}) + (\log N)(\text{df}_\Sigma + \text{df}_\Psi)$ balance goodness-of-fit with model complexity, with $\text{df}_\Sigma$ and $\text{df}_\Psi$ often approximated by the number of non-zero elements in the precision matrices. Stability selection chooses $\lambda$ to maximize the stability of selected edges across bootstrap samples, while empirical rules based on Gaussian graphical models suggest setting $\lambda = c\sqrt{\frac{\log d}{N}}$ where $d$ is the dimension and $c \in [0.5, 2]$.

## 4. Simulation Studies

### *4.1. Experimental Designs and Metrics*

We conduct comprehensive simulations across three distinct dimensional regimes to evaluate the standard and regularized Flip-Flop algorithms. Each design, summarized in Table 1, presents unique statistical challenges. We evaluate

Table 1. Simulation Designs and Parameter Specifications

| Feature | Design 1 (Low-dim) | Design 2 (Moderate-dim) | Design 3 (High-dim) |
|---|---|---|---|
| $(q, p)$ | $(20, 30)$ | $(50, 50)$ | $(100, 100)$ |
| $N$ | 50 | 30 | 20 |
| $\Sigma$ | AR(1), $\rho = 0.70$ | AR(1), $\rho = 0.30$ | Sparse precision (5% nonzero) |
| $\Psi$ | $I_p$ | Block-diagonal (5 blocks) | Sparse precision (5% nonzero) |
| Challenge | Baseline performance | Structured covariance recovery | $N \ll qp$, regularization need |

performance using multiple metrics: (1) *Covariance error*: relative Frobenius norm $\|\hat{\Sigma} \otimes \hat{\Psi} - \Sigma \otimes \Psi\|_F / \|\Sigma \otimes \Psi\|_F$; (2) *Predictive accuracy*: test negative log-likelihood (NLL); (3) *Structure recovery*: F1-scores for $\Sigma$ and $\Psi$ patterns; (4) *Numerical stability*: condition numbers of $\hat{\Sigma}$ and $\hat{\Psi}$; (5) *Computational time* in seconds. All results are based on 500 independent replications.

### *4.2. Results*

Table 2 presents the comprehensive performance comparison.

#### *4.2.1. Design 1: Low-Dimensional Regime*

In this well-conditioned setting, both algorithms performed well. The regularized variant significantly reduced covariance error by 21.3% (0.2151 vs. 0.2735) and improved numerical stability, lowering the $\Psi$ condition number from 3.35 to 1.10. It also correctly identified the identity structure of $\Psi$ (10% sparsity recovery). However, this came with a 10.7 times computational overhead. The standard algorithm remains an excellent choice when computational speed is prioritized.

#### *4.2.2. Design 2: Moderate-Dimensional with Block Structure*

Here we observe a striking pattern: the *standard* algorithm significantly outperformed the regularized version. It achieved 58.3% lower covariance error (0.2887 vs. 0.4570) and excelled at recovering the block-diagonal structure (F1-score: 0.951 vs. 0.225). The regularized algorithm was overly aggressive, destroying the block pattern while only moderately improving condition numbers. This demonstrates that for structured covariances in moderate dimensions, the separable MLE is inherently robust, and $\ell_1$ penalties can remove important signal.

#### *4.2.3. Design 3: High-Dimensional Sparse Regime*

In this challenging regime where $N \ll qp$, regularization proved essential for stability. While covariance errors were comparable, the regularized variant provided 3.4% better predictive performance (lower NLL) and dramatically improved numerical stability (condition numbers reduced by 87% for $\hat{\Sigma}$ and 73% for $\hat{\Psi}$). Neither algorithm recovered sparsity well (1.1% vs. true 5%), suggesting the need for stronger regularization. The computational cost of regularization increased by 3.6 times but remained feasible.

### *4.3. Comparison with Vectorized Approach*

To highlight the necessity of the matrix-variate framework, we compare against the naive vectorization approach (Table 3). The matrix-variate approach achieves dramatic error reductions (82-95%) in feasible settings. In Design 3, vectorization is completely infeasible, requiring estimation of a $10,000 \times 10,000$ covariance matrix from only 20 samples—a problem with 50 million parameters. This unequivocally demonstrates that exploiting the separable Kronecker structure is not merely beneficial but essential for matrix-structured data.

Table 2. Performance Comparison Across Simulation Designs (mean ± std)

| Metric | Design 1 | | Design 2 | | Design 3 | |
|---|---|---|---|---|---|---|
| | **Standard** | **Regularized** | **Standard** | **Regularized** | **Standard** | **Regularized** |
| Covariance Error | 0.2735 (0.0104) | **0.2151 (0.0120)** | **0.2887 (0.0060)** | 0.4570 (0.0030) | 0.3262 (0.0031) | 0.3571 (0.0063) |
| Test NLL | 673.12 (3.97) | **666.74 (3.75)** | **3030.47 (10.30)** | 3108.00 (10.03) | 7641.25 (21.00) | **7378.52 (19.31)** |
| Σ F1-score | 1.000 | 1.000 | 0.888 (0.023) | 0.000 (0.000) | 0.000 | 0.000 |
| Ψ F1-score | 0.000 | 0.000 | **0.951 (0.011)** | 0.225 (0.001) | 0.318 (0.026) | 0.318 (0.026) |
| Ψ Sparsity (%) | 0.005 (0.032) | **10.001 (0.053)** | **87.790 (0.369)** | 3.593 (0.555) | 1.125 (0.109) | 1.125 (0.109) |
| Σ Condition # | 31.07 (2.28) | **28.92 (1.76)** | 8.49 (0.51) | **1.12 (0.03)** | 10.39 (0.76) | **1.33 (0.04)** |
| Ψ Condition # | 3.35 (0.25) | **1.10 (0.02)** | 26.83 (1.87) | **23.48 (1.49)** | 10.41 (0.65) | **2.84 (0.29)** |
| Time (seconds) | **0.034 (0.025)** | 0.366 (0.264) | **0.206 (0.188)** | 4.264 (1.307) | **0.706 (0.297)** | 2.534 (3.482) |

Table 3. Comparison with Vectorized Approach

| **Design** | **Vectorized Error** | **Flip-Flop Error** | **Error Reduction** |
|---|---|---|---|
| Design 1 | 1.6123 (0.1521) | 0.2846 (0.0124) | 82.3% |
| Design 2 | 6.1226 (0.4318) | 0.2887 (0.0060) | 95.3% |
| Design 3 | Infeasible | 0.3262 (0.0031) | — |

## 5. Conclusion

This paper presents a framework for matrix-variate Gaussian models, leveraging a separable Kronecker product covariance structure to reduce parameter dimensionality. It compares the standard Flip-Flop algorithm with a regularized graphical lasso variant for estimation. Simulations show regularization is beneficial in low- and high-dimensional settings but can harm performance in moderate dimensions, where the standard MLE is robust. The matrix-variate approach significantly outperforms naive vectorization, offering substantial error reductions and enabling estimation in previously infeasible regimes.

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# Various rings of quotients of $C_c(X,F)$ and their connections

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**Abstract**

For a zero-dimensional topological space $X$ and a totally ordered field $F$ with interval topology, $C_c(X,F)$ denotes the ring consisting of ordered field-valued continuous functions with countable range on $X$. In this talk, we give some results on the rings of quotients of $C_c(X,F)$. $Q_c(X,F)$ (resp. $q_c(X,F)$), the maximal (resp. classical) ring of quotients of $C_c(X,F)$ as a modified countable analogue of $Q(X)$ (resp. $q(X)$), the maximal (resp. classical) ring of quotients of $C(X)$, are characterized. Also, the coincidence of rings of quotients of $C_c(X,F)$ is investigated.

## 1. Introduction

In this article, $X$ denotes a *zero-dimensional* topological space, i.e., a Hausdorff space with a base consisting of closed-open (clopen) sets. $F$ is a totally ordered field that is also a topological space under interval topology. $C(X)$ ($C^*(X)$) denotes the ring of all real-valued continuous (bounded) functions on a space $X$. $C_c(X)$ (resp, $C^F(X)$) is the subring of $C(X)$ consisting of those functions with countable (resp, finite) image. The latter ring is introduced and studied in [6], and, for more discussion on some topics related to this area, one can refer to articles [8, 9]. $C(X,F)$ is the set of all $F$-valued continuous functions on $X$. The latter set becomes a commutative lattice-ordered ring with identity, if the operations are defined pointwise on $X$. For more information in this regard, we refer the reader to articles [1, 3]. For each $f \in C(X,F)$, the *zero-set* of $f$, denoted by $Z(f)$, is the set of zeros of $f$ and $coz(f) = X \setminus Z(f)$ is the *cozero-set* of $f$. Let us define $C_c(X,F)$ as follows:

$$C_c(X,F) = \{f \in C(X,F) : f(X) \text{ is a countable subset of } F\}.$$

$C_c(X,F)$ is an $F$-subalgebra as well as a sublattice of $C(X,F)$. This ring has been introduced and investigated in [2]. $C_c^*(X,F)$ is a subset of $C_c(X,F)$ consisting of those elements that are bounded. Hence, $C_c^*(X,F) = C^*(X,F) \cap C_c(X,F)$. In the case that $F = \mathbb{R}$, we have $C(X,F) = C(X)$ and $C_c(X,F) = C_c(X)$. For an ideal $I$ of $C_c(X,F)$, we let $Z(I) = \bigcap_{f\in I} Z(f)$ and $coz(I) = X \setminus Z(I)$, which is equal to $\bigcup_{f\in I} coz(f)$.

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An ideal of a ring is called *dense* if its annihilator in the ring is 0. Suppose that $A$ and $B$ are commutative rings with identity. In [5, Definition 1.4], $B(\supseteq A)$ is called a *ring of quotients* of $A$, provided that for every $b \in B$, the ideal $b^{-1}A := \{a \in A : ab \in A\}$ is dense in $B$, that is to say, for each $0 \neq b' \in B$ there exists $a \in A$ such that $ba \in A$ and $b'a \neq 0$. Also, see [5, Lemma and Theorem 1.5].

## 2. Coincidence of rings of quotients of $C_c(X,F)$

In this section, we deal with the coincidence of rings of quotients of the $C_c(X,F)$. In particular, we are interested in cases when some of these rings of quotients coincide with $C_c(X,F)$, itself. $Q_c(X,F)$ and $q_c(X,F)$ are respectively maximal and classical ring of quotients of $C_c(X,F)$.
Let $S$ be a subring of $C_c(X,F)$ and $\mathcal{D}_0$ (resp. $\mathcal{D}$) be the family of all dense (resp. regular) ideals in $S$. Notice that $\mathcal{D}_0$ and $\mathcal{D}$ are closed under multiplication, i.e., if $D_1$ and $D_2$ are dense (resp. regular) ideals in $S$, then $D_1D_2$ is also a dense (resp. regular) ideal in $S$; and $(d) \in \mathcal{D}_0$ if and only if $(d) \in \mathcal{D}$. Furthermore, $d \in D$ gives $\mathrm{Hom}(D) \subseteq \mathrm{Hom}((d))$. Then $Q_c(S)$ (resp. $q_c(S)$), the maximal (resp. classical) ring of quotients of $S$ has been described based on the $S$-modules $\mathrm{Hom}(D)$, where $D \in \mathcal{D}_0$ (resp. $D \in \mathcal{D}$), i.e.,

$$Q_c(S) = \varinjlim_{D\in\mathcal{D}_0} Hom(D), \quad \text{and} \quad q_c(S) = \varinjlim_{D\in\mathcal{D}} Hom(D). \tag{1}$$

Observe that $Q_c(S)$ and $q_c(S)$ may be thought of as $\bigcup \mathrm{Hom}(D)$, where we identify $\varphi_1 \in \mathrm{Hom}(D_1)$ and $\varphi_2 \in \mathrm{Hom}(D_2)$ whenever $\varphi_1$ and $\varphi_2$ agree on $D_1D_2$ (see [5, 1.7]). Therefore,

$$Q_c(S) = \bigcup \{Hom(D) : D \in \mathcal{D}_0\}, \quad \text{and} \quad q_c(S) = \bigcup \{Hom(D) : D \in \mathcal{D}\}. \tag{2}$$

Applying the above discussion and [5, Lemma 2.5], maximal and classical ring of quotients of $C_c(X,F)$ are characterized as follows. Of course, more details of the proof can be found in [9, Theorem 2.19].

**Theorem 2.1.** *Let $Q_c(X,F)$ (resp. $q_c(X,F)$) be the maximal (resp. classical) ring of quotients of $C_c(X,F)$. Then*

*(i)* $Q_c(X,F) = \bigcup \{C_c(V,F) : V \text{ is a dense open subset of } X\}$.

*(ii)* $q_c(X,F) = \bigcup \{C_c(\mathrm{coz}(f),F) : f \in C_c(X,F) \text{ and } \overline{\mathrm{coz}(f)} = X\}$.

(Note that an equivalence relation on the right-hand unoins in $(i)$ and $(ii)$ is obtained by defining $f \in C_c(V_1,F)$ and $g \in C_c(V_2,F)$ to be equivalent if and only if the restriction of $f$ and $g$ to $V_1 \cap V_2$ are equal.)

**Definition 2.2.** A subset $S$ of a space $X$ is called *$C_cF$-embedded* (resp. *$C_c^*F$-embedded*) in $X$ if every function in $C_c(S,F)$ (resp. $C_c^*(S,F)$) can be extended to a function in $C_c(X,F)$ (resp. $C_c^*(X,F)$).

**Proposition 2.3.** *Every open set in $X$ is $C_cF$-embedded (resp. $C_c^*F$-embedded) if and only if every dense open set in $X$ is $C_cF$-embedded (resp. $C_c^*F$-embedded).*

From Theorem 2.1 and Proposition 2.3, we get:

**Proposition 2.4.** *$Q_c(X,F) = C_c(X,F)$ if and only if every open set in $X$ is $C_cF$-embedded.*

Notice that the "zero-dimensional" condition cannot be omitted from the above result. To see this, let $X = \mathbb{R}^2$, $F = \mathbb{R}$ and let $V = \mathbb{R}^2 \setminus \{(0,y) : y \in \mathbb{R}\}$. Remember that the relation $C_c(X,F) = F(= \mathbb{R})$ follows trivially from the connectedness of $X = \mathbb{R}^2$, and therefore, $Q_c(X,F) = Q(F) = F$. Now, let $f : V \to \mathbb{R}$ be defined by $f(x,y) = 1$, if $x > 0$, and $f(x,y) = -1$, if $x < 0$. Then $f \in C_c(V,F)$, but it cannot be extended to $X$, i.e., $V$ is not $C_cF$-embedded.

Using Proposition 2.4 and the fact that "$C_cF$-embedding gives $C_c^*F$-embedding", we obtain:

**Corollary 2.5.** *$Q_c(X,F) = C_c(X,F)$ implies that $Q_c^*(X,F) = C_c^*(X,F)$.*

The converse of the above corollary is not true in general, see Example 2.10.

**Theorem 2.6.** *For a zero-dimensional space $X$ and a totally ordered field $F$, the following are equivalent.*

$(i)$ $Q_c(X,F) = q_c(X,F)$.
$(ii)$ *Each $F$-valued continuous function on a dense open subset of $X$ agrees with a continuous function on a dense cozero-set of an element of $C_c(X,F)$.*
$(iii)$ $Q_c^*(X,F) = q_c^*(X,F)$.

*Proof.* $(i) \Leftrightarrow (ii)$. Evident by Theorem 2.1.
$(ii) \Rightarrow (iii)$. Let $f \in Q_c^*(X,F)$. Then for some dense open subset $V$ of $X$, we have $f \in C_c^*(V,F)$. By the assumption, there is $h \in C_c(X,F)$ and $g \in C_c(coz\,(h),F)$ such that $\overline{coz\,(h)} = X$ and $f$ agrees with $g$ on $V \cap coz(h)$. So $f \in q_c^*(X,F)$. The reverse inclusion is clear.
$(iii) \Rightarrow (i)$. If $f \in Q_c(X,F)$, then for some dense open subset $V$ of $X$, $f \in C_c(V,F)$. Take $g = (1+f^2)^{-1} = \frac{1}{1+f^2}$. Hence, $g, fg \in C_c^*(V,F) \subseteqq Q_c^*(X,F) = q_c^*(X,F)$. So there exist $h_1 \in C_c(X,F)$ and $k_1 \in C_c^*(coz\,(h_1),F)$ such that $\overline{coz\,(h_1)} = X$ and $fg$ agrees with $k_1$ on $V \cap coz\,(h_1)$. Also, we can take $h_2 \in C_c(X,F)$ and $k_2 \in C_c^*(coz\,(h_2),F)$ such that $\overline{coz\,(h_2)} = X$ and $g$ agrees with $k_2$ on $V \cap coz\,(h_2)$. Notice that $g^{-1} \geq 1$ and agrees with $k_2^{-1}$ on $V \cap coz\,(h_2)$, also, $\mathrm{coz}\,(h_1) \cap \mathrm{coz}\,(h_2) = \mathrm{coz}\,(h_1h_2)$ is dense. Therefore, $f = (fg)g^{-1} \equiv k_1k_2^{-1}$ on $V \cap \mathrm{coz}\,(h_1h_2)$. This yields $f \in q_c(X,F)$. The reverse inclusion is obvious.

□

Next, we need the following lemma. Its proof is straightforward.

**Lemma 2.7.** *Let $F$ be an infinite, totally ordered field and $(a_\lambda)_{\lambda\in\Lambda} \subseteq F$ be a net of nonzero elements. Then the following hold.*

$(i)$ *If $0 \neq a \in F$, then $a_\lambda \to a$ if and only if $a_\lambda^{-1} \to a^{-1}$.*
$(ii)$ *$a_\lambda \to 0$ if and only if the net $(a_\lambda^{-1})_{\lambda\in\Lambda}$ is unbounded.*

The $CP_F$-spaces are introduced and determined in [2, Section 4]. Now, with this terminology, we call a space $X$ an *almost $CP_F$-space*, if $int_X Z \neq \emptyset$, for each non-zero zero-set $Z \in Z_c(X,F)$.
Recall that $f \in C_c(X,F)$ is a zero-divisor if and only if $int_X Z(f) \neq \emptyset$ if and only if $\overline{coz\,(f)} \neq X$.

**Theorem 2.8.** *For a zero-dimensional space $X$ and a totally ordered field $F$, the following are equivalent.*

$(i)$ *Every non-unit element in $C_c(X,F)$ is a zero-divisor.*
$(ii)$ *There is no proper dense cozero-set in $X$.*
$(iii)$ $q_c(X,F) = C_c(X,F)$.
$(iv)$ *$X$ is almost $CP_F$-space.*

*Proof.* $(i) \Rightarrow (ii)$. Let $f \in C_c(X,F)$ and $\overline{coz\,(f)} = X$. Then $int_X Z(f) = \emptyset$ which means $f$ is a non-zero divisor. By the hypothesis, $f$ is a unit and thus $coz\,(f) = X$.
$(ii) \Rightarrow (iii)$. Since the only dense cozero-set in $X$ is itself, the result holds.
$(iii) \Rightarrow (iv)$. Let $Z = Z(f) \in Z_c(X,F)$ and $int_X Z = \emptyset$, i.e., $\overline{coz\,(f)} = X$. Notice that $f^{-1} \in C_c(coz\,(f),F)$. Therefore, by the assumption, $f^{-1}$ has an extension to $X$, say $\bar{f}$. For $x \in X$, there is a net $(x_\lambda)_{\lambda\in\Lambda} \subseteq coz\,(f)$ converging to $x$. Hence, $f(x_\lambda) \to f(x)$ and $\bar{f}(x_\lambda) \to \bar{f}(x)$, this means that $f^{-1}(x_\lambda) \to \bar{f}(x)$. We claim that $f(x) \neq 0$. Otherwise, Lemma 2.7 gives the net $(f^{-1}(x_\lambda))_{\lambda\in\Lambda}$ is unbounded, which is absurd, since $\bar{f}(x) \in F$. Consequently, $f(x) \neq 0$. This yields $coz\,(f) = X$, or $Z = \emptyset$.
$(iv) \Rightarrow (i)$. Let $f \in C_c(X,F)$ be a non-unit, i.e., $Z(f) \neq \emptyset$. By the assumption, $int_X Z(f) \neq \emptyset$, that is, $f$ is a zero-divisor.

□

An immediate conclusion of the above theorem is as follows.

**Corollary 2.9.** *$q_c(X,F) = C_c(X,F)$ implies that $q_c^*(X,F) = C_c^*(X,F)$.*

The converse of the above corollary is not generally true, see Example 2.10.

**Example 2.10.** Let $X = \Sigma = \mathbb{N} \cup \{\sigma\}$ (where $\sigma \notin \mathbb{N}$) be the space in [7, 4M] and let $F = \mathbb{R}$. We recall that the open neighborhoods of $\sigma$ in the space $X$ are of the form: $G \cup \{\sigma\}$, where $G$ is a member of a free ultrafilter on $\mathbb{N}$, and each point of $\mathbb{N}$ is isolated. If $f(n) = \frac{1}{n}$ and $f(\sigma) = 0$, then $f \in C_c(X,F) = C(X,F) = C(X)$. So $\{\sigma\}$ is a zero-set and thus $\mathbb{N}$ is a cozero-set in $X$. Notice that $X$ and $\mathbb{N}$ are precisely the dense open sets as well as the dense cozero-sets in $X$. Therefore, $C(X) \leqq C(\mathbb{N})$ and thus $Q(X) = q(X) = C(\mathbb{N})$ (note, $Q_c(X,F) = Q(X)$ and $q_c(X,F) = q(X)$). Hence, $Q^*(X) = q^*(X) = C^*(\mathbb{N})$. Moreover, $X$ is extremally disconnected, i.e., every open set of $X$ has an open closure in $X$ . So, by [7, 1H], every open subspace is $C^*$-embedded. This yields $C^*(X) = C^*(\mathbb{N})$. So $Q^*(X) = q^*(X) = C^*(X)$. Moreover, since $\mathbb{N}$ is not $C$-embedded in $X$ (consider $f(n) = n$), we obtain $C(X) \lneqq C(\mathbb{N})$.

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# When is $Q_c(R)$ included in $Q_c(X)$?

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**Abstract**

For a zero-dimensional Hausdorff topological space $X$, let $R$ be a subalgebra of $C_c(X)$, and let $Q_c(R)$ and $Q_c(X)$ be the maximal rings of quotients of $R$ and $C_c(X)$, respectively. We show that $Q_c(R) \subseteqq Q_c(X)$ if and only if every dense ideal in $R$ has a dense cozero-set in $X$.

## 1. Introduction

Let $X$ be a zero-dimensional Hausdorff space, $C_c(X) = \{f \in C(X) \mid f(X) \text{ is countable}\}$ and $C_c^*(X) = \{f \in C_c(X) \mid f \text{ is bounded}\}$. Let $R$ be a subring of $C_c(X)$ and $f \in R$. The zero-set of $f$ is $Z(f) = \{x \in X : f(x) = 0\}$ and its set-theoric complement is $coz\, f = X \setminus Z(f) = \{x \in X : f(x) \neq 0\}$. For an ideal $I$ of $R$; we let $Z(I) = \cap\{Z(f) : f \in I\}$ and $coz\, I = X \setminus Z(I) = \cup\{coz\, f : f \in I\}$. An ideal $I$ of $R$ is called *(algebraically) dense* in $R$ if its annihilator in $R$ is 0, i.e., $\{r \in R \mid \text{for every } i \in I,\ ri = 0\} = 0$. Also, a subset $U$ of $X$ is said to be *(topologically) dense* if $U \cap V \neq \emptyset$ for every non-void open subset $V$ of $X$. A subring $R$ of $C_c(X)$ is called $z_c$-*determining*, if the collection $\{Z(g) : g \in R\}$ is a base for closed sets of $X$. For instance, $C_c(X)$ and $C_c^*(X)$ are $z_c$-determinig subalgebras of $C_c(X)$. Using [3, Corollary 1.9] the following Characterization of $Q_c(R)$, the maximal ring of quotients of $R$, is as follows.

$$Q_c(R) = \bigcup \{\mathrm{Hom}(D) : D \text{ is a dense ideal in } R\}.$$

(We idetify $\varphi_1 \in \mathrm{Hom}(D_1)$ with $\varphi_2 \in \mathrm{Hom}(D_2)$ whenever $\varphi_1$ and $\varphi_2$ are agree on the dense ideal $D_1D_2$). Now, suppose that $R = C_c(X)$. Then $Q_c(R) = Q_c(C_c(X))$ ($Q_c(X)$, for short); the maximal ring of quotients of $C_c(X)$, has been characterized as the union of $C_c(V)$, where $V$ ranges over all dense open subsets of $X$, see [5, Theorem 2.12]. Therefore,

$$Q_c(X) = \bigcup \{\mathrm{Hom}(D) : D \text{ is a dense ideal in } C_c(X)\} \tag{1}$$

$$= \bigcup \{C_c(V) : V \text{ is a dense open subset of } X\}. \tag{2}$$

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(Note that an equivalence relation on the right-hand union is obtained by defining $f \in C_c(V_1, F)$ and $g \in C_c(V_2, F)$ to be equivalent if and only if the restriction of $f$ and $g$ to $V_1 \cap V_2$ are equal.)

## 2. Main Results

In this section, we show that $Q_c(R) \subseteqq Q_c(X)$ if and only if every dense ideal of $R$ has a dense cozero-set in $X$, where $R$ is a subring of $C_c(X)$.

**Lemma 2.1.** *Let $X$ be a zero-dimensional space and $R$ be a $z_c$-determining subalgebra of $C_c(X)$. Then an ideal $D$ in $R$ is a dense ideal if and only if $\operatorname{coz} D$ is a dense subset of $X$.*

*Proof.* First, suppose that $\operatorname{coz} D$ is dense in $X$. We must show that $D$ is a dense ideal in $R$. Let $g \in R$ such that $gD = 0$. So for each $d \in D$, $gd = 0$. Hence $\operatorname{coz} g \cap \operatorname{coz} d = \emptyset$. Therefore, $\operatorname{coz} g \cap \left(\operatorname{coz} D := \bigcup_{d\in D} \operatorname{coz} d\right) = \emptyset$. By our assumption, $\operatorname{coz} g = \emptyset$ and hence $g = 0$. This shows that $D$ is a dense ideal in $R$. For the converse, let $V$ be an open subset of $X$ such that $V \cap \operatorname{coz} D = \emptyset$. We must show that $V = \emptyset$. Since $R$ is $z_c$-determining, the set $\{\operatorname{coz} g : g \in R\}$ is a base for the topology on $X$. So, we let $V = \operatorname{coz} g$ for some $g \in R$. Since $\operatorname{coz} g \cap \operatorname{coz} D = \emptyset$, we have $\operatorname{coz}(gd) = \operatorname{coz} g \cap \operatorname{coz} d = \emptyset$ for every $d \in D$. Consequently, $gd = 0$. By density of $D$ (in $R$), we have $g = 0$ and hence $V = \operatorname{coz} g = \emptyset$. This completes the proof. □

The next example shows that if the "$z_c$-determining" condition is omitted, the necessity in the above lemma may be false. In other words, a cozero-set of a dense ideal of $R$ is not necessarily dense in $X$, even when $X$ is compact and $R$ is metrically dense in $C_c(X)$.

**Example 2.2.** Let $X = \{0\} \cup \{\pm\frac{1}{n} : n \in \mathbb{N}\}$. Then $X$ is a compact zero-dimensional space and hence $C(X) = C_c(X) = C_c^*(X) = C^*(X)$. Let $A_1 = \{-1, 0, 1\}$ and $A_m = \{\pm\frac{1}{mn} : n \in \mathbb{N}\} \setminus \bigcup_{2\le p<m} A_p$, where $p$ and $m$ are prime numbers. Define

$$d(x) = \begin{cases} 0 & x \in A_1 \\ \frac{1}{m} & x \in A_m. \end{cases}$$

Let $R$ be the subalgebra of $C(X)$ generated by all polynomials on $X$ and $d$, and let $(d)$ be the principal ideal of $R$ generated by $d$. Observe that

$$R = \Big\{ \sum_{i=0}^{k} p_i(x) d^i(x) : p_i(x) \text{ is a polynomial on } X \Big\}.$$

So, each element of $(d)$ is of the form of $\sum_{i=0}^{k} p_i(x) d^{i+1}(x)$. First, we show that $R$ is a metrically dense subspace of $C(X)$ under the *sup-norm* topology. To see this, let $g \in C(X)$ and $B_\epsilon(g) = \{f \in C(X) : \|f - g\| < \epsilon\}$ be a neighborhood at $g$. Since the closed subspace $X$ is $C$-embedded in $[-1, 1]$, the function $g$ can be extended to a continuous function on $[-1, 1]$, say $\overline{g}$ . Let $\{p_n\}$ be any sequence of polynomials converging to $\overline{g}$ uniformly. Take $p_k$ such that $\|p_k - g\| < \epsilon$. So $p_k \in B_\epsilon(g) \cap R$. It gives $R$ is dense in $C(X)$. Now, we show that $(d)$ is a dense ideal in $R$, while its cozero-set is not dense in $X$. Let $f \in R$ and $f.(d) = 0$. So $fd = 0$. Let

$$f(x) = \sum_{i=0}^{k} p_i(x) d^i(x) \text{ and then } f(x)d(x) = \sum_{i=0}^{k} p_i(x) d^{i+1}(x) = 0.$$

We can suppose that the degree of each $p_i$ is $n$ and it has the form below

$$p_i(x) = a_0^i + a_1^i x + a_2^i x^2 + \cdots + a_n^i x^n = \sum_{j=0}^{n} a_j^i x^j \quad (\text{where } 0 \le i \le k).$$

By a direct calculation, we now show that $f = 0$.

$$fd = (\sum_{j=0}^{n} a_j^0 x^j)d + (\sum_{j=0}^{n} a_j^1 x^j)d^2 + (\sum_{j=0}^{n} a_j^2 x^j)d^3 + \cdots + (\sum_{j=0}^{n} a_j^k x^j)d^{k+1} = 0.$$

If $x \in A_m$, then

$$\sum_{i=0}^{k} a_0^i (\frac{1}{m})^{i+1} + \Big(\sum_{i=0}^{k} a_1^i (\frac{1}{m})^{i+1}\Big)x + \Big(\sum_{i=0}^{k} a_2^i (\frac{1}{m})^{i+1}\Big)x^2 + \cdots + \Big(\sum_{i=0}^{k} a_n^i (\frac{1}{m})^{i+1}\Big)x^n = 0.$$

Hence for each $0 \leq j \leq n$, we have $\sum_{i=0}^{k} a_j^i (\frac{1}{m})^{i+1} = 0$. Now, consider the polynomial $q_j(t)$ as follows:

$$q_j(t) = \sum_{i=0}^{k} a_j^i t^{i+1} = a_j^0 t + a_j^1 t^2 + \ldots + a_j^k t^{k+1}.$$

Observe that $q_j(t)$ has an infinite number of zeros, such that $\frac{1}{m}$ for all $m \geq 2$, which is impossible. Hence $a_j^i = 0$ for each $0 \leq i \leq k$ and therefore $f = 0$. This yields the ideal $(d)$ is dense. On the other hand, $\operatorname{coz}(d) = X \setminus A_1$ is not dense in $X$ because the open set $\{1\}$ misses it.

**Theorem 2.3.** *Let $R$ be a subring of $C_c(X)$ and $\nu$ be the set consisting of all those dense open sets $V$ of $X$ such that $\operatorname{coz} D \subseteq V$, for some dense ideal $D$ of $R$. Let*

$$S = \bigcup \{C_c(V) : V \in \nu\} \text{ and } S' = \bigcup \{C_c(\operatorname{coz} D) : D \text{ is a dense ideal in } R\}.$$

*Then we have the following:*

*(1) $S$ and $S'$ are $\mathbb{R}$-algebras. Moreover, $S \subseteq Q_c(X)$ and $Q_c(R) \subseteq S'$.*
*(2) $S'$ is a ring homomorphic image of $S$.*

*Proof.* (1). We prove the theorem for $S$. The proof for $S'$ is similar. Let $f, g \in S$. Then, there exist $V_1, V_2 \in \nu$ and dense ideals $D_1, D_2$ in $R$ such that $f \in C_c(V_1)$, $g \in C_c(V_2)$ and; $\operatorname{coz}(D_1 \cap D_2) = \operatorname{coz} D_1 \cap \operatorname{coz} D_2 \subseteq V_1 \cap V_2$. Since $D_1 \cap D_2$ is a dense ideal in $R$, it follows that $V_1 \cap V_2 \in \nu$. This shows $fg, f + g \in S$. Furthermore, $S$ has the constant functions $1, -1$. So $S$ is a ring containing $\mathbb{R}$. According to $Q_c(X)$ (see Introduction), we get $S \subseteq Q_c(X)$. Also, [3, Lemma 2.5] implies that $Q_c(R) \subseteq S'$.
(2). Define $\varphi : S \to S'$ by $\varphi(f) = f|_{\operatorname{coz} D}$, where $f \in C_c(V)$, for some $V \in \nu$. It is clear that $\varphi$ is a ring homomorphism. If $g \in S'$ then for some dense ideal $D$ of $A$, we have $g \in C_c(\operatorname{coz} D)$. Recall that $\operatorname{coz} D \cup (X \setminus \overline{\operatorname{coz} D})$ is a dense open subset of $X$. Define

$$\overline{g}(x) = \begin{cases} g(x) & x \in \operatorname{coz} D \\ 0 & x \in X \setminus \overline{\operatorname{coz} D} \end{cases}.$$

Then $\overline{g} \in S$ and $\varphi(\overline{g}) = \overline{g}|_{\operatorname{coz} D} = g$. This gives $\varphi$ is onto. The result can be summarized as follows:

$$Q_c(R) \subseteq S' \xleftarrow{\varphi} S \subseteq Q_c(X).$$

□

**Theorem 2.4.** *Let $S, S'$ and $\varphi$ be as defined in Theorem 2.3. Then $\varphi$ is a one-to-one if and only if every dense ideal in $R$ has a dense cozero-set in $X$.*

*Proof.* ($\Rightarrow$) Let $D$ be a dense ideal in $R$ such that $\overline{\operatorname{coz} D} \neq X$. Consider the function $f$ below,

$$f(x) = \begin{cases} 0 & x \in \operatorname{coz} D \\ 1 & x \in X \setminus \overline{\operatorname{coz} D} \end{cases}.$$

Obviously, $f$ is a nonzero continuous function and $\varphi(f) = 0$. So $\varphi$ is not one-to-one. This proves the necessity. ($\Leftarrow$) Suppose that $f \in S$ and $\varphi(f) = 0$. Thus, for some $V \in \nu$ and a dense ideal $D$ of $R$, we have $f \in C_c(V)$ and $f|_{\operatorname{coz} D} = 0$. By our assumption, $\operatorname{coz} D$ is dense in $X$, which gives the desired result $f = 0$. This completes the proof. □

By combining Theorems 2.3 and 2.4, we present the main result of this note as follows.

**Corollary 2.5.** *$Q_c(R) \subseteq Q_c(X)$ if and only if all dense ideals in $R$ have dense cozero-sets in $X$.*

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# A Study on Generalized $m$-fold Symmetric Bi-Univalent Function Classes

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**Abstract**

In the present investigation, we introduce an interesting subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ of $m$-fold symmetric bi-univalent functions when both $f$ and $f^{-1}$ are $m$-fold symmetric in the open unit disk $\mathbb{D}$. Also, for functions of this subclass, we find bounds for the initial coefficients $|a_{m+1}|$, $|a_{2m+1}|$ and Fekete-Szegö problems. The results presented in this paper would generalize and improve some recent works of several earlier authors.

## 1. First Section

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ denote the class of functions in $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. It is well known (see [3]) that every function $f \in \mathcal{S}$ has an inverse map $f^{-1}$, defined by $f^{-1}(f(z)) = z, (z \in \mathbb{D})$ and $f(f^{-1}(w)) = w, \left(|w| < r_0(f),\ r_0(f) \geqq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$, if both $f$ and $f^{-1}$ are univalent in $\mathbb{D}$. Let $\sigma_B$ denote the class of bi-univalent functions.

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For each function $f \in \mathcal{S}$ function, the function

$$h(z) = \sqrt[m]{f(z^m)} \tag{3}$$

is univalent and maps the unit disk $\mathbb{D}$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric, if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in \mathbb{D}, m \in \mathbb{N}). \tag{4}$$

We denote by $\mathcal{S}_m$ the class of $m$-fold symmetric univalent functions in $\mathbb{D}$. In fact , the functions in the class $\mathcal{S}$ are $one$-fold symmetric. Each bi-univalent function generates an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given as in (4) and the series expansion for $f^{-1}$, which has been recently proven by Srivastava et al. [4], is given as follows:

$$\begin{aligned} f^{-1}(w) &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ &-[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots. \end{aligned} \tag{5}$$

We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $\mathbb{D}$. Thus, when $m = 1$, the formula (5) coincides with the formula (2).

The goal of the this paper is to introduce new subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda, \gamma)$ of function class $\Sigma_m$ by applying positive functions and obtain estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this subclass. Also, we find bounds on $|a_{2m+1} - \rho a_{m+1}^2|$ of functions in this subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

## 2. Second Section

**Definition 2.1.** Assume that the functions $h, p : \mathbb{D} \to \mathbb{C}$, analytic in $\mathbb{D}$, are given by

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots \quad \text{and} \quad p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots;$$

such that

$$\min \Re\,(h(z)) > 0 \quad and \quad \min \Re\,(p(w)) > 0.$$

A function $f$ given by (4) is said to be in the subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda, \gamma)(0 \leqq \lambda < 1,\ \gamma \in \mathbb{C} - \{0\})$, if the following conditions are satisfied:

$$f \in \Sigma_m, \quad 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right) \in h(\mathbb{D}) \quad (z \in \mathbb{D}), \tag{6}$$

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)g(w) + \lambda w g'(w)} - 1\right) \in p(\mathbb{D}) \quad (w \in \mathbb{D}), \tag{7}$$

where the function $g$ is given by the equality (5).

**Remark 2.2.** There are many choices of the functions $h$ and $p$ which would provide interesting subclasses of the general subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda, \gamma)$.

For example, if we take

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \ (0 < \alpha \leqq 1),$$

it can easily be verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. So, if we have $f \in \mathcal{S}_{\Sigma_m}^{h,p}(\lambda, \gamma)$, then

$$\left|\arg\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} - 1\right)\right)\right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left|\arg\left(1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)g(w) + \lambda w g'(w)} - 1\right)\right)\right| < \frac{\alpha\pi}{2}.$$

Therefore, for $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$ and $\gamma = 1$, the subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}(\alpha,\lambda)$, were defined by Altinkaya et al. [2].

Also, for $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha}$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}^{\alpha}$, were defined by Altinkaya et al. [2].

If we take

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots \ (0 \leqq \beta < 1),$$

it can easily be verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. So, if we have $f \in \mathcal{S}_{\Sigma_m}^{h,p}(\lambda,\gamma)$, then

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} - 1\right)\right) > \beta \ \text{ and } \ \Re\left(1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)} - 1\right)\right) > \beta.$$

Therefore, for $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ and $\gamma = 1$, the subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma_m}(\beta,\lambda)$ which was introduced and studied by Altinkaya et al. [2].

Also, for $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{S}_{\Sigma_m}^{h,p}$ which was introduced and studied by Hamidi et al. [5] and Altinkaya et al. [2].

**Remark 2.3.** For *one*-fold symmetric bi-univalent functions, we denote the subclass $\mathcal{S}_{\Sigma_1}^{h,p}(\lambda,\gamma) = \mathcal{S}_{\Sigma}^{h,p}(\lambda,\gamma)$. Special cases of this subclass illustrated below:

- By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{S}_{\Sigma}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma}^{*}(\alpha)$ of strongly bi-starlike functions of order $\alpha$.
- By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{S}_{\Sigma}^{h,p}(\lambda,\gamma)$ reduces to the subclass $\mathcal{S}_{\Sigma}(\beta)$ of bi-starlike functions of order $\beta$.

**Theorem 2.4.** *Let $f$ given by (4) be in subclass $\mathcal{S}_{\Sigma_m}^{h,p}(\lambda,\gamma)(0 \leqq \lambda < 1, \gamma \in \mathbb{C} - \{0\})$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{|\gamma||h_m|}{m(1-\lambda)}, \frac{1}{m(1-\lambda)}\sqrt{\frac{|\gamma|(|h_{2m}|+|p_{2m}|)}{2}}\right\},$$

$$|a_{2m+1}| \leqq \min\left\{\frac{|\gamma|^2(m+1)\left(|h_m|^2+|p_m|^2\right)}{4m^2(1-\lambda)^2} + \frac{\gamma(|h_{2m}|+|p_{2m}|)}{4m(1-\lambda)}, \frac{|\gamma|\left[(1+m(2-\lambda))\,|h_{2m}| + (1+m\lambda)|p_{2m}|\right]}{4m^2(1-\lambda)^2}\right\}$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_{2m+1} - \rho a_{m+1}^2| \leqq \begin{cases} \frac{|\gamma|}{4m(1-\lambda)}\{(k(\rho)+1)|h_{2m}| + (1-k(\rho))|p_{2m}|\}; & |k(\rho)| \leqq 1 \\ \frac{|\gamma|}{4m(1-\lambda)}\{|k(\rho)+1||h_{2m}| + |k(\rho)-1||p_{2m}|\}; & |k(\rho)| \geq 1, \end{cases}$$

*where $k(\rho) = \frac{m-2\rho+1}{m(1-\lambda)}$.*

## 3. Corollaries and Consequences

If we set

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^{\alpha} = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots \ (0 < \alpha \leqq 1)$$

and $\gamma = 1$ in Theorem 2.4, then Corollary 3.1 can be obtained.

**Corollary 3.1.** *Let $f$ given by (4) be in the subclass $\mathcal{S}_{\Sigma_m}(\alpha,\lambda)$. Then*

$$|a_{m+1}| \leqq \frac{\sqrt{2}\alpha}{m(1-\lambda)} \qquad |a_{2m+1}| \leqq \frac{(m+1)\alpha^2}{m^2(1-\lambda)^2}$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_{2m+1} - \rho a_{m+1}^2| \leqq \begin{cases} \frac{\alpha^2}{m(1-\lambda)} & ; \quad |m-2\rho+1| \leqq m(1-\lambda) \\ \frac{|m-2\rho+1|\alpha^2}{m^2(1-\lambda)^2}; & \quad |m-2\rho+1| \geqq m(1-\lambda). \end{cases}$$

**Remark 3.2.** Corollary 3.1 is an improvement of estimates obtained in [2, Theorem 3].

If we set $\lambda = 0$ in Corollary 3.1, then we have the following result.

**Corollary 3.3.** *Let $f$ given by (4) be in the subclass $\mathcal{S}_{\Sigma_m}^{\alpha}$. Then*

$$|a_{m+1}| \leqq \frac{\sqrt{2}\alpha}{m} \qquad |a_{2m+1}| \leqq \frac{(m+1)\alpha^2}{m^2}$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_{2m+1} - \rho a_{m+1}^2| \leqq \begin{cases} \frac{\alpha^2}{m} & ; \quad |m-2\rho+1| \leqq m \\ \frac{|m-2\rho+1|\alpha^2}{m^2}; & \quad |m-2\rho+1| \geqq m. \end{cases}$$

**Remark 3.4.** Corollary 3.3 is an improvement of estimates obtained in [2, Corollary 6].

If we set $m = 1$ in Corollary 3.3, then we have the following result.

**Corollary 3.5.** *Let $f$ given by (1) be strongly bi-starlike functions of order $\alpha$. Then*

$$|a_2| \leqq \sqrt{2}\alpha \qquad |a_3| \leqq 2\alpha^2$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_3 - \rho a_2^2| \leq \begin{cases} \alpha^2 & ; \quad |1-\rho| \leqq \frac{1}{2} \\ 2|1-\rho|\alpha^2; & \quad |1-\rho| \geqq \frac{1}{2}. \end{cases}$$

**Remark 3.6.** Corollary 3.5 is an improvement of estimates obtained in [1, Corollary 2.6].

If we set

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots \ (0 \leqq \beta < 1)$$

and $\gamma = 1$ in Theorem 2.4, then Corollary 3.7 can be obtained.

**Corollary 3.7.** *Let $f$ given by (4) be in the subclass $\mathcal{S}_{\Sigma_m}(\beta,\lambda)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)}{m(1-\lambda)}, \frac{\sqrt{2(1-\beta)}}{m(1-\lambda)}\right\},$$

$$|a_{2m+1}| \leqq \min\left\{\frac{2(m+1)(1-\beta)^2}{m^2(1-\lambda)^2} + \frac{1-\beta}{m(1-\lambda)}, \frac{(m+1)(1-\beta)}{m^2(1-\lambda)^2}\right\}$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_{2m+1} - \rho a_{m+1}^2| \leqq \begin{cases} \frac{1-\beta}{m(1-\lambda)} & ; \quad |m-2\rho+1| \leqq m(1-\lambda) \\ \frac{(1-\beta)|m-2\rho+1|}{m^2(1-\lambda)^2}; & \quad |m-2\rho+1| \geqq m(1-\lambda). \end{cases}$$

**Remark 3.8.** For the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$; it is easily seen that

$$\frac{2(1-\beta)}{m(1-\lambda)} \leqq \frac{\sqrt{2(1-\beta)}}{m(1-\lambda)}; \quad \beta \geqq \frac{1}{2},$$
$$\frac{(m+1)(1-\beta)}{m^2(1-\lambda)^2} \leqq \frac{2(m+1)(1-\beta)^2}{m^2(1-\lambda)^2} + \frac{1-\beta}{m(1-\lambda)}; \quad \beta \leqq \frac{1+m(2-\lambda)}{2(1+m)}.$$

Therefore, Corollary 3.7 is an improvement of estimates obtained in [2, Theorem 5].

If we set $\lambda = 0$ in Corollary 3.7, then we have the following result.

**Corollary 3.9.** *Let $f$ given by (4) be in the subclass of $m$-fold symmetric bi-starlike functions of order $\beta$. Then*

$$|a_{m+1}| \leqq \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}; & 0 \leqq \beta \leqq \frac{1}{2} \\ \frac{2(1-\beta)}{m}; & \frac{1}{2} \leqq \beta < 1, \end{cases}$$

$$|a_{2m+1}| \leqq \begin{cases} \frac{(m+1)(1-\beta)}{m^2}; & 0 \leqq \beta \leqq \frac{1+2m}{2(1+m)} \\ \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}; & \frac{1+2m}{2(1+m)} \leqq \beta < 1 \end{cases}$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_{2m+1} - \rho a_{m+1}^2| \leqq \begin{cases} \frac{1-\beta}{m}; & |m-2\rho+1| \leqq m \\ \frac{(1-\beta)|m-2\rho+1|}{m^2}; & |m-2\rho+1| \geqq m. \end{cases}$$

**Remark 3.10.** Corollary 3.7 is an improvement of estimates obtained in [2, Corollary 7].

If we set $m = 1$ in Corollary 3.9, then we have the following result.

**Corollary 3.11.** *Let $f$ given by (1) be bi-starlike functions of order $\beta$. Then*

$$|a_2| \leqq \begin{cases} \sqrt{2(1-\beta)}; & 0 \leq \beta \leqq \frac{1}{2} \\ 2(1-\beta); & \frac{1}{2} \leqq \beta < 1, \end{cases}$$

$$|a_3| \leqq \begin{cases} 2(1-\beta); & 0 \leqq \beta \leqq \frac{3}{4} \\ (1-\beta)(5-4\beta); & \frac{3}{4} \leqq \beta < 1 \end{cases}$$

*and for some $\rho \in \mathbb{R}$,*

$$|a_3 - \rho a_2^2| \leqq \begin{cases} 1-\beta; & |1-\rho| \leqq \frac{1}{2} \\ 2(1-\beta)|1-\rho|; & |1-\rho| \geqq \frac{1}{2}. \end{cases}$$

**Remark 3.12.** Corollary 3.11 is an improvement of estimates obtained in [1, Corollary 3.5].

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# A note on central reversible ring

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**Abstract**

In this article, we study conditions under which a group ring $R[G]$ of a group $G$ over a ring $R$ becomes a central reversible ring. The notion of central reversible ring is a generalization of reversible ring. We see that the group ring $K[G]$ of a group $G$ over a field $K$ of characteristic zero is reversible if and only if $K[G]$ is central reversible if and only if $K[G]$ is reduced. Also, if $K$ is a field of characteristic $p > 0$ and $G$ is a group which has no finite normal subgroup of order divisible by $p$, then $K[G]$ is reversible if and only if $K[G]$ is central reversible. Finally, we show that the group ring $K[Q_8]$ of quaternions $Q_8$ over a field $K$ of characteristic zero is reversible if and only if $K[Q_8]$ is central reversible if and only if the equation $1 + x^2 + y^2 = 0$ is not solvable in $K$.

## 1. Introduction

In this work, all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent element. If for any ideal $L$ of $R$, $L^2 = 0$ yields $L = 0$, then $R$ is called *semiprime*. A ring in which $rst = 0$ yields $rts = 0$, for all $r, s, t \in R$, is called *symmetric ring*. A ring $R$ is said to be *reversible* if for any $a, b$ of $R$, $ab = 0$ implies $ba = 0$. In this paper, we introduce central reversible rings as a generalization of reversible rings. A ring $R$ is called *central reversible* if $ab = 0$ implies $ba$ belongs to the center of $R$, for any $a, b \in R$. Clearly, reversible rings are central reversible. We see by an example every central reversible ring is not reversible, in general. In this paper, we study conditions under which a group ring $R[G]$ of a group $G$ over a ring $R$ becomes a central reversible ring. We see that the group ring $K[G]$ of a group $G$ over a field $K$ of characteristic zero is reversible if and only if $K[G]$ is central reversible. Also, if $K$ is a field with $char(K) = p > 0$ and $G$ is a group which has no finite normal subgroup of order divisible by $p$, then $K[G]$ is reversible if and only if $K[G]$ is central reversible if and only if $K[G]$ is reduced. Finally, we show that the group ring $K[Q_8]$ of quaternions $Q_8$ over a field $K$ of characteristic zero is reversible if and only if $K[Q_8]$ is central reversible if and only if the equation $1 + x^2 + y^2 = 0$ is not solvable in $K$.

### *1.1. Introduction to nil reversible ring*

In here, we study some properties of central reversible rings. First, remember the definition of central reversible ring.

**Definition 1.1.** A ring $R$ is called central reversible if for each $a$ and $b$ of $R$, $ab = 0$ implies $ba$ is a central element in $R$.

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**Example 1.2.** 1. Every commutative ring is central reversible.
2. reduced rings, central reduced rings, symmetric rings and reversible rings are central reversible.
3. In this part, we see a central reversible ring but is not reversible. Let $R$ be a commutative reduced ring and let

$$T = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid \ a, b, c, d \in R \right\}.$$

[2, Example 2.2] shows that $T$ is a central reversible but is not a reversible ring.
4. Let $K$ be a division ring, $R = K[x, y]$ and $I = (xy)$, where $xy \neq yx$. Since $R$ is a domain, then $R$ is central reversible. On the other hand, $(x + I)(y + I) = 0$ but $(y + I)(x + I) = yx + I \neq 0$ is not central in $\frac{R}{I}$. Hence $\frac{R}{I}$ is not central reversible.

In the following, we show that a set of central reversible rings is closed under finite direct sums.

**Proposition 1.3.** *Every direct sums of central reversible rings is a central reversible ring.*

**Proposition 1.4.** *Let $R$ be a ring and $I$ is a reduced ideal of $R$. If $\frac{R}{I}$ is a central reversible ring, then $R$ is a central reversible.*

## 2. Main results

In this section, we study conditions that the group ring $R[G]$ of a group $G$ over a ring $R$ is a central reversible ring. It is a classical result, the group ring over a field of characteristic zero is semiprime. The next Lemma points to this fact.

**Lemma 2.1.** *If $K$ is a field with $char(K) = 0$ and $G$ is a group, then the group ring $K[G]$ is semiprime.*

[3, Theorem 11.2] which we study in Proposition 2.2, is a result for the group rings of finite groups.

**Proposition 2.2.** *Let $K$ be a field with $char(K) = p > 0$ and $G$ be a group, then the group ring $K[G]$ is semiprime if and only if $G$ has no finite normal subgroup of order divisible by $p$.*

**Lemma 2.3.** *Let $R$ be a semiprime ring. Then, $R$ is central reversible ring if and only if $R$ is reversible.*

*Proof.* We know that every reversible ring is central reversible. Now, let $R$ is central reversible and $x, y \in R$ so that $xy = 0$. Since $R$ is central reversible, $yx$ is central. So $yxRyx = yxyxR = 0$ which yields $yx = 0$, because $R$ is a semiprime ring. Therefore $R$ is a reversible ring. □

Recall that a ring $R$ is called *semicommutative* if for any $x, y \in R$, $xy = 0$ implies $xRy = 0$.

**Theorem 2.4.** *Let $R$ be a semiprime ring. Then the following sets are equivalent:*

1. *$R$ is reversible.*
2. *$R$ is central reversible.*
3. *$R$ is reduced.*
4. *$R$ is semicommutative.*

**Theorem 2.5.** *Let $K$ be a field and $char(K) = p > 0$. If $G$ is a group which has no finite normal subgroup of order divisible by $p$, then the following sets coincide:*

1. *The group ring $K[G]$ is reduced.*
2. *The group ring $K[G]$ is semicommutative.*
3. *The group ring $K[G]$ is reversible.*
4. *The group ring $K[G]$ is central reversible.*

*Proof.* Since $char(K) = p > 0$ and $G$ has no finite normal subgroup of order divisible by $p$, the group ring $K[G]$ is semiprime, by Proposition 2.2. From this, we have this Theorem, by Theorem 2.4. □

**Theorem 2.6.** *Let $K$ be a field with $char(K) = 0$ and $G$ be a group. Then the following sets coincide:*

1. *The group ring $K[G]$ is reduced.*
2. *The group ring $K[G]$ is semicommutative.*
3. *The group ring $K[G]$ is reversible.*
4. *The group ring $K[G]$ is central reversible.*

**Corollary 2.7.** *If $K$ is a field such that $char(K) = 0$, then the following conditions are equivalent:*

1. *The group ring $K[Q_8]$ is a reversible ring.*
2. *The group ring $K[Q_8]$ is a central reversible ring.*
3. *The equation $1 + x^2 + y^2 = 0$ is not solvable in $K$.*

*Proof.* We obtain this result by [1, Theorem 2.1] and Theorem 2.6. □

**Theorem 2.8.** *Let $R$ be a ring and $G$ a group. If the right annihilator of each element of $R[G]$ is generated by an idempotent of $R[G]$, then $R[G]$ is central reversible ring if and only if $R$ is reversible.*

**Theorem 2.9.** *Let $R$ be a ring and $G$ a group. If the left annihilator of each element of $R[G]$ is generated by an idempotent of $R[G]$, then $R[G]$ is central reversible ring if and only if $R$ is reversible.*

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# Hyers–Ulam stability of orthogonally homo-derivations on Banach algebras

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**Abstract**

In this paper, employing the frameworks of additive and Jensen functional equations, we examine orthogonally additive–Jensen $s$-functional equations for a fixed complex constant $s$ with $|s| < 1$. We classify their solutions into two families of orthogonally additive mappings and show that these solutions are orthogonally $\mathbb{C}$-linear mappings on orthogonally Banach algebras. Next, we study the corresponding orthogonally homo-derivations on orthogonally Banach algebras. Finally, we establish the Hyers–Ulam stability of orthogonally additive–Jensen $s$-functional equations and of the associated orthogonally homo-derivations on orthogonally Banach algebras.

## 1. First Section

The study of stability for functional equations originates with Ulam's 1940 question concerning the proximity of approximate solutions to exact ones. Hyers answered this in 1941 by proving a foundational theorem for mappings between Banach spaces, thereby launching the modern theory of stability for functional equations. Rassias extended Hyers's result in 1978 by allowing the control function $\epsilon(\|t_1\|^r + \|t_2\|^r)$ with $\epsilon > 0$ and $r < 1$. Găvruta generalized the approach further in 1994 by replacing that specific control function with an arbitrary function $\varphi(t_1, t_2)$. These developments collectively constitute the Hyers–Ulam stability theory, which asserts that any mapping that approximately satisfies a given functional equations is close to an exact solution (see, e.g., [4] and [5]).
In 1993 M. Obłoza introduced the concept of Hyers–Ulam stability for linear differential equations, thereby initiating the study of stability for differential systems within this framework. Since Obłoza's work, Hyers–Ulam stability has attracted significant attention: numerous authors have investigated stability questions for linear differential equations, fractional differential equations, and related topics in fractional calculus. These contributions have clarified stability criteria for a variety of linear and fractional models and have stimulated further research in the fieldtions (see, e.g., [2] and [6]).

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### 1.1. Orthogonal Set

In 2017, M. Eshaghi Gordji et al. [3] introduced the notion of an orthogonal set on normed spaces as follows.
Let $\mathcal{M}$ be a nonempty set and let $t \in \mathcal{M}$. A relation $\perp \subseteq \mathcal{M} \times \mathcal{M}$ is said to satisfy the orthogonally set property if

$$\exists\, t_0 \in \mathcal{M} : \big(\forall t \in \mathcal{M},\ t \perp t_0\big) \quad \text{or} \quad \big(\forall t \in \mathcal{M},\ t_0 \perp t\big).$$

The pair $(\mathcal{M}, \perp)$ is called an orthogonal set.
Suppose $l : \mathcal{M} \to \mathcal{M}$ is a mapping and $(\mathcal{M}, \perp)$ is an orthogonal set. Then the following statements hold.

1. If $(\mathcal{M}, d)$ is a generalized metric space, then $(\mathcal{M}, \perp, d)$ is an *orthogonally generalized metric space*.
2. A sequence $\{t_i\}_{i\in\mathbb{N}}$ is an O-sequence if either $t_i \perp t_{i+1}$ for all $i$ or $t_{i+1} \perp t_i$ for all $i$. Every O-sequence $\{t_i\}$ in $\mathcal{M}$ that converges to $t$ is $\perp$-continuous.
3. The space $(\mathcal{M}, \perp, d)$ is *orthogonally complete* if every Cauchy O-sequence converges to a point in $\mathcal{M}$. If, in addition, $t_1 \perp t_2$ implies $l(t_1) \perp l(t_2)$, then $l$ is $\perp$-preserving.
4. The mapping $l : \mathcal{M} \to \mathcal{M}$ is a $\perp$-contraction if there exists a Lipschitz constant $0 < \tau < 1$ such that, for all $t_1, t_2$ with $t_1 \perp t_2$,
$$d\big(l(t_1), l(t_2)\big) \le \tau\, d(t_1, t_2).$$

In 2018, Bahraini et al. [1] employed the O-set framework to extend a fixed-point result of Margolis and Diaz. The following theorem states their generalization in the context of orthogonally structured generalized metric spaces.

**Theorem 1.1.** *Let $(\mathfrak{A}, d, \perp)$ be a complete generalized metric space equipped with an orthogonality relation that is preserved. Let $l : \mathfrak{A} \to \mathfrak{A}$ be a mapping that is $\perp$-preserving, $\perp$-continuous, and a $\perp$-contraction with constant $0 < \tau < 1$. Fix $t_0 \in \mathfrak{A}$ such that either $t_0 \perp t$ for all $t \in \mathfrak{A}$ or $t \perp t_0$ for all $t \in \mathfrak{A}$, and consider the O-sequence of successive approximations*

$$t_0,\ l(t_0),\ l^2(t_0),\ \ldots,\ l^i(t_0),\ \ldots$$

*Then exactly one of the following alternatives occurs:*

1. *For every integer $i \ge 0$ we have $d\big(l^i(t_0), l^{i+1}(t_0)\big) = \infty$; or*
2. *There exists $i_0 \in \mathbb{N}$ such that $d\big(l^i(t_0), l^{i+1}(t_0)\big) < \infty$ for all $i > i_0$. In this case:*
   (a) *The sequence $\{l^i(t_0)\}_{i\in\mathbb{N}}$ converges to a point $t^* \in \mathfrak{A}$ which is a fixed point of $l$.*
   (b) *The point $t^*$ is the unique fixed point of $l$ in the set*
   $$\mathfrak{A}^* = \big\{t \in \mathfrak{A} :\ d\big(l^i(t_0), t\big) < \infty\big\}.$$
   (c) *For every $t \in \mathfrak{A}$ with $d\big(l^i(t_0), t\big) < \infty$ one has the estimate*
   $$d(t, t^*) \le \frac{1}{1-\tau}\, d\big(t, l(t)\big).$$

### 1.2. The Basic Notions

In this subsection, let $\mathfrak{A}$ be an orthogonally Banach algebra.
Firstly, we now define orthogonally additive-Jensen $s$-functional equations. Suppose mappings $f, g : \mathfrak{A} \to \mathfrak{A}$ satisfy:

$$\begin{aligned} &\Big\|2f\Big(\frac{t_1+t_2}{2}\Big) - f(t_1) - f(t_2)\Big\| + \Big\|2g\Big(\frac{t_1+t_2}{2}\Big) - g(t_1) - g(t_2)\Big\| \\ &\le \Big\|s\big(f(t_1+t_2) - f(t_1) - f(t_2)\big)\Big\| + \Big\|s\big(g(t_1+t_2) - g(t_1) - g(t_2)\big)\Big\| \end{aligned} \tag{1}$$

for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$.

In the following definition, we proceed to introduce the notion of orthogonally homo-derivations in the context of Banach algebra.

**Definition 1.2.** Let $\mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ be an orthogonally derivation. Let an orthogonally $\mathbb{C}$-linear mapping $\mathfrak{H} : \mathfrak{A} \to \mathfrak{A}$. Such a map $\mathfrak{H}$ is called an orthogonally homo-derivation when it satisfies the following relation for every element $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$:

$$\mathfrak{H}(t_1 t_2) = \mathfrak{H}(t_1)t_2 + t_1\mathfrak{D}(t_2) + \mathfrak{H}(t_1)\mathfrak{H}(t_2).$$

This condition must hold for all $t_1, t_2 \in \mathfrak{A}$, and it encapsulates the defining property of an orthogonally homo-derivation on Banach algebra.

In the next section, we slove the orthogonally additive–Jensen $s$-functional equations for a fixed complex constant $s$ with $|s| < 1$ are orthogonally additive mappings, after that we show they are orthogonally $\mathbb{C}$-linear mappings on orthogonally Banach algebra. Finally, using the orthogonally fixed point method, we prove the Hyers–Ulam stability of orthogonally additive–Jensen $s$-functional equations and of the associated orthogonally homo-derivations on orthogonally Banach algebras.

## 2. Hyers-Ulam Stability

Let $\mathfrak{A}$ be an orthogonal Banach algebra. Fix $\lambda \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and a nonzero complex number $s$ with $|s| < 1$.
The first lemma of this section shows that the functional equation (1) implies orthogonal additivity.

**Lemma 2.1.** *If mappings $f, g : \mathfrak{A} \to \mathfrak{A}$ for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$ satisfy (1), then the mappings $f$ and $g$ are two implies orthogonal additivity.*

The next lemma examines when the functional equation (1) with $\lambda \in \mathbb{T}^1$ is orthogonally $\mathbb{C}$-linear.

**Lemma 2.2.** *If a mapping $f, g : \mathfrak{A} \to \mathfrak{A}$ for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$ satisfy*

$$\begin{aligned} &\left\|2f(\lambda\frac{t_1+t_2}{2}) - \lambda f(t_1) - \lambda f(t_2)\right\| + \left\|2g(\lambda\frac{t_1+t_2}{2}) - \lambda g(t_1) - \lambda g(t_2)\right\| \\ &\leq \left\|s\big(f(\lambda(t_1+t_2)) - \lambda f(t_1) - \lambda f(t_2)\big)\right\| + \left\|s\big(g(\lambda(t_1+t_2)) - \lambda g(t_1) - \lambda g(t_2)\big)\right\| \end{aligned} \tag{2}$$

*Then the mappings $f$ and $g$ are orthogonally $\mathbb{C}-$linear.*

In the next theorem, we investigate the Hyers-Ulam stability of the functional equation (1) on Banach algebra.

**Theorem 2.3.** *Let $f, g : \mathfrak{A} \to \mathfrak{A}$ are two mappings for which there exist function $\varphi : \mathfrak{A}^2 \to [0, \infty)$, satisfying for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$*

$$\varphi\left(\frac{t_1}{2}, \frac{t_2}{2}\right) \leq \frac{\tau}{2}\varphi(t_1, t_2) \tag{3}$$

$$\begin{aligned} &\left\|2f(\lambda\frac{t_1+t_2}{2}) - \lambda f(t_1) - \lambda f(t_2)\right\| + \left\|2g(\lambda\frac{t_1+t_2}{2}) - \lambda g(t_1) - \lambda g(t_2)\right\| \\ &\leq \left\|s\big(f(\lambda(t_1+t_2)) - \lambda f(t_1) - \lambda f(t_2)\big)\right\| + \left\|s\big(g(\lambda(t_1+t_2)) - \lambda g(t_1) - \lambda g(t_2)\big)\right\| + \varphi(t_1, t_2) \end{aligned} \tag{4}$$

*Then there exist two unique $\mathbb{C}-$linear mappings $\mathfrak{H}, \mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ such that for all $t \in \mathfrak{A}$*

$$\|f(t) - \mathfrak{H}(t)\| \leq \frac{\tau}{2(1-\tau)}\varphi(t, 0),$$
$$\|g(t) - \mathfrak{D}(t)\| \leq \frac{\tau}{2(1-\tau)}\varphi(t, 0).$$

Now, we can prove the Hyers-Ulam stability of generalized homo-derivation on Banach algebra.

**Theorem 2.4.** *Let $f, g : \mathfrak{A} \to \mathfrak{A}$ be mappings satisfying the conditions of Theorem 2.3. There exist function $\psi : \mathfrak{A}^2 \to [0, \infty)$, satisfying for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$*

$$\varphi\left(\frac{t_1}{2}, \frac{t_2}{2}\right) \leq \frac{\tau}{2^2}\psi(t_1, t_2) \tag{5}$$

$$\|f(t_1t_2) - f(t_1)t_2 - t_1f(t_2)\| \leq \psi(t_1, t_2) \tag{6}$$

*and*

$$\|g(t_1t_2) - g(t_1)t_2 - t_1f(t_2) - g(t_1)g(t_2)\| \leq \psi(t_1, t_2) \tag{7}$$

*then there exist unique an orthogonally derivation $\mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ such that $\mathfrak{H} : \mathfrak{A} \to \mathfrak{A}$ is an orthogonally homo-derivation.*

In the following corollaries, we discuss the Hyers-Ulam stability with the control function of Rassias.

**Corollary 2.5.** *Let $r > 1$ and $\alpha$ be non-negative real numbers and $f, g : \mathfrak{A} \to \mathfrak{A}$ be two mappings satisfying for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$*

$$\begin{aligned}&\left\|2f(\lambda\frac{t_1 + t_2}{2}) - \lambda f(t_1) - \lambda f(t_2)\right\| + \left\|2g(\lambda\frac{t_1 + t_2}{2}) - \lambda g(t_1) - \lambda g(t_2)\right\| \\ &\leq \left\|s\big(f(\lambda(t_1 + t_2)) - \lambda f(t_1) - \lambda f(t_2)\big)\right\| + \left\|s\big(g(\lambda(t_1 + t_2)) - \lambda g(t_1) - \lambda g(t_2)\big)\right\| + \alpha(\|t_1\|^r + \|t_2\|^r)\end{aligned}$$

*Then there exist two $\mathbb{C}-$linear mappings $\mathfrak{H}, \mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ such that*

$$\|f(t) - \mathfrak{H}(t)\| \leq \frac{2\alpha}{2^r - 2}\|t\|^r$$

$$\|g(t) - \mathfrak{D}(t)\| \leq \frac{2\alpha}{2^r - 2}\|t\|^r.$$

**Corollary 2.6.** *Let $r > 1$ and $\alpha$ be non-negative real numbers and $f, g : \mathfrak{A} \to \mathfrak{A}$ be mappings satisfying the conditions of Corollary 2.5. Assume the statement holds for all $t_1, t_2 \in \mathfrak{A}$ with $t_1 \perp t_2$ or $t_2 \perp t_1$*

$$\|f(t_1t_2) - f(t_1)t_2 - t_1f(t_2)\| \leq \alpha(\|t_1\|^r + \|t_2\|^r)$$

*and*

$$\|g(t_1t_2) - g(t_1)t_2 - t_1f(t_2) - g(t_1)g(t_2)\| \leq \alpha(\|t_1\|^r + \|t_2\|^r)$$

*then there exist unique an orthogonally derivation $\mathfrak{D} : \mathfrak{A} \to \mathfrak{A}$ such that $\mathfrak{H} : \mathfrak{A} \to \mathfrak{A}$ is an orthogonally homo-derivation.*

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# Coupled Best Proximity Points for KKM-type Theorem in Random Normed Spaces

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**Abstract**

We generalized the R-KKM theorem for the intersectionally closed-valued maps. We prove some best and coupled best proximity results for multivalued mappings on Random normed spaces.

## 1. Introduction and Preliminaries

The best proximity point theorems are basic part of nonlinear analysis and applications. In recent years many authors are studying the best proximiyu theorems and relatively result have been obtained in the metric spaces or normed spaces (see Guo et al. 2016, Sarma and Aemro 2017, Saleam et al. 2022, Sadigh Basha 2023) [1, 3–5]. In 2012, Raj et al. [2] introduced a new version of KKM-type theorem, called R-KKM mappings, and proved a best proximity point theorem in the normed linear spaces. They proved the Browder set-valued fixed point theorem by invoking the R-KKM technique.

**Definition 1.1.** mapping $F : (-\infty, \infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$. If in addition $F(0) = 0$, then $F$ is called a distance distribution function. The set of all distance distribution functions is denoted by $\Delta^+$.

**Definition 1.2.** A triangular norm (shorter $t$-norm) is a binary operation $*$ on $[0, 1]$, which satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0, 1]$;
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

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**Definition 1.3.** A random normed space (RN-space) is a triple $(V, \nu, *)$, where $V$ is a real vector space, $\nu$ is a mapping from $V$ into $\Delta^+$ (for $x \in V$, the distribution function $\nu(x)$ is denoted by $\nu_x$ , and $\nu_x(t)$ is the value $\nu_x$ at $t \in \mathbb{R}$) and $*$ is a $t-$norm satisfying the following conditions:
(RN-1) $\nu_x(0) = 0$;
(RN-2) $\nu_x(t) = 1$ for all $t > 0$ iff $x = 0$;
(RN-3) $\nu_{\alpha x}(t) = \nu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R} \setminus \{0\}$;
(RN-4) $\nu_{x+y}(s+t) \geq \nu_x(s) * \nu_y(t)$ for all $x, y \in V$ and $s, t \in [0,1]$.

Let $(V, \nu, *_1)$ and $(W, \omega, *_2)$ are RN-spaces, $A$ and $B$ be nonempty subsets of these spaces respectivly and $S : A \to B$ be a map. For each $t > 0$, we define the following notions, which will be used in the sequel.

$$\omega_{SA-B}(t) = \sup\{\omega_{Sx-y}(t) : (x, y) \in A \times B\},$$

$$A_0^S = \{x \in A : \omega_{Sx-y}(t) = \omega_{SA-B}(t) \, for \, some \, y \in B\},$$

$$B_0^S = \{y \in B : \omega_{Sx-y}(t) = \omega_{SA-B}(t) \, for \, some \, x \in A\}.$$

If $(V, \nu, *_1) = (W, \omega, *_2)$ and for all $x \in A$, $Sx = x$ then we denote $A_0^S$ and $B_0^S$ with $A_0$ and $B_0$ respectivly.

**Definition 1.4.** Let $(V, \nu, *_1)$ and $(W, \omega, *_2)$ are RN-spaces, $A$ and $B$ be nonempty subsets of these spaces respectivly and $S : A \to B$ be a map.Then the pair $(A, B)$ is said to be a proximal pair with respect to $S$ if for each $t > 0$ and $(x, y) \in A \times B$ there exists $(\tilde{x}, \tilde{y}) \in A \times B$ such that

$$\omega_{Sx-\tilde{y}}(t) = \omega_{S\tilde{x}-y}(t) = \omega_{SA-B}(t).$$

If $(V, \nu, *_1) = (W, \omega, *_2)$ and for all $x \in A$, $Sx = x$ then the pair $(A, B)$ is called a proximal pair. Note that a pair $(A, B)$ is a proximal pair if and only if $A = A_0$ and $B = B_0$.

## 2. Main results

Now let us define the generalized R-KKM mappings on RN-spaces and state our main results.

**Definition 2.1.** Let $(V, \nu, *_1)$ and $(W, \omega, *_2)$ are RN-spaces, $A$ and $B$ be nonempty subsets of these spaces respectivly and $S : A \to B$ be a map. A set-valued map $G : A \to 2^B$ is said to be a generalized $R\text{-}KKM$ map if, for any finite set $\{x_1, ..., x_n\} \subseteq A$, there exists a finite subset $\{y_1, ..., y_n\} \subseteq B$ such that $\{y_{i_1}, ..., y_{i_k}\} \subseteq \{y_1, ..., y_n\}$, for all $\{i_1, i_2, ..., i_k\} \subseteq \{1, 2, ..., n\}$, with $\omega_{Sx_{i_j}-y_{i_j}}(t) = \omega_{SA-B}(t)$ , for $1 \leq j \leq k$, and $t > 0$, we have

$$co(\{y_{i_1}, ..., y_{i_k}\}) \subseteq \bigcup_{j=1}^{k} G(x_{i_j}).$$

If $(V, \nu, *_1) = (W, \omega, *_2)$ and $\{i_1, i_2, ..., i_k\} = \{1, 2, ..., n\}$ and for all $x \in A$ , $Sx = x$. Then a set-valued map $G : A \to 2^B$ with above condition is called R-KKM map.

**Theorem 2.2.** *Let $(V, \nu, *_1)$ and $(W, \omega, *_2)$ are RN-spaces, A and B be nonempty subsets of these spaces respectivly and $S : A \to B$ be a map. Let $(A, B)$ be a proximal pair with respect to S and $G : A \to 2^B$ be a generalized R-KKM map. If for each $x \in A$, $G(x)$ is a $\omega$-closed and there exists at least an $x_0 \in A$ such that $G(x_0)$ is $\omega$-compact. Then $\bigcap\{G(x) : x \in A\}$ is nonempty.*

*Proof.* First we prove that $\{G(x) : x \in A\}$ has the finite intersection property. Suppose it is not true, then there exists $\{x_1, ..., x_n\} \subseteq A$ such that $\bigcap_{i=1}^{n} G(x_i) = \emptyset$. For a given finite set $\{x_1, ..., x_n\}$ there exists a subset $\{y_1, ..., y_n\} \subseteq B$ such that for any $\{y_{i_1}, ..., y_{i_k}\} \subseteq \{y_1, ..., y_n\}$, with $\omega_{Sx_{i_j}-y_{i_j}}(t) = \omega_{SA-B}(t)$, for $1 \leq j \leq k$ and each $t > 0$, we have

$$co(\{y_{i_1}, ..., y_{i_k})\} \subseteq \bigcup_{j=1}^{k} G(x_{i_j}).$$

In particular, we have $co(\{y_1,\dots,y_n\}) \subset \bigcup_{i=1}^{n} G(x_i)$. let $C = co(\{y_1,\dots,y_n\})$, since $\bigcap_{i=1}^{n} G(x_i) = \emptyset$, then for every $c \in C$, there exists $i_0 \in \{1,\dots,n\}$ such $c \notin G(x_{i_0})$. Since $G(x_{i_0})$ is $\omega$-closed, by Lemma 2.5, $\omega_{c-G(x_{i_0})}(t) < 1$ for $t > 0$. Thus $(1 - \omega_{c-G(x_i)}(t)) > 0$. Therefore if $\alpha(c) = \sum_{i=1}^{n}(1 - \omega_{c-G(x_i)}(t))$, then $\alpha(c) \neq 0$, for every $c \in C$. Define the mapping $f : C \to C$ by setting

$$f(c) = \frac{1}{\alpha(c)} \sum_{i=1}^{n} (1 - \omega_{c-G(x_i)}(t)) y_i \,.$$

It is clear that $f$ is a well-defined and continuous map from $C$ to $C$. By Corollary 2.4, since $C$ is $\omega$- compact, therefore by the Browder fixed point theorem for single-valued map, there exists $c_0 \in C$ such that $f(c_0) = c_0$.
Set $I := \{i \in \{1,\dots,n\} : (1 - \omega_{c_0-G(x_i)}(t)) > 0\}$. Then for each $i \in I$, $c_0 \notin G(x_i)$. So we have $c_0 \notin \bigcup_{i\in I} G(x_i)$. On the other hand,

$$c_0 = \frac{1}{\alpha(c_0)} \sum_{i\in I} (1 - \omega_{c_0-G(x_i)}(t)) y_i \,.$$

And so $c_0 \in co(\{y_i : i \in I\})$. But $G$ is a generalized R-KKM mapping and we have,

$$c_0 \in co(\{y_i : i \in I\}) \subset \bigcup_{i\in I} G(x_i) \,,$$

Which is a contradiction. Hence $\{G(x) : x \in X\}$ has the finite intersection property.
However since $\{G(x) \cap G(x_0) : x \in X\}$ is a family of $\omega$-closed sets in $G(x_0)$, by the property of compact sets we have

$$\emptyset \neq \bigcap_{x\in X} \{G(x) \cap G(x_0)\} = G(x_0) \cap \bigcap_{x\in X} G(x) \subseteq \bigcap_{x\in X} G(x) \,.$$

This complete the proof. □

**Corollary 2.3.** *Let $(A,B)$ be a proximal pair in RN-space $(V,\nu,*)$ and $G : A \to 2^B$ be an R-KKM map. If for each $x \in A$, $G(x)$ is a $\nu$-closed and there exists at least one $x_0 \in A$ such that $G(x_0)$ is $\nu$-compact, then $\bigcap\{G(x) : x \in A\}$ is nonempty.*

Note that, if $G : A \to 2^B$ be an R-KKM map on RN-space $(V,\nu,*)$, then $\nu_{x-G(x)}(t) = \nu_{A-B)}(t)$, for any $x \in A$ and $t > 0$. Also if $A = B$, then we have the following definition and corollary.

**Definition 2.4.** Let $(V,\nu,*)$ be a RN-space, $A \subseteq V$ a nonempty subset. A set-valued $G : A \to 2^V$ is called KKM, if $co(\{x_1,\dots,x_n\}) \subseteq \bigcup_{i=1}^{n} G(x_i)$ for each finite subset $\{x_1,\dots,x_n\} \subseteq A$.

**Corollary 2.5.** (KKM principle) Let $(V,\nu,*_1)$ be a RN-space, $A \subseteq V$ be a arbitrary subset, and $G : A \to 2^V$ be a KKM map. If all the sets $G(x)$, $x \in A$ are $\nu$-closed and if one is $\nu$-compact, then $\bigcap\{G(x) : x \in A\} \neq \emptyset$.

**Theorem 2.6.** *Let $A$ and $B$ be a nonempty, $\nu$-compact convex subset of RN-space $(V,\nu,*)$ and $S : A \to B$ is a affine mapping. Suppose $(A,B)$ is a nonempty proximal pair with respect to $S$. Let $F : A \to 2^B$ be a set-valued map that satisfies the following:*

1. *for each $x \in A$, $F(x)$ is a convex subset of $B$;*
2. *for each $y \in B$, $F^{-1}(y) = \{x \in A : y \in F(x)\}$ is a nonempty open set.*

*Then there is $x^* \in A_0^S$ such that for $t > 0$,*

$$\nu_{Sx^*-Fx^*}(t) = \nu_{SA-B}(t) \,.$$

*Proof.* Let us define $G : B \to 2^A$ by $G(y) = A \setminus F^{-1}(y)$, $y \in B$. If $G(y) = \emptyset$ for some $y \in B_0^S$, that is $F^{-1}(y) = A$, then $y \in F(x)$ for all $x \in A$. Since $y \in B$, there exists $x_0 \in A$ such that for $t > 0$, $\nu_{Sx_0-y}(t) = \nu_{SA-B}(t)$. In particular $y \in F(x_0)$. Then for $t > 0$,

$$\nu_{SA-B}(t) \geq \nu_{Sx_0-F(x_0)}(t) \geq \nu_{Sx_0-y}(t) = \nu_{SA-B}(t)\,,$$

which implies that $\nu_{Sx_0-F(x_0)}(t) = \nu_{SA-B}(t)$ for $t > 0$. Assume that $G(y)$ is nonempty. By the hypothesis we have that $G$ is a nonempty closed set-valued on $B$. Also, $\bigcup\{F^{-1}(y) : y \in B\}$ is an open cover for $A$. Then

$$\bigcap_{y\in B} G(y) = \bigcap_{y\in B}(A \setminus F^{-1}(y)) = A \setminus (\bigcup_{y\in B} F^{-1}(y)) = \emptyset\,.$$

From Theorem 3.2 we conclude that $G$ is not a generalized $R$-$KKM$ map. Therefore, there exist $\{y_1, ..., y_n\} \subseteq B$ and $\{x_1, ..., x_n\} \subseteq A$ such that for $\{x_{i_1}, ..., x_{i_k}\} \subseteq \{x_1, ..., x_n\}$ with $\nu_{Sx_{i_j}-y_{i_j}}(t) = \nu_{SA-B}(t)$ for $1 \leq j \leq k$ and $t > 0$, such that $co(\{x_{i_1}, ..., x_{i_k}\})$ is not contained in $\bigcup_{j=1}^k G(y_{i_j})$.

Choose $x^* = \sum_{j=1}^k \lambda_j x_{i_j} \in co(\{x_{i_1}, ..., x_{i_k}\})$, but $x^* \notin \bigcup_{j=1}^k G(y_{i_j}) = A \setminus \bigcap_{j=1}^k F^{-1}(y_{i_j})$. Therefore, $x^* \in F^{-1}y_{i_j}$, for all $1 \leq j \leq k$. That is $y_{i_j} \in F(x^*)$ for all $1 \leq j \leq k$. Put $z = \sum_{j=1}^k \lambda_j y_{i_j}$. Since $F(x^*)$ is convex and $y_{i_j} \in F(x^*)$, for each $1 \leq j \leq k$, we conclude that $z \in F(x^*)$. Then

$$\nu_{SA-B}(t) \geq \nu_{Sx^*-Fx^*}(t) \geq \nu_{Sx^*-z}(t) = \nu_{S\sum_{j=1}^k \lambda_j x_{i_j} - \sum_{j=1}^k \lambda_j y_{i_j}}(t) = \nu_{SA-B}(t)\,.$$

Therefore for $t > 0$, $\nu_{Sx^*-Fx^*}(t) = \nu_{SA-B}(t)$. □

Now from Theorem 3.6, we immediatly get the following best proximity result on RN-spaces.

**Corollary 2.7.** *Let $(A, B)$ be a nonempty compact convex proximal pair with $A_0 \neq \emptyset$ on a RN- space $(V, \nu, *)$ and $F : A \to 2^B$ be a set-valued map that satisfies the following:*

1. *for each $x \in A$, $F(x)$ is a convex subset of $B$;*
2. *for each $y \in B$, $F^{-1}(y) = \{x \in A : y \in F(x)\}$ is an nonempty open set.*

*Then there is $x^* \in A_0$ such that for $t > 0$,*

$$\omega_{x^*-Fx^*}(t) = \omega_{A-B}(t)\,.$$

The following corollary is a special case for fixed point theorem of Browder for set-valued maps on $RN$-spaces.

**Corollary 2.8.** *Let $A$ be a nonempty compact convex subset of $RN$-space $(V, \nu, *)$. If $F : A \to 2^A$ is a convex valued with open fibers, then there exists $x^* \in A$ such $x^*$ is a fixed point of $F$.*

**Remark 2.9.** It is easy to see if $(X, N', *)$ and$(Y, N'', *)$ are two $RN$-spaces, so is $X \times Y$ equipped with $N_{(x,y)}(t) = N'_x * N''_y$, where $(x, y) \in X \times Y$ and $t > 0$.

**Theorem 2.10.** *Let $C$ and $D$ be nonempty compact convex subset of $RN$-space $(X, N', *)$ and $S : C \to D$ is a affine mapping. Let $(C, D)$ be a proximal pair with respect to $S$ and $T : C \times C \to 2^D$ be a set-valued map that satisfies the following:*

1. *for each $(x, y) \in C \times C$, $T(x, y)$ is a convex subset of $D$;*
2. *for each $z \in D$, $T^{-1}(z) = \{(x, y) \in C \times C : z \in T(x, y)\}$ is an nonempty open set.*

*Then there is $(x^*, y^*) \in C_0^S \times C_0^S$ such that for $t > 0$*

$$N'_{Sx^*-T(x^*,y^*)}(t) * N'_{Sy^*-T(y^*,x^*)}(t) = \sup_{(x,y)\in C\times C} \{N'_{Sx-D}(t) * N'_{Sy-D}(t)\}$$

*Proof.* We consider RN-spaces $(W, N, *)$, where $W = X \times X$ and for $t > 0$, $N_{(u,v)}(t) = N'_u(t) * N'_v(t)$ for any $(u, v) \in X \times X$. Let $A = C \times C$ and $B = D \times D$, define the mappings $F : A \to 2^B$ and $\tilde{S} : A \to B$ respectively by $F(x, y) = (T(x, y), T(y, x))$ and $\tilde{S}(x, y) = (Sx, Sy)$. Then $(A, B)$ be a nonempty compact convex proximal pair with respect to $\tilde{S}$ on a RN-spaces $(W, N, *)$. Then all of the conditions of Theorem 3.6 are satisfied and so the conclusion follows. □

Now, from Theorem 3.10 we immediatly get the following coupled best proximity points theorem on RN-spaces.

**Corollary 2.11.** *Let $C$ and $D$ be nonempty compact convex subsets of RN-space $(X, N, *)$ and $(C, D)$ be a proximal pair and $T : C \times C \to 2^D$ be a set-valued map that satisfies the following:*

1. *for each $(x, y) \in C \times C$, $T(x, y)$ is a convex subset of $D$;*
2. *for each $z \in D$, $T^{-1}(z) = \{(x, y) \in C \times C : z \in T(x, y)\}$ .*

*Then there is $(x^*, y^*) \in C_\circ \times C_\circ$ such that for $t > 0$*

$$N_{x^* - T(x^*, y^*)}(t) * N_{y^* - T(y^*, x^*)}(t) = \sup_{(x,y) \in C \times C} \{N_{x-D}(t) * N_{y-D}(t)\}.$$

**Corollary 2.12.** *Let $C$ be nonempty compact convex subset of RN-spaces $(X, N, *)$ and $(C, D)$ be a proximal pair with respect to affine mapping $S$ and $T : C \times C \to 2^D$ be a set-valued map that satisfies the following:*

1. *for each $(x, y) \in C \times C$, $T(x, y)$ is a convex subset of $D$;*
2. *for each $z \in D$, $T^{-1}(z) = \{(x, y) \in C \times C : z \in T(x, y)\}$ is an nonempty open set .*

*Then $T$ and $S$ have coupled coincidence point .*

The following corollary is a special case of a coupled fixed point theorem on RN-spaces.

**Corollary 2.13.** *Let $(X, \mu, *)$ be a RN-space and $C$ be a nonempty compact convex subset of $X$. If $T : C \times C \to 2^C$ is a convex valued set-valued map with open fibers, then there exists $(x^*, y^*) \in C \times C$ such that $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$.*

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# A Nonstandard Finite Difference Approach for Solving the Burgers–Fisher Equation

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**Abstract**

This study proposes a numerical method for solving the Burgers–Fisher (BF) equation by using a nonstandard finite difference (NSFD) scheme approach. First, an exact NSFD scheme for the BF equation is derived. Then, an NSFD scheme is constructed to solve this equation. The positivity and boundedness properties of the proposed NSFD scheme are thoroughly analyzed. Numerical simulations demonstrate the remarkable accuracy and computational efficiency of the NSFD scheme.

## 1. Introduction

Partial differential equations (PDEs) are of fundamental importance in diverse areas of physical sciences and engineering. The BF equation represents one of the significant nonlinear PDEs that arises in a wide range of applications, such as fluid mechanics, thermal conduction and gas dynamics. In this work, we study the BF equation given by

$$U_t(x,t) + \alpha U(x,t)U_x(x,t) - \beta U_{xx}(x,t) = \gamma U(x,t)(1-U(x,t)), \quad x \in [0,1], \quad t \in [0,T], \tag{1}$$

wherein $\alpha$, $\beta$, and $\gamma$ are prescribed constants satisfying $\alpha \geq 0$, $\gamma \geq 0$, and $\beta > 0$. Mickens developed NSFD schemes for solving differential equations [1], and these have proven highly effective [4, 5]. NSFD schemes avoid instabilities found in standard finite difference methods by using special denominator functions instead of regular stepsizes [3]. The remainder of this paper is structured as follows: Section 2 briefly reviews NSFD construction for differential equations and derives an explicit exact NSFD scheme for the BF equation. Section 3 focuses on the development of an NSFD scheme for the BF equation, where the fundamental properties of positivity and boundedness associated with the scheme are proven. Finally, numerical simulations in the last section verify the scheme's efficiency.

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## 2. NSFD schemes for ordinary differential equations

In this section, we review the fundamental principles NSFD schemes. Mickens introduced NSFD schemes for ordinary differential equations [1]. These schemes were developed to address the numerical instabilities that emerge when standard finite difference methods are applied. Through the selection of an appropriate denominator function instead of the standard stepsize, combined with the implementation of nonlocal discretization techniques, NSFD schemes are capable of preserving essential properties including positivity, boundedness, and stability, when larger stepsizes are employed. To illustrate the construction of an NSFD scheme, consider the following initial value problem

$$Y'(t) = f(Y(t)), \quad Y(0) = Y_0, \quad t \in [0, t_f].$$

The formulation of an NSFD scheme can be done through at least one of the two procedures outlined below [5]:
1. The continuous derivative is approximated using the discrete form $Y'(t_k) \approx \frac{Y_{k+1}-Y_k}{\phi(\Delta t)}$, where $Y_k$ represents the numerical approximation of $Y(t_k)$. The nontrivial denominator function $\phi(\Delta t)$ is required to be continuous and increasing, while satisfying:

$$\phi(\Delta t) = \Delta t + O(\Delta t^2), \quad 0 < \phi(\Delta t) < 1, \quad \Delta t \to 0.$$

2. Nonlocal approximations are used for both linear and nonlinear terms. Examples of suitable denominator functions are [4, 5]

$$\frac{1 - e^{-\lambda \Delta t}}{\lambda}, \quad \sin(\Delta t).$$

In [2] Wazwaz derived the exact solution to the BF equation (1) using nonlinear transformations for the initial condition:

$$U(x,0) = \frac{1}{1 + e^{\frac{\alpha}{2\beta}x}}, \quad x \in [0,1],$$

with boundary conditions:

$$U(0,t) = \frac{1}{1 + e^{-\frac{\alpha^2+4\beta\gamma}{4\beta}t}}, \quad U(1,t) = \frac{1}{1 + e^{\frac{2\alpha-(\alpha^2+4\beta\gamma)t}{4\beta}}}, \quad t \geq 0.$$

The exact analytical solution of the BF equation (1) is expressed as:

$$U(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\alpha}{4\beta}(x - \frac{\alpha^2+4\beta\gamma}{2\alpha}t)\right) = \frac{1}{1 + e^{\frac{\alpha}{2\beta}(x - \frac{\alpha^2+4\beta\gamma}{2\alpha}t)}}. \tag{2}$$

Based on the exact solution (2) of the BF equation and the fundamental concepts for constructing NSFD schemes, we arrive at the following theorem.

**Theorem 2.1.** *The explicit exact NSFD scheme associated with the BF equation (1) takes the form:*

$$\frac{U_j^{n+1} - U_j^n}{\varphi_1} + \alpha U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\psi_1} - \beta \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\psi_1 \psi_2}$$
$$= \gamma U_j^n (1 - U_{j-1}^n)$$

*in which the spatial stepsize $h$ and the time stepsize $\Delta t$ are required to fulfill the condition $h = \frac{\alpha^2+4\beta\gamma}{2\alpha}\Delta t$, and the denominator functions are specified as follows:*

$$\varphi_1 = \frac{e^{\frac{\alpha^2+4\beta\gamma}{4\beta}\Delta t} - 1}{\frac{\alpha^2+4\beta\gamma}{4\beta}}, \quad \psi_1 = \frac{e^{\frac{\alpha}{2\beta}h} - 1}{\frac{\alpha}{2\beta}}, \quad \psi_2 = \frac{1 - e^{-\frac{\alpha}{2\beta}h}}{\frac{\alpha}{2\beta}}.$$

## 3. On the analysis of a NSFD scheme for the BF equation

The aim of this part is to construct an NSFD scheme for the BF model (1) and analyzing its fundamental characteristics. We consider the following NSFD scheme for the BF model (1).

$$\frac{U_j^{n+1}-U_j^n}{\phi}+\alpha U_j^{n+1}\frac{U_{j+1}^n-U_{j-1}^n}{\Gamma}-\beta\frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{\psi}=\gamma U_j^{n+1}-\gamma U_j^{n+1}U_j^n, \tag{3}$$

where $U_j^n$ stands for the numerical approximation of the exact solution $U(x_j,t_n)$ at the mesh point $(x_j,t_n)$, and the denominator functions $\phi,\Gamma,$ and $\psi$ are given by

$$\phi=\frac{1-e^{-\frac{\alpha^2+4\beta\gamma}{4\beta}\Delta t}}{\frac{\alpha^2+4\beta\gamma}{4\beta}},\qquad \Gamma=\psi_1+\psi_2,\qquad \psi=\psi_1\psi_2.$$

By comparing (3) with (1), one deduces that the linear and nonlinear terms on the right–hand–side of the BF equation (1) can be approximated through nonlocal discretization as

$$U(x_j,t_n)\approx U_j^{n+1},\qquad U^2(x_j,t_n)\approx U_j^{n+1}U_j^n. \tag{4}$$

It is straightforward to verify that the scheme (3) can be written in explicit form. Letting $R=\frac{\phi}{\psi}$ and $r=\frac{\phi}{\Gamma}$, we can express (3) as

$$U_j^{n+1}=\frac{2R\beta(U_{j+1}^n+U_{j-1}^n)+(2-4R\beta)U_j^n}{2+\alpha r(U_{j+1}^n-U_{j-1}^n)+2\gamma\phi(U_j^n-1)}.$$

As evident from (2), the analytical solution (1) fulfills $0<U(x,t)<1$, indicating positivity and boundedness. The NSFD scheme (4) preserves both properties. The following theorem demonstrates these characteristics.

**Theorem 3.1.** *Let the condition $\alpha r\le 2R\beta\le 1-\gamma\phi$ hold. Then the numerical solution (4) preserves positivity and boundedness, satisfying*

$$0<U_j^n<1\implies 0<U_j^{n+1}<1$$

*for all values of $n$ and $j$.*

*Proof.* Suppose that the theorem's hypotheses are fulfilled, it then follows that $1-2R\beta\ge 0$ and $2-2\gamma\phi-\alpha r\ge 0$. Under these conditions, the following inequalities hold:

$$\begin{cases}2R\beta(U_{j+1}^n+U_{j-1}^n)+(2-4R\beta)U_j^n>0,\\ 2+\alpha r(U_j^n-U_{j-1}^n)+2\gamma\phi(U_j^n-1)>0.\end{cases} \tag{5}$$

Inequalities (5) yield $U_j^{n+1}>0$. Additionally, under the assumptions $1-2R\beta-\gamma\phi\ge 0$ and $2R\beta-\alpha r\ge 0$, one obtains

$$\begin{aligned}&2R\beta(U_{j+1}^n+U_{j-1}^n)+(2-4R\beta)U_j^n-\alpha r(U_{j+1}^n-U_{j-1}^n)\\&\quad-2\gamma\phi(U_j^n-1)\\&=(2R\beta-\alpha r)U_{j+1}^n+(2R\beta+\alpha r)U_{j-1}^n\\&\quad+(2-4R\beta-2\gamma\phi)U_j^n+2\gamma\phi\\&<2R\beta-\alpha r+2R\beta+\alpha r+2-4R\beta-2\gamma\phi+2\gamma\phi\\&=2.\end{aligned} \tag{6}$$

Consequently, bounds (5) and (6) verify that the NSFD scheme (4) preserves positivity and satisfies boundedness. □

## 4. Numerical results and error analysis

To validate the effectiveness and accuracy of the proposed NSFD scheme, we consider the following initial–value problem modeled by the BF equation:

$$U_t(x,t) + \alpha U(x,t)U_x(x,t) - \beta U_{xx}(x,t) = \gamma U(x,t)(1 - U(x,t)), \qquad x \in [0,1], \qquad t \geq 0,$$

with the initial condition

$$U(x,0) = \frac{1}{1 + e^{\frac{\alpha}{2\beta}x}}, \qquad x \in [0,1],$$

along with the boundary conditions

$$U(0,t) = \frac{1}{1 + e^{-\frac{\alpha^2+4\beta\gamma}{4\beta}t}}, \qquad t \geq 0,$$

$$U(1,t) = \frac{1}{1 + e^{\frac{2\alpha-(\alpha^2+4\beta\gamma)t}{4\beta}}}, \qquad t \geq 0.$$

We apply the NSFD scheme (3) together with the following initial and boundary conditions

$$U_j^0 = \frac{1}{1 + e^{\frac{\alpha}{2\beta}x_j}}, \qquad j = 0, 1, \dots, M,$$

$$U_0^n = \frac{1}{1 + e^{-\frac{\alpha^2+4\beta\gamma}{4\beta}t_n}}, \qquad n = 0, 1, \dots, N,$$

$$U_M^n = \frac{1}{1 + e^{\frac{\alpha}{2\beta}(x_M - \frac{\alpha^2+4\beta\gamma}{2\alpha}t_n)}}, \qquad n = 0, 1, \dots, N,$$

where $x_j = jh$ and $t_n = n\Delta t$ for $0 \leq j \leq M$ and $0 \leq n \leq N$, with $h = \frac{1}{M}$ and $\Delta t = \frac{T}{N}$. In order to evaluate the effectiveness and accuracy of the constructed NSFD scheme for the given problem, we calculate the absolute error, given by $|U(x_j, t_n) - U_j^n|$, at the node $(x_j, t_n)$ for different values of $\alpha, \beta$ and $\gamma$. Tables 1–2 provide the differences between the numerically computed solution and the corresponding exact solution for selected values of the constant $\alpha, \beta$ and $\gamma$.

| $x$ | $t$ | $\gamma = 0.01$ | $\gamma = 1$ | $\gamma = 10$ |
|---|---|---|---|---|
| | 0.01 | 1.7656E-17 | 1.9320E-13 | 3.5132E-12 |
| 0.1 | 0.5 | 7.5111E-17 | 5.3267E-12 | 8.6990E-11 |
| | 1 | 4.2132E-17 | 8.3211E-12 | 5.4022E-13 |
| | 0.01 | 5.6214E-17 | 3.6948E-13 | 6.2997E-12 |
| 0.5 | 0.5 | 4.8049E-16 | 5.8221E-12 | 4.7719E-11 |
| | 1 | 2.2333E-16 | 2.4921E-11 | 3.4562E-14 |
| | 0.01 | 8.3073E-17 | 7.4152E-13 | 4.7922E-12 |
| 0.9 | 0.5 | 3.5716E-16 | 9.7639E-11 | 7.3970E-13 |
| | 1 | 4.9044E-16 | 4.8127E-13 | 1.3210E-15 |

Table 1. Absolute erros between numerical and exact solution for selected values of $\gamma, x$ and $t$ with $\beta = 2$ and $\alpha = 0.01$.

| $x$ | $t$ | $\beta = 0.01$ | $\beta = 0.1$ | $\beta = 1$ |
|---|---|---|---|---|
| | 0.01 | 2.2042E-14 | 6.4275E-16 | 2.7643E-15 |
| 0.1 | 0.5 | 6.7573E-13 | 9.7523E-14 | 5.6451E-13 |
| | 1 | 3.6734E-10 | 4.0479E-13 | 7.3283E-13 |
| | 0.01 | 3.3370E-14 | 3.6369E-16 | 4.1733E-15 |
| 0.5 | 0.5 | 7.4245E-12 | 5.4739E-13 | 2.8293E-13 |
| | 1 | 5.2379E-10 | 2.1642E-12 | 3.3245E-10 |
| | 0.01 | 3.3636E-14 | 3.9665E-16 | 1.9462E-15 |
| 0.9 | 0.5 | 4.5521E-13 | 9.7175E-11 | 3.5243E-13 |
| | 1 | 7.8637E-10 | 4.3568E-10 | 6.5613E-12 |

Table 2. Absolute errors between numerical and exact solutions for values of $\beta, x$ and $t$ with $\gamma = 1$ and $\alpha = 0.001$.

The numerical results from these tables reveal that the NSFD scheme has good accuracy and agreement with exact solutions. The exact solution and the NSFD scheme are depicted in Figure 1 for the stepsizes $h = 0.1$ and $\Delta t = 0.2$. In addition, a comparison of the absolute errors is performed at $x = 0.5$ with the values $\alpha = 0.1$, $\beta = 0.4$, and $\gamma = 1$. This figure confirms that the NSFD scheme possesses good effeciency even when using large stepsizes.

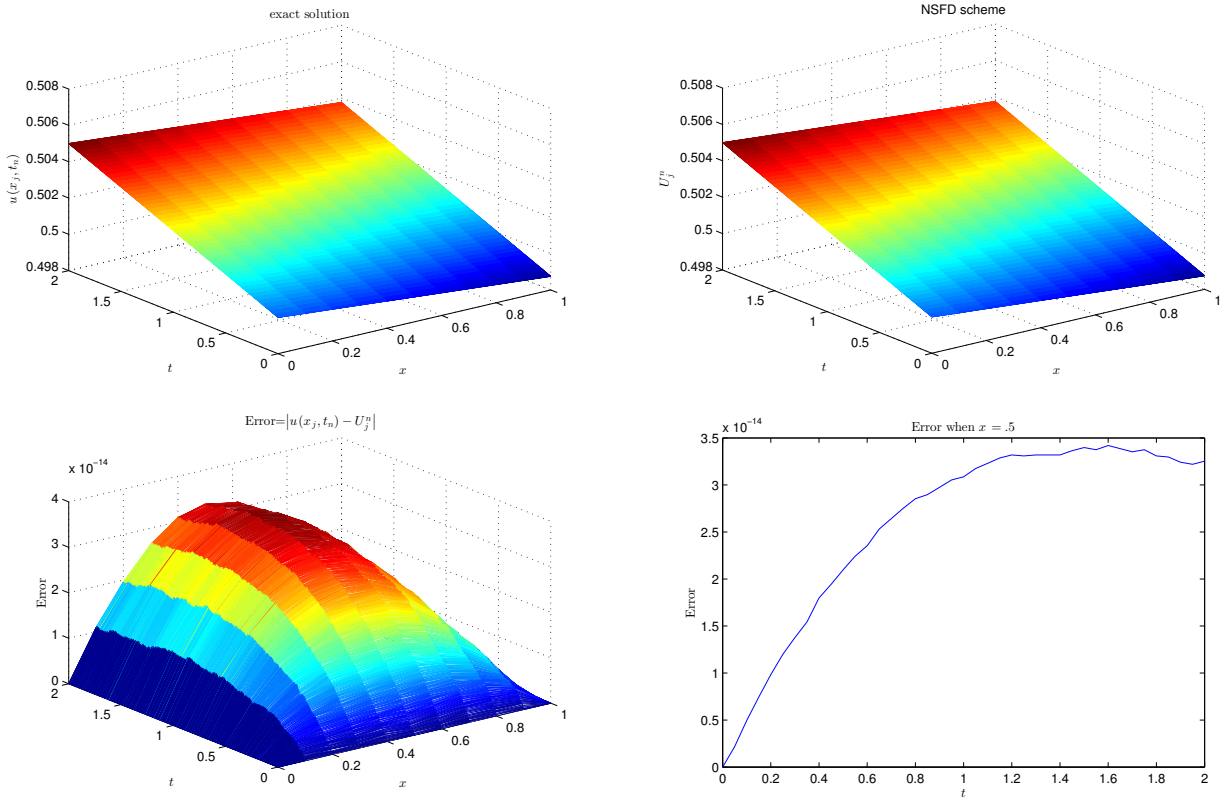


Fig. 1: Comparison of the NSFD scheme and the exact solution for the BF equation with $\alpha = 0.1, \beta = 0.4, \gamma = 1, h = 0.1,$ and $\Delta t = 0.2$.

## 5. Conclusions

In this paper, an exact NSFD scheme for the BF equation was constructed based on its analytical solution. Additionally, an efficient NSFD scheme was proposed for the numerical solution of the model, and its positivity and boundedness properties were studied. The numerical results demonstrated that the NSFD scheme yields solutions in excellent agreement with the exact solutions of the continuous model. Future research will focus on the derivation of exact NSFD schemes and development of effective NSFD schemes for various classes of deterministic and stochastic partial differential equations.

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# Weakly Laskerian (FSF) modules

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**Abstract**

Let $R$ be a commutative Noetherian ring and $M$ be an $R$-module such that the set of associated prime ideals of the quotient module $M/L$ is finite for all submodules $L$ of $M$. In this paper, we find a finitely generated submodule $N$ of $M$ such that the set of associated primes of $M/N$ and the support of $M/N$ are equal.

## 1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with nonzero identity. Let $N$ be a proper submodule of an $R$-module $M$. Then $N$ is called a *primary submodule* of $M$ when for all $r \in R$ and $x \in M$ if $rx \in N$, then $x \in N$ or $r^n M \subseteq N$ for some $n \in \mathbb{N}$. If $N$ is a primary submodule of $M$, then $\mathfrak{p} := \sqrt{(\mathrm{Ann}_R(M/N))}$ is a prime ideal of $R$ and $N$ is called a $\mathfrak{p}$-primary submodule of $M$. An expression of $N$ as an intersection of finitely many primary submodules of $M$ is called a *primary decomposition* of $N$ in $M$. Such a primary decomposition

$$N = M_1 \cap \cdots \cap M_n \quad \text{with } M_i \ \ \mathfrak{p}_i\text{-primary in } M \ (1 \leq i \leq n)$$

of $N$ in $M$ is said to be minimal when $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are distinct and $\bigcap_{1 \leq j \neq i \leq n} M_j \nsubseteq M_i$ for all $1 \leq i \leq n$. In this situation, we have $\mathrm{Ass}_R(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$; see [4] for more details. An $R$-module $M$ is said to be *Laskerian* if each proper submodule of $M$ has a primary decomposition in $M$.

Divaani–Aazar and Mafi in [2] defined an $R$-module $M$ to be *weakly Laskerian*, if $\mathrm{Ass}_R(M/N)$ is a finite set for each submodule $N$ of $M$. Therefore Laskerian modules are weakly Laskerian.

On the other hand, Quy in [5] introduced the class of FSF modules. An $R$-module $M$ is called an FSF module if there is a **F**initely generated submodule $N$ of $M$ such that the **S**upport of the quotient module $M/N$ is **F**inite. It is easy to see that an FSF module is weakly Laskerian. Bahmanpour in [1] proved that the converse statement is also true and so the class of FSF $R$-modules and the class of weakly Laskerian $R$-modules are equal. In this short note, we improve this result by proving that if $M$ is weakly Laskerian, then $\mathrm{Supp}_R(M/N) = \mathrm{Ass}_R(M/N)$ for some finitely generated submodule $N$ of $M$ (and so $M$ is an FSF module).

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## 2. Main Result

**Theorem 2.1.** *Let $M$ be an $R$-module. Then the following statements are equivalent:*

(i) *$M$ is a weakly Laskerian $R$-module;*
(ii) *there is a finitely generated submodule $N$ of $M$ such that* $\mathrm{Supp}_R(M/N) = \mathrm{Ass}_R(M/N)$ *and* $\mathrm{Supp}_R(M/N)$ *is a finite set;*
(iii) *$M$ is an FSF $R$-module.*

*Proof.* The implication (ii)⇒(iii) is clear. To prove the implication (iii)⇒(i), assume that $N$ is a finitely generated submodule of $M$ such that $\mathrm{Supp}_R(M/N)$ is finite. Suppose that $L$ is an arbitrary submodule of $M$. It follows from the exact sequence

$$0 \to (N+L)/L \to M/L \to M/(N+L) \to 0$$

that $\mathrm{Ass}_R(M/L) \subseteq \mathrm{Ass}_R((M/N)/((N+L)/N)) \cup \mathrm{Ass}_R(N/(N\cap L))$. Therefore $\mathrm{Ass}_R(M/L)$ is a finite set, and so $M$ is weakly Laskerian.
(i)⇒(ii). Assume that $M$ is weakly Laskerian and to prove (ii) it is sufficient for us to show that $\mathrm{Supp}_R(M/N) = \mathrm{Ass}_R(M/N)$ for some finitely generated submodule $N$ of $M$.
We set $M_0 := 0$ (the zero submodule of $M$). Assume $i \in \mathbb{N}$. If $\mathrm{Supp}_R(M/M_{i-1}) = \mathrm{Ass}_R(M/M_{i-1})$, then we set $N := M_{i-1}$ and we end the process. Otherwise, we construct $x_i, M_i, \mathfrak{q}_i, \mathfrak{p}_i, \Sigma_i$ as follows. We set $\Sigma_i := \mathrm{Supp}_R(M/M_{i-1}) \setminus \mathrm{Ass}_R(M/M_{i-1})$. Since $\Sigma_i$ is a non-empty set of ideals of $R$ and $R$ is Noetherian, $\Sigma_i$ has a maximal element under inclusion, say $\mathfrak{p}_i$. There exist $\mathfrak{q}_i \in \mathrm{Ass}_R(M/M_{i-1})$ and $x_i \in M$ such that $\mathfrak{q}_i = (0 :_R x_i + M_{i-1}) \subset \mathfrak{p}_i$ (note that $\mathfrak{q}_i \in \mathrm{Ass}_R(M/M_{i-1})$ while $\mathfrak{p}_i \notin \mathrm{Ass}_R(M/M_{i-1})$ and so $\mathfrak{q}_i \neq \mathfrak{p}_i$). Now, we set $M_i := M_{i-1} + \mathfrak{p}_i x_i$. It is clear that $M_i$ is a finitely generated submodule of $M$. We claim that after a finite number of steps the procedure must stop. Assume for the sake of contradiction that $\mathrm{Supp}_R(M/M_i) \neq \mathrm{Ass}_R(M/M_i)$ for all $i$ and $x_i, M_i, \mathfrak{q}_i, \mathfrak{p}_i, \Sigma_i$ are constructed for all $i \in \mathbb{N}$ as above. Before continuing the proof, we need the following two lemmas.

**Lemma 2.2.** *For each $j \in \mathbb{N}$,* $\mathrm{Ass}_R(M_j/M_{j-1}) = \{\mathfrak{q}_j\}$.

*Proof.* Since $\mathfrak{p}_j \nsubseteq \mathfrak{q}_j$, we obtain $M_{j-1} \subset M_j$ and so $\mathrm{Ass}_R(M_j/M_{j-1}) \neq \emptyset$. Now, suppose that $\mathfrak{p} \in \mathrm{Ass}_R(M_j/M_{j-1})$. Then $\mathfrak{p} = (0 :_R a_j x_j + M_{j-1})$ for some $a_j \in \mathfrak{p}_j$. Therefore $\mathfrak{p} a_j \subseteq \mathfrak{q}_j$ and so $\mathfrak{p} \subseteq \mathfrak{q}_j$. The reverse inclusion is clear and hence $\mathfrak{p} = \mathfrak{q}_j$. □

**Lemma 2.3.** *If $1 \leq i \leq j$, then $\mathfrak{p}_i = (0 :_R x_i + M_j) \in \mathrm{Ass}_R(M/M_j)$ and $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ whenever $1 \leq i < j$.*

*Proof.* First, we prove that $\mathfrak{p}_j = (0 :_R x_j + M_j)$ for all $j \in \mathbb{N}$. Assume that $r \in (0 :_R x_j + M_j)$. Therefore $rx_j = m_{j-1} + a_j x_j$ for some $m_{j-1} \in M_{j-1}, a_j \in \mathfrak{p}_j$. It follows that $r - a_j \in \mathfrak{q}_j$ and so $r \in \mathfrak{p}_j$. This shows that $(0 :_R x_j + M_j) \subseteq \mathfrak{p}_j$. The reverse inclusion is clear and so $\mathfrak{p}_j = (0 :_R x_j + M_j) \in \mathrm{Ass}_R(M/M_j)$ for all $1 \leq j$.
Next, we show that if $1 \leq i < j$, then $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$. Assume for the sake of contradiction that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for some $i, j$ with $i < j$. We can assume that $j$ is the least positive integer with the property that there exists $i$ with $i < j$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$. If $\mathfrak{p}_i = \mathfrak{p}_j$, then since $\mathfrak{p}_i = (0 :_R x_i + M_i)$ we have $\mathfrak{p}_j \in \mathrm{Ass}_R(M/M_i)$ while $\mathfrak{p}_j \notin \mathrm{Ass}_R(M/M_{j-1})$ by definition. Therefore $i < j-1$ and there exists $i+1 \leq k < j$ such that $\mathfrak{p}_j \in \mathrm{Ass}_R(M/M_{k-1})$ and $\mathfrak{p}_j \notin \mathrm{Ass}_R(M/M_k)$. Also, if $\mathfrak{p}_i \subset \mathfrak{p}_j$, then, by the maximality of $\mathfrak{p}_i$ in $\Sigma_i$, $\mathfrak{p}_j \in \mathrm{Ass}_R(M/M_{i-1})$ while $\mathfrak{p}_j \notin \mathrm{Ass}_R(M/M_{j-1})$ by definition. Therefore there exits $i \leq k < j$ such that $\mathfrak{p}_j \in \mathrm{Ass}_R(M/M_{k-1})$ and $\mathfrak{p}_j \notin \mathrm{Ass}_R(M/M_k)$. In both cases, we deduce from the exact sequence

$$0 \to M_k/M_{k-1} \to M/M_{k-1} \to M/M_k \to 0$$

that $\mathfrak{p}_j \in \mathrm{Ass}_R(M_k/M_{k-1})$ and so Lemma 2.2 implies that $\mathfrak{p}_j = \mathfrak{q}_k \subset \mathfrak{p}_k$. Since $\mathfrak{p}_i \subseteq \mathfrak{p}_j$, we obtain $\mathfrak{p}_i \subset \mathfrak{p}_k$, which is impossible by the minimality of $j$. Therefore $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ for all $1 \leq i < j$.
Finally, we prove by induction on $j$ that $\mathfrak{p}_i = (0 :_R x_i + M_j)$ for all $1 \leq i \leq j$. If $j = 1$, then we have $\mathfrak{p}_1 = (0 :_R x_1 + M_1)$. Now assume that $j > 1$ and the result has been proved for the smaller values of $j$. The case $i = j$ is proved at the beginning of the proof of this lemma. So assume that $i < j$. By the above proof, we have $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ and hence there exists $s \in \mathfrak{p}_i \setminus \mathfrak{p}_j$. Suppose that $r \in (0 :_R x_i + M_j)$. Therefore $rx_i = m_{j-1} + a_j x_j$ for some $m_{j-1} \in M_{j-1}, a_j \in \mathfrak{p}_j$. Hence $srx_i = sm_{j-1} + sa_j x_j$ and so $sa_j x_j \in M_{j-1}$. It follows that $sa_j \in \mathfrak{q}_j$ and so $a_j \in \mathfrak{q}_j$. Therefore $a_j x_j \in M_{j-1}$ and so $rx_i \in M_{j-1}$. By the inductive hypothesis, we have $(0 :_R x_i + M_{j-1}) = \mathfrak{p}_i$ and hence $r \in \mathfrak{p}_i$. Since $r$ is

an arbitrary element of $(0 :_R x_i + M_j)$, we obtain $(0 :_R x_i + M_j) \subseteq \mathfrak{p}_i$. The revers inclusion is clear. Therefore $(0 :_R x_i + M_j) = \mathfrak{p}_i$, as required. This completes the proof of the lemma. □

Now, we continue the proof of the theorem. We set $M_\infty := \bigcup_{i=1}^{\infty} M_i$ (note that $M_1 \subset M_2 \subset M_3 \subset \ldots$). We show that $(0 :_R x_i + M_\infty) = \mathfrak{p}_i$ for all $i \in \mathbb{N}$. Assume $r \in (0 :_R x_i + M_\infty)$. Therefore $rx_i \in M_j$ for some sufficiently large positive integer $j$. We can assume that $i \leq j$ and so $r \in (0 :_R x_i + M_j) = \mathfrak{p}_i$ by above lemma. This shows that $(0 :_R x_i + M_\infty) \subseteq \mathfrak{p}_i$. The reverse inclusion is clear and so, in view of the above lemma, $\mathrm{Ass}_R(M/M_\infty)$ contains the distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$, which is impossible because $M$ is weakly Laskerian. Therefore $\mathrm{Supp}_R(M/M_i) = \mathrm{Ass}_R(M/M_i)$ for some $i$, as required. This proves the implication (i)⇒(ii) and so the proof of the theorem is completed. □

**Remark 2.4.** In [3], we use [1, Theorem 3.3] to prove the above theorem. But here we provide an independent proof.

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# Statistical Bounds for the Laplacian Energy of Graphs

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**Abstract**

In this paper, we investigate bounds for the Laplacian energy $LE(G)$ of graph $G$ using some statistical and mathematical tools including arithmetic-geometric mean inequalities, variance, and Jensen-type inequalities. These bounds capture not only central tendencies but also the dispersion of Laplacian eigenvalues around their mean degree. Furthermore, we explore asymptotic behavior for large random graphs and provide illustrative examples demonstrating the sharpness of our results.

## 1. Introduction

Assume $G$ be a simple graph with $n$ vertices and $m$ edges. Also, let $d_i$ be the degree of the $i$-th vertex of $G$, for $i = 1, 2, \dots, n$, then the spectrum of the adjacency matrix of the graph $G$ is displayed by the numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, and the spectrum of its Laplacian matrix by $\mu_1, \mu_2, \dots, \mu_n$ [1, 3].
The concept of graph energy, $E(G) = \sum_{i=1}^{n} |\lambda_i|$, was introduced in the context of theoretical chemistry as a measure of the total $\pi$-electron energy in conjugated hydrocarbon molecules [2, 5].
In parallel, the Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the degree matrix and $A(G)$ the adjacency matrix, plays a fundamental role in spectral graph theory. Its eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ encode essential structural information about the graph, such as connectivity, expansion properties, and random-walk behavior [4].
Motivated by the success and utility of the graph energy $E(G)$, researchers proposed the Laplacian energy as a spectral invariant derived from the Laplacian matrix:

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

The present work aims to strengthen the bridge between the theories of $E(G)$ and $LE(G)$ by developing a unified framework for deriving statistical bounds. We adapt and extend these techniques to the Laplacian case. Our approach uses inequalities like the arithmetic-geometric mean (AM-GM) inequality, weighted mean inequalities, and Jensen's inequality to produce both upper and lower bounds for $LE(G)$. These bounds incorporate information about the distribution and dispersion of the eigenvalues $\mu_i$, not just their extreme values.

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## 2. Preliminaries and Notation

Throughout this paper, $G$ denotes a simple, undirected graph with $n$ vertices and $m$ edges. The adjacency matrix $A(G) = [a_{ij}]$ is the $n \times n$ symmetric matrix with $a_{ij} = 1$ if vertices $i$ and $j$ are adjacent, and $a_{ij} = 0$ otherwise. Its eigenvalues are denoted $\lambda_1, \lambda_2, \dots, \lambda_n$ and are real, satisfying $\sum_{i=1}^{n} \lambda_i = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 2m$.
The Laplacian matrix of $G$ is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees. $L(G)$ is positive semidefinite, and its eigenvalues are denoted $0 = \mu_1 \le \mu_2 \le \cdots \le \mu_n$. They satisfy $\sum_{i=1}^{n} \mu_i = 2m$ and $\sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2$, where $d_i$ is the degree of vertex $i$.
For a given graph $G$, we define the following key quantities:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|, \quad \text{(the energy of } G\text{)},$$

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|, \quad \text{(the Laplacian energy of } G\text{)}.$$

Let $x_i = |\lambda_i|$ for $i = 1, \dots, n$ and $z_i = \left| \mu_i - \frac{2m}{n} \right|$ for $i = 1, \dots, n$. Then $E(G) = \sum x_i$ and $LE(G) = \sum z_i$. Based on the [5], we denote the arithmetic mean of a sequence $\{z_i\}$ by $\mu = \frac{1}{n} \sum_{i=1}^{n} z_i$, its geometric mean by $R_g = (\prod_{i=1}^{n} z_i)^{1/n}$, and its variance by $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (z_i - \mu)^2$.

## 3. Definitions and Basic Inequalities

We recall some definitions and known inequalities that will be used in the sequel.

**Theorem 3.1** (AM-GM Inequality). *For nonnegative numbers $x_1, x_2, \dots, x_n$,*

$$\left( \prod_{i=1}^{n} x_i \right)^{1/n} \le \frac{1}{n} \sum_{i=1}^{n} x_i,$$

*with equality if and only if $x_1 = x_2 = \cdots = x_n$.*

**Theorem 3.2** (Jensen's Inequality). *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I$, and let $x_1, x_2, \dots, x_m \in I$ and weights $\alpha_i \ge 0$ with $\sum_{i=1}^{m} \alpha_i = 1$. Then*

$$f\left( \sum_{i=1}^{m} \alpha_i x_i \right) \le \sum_{i=1}^{m} \alpha_i f(x_i).$$

*If $f$ is concave, the inequality reverses [8].*

## 4. Statistical Bounds for Laplacian Energy

This section presents our main results as a series of upper and lower bounds for $LE(G)$ derived using statistical inequalities.

**Theorem 4.1.** *For any graph $G$, the Laplacian energy satisfies*

$$LE(G) \ge n \left( \prod_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \right)^{1/n} = nR_g(\mathbf{Z}),$$

*where $z_i = |\mu_i - 2m/n|$. Equality holds if and only if all $z_i$ are equal.*

*Proof.* Set $z_i = |\mu_i - 2m/n| \geq 0$. By the AM-GM inequality,

$$\left(\prod_{i=1}^{n} z_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} z_i = \frac{LE(G)}{n}.$$

Multiplying by $n$ gives $nR_g(\mathbf{Z}) \leq LE(G)$. Equality in AM-GM occurs iff $z_1 = z_2 = \cdots = z_n$. □

**Theorem 4.2.** *Let $G$ be a graph and define $z_i = |\mu_i - 2m/n|$ for $i = 1, \ldots, n$. Let $\mathbf{Z}^{1/2} = (z_1^{1/2}, \ldots, z_n^{1/2})$. Then*

$$LE(G) \geq n\left(R_g(\mathbf{Z}) + \sigma^2(\mathbf{Z}^{1/2})\right),$$

*where*

$$\sigma^2(\mathbf{Z}^{1/2}) = \frac{1}{n}\sum_{i=1}^{n}\left(z_i^{1/2} - \frac{1}{n}\sum_{k=1}^{n} z_k^{1/2}\right)^2.$$

*Proof.* Using Theorem 2.8 of [6] (or Aldaz [7]) with equal weights $\alpha_i = 1/n$, we have

$$\prod_{i=1}^{n} z_i^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} z_i - \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{z_i} - \frac{1}{n}\sum_{k=1}^{n}\sqrt{z_k}\right)^2.$$

The left side is $R_g(\mathbf{Z})$, the first term on the right is $LE(G)/n$, and the second term is exactly $\sigma^2(\mathbf{Z}^{1/2})$. Rearranging gives the desired inequality. □

The following theorem provides bounds involving the variance of $z_i$ itself.

**Theorem 4.3.** *Let $z_i = |\mu_i - 2m/n|$, $M_1 = \min\{z_1, \ldots, z_n\}$, and $M_2 = \max\{z_1, \ldots, z_n\}$. Then*

$$n\left(R_g(\mathbf{Z}) + \frac{\sigma^2(\mathbf{Z})}{2M_2}\right) \leq LE(G) \leq n\left(R_g(\mathbf{Z}) + \frac{\sigma^2(\mathbf{Z})}{2M_1}\right),$$

*where $\sigma^2(\mathbf{Z})$ is the variance of the sequence $\{z_i\}$.*

*Proof.* From Remark 2.9 of [7] (or Remark 2.10 of [6]) with equal weights $\alpha_i = 1/n$, we have

$$\frac{1}{2M_2}\sigma^2(\mathbf{Z}) \leq \frac{1}{n}\sum_{i=1}^{n} z_i - R_g(\mathbf{Z}) \leq \frac{1}{2M_1}\sigma^2(\mathbf{Z}).$$

Since $\frac{1}{n}\sum z_i = LE(G)/n$, the result follows by adding $R_g(\mathbf{Z})$ and multiplying by $n$. □

Also we provide bounds via Generalized Means and Jensen's Inequality. For $r \geq 1$, the power mean of order $r$ is $M_r(z) = \left(\frac{1}{n}\sum_{i=1}^{n} z_i^r\right)^{1/r}$.

**Theorem 4.4.** *For any $r \geq 1$, the Laplacian energy satisfies*

$$LE(G) \leq n \cdot M_r(z) = n\left(\frac{1}{n}\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|^r\right)^{1/r}.$$

*Proof.* The function $f(x) = x^r$ is convex for $x \geq 0$ and $r \geq 1$. By Jensen's inequality with equal weights,

$$\left(\frac{1}{n}\sum_{i=1}^{n} z_i\right)^r \leq \frac{1}{n}\sum_{i=1}^{n} z_i^r.$$

Taking the $r$-th root (which is increasing) gives

$$\frac{1}{n}\sum_{i=1}^{n} z_i \leq \left(\frac{1}{n}\sum_{i=1}^{n} z_i^r\right)^{1/r}.$$

Since $\sum z_i = LE(G)$, the theorem follows. □

## 5. Numerical Example

We illustrate the bounds obtained in Theorems 4.1, 4.3, and 4.4 with a simple graph.

**Example 5.1.** Consider the cycle graph $C_4$ with $n = 4$ vertices and $m = 4$ edges. The Laplacian eigenvalues of $C_4$ are

$$\mu_1 = 0, \quad \mu_2 = 2, \quad \mu_3 = 2, \quad \mu_4 = 4.$$

The average degree is $\frac{2m}{n} = 2$. Thus,

$$z_i = |\mu_i - 2| : \quad z_1 = 2,\ z_2 = 0,\ z_3 = 0,\ z_4 = 2.$$

Hence,

$$LE(C_4) = \sum_{i=1}^{4} z_i = 2 + 0 + 0 + 2 = 4.$$

Now we compute the bounds:

1. **By theorem 4.1 (AM-GM bound):**

$$R_g(z) = (2 \cdot 0 \cdot 0 \cdot 2)^{1/4} = 0, \quad \text{so} \quad nR_g(z) = 0 \leq LE = 4.$$

2. **By theorem 4.3 (variance bound):**

$$\mu = \frac{4}{4} = 1, \quad \sigma^2(\mathbf{Z}) = \frac{1}{4}[(2-1)^2 + (0-1)^2 + (0-1)^2 + (2-1)^2] = 1.$$

$$M_1 = \min\{2, 0, 0, 2\} = 0, \quad M_2 = \max\{2, 0, 0, 2\} = 2.$$

Since $M_1 = 0$, the upper bound in Theorem 4.3 is infinite (not informative). The lower bound is:

$$n\left(R_g(z) + \frac{\sigma^2(z)}{2M_2}\right) = 4\left(0 + \frac{1}{2 \cdot 2}\right) = 1.$$

Thus, $1 \leq LE = 4$.

3. **By theorem 4.4 with $r = 2$ (quadratic mean bound):**

$$M_2(z) = \left(\frac{1}{4}[2^2 + 0^2 + 0^2 + 2^2]\right)^{1/2} = \left(\frac{8}{4}\right)^{1/2} = \sqrt{2} \approx 1.414.$$

Hence,

$$LE(G) \leq n \cdot M_2(z) = 4 \cdot 1.414 = 5.656.$$

So we have $4 \leq 5.656$, which is true.

The bounds obtained are consistent with the exact value $LE(C_4) = 4$.

## 6. Conclusion

In this paper, we established several statistical bounds for the Laplacian energy of a graph, using statistical and mathematical tools. The bounds incorporate central tendencies (arithmetic and geometric means) as well as dispersion measures (variance) of the centered Laplacian eigenvalues. The derived inequalities are illustrated through examples and shown to be sharp under certain conditions. Future work may involve applying these bounds to specific graph classes, investigating extremal graphs, and exploring connections with other spectral invariants.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

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# Discontinuous Fourth Order Differential Operator with Eigenparameter Dependent Boundary and Transmission Conditions

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**Abstract**

In this paper, we consider a fourth order differential operator with eigenparameter dependent boundary conditions and transmission conditions at one interior point. A self-adjoint linear operator is defined in a suitable Hilbert space such that the eigenvalues of such a problem coincide with those of the operator associated with the problem. Then we construct fundamental solutions of the problem and investigate some properties of its spectrum.

## 1. Introduction

In recent years, the differential operators with boundary conditions depending on eigenparameter are also widely used in acoustic scattering, quantum mechanics theory and so on. As far as we know, some partial differential equations can be transformed into differential operator problems with eigenparameter dependent boundary conditions through the method of separation of variables. Particularly, the fourth order differential operators with eigenparameter dependent boundary conditions appear in elastic beam models, free bending vibrations of rod and so on [1, 3]. Up to now, the differential operators with boundary conditions depending on eigenparameter have become an important research topic and some excellent results have been obtained [2, 4–6].
In this paper, we shall consider the following fourth order boundary value transmission problem

$$\ell y := y^{(4)}(x) + q(x)y(x) = \lambda y(x), \quad x \in J := [a, c) \cup (c, b], \tag{1}$$

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subject to the eigenparameter dependent boundary conditions at end pints

$$l_1(y) := \lambda\left(\alpha_1' y(a) - \alpha_2' y'''(a)\right) + \left(\alpha_1 y(a) - \alpha_2 y'''(a)\right) = 0 \tag{2}$$
$$l_2(y) := \lambda\left(\beta_1' y'(a) - \beta_2' y''(a)\right) - \left(\beta_1 y'(a) - \beta_2 y''(a)\right) = 0 \tag{3}$$
$$l_3(y) := \lambda\left(\gamma_1' y(b) - \gamma_2' y'''(b)\right) - \left(\gamma_1 y(b) - \gamma_2 y'''(b)\right) = 0 \tag{4}$$
$$l_4(y) := \lambda\left(\tau_1' y'(b) - \tau_2' y''(b)\right) + \left(\tau_1 y'(b) - \tau_2 y''(b)\right) = 0, \tag{5}$$

and the eigenpapameter dependent transmission conditions at interior discontinuity

$$l_5(y) := y(c+0) - y(c-0) = 0, \tag{6}$$
$$l_6(y) := y'(c+0) - y'(c-0) = 0, \tag{7}$$
$$l_7(y) := \eta_2 y'''(c+0) - \eta_1 y'''(c-0) + (\lambda\delta_1 + \delta_2) y(c) = 0, \tag{8}$$
$$l_8(y) := \theta_2 y''(c+0) - \theta_1 y''(c-0) + (\lambda\sigma_1 + \sigma_2) y'(c) = 0, \tag{9}$$

where $q(x) \in L^1[J,R]$ and has finite limits $q(c \pm 0) := \lim_{x\to c\pm 0} q(x)$; $\lambda$ is complex eigenparameter; $\alpha_i$, $\alpha_i'$, $\beta_i$, $\beta_i'$, $\gamma_i$, $\gamma_i'$, $\tau_i$, $\tau_i'$, $\eta_i$, $\eta_i'$, $\theta_i$, $\theta_i'$, $\delta_i$, $\sigma_i$ $(i = 1,2)$ are real numbers. Throughout this paper we shall assume that the coefficients $\eta_1, \eta_2, \theta_1, \theta_2, \delta_1, \sigma_1$ have the same sign ( without lose of generality, we shall assume that $\eta_1, \eta_2, \theta_1, \theta_2, \delta_1, \sigma_1$ are positive), and

$$\rho_1 := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} > 0,\ \rho_2 := \begin{vmatrix} \beta_1' & \beta_1 \\ \beta_2' & \beta_2 \end{vmatrix} > 0,\ \rho_3 := \begin{vmatrix} \gamma_1 & \gamma_1' \\ \gamma_2 & \gamma_2' \end{vmatrix} > 0,$$

$$\rho_4 := \begin{vmatrix} \tau_1' & \tau_1 \\ \tau_2' & \tau_2 \end{vmatrix} > 0,\ \begin{vmatrix} \eta_1 & \theta_1 \\ \eta_2 & \theta_2 \end{vmatrix} = 0.$$

## 2. Operator theoretic formulation in a suitable Hilbert space

Firstly, a convenient special Hilbert space for given problem and then a linear operator owning same eigenvalues with given boundary value transmission problem in this space are constructed. We define the inner product in $L^2(J) := L^2(a,c) \oplus L^2(c,b)$ as

$$\langle f, g\rangle_1 := \eta_1\theta_1 \int_a^c f(x)\overline{g(x)}dx + \eta_1\theta_2 \int_c^b f(x)\overline{g(x)}dx \tag{10}$$

for all $f, g \in L^2(J)$. It is easy to verify that $H_1 := \left(L^2(J), \langle\cdot,\cdot\rangle_1\right)$ is a Hilbert space.
Now, we introduce a special inner product in the Hilbert space $H := H_1 \oplus \mathbb{C}^6$, where $\mathbb{C}^6$ denotes six-dimension complex vector space, and a symmetric linear operator $A$ defined on this Hilbert space such that the problem (1)–(9) can be considered as the eigenvalue problem of this operator. Namely, we define an inner product on $H$ by

$$\langle F, G\rangle := \langle f, g\rangle_1 + \frac{\eta_1\theta_1}{\rho_1} f_1\overline{g_1} + \frac{\eta_1\theta_1}{\rho_2} f_2\overline{g_2} + \frac{\eta_1\theta_2}{\rho_3} f_3\overline{g_3} \tag{11}$$
$$+ \frac{\eta_1\theta_2}{\rho_4} f_4\overline{g_4} + \frac{\theta_1}{\delta_1} f_5\overline{g_5} + \frac{\eta_1}{\sigma_1} f_6\overline{g_6} \tag{12}$$

for $F = (f(x), f_1, f_2, f_3, f_4, f_5, f_6), G = (g(x), g_1, g_2, g_3, g_4, g_5, g_6) \in H$.
For convenience, we shall use the following notations:

$$\begin{aligned}
&B_1(f) := \alpha_1 f(a) - \alpha_2 f'''(a), && B_1'(f) := \alpha_1' f(a) - \alpha_2' f'''(a),\\
&B_2(f) := \beta_1 f'(a) - \beta_2 f''(a), && B_2'(f) := \beta_1' f'(a) - \beta_2' f''(a),\\
&B_3(f) := \gamma_1 f(b) - \gamma_2 f'''(b), && B_3'(f) := \gamma_1' f(b) - \gamma_2' f'''(b),\\
&B_4(f) := \beta_1 f'(a) - \beta_2 f''(a), && B_4'(f) := \beta_1' f'(a) - \beta_2' f''(a),\\
&T_1(f) := \eta_2 f'''(c+0) - \eta_1 f'''(c-0) + \delta_2 f(c), && T_1'(f) := -\delta_1 f(c),\\
&T_2(f) := \theta_2 f''(c+0) - \theta_1 f''(c-0) + \sigma_2 f'(c), && T_2'(f) := -\sigma_1 f'(c).
\end{aligned}$$

In the Hilbert space $H$, we consider the operator $A$ with domain

$$D(A) := \{F = (f(x), f_1, f_2, f_3, f_4, f_5, f_6) \in H | f, f', f'', f''' \in AC_{loc}(J), \\ \text{and have one-hand sides limits } f(c \pm 0), f'(c \pm 0), f''(c \pm 0), f'''(c \pm 0), \\ \ell f \in L^2(J),\ l_k(f) = 0,\ k = 5, 6,\ f_i = B_i'(f),\ i = 1, 2, 3, 4, f_5 = T_1'(f),\ f_6 = T_2'(f)\}$$

and the rule

$$AF = (\ell f, B_1(f), B_2(f), B_3(f), B_4(f), T_1(f), T_1(f)) \tag{13}$$

for $F = (f(x), f_1, f_2, f_3, f_4, f_5, f_6) \in D(A)$.
Now, we rewrite the problem (1)–(9) in the operator form

$$AF = \lambda F$$

for $F = (f(x), f_1, f_2, f_3, f_4, f_5, f_6) \in D(A)$.

**Lemma 2.1.** *The eigenvalues of the problem (1)–(9) coincide with those of A and its eigenfunctions are the first components of the corresponding eigenelements of the operator A.*

**Lemma 2.2.** *The domain $D(A)$ is dense n $H$.*

**Theorem 2.3.** *The operator A is self-adjoint in $H$.*

**Corollary 2.4.** *All the eigenvalues of the problem (1)–(9) are real.*

**Corollary 2.5.** *Let $\lambda_1$ and $\lambda_2$ be two different eigenvalues of the problem (1)–(9). Then, the corresponding eigenfunctions $f$ and $g$ are orthogonal in the sense that*

$$\eta_1\theta_1 \int_a^c f(x)g(x)dx + \eta_1\theta_2 \int_c^b f(x)g(x)dx + \frac{\eta_1\theta_1}{\rho_1} B_1'(f)B_1'(g) + \frac{\eta_1\theta_1}{\rho_2} B_2'(f)B_2'(g) + \frac{\eta_1\theta_2}{\rho_3} B_3'(f)B_3'(g) \\ + \frac{\eta_1\theta_1}{\rho_1} B_1'(f)B_1'(g) + \frac{\theta_1}{\delta_1} T_1'(f)T_1'(g) + \frac{\eta_1}{\sigma_1} T_2'(f)T_2'(g) = 0.$$

## 3. The fundamental solutions of the problem

**Lemma 3.1.** *Let $q(x) \in L^1(a, b)$ be a real-valued and $f_i(\lambda)$ $(i = 1, 2, 3, 4)$ are given entire functions. Then, for all $\lambda \in \mathbb{C}$,the equation (1) has a unique solution $y(x, \lambda)$ satisfying the initial conditions*

$$y(a) = f_1(\lambda), \quad y'(a) = f_2(\lambda), \quad y''(a) = f_3(\lambda), \quad y'''(a) = f_4(\lambda).$$

*For each fixed $x \in [a, b]$, $y(x, \lambda)$, $y(x, \lambda)$ is an entire function of $\lambda$.*

We shall define four solutions

$$\varphi_i(x, \lambda) = \begin{cases} \varphi_{i1}(x, \lambda), & x \in [a, c) \\ \varphi_{i2}(x, \lambda), & x \in (c, b] \end{cases}, \quad \psi_i(x, \lambda) = \begin{cases} \varphi_{i1}(x, \lambda), & x \in [a, c) \\ \varphi_{i2}(x, \lambda), & x \in (c, b] \end{cases}, \quad i = 1, 2$$

as follows:
Let $\varphi_{11}(x, \lambda)$ be the solution of equation (1) on the interval $[a, c)$ satisfying the initial conditions

$$\varphi_{11}(a, \lambda) = \alpha_2 + \lambda\alpha_2', \quad \varphi_{11}'(a, \lambda) = \varphi_{11}''(a, \lambda) = 0, \quad \varphi_{11}'''(a, \lambda) = \alpha_1 + \lambda\alpha_1'. \tag{14}$$

Now, by virtue of Lemma 3.1, we define the solution $\varphi_{12}(x,\lambda)$ of equation (1) by the initial conditions

$$\varphi_{12}(c+0,\lambda) = \varphi_{11}(c-0,\lambda), \tag{15}$$
$$\varphi'_{12}(c+0,\lambda) = \varphi'_{11}(c-0,\lambda), \tag{16}$$
$$\varphi''_{12}(c+0,\lambda) = \frac{\theta_1}{\theta_2}\varphi''_{11}(c-0,\lambda) - \left(\frac{\lambda\sigma_1+\sigma_2}{\theta_2}\right)\varphi'_{11}(c-0,\lambda), \tag{17}$$
$$\varphi'''_{12}(c+0,\lambda) = \frac{\eta_1}{\eta_2}\varphi'''_{11}(c-0,\lambda) - \left(\frac{\lambda\delta_1+\delta_2}{\eta_2}\right)\varphi_{11}(c-0,\lambda). \tag{18}$$

Again let $\varphi_{21}(x,\lambda)$ still be the solution of equation (1) on the interval $[a,c)$, satisfying the initial conditions

$$\varphi_{21}(a,\lambda) = 0, \quad \varphi'_{21}(a,\lambda) = -\beta_2 + \lambda\beta'_2, \quad \varphi''_{21}(a,\lambda) = -\beta_1 + \lambda\beta'_1, \quad \varphi'''_{21}(a,\lambda) = 0. \tag{19}$$

After defining this solution we can also define the solution $\varphi_{22}(x,\lambda)$ of equation (1) on the interval $(c,b]$ by the initial conditions

$$\varphi_{22}(c+0,\lambda) = \varphi_{21}(c-0,\lambda), \tag{20}$$
$$\varphi'_{22}(c+0,\lambda) = \varphi'_{21}(c-0,\lambda), \tag{21}$$
$$\varphi''_{22}(c+0,\lambda) = \frac{\theta_1}{\theta_2}\varphi''_{21}(c-0,\lambda) - \left(\frac{\lambda\sigma_1+\sigma_2}{\theta_2}\right)\varphi'_{21}(c-0,\lambda), \tag{22}$$
$$\varphi'''_{22}(c+0,\lambda) = \frac{\eta_1}{\eta_2}\varphi'''_{21}(c-0,\lambda) - \left(\frac{\lambda\delta_1+\delta_2}{\eta_2}\right)\varphi_{21}(c-0,\lambda). \tag{23}$$

analogously we shall define the solutions $\psi_{12}(x,\lambda)$, $\psi_{11}(x,\lambda)$ by the initial conditions

$$\psi_{12}(b,\lambda) = -\gamma_2 + \lambda\gamma'_2, \quad \psi'_{12}(b,\lambda) = \psi''_{12}(b,\lambda) = 0, \quad \varphi'''_{12}(b,\lambda) = -\gamma_1 + \lambda\gamma'_1 \tag{24}$$

and

$$\psi_{11}(c-0,\lambda) = \psi_{12}(c+0,\lambda), \tag{25}$$
$$\psi'_{11}(c-0,\lambda) = \psi'_{21}(c+0,\lambda), \tag{26}$$
$$\psi''_{11}(c-0,\lambda) = \frac{\theta_2}{\theta_1}\psi''_{12}(c+0,\lambda) - \left(\frac{\lambda\sigma_1+\sigma_2}{\theta_1}\right)\psi'_{12}(c+0,\lambda), \tag{27}$$
$$\psi'''_{11}(c-0,\lambda) = \frac{\eta_2}{\eta_1}\psi'''_{12}(c+0,\lambda) - \left(\frac{\lambda\delta_1+\delta_2}{\eta_2}\right)\psi_{12}(c+0,\lambda). \tag{28}$$

In addition, we shall define the solution $\psi_{22}(x,\lambda)$, $\psi_{21}(x,\lambda)$, satisfying the initial conditions

$$\psi_{22}(b\psi,\lambda) = 0, \quad \psi'_{22}(b,\lambda) = \tau_2 + \lambda\tau'_2, \quad \psi''_{22}(b,\lambda) = \tau_1 + \lambda\tau'_1, \quad \psi'''_{22}(b,\lambda) = 0 \tag{29}$$

and

$$\psi_{21}(c-0,\lambda) = \psi_{22}(c+0,\lambda),$$
$$\psi'_{21}(c-0,\lambda) = \psi'_{22}(c+0,\lambda),$$
$$\psi''_{21}(c-0,\lambda) = \frac{\theta_2}{\theta_1}\psi''_{22}(c+0,\lambda) - \left(\frac{\lambda\sigma_1+\sigma_2}{\theta_1}\right)\psi'_{22}(c+0,\lambda),$$
$$\psi'''_{21}(c-0,\lambda) = \frac{\eta_2}{\eta_1}\psi'''_{22}(c+0,\lambda) - \left(\frac{\lambda\delta_1+\delta_2}{\eta_1}\right)\psi_{22}(c+0,\lambda).$$

Let us consider the Wronskians

$$W_i(\lambda) := \begin{vmatrix} \varphi_{1i}(x,\lambda) & \varphi_{2i}(x,\lambda) & \psi_{1i}(x,\lambda) & \psi_{2i}(x,\lambda) \\ \varphi'_{1i}(x,\lambda) & \varphi'_{2i}(x,\lambda) & \psi'_{1i}(x,\lambda) & \psi'_{2i}(x,\lambda) \\ \varphi''_{1i}(x,\lambda) & \varphi''_{2i}(x,\lambda) & \psi''_{1i}(x,\lambda) & \psi''_{2i}(x,\lambda) \\ \varphi'''_{1i}(x,\lambda) & \varphi'''_{2i}(x,\lambda) & \psi'''_{1i}(x,\lambda) & \psi'''_{2i}(x,\lambda) \end{vmatrix}, \quad i = 1,2, \tag{30}$$

which are independent of $x \in \Omega_i$ $(i = 1, 2)$ and are entire functions of $\lambda$, where $\Omega_1 = [a, c)$, $\Omega_2 = (c, b]$. It is not difficult to see that

$$\eta_1\theta_1 W_1(\lambda) = \eta_2\theta_2 W_2(\lambda).$$

Let us introduce the characteristic function

$$\omega(\lambda) := \eta_1\theta_1 W_1(\lambda) = \eta_2\theta_2 W_2(\lambda).$$

**Theorem 3.2.** *The eigenvalues of the problem* (1)–(9) *are consist of the zeros of the functions* $\omega(\lambda)$ *and*

$$\Delta(\lambda) := \frac{\eta_2\theta_2}{\eta_1\theta_1}\omega(\lambda) + (\lambda\delta_1 + \delta_2)(\lambda\sigma_1 + \sigma_2)\varphi_{12}(c, \lambda)\varphi_{22}(c, \lambda)\psi_{12}(c, \lambda)\varphi_{22}(c, \lambda).$$

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# On Finitely Cogenerated and Po-artinian $S$-posets over pomonoids

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**Abstract**

In this paper, we introduce the notion of finitely cogenerated $S$-poset and characterize them as those whose socle is both finitely cogenerated and large. Furthermore, we define po-artinian $S$-posets based on descending chain conditions on order congruences. Using the concept of finite cogenerated, we provide a characterization of po-artinian $S$-posets and investigate their general properties.

## 1. Introduction

The property of being artinian plays a fundamental role in abstract algebra. Artinian rings and modules have been widely studied, for example [1]. As duals to generating sets and finitely generated $S$-acts, the concepts of cogenerating classes of $S$-acts and finitely (Rees) cogenerated $S$-acts were studied in [4]. Then artinian $S$-acts were investigated in [6]. In this paper, we focus on the notions of finitely generated and po-artinian in the category of $S$-posets.
A monoid $S$ is called a *pomonoid* if it is equipped with a partial order such that the binary operation and the order relation are compatible. A *right $S$-poset* $A_S$ is a right $S$-act $A_S$ equipped with a partial order $\leq$, and furthermore,, for every $s, t \in S$ and $a, b \in A$, if $s \leq t$, then $as \leq at$, and if $a \leq b$, then $as \leq bs$. An *$S$-subposet* of a right $S$-poset $A$ is a subset of $A$ that is closed under the $S$-action. An $S$-subposet $B$ of an $S$-poset $A_S$ is called *strongly convex* if $B = (B]$ where

$$(B] = \{a \in A \mid \exists\, b \in B,\ a \leq b\}.$$

Recall that the *product* of a family of $S$-posets is their Cartesian product, equipped with componentwise action and order. The *coproduct* corresponds to their disjoint union, endowed with the natural action and componentwise order.
Let $A$ be a right $S$-poset. An *$S$-poset congruence* $\theta$ on $A$ is a right $S$-act congruence with the property that the $S$-act $A/\theta$ can be made into an $S$-poset in such a manner that the natural map $A \longrightarrow A/\theta$ is an $S$-poset map. For an $S$-act congruence $\theta$ on $A$ we write $a \leq_\theta a'$ if the so-called $\theta$-chain

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$$a \leq a_1\ \theta b_1 \leq a_2 \theta b_2\ \ldots\ \leq a_n \theta b_n \leq a'$$

from $a$ to $a'$ exists in $A$, where $a_i, b_i \in A,\ 1 \leq i \leq n$. It can be shown that an $S$-act congruence $\theta$ on a right $S$-poset $A$ is an $S$-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$.

Let $H \subseteq A \times A$. Then $a \leq_{\alpha(H)} b$ if and only if $a \leq b$ or there exist $n \geq 1, (c_i, d_i) \in H, s_i \in S, 1 \leq i \leq n$ such that

$$a \leq c_1 s_1 \quad d_1 s_1 \leq c_2 s_2 \quad \ldots \quad d_n s_n \leq b.$$

The relation $\nu(H)$ given by $a\ \nu(H)\ b$ if and only if $a \leq_{\alpha(H)} b \leq_{\alpha(H)} a$ is *the S-poset congruence induced by H*. We have $[a]_{\nu(H)} \leq [b]_{\nu(H)}$ if and only if $a \leq_{\alpha(H)} b$. Moreover, $\nu(H \cup H^{-1})$ is denoted $\theta(H)$ and is called *the S-poset congruence generated by H*. The set of all congruences on $A_S$ is denoted by $Con(A)$. For more information on $S$-posets, refer to [3].

An $S$-subposet $B$ of $A$ is called *convex* if $B = [B]$ where

$$[B] = \{a \in A \mid \exists\ b, b' \in B,\ b \leq a \leq b'\}.$$

Let $B$ be a proper convex right ideal of an $S$-poset $A$. We define now the congruence $\rho_B$ on $S$ such that $a\rho_B a'$ if $a = a'$ or $a, a' \in B$. The quotient $A/\rho_B$ is called the *Rees factor* and denoted by $A/B$. Then $[a]_{\rho_B} \leq [a']_{\rho_B}$ if and only if $a \leq a'$, or there exists $b, b' \in B$ such that $a \leq b$ and $b' \leq a'$.

The map $f : A \to B$, where $A$ and $B$ are $S$-posets, is called an *S-morphism* if it is monotonic (i.e., $a \leq a' \Rightarrow f(a) \leq f(a')$) and respects the $S$-action (i.e., $f(as) = f(a)s$ for all $a \in A, s \in S$). The $S$-morphism $f$ is said to be a *regular monomorphism* if it is order embedding; i.e., $f(a) \leq f(a') \Rightarrow a \leq a'$. The *subkernel* of an $S$-poset morphism $f$ is defined by

$$\overrightarrow{kerf} := \{(a, a') \in A \times A : f(a) \leq f(a')\}.$$

Then $\nu(\overrightarrow{kerf}) = kerf := \{(a, a') \in A \times A : f(a) = f(a')\}$, and in $A_S/kerf$, we have

$$[a]_{kerf} \leq [a']_{kerf} \text{ if and only if } f(a) \leq f(a').$$

Moreover, the mapping $\overline{f} : A_S/kerf \longrightarrow B_S$, defined by $\overline{f}([a]_{kerf}) = f(a)$ for $a \in A$, is an $S$-isomorphism. For more information see [2]. From now on we denote the subkernel of $f$ briefly by $\mathcal{K}_f$. Obviously, $f$ is a regular monomorphism if and only if

$$\mathcal{K}_f = \xi_A = \{(a, a') \in A \times A |\ a \leq a'\}.$$

Recall from [7] that an order embedding $S$-morphism $f : A \longrightarrow B$ is said to be *regular-essential* if for each $S$-morphism $g : B \longrightarrow C$, the composition $gf$ being an order embedding implies that $g$ itself is an order embedding. If $f$ is an inclusion map, then $B$ is said to be an *regular-essential extension* of $A$, or $A$ is called *regular-large* in $B$. In this situation, we write $A \subseteq' B$. It follows from [7, Proposition 2.2] that $A \subseteq' B$ if and only if for every non-trivial $\theta \in Con(B)$, $\leq_\theta \cap (A \times A) \neq \xi_A$. For an $S$-poset $A$, we define *regular-socle* of $A$ as

$$\mathrm{r-Soc}(A) = \cap\{L \subseteq A |\ L \subseteq' A\}.$$

## 2. Main results

In this section, we study two important types of $S$-posets entitled finitely cogenerated and po-artinian. We begin by defining finite cogenerated $S$-posets and give several useful characterizations. Then, we introduce po-artinian $S$-posets using chain conditions on congruences and explore how these properties behave under substructures, quotients, and exact sequences.

**Definition 2.1.** An $S$-poset $A_S$ is called *finitely cogenerated* if for every regular monomorphism $A \xrightarrow{f} \prod_{i \in I} A_i$,

$$A \xrightarrow{f} \prod_{i \in I} A_i \xrightarrow{\pi} \prod_{j \in J} A_j$$

is also a regular monomorphism for some finite subset $J$ of $I$.

**Proposition 2.2.** *For any S-poset $A_S$, the following are equivalent.*

(i) *$A_S$ is finitely cogenerated.*

(ii) *For every family of S-morphisms $\{f_i : A \longrightarrow A_i\}$ with $\cap_{i \in I} \mathcal{K}_{f_i} = \xi_A$, there is a finite subset $J$ of $I$ with $\cap_{j \in J} \mathcal{K}_{f_j} = \xi_A$.*

(iii) *For any family of congruences* $\{\rho_i | \ i \in I\}$ *on* $A_S$*, if* $\cap_{i\in I} \leq_{\rho_i} = \xi_A$*, then* $\cap_{j\in J} \leq_{\rho_j} = \xi_A$ *for some finite subset* $J$ *of* $I$.

(iv) *Every* $S$*-subposet of* $A_S$ *is finitely cogenerated.*

**Proposition 2.3.** (*[4]*) *Let* $A_S$ *be an* $S$*-poset, and* $\theta \in Con(A)$*. Then,* $A/\theta$ *is finitely cogenerated if and only if for any family of congruences* $\{\rho_i | \ i \in I\}$ *on* $A_S$*, if* $\cap_{i\in I} \leq_{\rho_i} = \leq_\theta$*, then* $\cap_{j\in J} \leq_{\rho_j} = \leq_\theta$ *for some finite subset* $J$ *of* $I$.

According to [8, Theorem 2.3], every $S$-poset is uniquely decomposable into a coproduct of strongly convex indecomposable $S$-subposets. Next, we discuss this structure for finitely cogenerated $S$-posets.

**Proposition 2.4.** *Every finitely cogenerated* $S$*-poset is a finite coproduct of strongly convex indecomposable* $S$*-subposets.*

**Proposition 2.5.** *Every regular-essential extension of a finitely cogenerated* $S$*-act is again finitely cogenerated.*

The next theorem connects the notion of finite cogenerated with the regular socle of an $S$-poset.

**Theorem 2.6.** *An* $S$*-poset* $A_S$ *is finitely cogenerated if and only if* $r - Soc(A)$ *is a finitely cogenerated regular-large* $S$*-subposet of* $A_S$.

**Definition 2.7.** Let $A_S$ be an $S$-poset. We say that $A_S$ *po-artinian* if the set of congruences $Con(A)$ satisfies the descending chain condition.

**Theorem 2.8.** *For an* $S$*-poset* $A_S$*, the following statements are equivalent.*

(i) $A_S$ *is po-artinian.*

(ii) *Every factor* $S$*-subposet of* $A_S$ *is finitely cogenerated.*

(iii) *Every non-empty subset of* $Con(A)$ *contains a minimal element.*

**Lemma 2.9.** *For a pomonoid* $S$*, the following statements hold.*

(i) *Every* $S$*-subposet of a po-artinian* $S$*-poset is itself po-artinian.*

(ii) *Every factor* $S$*-poset of a po-artinian* $S$*-poset remains po-artinian.*

**Corollary 2.10.** *Let* $S$ *be a pomonoid. Then,* $S_S$ *is po-artinian if and only if every cyclic* $S$*-poset is po-artinian.*

Recall from [5] that if $f : A \longrightarrow B$, $g : B \longrightarrow C$ are order preserving maps, then the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* at $B$ if $\mathcal{I}_f = \mathcal{K}_g$, (i.e., $\leq_{\mathcal{I}m_f} = \leq_{ker g}$). It is also called a *short exact sequence* if $g$ is surjective, $f$ is a regular monomorphism, and $\mathcal{I}_f = \mathcal{K}_g$. The next theorem investigates how the property of being po-artinian behaves under Rees short exact sequences.

**Theorem 2.11.** *Let* $A \longrightarrow B \longrightarrow C$ *be a Rees short exact sequence of* $S$*-posets. Then,* $B_S$ *is po-artinian if and only if both* $A_S$ *and* $C_S$ *are po-artinian.*

**Proposition 2.12.** *For a pomonoid* $S$*, the following statements are true.*

(i) *If* $A = \prod_{i\in I} A_i$ $(A = \coprod_{i\in I} A_i)$ *is po-artinian, then* $I$ *is finite and each* $A_i$ *is po-artinian.*

(ii) $A_S$ *is po-artinian if and only if* $A \coprod \Theta$ *is po-artinian.*

(iii) *If* $A_S$ *is po-artinian, then* $\coprod_{i=1}^{i=n} A$ *is po-artinian for each* $n \in \mathbb{N}$.

**Proposition 2.13.** *Let* $f : S \longrightarrow T$ *be a pomonoid homomorphism, and let* $A$ *be a* $T$*-act. If* $A$ *is a right po-artinian* $S$*-poset, then it is also a right po-artinian* $T$*-poset. If* $f$ *is an epimorphism, the converse is also true.*

The following corollary shows that po-artinian property is closed under quotients.

**Corollary 2.14.** *If* $\rho$ *is a congruence on a right po-artinian pomonoid* $S$*, then* $S/\rho$ *is a right po-artinian pomonoid.*

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# A Generalization of Chain Conditions in Abelian Groups

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**Abstract**

In this article we study deviation (co-deviation) of an Abelian group $A$ as a generalazition of descending (ascending) chain cindition of subgroups of $A$ and finde some properties and structures of Abelian groups with deviation (co-deviation). Also we show some relationships between deviation (co-deviation) of partialy ordered set of essential subgroup of an Abelian group $A$ and its deviation (co-deviation) of $A$.

## 1. Introduction

The study of chain conditions in algebraic structures has long been a cornerstone of ring and module theory, providing deep insights into the finiteness and structural properties of these objects. In the context of Abelian groups, classical conditions such as the ascending chain condition (ACC) and descending chain condition (DCC) have been extensively investigated, leading to fundamental results on Artinian and Noetherian groups. However, many natural Abelian groups—such as divisible groups, torsion-free groups, and infinite direct sums—fail to satisfy these conventional chain conditions, prompting the need for more refined invariants that capture their complexity in a graded or ordinal-valued manner.

This paper introduces and systematically develops the notions of *deviation* and *co-deviation* for Abelian groups, which serve as natural ordinal-valued generalizations of the DCC and ACC, respectively. By extending the concept of deviation from partially ordered sets to the lattice of subgroups of an Abelian group, we provide a flexible framework for measuring the extent to which chain conditions may fail. These invariants allow us to classify groups beyond the classical Artinian and Noetherian settings, offering a finer stratification of infinite Abelian groups based on their chain-theoretic behavior.

Our work is structured into three principal parts. First, we define deviation and co-deviation for Abelian groups and establish their fundamental properties, including behavior under subgroups, quotients, and extensions. Key lemmas demonstrate how these invariants interact with group-theoretic constructions, providing tools for inductive analysis and classification. Second, we introduce the concept of *atomic* and *conotable* groups—groups whose deviation or co-deviation is minimal relative to their proper subgroups or quotients—and characterize them in terms of familiar

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algebraic structures such as $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$. Finally, we turn our attention to *essential deviation*, a localized version of deviation that focuses on the poset of essential subgroups. This leads to a delicate interplay between deviation, essential deviation, and the Goldie dimension of the group, culminating in a characterization theorem that precisely delineates when the existence of essential deviation implies global deviation.

## 2. Deviation and Co-Deviation of Abelian Groups

**Definition 2.1.** The deviation (co-deviation) of a partially ordered set $S$ is denoted by $dev(S)$ $(co-dev(S))$ is defined as follows: If $S$ is trivial then $dev(S) = -1$ $(co - dev(S) = -1)$. If $S$ is nontrivial but satisfies $d.c.c$ $(a.c.c)$ then $dev(S) = 0$ $(co - dev(S) = 0)$. For a general ordinal $\alpha$, we define $dev(S) = \alpha$ $(co - dev(S) = \alpha)$ provide:

1. $dev(S) \neq \beta < \alpha$ $(co - dev(S) \neq \beta < \alpha)$.
2. if any descending (ascending) chain of elements of $S$ but finitely many factors have deviation (co-deviation) less than $\alpha$. In the other words, for any descending (ascending) chain

$$s_1 \geq s_2 \geq s_3 \geq \dots \ (s_1 \leq s_2 \leq s_3 \leq \dots)$$

of elements of $S$, there is $n \in \mathbb{N}$ such that for all $i \geqslant n$, $dev(\frac{s_i}{s_{i+1}}) < \alpha$ $(co - dev(\frac{s_{i+1}}{s_i}) < \alpha)$, where $\frac{a}{b} = \{x \in S | a \leqslant x \leqslant b\}$.

**Definition 2.2.** Let $A$ be an Abelian group, deviation (co-deviation) of its set of subgroups is called deviation (co-deviation) of $A$ and denoted it by $dev(A)$ $(co - dev(A))$.

It is clear that deviation (co-deviation) of an Abelian group is 0 if and only if, it has descending (asscending) chain condition on its subgroups.

**Theorem 2.3.** *Let $S$ be a partially ordered set, then the following are equivalent:*

1. *$S$ has deviation.*
2. *$S$ has co-deviation.*

**Lemma 2.4.** *Let $A$ be an Abelian group and $dev(A)$ $(co - dev(A))$ exists. Then for every $B \subseteq A$, $dev(B)$ $(co - dev(B))$and $dev(\frac{A}{B})$ $(co - dev(\frac{A}{B}))$ exist and they are less than or equal to $dev(A)$.*

**Lemma 2.5.** *Let $A$ be an Abelian group and each proper subgroup $B$ of $A$ has deviation. Then so does $A$ and $dev(A) = \sup\{dev(B) | B \subset A\}$.*

**Lemma 2.6.** *If $A$ is an Abelian group and each $B \subset A$, $\frac{A}{B}$ has deviation, then so does $A$ and $dev(A) \leq \sup\{dev(\frac{A}{B}) | B \subset A\} + 1$.*

**Lemma 2.7.** *Let $A$ be an Abelian group. For any subgroup $B$ of $A$. Then $dev(A) = \sup\{dev(A), dev(\frac{A}{B})\}$, if either side exists.*

**Lemma 2.8.** *Let $A$ be an abelian group. If for any subgroup $B$ of $A$ either $B$ or $\frac{A}{B}$ has deviation, then so does $A$.*

**Definition 2.9.** An Abelian group $A$ is called $\alpha-$atomic if $devv(A) = \alpha$ and $dev(\frac{A}{B}) < \alpha$ for all nonzero subgroup $B$ of $A$. An Abelian group is calle atomic if it is atomic for some $\alpha$

**Lemma 2.10.** *Any subgroup of an $\alpha-$atomic group is $\alpha-$atomic.*

**Proposition 2.11.** *Any Abelian group $A$ with deviation has an $\alpha-$atomic subgroup.*

**Lemma 2.12.** *If every proper factor group of an Abelian group $A$ has co-deviation, then so dose $A$ and $dev(A) = \sup\{dev(\frac{A}{B}) | 0 \neq B \subseteq A\}$.*

**Lemma 2.13.** *If $A$ is an Abelian group and $B$ is a subgroup of $A$, then $co-dev(A) = \sup\{co-dev(A), co-dev(\frac{A}{B})\}$, if either side exists.*

**Lemma 2.14.** *Let $A$ be an Abelian group. If fpr every subgroup $B$ of $A$ either $B$ or $\frac{A}{B}$ has co-deviation, then so does $A$.*

**Proposition 2.15.** *Let $A$ be an Abelian group. Then $co-dev(A)=\alpha$ if and only if, $co-dev(A)\not<\alpha$ and for each family of subgroup of $A$ such $F$, there exists $N\in F$ such that for each $B\subseteq C\in F$, $co-dev(\frac{C}{B})<\alpha$.*

**Definition 2.16.** An Abelian group $A$ is said to be $\alpha-$conotable when $co-dev(A)=\alpha$ and $B$ is a proper subgroup of $A$ implies $co-dev(B)<\alpha$. Further $A$ is said to be conotable when $A$ is $\alpha-$conotable for some $\alpha$.

**Example 2.17.** $\mathbb{Z}_{p}\infty$ is an $1-$conotabl Abelian group and each $1-$conotabl Abelian group is isomorphic to $\mathbb{Z}_{p}\infty$.

**Lemma 2.18.** *Any factor group of an $\alpha-$conotabl group is $\alpha-$conotabl.*

**Proposition 2.19.** *Let $A$ be an Abelian group with $co-dev(A)=\alpha$, then $A$ has a nonzero factor group which is conotabl. However, $A$ need not have a conotabl factor group with the same co-deviation as $A$.*

**Lemma 2.20.** *If $A$ ia a conotabl Abelian group, then $A$ is a divisible group.*

**Lemma 2.21.** *If $A$ is a(n) conotabl (atomic) Abelian group, then $A$ has no direct summand.*

**Proposition 2.22.** *If $A$ is an conotabl torsion free Abelian group, then $A$ is isomorphic to the $\mathbb{Q}$.*

**Proposition 2.23.** *If $A$ is an Abelian group with $co-dev(A)=1$, then $A$ has a proper subgroup $B$ such that $\frac{A}{B}$ has descending chain condition on its subgroup or $\frac{A}{B}$ is isomorphic to $\mathbb{Q}$.*

## 3. Essential Deviation of an Abelian Group

**Definition 3.1.** An essential subgroup of an Abelian group $A$ is any subgroup $E$ which has nonzero intersection with every nonzero subgroup of $A$. We write $E\subseteq_{e}A$ to denote this situation.

**Example 3.2.** Since any two nonzero subgroups of $\mathbb{Q}$ have nonzero intersection, all nonzero subgroups of Q are essential. For instance, $\mathbb{Z}\subset_{e}\mathbb{Q}$

**Example 3.3.** Let $G=\mathbb{Z}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}}$ and $A, B_1, B_2$ denote the cyclic subgroups of $G$ generated by $(2,0)$, $(1,0)$, $(1,1+2Z)$, respectively. Then $A\subset_{e}B_1$ and $A\subset_{e}B_2$ but $A\not\subseteq_{e}B_1+B_2$.

**Lemma 3.4.** *Let $B$ and $C$ be submodules of an infinite Abelian group $A$, and suppose that $B$ is maximal with respect to the property $C\cap B=0$. Then $B\oplus C\subseteq_{e}A$.*

**Corollary 3.5.** *Let $A$ be an Abelian group and $B\subseteq A$. Then there exists proper subgroup $C$ of $A$ such that $B\oplus C\subseteq_{e}A$.*

**Corollary 3.6.** *Let $A$ be an Abelian group and $B\subseteq A$. Then there exists proper subgroup $C$ of $A$ such that $B\oplus C\subseteq_{e}A$.*

**Lemma 3.7.** *If $A$ is an Abelian group and $C\subseteq_{e}B\subseteq_{e}A$, then $C\subseteq_{e}A$.*

**Definition 3.8.** A uniform Abelian group is a nonzero Abelian group $A$ such that the intersection of any two nonzero subgroups of $A$ is nonzero or equivalently, every nonzero subgroup of $A$ is essential in $A$.

**Definition 3.9.** An Abelian group $A$ has finite Goldie dimension if $A$ has an essential subgroup which is a finite direct sum of uniform subgroups. Goldie dimension denoted by $G-dim(A)$.

**Lemma 3.10.** *Let $A$ be an Abelian group and $n$ a non negative integer. Then the following conditions are equivalent:*

1. $G-dim(A)=n$.
2. *$A$ contains a direct sum of $n$ nonzero subgroups but no direct sum of $n+1$ nonzero subgroups.*
3. *For every ascending chain $A_0\subseteq A_1\subseteq A_2\subseteq\ldots$ of subgroups of $A$ there is aninteger $m$ such that $A_j\subseteq_{e}A_{j+1}$ for every $j\geqslant k$.*

4. *For every descending chain $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ of subgroups of $A$ there is an integer $m$ such that $A_{j+1} \subseteq_e A_j$ for every $j \geqslant k$.*

**Example 3.11.** $\mathbb{Z}$, $\mathbb{Q}$ and $Z_{p^\infty}$ are uniform Abelian groups but $A = \mathbb{Z} \oplus \mathbb{Q} \oplus Z_{p^\infty}$ is not uniform group and $G - dim(A) = 3$.

**Lemma 3.12.** *Every module with deviation has finite Goldie dimension.*

**Definition 3.13.** The socle of an Abelian group $A$ is the sum of all simple subgroups of $A$ and is denoted by $Soc(A)$.

**Lemma 3.14.** *The intersection of the essential subgroups of $A$ equals $Soc(A)$.*

**Definition 3.15.** An Abelian group $A$ has essential deviation if the partially ordered set of its essential subgroups has deviation and denoted it by $e.dev(A)$

**Lemma 3.16.** *Let $A$ be an Abelian group and $e.dev(A) = \alpha$, then for every $B \subseteq A$, $e.dev(B) \leqslant \alpha$.*

**Lemma 3.17.** *Let $A$ be an Abelian group with essential deviation $\alpha$ and $B \subset A$. Then $e.dev(\frac{A}{B}) \leqslant \alpha$.*

**Lemma 3.18.** *If the Abelian group $A$ has deviation, then $A$ has essential deviation and $e.dev(A) \leqslant dev(A)$.*

The following example shows that the converse of the previous lemma does not hold in general.

**Example 3.19.** $A = \oplus_{n\in\mathbb{N}} \mathbb{Q}$ has essential deviation but has no deviation.

**Lemma 3.20.** *Let $A$ be an Abelian group and for every proper essential subgroup $B$ of $A$ has essential deviation, then so does $A$ and $e.dev(A) = \sup\{e.dev(B) | B \subset_e A\}$.*

**Lemma 3.21.** *If for every essential subgroup $E$ of $A$, $\frac{A}{E}$ has essential deviation, then $A$ has essential deviation and $e.dev(A) \leqslant \sup\{e.dev(\frac{A}{B}) | B \subset_e A\} + 1$.*

**Lemma 3.22.** *Let $A$ be an Abelian group. If for any essential subgroup $B$ of $A$, either $B$ or $\frac{A}{B}$ has essential deviation, then so does $A$.*

**Theorem 3.23.** *Let $A$ be an Abelian groups. Then $dev(A)$ exists if and only if, $e.dev(A)$ exists and $G - dim(A) < \infty$. In this case $dev(A) = e.dev(A)$.*

**Proposition 3.24.** *Let $A$ be an Abelian group. Then $\frac{A}{Soc(A)}$ has deviation if and only if, for every essential subgroup $B$ of $A$, $\frac{A}{B}$ has deviation.*

**Theorem 3.25.** *Let $A$ be an Abelian group. Then $\frac{A}{Soc(A)}$ has deviation if and only if, every essential subgroup $B$ of $A$, has essential deviation.*

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# Some Results on $n$-Absorbing Submodules

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**Abstract**

This paper investigates the theory of $n$-absorbing submodules and ideals, with a focus on their structural properties and interplay in commutative algebra. We establish fundamental characterizations, demonstrating that for a proper submodule $B$ of an $R$-module $A$, the ideal $(B :_R A)$ is $n$-absorbing (resp. strongly $n$-absorbing) precisely when $B$ is an $n$-absorbing (resp. strongly $n$-absorbing) submodule under certain ring-theoretic conditions, particularly when $R$ is a domain with dimension at most one or a zero-dimensional ring. Special attention is given to multiplication modules, where stronger equivalences hold, including the relationship between the $n$-absorbing property and primary decomposition. We also introduce the invariant $m_A(B)$, the minimal $n$ for which $B$ is $n$-absorbing, and relate it to powers of prime ideals. The results generalize and unify known theorems on absorbing structures, offering new insights into the module-theoretic analogues of $n$-absorbing ideals.

## 1. Introduction

The study of absorbing ideals in commutative rings, initiated by Badawi in the context of 2-absorbing ideals and later extended to $n$-absorbing ideals, has grown into a rich area of research with applications in factorization theory, multiplicative ideal theory, and module theory. In parallel, the module-theoretic counterpart—$n$-absorbing submodules—has emerged as a natural generalization, prompting investigations into the transfer of absorbing properties between submodules and their colon ideals.

This paper contributes to this line of inquiry by systematically exploring conditions under which the $n$-absorbing property of a submodule $B$ of an $R$-module $A$ is equivalent to the $n$-absorbing property of the ideal $(B :_R A)$. We show that, in general, the implication holds only in one direction, but under suitable dimensionality assumptions—such as when $R$ is a domain of dimension at most one or a zero-dimensional ring—a full equivalence is restored. Furthermore, we prove that in the class of multiplication modules, the notions of $n$-absorbing and strongly $n$-absorbing submodules coincide, leading to simplified characterizations in terms of prime ideal powers.

We also examine the invariant $m_A(B)$, the smallest integer $n$ for which $B$ is $n$-absorbing, and establish its connection to the structure of $(B :_R A)$. In particular, for multiplication modules, we show that $m_A(B) = m_R((B :_R A))$, generalizing known results for ideals.

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The paper is organized as follows: after recalling essential definitions and preliminary results, we present a series of propositions and theorems that clarify the relationship between $n$-absorbing submodules and ideals, with illustrative examples and counterexamples. Special cases, such as faithful modules and modules over principal ideal domains, are treated in detail. Our work unifies and extends several existing results in the literature, providing a clearer picture of the behavior of absorbing structures in both ring and module theory.

## 2. Main Results

**Proposition 2.1.** *Let $A$ be a module over a ring $R$. If $B$ is an $n$-absorbing submodule of $A$, then $(B :_R A)$ is an $n$-absorbing ideal of $R$.*

**Example 2.2.** Let $R$ be a ring containing two distinct nonzero prime ideals $P \subset Q$. By setting $A = R \times R$ and $B = P \times Q$, one obtains that $(B :_R A) = P$ is an 1-absorbing ideal of R. However, $B$ is not an 1-absorbing submodule of $A$ because for every $q \in Q \setminus P$, $q \notin (B :_R A)$, $m = (0,1) \notin B$ but $qm = (0,q) \in B$.

**Proposition 2.3.** *If for a ring $R$ a proper submodule $B$ of an $R$-module $A$ is strongly $n$-absorbing, then $(B :_R A)$ is a strongly $n$-absorbing ideal of $R$.*

**Proposition 2.4.** *Let $A$ be a module over a ring $R$. $B$ is an $n$-absorbing submodule of $A$ if and only if $B$ is a strongly $n$-absorbing submodule of $A$.*

**Corollary 2.5.** *An ideal $I$ of a ring $R$ is $n$-absorbing if and only if $I$ is strongly $n$-absorbing.*

**Proposition 2.6.** *Let $R$ be an integral domain. The following assertions are equivalent:*

1. $dim(R) \leqslant 1$.
2. *For every $R$-module $A$ and every proper submodule $B$ of $A$ such that $(B :_R A) \neq 0$, $B$ is an $n$-absorbing submodule of $A$ if and only if $(B :_R A)$ is an $n$-absorbing ideal of $R$.*

**Proposition 2.7.** *For every ring $R$, the following hold:*

1. *the radical of every nonzero ideal of $R$ is a prime ideal.*
2. *every $n$-absorbing ideal $I$ of $R$ is $P$-primary for some prime ideal $P$ of $R$.*

**Definition 2.8.** For some positive integer $n$ and any $n$-absorbing submodule $B$ of an $R$-module $A$,

$$m_A(B) = \min\{n \mid B \text{ is an } n-\text{absorbing submodule of } A\}.$$

**Proposition 2.9.** *Let $R$ be a ring, $n > 1$ an integer and $I$ a nonzero ideal of $R$. Then $m_R(I) = n$ if and only if $I = P^n$ for some principal prime ideal $P$ of $R$ such that $P^n \neq 0$.*

**Proposition 2.10.** *Let $R$ be a ring. The following assertions are equivalent:*

1. *$R$ is zero-dimensional or $R$ is a domain with $dim(R) \leqslant 1$.*
2. *For every $R$-module $A$ and every proper submodule $B$ of $A$ such that $(B :_R A) \neq 0$, $B$ is an $n$-absorbing submodule of $A$ if and only if $(B :_R A)$ is an $n$-absorbing ideal of $R$.*

**Theorem 2.11.** *Let $R$ be a ring, $n$ a positive integer, $A$ an $R$-module and $B$ a proper submodule of $A$ such that $(B :_R A) \neq 0$. Assume that $R$ is zero-dimensional or a domain with $dim(R) \leqslant 1$. Then $B$ is $n$-absorbing if and only if $(B :_R A) = P^k$, for some $1 \leqslant k \leqslant n$ and prime ideal $P$ of $R$.*

**Remark 2.12.** Let $A$ be a multiplication $R$-module. In what follows for all ideal $I$ and $J$ of $R$ and all submodules $N$ and $M$ of $A$ such that $N = IA$ and $N = JM$, $NM := IJA$ and for every $a, b \in A$, $ab$ denotes the submodule $(Ra)(Rb)$.

**Proposition 2.13.** *Let $A$ be a multiplication $R$-module for some ring $R$, $B$ a proper submodule of $A$ and $n$ a nonnegative integer. Then $B$ is a strongly $n$-absorbing submodule of $A$ if and only if $(B :_R A)$ is a strongly $n$-absorbing ideal of $R$.*

**Proposition 2.14.** *Let $A$ be a multiplication $R$-module for some ring $R$ and $B$ a proper submodule of $A$. Then $N$ is 2-absorbing if and only if $B$ is strongly 2-absorbing.*

**Proposition 2.15.** *Let $A$ be a multiplication $R$-module and $B$ a $n$-absorbing submodule of $A$. Then $(A\backslash rad(B))^n \subseteq N$.*

**Proposition 2.16.** *Let $A$ be a multiplication module over a ring $R$, $B$ a proper submodule of $A$ and $n$ a nonnegative integer. Then:*

1. *$B$ is a $n$-absorbing submodule of $A$ if and only if for every submodules $M_1, \dots, M_{n+1}$ of $A$ such that $M_1, \dots, M_{n+1} \subseteq B$, there are $n$ of the $M_i$'s whose product is contain in $B$.*
2. *If $B$ is a $n$-absorbing submodule of $A$, then for every $a_1, \dots, a_{n+1} \in A$ such that $a_1 \dots a_{n+1} \subseteq B$, there are $n$ of the $a_i$'s whose product is included in $B$.*

**Proposition 2.17.** *Let $R$ be a ring. Then for every multiplication $R$- module $A$ and every proper submodule $B$ of $A$, $m_A(B) = m_R(B :_R A)$.*

**Proposition 2.18.** *Let $R$ be a ring, $n$ a positive integer and $A$ a multiplication $R$-module. Then we have the following equivalences for every proper submodule $B$ of $A$:*

1. *$B$ is an $n$-absorbing submodule of $A$ if and only if $B = P^k A$ for a prime ideal $P$ of $R$ and a nonnegative integer $k \leqslant n$.*
2. *If $A$ is a faithful module, then $B$ is an $n$-absorbing submodule of $A$ if and only if $B = P^k$ for some $1 \leqslant k \leqslant n$ and prime submodule $P$ of $A$ containing $B$.*

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# The Rational Character Table and Permutation and Quasi-Permutation Representations for the Group $SL(2,q)$ when Extended by a certain Group of order Two

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**Abstract**

Let $G$ be a finite group of degree $n$. We shall say that $G$ is a quasi-permutation group, if the trace of every element of $G$ is a non-negative rational integer. By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. For a given finite group $G$, let $p(G)$ denote the minimal degree of a faithful permutation representation of $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field $Q$, and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of $G$. The purpose of this paper is to calculate $q(G), c(G), r(G)$ and $p(G)$ for the group $K_2^2(2^n)$. Also in this paper we will give the character table of the irreducible rational representations of $K_2^2(2^n)$, by using the ordinary character table and the Schur indices of this group.

## 1. Introduction

Let $G$ be a finite linear group of degree $n$; that is, a finite group of automorphisms of an $n$-dimensional complex vector space. We shall say that $G$ is a quasi-permutation group if the trace of every element of $G$ is a non-negative rational integer. The reason for this terminology is that, if $G$ is a permutation group of degree $n$, its elements, considered as acting on the elements of a basis of an $n$-dimensional complex vector space $V$, induce automorphisms of $V$ forming a group isomorphic to $G$. The trace of the automorphism corresponding to an element $x$ of $G$ is equal to the number of letters left fixed by $x$, and so is a non-negative integer. Thus, a permutation group of degree $n$ has a representation as a quasi-permutation group of degree $n$ .See [14]. In [3] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results.
By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. For a given finite group $G$, let $p(G)$ denote the minimal degree of a faithful permutation representation of $G$ , let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field Q and, let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices and finally let

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$r(G)$ denote the minimal degree of a faithful rational valued character of $G$. It is easy to see that for a finite group $G$ the following inequalities hold

$$r(G) < c(G) \leq q(G) \leq p(G).$$

Let $G$ be a finite group and $\chi$ be an irreducible complex character of $G$. Let $m_Q(\chi)$ denote the Schur index of $\chi$ over $Q$ and $\Gamma(\chi)$ be the Galois group $Q(\chi)$ over $Q$. It is known that

$$\sum_{\alpha\in\Gamma(\chi)} m_Q(\chi)\chi^\alpha \qquad (*)$$

is a character of an irreducible $Q(G)$-module [[12], Corollary 10.2(b)]. So by knowing the character table of a group and Suchr indices of each of the irreducible characters of $G$, we can find the irreducible rational characters of $G$.
Finding the above quantities have been carried out in some papers, for example in [4], [5], [6] and [10] we found these for the groups $GL(2,q)$, $SU(3,q^2)$, $PSU(3,q^2)$, $SL(3,q)$, $PSL(3,q)$ and $G_2(2^n)$ respectively.
In this paper we will apply the algorithms in [1] for the group $K_2^2(2^n)$, where

$$K_2^2(q) =< SL(2,q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} > .$$

## 2. Notation and preliminary results

We can see all the following statements in [1], [2], [11].

**Lemma 2.1.** *Every proper normal subgroup of $G = SL(n,F)$ is in $Z(G)$ except when $n = 2$ and $| F | = 2$ or 3.*

**Lemma 2.2.** *Let $G = SL(2,F)$ and $char(F) \neq 2$. Then $G$ has a unique involution.*

**Lemma 2.3.** *Let $G = SL(2,F)$ and $char(F) \neq 2$. Then $Z(G) = \pm I$ and $|Z(G)| = 2$. Moreover $Z(G)$ is the unique minimal normal subgroup of $G$ and the core of any subgroup of even order is non-trivial.*

**Lemma 2.4.** *Let $G = SL(2,q)$, where $q = 2^n$. Then $SL(2,q)$ is a simple group when $n \neq 1$, and when $n = 1$ it has a unique minimal normal subgroup , which has order 3.*

Let $\chi$ be a character of $G$ such that, for all $g \in G, \chi(g) \in Q$ and $\chi(g) \geq 0$. Then we say that $\chi$ is a non-negative rational valued character.

**Definition 2.5.** Let $\Gamma(\chi)$ be the Galois group of $Q(\chi)$ over $Q$.
Let $G$ be a finite group and $\chi$ be an irreducible complex character of $G$. Then we define

**1)** $d(\chi) = |\Gamma(\chi)|\chi(1)$

**2)** $m(\chi) = \begin{cases} 0 & \chi = 1_G \\ | \min\{\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha(g) : g \in G\}| & otherwise \end{cases}$

**3)** $c(\chi) = \sum_{\alpha\in\Gamma(\chi)}\chi^\alpha + m(\chi)1_G$.

**Lemma 2.6.** *Let $\chi$ be a character of $G$. Then $Ker\chi = Ker\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$. Moreover $\chi$ is faithful if and only if $\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$ is faithful.*

**Lemma 2.7.** *Let $\chi \in Irr(G)$, then $\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha$ is a rational valued character of $G$. Moreover $c(\chi)$ is a non-negative rational valued character of $G$ and $c(\chi)(1) = d(\chi) + m(\chi)$.*

**Lemma 2.8.** *Let $\chi \in Irr(G), \chi \neq 1_G$. Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.*

If the Schur index of each non-principal irreducible character of $G$ over $Q$ is equal to $m$, then from [1] Corollary 3.15 we have $q(G) = mc(G)$.
The proof of the following facts may be found in [2].
Let $\epsilon$ be a primitive $n$-th root of unity in $C$. Then $\epsilon + \epsilon^{-1}$ is rational if and only if $n = 1, 2, 3, 4, 6$.
Also $\epsilon^j + \epsilon^{-j}, 1 \leq j \leq n$ is rational if and only if $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$.
In this case if $i \in Z$ and $d_i = (i,n)$, and $n > 2d_i$, then $[Q(\epsilon^i + \epsilon^{-i}) : Q] = \frac{1}{2}\varphi(\frac{n}{d_i})$ , and if $n \neq d_i, 2d_i$, then $\sum_{\alpha\in\Gamma_i}(\epsilon^i + \epsilon^{-i})^\alpha = \mu(\frac{n}{d_i})$, where $\Gamma_i = (Q(\epsilon^i + \epsilon^{-i} : Q)$ and $\mu$ is the Möbius function.
With the above assumption if we set $\Gamma = (Q(\epsilon + \epsilon^{-1}) : Q)$, then $\sum_{\alpha\in\Gamma}(\epsilon^i + \epsilon^{-i})^\alpha = \frac{\varphi(n)}{\varphi(\frac{n}{d_i})}\mu(\frac{n}{d_i})$, where $d_i = (i,n)$.

**Lemma 2.9.** *Let $G$ be a finite group and let $\chi \in Irr(G)$ .Then $m_Q(\chi) \mid \chi(1)$. Moreover when $\chi$ is linear we have $m_Q(\chi) = 1$.*

### 3. Permutation and quasi-permutation representations

Let $SL(n,q)$ denote the general linear group of a vector space of dimension $n$ over a field with $q$ elements. Let $\theta : SL(n,q) \longrightarrow SL(n,q)$ be the automorphism of $SL(n,q)$ given by $\theta(A) = (A^t)^{-1}$ , where $A^t$ denotes the transpose of the matrix $A \in SL(n,q)$. In this case one can define the split extension $SL(n,q).<\theta>$, that following the notations used in [9] is denoted by $K_n^2(q)$. Therefore we have $K_n^2(q) =< SL(n,q), \theta | \theta^2 = 1, \theta^{-1}A\theta = (A^t)^{-1} >$, see[7]. Now by [7] we have important lemma as follows:

**Lemma 3.1.** *Let $G = K_2^2(q)$, if $q$ is even, then $K_2^2(q) \cong SL(2,q) \times <\theta>$ and if $q$ is odd, then $K_2^2(q) \cong SL(2,q) \circ 4$ a central product of $SL(2,q)$ with the cyclic group of order 4.*

**Remark 3.2.** In the case of $q$ even $<\theta J>$ has order 2 and its irreducible characters are denoted by $\mu_0$ and $\mu_1$ where $\mu_0$ is the identity character. Regarding the structure of $K_2^2(2^n)$, Lemma 2.11 and [8 Theorem 38.1], the irreducible characters of $K_2^2(q)$ in the case of $q$ even are $\mu_k \times 1_G$ , $\mu_k \times \psi$ , $\mu_k \times \chi_i$ and $\mu_k \times \theta_j$ where $k = 0,1$ and $1 \le i \le \frac{q-2}{2}, 1 \le j \le \frac{q}{2}$.

By [13] we have the following lemma:

**Lemma 3.3.** *Let $G = SL(2,q)$, if $q$ is a power of 2 then the Schur index of any irreducible character of $G$ over the rational numbers $Q$ is 1.*

**Lemma 3.4.** *Let $G = K_2^2(2^n)$ , then the Schur index of any irreducible character of $G$ over the rational numbers $Q$ is 1.*

**Lemma 3.5.** *Let $G = K_2^2(q)$, where $q = 2^n$ and $n \ge 2$, then $\mu_k \times \theta_j$ ; $k = 0,1$ , $1 \le j \le q/2$ is rational if and only if $q \equiv -1$ (mod 3) and $j = \frac{q+1}{3}$.*

*Proof.* As $1 \le j \le \frac{q}{2} < \frac{q+1}{2}$ and as $\sigma$ is a primitive $(q+1)$-th root of unity , by section 2, $\mu_k \times \theta_j$ is rational if and only if $j = \frac{q+1}{6}, \frac{q+1}{4}, \frac{q+1}{3}$. Since $q+1$ is odd , $\frac{q+1}{6}$ and $\frac{q+1}{4}$ are not integers . Thus , $\sigma^j + \sigma^{-j} \in Q$ if and only if $3 | q+1$ and $j = \frac{q+1}{3}$. In this case, $\sigma^{jm} + \sigma^{-jm} = \begin{cases} -1 & if\ m\ is\ odd \\ 1 & if\ m\ is\ even \end{cases}$ □

**Lemma 3.6.** *Let $G = K_2^2(q)$ where $q = 2^n$ and $n \ge 2$. Then for each $j, k$ , $1 \le j \le q/2$ and $k = 0,1$ we have*
*(1) $d(\mu_k \times \theta_j) \ge q - 1$ and equality holds if $\mu_k \times \theta_j$ is rational ;*
*(2) $c(\mu_k \times \theta_j) \ge 2(q-1)$ and equality holds if $\mu_k \times \theta_j$ is rational.*

**Theorem 3.7.** *Let $G = K_2^2(2)$, then $r(G) = 2$ and $c(G) = q(G) = 4$.*

*Proof.* By Lemma 3.4, Schur index of each irreducible characters of the group $K_2^2(2^n)$ is 1 and so by Corollary 2.15 of [1] we have $c(G) = q(G)$. Since the only faithful irreducible character of $G$ is $\mu_1 \times \psi$ so by the character table of the group $K_2^2(2)$ the result follows. □

**Theorem 3.8.** *Let $G = K_2^2(q)$, where $q = 2^n > 2$. Then*
**1)** *If $q \equiv -1$ (mod 3), then $r(G) = q - 1$, $c(G) = q(G) = 2(q-1)$*
**2)** *Otherwise $r(G) = q$, $c(G) = q(G) = 2q$*

By [1] we have the following lemma:

**Lemma 3.9.** *Let $G$ be a finite group with a unique minimal normal subgroup. Then $p(G)$ is the smallest index of a subgroup with trivial core (that is, containing no non-trivial normal subgroup).*

**Theorem 3.10.** *Let $G = K_2^2(q) = SL(2,q).<\theta>$, where $q = 2^n$, Then $p(G) = 2(q+1)$.*

## 4. The rational character table

**Lemma 4.1.** *Let $G = K_2^2(2^n)$, then the Galois orbit sums in $Irr(G)$ are as follows:*
**a)** $\mu_k \times \chi_e = \sum_{\alpha\in\Gamma}(\mu_k \times \chi_i)^\alpha$
*where,* $e = (i, q-1)$ *and* $1 \le i \le \frac{q-2}{2}$ , $k = 0, 1$ *and* $\Gamma = \Gamma(Q(\mu_k \times \chi_i) : Q) = \Gamma(Q(\chi_i) : Q)$.
**b)** $\mu_k \times \theta_f = \sum_{\alpha\in\Gamma}(\mu_k \times \theta_j)^\alpha$
*where,* $f = (j, q+1)$ *and* $1 \le j \le \frac{q}{2}$ , $k = 0, 1$ *and* $\Gamma = \Gamma(Q(\mu_k \times \theta_j) : Q) = \Gamma(Q(\theta_j) : Q)$.
**c)** $\mu_k \times \theta_{(\frac{q+1}{4})}$ *when,* $q \equiv -1 \pmod 3$.
**d)** $1_G$, $\psi$.

By Lemma 2.8 we know:
If $\chi_i, \theta_j$ are irreducible characters of $G$, then $\mu_k \times \chi_e = \sum_{\alpha\in\Gamma_1}(\mu_k \times \chi_i)^\alpha$ and $\mu_k \times \theta_f = \sum_{\alpha\in\Gamma_2}(\mu_k \times \theta_j)^\alpha$ are rational valued characters of $G$, where $\Gamma_1 = \Gamma(Q(\chi_i) : Q)$ and $\Gamma_2 = \Gamma(Q(\theta_j) : Q)$.

**Corollary 4.2.** *Let $G = K_2^2(2^n)$ ,then characters of an irreducible $Q(G)$-module are as follows:*
$\mu_k \times \chi_e = \sum_{\alpha\in\Gamma_1}(\mu_k \times \chi_i)^\alpha$ , $\mu_k \times \theta_f = \sum_{\alpha\in\Gamma_2}(\mu_k \times \theta_j)^\alpha$ , $\mu_k \times \theta_{(\frac{q+1}{4})}$ , $1_G$, $\psi$
*where,* $e = (i, q-1)$ , $1 \le i \le \frac{q-2}{2}$ *and* $k = 0, 1$ ; $f = (j, q+1)$ , $1 \le j \le \frac{q}{2}$ *and* $\Gamma_1 = \Gamma(Q(\mu_k \times \chi_i) : Q)$ , $\Gamma_2 = \Gamma(Q(\mu_k \times \theta_j) : Q)$.

*Proof.* By lemma 3.4, Schur index of each irreducible characters is 1. So by $(*)$ and Lemmas 2.8, 4.1 the result follows. □

**Lemma 4.3.** *Let $G = K_2^2(2^n)$, then for the irreducible characters $\mu_k \times \chi_e, \mu_k \times \theta_f$ where, $k = 0, 1$ of rational representations of $G$ we have*
**a)** $(\mu_0 \times \chi_e)(1,1) = (\mu_0 \times \chi_e)(x,1) = (\mu_1 \times \chi_e)(1,1) = \frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$
$(\mu_0 \times \chi_e)(1,c) = (\mu_0 \times \chi_e)(x,c) = (\mu_1 \times \chi_e)(1,c) = \frac{1}{2}\varphi(\frac{q-1}{e})$
$(\mu_0 \times \chi_e)(1,a^l) = (\mu_0 \times \chi_e)(x,a^l) = (\mu_1 \times \chi_e)(1,a^l) = S_1(e,e')$
$(\mu_0 \times \chi_e)(1,b^m) = (\mu_0 \times \chi_e)(x,b^m) = (\mu_1 \times \chi_e)(1,b^m) = 0$
$(\mu_1 \times \chi_e)(x,1) = -\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$
$(\mu_1 \times \chi_e)(x,c) = -\frac{1}{2}\varphi(\frac{q-1}{e})$
$(\mu_1 \times \chi_e)(x,a^l) = -S_1(e,e')$
$(\mu_1 \times \chi_e)(x,b^m) = 0.$
**b)** $(\mu_0 \times \theta_f)(1,1) = (\mu_0 \times \theta_f)(x,1) = (\mu_1 \times \theta_f))(1,1) = \frac{1}{2}(q-1)\varphi(\frac{q+1}{e})$
$(\mu_0 \times \theta_f))(1,c) = (\mu_0 \times \theta_f))(x,c) = (\mu_1 \times \theta_f))(1,c) = -\frac{1}{2}\varphi(\frac{q+1}{e})$
$(\mu_0 \times \theta_f))(1,a^l) = (\mu_0 \times \theta_f))(x,a^l) = (\mu_1 \times \theta_f))(1,a^l) = 0$
$(\mu_0 \times \theta_f))(1,b^m) = (\mu_0 \times \theta_f))(x,b^m) = (\mu_1 \times \theta_f))(1,b^m) = -S_2(e,e')$
$(\mu_1 \times \theta_f)(x,1) = -\frac{1}{2}(q-1)\varphi(\frac{q+1}{e})$
$(\mu_1 \times \theta_f))(x,c) = \frac{1}{2}\varphi(\frac{q+1}{e})$
$(\mu_1 \times \theta_f))(x,a^l) = S_2(e,e')$
$(\mu_1 \times \theta_f))(x,b^m) = 0.$

*Proof.* These follows from Corollary 4.2 , Remarks 3.2, 4.3 and section 2. □

Now in the above notation we have:

**Theorem 4.4.** *The character table of the irreducible rational representation of $K_2^2(2^n)$ is as follow:*

**Table (II)**
**The irreducible rational characters of $K_2^2(q)$ , $q \neq -1 \ mod 3$**

| | $\mu_0 \times 1_G$ | $\mu_0 \times \psi$ | $\mu_0 \times \chi_e$ $1 \le i \le \frac{q-2}{2}$ | $\mu_0 \times \theta_f$ $1 \le j \le \frac{q}{2}$ | $\mu_1 \times 1_G$ | $\mu_1 \times \psi$ | $\mu_1 \times \chi_e$ $1 \le i \le \frac{q-2}{2}$ | $\mu_1 \times \theta_f$ $1 \le j \le \frac{q}{2}$ |
|---|---|---|---|---|---|---|---|---|
| $(1,1)$ | 1 | $q$ | $\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$ | $\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$ | 1 | $q$ | $\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$ | $\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$ |
| $(1,c)$ | 1 | 0 | $\frac{1}{2}\varphi(\frac{q-1}{e})$ | $-\frac{1}{2}\varphi(\frac{q+1}{f})$ | 1 | 0 | $\frac{1}{2}\varphi(\frac{q-1}{e})$ | $-\frac{1}{2}\varphi(\frac{q+1}{f})$ |
| $(1,a^l)$ $1 \le l \le \frac{q-2}{2}$ | 1 | 1 | $S_1(e,e')$ | 0 | 1 | 1 | $S_1(e,e')$ | 0 |
| $(1,b^m)$ $1 \le m \le \frac{q}{2}$ | 1 | $-1$ | 0 | $-S_2(f,f')$ | 1 | $-1$ | 0 | $-S_2(f,f')$ |
| $(x,1)$ | 1 | $q$ | $\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$ | $\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$ | $-1$ | $-q$ | $-\frac{1}{2}(q+1)\varphi(\frac{q-1}{e})$ | $-\frac{1}{2}(q-1)\varphi(\frac{q+1}{f})$ |
| $(x,c)$ | 1 | 0 | $\frac{1}{2}\varphi(\frac{q-1}{e})$ | $-\frac{1}{2}\varphi(\frac{q+1}{f})$ | $-1$ | 0 | $-\frac{1}{2}\varphi(\frac{q-1}{e})$ | $\frac{1}{2}\varphi(\frac{q+1}{f})$ |
| $(x,a^l)$ $1 \le l \le \frac{q-2}{2}$ | 1 | 1 | $S_1(e,e')$ | 0 | $-1$ | $-1$ | $-S_1(e,e')$ | 0 |
| $(x,b^m)$ $1 \le m \le \frac{q}{2}$ | 1 | $-1$ | 0 | $-S_2(f,f')$ | $-1$ | 1 | 0 | $S_2(f,f')$ |

*For the case $q \equiv -1 \pmod 3$, we have the above table with two extra columns , as follow:*

**Table (III)**

$q \equiv -1\ (mod 3)$

| | $\mu_0 \times \theta_{\frac{q+1}{3}}$ | $\mu_1 \times \theta_{\frac{q+1}{3}}$ |
|---|---|---|
| $(1,1)$ | $q-1$ | $q-1$ |
| $(1,c)$ | $-1$ | $-1$ |
| $(1,a^l)$ $1 \le l \le \frac{q-2}{2}$ | 0 | 0 |
| $(1,b^m)$ $1 \le m \le \frac{q}{2}$ | $(-1)^{m+1}$ | $(-1)^{m+1}$ |
| $(x,1)$ | $q-1$ | $-(q-1)$ |
| $(x,c)$ | $-1$ | 1 |
| $(x,a^l)$ $1 \le l \le \frac{q-2}{2}$ | 0 | 0 |
| $(x,b^m)$ $1 \le m \le \frac{q}{2}$ | $(-1)^{m+1}$ | $(-1)^m$ |

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# Computing GCD and LCM of Parametric Polynomials

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**Abstract**

This paper presents a generalization of the computation of greatest common divisors (GCD) and least common multiples (LCM) for polynomials with parametric coefficients. While classical methods rely on Gröbner bases and ideal intersections for polynomials with numeric coefficients, these approaches sometimes fail in the parametric case due to specialization issues. We introduce *LCM/GCD systems* using the CIS algorithm [7], derived from Gröbner systems, to compute GCD and LCM under parametric variations. The proposed method ensures correctness for all parameter specializations, overcoming the limitations of existing symbolic computation tools such as `Maple`. An algorithm is presented, along with illustrative examples computed in `Maple`, that demonstrate the effectiveness of the approach.

## 1. Introduction

The computation of greatest common divisors (GCD) and least common multiples (LCM) of polynomials is a fundamental problem in symbolic computation. For univariate polynomials with constant coefficients, the Euclidean algorithm provides an efficient and well-understood solution. In the multivariate setting with constant coefficients, namely in $\mathbb{K}[x_1, \dots, x_n]$, Gröbner bases[1] offer a powerful algebraic framework for ideal-theoretic computations, including GCD and LCM through the well-known relationships: $\langle \mathrm{LCM}(f,g)\rangle = \langle f\rangle \cap \langle g\rangle$, $\mathrm{GCD}(f,g)\cdot\mathrm{LCM}(f,g) = f\cdot g$. However, when coefficients are symbolic parameters– elements of a polynomial ring $\mathbb{K}[a_1, \dots, a_m]$– the situation becomes considerably more complex. Let $f, g \in \mathbb{K}[a_1, \dots, a_m][x_1, \dots, x_n]$ be parametric polynomials, and let $\sigma : \mathbb{K}[a_1, \dots, a_m] \to \mathbb{K}'$ be a *specialization* (or evaluation) of the parameters to specific values. A natural question arises: Does specialization commute with the computation of LCM? That is, for parametric polynomials $f$ and $g$, is it always true that $\sigma\big(\mathrm{LCM}(f,g)\big) = \mathrm{LCM}\big(\sigma(f), \sigma(g)\big)$? As we demonstrate through counterexamples in Section 2, the answer is generally *no*. The fundamental issue is that specialization does not commute with polynomials LCM in the parametric setting:

$$\sigma\big(\mathrm{LCM}(f,g)\big) \neq \mathrm{LCM}\big(\sigma(f), \sigma(g)\big),$$

This inconsistency renders classical Gröbner basis methods unreliable for parametric GCD/LCM computation, leading to incorrect results in symbolic computation systems such as `Maple` when parameters are specialized.

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[1] Gröbner bases were introduced by Buchberger [1] and have since become a cornerstone of computational algebra [2].

To address this fundamental problem, we introduce in this paper the concept of a *least common multiple system* (LCMS) based on *comprehensive intersection systems* (CIS). This approach builds upon our previous work on comprehensive intersection systems for parametric ideal intersection [7], where we developed the theoretical foundation for handling parametric ideal intersections in a specialization-consistent manner. In this paper, we extend the framework specifically to GCD and LCM computation. We show how a CIS for the intersection ideal $\langle f\rangle \cap \langle g\rangle$ naturally induces an LCMS that correctly computes $\mathrm{LCM}(f,g)$ across all parameter values. The GCD is then recovered through the standard relationship $\mathrm{GCD}(f,g)\cdot \mathrm{LCM}(f,g) = f\cdot g$, applied branch-wise according to the LCMS structure. Our approach ensures that for every parameter specialization $\sigma$, the specialized GCD/LCM matches the result obtained by first specializing and then computing GCD/LCM.

## 2. Gröbner Systems and Preliminaries

The parametric extension arose from the need to handle symbolic design parameters in fields like control theory, robotics, and geometric modeling. A key challenge was the specialization problem, where substituting parameter values before computation produced different results. The transition from Gröbner bases to Gröbner systems marks a major advance in computational algebra.
Volker Weispfenning's seminal 1992 paper [19] introduced *Gröbner systems*, providing the first systematic framework for handling parametric polynomials. His key insight was to partition the parameter space into regions where specialized Gröbner bases could be uniformly described. Building on Weispfenning's foundation, Antonio Montes developed the DisPGB algorithm [17] (see also [3]), introducing the "discuss" paradigm that became influential in later implementations. Subsequent work by Kapur [16], Suzuki and Sato [18], and others refined both theory, algorithms, and applications [4–6, 8–15].
Let $\mathbb{K}$ be a field of characteristic zero. Let $\mathbf{a} = (a_1, \dots, a_m)$ be parameters and $\mathbf{x} = (x_1, \dots, x_n)$ be variables. We work in the polynomial ring $S = \mathbb{K}[\mathbf{a}, \mathbf{x}] = \mathbb{K}[a_1, \dots, a_m, x_1, \dots, x_n]$. A *specialization* is a ring homomorphism $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}'$, where $\mathbb{K}'$ is a field extension of $\mathbb{K}$. For $f \in S$, we denote $\sigma(f)$ as the polynomial obtained by applying $\sigma$ to the coefficients of $f$.

**Definition 2.1.** Given an ideal $I \subset S$ and two monomial ordering $\prec_{\mathbf{x}}, \prec_{\mathbf{a}}$, a *Gröbner system* for $I$ is a finite set: $\mathcal{G} = \{(N_1, W_1, G_1), \dots, (N_l, W_l, G_l)\}$, where:

- $N_i, W_i \subset \mathbb{K}[\mathbf{a}]$ are finite sets defining parametric conditions
- $G_i \subset S$ is a finite set of polynomials s.t. for any specialization $\sigma$ satisfying:

$$\sigma(p) = 0 \ \forall p \in W_i, \quad \sigma(q) \neq 0 \ \forall q \in N_i,$$

  the set $\sigma(G_i)$ is a Gröbner basis of $\sigma(I)$ w.r.t. $\prec_{\mathbf{x}}$ in $\mathbb{K}'[\mathbf{x}]$.

Based on the above definition, for every specialization $\sigma$, there exists exactly one branch $(N_i, W_i, G_i)$ whose conditions are satisfied by $\sigma$.

## 3. Comprehensive Intersection Systems (CIS)

In our previous work [7], we introduced *comprehensive intersection systems* (CIS) as a computational framework for intersecting parametric polynomial ideals in a way that respects specialization. Traditional methods for ideal intersection, such as the elimination technique using an auxiliary variable $t$-fail when coefficients are parameters, as shown in the following example.

**Example 3.1** (Failure of classical intersection under specialization)**.** Let

$$I = \langle -ay + x^2,\ cy - z,\ bz^2 + x\rangle, \quad J = \langle bx - y^2,\ xz - a\rangle$$

in $\mathbb{C}[a,b,c][x,y,z]$. Using the standard `Maple` `PolynomialIdeals:-Intersect` function yields a large ideal for $I \cap J$. However, specializing at $(a,b,c) = (0,1,1)$ gives an incorrect result when compared to computing $\sigma(I) \cap \sigma(J)$ directly. This inconsistency motivates the need for a *specialization-commuting* description of the intersection.

**Definition 3.2** (Comprehensive Intersection System)**.** Let $I, J \subset S = \mathbb{K}[\mathbf{a}, \mathbf{x}]$ be parametric polynomial ideals. A finite set $\mathcal{G} = \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$ is called a *comprehensive intersection system (CIS)* for $I \cap J$ if for every specialization $\sigma : \mathbb{K}[\mathbf{a}] \to \mathbb{K}' \supseteq \mathbb{K}$ satisfying $(N_i, W_i)$, $\langle \sigma(G_i) \rangle = \sigma(I) \cap \sigma(J)$. Here $(N_i, W_i)$ are parametric constraints (equations and inequations on $\mathbf{a}$) and $G_i \subset \mathbb{K}[\mathbf{a}, \mathbf{x}]$.

Given $I = \langle f_1, \dots, f_m \rangle$ and $J = \langle g_1, \dots, g_n \rangle$, define $H = \langle tf_1, \dots, tf_m,\ (1-t)g_1, \dots, (1-t)g_n \rangle \subset S[t]$. Compute a Gröbner system $\mathcal{H} = \{(N_i, W_i, H_i)\}$ of $H$ with respect to the lexicographic order $\mathbf{a} \prec_{\text{lex}} \mathbf{x} \prec_{\text{lex}} t$. Then, for each branch, let $G_i = \{h \in H_i \mid t \notin \text{vars}(h)\}$, i.e., the polynomials in $H_i$ that do not involve $t$. The triple $(N_i, W_i, G_i)$ forms part of the CIS for $I \cap J$.

**Algorithm 1** INTERSECT

**Require:** $I = \langle f_1, \dots, f_m \rangle, J = \langle g_1, \dots, g_n \rangle \subset \mathbb{K}[\mathbf{a}, \mathbf{x}]$
**Ensure:** $\mathcal{G} = \{(N_i, W_i, G_i)\}_{i=1}^{\ell}$, a CIS for $I \cap J$

```
H ← ⟨tf_1, ..., tf_m, (1−t)g_1, ..., (1−t)g_n⟩ ⊂ K[a, x, t]
Compute Gröbner system H = {(N_i, W_i, H_i)} of H w.r.t. a ≺_lex x ≺_lex t
for (N_i, W_i, H_i) in H do
    if H_i ≠ {1} then
        G_i ← {h ∈ H_i | t ∉ vars(h)}
        Add (N_i, W_i, G_i) to G
    end if
end for
return G
```

Applying Algorithm 1 to the ideals $I$ and $J$ from the example above yields a CIS with several branches. For instance, one branch corresponds to the condition $(a, b, c) = (0, 1, 1)$. Under this specialization,

$$G_7|_{(0,1,1)} = \langle y^2z^2,\ y^3z - y^2z^2,\ y^2z + xz,\ -y^3 + 2y^2z + xy,\ -y^2z^2 - xy^2 + x^2 \rangle,$$

which indeed equals $\sigma(I) \cap \sigma(J)$.
The method generalizes to $r > 2$ ideals. For $I_1, \dots, I_r \subset \mathbb{K}[\mathbf{a}, \mathbf{x}]$, introduce $r$ new variables $t_1, \dots, t_r$ and define

$$J = \langle t_1 I_1, \dots, t_r I_r,\ 1 - \textstyle\sum_{i=1}^{r} t_i \rangle \subset \mathbb{K}[\mathbf{a}, \mathbf{x}, \mathbf{t}].$$

A Gröbner system of $J$ with respect to $\mathbf{a} \prec_{\text{lex}} \mathbf{x} \prec_{\text{lex}} \mathbf{t}$ yields, after removing polynomials containing any $t_i$, a CIS for $I_1 \cap \cdots \cap I_r$.

## 4. Least Common Multiple System (LCMS)

The connection to LCM computation follows directly from the ideal-theoretic characterization:

$$\langle \text{LCM}(f, g) \rangle = \langle f \rangle \cap \langle g \rangle.$$

However, when coefficients are parameters, i.e., elements of $\mathbb{K}[a_1, \dots, a_m]$, classical methods fail because specialization does not commute with LCM.

**Example 4.1.** Consider the following two parametric polynomials in $\mathbb{Q}[a, b][x, y]$:

$$f = ax^2 - c + y, \quad g = bx^2 + cy - d.$$

Let $I = \langle f \rangle$ and $J = \langle g \rangle$. Using `Maple's lcm` function, we compute the intersection ideal:

$$\text{LCM}(f, g) = acx^2y + bx^4 - adx^2 - bcx^2 + bx^2y - c^2y + cy^2 + cd - dy.$$

Now specialize the parameters to $a = b = c = d = 1$: $\sigma(f) = x^2 + y - 1$, $\sigma(g) = x^2 + y - 1$. Computing the LCM of the specialized polynomials gives: $\text{LCM}(\sigma(f), \sigma(g)) = x^2 + y - 1$. However, specializing $\text{LCM}(f, g)$ directly yields: $\text{LCM}(f, g) = x^4 + 2x^2y - 2x^2 + y^2 - 2y + 1$. Thus, $\sigma\big(\text{LCM}(f, g)\big) \neq \text{LCM}\big(\sigma(f), \sigma(g)\big)$, demonstrating that specialization does not commute with polynomials LCM in the parametric case.

This leads naturally to our definition of a *least common multiple system* (LCMS) as a CIS specifically for principal ideals generated by single polynomials. The GCD is then recovered through the fundamental relationship:

$$\mathrm{GCD}(f,g)\cdot\mathrm{LCM}(f,g)=f\cdot g,$$

applied branch-wise according to the LCMS structure. This framework, developed in our previous work and extended here, provides a complete solution to the parametric GCD/LCM problem while ensuring specialization-commuting behavior.

**Definition 4.2** (LCM/GCD System)**.** Let $f,g\in\mathbb{K}[\mathbf{a},\mathbf{x}]$ be parametric polynomials. An *LCM/GCD system* for $f$ and $g$ is a finite set of quadruples $\mathcal{L}=\{(N_i,W_i,\mathrm{LCM}_i,\mathrm{GCD}_i)\}_{i=1}^{l}$ where:

- $N_i,W_i\subset\mathbb{K}[\mathbf{a}]$ are finite sets of parametric constraints (null and non-conditions respectively),
- $\mathrm{LCM}_i,\mathrm{GCD}_i\in S=\mathbb{K}[\mathbf{a},\mathbf{x}]$ are polynomials,
- For any specialization $\sigma:\mathbb{K}[\mathbf{a}]\to\mathbb{K}'$ satisfying $\sigma(p)=0$ for all $p\in W_i$ and $\sigma(q)\neq 0$ for all $q\in N_i$, we have:

$$\sigma(\mathrm{LCM}_i)=\mathrm{LCM}(\sigma(f),\sigma(g)),$$
$$\sigma(\mathrm{GCD}_i)\cdot\sigma(\mathrm{LCM}_i)=\sigma(f)\cdot\sigma(g).$$

The tuples $(N_i,W_i,\mathrm{LCM}_i,\mathrm{GCD}_i)$ correspond exactly to the branches produced by Algorithm 2.

The main contribution of this work is to generalize GCD/LCM computation to parametric polynomials. We adapt the INTERSECT algorithm from the extended abstract to compute parametric LCM via ideal intersection, and derive GCD through the relation:

$$\mathrm{GCD}(f,g)\cdot\mathrm{LCM}(f,g)=f\cdot g.$$

Given a CIS for $\langle f\rangle\cap\langle g\rangle$, each branch provides:

- Parametric conditions $(N_i,W_i)$
- A generator $h_i$ for the intersection in that branch
- Thus, $h_i$ serves as $\mathrm{LCM}(f,g)$ under those conditions

**Algorithm 2** Parametric LCM and GCD

**Require:** $f,g\in\mathbb{K}[\mathbf{a},\mathbf{x}]$
**Ensure:** CIS for $\mathrm{LCM}(f,g)$ and $\mathrm{GCD}(f,g)$
1: $I\leftarrow\langle f\rangle, J\leftarrow\langle g\rangle$
2: Compute CIS of $I$ and $J$ via INTERSECT$(I,J)=\{(N_i,W_i,G_i)\}_{i=1}^{l}$
3: **for** each branch $(N_i,W_i,G_i)$ **do**
4: Let $h_i$ be a generator of $\langle G_i\rangle$; ($G_i$ is principal)
5: $\mathrm{LCM}_i\leftarrow h_i$
6: $\mathrm{GCD}_i\leftarrow(f\cdot g)/h_i$
7: **end for**
8: **return** $\{(N_i,W_i,\mathrm{LCM}_i,\mathrm{GCD}_i)\}_{i=1}^{l}$

**Theorem 4.3.** *Let $f,g\in\mathbb{K}[\mathbf{a},\mathbf{x}]$ be parametric polynomials. Then Algorithm 2 terminates, and its output is an LCM/GCD system for $f$ and $g$.*

*Proof.* Termination follows from the termination of the underlying parametric Gröbner system computation (Algorithm 1), which is known to terminate for polynomial ideals. For correctness, observe that Algorithm 1 produces a CIS

$\{(N_i, W_i, G_i)\}$ for $\langle f\rangle \cap \langle g\rangle$. Since $\langle f\rangle$ and $\langle g\rangle$ are principal, each $G_i$ generates a principal ideal; let $h_i$ be a generator of $\langle G_i\rangle$. By construction of the CIS, for any specialization $\sigma$ satisfying $(N_i, W_i)$,

$$\langle \sigma(h_i)\rangle = \langle \sigma(f)\rangle \cap \langle \sigma(g)\rangle.$$

Hence $\sigma(h_i)$ is a least common multiple of $\sigma(f)$ and $\sigma(g)$ up to a unit in $\mathbb{K}'[\mathbf{x}]$.
Define $\mathrm{LCM}_i = h_i$ and $\mathrm{GCD}_i = (f \cdot g)/h_i$ (where division is performed in the fraction field $\mathbb{K}(\mathbf{a})[\mathbf{x}]$ and yields a polynomial due to the identity $\mathrm{GCD} \cdot \mathrm{LCM} = f \cdot g$). Then for any such $\sigma$; $\sigma(\mathrm{GCD}_i) \cdot \sigma(\mathrm{LCM}_i) = \sigma(f) \cdot \sigma(g)$, and $\sigma(\mathrm{GCD}_i)$ is a greatest common divisor of $\sigma(f)$ and $\sigma(g)$. Thus each $(N_i, W_i, \mathrm{LCM}_i, \mathrm{GCD}_i)$ satisfies the conditions of Definition 4.2, and $\mathcal{L}$ is an LCM/GCD system. □

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# Some notes on the multiplicative commutators of a division ring

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**Abstract**

Let $D$ be a division ring with center $F$ and $T(D)$ be the vector space over $F$ generated by all multiplicative commutators in $D$. In this paper, it is shown that if $D$ is algebraic over $F$ and $Char(D) = 0$, then $D = T(D)$. Among other results it is shown that in characteristic zero if $T(D)$ is algebraic over $F$, then $D$ is algebraic over $F$.

## 1. Introduction

Throughout this paper $D$ is a division ring with center $F$. An element of the form $xyx^{-1}y^{-1} \in D$ is called a *multiplicative commutator*, and $D'$ and $[D, D]$ denote the multiplicative and additive commutator subgroup of $D$, respectively. Also we denote by $T(D)$ the vector space generated by the set of all multiplicative commutators of $D$ over $F$. An element $a \in D$ is said to be *algebraic* over $F$ if $a$ satisfies a non-zero polynomial in $F[x]$. A set $S \subseteq D$ is called *algebraic* if each of its elements is algebraic over $F$. When $K$ is a finite dimensional extension of $F$, then we denote by $Tr_{K/F}$, the regular trace of $K$ over $F$. If $a \in D$, then $F(a)$ denotes the subfield of $D$ generated by $F$ and $\{a\}$. An element $a \in D$ is said to be *radical* over $F$ if there exists an integer $n = n(a)$ such that $a^n \in F$. A set $S \subseteq D$ is said to be radical over $F$, if each element of $S$ is radical over $F$. Also we denote by $Char(D)$ the characteristic of $D$.

The division ring generated by additive commutators or multiplicative commutators of $D$ is the whole $D$ [3, pp. 205, 211]. In the algebraic and zero characteristic case, it was proved that $D$ is generated as a vector space over the center by the union of its additive commutators and the unity, see [1]. As a generalization we provide another generating structure for a division ring as a vector space generated by all multiplicative commutators over its center. We prove this in the algebraic case when the characteristic is zero. Besides, consider a special property P in a ring (for example commutativity, algebricity or some finiteness conditions), there are a lot of studies in literature to specify a set or a substructure S, such that the property P for S implies the property P for the whole ring, for instance see [1, 2] and [4–6].
Most of these studies have focused on the set of multiplicative and additive commutators and their generating subgroups $D'$ and $[D, D]$ in division rings. We show that the subspace $T(D)$ reflects some properties to the whole division

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ring.

## 2. Main results

We start this section with the following lemma.

**Lemma 2.1.** *Let $D$ be a division ring with center $F$. Then for each algebraic element $a \in D$, there exists an element $d \in T(D) \cap F(a)$ such that $Tr_{F(a)/F}(a) = ad$.*

The following lemma is an immediate consequence of the previous lemma.

**Lemma 2.2.** *Let $D$ be a division ring with center $F$ and $a \in D$ be an algebraic element over $F$ such that $Tr_{F(a)/F}(a) \neq 0$. Then $a^{-1} \in T(D)$.*

Now, we have the following result.

**Theorem 2.3.** *Let $D$ be a division ring with center $F$. Then $T(D)$ contains all separable elements of $D$.*

The following is a simple corollary of the previous result.

**Theorem 2.4.** *Let $D$ be an algebraic division ring over its center $F$. If $Char(D) = 0$, then $T(D) = D$.*

A theorem due to Kaplansky [3, p. 246] states that if $D$ is radical over $F$, then $D = F$. There are various kinds of generalizations of this theorem. The next theorem is another one.

**Theorem 2.5.** *Let $D$ be a division ring with center $F$. If $T(D)$ is radical over $F$, then $D = F$.*

Mahdavi-Hezavehi together with his colleagues in [6] conjectured that one could conclude the algebracity of a division ring over its center from the algebracity of all its multiplicative commutators. They were able to deduce the conjecture is true in the case $D'$ is algebraic over the center, but in general the problem is still open. Also it is proved that when $[D, D]$ is algebraic over $F$, then $D$ is algebraic over $F$, see [1]. In what follows we give an affirmative answer to the conjecture when the center of division ring is uncountable.

**Theorem 2.6.** *Let $D$ be a division ring with uncountable center $F$ such that all of its multiplicative commutators are algebraic over $F$. Then $D$ is algebraic over $F$. In particular, if $D$ is a division ring with uncountable center $F$ and $T(D)$ is algebraic over $F$, then $D$ is algebraic over $F$.*

Now, a question is naturally proposed: Whether a division ring is finite dimensional when $T(D)$ is a finite dimensional vector space over $F$? This is the content of the following theorem. This theorem is along of some analogous results which state that if each element of a specific set has a minimal polynomial of bounded degree, then $D$ is finite dimensional over $F$ [6]. Note that finite dimensionality of $T(D)$ as a vector space does not imply that all elements of $T(D)$ or multiplicative commutators are algebraic over $F$.

**Theorem 2.7.** *Let $D$ be a division ring with center $F$. If $dim_F T(D) = n < \infty$, then $dim_F D < \infty$.*

Now, we would like to prove that if $D$ is a division ring with center $F$ and $T(D)$ is algebraic over $F$, then $D$ is algebraic over $F$. Before stating the proof we need a lemma.

**Lemma 2.8.** *Let $D$ be a division ring with center $F$, $T(D)$ is algebraic over $F$ and $Char(D) = 0$. Then for any two algebraic elements $a, b \in D$, the set $S = \{a + b, aba, a^2b\}$ is algebraic over $F$.*

**Theorem 2.9.** *Let $D$ be a division ring with center $F$ and $Char(D) = 0$. Then $T(D)$ is algebraic over $F$ if and only if $D$ is algebraic over $F$.*

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# Cohen-Macaulay $r$-partite graphs with minimal clique cover

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**Abstract**

In this paper, we give some necessary conditions for an $r$-partite graph such that the edge ring of the graph is Cohen-Macaulay. It is proved that if there exists a cover of an $r$-partite Cohen-Macaulay graph by disjoint cliques of size $r$, then such a cover is unique.

## 1. Introduction and preliminaries

A simple graph is an undirected graph that has no loop and multiple edge. A finite graph is denoted by $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. Let $|V(G)| = n$. We use $[n] = \{1, 2, \dots, n\}$ as vertices of $G$. The complementary graph of $G$ is the graph $\bar{G}$ on $[n]$ whose edge set $E(\bar{G})$ consists of those edges $\{i, j\}$ which are not in $E(G)$. An independent set of vertices is a set of pairwise nonadjacent vertices. An $r$-partite graph is a graph that the set of its vertices can be partitioned into $r$ disjoint subsets such that each set is independent. A subset $A \subset [n]$ is a minimal vertex cover of $G$ if (i) each edge of $G$ is incident with at least one vertex in $A$, and (ii) there is no proper subset of $A$ with property (i). If $A$ satisfies condition (i) only, then $A$ is called a vertex cover for $G$. It is easy to check that any minimal vertex cover of a graph is complement set of a maximal independent set of the graph. A graph $G$ is called unmixed (well-covered) if any two minimal vertex covers of $G$ have the same cardinality. A clique in a graph is a set of pairwise adjacent vertices, and by an $r$-clique we mean a clique of size $r$. An $r$-matching is a set of pairwise disjoint $r$-cliques and a perfect $r$-matching is an $r$-matching which covers all vertices of $G$.
Let $\omega(G)$ denote the maximum size of cliques in $G$, which is called the clique number of $G$. Let $f : V(G) \to [k]$ be a map such that if $v_1$ is adjacent to $v_2$ then $f(v_1) \neq f(v_2)$. If such a map exists, we say that $G$ is colorable by $k$ colors. The smallest such $k$ is called the chromatic number of the graph and is denoted by $\chi(G)$. A graph $G$ is called perfect if $\omega(H) = \chi(H)$ for each induced subgraph $H$ of $G$. The class of perfect graphs plays an important role in graph theory and most of computations in this class can be done by fast algorithms.
Let $G$ be a graph on $[n]$. Let $S = K[x_1, \dots, x_n]$, the polynomial ring over a field $K$. The edge ideal $I(G)$ of $G$ is defined to be the ideal of $S$ generated by all square-free monomials $x_i x_j$ provided that $i$ is adjacent to $j$ in $G$. The quotient ring $R(G) = S/I(G)$ is called the edge ring of $G$.

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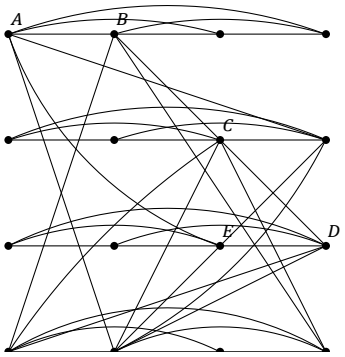


Fig. 1. A graph in class $\mathcal{G}$ which is not perfect

Let $R$ be a commutative ring with an identity. The depth of $R$, denoted by depth($R$), is the largest integer $r$ such that there is a sequence $f_1, \dots, f_r$ of elements of $R$ such that $f_i$ is not a zero-divisor in $R/(f_1, \dots, f_{i-1})$ for all $1 \leq i \leq r$, and $(f_1, \dots, f_r) \neq R$. Such a sequence is a called a regular sequence. The depth is an important invariant of a ring. It is bounded by another important invariant, the Krull dimension, the length of the longest chain of prime ideals in the ring. A ring $R$ is called Cohen-Macaulay if depth($R$)=dim($R$). A graph $G$ is called Cohen-Macaulay if the ring $R(G)$ is Cohen-Macaulay.

## 2. Main results

M. Estrada and R. H. Villarreal in [1] have proved that if $G$ is a Cohen-Macaulay bipartite graph and has at least one vertex of positive degree, then there is a vertex $v$ such that $\deg(v) = 1$. By $\deg(v)$ we mean the number of vertices adjacent to $v$. J. Herzog and T. Hibi in [2] have proved that a bipartite graph $G$ with parts $V_1$ and $V_2$ is Cohen-Macaulay if and only if, $|V_1| = |V_2|$ and there is an order on the vertices of $V_1$ and $V_2$ as $x_1, \dots, x_n$ and $y_1, \dots, y_n$ respectively, such that:
1) $x_i \sim y_i$ for $i = 1, \dots, n$,
2) if $x_i \sim y_j$, then $i \leq j$,
3) for each $1 \leq i < j < k \leq n$ if $x_i \sim y_j$ and $x_j \sim y_k$, then $x_i \sim y_k$.
R. Zaare-Nahandi in [4] has proved that a well-covered bipartite graph $G$ is Cohen-Macaulay if and only if there is a unique perfect 2- matching in $G$.
Let $\alpha(G)$ denote the maximum cardinality of independent sets of vertices of $G$. Let $\mathcal{G}$ be the class of graphs such that for each $G \in \mathcal{G}$ there are $k = \alpha(G)$ cliques in $G$ covering all its vertices. For each $G \in \mathcal{G}$ and cliques $Q_1, \dots, Q_k$ such that $V(Q_1) \cup \cdots \cup V(Q_k) = V(G)$, we may take $Q'_1 = Q_1$ and for $i = 2, \dots, k$, $Q'_i$ the induced subgraph on the vertices $V(Q_i) \setminus (V(Q_1) \cup \cdots \cup V(Q_{i-1}))$. Then $Q'_1, \dots, Q'_k$ are $k$ disjoint cliques covering all vertices of $G$. We call such a set of cliques, a basic clique cover of the graph $G$. Therefore any graph in the class $\mathcal{G}$ has a basic clique cover.

**Proposition 2.1.** *Let $G$ be an $r$-partite, unmixed and perfect graph such that all maximal cliques are of size $r$. Then $G$ is in the class $\mathcal{G}$.*

The converse of the above proposition is not true in general as shown in the following example.

**Example 2.2.** Let $G$ be the graph in Figure 1. Then $G$ is a graph in class $\mathcal{G}$ which is 4-partite, unmixed and all maximal cliques are of size 4. But $G$ is not perfect because the induced subgraph on $\{A, B, C, D, E\}$ is a cycle of length 5.

Let $H$ be a graph and $v$ be a vertex of $H$. Let $N(v)$ be the set of all vertices of $H$ adjacent to $v$.

**Theorem 2.3.** [3, Proposition 6.2.4] If $H$ is Cohen-Macaulay and $v$ is a vertex of $H$, then $H \setminus (v, N(v))$ is Cohen-Macaulay.

**Theorem 2.4.** [5] Let $G$ be an $r$-partite unmixed graph such that all maximal cliques are of size $r$. Then all parts have the same cardinality and there is a perfect 2-matching between each two parts.

Now, we present the main theorem of this paper which is a generalization of [1, Theorem 2.4].

**Theorem 2.5.** *Let $G$ be an $r$-partite graph in the class $\mathcal{G}$ such that each maximal clique is of size $r$. If $G$ is Cohen-Macaulay then there is a vertex of degree $r-1$ in $G$.*

Theorem 2.5 implies that the perfect $r$-matching in a Cohen-Macaulay $r$-partite graph is unique.

**Corollary 2.6.** *Let $G$ be an $r$-partite graph in the class $\mathcal{G}$ such that all maximal cliques are of size $r$. If $G$ is Cohen-Macaulay, then there is a unique perfect $r$-matching in $G$.*

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# The Relation Between Graphs and Quasi-Forest Simplicial Complexes

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**Abstract**

Let $G$ be a chordal graph, namely each cycle of four or more vertices have a chord. It is shown that if $\bar{G}$ is a chordal graph then $I(G)$ is weakly polymatroidal, where the Stanley-Reisner ideal associated to $\Delta$ and edge ideal $I(G)$ are equal. It is shown that if $\Delta$ is a quasi-forest simplicial complex, then the Stanley-Reisner ideal of $\Delta$ is weakly polymatroidal and Stanley's conjecture holds for $K[\Delta^v]$. We prove that if $\bar{G}$ is a chordal graph then $\bar{G}$ and $\Delta(G)$ are vertex decomposable. Also we show that if $\Delta$ is a quasi-forest simplicial complex then Castelnuovo-Mumford regularity and initial degree of the Stanley-Reisner ideal associated to $\Delta$ are equal.

## 1. Introduction and Preliminaries

Monomials are the link between commutative algebra and combinatorics. If $\Delta$ is a simplicial complex on the vertex set $[n]$, then one can associate the Stanley-Reisner ideal $I_\Delta$ to this complex, where the generators correspond to the non-faces of $\Delta$. In combinatorial commutative algebra, using the combinatorial properties of simplicial complexes, researchers study the algebraic properties of ideals associated with these combinatorial objects. One of the interesting problems in combinatorial commutative algebra that has attracted much attention is Richard Stanley's conjectures. Suppose $R$ is a $\mathbb{N}^n$-graded ring and M is a $\mathbb{Z}^n$-graded R-module. In 1982, Stanley conjectured that $depth(M) \leq sdepth(M)$. A year later, in his book [7], Stanley conjectured that every Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi [2] showed that the partitionability conjecture of simplicial complexes is a special case of Stanley's first conjecture. The intriguing nature of Stanley's conjectures motivated us to investigate the Stanley conjecture on the Alexander duals of quasi-forest simplicial complexes.
In this Section, we reference some basic definitions and theorems that we will need later. In Section 2, we first prove that if $\Delta$ is a quasi-forest simplicial complex, then the Stanley-Reisner ideal associated with $\Delta$ is weakly polymatroidal, and then we study the Stanley conjecture on the Alexander dual of the quasi-forest simplicial complex $\Delta$. We then consider the graph $G$ such that the edge ideal $I(G)$ equals the Stanley-Reisner ideal associated with the simplicial

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complex $\Delta$. In this case, we prove that if the complement of the graph $G$ is a chordal graph, then the edge ideal $I(G)$ is weakly polymatroidal. Finally, we prove that if $\bar{G}$ is a chordal graph, then both $\bar{G}$ and $\Delta(G)$ are vertex decomposable. In this article, we denote the polynomial ring over the field $K$ by $S = K[x_1, x_2, \dots, x_n]$. A simplicial complex $\Delta$ on the vertex set $[n]$ is a collection of subsets of $[n]$ such that if $F \in \Delta$, then all subsets of $F$ are also in $\Delta$. Each member of $\Delta$ is called a face. The maximal faces of $\Delta$ under inclusion are called facets. The set of all facets of $\Delta$ is denoted by $\mathcal{F}(\Delta)$. The dimension of a face $F$ is denoted by $dimF$ and is defined as $dimF = |F| - 1$, where $|F|$ is the number of vertices in the face $F$. The maximum dimension of the facets of $\Delta$ is called the dimension of the simplicial complex. If all facets of $\Delta$ have the same dimension, then $\Delta$ is called pure. A simplicial complex $\Delta$ with facets $F_1, F_2 \dots, F_t$ is denoted by $\Delta = \langle F_1, \dots, F_t \rangle$. A simplicial complex with only one facet is called a simplex. For a face $F$ in $\Delta$, the sets $link_\Delta F$ and $\Delta \setminus F$ are defined as $link_\Delta F = \{G \in \Delta : F \cap \Delta = \emptyset\ ,\ F \cup G \in \Delta\}$ and $\Delta \setminus F = \{G \in \Delta : F \nsubseteq G\}$.
The Stanley-Reisner ideal of a simplicial complex $\Delta$ on the vertex set $[n]$ is denoted by $I_\Delta$, and $I_\Delta$ is defined as the ideal generated by all square-free monomials corresponding to the non-faces of $\Delta$. A simplicial complex $\Delta$ is called Cohen-Macaulay if the ring $K[x_1, x_2, \dots, x_n]/I_\Delta$ is Cohen-Macaulay. The facet ideal of $\Delta$ is denoted by $I(\Delta)$ and is defined as the ideal generated by square-free monomials of the form $x_F = \prod_{i \in F} x_i$, where $F$ is a facet of $\Delta$. The complement of a facet $F$ is denoted by $F^c$ and is defined as $[n] \setminus F$. The complement of the simplicial complex $\Delta = \langle F_1, \dots, F_t \rangle$ is defined as $\Delta^c = \langle [\![F_1]\!]^c, \dots, [\![F_t]\!]^c \rangle$. The Alexander dual of $\Delta$ is defined as $\Delta^\vee = \{F^c : F \notin \Delta\}$.

**Example 1.1.** Suppose $\Delta$ is a simplicial complex on the vertex set [5] and $\mathcal{F}(\Delta) = \{\{1,2,3\}, \{2,3,4\}, \{4,5\}\}$. Then $\Delta^\vee = \langle \{2,3,5\}, \{2,3,4\}, \{1,3,4\}, \{1,2,4\} \rangle$ and $I_\Delta = (x_1x_4, x_1x_5, x_2x_5, x_3x_5)$.

If $\Delta$ is a simplicial complex, then $I_{\Delta^\vee} = I(\Delta^c)$ [4]. A simplicial complex $\Delta$ on the vertex set $[n]$ is vertex decomposable if $\Delta$ is a simplex or there exists a vertex $v$ such that both $\Delta \setminus v$ and $[\![link]\!]_\Delta v$ are vertex decomposable, and no face of $[\![link]\!]_\Delta v$ is a facet of $\Delta \setminus v$. In this case, the vertex $v$ is called a shedding vertex. A simplicial complex $\Delta$ is called $k$-decomposable (recursively) if $\Delta$ is a simplex or there exists a shedding face $F$ such that $dim(F) \leq k$ and both $\Delta \setminus F$ and $[\![link]\!]_\Delta F$ are $k$-decomposable. It can be observed that every 0-decomposable simplicial complex is vertex decomposable. Let $\Delta$ be a simplicial complex. A facet $F$ of $\Delta$ is called a leaf if $F$ is the only facet of $\Delta$ or there exists another facet $H$ in $\Delta$ such that $H \neq F$ and for every $G \in F(\Delta)$ with $G \neq F$, it holds that $F \cap G \subseteq F \cap H$. In this case, the facet $H$ is called a branch of $F$ in $\Delta$.

**Example 1.2.** Suppose $\Delta$ is a simplicial complex on the vertex set [7] and $\Delta = \langle \{1,2,3\}, \{1,3,4\}, \{3,4,5\}, \{4,6,7\} \rangle$. Then $F = \{3,4,5\}$ is a leaf of $\Delta$ and $H = \{1,3,4\}$ is a branch of $F$ in $\Delta$.

A simplicial complex $\Delta$ is called a quasi-forest if the facets of $\Delta$ can be ordered as $F_1, F_2, \dots, F_m$ such that for each $i = 2, \dots, m$ the facet $F_i$ is a leaf of $[\![\langle F_1, \dots, F_{i-1} \rangle]\!]$.

**Lemma 1.3.** [8] *A simplicial complex $\Delta$ is a quasi-forest if and only if $pd(I(\Delta^c)) = 1$.*

A simplicial complex $\Delta$ is called a flag if all its minimal non-faces have two members. In other words, $I_\Delta$ is generated by square-free monomials of degree two.

**Proposition 1.4.** [4] *Every quasi-forest simplicial complex is a flag complex.*

## 2. Main Results

In this section, we first prove that if $\Delta$ is a quasi-forest simplicial complex, then the Stanley-Reisner ideal associated with $\Delta$ is weakly polymatroidal, and then we show that the Stanley conjecture holds for $K[\Delta^\vee]$. We also examine other results. Before proving the main result of this article, we need a series of definitions and theorems. An ideal $I$ generated by monomials is called weakly polymatroidal if for any two square-free monomials $u = [\![x_1]\!]^{a_1} \cdots [\![x_n]\!]^{a_n} > v = x_1^{b_1} \cdots [\![x_n^{b_n}$ in $G(I)$ such that $a_1 = b_1, \dots, a_{t-1} = b_{t-1}$ and $a_t > b_t$, there exists $j > t$ such that $x_t(v/x_j) \in I$.

**Example 2.1.** Suppose $I = (x_1x_3x_5, x_1x_3x_6, x_1x_4x_6, x_2x_4x_6)$ is a square-free monomial ideal in the ring $K[x_1, \dots, x_6]$. Then $I$ is weakly polymatroidal.

**Theorem 2.2.** [6] *Let $\Delta$ be a $d$-dimensional simplicial complex on the vertex set $[n]$. Then $\Delta$ is $k$-decomposable if and only if $I_{\Delta^\vee}$ is a $k$-decomposable ideal where $k \leq d$.*

**Theorem 2.3.** [5] *If $I \subset R$ is a weakly polymatroidal ideal, then $I$ is a 0-decomposable ideal.*

**Lemma 2.4.** [3] *If $I$ is a square-free monomial ideal generated by quadratic forms with a linear resolution, then after suitable indexing of variables, we have the following property: If $x_i x_j \in I$ where $i \neq j$, $k > i$, and $k > j$ then $x_i x_k \in I$ or $x_j x_k \in I$.*

Now we are ready to prove the main result of this article.

**Theorem 2.5.** *Let $\Delta$ be a simplicial complex that is not a simplex. If $\Delta$ is a quasi-forest, then $I_\Delta$ is weakly polymatroidal.*

*Proof.* We know that every quasi-forest simplicial complex is a flag complex. Thus, $I_\Delta$ is generated by square-free monomials of degree two. So $ht(I_{\Delta^\vee}) = 2$. Since $\Delta$ is a quasi-forest, by Lemma 1.3, we have $pd(K[\Delta^\vee]) = 2$. Therefore, $\Delta^\vee$ is Cohen-Macaulay and $I_\Delta$ has a linear resolution. Suppose the monomials in $G(I_\Delta)$ are lexicographically ordered such that $x_n > x_{n-1} > \cdots > x_1$. We now prove that $I_\Delta$ is weakly polymatroidal. Suppose $u = x_s x_t > v = x_i x_j$ are square-free monomials in $G(I_\Delta)$ such that $s < t$ and $i < j$. We will show that $t \geq j$. If $t = j$ then $x_s(v/x_i) = u \in G(I_\Delta)$. If $t > j$ then by Lemma 2.4, $x_i x_t \in G(I_\Delta)$ or $x_j x_t \in G(I_\Delta)$, which completes the proof. □

Herzog, Soleyman Jahan and Yassemi [2] showed that the conjecture of partitionability of simplicial complexes is a special case of Stanley's first conjecture. We know that every vertex decomposable simplicial complex leads to shellability, and every shellable complex is partitionable and satisfies Stanley's first conjecture.

**Corollary 2.6.** *If $\Delta$ is a quasi-forest simplicial complex, then the Stanley conjecture holds for $K[\Delta^\vee]$.*

*Proof.* Since $\Delta$ is a quasi-forest, by Theorem 2.5, $I_\Delta$ is weakly polymatroidal. Thus, by Theorem 2.3, the ideal $I_\Delta$ is a 0-decomposable ideal and by Theorem 2.2, the complex $\Delta^\vee$ is 0-decomposable, indicating that $\Delta^\vee$ is vertex decomposable, from which it follows that the Stanley conjecture holds for $K[\Delta^\vee]$. □

Suppose $G$ is a simple graph such that $I_\Delta = I(G)$. Herzog, Hibi and Zheng [6] proved that if $\bar{G}$ is a chordal graph, then $\Delta$ is a quasi-forest, and vice versa.

**Corollary 2.7.** *Suppose $G$ is a graph such that $I_\Delta = I(G)$ is an edge ideal. If $\bar{G}$ is a chordal graph, then $I(G)$ is weakly polymatroidal.*

*Proof.* Since $\bar{G}$ is a chordal graph, $\Delta$ is a quasi-forest simplicial complex. Thus, by Theorem 2.5, $I(G)$ is weakly polymatroidal. □

A graph $G$ is called vertex decomposable if $Ind(G)$ is vertex decomposable. Dochtermann and his colleagues [1] proved that if $G$ is a chordal graph, then $Ind(G)$ is vertex decomposable. Based on this, we have the following theorem.

**Theorem 2.8.** *If $\bar{G}$ is a chordal graph, then both $\bar{G}$ and $\Delta(G)$ are vertex decomposable.*

*Proof.* By Theorem 4-1 in [1], $Ind(\bar{G})$ is vertex decomposable. Moreover, it can be easily seen that $\Delta(G) = Ind(\bar{G})$. Thus, $\bar{G}$ and $\Delta(G)$ are both vertex decomposable. □

Suppose $M$ is a finitely generated graded $R$-module. The Castelnuovo-Mumford regularity of $M$ is defined as $regM = max\{j - i \mid \beta_{i,i+j}(M) \neq 0\}$. Terai [5] defined the initial degree of the module M of the form $indeg(M) = min\{j - i \mid M_j \neq 0\} = min\{j - i \mid \beta_{0,j}(M) \neq 0\}$. Also Terai [5] proved that if $\Delta$ is a simplicial complex on the vertex set $[n]$, then $reg(I_\Delta) - indeg(I_\Delta) = dim\,(K[\Delta^\vee]) - depth(K[\Delta^\vee])$. Now we can show the following result.

**Corollary 2.9.** *If $\Delta$ is a quasi-forest simplicial complex on the vertex set $[n]$, then $reg(I_\Delta) = indeg(I_\Delta)$.*

*Proof.* In the proof of Theorem 2.5, we showed that $\Delta^\vee$ is Cohen-Macaulay. Thus, $\dim(K[\Delta^\vee]) = depth(K[\Delta^\vee])$. On the other hand, we know that $reg(I_\Delta) - indeg(I_\Delta) = dim\,(K[\Delta^\vee]) - depth(K[\Delta^\vee])$. Therefore, $reg(I_\Delta) = indeg(I_\Delta)$. □

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# Some notes on the Nakayama Lemma in the Category **Act-$S$**

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**Abstract**

In this article, we shall investigate some forms of Nakayama's lemma in the category of right acts over a given monoid $S$ with identity 1. For this, we introduce quasi-strongly faithful property for an $S$-act.

## 1. Introduction

Nakayama's Lemma was first discovered in the special case of ideals in a commutative ring by W. Krull and then in general by G. Azumaya [4]. The lemma is named after the Japanese mathematician T. Nakayama and introduced in its present form in [8]. It is a significant tool in algebraic geometry, because it allows local data on algebraic varieties, in the form of modules over local rings, to be studied pointwise as vector spaces over the residue field of the ring.
Let $S$ be a monoid with identity 1. Recall that a *(right) $S$-act* is a nonempty set $A$ equipped with a map $\mu : A \times S \to A$ called its action, such that, denoting $\mu(a, s)$ by $as$, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s, t \in S$. An element $\theta \in A$ is called a *zero element* of $A$ if $\theta s = \theta$ for any $s \in S$. Let $A$ be an $S$-act and $B \subseteq A$ a non-empty subset. Then $B$ is called a *subact* of $A$ if $bs \in B$ for all $s \in S$ and $b \in B$. In particular, if $I$ is a (proper) ideal of $S$, then $AI := \{as \mid a \in A, s \in I\}$ is a subact of $A$. The category of all $S$-acts, with action-preserving ($S$-act) maps ($f : A \to B$ with $f(as) = f(a)s$, for $s \in S$, $a \in A$) between them, is denoted by **Act-$S$**. Clearly $S$ itself is an $S$-act with its operation as the action. Throughout this paper, $S$ is a monoid with at least a right non-invertible element, all $S$-acts will be right $S$-acts and all ideals of $S$ are right ideals, zero element of an $S$-act, if it exists, is unique. If $S$-act $A$ has a unique zero element $\theta$, then $\theta \in B$ for any subact $B$ of $A$. It is known that the set

$$\{s \mid s \text{ is a right non-invertible element of } S\}$$

is the only maximal right ideal of $S$. In this note, we reserve $\mathfrak{M}$ to denote, always, this unique maximal right ideal of $S$. For more information on $S$-acts we refer the reader to [7].

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We therefore consider Nakayama's Lemma in **Act**-$S$ where $S$ is a monoid with a unique two-sided maximal ideal $\mathfrak{M}$. In [7, Example 3.18.10], the authors present a monoid $S$ in which $\mathfrak{M}$ is not a two-sided ideal. Moreover, there are examples of monoids $S$ that are not commutative, but their maximal ideals are two-sided. For example, given $S = (M_n(\mathbb{R}), \cdot)$, the monoid of all $n \times n$ matrices with real number entries under usual multiplication of matrices. Since $ab = 1$ implies $ba = 1$ for $a, b \in S$, the unique maximal ideal of $S$ is two-sided (see Lemma 2.1 of this paper). Besides, there are many examples of finitely generated $S$-acts $A$ with a zero element $\theta$ in which for a proper ideal $I$ of $S$, $AI = A$, but $A \neq \{\theta\}$; take any monoid $S$ and an arbitrary finite set $A$ with $|A| > 1$. Then $A$ becomes a right $S$-act by trivial action, i.e., $as = a$ for all $a \in A, s \in S$. Therefore, $AI = A$ for every proper ideal $I$, although, $A \neq \{\theta\}$. As another example of this situation, we will provide in Example **??** of next section, a finitely generated $\mathbb{N}$-act with a unique zero for which $A(2\mathbb{N}) = A$, but $A \neq \{\theta\}$.
In this paper, we are going to study some forms of Nakayama's Lemma on **Act**-$S$. More precisely, in Theorem 2.8 of next section, we show that equality $AI = A$ for some proper ideal $I$ of $S$ implies $A = \{\theta\}$, when $A$ is a finitely generated quasi-strongly faithful $S$-act with a unique zero element $\theta$ and $S$ is a monoid in which the unique maximal right ideal $\mathfrak{M}$ is two-sided.

## 2. Main results

In this section first we present a lemma which determines monoids in which their unique maximal ideals are two-sided.

**Lemma 2.1.** *Let $S$ be a monoid. Then the following statements are equivalent:*
1) $\mathfrak{M}$ *is a two-sided ideal of* $S$.
2) $st = 1$ *implies* $ts = 1$, *for all* $s, t \in S$.

The next lemma guarantees that every finitely generated $S$-act with a unique zero element has a maximal subact.

**Lemma 2.2.** *Let $S$ be a monoid and let $A$ be a finitely generated $S$-act with a unique zero element $\theta$ and $A \neq \{\theta\}$. Then every proper subact of $A$ is contained in a maximal subact. In particular, $A$ has a maximal subact.*

Now we are ready to state the first version of Nakayama's lemma.

**Theorem 2.3.** *Let $S$ be a monoid in which its unique maximal right ideal $\mathfrak{M}$ is two-sided. Moreover, let $A$ be an $S$-act and $B$ a maximal subact of $A$ in which there exists $a \in A \setminus B$ such that $\mathfrak{M} = \{s \in S \mid as \in B\}$. Then for every proper ideal $I$ of $S$ we have $AI \neq A$.*

The next example illustrates Theorem 2.3.

**Example 2.4.** Let $S = (\mathbb{N}, \cdot)$ be the monoid of natural numbers with the usual multiplication. Then $A = \{2, 3, \cdots\}$ is a subact of $\mathbb{N}$. Let $U$ be the set of all prime numbers. Then $U$ is a set of generating elements of $A$. Note that $U$ is the least generating set of $A$, i.e. $A$ is not finitely generated. The set $B = \{3, 4, \cdots\}$ is a maximal subact of $A$ in which $\mathfrak{M} = \{n \in \mathbb{N} \mid 2n \in B\}$. All assumptions of Theorem 2.3 hold for $A$. Therefore for any proper ideal $I$ of $S$, $AI \neq A$.

In the next lemma we will see that the second condition of Theorem 2.3 is equivalence to the implication that '$as = a$ implies $s \notin \mathfrak{M}$'. More precisely, we have:

**Lemma 2.5.** *Let $A$ be an $S$-act and $B$ a maximal subact of $A$. Then for every $a \in A \setminus B$ the following statements are equivalent:*
1) $\mathfrak{M} = \{s \in S \mid as \in B\}$.
2) $as = a$ *implies* $s \notin \mathfrak{M}$.

**Definition 2.6.** Let $A$ be an $S$-act. A nonzero element $a \in A$ is called *quasi-strongly faithful*, if for $s \in S$ the equality $as = a$ implies that $s \notin \mathfrak{M}$. One calls $A$ a *quasi-strongly faithful $S$-act* if all of its nonzero elements are quasi-strongly faithful.

**Proposition 2.7.** *Let $S$ be a monoid in which its unique maximal right ideal $\mathfrak{M}$ is two-sided.*
(i) *Let $A$ be an $S$-act and $B$ a maximal subact of $A$ such that there exists a quasi-strongly faithful element in $A \setminus B$. Then:*

a) $AI = A$ *if and only if* $I = S$;
b) *For every ideal* $I$ *of* $S$ *with* $I^2 = I$, $AI \cong A$ *if and only if* $I = S$.
(ii) *Let* $A$ *be a quasi-strongly faithful* $S$*-act and* $B$ *a maximal subact of* $A$. *Then* $AI = A$ *if and only if* $I = S$.

Now we state a second version of Nakayama's Lemma which is quite similar to module theory. Also, in the next corollary we give another similar consequence.

**Theorem 2.8.** *Let S be a monoid in which its unique maximal right ideal* $\mathfrak{M}$ *is two-sided. Let A be a finitely generated quasi-strongly faithful S-act with a unique zero element* $\theta$. *If* $AI = A$ *for some proper ideal I of S, then* $A = \{\theta\}$.

**Corollary 2.9.** Let S be a monoid in which its unique maximal right ideal $\mathfrak{M}$ is two-sided. Let $A$ be a finitely generated quasi-strongly faithful $S$-act. If $B \cup AI = A$ for some proper ideal $I$ of $S$ and some subact $B$ of $A$, then $A = B$.

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# Morphic Acts over Monoids

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**Abstract**

In this manuscript, we introduce and study morphic $S$-acts in the theory of acts over monoids. Thus we consider connection between morphic $S$-acts and Dedekind-finite $S$-acts.

## 1. Introduction

Let $S$ be a monoid with 0 and identity 1. Recall that a (*right*) $S$-*acts* is a non-empty set $A$ equipped with a map $\mu : A \times S \to A$ called its action, such that, denoting $\mu(a,s)$ by $as$, we have $a1 = a$ and $a(st) = (as)t$, for all $a \in A$, and $s,t \in S$. An element $\theta \in A$ is called a *zero* of $A$ if $\theta s = \theta$ for every $s \in S$. Let $A$ be an $S$-acts and $B \subseteq A$ a non-empty subset. Then $B$ is called a *subact* of $A$ if $bs \in B$ for all $s \in S$ and $b \in B$. The category of all $S$-acts, with $S$-homomorphisms is denoted by **Act**-$S$. Clearly $S$ itself is an $S$-acts with its operation as the action. An equivalence relation $\rho$ on an $S$-acts $A$ is called a *congruence* on $A$ if $a\rho a'$ implies $(as)\rho(a's)$ for $a,a' \in A$ and $s \in S$. Let $f : A \to B$ be an $S$-homomorphism. Then the kernel equivalence relation of $f$, $\ker f$, defined by $a(\ker f)a'$ if and only if $f(a) = f(a')$ for $a,a' \in A$, is an act congruence which is called the *kernel congruence* of $f$. Any subact $B \subseteq A$ defines the *Rees congruence* $\rho_B$ on A, by setting $a\rho_B a'$ if $a,a' \in B$ or $a = a'$. We denote the resulting factor act by $A/B$ and call it the *Rees factor* act of $A$ by the subact $B$. Clearly, $A/B$ has a zero which is the class consisting of $B$, all other classes are one-element sets. Let $A$ be an $S$-acts and $B$ be a subact of $A$. Then $B \times B \cup \Delta_A$ is a congruence on $A$ and throughout this paper is denoted by $[B]$.
Throughout this paper, $S$ is a monoid with 0, all $S$-acts will be right $S$-acts with unique zero $\theta$ and any subact of an $S$-acts contains the zero $\theta$. The category of all $S$-acts with unique zero $\theta$ and $S$-homomorphisms preserving zero (i.e., $f : A \to B$ with $f(as) = f(a)s$, for $s \in S$, $a \in A$ and $f(\theta) = \theta$), is denoted by $\mathbf{Act}_0$-$S$. Note that for each $A \in \mathbf{Act}_0$-$S$ we have $A0 = \{\theta\}$. Let $B$ be a subset of an $S$-acts $A$. Then the subact generated by the set $B$ is denoted by $\langle B \rangle$ which is the smallest subact of $A$ containing $B$.

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Recall that the category $\mathbf{Act}_0$-$S$ has coproducts of any non-empty families of $S$-acts. More precisely, if $I$ is a non-empty set and $X_i \in \mathbf{Act}_0$-$S, i \in I$ and $\theta_i$ be the zero of $X_i$, then by [6, Proposition 2.1.15] the coproduct of $\{X_i : i \in I\}$ is $(\dot{\cup}_{i\in I}(X_i\backslash\{\theta_i\}))\dot{\cup}\{\theta\}$ with $x_i s = \theta$, if $x_i s = \theta_i$ in $X_i$, $\theta s = \theta$ for $s \in S$. Likewise, we recall that an $S$-acts $A$ *decomposable* if there exist two subacts $B, C \subseteq A$ such that $A = B \cup C$ and $B \cap C = \{\theta\}$. In this case $A = B \cup C$ is called a *decomposition* of $A$. Otherwise, $A$ is called *indecomposable*. By [6, Theorem 1.5.10], every $S$-acts $A$ has a unique decomposition into indecomposable subacts. For more information about $S$-acts we encourage the reader to see [6].
Let $A$ be an $S$-acts. Then $A$ is called cohopfian (hopfian) if every injective endomorphism (surjective endomorphism) of $A$ is an automorphism [1]. By a *Dedekind-finite* $S$-acts we mean an $S$-acts $A$ which is not isomorphic to any proper direct summand of itself. Equivalently, $A$ is Dedekind-finite if and only if $B = \{\theta\}$ is the only $S$-acts for which $A\dot{\cup}B \cong A$.
The aim of this paper is to study morphic $S$-actss and examine their relationship with hopfian, Dedekind-finite and cancellable $S$-acts. At first we show that if $S$-acts $A$ is morphic then hopfian and cohopfian are equivalent. Then in corollary 2.7, we prove that if $K$ is a subact of morphic $S$-acts $A$ which is isomorphic to $A$, then $A = K$. This means morphicness implies Dedekind-finiteness.

## 2. Morphic Acts

We begin with a definition.

**Definition 2.1.** Let $A$ be an $S$-acts with unique zero $\theta$ and let $\alpha : A \to A$ be a homomorphism. Then $\alpha$ is called *morphic* if $\frac{A}{[\alpha(A)]} \cong \langle K_\alpha \rangle$, where $[\alpha(A)] = \alpha(A) \times \alpha(A) \cup \Delta_A$ and $K_\alpha = \{a \in A;\ \exists a' \neq a \quad \alpha(a) = \alpha(a')\} \cup \{\theta\}$ and $\langle K_\alpha \rangle$ is the subact generated by $K_\alpha \subseteq A$. One calls $A$ is *morphic act* if every endomorphism of $A$ is a morphic.

**Lemma 2.2.** *Let $A$ be an $S$-acts with unique zero $\theta$ and let $\alpha : A \to A$ be a homomorphism. Then the following statements are equivalent.*
i) $\frac{A}{[\alpha(A)]} \cong \langle K_\alpha \rangle$.
ii) *There exists a homomorphism $\beta : A \to A$ such that $\beta(A) = \langle K_\alpha \rangle$ and $ker\beta = [\alpha(A)]$.*
iii) *There exists a homomorphism $\beta : A \to A$ such that $\beta(A) \cong \langle K_\alpha \rangle$ and $ker\beta = [\alpha(A)]$.*

In order to see the connection between hopfian (cohopfian) $S$-acts and morphic $S$-acts, the following proposition is needed.

**Proposition 2.3.** *Let $\alpha : A \to A$ be a morphic. Then $\alpha$ is one to one if and only if $\alpha$ is onto.*

**Theorem 2.4.** *Let $A$ be a morphic $S$-acts. Then $A$ is hopfian if and only if $A$ is cohopfian.*

Recall [6] that an $S$-acts $A$ is strongly faithful if for $s, t \in S$ the equality $as = at$ for some nonzero element $a \in A$ implies that $s = t$.

**Lemma 2.5.** *Let $\alpha \in End(A)$ be a morphic. If $A$ is strongly faithful and $\sigma : A \to A$ is an automorphism then $\sigma\alpha$ and $\alpha\sigma$ are morphic.*

**Theorem 2.6.** *Let $A$ be an $S$-acts with zero element. Then the following statements are equivalent.*
i) *$A$ is morphic.*
ii) *If $\frac{A}{\rho} \cong N$, where $N \leq A$ and $\rho$ is a congruence relation on $A$ ,then $\frac{A}{[N]} \cong \langle A_\rho \rangle$.*

**Corollary 2.7.** *Let $A$ is a morphic $S$-acts and $K \leq A$.*
*i) If $A \cong K$ then $A = K$.*
*ii) If $\frac{A}{[K]} \cong A$ then $[K] = I_A$.*

**Proposition 2.8.** *Let $A$ be a morphic $S$-acts. The following statements are equivalent.*
*i) For any subact $K \leq A$ there exists an $S$-acts $N$ of $A$ such that $K \cong \frac{A}{[N]}$.*
*ii) For any subact $N$ of $A$ there exists an $S-$subact $K$ of $A$ such that $K \cong \frac{A}{[N]}$.*

**Theorem 2.9.** *Let $A$ be a morphic $S$-acts and one of the conditions of the proposition 2.8 holds. Then for any $N, N' \leq A$ we have $\frac{A}{[N]} \cong \frac{A}{[N']}$ if and only if $N \cong N'$.*

**Proposition 2.10.** *If $A$ be a morphic $S$-acts then $A$ is Dedekind-finite.*

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# Artinianness of Local Cohomology for Modules with Minimax Cousin Cohomologies

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**Abstract**

Let $R$ be a commutative Noetherian ring with non-zero identity and let $X$ be an arbitrary $R$-module. In this paper, we show that if all the cohomology modules of the Cousin complex for $X$ are minimax, then the $n$th local cohomology module of $X_{\mathfrak{p}}$ with respect to $\mathfrak{p}R_{\mathfrak{p}}$ is Artinian for any prime ideal $\mathfrak{p}$ of $R$ and for every integer $n$ less than $X$-height of $\mathfrak{p}$.

## 1. Introduction

Throughout $R$ will denote a commutative Noetherian ring with non-zero identity, $X$ an arbitrary $R$-module which is not necessarily finite (i.e., finitely generated), and $M$ a non-zero finite $R$-module. For basic results, notations and terminology not given in this paper, the reader is referred to [2, 3, 12].

The notion of Cousin complex for an $R$-module $X$ was introduced by Sharp [13] as an analogue of Hartshorne [8]. The Cousin cohomologies (i.e., the cohomology modules of the Cousin complex) have been studied by several authors. Sharp used the vanishing of Cousin cohomologies for investigating the Cohen-Macaulay property, Serre's $S_n$-condition, and the vanishing of Bass numbers of $X$ in [13–15]. Dibaei, Tousi, Jafari, and Kawasaki, in [4–7, 10], worked on the finiteness of Cousin cohomologies and, in [11, Proposition 9.3.5], Lipman, Nayak, and Sastry generalized their results to complexes on formal schemes.

Sharp, in [14, Theorem 2.4], showed that $M$ is Cohen-Macaulay if and only if the Cousin complex for $M$ is exact. Thus we get the following theorem.

**Theorem 1.1.** *0.Le1t $M$ be a non-zero finite $R$-module such that all the cohomology modules of the Cousin complex for $M$ are zero. Then the $n$th local cohomology module of $M_{\mathfrak{p}}$ with respect to $\mathfrak{p}R_{\mathfrak{p}}$ is zero for any prime ideal $\mathfrak{p}$ of $R$ and for every integer $n$ less than $X$-height of $\mathfrak{p}$.*

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Now, it is natural to ask whether a similar statement is valid if 'zero' is replaced by 'Artinian'.

**Question 1.2.** Let $X$ be an arbitrary $R$-module such that all the cohomology modules of the Cousin complex for $X$ are Artinian. Is the $n$th local cohomology module of $X_\mathfrak{p}$ with respect to $\mathfrak{p}R_\mathfrak{p}$ Artinian for any prime ideal $\mathfrak{p}$ of $R$ and for every integer $n$ less than $X$-height of $\mathfrak{p}$?

In this paper, we answer the above question. We show that Question 1.2 is true. We prove that if all the cohomology modules of the Cousin complex for $X$ are minimax, then the $n$th local cohomology module of $X_\mathfrak{p}$ with respect to $\mathfrak{p}R_\mathfrak{p}$ is Artinian for any prime ideal $\mathfrak{p}$ of $R$ and for every integer $n$ less than $X$-height of $\mathfrak{p}$.

## 2. Main Results

Suppose that $X$ is an arbitrary $R$-module. Recall that, for a prime ideal $\mathfrak{p}$ of $\operatorname{Supp}_R(X)$, the $X$-height of $\mathfrak{p}$ is defined to be $\operatorname{ht}_X(\mathfrak{p}) = \dim_{R_\mathfrak{p}}(X_\mathfrak{p})$. Let $i$ be a non-negative integer and set $\mathrm{U}^i(X) = \{\mathfrak{p} \in \operatorname{Supp}_R(X) : \operatorname{ht}_X(\mathfrak{p}) \geq i\}$. Then $\operatorname{Supp}_R(X) = \mathrm{U}^0(X)$, $\mathrm{U}^i(X) \supseteq \mathrm{U}^{i+1}(X)$, and $\mathrm{U}^i(X) - \mathrm{U}^{i+1}(X) = \{\mathfrak{p} \in \operatorname{Supp}_R(X) : \operatorname{ht}_X(\mathfrak{p}) = i\}$ is low with respect to $\mathrm{U}^i(X)$ (i.e., each member of $\mathrm{U}^i(X) - \mathrm{U}^{i+1}(X)$ is a minimal member of $\mathrm{U}^i(X)$ with respect to inclusion). The Cousin complex $\mathrm{C}_R(X)$ for $X$ is of the form

$$\mathrm{C}_R(X) : 0 \xrightarrow{d^{-2}} X \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} \cdots$$

where, for all $i \geq 0$,

- $X^i = \bigoplus_{\mathfrak{p} \in \mathrm{U}^i(X) - \mathrm{U}^{i+1}(X)} (\operatorname{Coker} d^{i-2})_\mathfrak{p}$ and
- $d^{i-1}(x) = \{(x + \operatorname{Im} d^{i-2})/1\}_{\mathfrak{p} \in \mathrm{U}^i(X) - \mathrm{U}^{i+1}(X)}$ for every element $x$ of $X^{i-1}$;

and satisfies

- $\operatorname{Supp}_R(X^i) \subseteq \mathrm{U}^i(X)$,
- $\operatorname{Supp}_R(\operatorname{Coker} d^{i-2}) \subseteq \mathrm{U}^i(X)$, and
- $\operatorname{Supp}_R(\mathrm{H}^{i-1}(\mathrm{C}_R(X))) \subseteq \mathrm{U}^{i+1}(X)$

(see [13] for details). Here, we use the notations $\mathrm{C}^{i-2} := \operatorname{Coker} d^{i-2}$ and $\mathrm{H}^{i-1} := \mathrm{H}^{i-1}(\mathrm{C}_R(X))$ for all $i \geq 0$.
Recall that an $R$-module $X$ is said to be minimax, if there is a finite submodule $X'$ of $X$ such that $X/X'$ is Artinian [16]. Thus the class of minimax modules includes all finite and all Artinian modules. Note that, for any short exact sequence

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

of $R$-modules, $X$ is minimax if and only if $X'$ and $X''$ are both minimax [1, Lemma 2.1].
For an $R$-module $X$ and an ideal $\mathfrak{a}$ of $R$, we write $\mathrm{H}^n_\mathfrak{a}(X)$ as the $n$th local cohomology module of $X$ with respect to $\mathfrak{a}$. An important problem in commutative algebra is to determine when $\mathrm{H}^n_\mathfrak{a}(X)$ is Artinian. In the main result of this paper, we show that for an arbitrary $R$-module $X$ (not necessarily finite) with minimax Cousin cohomologies, $\mathrm{H}^n_{\mathfrak{p}R_\mathfrak{p}}(X_\mathfrak{p})$ is Artinian for all prime ideals $\mathfrak{p}$ of $R$ and all $n < \operatorname{ht}_X(\mathfrak{p})$, which is related to the third of Huneke's four problems in local cohomology modules [9].

**Theorem 2.1.** *Let $X$ be an arbitrary $R$-module such that $\mathrm{H}^i$ is minimax for all $i$. Then $\mathrm{H}^n_{\mathfrak{p}R_\mathfrak{p}}(X_\mathfrak{p})$ is Artinian for all prime ideals $\mathfrak{p}$ of $R$ and all $n < \operatorname{ht}_X(\mathfrak{p})$.*

*Proof.* Let $\mathfrak{p}$ be a prime ideal of $R$ and let $n < \operatorname{ht}_X(\mathfrak{p})$. Let $i$ be an integer such that $0 \leq i \leq n$. By considering the short exact sequences

$$0 \longrightarrow \mathrm{C}^{i-2} / \mathrm{H}^{i-1} \longrightarrow X^i \longrightarrow \mathrm{C}^{i-1} \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{H}^{i-1} \longrightarrow \mathrm{C}^{i-2} \longrightarrow \mathrm{C}^{i-2} / \mathrm{H}^{i-1} \longrightarrow 0,$$

we have the long exact sequences

$$\begin{array}{lll} 0 \longrightarrow \square_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \square_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}^{i}) & \longrightarrow \square_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \mathrm{H}^{1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \mathrm{H}^{1}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}^{i}) & \longrightarrow \mathrm{H}^{1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \cdots & & \\ \longrightarrow \mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}^{i}) & \longrightarrow \mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}^{i}) & \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \cdots & & \end{array}$$

and

$$\begin{array}{lll} 0 \longrightarrow \square_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \square_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}) & \longrightarrow \square_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \mathrm{H}^{1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \mathrm{H}^{1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}) & \longrightarrow \mathrm{H}^{1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \cdots & & \\ \longrightarrow \mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}) & \longrightarrow \mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1}) & \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}) & \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2}/\mathrm{H}_{\mathfrak{p}}^{i-1}) \\ \longrightarrow \cdots. & & \end{array}$$

Since $\mathrm{H}^{i}$ is minimax for all $i$, there is a finite submodule $\mathrm{H}^{i'}$ of $\mathrm{H}^{i}$ such that $\mathrm{H}^{i}/\mathrm{H}^{i'}$ is Artinian. Therefore, from the exact sequence

$$\mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1'}) \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1}) \longrightarrow \mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1}/\mathrm{H}_{\mathfrak{p}}^{i-1'}),$$

$\mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{H}_{\mathfrak{p}}^{i-1})$ is Artinian for all $0 \leq i \leq n$. On the other hand, by [13, Lemma 4.5], for all $0 \leq i \leq n$ and all $j \geq 0$, $\mathrm{Ext}_{R}^{n-i}(R/\mathfrak{p}^{j}, X^{i}) = 0$ and so $\mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}}^{i}) \cong (\mathrm{H}_{\mathfrak{p}}^{n-i}(X^{i}))_{\mathfrak{p}} = 0$ because

$$\mathrm{H}_{\mathfrak{p}}^{n-i}(X^{i}) \cong \varinjlim_{j\geq 0} \mathrm{Ext}_{R}^{n-i}(R/\mathfrak{p}^{j}, X^{i}).$$

Thus, from the above long exact sequences, $\mathrm{H}^{n-i}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-2})$ is Artinian whenever $\mathrm{H}^{n-i-1}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{i-1})$ is Artinian. Hence $\mathrm{H}^{n}_{\mathfrak{p}R_{\mathfrak{p}}}(\mathrm{C}_{\mathfrak{p}}^{-2})$ is Artinian. Therefore $\mathrm{H}^{n}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ is Artinian. □

The following corollaries are immediate applications of the above theorem.

**Corollary 2.2.** *Let $X$ be an arbitrary $R$-module such that $\mathrm{H}^{i}$ is finite for all $i$. Then $\mathrm{H}^{n}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ is Artinian for all prime ideals $\mathfrak{p}$ of $R$ and all $n < \mathrm{ht}_{X}(\mathfrak{p})$.*

**Corollary 2.3.** *Let $X$ be an arbitrary $R$-module such that $\mathrm{H}^{i}$ is Artinian for all $i$. Then $\mathrm{H}^{n}_{\mathfrak{p}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ is Artinian for all prime ideals $\mathfrak{p}$ of $R$ and all $n < \mathrm{ht}_{X}(\mathfrak{p})$.*

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# Cofinite Properties of Generalized Local Cohomology Modules

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**Abstract**

Let $R$ be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of $R$, and $M$ a finitely generated $R$-module. In this paper, for an arbitrary $R$-module $X$ which is not necessarily finitely generated, we prove that $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is $\mathfrak{a}$-cofinite for all $i < f^2_{\mathfrak{a}}(M,X)$ and $\mathrm{Ass}_R(\mathrm{H}^{f^2_{\mathfrak{a}}(M,X)}_{\mathfrak{a}}(M,X))$ is finite if $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is finitely generated for all $i \leq f^2_{\mathfrak{a}}(M,X)$. Here, $f^2_{\mathfrak{a}}(M,X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq 2\}$ is the second finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$ and $f_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a finitely generated } R\text{-module}\}$ is the finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$.

## 1. Introduction

Throughout $R$ is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of $R$, $M$ is a finite (i.e., finitely generated) $R$-module, and $n$ is a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [8, 9].
Let $N$ be a finite $R$-module. Grothendieck, in [14], proposed the following conjecture.

**Conjecture 1.1.** $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^i_{\mathfrak{a}}(N))$ is a finite $R$-module for all $i$.

Hartshorne gave a counterexample to this conjecture in [15] and defined an $R$-module $X$ to be $\mathfrak{a}$-cofinite if $\mathrm{Supp}_R(X) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$ and $\mathrm{Ext}^i_R(R/\mathfrak{a},X)$ is a finite $R$-module for all $i$. He also asked the following question.

**Question 1.2.** When is $\mathrm{H}^i_{\mathfrak{a}}(N)$ an $\mathfrak{a}$-cofinite $R$-module for all $i$?

The following question is also an important problem in commutative algebra (see [17, Problem 4]).

**Question 1.3.** When is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(N))$ a finite set for all $i$?

The $n$th generalized local cohomology module $\mathrm{H}^n_{\mathfrak{a}}(X,Y) \cong \varinjlim_{i\in\mathbb{N}} \mathrm{Ext}^n_R(X/\mathfrak{a}^iX,Y)$ of arbitrary $R$-modules $X$ and $Y$ with respect to $\mathfrak{a}$ was introduced by Herzog in [16]. It is clear that $\mathrm{H}^n_{\mathfrak{a}}(R,Y)$ is just the $n$th ordinary local cohomology module $\mathrm{H}^n_{\mathfrak{a}}(Y)$ of arbitrary $R$-module $Y$ with respect to $\mathfrak{a}$. As generalizations of Conjecture and Questions 1.1-1.3, we have the following questions (see [6, Question 1.1] and [23, Question 2.7]).

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**Question 1.4.** When is $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_\mathfrak{a}(M,N))$ a finite $R$-module for all $i$?

**Question 1.5.** When is $\mathrm{H}^i_\mathfrak{a}(M,N)$ an $\mathfrak{a}$–cofinite $R$-module for all $i$?

**Question 1.6.** When is $\mathrm{Ass}_R(\mathrm{H}^i_\mathfrak{a}(M,N))$ a finite set for all $i$?

These questions were studied by several authors. In this direction, for an arbitrary $R$-module $X$ which is not necessarily finite and for a non-negative integer $t$, we show that if $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i \leq t$ and $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i < t$, then $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$-cofinite $R$-module for all $i < t$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M,X))$ is a finite $R$-module, and $\mathrm{Ass}_R(\mathrm{H}^t_\mathfrak{a}(M,X))$ is a finite set. This generalizes and improves all of the previous results concerning Conjecture and Questions 1.1-1.6 (see e.g., [1, 3–7, 10–13, 18–21, 24, 25]). Recall that an arbitrary $R$-module $X$ is said to be an $\mathrm{FD}_{<n}$ $R$-module if there exists a finite submodule $X'$ of $X$ such that $\dim_R(X/X') < n$ [1, 2]. We observe that $\mathrm{H}^i_\mathfrak{a}(M,X)$ is $\mathfrak{a}$-cofinite for all $i < f^2_\mathfrak{a}(M,X)$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{f^2_\mathfrak{a}(M,X)}_\mathfrak{a}(M,X))$ is finite, and $\mathrm{Ass}_R(\mathrm{H}^{f^2_\mathfrak{a}(M,X)}_\mathfrak{a}(M,X))$ is finite if $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is finite for all $i \leq f^2_\mathfrak{a}(M,X)$ where $f^2_\mathfrak{a}(M,X) = \inf\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq 2\}$ is the second finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$ and $f_\mathfrak{a}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_\mathfrak{a}(M,X) \text{ is not a finite } R\text{-module}\}$ is the finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$.

## 2. Main Results

The following lemma is needed in the proof of the main result of this paper.

**Lemma 2.1.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $s, t$ non-negative integers such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $t \leq i \leq s+t+1$, $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$–cofinite $R$-module for all $i < t$, and $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $t \leq i \leq s+t$. Then $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$–cofinite $R$-module for all $i \leq s+t$.*

*Proof.* We prove by using induction on $s$. Let $s = 0$. Since $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M,X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M,X))$ are finite from [22, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{H}^t_\mathfrak{a}(M,X)$ is $\mathfrak{a}$–cofinite by [1, Theorem 3.1]. Suppose that $s > 0$ and that $s-1$ is settled. It is enough to show that $\mathrm{H}^{s+t}_\mathfrak{a}(M,X)$ is $\mathfrak{a}$–cofinite because $\mathrm{H}^i_\mathfrak{a}(M,X)$ is $\mathfrak{a}$–cofinite for all $i \leq s+t-1$ from the induction hypothesis on $s-1$. By [22, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_\mathfrak{a}(M,X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_\mathfrak{a}(M,X))$ are finite. Thus $\mathrm{H}^{s+t}_\mathfrak{a}(M,X)$ is $\mathfrak{a}$–cofinite from [1, Theorem 3.1]. □

We prove the main result of this paper which generalizes and improves all of the previous results concerning Conjecture and Questions 1.1-1.6 (see e.g., [12, Theorem 1], [25, Theorem 1.1], [18, Theorem B], [7, Theorem 2.2], [24, Theorem 2.1], [3, Theorem 2.5], [4, Theorem 2.6], [10, Theorem 2.5], [20, Theorem 3.2], [6, Theorem 3.6], [19, Theorem 2.10], [13, Theorem 2.5], [5, Theorems 2.3 and 3.2], [1, Theorem 3.4], and [11, Theorem 1.2]).

**Theorem 2.2.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i \leq t$ and $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i < t$. Then the following statements hold true:*

(i) *$Y_i$ and $\mathrm{H}^i_\mathfrak{a}(M,X)/Y_i$ are $\mathfrak{a}$-cofinite $R$-modules for all $i < t$ and every $\mathrm{FD}_{<1}$ $R$-submodule $Y_i$ of $\mathrm{H}^i_\mathfrak{a}(M,X)$. In particular, $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$-cofinite $R$-module for all $i < t$;*

(ii) *Let $N$ be a finite $R$-module. Then $\mathrm{Ext}^j_R(N, Y_i)$, $\mathrm{Tor}^R_j(N, Y_i)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\mathfrak{a}(M,X)/Y_i)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\mathfrak{a}(M,X)/Y_i)$ are $\mathfrak{a}$-cofinite $R$-modules for all $i < t$, all $j$, and every $\mathrm{FD}_{<1}$ $R$-submodule $Y_i$ of $\mathrm{H}^i_\mathfrak{a}(M,X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\mathfrak{a}(M,X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\mathfrak{a}(M,X))$ are $\mathfrak{a}$-cofinite $R$-modules for all $i < t$ and all $j$;*

(iii) *$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M,X)/Y)$ is a finite $R$-module for every $\mathrm{FD}_{<1}$ $R$-submodule $Y$ of $\mathrm{H}^t_\mathfrak{a}(M,X)$. In particular, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M,X))$ is a finite $R$-module;*

(iv) *$\mathrm{Ass}_R(\mathrm{H}^t_\mathfrak{a}(M,X)/Y)$ is a finite set for every $\mathrm{FD}_{<1}$ $R$-submodule $Y$ of $\mathrm{H}^t_\mathfrak{a}(M,X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^t_\mathfrak{a}(M,X))$ is a finite set;*

(v) *Assume that* $\operatorname{Ext}_R^{t+1}(M/\mathfrak{a}M,X)$ *is finite. Then* $\operatorname{Ext}_R^1(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^t(M,X)/Y)$ *is a finite* $R$*-module for every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\operatorname{H}_\mathfrak{a}^t(M,X)$. *In particular,* $\operatorname{Ext}_R^1(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^t(M,X))$ *is a finite* $R$*-module.*

*Proof.* (i). Since $\operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^0(M,X))$ and $\operatorname{Ext}_R^1(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^0(M,X))$ are finite by [22, Corollary 2.2, Theorem 2.3, and Theorem 2.7], $\operatorname{H}_\mathfrak{a}^0(M,X)$ is $\mathfrak{a}$-cofinite from [1, Theorem 3.1], and so $\operatorname{H}_\mathfrak{a}^i(M,X)$ is $\mathfrak{a}$-cofinite for all $i<t$ by Lemma 2.1. Let $i<t$ and let $Y_i$ be an $\operatorname{FD}_{<1}$ $R$-submodule of $\operatorname{H}_\mathfrak{a}^i(M,X)$. Then $\operatorname{Hom}_R(R/\mathfrak{a},Y_i)$ is finite and so $Y_i$ is $\mathfrak{a}$-cofinite from [1, Lemma 3.3]. Thus $\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i$ is $\mathfrak{a}$-cofinite by the short exact sequence

$$0\longrightarrow Y_i\longrightarrow \operatorname{H}_\mathfrak{a}^i(M,X)\longrightarrow \operatorname{H}_\mathfrak{a}^i(M,X)/Y_i\longrightarrow 0.$$

(ii). It follows from the first part and [1, Theorem 3.7].
(iii). Let $Y$ be an $\operatorname{FD}_{<1}$ $R$-submodule of $\operatorname{H}_\mathfrak{a}^t(M,X)$. From the first part and [22, Corollary 2.2 and Theorem 2.3], $\operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^t(M,X))$ is finite. Thus $\operatorname{Hom}_R(R/\mathfrak{a},Y)$ is finite and so $Y$ is $\mathfrak{a}$-cofinite by [1, Lemma 3.3]. Hence, from the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^t(M,X))\longrightarrow \operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^t(M,X)/Y)\longrightarrow \operatorname{Ext}_R^1(R/\mathfrak{a},Y),$$

$\operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^t(M,X)/Y)$ is finite.
(iv). Follows by the third part and [9, Exercise 1.2.28].
(v). This is similar to the proof of the third part. □

**Corollary 2.3.** *Let* $M$ *be a finite* $R$*-module and let* $X$ *be an arbitrary* $R$*-module such that* $\operatorname{Ext}_R^i(M/\mathfrak{a}M,X)$ *is a finite* $R$*-module for all* $i\le f_\mathfrak{a}^2(M,X)$. *Then the following statements hold true:*

(i) $Y_i$ *and* $\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i<f_\mathfrak{a}^2(M,X)$ *and every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\operatorname{H}_\mathfrak{a}^i(M,X)$. *In particular,* $\operatorname{H}_\mathfrak{a}^i(M,X)$ *is an* $\mathfrak{a}$*-cofinite* $R$*-module for all* $i<f_\mathfrak{a}^2(M,X)$;

(ii) *Let* $N$ *be a finite* $R$*-module. Then* $\operatorname{Ext}_R^j(N,Y_i)$, $\operatorname{Tor}_j^R(N,Y_i)$, $\operatorname{Ext}_R^j(N,\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i)$, *and* $\operatorname{Tor}_j^R(N,\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i<f_\mathfrak{a}^2(M,X)$, *all* $j$, *and every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\operatorname{H}_\mathfrak{a}^i(M,X)$. *In particular,* $\operatorname{Ext}_R^j(N,\operatorname{H}_\mathfrak{a}^i(M,X))$ *and* $\operatorname{Tor}_j^R(N,\operatorname{H}_\mathfrak{a}^i(M,X))$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i<f_\mathfrak{a}^2(M,X)$ *and all* $j$;

(iii) $\operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X)/Y)$ *is finite for every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X)$. *In particular,* $\operatorname{Hom}_R(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X))$ *is a finite* $R$*-module;*

(iv) $\operatorname{Ass}_R(\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X)/Y)$ *is a finite set for every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X)$. *In particular,* $\operatorname{Ass}_R(\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X))$ *is a finite set;*

(v) *Assume that* $\operatorname{Ext}_R^{f_\mathfrak{a}^2(M,X)+1}(M/\mathfrak{a}M,X)$ *is finite. Then* $\operatorname{Ext}_R^1(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X)/Y)$ *is finite for every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X)$. *In particular, the* $R$*-module* $\operatorname{Ext}_R^1(R/\mathfrak{a},\operatorname{H}_\mathfrak{a}^{f_\mathfrak{a}^2(M,X)}(M,X))$ *is finite.*

*Proof.* Since $f_\mathfrak{a}^2(M,X)=\inf\{i\in\mathbb{N}_0:\operatorname{H}_\mathfrak{a}^i(M,X)\text{ is not an }\operatorname{FD}_{<2}\ R\text{-module}\}$, this follows from Theorem 2.2. □

**Corollary 2.4.** *Let* $M$ *and* $X$ *be finite* $R$*-modules such that* $\dim_R((M\otimes_R X)/\mathfrak{a}(M\otimes_R X))\le 1$ (*e.g.,* $\dim(R/\mathfrak{a})\le 1$). *Then the following statements hold true:*

(i) $Y_i$ *and* $\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i$ *and every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\operatorname{H}_\mathfrak{a}^i(M,X)$. *In particular,* $\operatorname{H}_\mathfrak{a}^i(M,X)$ *is an* $\mathfrak{a}$*-cofinite* $R$*-module for all* $i$;

(ii) *Let* $N$ *be a finite* $R$*-module. Then* $\operatorname{Ext}_R^j(N,Y_i)$, $\operatorname{Tor}_j^R(N,Y_i)$, $\operatorname{Ext}_R^j(N,\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i)$, *and* $\operatorname{Tor}_j^R(N,\operatorname{H}_\mathfrak{a}^i(M,X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i$, *all* $j$, *and every* $\operatorname{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\operatorname{H}_\mathfrak{a}^i(M,X)$. *In particular,* $\operatorname{Ext}_R^j(N,\operatorname{H}_\mathfrak{a}^i(M,X))$ *and* $\operatorname{Tor}_j^R(N,\operatorname{H}_\mathfrak{a}^i(M,X))$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i$ *and all* $j$;

(iii) $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)/Y_i)$ *is a finite set for all* $i$ *and every* $\mathrm{FD}_{<1}$ *R-submodule* $Y_i$ *of* $\mathrm{H}^i_{\mathfrak{a}}(M,X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X))$ *is a finite set for all* $i$.

*Proof.* It follows from Theorem 2.2. □

For the ordinary local cohomology modules, we have the following results.

**Corollary 2.5.** *Let* $X$ *be an arbitrary* $R$*-module and let* $t$ *be a non-negative integer such that* $\mathrm{Ext}^i_R(R/\mathfrak{a},X)$ *is a finite* $R$*-module for all* $i \leq t$ *and* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathrm{FD}_{<2}$ $R$*-module for all* $i < t$. *Then the following statements hold true:*

(i) $Y_i$ *and* $\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i < t$ *and every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* $R$*-module for all* $i < t$;

(ii) *Let* $N$ *be a finite* $R$*-module. Then* $\mathrm{Ext}^j_R(N,Y_i)$, $\mathrm{Tor}^R_j(N,Y_i)$, $\mathrm{Ext}^j_R(N,\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$, *and* $\mathrm{Tor}^R_j(N,\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i < t$, *all* $j$, *and every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ext}^j_R(N,\mathrm{H}^i_{\mathfrak{a}}(X))$ *and* $\mathrm{Tor}^R_j(N,\mathrm{H}^i_{\mathfrak{a}}(X))$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i < t$ *and all* $j$;

(iii) $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ *is a finite* $R$*-module for every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\mathrm{H}^t_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(X))$ *is a finite* $R$*-module;*

(iv) $\mathrm{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ *is a finite set for every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\mathrm{H}^t_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}^t_{\mathfrak{a}}(X))$ *is a finite set;*

(v) *Assume that* $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a},X)$ *is finite. Then* $\mathrm{Ext}^1_R(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ *is a finite* $R$*-module for every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\mathrm{H}^t_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ext}^1_R(R/\mathfrak{a},\mathrm{H}^t_{\mathfrak{a}}(X))$ *is a finite* $R$*-module.*

We write $f^2_{\mathfrak{a}}(X) = f^2_{\mathfrak{a}}(R,X)$ which is called the second finiteness dimension of $X$ with respect to $\mathfrak{a}$.

**Corollary 2.6.** *Let* $X$ *be an arbitrary* $R$*-module such that* $\mathrm{Ext}^i_R(R/\mathfrak{a},X)$ *is a finite* $R$*-module for all* $i \leq f^2_{\mathfrak{a}}(X)$. *Then the following statements hold true:*

(i) $Y_i$ *and* $\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i < f^2_{\mathfrak{a}}(X)$ *and every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* $R$*-module for all* $i < f^2_{\mathfrak{a}}(X)$;

(ii) *Let* $N$ *be a finite* $R$*-module. Then* $\mathrm{Ext}^j_R(N,Y_i)$, $\mathrm{Tor}^R_j(N,Y_i)$, $\mathrm{Ext}^j_R(N,\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$, *and* $\mathrm{Tor}^R_j(N,\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i < f^2_{\mathfrak{a}}(X)$, *all* $j$, *and every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ext}^j_R(N,\mathrm{H}^i_{\mathfrak{a}}(X))$ *and* $\mathrm{Tor}^R_j(N,\mathrm{H}^i_{\mathfrak{a}}(X))$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i < f^2_{\mathfrak{a}}(X)$ *and all* $j$;

(iii) $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ *is finite for every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Hom}_R(R/\mathfrak{a},\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ *is a finite* $R$*-module;*

(iv) $\mathrm{Ass}_R(\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ *is a finite set for every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ *is a finite set;*

(v) *Assume that* $\mathrm{Ext}^{f^2_{\mathfrak{a}}(X)+1}_R(R/\mathfrak{a},X)$ *is finite. Then* $\mathrm{Ext}^1_R(R/\mathfrak{a},\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ *is finite for every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y$ *of* $\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. *In particular, the* $R$*-module* $\mathrm{Ext}^1_R(R/\mathfrak{a},\mathrm{H}^{f^2_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ *is finite.*

**Corollary 2.7.** *Let* $X$ *be a finite* $R$*-module such that* $\dim_R(X/\mathfrak{a}X) \leq 1$. *Then the following statements hold true:*

(i) $Y_i$ *and* $\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i$ *are* $\mathfrak{a}$*-cofinite* $R$*-modules for all* $i$ *and every* $\mathrm{FD}_{<1}$ $R$*-submodule* $Y_i$ *of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathfrak{a}$*-cofinite* $R$*-module for all* $i$;

(ii) *Let $N$ be a finite $R$-module. Then* $\mathrm{Ext}^j_R(N, Y_i)$, $\mathrm{Tor}^R_j(N, Y_i)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$, *and* $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$ *are $\mathfrak{a}$-cofinite $R$-modules for all $i$, all $j$, and every* $\mathrm{FD}_{<1}$ *$R$-submodule $Y_i$ of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ *and* $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ *are $\mathfrak{a}$-cofinite $R$-modules for all $i$ and all $j$;*

(iii) $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)/Y_i)$ *is a finite set for all $i$ and every* $\mathrm{FD}_{<1}$ *$R$-submodule $Y_i$ of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ *is a finite set for all $i$.*

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# On the First Finiteness Dimensions and Minimaxness of Generalized Local Cohomology Modules

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**Abstract**

Let $R$ be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of $R$, and $M$ a finitely generated $R$-module. In this paper, for an arbitrary $R$-module $X$ which is not necessarily finitely generated, we prove that $f_{\mathfrak{a}}^{1}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not an FD}_{<1}\ R\text{-module}\} = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a minimax } R\text{-module}\}$ if $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is finitely generated for all $i$. Here, $f_{\mathfrak{a}}^{1}(M,X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq 1\}$ is the first finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$ and $f_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a finitely generated } R\text{-module}\}$ is the finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$.

## 1. Introduction

Throughout $R$ is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of $R$, $M$ is a finite (i.e., finitely generated) $R$-module, and $n$ is a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [4, 5].

An important problem in local cohomology is to investigate finiteness of local cohomology modules (see [10, Problem 2]). Let $N$ be a finite $R$-module. The following theorem is an important result in local cohomology and known as Faltings' Local-global Principle for the finiteness of local cohomology modules (see [6, Satz 1] or [4, Theorem 9.6.1]).

**Theorem 1.1.** *Let $t$ be a non-negative integer. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(N)$ *is a finite $R$-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ *is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and all $i \leq t$.*

Another formulation of Faltings' Local-global Principle is in terms of the finiteness dimension $f_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N) \text{ is not a finite } R\text{-module}\}$ of $N$ with respect to $\mathfrak{a}$ with the usual convention that the infimum of the empty set is interpreted as $\infty$. In this formulation, Faltings' Local-global Principle says that $f_{\mathfrak{a}}(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)\}$. Bahmanpour et al., in [3], introduced the notion of the $n$th finiteness dimension of $N$ with respect to $\mathfrak{a}$ by $f^{n}_{\mathfrak{a}}(N) =$

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$\inf\{f_{\mathfrak{a}R_\mathfrak{p}}(N_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$. Thus Faltings' Local-global Principle states that $f_\mathfrak{a}(N) = f_\mathfrak{a}^0(N)$, that is, $f_\mathfrak{a}^0(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(N) \text{ is not a finite } R\text{-module}\}$. In [3, Corollary 2.4], the authors obtained that

$$f_\mathfrak{a}^1(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(N) \text{ is not a minimax } R\text{-module}\}. \tag{1}$$

Recall that an arbitrary $R$-module $X$ is said to be minimax if there exists a finite submodule $X'$ of $X$ such that $X/X'$ is Artinian [13]. Also, by [11, Theorem 2.10], we have

$$f_\mathfrak{a}^1(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(N) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\} \tag{2}$$

(see also [2, Theorem 2.5]). Recall that an arbitrary $R$-module $X$ is said to be an $\mathrm{FD}_{<1}$ $R$-module if there exists a finite submodule $X'$ of $X$ such that $\dim_R(X/X') < 1$ [1, 2].
The $n$th generalized local cohomology module

$$\mathrm{H}_\mathfrak{a}^n(X, Y) \cong \varinjlim_{i \in \mathbb{N}} \operatorname{Ext}_R^n(X/\mathfrak{a}^i X, Y)$$

of arbitrary $R$-modules $X$ and $Y$ with respect to $\mathfrak{a}$ was introduced by Herzog in [8]. It is clear that $\mathrm{H}_\mathfrak{a}^n(R, Y)$ is just the $n$th ordinary local cohomology module $\mathrm{H}_\mathfrak{a}^n(Y)$ of arbitrary $R$-module $Y$ with respect to $\mathfrak{a}$. In [9, Definition 2.3 and Theorem 2.4], Hoang introduced the notion of the $n$th finiteness dimension $f_\mathfrak{a}^n(M, N)$ of $M$ and $N$ with respect to $\mathfrak{a}$ by $f_\mathfrak{a}^n(M, N) = \inf\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$, where $f_\mathfrak{a}(M, N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(M, N) \text{ is not a finite } R\text{-module}\}$, and generalized (2) by showing that

$$f_\mathfrak{a}^1(M, N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(M, N) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\}. \tag{3}$$

Let $X$ be an arbitrary $R$-module which is not necessarily finite. We set $f_\mathfrak{a}(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(M, X) \text{ is not a finite } R\text{-module}\}$ and $f_\mathfrak{a}^n(M, X) = \inf\{f_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$. When $M = R$, we write $f_\mathfrak{a}(X) = f_\mathfrak{a}(R, X)$ and $f_\mathfrak{a}^n(X) = f_\mathfrak{a}^n(R, X)$. Thus $f_\mathfrak{a}(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(X) \text{ is not a finite } R\text{-module}\}$ and $f_\mathfrak{a}^n(X) = \inf\{f_{\mathfrak{a}R_\mathfrak{p}}(X_\mathfrak{p}) : \mathfrak{p} \in \operatorname{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$. In this paper, we improve the equality (3). We show that the equality

$$f_\mathfrak{a}^1(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(M, X) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\}$$

holds if $\operatorname{Ext}_R^i(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i$. We also generalize and improve the equality (1). We prove that if $\operatorname{Ext}_R^i(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i$, then

$$f_\mathfrak{a}^1(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_\mathfrak{a}^i(M, X) \text{ is not a minimax } R\text{-module}\}.$$

## 2. Main Results

The following lemmas are needed in the proof of the main result of this paper.

**Lemma 2.1.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\operatorname{Ext}_R^i(M/\mathfrak{a}M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module for all $i \leq t$. Then $\operatorname{Ext}_R^i(\operatorname{Tor}_j^R(R/\mathfrak{a}, M), X)$ is an $\mathrm{FD}_{<1}$ $R$-module for all $i \leq t$ and all $j$.*

*Proof.* The proof is similar to that of [12, Lemma 2.1 and Corollary 2.2] and left to the reader. □

**Lemma 2.2.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that*

(i) $\operatorname{Ext}_R^{t-i}(\operatorname{Tor}_i^R(R/\mathfrak{a}, M), X)$ *is an* $\mathrm{FD}_{<1}$ *$R$-module for all $i \leq t$, and*
(ii) $\operatorname{Ext}_R^{t+1-i}(R/\mathfrak{a}, \mathrm{H}_\mathfrak{a}^i(M, X))$ *is an* $\mathrm{FD}_{<1}$ *$R$-module for all $i < t$.*

*Then $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}_\mathfrak{a}^t(M, X))$ is an $\mathrm{FD}_{<1}$ $R$-module.*

*Proof.* This is sufficiently similar to that of [12, Theorem 2.3] to be omitted. We leave the proof to the reader. □

**Lemma 2.3.** *Let $X$ be an $\mathfrak{a}$-torsion $R$-module such that $X_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 1$ and $\operatorname{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ $R$-module. Then $X$ is an $\mathrm{FD}_{<1}$ $R$-module.*

*Proof.* Suppose, on the contrary, that $X$ is not an $\mathrm{FD}_{<1}$ $R$-module and seek a contradiction. Let $A_1 = \{\mathfrak{p} \in \operatorname{Ass}_R(X) : \dim_R(R/\mathfrak{p}) \geq 1\}$ and $\mathfrak{a}_1 = \bigcap_{\mathfrak{p}\in A_1} \mathfrak{p}$. Since $X$ is not an $\mathrm{FD}_{<1}$ $R$-module, $\dim_R(X) \geq 1$. Thus $A_1$ is a non-empty and finite set because $\operatorname{Ass}_R(X) = \operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, X))$ and $\operatorname{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ $R$-module. For all $\mathfrak{p} \in A_1$, $X_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module and so there exists a finite submodule $N(\mathfrak{p})$ of $X$ such that $(N(\mathfrak{p}))_\mathfrak{p} = X_\mathfrak{p}$. Let $N_1 = \sum_{\mathfrak{p}\in A_1} N(\mathfrak{p})$. Then $N_1$ is a finite submodule of $X$ such that $A_1 \cap A_2 = \emptyset$ and $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$, where $A_2 = \{\mathfrak{p} \in \operatorname{Ass}_R(X/N_1) : \dim_R(R/\mathfrak{p}) \geq 1\}$ and $\mathfrak{a}_2 = \bigcap_{\mathfrak{p}\in A_2} \mathfrak{p}$. Since $X$ is not an $\mathrm{FD}_{<1}$ $R$-module, $\dim_R(X/N_1) \geq 1$. Note that $X/N_1$ is an $\mathfrak{a}$-torsion $R$-module and $\operatorname{Hom}_R(R/\mathfrak{a}, X/N_1)$ is an $\mathrm{FD}_{<1}$ $R$-module from the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, X) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, X/N_1) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, N_1).$$

Thus $A_2$ is a non-empty and finite set, and so $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2$.
Therefore, using the above method on the $R$-module $X/N_1$, there is a finite submodule $N_2$ $(\supseteq N_1)$ of $X$ such that $A_2 \cap A_3 = \emptyset$ and $\mathfrak{a}_2 \subsetneq \mathfrak{a}_3$, where $A_3 = \{\mathfrak{p} \in \operatorname{Ass}_R(X/N_2) : \dim_R(R/\mathfrak{p}) \geq 1\}$ and $\mathfrak{a}_3 = \bigcap_{\mathfrak{p}\in A_3} \mathfrak{p}$.
Proceeding in the same way, there is an ascending chain of ideals of Noetherian ring $R$,

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots \subsetneq \mathfrak{a}_i \subsetneq \cdots,$$

which is not stable. This contradiction shows that $X$ is an $\mathrm{FD}_{<1}$ $R$-module, as we desired. □

Recall that an arbitrary $R$-module $X$ is said to be $\mathfrak{a}$-cofinite if $\operatorname{Supp}_R(X) \subseteq \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite $R$-module for all $i$ [7].

**Lemma 2.4.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $s, t$ non-negative integers such that $\operatorname{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $t \leq i \leq s+t+1$, $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathfrak{a}$-cofinite $R$-module for all $i < t$, and $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module for all $t \leq i \leq s+t$. Then $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathfrak{a}$-cofinite $R$-module for all $i \leq s+t$.*

*Proof.* We prove by using induction on $s$. Let $s = 0$. Since $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M, X))$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M, X))$ are finite from [12, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{H}^t_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite by [1, Theorem 3.1]. Suppose that $s > 0$ and that $s-1$ is settled. It is enough to show that $\mathrm{H}^{s+t}_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite because $\mathrm{H}^i_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite for all $i \leq s+t-1$ from the induction hypothesis on $s-1$. By [12, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_\mathfrak{a}(M, X))$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_\mathfrak{a}(M, X))$ are finite. Thus $\mathrm{H}^{s+t}_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite from [1, Theorem 3.1]. □

In the main result of this paper, we generalize and improve [3, Proposition 2.2 and Corollary 2.4].

**Theorem 2.5.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\operatorname{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i \leq t+1$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_\mathfrak{a}(M, X)$ *is a minimax $R$-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_\mathfrak{a}(M, X)$ *is an $\mathrm{FD}_{<1}$ $R$-module for all $i \leq t$;*

(iii) $\mathrm{H}^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p})$ *is a finite $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 1$ and for all $i \leq t$.*

*Proof.* (i)⇒(ii). This is clear.
(ii)⇒(iii). Let $\mathfrak{p}$ be a prime ideal of $R$ with $\dim_R(R/\mathfrak{p}) \geq 1$ and let $i \leq t$. Since $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module, there exists a finite submodule $N_i$ of $\mathrm{H}^i_\mathfrak{a}(M, X)$ such that $\dim_R(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i) < 1$. Thus $(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i)_\mathfrak{p} = 0$ and so $\mathrm{H}^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p}) \cong (\mathrm{H}^i_\mathfrak{a}(M, X))_\mathfrak{p} = (N_i)_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module.
(iii)⇒(ii). We prove by using induction on $t$. Let $t = 0$. Since $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^0_\mathfrak{a}(M, X))$ is an $\mathrm{FD}_{<1}$ $R$-module from Lemma 2.2, $\mathrm{H}^0_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module by Lemma 2.3. Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^t_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module because $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module for all $i \leq t-1$ from the induction hypothesis on $t-1$. Thus, by Lemmas 2.1 and 2.2, $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M, X))$ is an $\mathrm{FD}_{<1}$ $R$-module. Hence $\mathrm{H}^t_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<1}$ $R$-module from Lemma 2.3.

(ii)⇒(i). Let $i \leq t$. Since $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is an $\mathrm{FD}_{<1}$ $R$-module, there exists a finite submodule $N_i$ of $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ such that $\dim_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i) < 1$. On the other hand, $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is an $\mathfrak{a}$-cofinite $R$-module by Lemma 2.4, and hence $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i)$ is finite. Now, since $\dim_R(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i)) < 1$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i)$ is Artinian. Thus $\mathrm{H}^i_{\mathfrak{a}}(M,X)/N_i$ is Artinian by [4, Theorem 7.1.2], and so $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is minimax. □

**Corollary 2.6.** *Let $M$ be a finite $R$-module and let $X$ be an arbitrary $R$-module such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is a finite $R$-module for all $i$ (in fact, for all $i \leq f^1_{\mathfrak{a}}(M,X)+1$). Then*

$$\begin{aligned} f^1_{\mathfrak{a}}(M,X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a minimax } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\}. \end{aligned}$$

*Proof.* This follows from Theorem 2.5. □

We have the following results for the ordinary local cohomology modules.

**Corollary 2.7.** *Let $X$ be an arbitrary $R$-module and let $t$ be a non-negative integer such that $\mathrm{Ext}^i_R(R/\mathfrak{a},X)$ is a finite $R$-module for all $i \leq t+1$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is a minimax $R$-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an $\mathrm{FD}_{<1}$ $R$-module for all $i \leq t$;*

(iii) $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ *is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 1$ and for all $i \leq t$.*

**Corollary 2.8.** *Let $X$ be an arbitrary $R$-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a},X)$ is a finite $R$-module for all $i$ (in fact, for all $i \leq f^1_{\mathfrak{a}}(X)+1$). Then*

$$\begin{aligned} f^1_{\mathfrak{a}}(X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not a minimax } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not an } \mathrm{FD}_{<1} \ R\text{-module}\}. \end{aligned}$$

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# On the Second Finiteness Dimensions and Weakly Laskerianness of Generalized Local Cohomology Modules

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**Abstract**

Let $R$ be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of $R$, and $M$ a finitely generated $R$-module. In this paper, for an arbitrary $R$-module $X$ which is not necessarily finitely generated, we prove that $f_{\mathfrak{a}}^2(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X)$ is not an $\mathrm{FD}_{<2}$ $R$-module$\} = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X)$ is not a weakly Laskerian $R$-module$\}$ if $R$ is semi-local and $\mathrm{Ext}^i_R(M/\mathfrak{a}M,X)$ is finite for all $i$. Here, $f_{\mathfrak{a}}^2(M,X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}},X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)$ and $\dim_R(R/\mathfrak{p}) \geq 2\}$ is the second finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$ and $f_{\mathfrak{a}}(M,X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X)$ is not a finitely generated $R$-module$\}$ is the finiteness dimension of $M$ and $X$ with respect to $\mathfrak{a}$.

## 1. Introduction

Throughout $R$ is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of $R$, $M$ is a finite (i.e., finitely generated) $R$-module, and $n$ is a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [5, 6].

An important problem in local cohomology is to investigate finiteness of local cohomology modules (see [12, Problem 2]). Let $N$ be a finite $R$-module. The following theorem is an important result in local cohomology and known as Faltings' Local-global Principle for the finiteness of local cohomology modules (see [8, Satz 1] or [5, Theorem 9.6.1]).

**Theorem 1.1.** *Let $t$ be a non-negative integer. Then $\mathrm{H}^i_{\mathfrak{a}}(N)$ is a finite $R$-module for all $i \leq t$ if and only if $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and all $i \leq t$.*

Another formulation of Faltings' Local-global Principle is in terms of the finiteness dimension $f_{\mathfrak{a}}(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(N)$ is not a finite $R$-module$\}$ of $N$ with respect to $\mathfrak{a}$ with the usual convention that the infimum of the empty set is interpreted as $\infty$. In this formulation, Faltings' Local-global Principle says that $f_{\mathfrak{a}}(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)\}$. Bahmanpour et al., in [4], introduced the notion of the $n$th finiteness dimension of $N$ with respect to $\mathfrak{a}$ by $f_{\mathfrak{a}}^n(N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R)$ and $\dim_R(R/\mathfrak{p}) \geq n\}$. Thus Faltings' Local-global Principle states that $f_{\mathfrak{a}}(N) = f_{\mathfrak{a}}^0(N)$,

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that is, $f_{\mathfrak{a}}^0(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(N) \text{ is not a finite } R\text{-module}\}$. In [4, Proposition 3.7], the authors obtained that if $R$ is a semi-local ring, then

$$f_{\mathfrak{a}}^2(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(N) \text{ is not a weakly Laskerian } R\text{-module}\}. \tag{1}$$

Recall that an arbitrary $R$-module $X$ is said to be weakly Laskerian if the set of associated prime ideals of any quotient module of $X$ is finite [7]. Also, by [13, Theorem 2.10], we have

$$f_{\mathfrak{a}}^2(N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(N) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\} \tag{2}$$

(see also [2, Theorem 2.5]). Recall that an arbitrary $R$-module $X$ is said to be an $\mathrm{FD}_{<2}$ $R$-module if there exists a finite submodule $X'$ of $X$ such that $\dim_R(X/X') < 2$ [1, 2].
The $n$th generalized local cohomology module

$$\mathrm{H}_{\mathfrak{a}}^n(X, Y) \cong \varinjlim_{i \in \mathbb{N}} \mathrm{Ext}_R^n(X/\mathfrak{a}^i X, Y)$$

of arbitrary $R$-modules $X$ and $Y$ with respect to $\mathfrak{a}$ was introduced by Herzog in [10]. It is clear that $\mathrm{H}_{\mathfrak{a}}^n(R, Y)$ is just the $n$th ordinary local cohomology module $\mathrm{H}_{\mathfrak{a}}^n(Y)$ of arbitrary $R$-module $Y$ with respect to $\mathfrak{a}$. In [11, Definition 2.3 and Theorem 2.4], Hoang introduced the notion of the $n$th finiteness dimension $f_{\mathfrak{a}}^n(M, N)$ of $M$ and $N$ with respect to $\mathfrak{a}$ by $f_{\mathfrak{a}}^n(M, N) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$, where $f_{\mathfrak{a}}(M, N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M, N) \text{ is not a finite } R\text{-module}\}$, and generalized (2) by showing that

$$f_{\mathfrak{a}}^2(M, N) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M, N) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\}. \tag{3}$$

Let $X$ be an arbitrary $R$-module which is not necessarily finite. We set $f_{\mathfrak{a}}(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M, X) \text{ is not a finite } R\text{-module}\}$ and $f_{\mathfrak{a}}^n(M, X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$. When $M = R$, we write $f_{\mathfrak{a}}(X) = f_{\mathfrak{a}}(R, X)$ and $f_{\mathfrak{a}}^n(X) = f_{\mathfrak{a}}^n(R, X)$. Thus $f_{\mathfrak{a}}(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(X) \text{ is not a finite } R\text{-module}\}$ and $f_{\mathfrak{a}}^n(X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim_R(R/\mathfrak{p}) \geq n\}$. In this paper, we improve the equality (3). We show that the equality

$$f_{\mathfrak{a}}^2(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M, X) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\}$$

holds if $\mathrm{Ext}_R^i(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i$. We also generalize and improve the equality (1). We prove that if $R$ is a semi-local ring and $\mathrm{Ext}_R^i(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i$, then

$$f_{\mathfrak{a}}^2(M, X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}_{\mathfrak{a}}^i(M, X) \text{ is not a weakly Laskerian } R\text{-module}\}.$$

## 2. Main Results

The following lemmas are needed in the proof of the main result of this paper.

**Lemma 2.1.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\mathrm{Ext}_R^{t-i}(\mathrm{Tor}_i^R(R/\mathfrak{a}, M), X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i \leq t$ and $\mathrm{Ext}_R^{t+1-i}(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^i(M, X))$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i < t$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^t(M, X))$ is an $\mathrm{FD}_{<2}$ $R$-module.*

*Proof.* This is sufficiently similar to that of [14, Theorem 2.3] to be omitted. We leave the proof to the reader. □

**Lemma 2.2.** *Let $X$ be an $\mathfrak{a}$-torsion $R$-module such that $X_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 2$ and $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<2}$ $R$-module. Then $X$ is an $\mathrm{FD}_{<2}$ $R$-module.*

*Proof.* Suppose, on the contrary, that $X$ is not an $\mathrm{FD}_{<2}$ $R$-module and seek a contradiction. Let $A_1 = \{\mathfrak{p} \in \mathrm{Ass}_R(X) : \dim_R(R/\mathfrak{p}) \geq 2\}$ and $\mathfrak{a}_1 = \bigcap_{\mathfrak{p} \in A_1} \mathfrak{p}$. Since $X$ is not an $\mathrm{FD}_{<2}$ $R$-module, $\dim_R(X) \geq 2$. Thus $A_1$ is a non-empty and finite set because $\mathrm{Ass}_R(X) = \mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, X))$ and $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<2}$ $R$-module. For all $\mathfrak{p} \in A_1$, $X_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module and so there exists a finite submodule $N(\mathfrak{p})$ of $X$ such that $(N(\mathfrak{p}))_{\mathfrak{p}} = X_{\mathfrak{p}}$. Let $N_1 = \sum_{\mathfrak{p} \in A_1} N(\mathfrak{p})$. Then $N_1$ is a finite submodule of $X$ such that $A_1 \cap A_2 = \emptyset$ and $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$, where $A_2 = \{\mathfrak{p} \in \mathrm{Ass}_R(X/N_1) : \dim_R(R/\mathfrak{p}) \geq 2\}$ and $\mathfrak{a}_2 = \bigcap_{\mathfrak{p} \in A_2} \mathfrak{p}$. Since $X$ is not an $\mathrm{FD}_{<2}$ $R$-module, $\dim_R(X/N_1) \geq 2$. Note that $X/N_1$ is an $\mathfrak{a}$-torsion $R$-module and $\mathrm{Hom}_R(R/\mathfrak{a}, X/N_1)$ is an $\mathrm{FD}_{<2}$ $R$-module from the exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, X) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, X/N_1) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, N_1).$$

Thus $A_2$ is a non-empty and finite set, and so $\mathfrak{a}_1 \subsetneq \mathfrak{a}_2$.
Therefore, using the above method on the $R$-module $X/N_1$, there is a finite submodule $N_2$ $(\supseteq N_1)$ of $X$ such that $A_2 \cap A_3 = \emptyset$ and $\mathfrak{a}_2 \subsetneq \mathfrak{a}_3$, where $A_3 = \{\mathfrak{p} \in \mathrm{Ass}_R(X/N_2) : \dim_R(R/\mathfrak{p}) \geq 2\}$ and $\mathfrak{a}_3 = \bigcap_{\mathfrak{p}\in A_3} \mathfrak{p}$.
Proceeding in the same way, there is an ascending chain of ideals of Noetherian ring $R$,

$$\mathfrak{a}_1 \subsetneq \mathfrak{a}_2 \subsetneq \cdots \subsetneq \mathfrak{a}_i \subsetneq \cdots,$$

which is not stable. This contradiction shows that $X$ is an $\mathrm{FD}_{<2}$ $R$-module, as we desired. □

**Lemma 2.3.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i \leq t$. Then $\mathrm{Ext}^i_R(\mathrm{Tor}^R_j(R/\mathfrak{a}, M), X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i \leq t$ and all $j$.*

*Proof.* The proof is similar to that of [14, Lemma 2.1 and Corollary 2.2] and left to the reader. □

Recall that an arbitrary $R$-module $X$ is said to be $\mathfrak{a}$-cofinite if $\mathrm{Supp}_R(X) \subseteq \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite $R$-module for all $i$ [9].

**Lemma 2.4.** *Let $M$ be a finite $R$-module, $X$ an arbitrary $R$-module, and $s, t$ non-negative integers such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $t \leq i \leq s+t+1$, $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathfrak{a}$-cofinite $R$-module for all $i < t$, and $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $t \leq i \leq s+t$. Then $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathfrak{a}$-cofinite $R$-module for all $i \leq s+t$.*

*Proof.* We prove by using induction on $s$. Let $s = 0$. Since $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M, X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M, X))$ are finite from [14, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{H}^t_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite by [1, Theorem 3.1]. Suppose that $s > 0$ and that $s-1$ is settled. It is enough to show that $\mathrm{H}^{s+t}_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite because $\mathrm{H}^i_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite for all $i \leq s+t-1$ from the induction hypothesis on $s-1$. By [14, Corollary 2.2, Theorem 2.3, Theorem 2.7, and Corollary 2.14], $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_\mathfrak{a}(M, X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_\mathfrak{a}(M, X))$ are finite. Thus $\mathrm{H}^{s+t}_\mathfrak{a}(M, X)$ is $\mathfrak{a}$-cofinite from [1, Theorem 3.1]. □

In the main result of this paper, we generalize and improve [4, Proposition 3.7].

**Theorem 2.5.** *Let $R$ be a semi-local ring, $M$ a finite $R$-module, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i \leq t+1$. Then the following statements are equivalent:*

(i) *$\mathrm{H}^i_\mathfrak{a}(M, X)$ is a weakly Laskerian $R$-module for all $i \leq t$;*

(ii) *$\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i \leq t$;*

(iii) *$\mathrm{H}^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p})$ is a finite $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 2$ and for all $i \leq t$.*

*Proof.* (i)⇒(ii). This is clear from [3, Theorem 3.3].
(ii)⇒(iii). Let $\mathfrak{p}$ be a prime ideal of $R$ with $\dim_R(R/\mathfrak{p}) \geq 2$ and let $i \leq t$. Since $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module, there exists a finite submodule $N_i$ of $\mathrm{H}^i_\mathfrak{a}(M, X)$ such that $\dim_R(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i) < 2$. Thus $(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i)_\mathfrak{p} = 0$ and so $\mathrm{H}^i_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p}, X_\mathfrak{p}) \cong (\mathrm{H}^i_\mathfrak{a}(M, X))_\mathfrak{p} = (N_i)_\mathfrak{p}$ is a finite $R_\mathfrak{p}$-module.
(iii)⇒(ii). We prove by using induction on $t$. Let $t = 0$. Since $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^0_\mathfrak{a}(M, X))$ is an $\mathrm{FD}_{<2}$ $R$-module from Lemma 2.1, $\mathrm{H}^0_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module by Lemma 2.2. Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^t_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module because $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module for all $i \leq t-1$ from the induction hypothesis on $t-1$. Thus, by Lemmas 2.3 and 2.1, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(M, X))$ is an $\mathrm{FD}_{<2}$ $R$-module. Hence $\mathrm{H}^t_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module from Lemma 2.2.
(ii)⇒(i). Let $i \leq t$. Since $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathrm{FD}_{<2}$ $R$-module, there is a finite submodule $N_i$ of $\mathrm{H}^i_\mathfrak{a}(M, X)$ such that $\dim_R(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i) < 2$. On the other hand, $\mathrm{H}^i_\mathfrak{a}(M, X)$ is an $\mathfrak{a}$-cofinite $R$-module by Lemma 2.4, and so the set $\mathrm{Ass}_R(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i)$ is finite. Thus $\mathrm{Supp}_R(\mathrm{H}^i_\mathfrak{a}(M, X)/N_i)$ is a finite set. Hence, by [3, Theorem 3.3], $\mathrm{H}^i_\mathfrak{a}(M, X)$ is weakly Laskerian. □

**Corollary 2.6.** *Let $R$ be a semi-local ring, $M$ a finite $R$-module, and $X$ an arbitrary $R$-module such that $\operatorname{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite $R$-module for all $i$ (in fact, for all $i \leq f^2_{\mathfrak{a}}(M,X)+1$). Then*

$$\begin{aligned} f^2_{\mathfrak{a}}(M,X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not a weakly Laskerian } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(M,X) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\}. \end{aligned}$$

*Proof.* This follows from Theorem 2.5. □

We have the following results for the ordinary local cohomology modules.

**Corollary 2.7.** *Let $R$ be a semi-local ring, $X$ an arbitrary $R$-module, and $t$ a non-negative integer such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite $R$-module for all $i \leq t+1$. Then the following statements are equivalent:*

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is a weakly Laskerian $R$-module for all $i \leq t$;*

(ii) $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an $\mathrm{FD}_{<2}$ $R$-module for all $i \leq t$;*

(iii) $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ *is a finite $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim_R(R/\mathfrak{p}) \geq 2$ and for all $i \leq t$.*

**Corollary 2.8.** *Let $R$ be a semi-local ring and let $X$ be an arbitrary $R$-module such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite $R$-module for all $i$ (in fact, for all $i \leq f^2_{\mathfrak{a}}(X)+1$). Then*

$$\begin{aligned} f^2_{\mathfrak{a}}(X) &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not a weakly Laskerian } R\text{-module}\} \\ &= \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not an } \mathrm{FD}_{<2} \ R\text{-module}\}. \end{aligned}$$

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# Existence results for nonlocal systems via sub and supersolutions method

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**Abstract**

Using the method of sub-super solutions, we study the existence of positive solutions for a class of infinite semipositone problems involving nonlocal operator.

## 1. Introduction

In this paper, we are interested in the existence of positive solutions for the Kirchhoff-type problem

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = au - bu^\gamma - f(u) - \frac{c}{u^\alpha},\ x\in\Omega, & x\in\Omega,\\ u = 0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where $\Delta$ denotes the Laplacian operator defined by $\Delta z = \mathrm{div}(\nabla z)$, $\gamma > 1$, $\alpha \in (0,1)$, $a, b$ and $c$ are positive constants, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $f : [0,\infty) \to \mathbb{R}$ is a continuous function. This model describes the steady states of a logistic growth model with grazing and constant yield harvesting. It also describes the dynamics of the fish population with natural predation and constant yield harvesting.
We make the following assumptions:

$(H_1)$ $f : [0,\infty) \to \mathbb{R}$ is a continuous function such that $\lim_{s\to\infty} f(s) = \infty$.

$(H_2)$ There exist $A > 0$ and $\beta > 1$ such that $f(s) \le As^\beta$, for all $s \ge 0$.

$(H_3)$ $M : [0,\infty) \to \mathbb{R}$ is a continuous and increasing function such that $0 < M_0 \le M(t) \le M_\infty$ for all $t$.

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In [12], the authors have studied the equation $-\Delta u = g(u) - (c/u^{\alpha})$ with Dirichlet boundary conditions, where $g$ is nonnegative and nondecreasing and $\lim_{u\to\infty} g(u) = \infty$. The case $g(u) := au - f(u)$ has been study in [10], where $f(u) \geq au - N$ and $f(u) \leq Au^{\beta}$ on $[0,\infty)$ for some $N, A > 0, \beta > 1$ and this $g$ may have a falling zero. Here $u$ is the population density and $au - bu^{\gamma} - f(u)$ represents logistics growth. This model describes grazing of a fixed number of grazers on a logistically growing species (see [11]). The herbivore density is assumed to be a constant which is a valid assumption for managed grazing systems and the rate of grazing is given by $\frac{c}{u^{\alpha}}$. At high levels of vegetation density this term saturates to $c$ as the grazing population is a constant. This model has also been applied to describe the dynamics of fish populations (see [11, 14]). The diffusive logistic equation with constant yield harvesting, in the absence of grazing was studied in [13]. More recently, reaction-diffusion models have been used to describe spatiotemporal phenomena in disciplines other than ecology, such as physics,chemistry (see [15, 16]). In this paper, we study the equation

$$-M\left(\int_{\Omega} |\nabla u|^2\, dx\right)\Delta u = au - bu^{\gamma} - f(u) - (c/u^{\alpha}) \tag{2}$$

with Dirichlet boundary conditions. Let $F(u) := au - bu^{\gamma} - f(u) - (c/u^{\alpha})$, then $\lim_{u\to 0^+} F(u) = -\infty$ and hence we refer to (1) as an infinite semipositone problem. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [1, 2] in which the authors have used variational method and topological method to get the existence of solutions for (1). In this paper, motivated by the ideas introduced in [6] and the properties of Kirchhoff type operators in [8, 9], we study problem (1) in the semipositone case ($F(0) < 0$ but finite) (see [3–5]). The main tool used in this study is the method of sub- and super solutions. Our result in this note improves the previous one [6] in which $M(t) \equiv 1$. To our best knowledge, this is a new research topic for nonlocal problems, see [1, 9].

## 2. Main result

A function $\psi$ is said to be a subsolution of problem (1) if it is in $W^{1,2}(\Omega)$ such that $\psi = 0$ on $\partial\Omega$ and satisfies

$$M\left(\int_{\Omega} |\nabla\psi|^2\, dx\right)\int_{\Omega} \nabla\psi \cdot \nabla w\, dx \leq \int_{\Omega} \left(a\psi - b\psi^{\gamma} - f(\psi) - \frac{c}{\psi^{\alpha}}\right) w\, dx, \quad \forall w \in W, \tag{3}$$

where $W := \{w \in C_0^{\infty}(\Omega) : w \geq 0 \text{ in } \Omega\}$. A function $z \in W^{1,2}(\Omega)$ is said to be a supersolution if $z = 0$ on $\partial\Omega$ and satisfies

$$M\left(\int_{\Omega} |\nabla z|^2\, dx\right)\int_{\Omega} \nabla z \cdot \nabla w\, dx \geq \int_{\Omega} \left(az - bz^{\gamma} - f(z) - \frac{c}{z^{\alpha}}\right) w\, dx, \quad \forall w \in W. \tag{4}$$

Our main result is given by the following theorem.

**Theorem 2.1.** *Let (H1), (H2) and (H3) hold. If $a > (\frac{2}{1+\alpha})\lambda_1$, then there exists a positive constant $c^* := c^*(a, A, \alpha, \beta, \gamma, \Omega)$ such that for $c \leq c^*$, problem (1) has a positive solution, where $\lambda_1$ is the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions.*

*Proof.* We shall establish our result by constructing positive sub-supersolutions to equation (1). From an anti-maximum principle (see [7, pages 155-156]), there exists a $\sigma(\Omega) > 0$ such that the solution $z_{\lambda}$ of

$$\begin{cases} -\Delta z - \lambda z = -1, & x \in \Omega, \\ z = 0 & x \in \partial\Omega \end{cases}$$

for $\lambda \in (\lambda_1, \lambda_1 + \sigma)$ is positive in $\Omega$ and is such that $\frac{\partial z}{\partial v} < 0$ on $\partial\Omega$, where $v$ is outward normal vector on $\partial\Omega$. Fix $\lambda^* \in (\lambda_1, \min\{\lambda_1 + \sigma, (\frac{1+\alpha}{2})a\})$ and let

$$K := \min\left\{\left(\frac{M_\infty(2+\alpha)}{2b\|z_{\lambda^*}\|_\infty^{\frac{\gamma(\alpha-1)}{1+\alpha}}}\right)^{\frac{1}{\gamma-1}}, \left(\frac{a-(\frac{2}{1+\alpha})\lambda^*}{3M_\infty b\|z_{\lambda^*}\|_\infty^{\frac{2(\gamma-1)}{1+\alpha}}}\right)^{\frac{1}{\gamma-1}}, \left(\frac{M_\infty(2+\alpha)}{2A\|z_{\lambda^*}\|_\infty^{\frac{\beta(\alpha-1)}{1+\alpha}}}\right)^{\frac{1}{\beta-1}}, \left(\frac{a-(\frac{2}{1+\alpha})\lambda^*}{3M_\infty A\|z_{\lambda^*}\|_\infty^{\frac{2(\beta-1)}{1+\alpha}}}\right)^{\frac{1}{\beta-1}}\right\}.$$

Define $\psi = Kz_{\lambda^*}^{\frac{2}{1+\alpha}}$. Then

$$\nabla\psi = K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}}\nabla z_{\lambda^*}$$

and

$$\begin{aligned}
&M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx\\
&= M\left(\int_\Omega |\nabla\psi|^2\,dx\right)K(\frac{2}{1+\alpha})\int_\Omega\left[\left\{(\frac{1-\alpha}{1+\alpha})z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}}|\nabla z_{\lambda^*}|^2 + z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}}\Delta z_{\lambda^*}\right\}\right]w\,dx\\
&= M\left(\int_\Omega |\nabla\psi|^2\,dx\right)K(\frac{2}{1+\alpha})\int_\Omega\left[\left\{(\frac{1-\alpha}{1+\alpha})z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}}|\nabla z_{\lambda^*}|^2 + z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}}(1-\lambda^* z_{\lambda^*})\right\}\right]w\,dx\\
&\le M_\infty K(\frac{2}{1+\alpha})\int_\Omega\left[\left\{\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right\}\right]w\,dx.
\end{aligned}$$

Let $\delta > 0$, $\mu > 0$ and $m > 0$ be such that $|\nabla z_{\lambda^*}|^2 \ge m$ in $\overline{\Omega}_\delta$ and $z_{\lambda^*} \ge \mu$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta := \{x \in \Omega : d(x, \partial\Omega) \le \delta\}$. Let

$$c^* := K^{1+\alpha}\min\left\{M_\infty(\frac{2}{1+\alpha})\left(\frac{1-\alpha}{1+\alpha}\right)m^2, \frac{1}{3}\mu^2\left(a - M_\infty(\frac{2}{1+\alpha})\lambda^*\right)\right\}.$$

Let $x \in \overline{\Omega}_\delta$ and $c \le c^*$. Since $(\frac{2}{1+\alpha})\lambda^* < a$ , we have

$$M_\infty K(\frac{2}{1+\alpha})\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} < a\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right). \tag{5}$$

From the choice of $K$, we have

$$\frac{M_\infty}{2}(\frac{2}{1+\alpha}) \ge bK^{\gamma-1}\|z_{\lambda^*}\|_\infty^{\frac{2\gamma(\alpha-1)}{1+\alpha}}, \tag{6}$$

$$\frac{M_\infty}{2}(\frac{2}{1+\alpha}) \ge AK^{\beta-1}\|z_{\lambda^*}\|_\infty^{\frac{2\beta(\alpha-1)}{1+\alpha}}. \tag{7}$$

By (6), (7) and (H2), we know that

$$-\frac{M_\infty}{2}K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \le -b\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\gamma, \tag{8}$$

$$-\frac{M_\infty}{2}K(\frac{2}{1+\alpha})z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \le -A\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\beta \le -f\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right). \tag{9}$$

Since $|\nabla z_{\lambda^*}|^2 \geq m$ in $\overline{\Omega}_\delta$, from the choice of $c^*$ we have

$$\begin{aligned} & -M_\infty K\left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\ & \leq M_\infty K\left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right)\frac{m^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\ & \leq -\frac{c}{\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\alpha}. \end{aligned} \tag{10}$$

Hence for $c^*$, combining (5), (8), (9), (10) and (H3) we have

$$\begin{aligned} & M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx \\ = \; & M\left(\int_\Omega |\nabla\psi|^2\,dx\right)K\left(\frac{2}{1+\alpha}\right) \\ \times \; & \int_\Omega\left(\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} - \left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right)w\,dx, \\ & \leq M_\infty K\left(\frac{2}{1+\alpha}\right)\int_\Omega\left\{\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - \frac{1}{2}K\left(\frac{2}{1+\alpha}\right)z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}}\right. \\ & -\frac{1}{2}K\left(\frac{2}{1+\alpha}\right)z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \\ & \left. K\left(\frac{2}{1+\alpha}\right)\left(\frac{1-\alpha}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right\}w\,dx, \\ & \leq \int_\Omega\left(a\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - b\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\gamma - f\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - \frac{c}{\left(K z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\alpha}\right)w\,dx, \\ & \leq \int_\Omega\left(a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^\alpha}\right)w\,dx, \quad \forall w\in W,\ x\in\overline{\Omega}_\delta. \end{aligned}$$

Next in $\Omega\setminus\overline{\Omega}_\delta$, for $c\leq c^*$ from the choice of $c^*$ and $K$, we know that

$$\frac{c}{K^\alpha} \leq \frac{1}{3}K z_{\lambda^*}^2\left(a - \left(\frac{2}{1+\alpha}\right)\lambda^*\right), \tag{11}$$

$$b K^{\gamma-1} z_{\lambda^*}^{\frac{p(\gamma-1)}{1+\alpha}} \leq \frac{1}{3}\left(a - \left(\frac{2}{1+\alpha}\right)\lambda^*\right), \tag{12}$$

and

$$A K^{\beta-1} z_{\lambda^*}^{\frac{2(\beta-1)}{1+\alpha}} \leq \frac{1}{3}\left(a - \left(\frac{2}{1+\alpha}\right)\lambda^*\right) \tag{13}$$

By combining (11), (12) and (13), we have

$$
\begin{aligned}
& M\left(\int_\Omega |\nabla\psi|^2\,dx\right)\int_\Omega \nabla\psi\cdot\nabla w\,dx \\
& = M\left(\int_\Omega |\nabla\psi|^2\,dx\right)K\left(\frac{2}{1+\alpha}\right) \\
\times\ & \int_\Omega\left(\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - z_{\lambda^*}^{\frac{(1-\alpha)}{1+\alpha}} - \left(\frac{(1-\alpha)}{1+\alpha}\right)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\right)w\,dx, \\
& \le \int_\Omega\left(M_\infty k\left(\frac{2}{1+\alpha}\right)\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}}\right)w\,dx, \\
& = \int_\Omega \frac{M_\infty}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\sum_{i=1}^{3}\left(\frac{1}{3}k\left(\frac{2}{1+\alpha}\right)\lambda^* z_{\lambda^*}^2\right)w\,dx, \\
& \le \int_\Omega\left\{\frac{1}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\left(\frac{1}{3}kz_{\lambda^*}^2 a - \frac{c}{k^\alpha}\right) + kz_{\lambda^*}^2\left(\frac{1}{3}a - bk^{\gamma-1}z_{\lambda^*}^{\frac{2(\gamma-1)}{1+\alpha}}\right)\right. \\
& \left. + kz_{\lambda^*}^2\left(\frac{1}{3}a - Ak^{\beta-1}z_{\lambda^*}^{\frac{2(\beta-1)}{1+\alpha}}\right)\right\}w\,dx, \\
& \le \int_\Omega\left(a\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - b\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\gamma - f\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right) - \frac{c}{\left(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}\right)^\alpha}\right)w\,dx, \\
& \le \int_\Omega\left(a\psi - b\psi^\gamma - f(\psi) - \frac{c}{\psi^\alpha}\right)w\,dx, \quad \forall w\in W,\ x\in\Omega\setminus\overline{\Omega}_\delta.
\end{aligned}
$$

Thus $\psi$ is a positive subsolution of (1). From $(H_1)$ and $\gamma > 1$, it is obvious that $z = M$, where $M$ is sufficiently large constant is a supersolution of (1) with $z \ge \psi$. Thus, the proof is complete. □

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# Review on frames for operators on Hilbert spaces

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**Abstract**

In this paper, we aim to introduce some recent achievements and properties of frames of operators, or briefly $K$-frames, for separable Hilbert spaces. Also, the method of constructing k-frames in different modes will be stated , and the duality of these frames is characterized. Finally, equal-norm $K$-frames will be presented.

## 1. Introduction

The focus of this paper is mainly on the frames of operators or $K$-frames on Hilbert spaces. This frames, which have been introduced by Găvruţa in [3], as a new generalization of frames have important applications especially in sampling theory and help us to reconstruct elements from the range of a bounded linear operator $K$ in a separable Hilbert space. Throughout this paper, $H$, $H_1$, and $H_2$ are separable Hilbert spaces and $\mathcal{B}(H_1, H_2)$ is the collection of all the bounded linear operators of $H_1$ into $H_2$. If $H_1 = H_2 = H$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$. The notation $W \prec H$ means that $W$ is a closed subspace of $H$, also $P_W$ is the orthogonal projection from $H$ onto the closed subspace $W \prec H$ and $\{H_j\}_{j\in\mathbb{J}}$ is a sequence of Hilbert spaces where $\mathbb{J}$ is a subset of $\mathbb{Z}$. When $\dim H < \infty$, we say that $H$ is the finite Hilbert space. Also, we define

$$\ell^2(\mathbb{J}) = \Big\{\{x_j\}_{j\in\mathbb{J}}\,, \quad x_j \in \mathbb{C}, \sum_{j\in\mathbb{J}} |x_j|^2 < \infty\Big\}.$$

## 2. Review of $K$-frames

**Definition 2.1** ($K$-frame)**.** [3] Let $F := \{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of $H$ and $K \in \mathcal{B}(H)$. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$ if there exist $0 < A \leq B < \infty$ such that for each $f \in H$,

$$A\|K^*f\|^2 \leq \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \leq B\|f\|^2. \tag{1}$$

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If $K = Id_H$, then $F$ is an ordinary frame for $H$. We call $A$ and $B$ the lower and the upper frame bounds for $K$-frame $F$. If only the right hands of (1) holds, then $F$ is said a Bessel sequence. We say that $F$ is a tight $K$-frame for $H$, if for each $f \in H$, we have

$$\sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 = A\|K^* f\|^2.$$

When $A = 1$, the set $\{f_j\}_{j\in\mathbb{J}}$ is called a Parseval $K$-frame.

**Example 2.2.** [4] Let $H = \mathbb{C}^3$ and $\{e_1, e_2, e_3\}$ is an orthonormal basis for $H$. Define

$$K : H \to H$$
$$Ke_1 = e_1, Ke_2 = e_1, Ke_3 = e_2.$$

Then $\{f_i\}_{i=1}^3 := \{e_1, e_1, e_2\}$ is a $K$-frame for $H$.

## 3. Frame operators

Since every $K$-frame is a Bessel sequence, then (as frames) the synthesis and analysis operators is defined by

$$T_F : \ell^2(\mathbb{N}) \to H, \qquad T_F\{c_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} c_j f_j, \tag{2}$$

and

$$T_F^* : H \to \ell^2(\mathbb{N}), \qquad T_F^* f = \{\langle f, f_j\rangle\}_{j\in\mathbb{J}}. \tag{3}$$

Now, the frame operator is defined by $S_F = T_F T_F^*$ and for each $f \in H$ we obtain

$$S_F f = \sum_{j\in\mathbb{J}} \langle f, f_j\rangle f_j,$$

and

$$\langle S_F f, f\rangle = \sum_{j\in\mathbb{J}} |\langle f, f_j\rangle|^2 \geq A\|K^* f\|^2 = \langle AKK^* f, f\rangle.$$

It follows that $S_F \geq AKK^*$ and so the frame operator is not invertible on $H$. However, $S_F : \mathcal{R}(K) \to S_F\big(\mathcal{R}(K)\big)$ is invertible when the operator $K$ has closed range ([6]). So, we can construct a Parseval frame for $\mathcal{R}(K)$ by the following result.

**Proposition 3.1.** *([5]) Let $\{f_j\}_{j\in\mathbb{J}}$ be a $K$-frame for $H$ and $K$ has closed range. If $S_F\big(\mathcal{R}(K)\big) = \mathcal{R}(K)$, then*

*(I)* $\{S_F^{\frac{1}{2}} P_{\mathcal{R}(K)} f_j\}_{j\in\mathbb{J}}$ *is a Parseval frame for* $\mathcal{R}(K)$.

*(II)* $\{S_F^{\frac{1}{2}} P_{\mathcal{R}(K)} f_j\}_{j\in\mathbb{J}}$ *is a Parseval* $P_{\mathcal{R}(K)}$*-frame for* $H$.

Also, we can extract a relation between the eigenvalues of the operator $S_F$ for Parseval $K$-frames in finite case:

**Proposition 3.2.** *([5]) Let $F = \{f_j\}_{j\in\mathbb{J}}$ be a Parseval $K$-frame for $H$ and $\{\lambda_k\}_{k=1}^n$ be the eigenvalues for the frame operator $S$ where $n = \dim H$. Then $\lambda_k > 0$ for each $k = 1, 2, \cdots, n$ and also*

$$\sum_{j=1}^{n} \lambda_j = \sum_{i\in\mathbb{J}} \|f_i\|^2 = n\|K^* e_k\|^2, \tag{4}$$

*where $\{e_k\}_{k=1}^n$ is an orthonormal basis for $H$.*

Suppose that $F = \{f_j\}_{j\in\mathbb{J}}$ is a Parseval $K$-frame for $H$ and $K$ has closed range. It is obvious that $\ker KK^* = \ker K^*$, therefore we get $\mathcal{R}(KK^*) = \mathcal{R}(K)$. Then $\langle S_F f, f\rangle = \|K^* f\|^2$. So, $S_F = KK^*$ and we conclude that $\mathcal{R}(S_F) = \mathcal{R}(K)$. Now, we can obtain the optimal lower and upper frame bounds. Assume that $F$ is a $K$-frame for $H$. If $B_{op}$ is the optimal upper frame bound of $\{f_j\}_{j\in\mathbb{J}}$, then it is obvious that $B_{op} = \|S_F\|$. For the optimal lower bound, we can show that $A_{op} = \|X\|^{-2}$, where $X \in \mathcal{B}(H, \ell^2(\mathbb{J}))$ such that $T_F X = K$ and (for more details, we refer to [2]),

$$\|X\|^2 = \inf\{\alpha > 0, \quad KK^* \leq \alpha T_F T_F^*\}.$$

We notice that, if $K = Id_H$, then $X = T_F^* S_F$ and so $\|X\|^{-2} = \|S_F^{-1}\|^{-1}$.

## 4. Some properties of $K$-frames

First, we aim to introduce several methods to construct $K$-frames. In the following, $K$-frames is given by ordinary frames.

**Proposition 4.1.** *([6]) If $\{f_j\}_{j\in\mathbb{J}}$ is an ordinary frame for $H$, then $\{Kf_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$.*

Thus, if $\{e_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for $H$, then $\{Ke_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$.

**Proposition 4.2.** *If $U \in \mathcal{B}(H)$ and $\{f_j\}_{j\in\mathbb{J}}$ is $K$-frame for $H$, then $\{Uf_j\}_{j\in\mathbb{J}}$ is a $UK$-frame for $H$.*

We give a characterization of $K$-frames using linear bounded operators.

**Theorem 4.3.** *([3]) The set $F = \{f_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$ if and only if there exists an operator $U \in \mathcal{B}(\ell^2(\mathbb{J}), H)$ such that $f_j = U\delta_j$ and $\mathcal{R}(K) \subseteq \mathcal{R}(U)$, where $\{\delta_j\}_{j\in\mathbb{J}}$ is an orthonormal basis for $\ell^2(\mathbb{J})$.*

Notice that, in the proof of Theorem 4.3, we can check that the operator $U$ is the synthesis operator of $F$. In next result, a new $K$-frame is constructed by arbitrary $K$-frame.

**Theorem 4.4.** *([6]) Suppose that $F := \{f_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$ with frame bounds $A, B$, and $\alpha, \beta, \gamma \geq 0$ such that $\max\{\alpha + \gamma\sqrt{A^{-1}}\|K^\dagger\|, \beta\} < 1$, where $K^\dagger$ is the pseudo inverse of $K$. If $G := \{g_j\}_{j\in\mathbb{J}} \subset H$ and satisfy*

$$\Big\|\sum_{k=1}^{n} c_k(f_k - g_k)\Big\| \leq \alpha\Big\|\sum_{k=1}^{n} c_k f_k\Big\| + \beta\Big\|\sum_{k=1}^{n} c_k g_k\Big\| + \gamma\sqrt{\sum_{k=1}^{n}|c_k|^2},$$

*for any $c_k$, then $\{g_j\}_{j\in\mathbb{J}}$ is a $P_{Q(\mathcal{R}(K))}K$-frame for $H$, where $P_{Q(\mathcal{R}(K))}$ is the orthogonal projection from $H$ to $Q(\mathcal{R}(K))$, and $Q := T_G T_F^*$ where $T_F$ and $T_G$ are the synthesis operators for $F$ and $G$, respectively.*

Now, we want to sum $K$-frames and get a new $K$-frame.

**Theorem 4.5.** *([4]) Let $F = \{f_j\}_{j\in\mathbb{J}}$ and $G = \{g_j\}_{j\in\mathbb{J}}$ be $K$-frames for $H$ with the synthesis operators $T_F$ and $T_G$, respectively. If $T_F T_G^*$ and $T_G T_F^*$ are positive operators, then $\{f_j + g_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$.*

**Theorem 4.6.** *([4]) Let $F = \{f_j\}_{j\in\mathbb{J}}$ and $G = \{g_j\}_{j\in\mathbb{J}}$ be $K$-frames for $H$ with the synthesis operators $T_F$ and $T_G$, respectively such that $T_F T_G^* = 0$. If $U_1, U_2 \in \mathcal{B}(H)$ with $\mathcal{R}(T_F) \subset \mathcal{R}(U_1 T_F)$ and $\mathcal{R}(T_G) \subset \mathcal{R}(U_2 T_G)$, then $\{U_1 f_j + U_2 g_j\}_{j\in\mathbb{J}}$ is a $K$-frame for $H$.*

Suppose that $I \subset \mathbb{J}$ and let

$$S_I f = \sum_{j\in I}\langle f, f_j\rangle f_j, \quad (f \in H), \tag{5}$$

then, we can obtain some inequalities for $K$-frames. Let $F = \{f_j\}_{j\in\mathbb{J}}$ be a Parseval $K$-frame for $H$, $E \subseteq I^c$ and $f \in H$. Then ([2])

$$\|S_{I\cup E} f\|^2 - \|S_{I^c\setminus E} f\|^2 = \|S_I f\|^2 - \|S_{I^c} f\|^2 + 2\mathrm{Re}\sum_{j\in E}\langle f, f_j\rangle\overline{\langle KK^* f, f_j\rangle}.$$

Recall that a continuous function $h$ defined on a interval $I$ is said to be operator convex if

$$h(\lambda A + (1-\lambda)B) \leq \lambda h(A) + (1-\lambda)h(B),$$

for all $0 \leq \lambda \leq 1$ and all self-adjoint operators $A, B$ with spectra in $I$.

**Theorem 4.7.** *([2]) Let $\{f_j\}_{j\in\mathbb{J}}$ be a Parseval $K$-frame for $H$. Then for any $I \subset \mathbb{J}$,*

1. *if $h : [0, \|K\|^2] \to \mathbb{R}$ is an operator convex function, then*

$$h\Big(\frac{KK^*}{2}\Big) \leq \frac{h(S_I + S_{I^c})}{2};$$

2. *if $h : [0, \|K\|^2] \to \mathbb{R}$ is a convex function, then*

$$h\Big(\|K\|^2 Id_H - \frac{KK^*}{2}\Big) \leq h(\|K\|^2)Id_H - \frac{h(S_I + S_{I^c})}{2}.$$

The following result shows a natural one-to-one correspondence between certain subspaces of $H$ and Parseval $K$-frames.

**Theorem 4.8.** *([5]) Let $K$ be closed range, $P$ be an orthonormal projection on $H$ onto $\mathcal{R}(K^\dagger)$ such that $KP$ be a rank-$m$ and* $\dim H < \infty$. *Define*

$$\mathcal{K} = \{\text{all Parseval } K\text{-frames for } KPH\}$$
$$\mathcal{M} = \{W \prec H, \quad \dim W = m\}.$$

*Then there exists a natural one-to-one correspondence between $\mathcal{K}$ and $\mathcal{M}$.*

## 5. $K$-duals and $K$-norms

In this section, first we introduce the notion of $K$-duals for $K$-frames and characterize such duals. Then, we will define $K$-norm vectors and present some properties about this vectors and $K$-duals.

**Definition 5.1.** Let $F := \{f_j\}_{j\in\mathbb{J}}$ be a $K$-frame and $G := \{g_j\}_{j\in\mathbb{J}}$ be a Bessel sequence for $H$ with synthesis operators $T_F$ and $T_G$ respectively. We say that $G$ is a $K$-dual for $F$ if

$$T_F T_G^* = K. \tag{6}$$

In this case, we can write for each $f \in H$,

$$Kf = \sum_{j\in\mathbb{J}} \langle f, g_j\rangle f_j, \tag{7}$$

and it is easy to see that $G$ is a $K^*$-frame for $H$. In the following, we can study to characterize all $K$-dual of a $K$-frame.

**Theorem 5.2.** *([1]) Let $F := \{f_j\}_{j\in\mathbb{J}}$ be a $K$-frame. Then $\{g_j\}_{j\in\mathbb{J}}$ be a $K$-dual of $F$ if and only if $\{g_j\}_{j\in\mathbb{J}} = \{V\delta_j\}_{j\in\mathbb{J}}$ where $\{\delta_j\}_{j\in\mathbb{J}}$ is the standard orthonormal basis of $\ell^2(\mathbb{J})$ and $V : \ell^2(\mathbb{J}) \to H$ is a bounded operator such that $TV^* = K$. In this case, $\{g_j\}_{j\in\mathbb{J}}$ is in fact a Parseval $V$-frame.*

**Theorem 5.3.** *([1]) Let $F := \{f_j\}_{j\in\mathbb{J}}$ be a $K$-frame and $K$ has closed range. Then $\{K^* S_F^{-1} P_{S_F(\mathcal{R}(K))} f_j\}_{j\in\mathbb{J}}$ is a $K$-dual of $P_{\mathcal{R}(K)} F$.*

The set $\{K^* S_F^{-1} P_{S_F(\mathcal{R}(K))} f_j\}_{j\in\mathbb{J}}$ in Theorem 5.3 is called the *canonical $K$-dual* of $P_{\mathcal{R}(K)}F$. In the next, we present several equalities and inequalities for $K$-frames, $K$-duals.

**Theorem 5.4.** *([2]) Let $F = \{f_j\}_{j\in\mathbb{J}}$ be a $K$-frame for $H$ and $G = \{g_j\}_{j\in\mathbb{J}}$ be a $K$-dual of $F$. Then for each $I \subseteq \mathbb{J}$ and $f \in H$,*

$$\Big(\sum_{j\in I} \langle f, g_j\rangle \overline{\langle Kf, f_j\rangle}\Big) - \Big\|\sum_{j\in I} \langle f, g_j\rangle f_j\Big\|^2 = \Big(\sum_{j\in I^c} \overline{\langle f, g_j\rangle}\langle Kf, f_j\rangle\Big) - \Big\|\sum_{j\in I^c} \langle f, g_j\rangle f_j\Big\|^2.$$

We say that $F = \{f_j\}_{j\in\mathbb{J}}$ is an *equal-norm-K-frame* for $H$, when $F$ is a $K$-frame and $\|f_j\| = c$ for every $j \in \mathbb{J}$. When $c = \|K\|$, we call $\{f_j\}_{j\in\mathbb{J}}$ a *K-norm-frame*. Via Proposition 3.2, if $F$ is an equal-norm-Parseval $K$-frame for $H$, then for each $k = 1, \cdots, n$, we can get

$$\|f_j\|^2 = \frac{n}{m}\|K^* e_k\|^2, \qquad (j = 1, \cdots, m).$$

The following result shows that each finite set of $K$-norm vectors can be extended to become a $K$-norm of a $K$-frame.

**Theorem 5.5.** *([5]) If $F := \{f_j\}_{j=1}^m$ is a set of K-norm vectors in H, then there is a K-norm-frame for H which contains the set F.*

In the next, we characterize all Parseval $K$-frames as a projection of an orthonormal basis from a larger space.

**Theorem 5.6.** *([5]) Let $F = \{f_j\}_{j\in\mathbb{J}}$ be a Parseval K-frame for H and K be closed range. Then there is a larger Hilbert space $M \supseteq \mathcal{R}(K^\dagger)$ and an orthonormal basis $\{e_j\}_{j\in\mathbb{J}}$ for M so that the orthonormal projection P of M onto $\mathcal{R}(K^\dagger)$ satisfies $f_j = KPe_j$ for each $j \in \mathbb{J}$.*

Finally, the following is given a interesting result about equal-norm-Parseval $K$-frames and the number of their duality. First, we define the set $K^\natural$ as a left inverse of $K$ and $\{\delta_j\}_{j=1}^m$ is the orthonormal basis for $\ell^2(\mathbb{J})$.

**Theorem 5.7.** *([5]) Let $F = \{f_j\}_{j=1}^m$ be an equal-norm-Parseval K-frame for the finite Hilbert space H with the synthesis operator $T_F$ such that $T_F^* H \perp U^* H$ and $K^\natural f_j \perp U\delta_j$ for each $1 \le j \le m$ where $U \in \mathcal{B}(\ell^2(\mathbb{J}), H)$ and $K^\natural|_F$ are two isometry. Then F has infinitely many dual equal-norm K-frames.*

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# Coupled fixed point theorems for contractive type mappings in partially Menger probabilistic metric spaces

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**Abstract**

In this paper, we introduce a new type of contraction mapping and obtain certain fixed point, coupled fixed point, tripled fixed point results on Menger and partially ordered Menger spaces.This work is in line with the research for extending and improving the Banach contraction principle on menger spaces.

## 1. Introduction and Preliminaries

Coupled fixed point results constitute a domain in metric fixed point theory which has experienced rapid development in recent times. The consept of couple fixed point was introduced some time back in 1989 by Guo et al. [6]. But only after the publication of the work of Bhaskar et al. [3] a large number of papers have been written on this topics related to it [1, 2] It is known that Menger PM-space is a significant generalization of ordinary metric spaces. Numerous fixed point results werw obtained by authors wiith respect to different contractive conditions for self mappings in Menger PM-space; see e.g., [4, 5, 9] The notion of a probabilistic metric spaces is introduced in 1942 by K. Menger.
A Menger space is a triple $(V, \nu, *)$, where $V$ is a non-empty set, $\nu$ is a mapping from $V \times V$ into $\Delta^+$ (for $(x, y) \in V \times V$, the distribution function $\nu(x, y)$ is denoted by $\nu_{x,y}$ , and $\nu_{x,y}(t)$ is the value $\nu_{x,y}$ at $t \in \Re$) and $*$ is a $t-$norm satisfying the following conditions:
(M-1) $\nu_{x,y}(0) = 0$ for all $x, y \in V$ ;
(M-2) $\nu_{x,y}(t) = 1 \quad$ for all $\quad t > 0 \quad$ iff $\quad x = y$;
(M-3) $\nu_{x,y}(t) = \nu_{y,x}(t)$, for all $x, y \in V$;
(M-4) $\nu_{x,y}(s + t) \geq \nu_{x,z}(s) * \nu_{z,y}(t)$ for all $x, y, z \in V$ and $s, t \in [0, 1]$ .

**Definition 1.1.** A triangular norm $*$ is said to be of $H$-type if a family of functions $\{*^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t = 1$, that is,

$$\forall \epsilon \in (0, 1)\ \exists\, \delta \in (0, 1)\ :\ t > 1 - \delta \Rightarrow\ *^n(t) > 1 - \epsilon \quad (n \geq 1)\,.$$

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## 2. Main results

In this section, we introduce a concept of $\alpha$-$\psi$-contractive type mappings and establish various fixed point, coupled fixed point and tripled fixed point theorems for such mapping in complete Menger spaces. Denote with $\Psi$ the family of functions $\psi : [0,1] \to [0,1]$ such that $\psi(t) \geq t$ for each $t \geq 0$ .

**Definition 2.1** Let$(V, \nu, *)$ be Menger space and $F : V \to V$ be a given mapping. We say that $F$ is an $\alpha$-$\psi$- contractive mapping if there exists function $\alpha : V \times V \to [0, +\infty)$ such that

$$\alpha(x,y)(\nu_{Fx,Fy}(kt)) \geq \psi(\nu_{x,y}(t)), \qquad (1)$$

for all $x, y \in V$ and $t > 0$, where $0 < k < 1$.

**Theorem 2.1** Let $(V, \nu, *)$ be a complete Menger space such that the $t$-norm $*$ is of $H$-type. Let $F : V \to V$ be an $\alpha$-$\psi$-contractive mapping satisfing the following conditions:
(i) if there exists $x, y \in V$ such that $\alpha(x,y) \leq 1$, then $\alpha(Fx, Fy) \leq 1$;
(ii) there exists $x_0 \in V$ such that $\alpha(x_0, Fx_0) \leq 1$;
(iii) $F$ is continuous.
Then, $F$ has a fixed point .

**Proof:** Let $x_0 \in V$ such that $\alpha(x_0, Fx_0) \leq 1$. Define the sequence $\{x_n\}$ in $V$ by $x_{n+1} = Fx_n$, for all $n \in N$. If $x_n = x_{n+1}$ for some $n \in N$, then $x^* = x_n$ is a fixed point for $F$. Assume that $x_n \neq x_{n+1}$.By induction, we get

$$\alpha(x_n, x_{n+1}) \leq 1, \quad for\ all \quad n \in N. \qquad (2)$$

Now put $\delta_n(t) := \nu_{x_n,x_{n+1}}(t)$. Since $F$ is $\alpha$-$\psi$-contractive, we have

$$\begin{aligned} \nu_{x_n,x_{n+1}}(kt) = \nu_{Fx_{n-1},Fx_n}(kt) &\geq \alpha(x_{n-1}, x_n)\nu_{Fx_{n-1},Fx_n}(kt) \\ &\geq \psi(\nu_{x_{n-1},x_n}(t)) \geq \nu_{x_{n-1},x_n}(t) = \delta_{n-1}(t). \end{aligned}$$

Thus, it follows that $\delta_n(kt) \geq \delta_{n-1}(t)$, and so $\delta_n(t) \geq \delta_{n-1}(\frac{t}{k}) \geq \delta_0(\frac{t}{k^n})$. (3)

$$\begin{aligned} \nu_{x_n,x_m}(t) &\geq \nu_{x_n,x_m}(t(1-k)(1+k+\ldots+k^{m-n-1})) \\ &\geq \nu_{x_n,x_{n+1}}(t(1-k)) * \nu_{x_{n+1},x_m}(t(1-k)(1+k+\ldots+k^{m-n-1})) \\ &\geq \nu_{x_n,x_{n+1}}(t(1-k)) * \nu_{x_{n+1},x_{n+2}}(t(1-k)k) * \ldots * \nu_{x_{m+1},x_m}(t(1-k)k^{m-n-1}) \qquad (4) \end{aligned}$$

It follows from (3) and (4) that

$$\nu_{x_n,x_m}(t) \geq [\delta_0(\frac{t(1-k)}{k^n})] * \ldots * [\delta_0(\frac{t(1-k)}{k^n})] = *^{m-n}\delta_0(\frac{t(1-k)}{k^n})$$

By the hypothesis, the $t$-norm $*$ is of $H$-type; for all $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that $*^n(s) > 1 - \varepsilon$, for all $s \in (1-\delta, 1]$ and for all $n$. Note that $\lim_{n\to\infty} \delta_0(\frac{t(1-k)}{k^n}) = 1$ ,
for all $t > 0$ and $0 < k < 1$, we have there exists $n_0$ such that $\nu_{x_n,x_m}(t) > 1 - \varepsilon$, for all $m > n > n_0$. Thus $\{x_n\}$ is a cauchy sequence. Since $(V, \nu, *)$ is complete Menger space, there exists $x^* \in V$ such that $x_n \xrightarrow{\nu} x^*$. From the continuity of $F$, we get that $x_{n+1} = Fx_n \xrightarrow{\nu} Fx^*$. By the uniqueness of the convergent point, we get $x^* = Fx^*$, that is, $x^*$ is a fixed point of $F$. □

In the next theorem, we omit the continuity hypothesis of $F$ and proved the version of Nieto and Rodriguez-Lopez [7] 's main theorem in Menger spaces.

**Theorem 2.2** Let $(V, \nu, *)$ be a complete Menger space with a t-norm $*$ defined as $a * b = \min\{a, b\}$. Let $F : V \to V$ be an $\alpha$-$\psi$-contractive mapping satisfing the following conditions:
(i) if there exist $x, y \in V$ such that $\alpha(x,y) \leq 1$ then $\alpha(Fx, Fy) \leq 1$;
(ii) there exists $x_0 \in V$ such that $\alpha(x_0, Fx_0) \leq 1$;
(iii) if $\{x_n\}$ is a sequence in $V$ such that $\alpha(x_n, x_{n+1}) \leq 1$, for all $n$ and $x_n \xrightarrow{\nu} x \in V$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n$.
Then, $F$ has a fixed point.

Now, we will show that our previous obtained results allow us to derive coupled fixed point theorems on complete Menger PM-spaces. At first we recall the following definition.

**Definition 2.2**(Bhaskar and Lakshmikantham [3]) Let $F : V \times V \to V$ be a given mapping. We say that $(x, y) \in V \times V$ is a coupled fixed point of $F$ if

$$F(x, y) = x \quad and \quad F(y, x) = y .$$

Our results are based on the following simple lemma.

**Lemma 2.1(A coupled fixed point is a fixed point)** Let $F : V \times V \to V$ be a given mapping. Define the mapping $T : V \times V \to V \times V$ by

$$T(x, y) = (F(x, y), F(y, x)), \quad for\ all\ (x, y) \in V \times V .$$

Then, $(x, y)$ is a coupled fixed point of $F$ if and only if $(x, y)$ is a fixed point of $T$.

**Theorem 2.3** Let $(V, \nu, *)$ be a complete Menger space with a t-norm $*$ is of $H$-type, and $F : V \times V \to V$ be a given mapping. suppose that there exists $\psi \in \Psi$ and function $\alpha : V^2 \times V^2 \to [0, +\infty)$ such that

$$\alpha((x, y), (u, v))(\nu_{F(x,y)-F(u,v)}(kt)) \geq \psi(\nu_{x-u}(t) * \nu_{y-v}(t)) , \qquad (6)$$

for all $(x, y), (u, v) \in V \times V$ and $t > 0$, where $0 < k < 1$.

Suppose also that

(i) For all $(x, y), (u, v) \in V \times V$ , we have

$$\alpha((x, y), (u, v)) \leq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \leq 1 ;$$

(ii) there exists $(x_0, y_0) \in V \times V$ such that

$$\alpha((x_0, y_0), (F(x_0, y_0), F(y_0, x_0)) \leq 1 \quad and \quad \alpha((F(y_0, x_0), F(x_0, y_0)), ((y_0, x_0),) \leq 1 ;$$

(iii) $F$ is continuous

Then $F$ has a coupled fixed point .

**Proof:** The idea consists in transporting the problem to the complete Menger space $(W, \tilde{\nu}, *)$, where $W = V \times V$ and $\tilde{\nu}_{(x,y)(u,v)}(t) = \nu_{x,u}(t) * \nu_{y,v}(t)$ for any $(x, y) \in V \times V$ and any $t > 0$. Since $(V, \nu, *)$ is a complete Menger space , we can easily prove that $(W, \tilde{\nu}, *)$ is a complete Menger space. From (6), we have

$$\alpha((x, y), (u, v))(\nu_{F(x,y),F(u,v)}(kt)) \geq \psi(\nu_{x,u}(t) * (\nu_{y,v}(t)) \qquad (7)$$

and

$$\alpha((v, u), (y, x))(\nu_{F(v,u)-F(y,x)}(kt)) \geq \psi(\nu_{x,u}(t) * (\nu_{y,v}(t)) \qquad (8)$$

We define a mapping $\tilde{F} : V \times V \to V \times V$ by $\tilde{F}(x, y) = (F(x, y), F(y, x)), \quad \forall (x, y) \in V \times V$ .

If $\zeta = (x, y), \eta = (u, v) \in W$, and $\beta : W \times W \to [0, +\infty)$ is the function define by

$$\beta((x, y), (u, v)) = \max\{\alpha((x, y), (u, v)), \alpha((v, u), (y, x))\} ;$$

then by (7) and (8) we get $\beta(\zeta, \eta)(\tilde{\nu}_{\tilde{F}\zeta,\tilde{F}\eta}(kt)) \geq \psi(\tilde{\nu}_{\zeta,\eta}(t))$ .

Then $\tilde{F}$ is continuous and $\beta$-$\psi$-contractive mapping.

Let $\beta(\zeta, \eta) \leq 1$. Using condition (i), we obtain that $\beta(\tilde{F}\zeta, \tilde{F}\eta) \leq 1$. Then $\tilde{F}$ satisfies in condition (i) of Theorem 2.1. Moreover, from condition (ii), we know there exists $(x_0, y_0) \in W$ such that $\beta((x_0, y_0), \tilde{F}(x_0, y_0)) \leq 1$. All the hypotheses of Theorem 2.1 are satisfied, and so we deduce the existence of a fixed point of $\tilde{F}$ that gives us from Lemma 2.1 the existence of a couple fixed point of $F$. □

Now, we omit the continuity hypothesis of $F$.

**Theorem 2.4** Let $(V, \nu, *)$ be a complete Menger space with a t-norm $*$ defined as $a * b = \min\{a, b\}$, and $F : V \times V \to V$ be a given mapping. suppose that there exists $\psi \in \Psi$ and function $\alpha : V^2 \times V^2 \to [0, +\infty)$ such that

$$\alpha((x, y), (u, v))(\nu_{F(x,y),F(u,v)}(kt)) \geq \psi(\nu_{x,u}(t) * \nu_{y,v}(t)) ,$$

for all $(x,y),(u,v) \in V \times V$ and $t > 0$, where $0 < k < 1$.
Suppose also that
(i) For all $(x,y),(u,v) \in V \times V$, we have

$$\alpha((x,y),(u,v)) \leq 1 \Rightarrow \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u))) \leq 1\,;$$

(ii) there exists $(x_0,y_0) \in V \times V$ such that

$$\alpha((x_0,y_0),(F(x_0,y_0),F(y_0,x_0)) \leq 1 \quad and \quad \alpha((F(y_0,x_0),F(x_0,y_0)),((y_0,x_0),) \leq 1\,;$$

(iii) if $\{x_n\}$ and $\{y_n\}$ are sequences in $V$ such that $\alpha((x_n,y_n),(x_{n+1},y_{n+1})) \leq 1$ and $\alpha((y_{n+1},x_{n+1}),(y_n,x_n)) \leq 1$, $x_n \xrightarrow{\nu} x \in V$ and $y_n \xrightarrow{\nu} y \in V$, then

$$\alpha((x_n,y_n),(x,y)) \leq 1 \quad and \quad \alpha((y,x),(y_n,x_n)) \leq 1 \quad \forall n \in N\,.$$

Then , $F$ has a couple fixed point.

## 3. Fixed Point Theorems in Partially Ordered Set

The existence of fixed point in partially ordered sets has been considered in [8]. In this section, we will show that fixed point results in partially ordered sets for Menger spaces. The Banach-Caccioppoli type principle in orded metric spaces proved by Ran and Reurings [8]. The following is the version of their main theorem in Menger space.
**Theorem 3.1** Let $(V,\sqsubseteq)$ be a partially ordered set and $(V,\nu,*)$ be a complete Menger space with a t-norm $*$ defined as $a*b = \min\{a,b\}$. Let $F : V \to V$ be a continuous and nondecreasing mapping $w.r.t\ \sqsubseteq$. Suppose that the following hold:
(i) There exists $k \in [0,1)$ such that

$$\nu_{Fx-Fy}(kt) \geq \nu_{x-y}(t)$$

for all $x,y \in V, t > 0$ with $x \sqsubseteq y$;
(ii) There exists $x_0 \in V$ such that $x_0 \sqsubseteq Fx_0$;
(iii) $F$ is continuous.
Then $F$ has a fixed point.
**Proof:** Define the mapping $\alpha : V \times V \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \sqsubseteq y \\ 2 & otherwise \end{cases}$$

From (i) we have

$$\alpha(x,y)(\nu_{Fx,Fy}(t)) \geq \psi(\nu_{x,y}(t)) \qquad \forall x,y \in V\,.$$

Then $F$ is an $\alpha$-$\psi$-contractive mapping with $\psi(t) = t$ for all $t > 0$. Now let $x,y \in V$ such that $\alpha(x,y) \leq 1$. By the definition of $\alpha$, this implies that $x \sqsubseteq y$. Since $F$ is a nondecreasing mapping $w.r.t\ \sqsubseteq$, we have $Fx \sqsubseteq Fy$, which gives us that $\alpha(Fx,Fy) = 1$. Also From (ii), there exists $x_0 \in V$ such that $x_0 \sqsubseteq Fx_0$. This implies that $\alpha(x_0,Fx_0) = 1$. Therefore, all the hypotheses of Theorem 2.1 are satisfied and so $F$ has a fixed point. □
Bhaskar and Lakshmikantham [3] introduced the following notions of a mixed monotone property for operators of the form $F : X \times X \to X$, where $X$ is a partially ordered metric space.
**Definition 3.1** Let $(V,\sqsubseteq)$ be a partially ordered set and $F : V \times V \to V$ be a given mapping. We say that $F$ has the mixed monotone property if for all $x,y \in V$, we have

$$x_1,x_2 \in V, \quad x_1 \sqsubseteq x_2 \Longrightarrow F(x_1,y) \sqsubseteq F(x_2,y)$$

and

$$y_1,y_2 \in V, \quad y_1 \sqsubseteq y_2 \Longrightarrow F(x,y_2) \sqsubseteq F(x,y_1)\,.$$

The main theoretical results of Bhaskar and Lakshmikantham [3] in Menger space are the following coupled fixed point theorem.

**Theorem 3.2** Let $(V, \sqsubseteq)$ be a partially ordered set and $(V, \nu, *)$ be a complete Menger space with a t-norm $*$ defined as $a * b = \min\{a, b\}$. Let $F : V \times V \to V$ be a continuous mapping having the mixed monotone property on $V$. Assume that there exists $k \in [0, 1)$ with

$$\nu_{F(x,y),F(u,v)}(kt) \geq \nu_{x,u}(t) * \nu_{y,v}(t) \qquad (14)$$

for all $x, y, u, v \in V$ such that $x \sqsupseteq u$ and $y \sqsubseteq v$. If there exists $(x_0, y_0) \in V \times V$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $y_0 \sqsupseteq F(y_0, x_0)$. Then $F$ has a coupled fixed point.

**Proof:** Define the mapping $\alpha : V^2 \times V^2 \to [0, +\infty)$ by

$$\alpha((x, y), (u, v)) = \begin{cases} 1 & \text{if } x \sqsupseteq u, y \sqsubseteq v, \\ 2 & otherwise\,. \end{cases} \qquad (15)$$

It follows from (14) that

$$\alpha((x, y), (u, v))(\nu_{F(x,y),F(u,v)}(kt)) \geq \psi(\nu_{x,u}(t) * \nu_{y,v}(t))$$

for all $x, y, u, v \in V$. It implies that $T$ is a $\alpha$-$\psi$-contractive mapping with $\psi(t) = t$ for all $t \geq 0$. Let $(x, y), (u, v) \in V \times V$ such that $\alpha((x, y), (u, v)) \geq 1$. By the definition of $\alpha$, this implies that $x \sqsupseteq u$ , $y \sqsubseteq v$. Since $F$ satisfies the mixed monotone property, we have $F(x, y) \sqsupseteq F(u, v)$ and $F(y, x) \sqsubseteq F(v, u)$. Then we get

$$\alpha(((x, y), (y, x)), (F(u, v), F(v, u))) \leq 1$$

By hypothesis, we know that there exists $(x_0, y_0) \in V \times V$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $y_0 \sqsupseteq F(y_0, x_0)$. Now by the definition of $\alpha$, this implies that

$$\alpha((a, b), (F(a, b), F(b, a))) \leq 1\,,$$

$$\alpha((F(b, a), F(a, b)), (b, a)) \leq 1$$

where $a = x_0$, $b = y_0$. All the hypotheses of Theorem (2.3) are satisfied, then $F$ has a coupled fixed point. □

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# Well-posed of Benson properly efficient solutions for set-valued equilibrium problems

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**Abstract**

In this paper, we define some notions of well-posedness for type I-E-Benson properly efficient solution and type I-E-Benson properly efficient solution for the set-valued equilibrium problems and obtain some of their properties. In the end, we attain sufficient and necessary conditions for well-posedness of these problems.

## 1. Introduction

We show by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and all positive real numbers. Let $X$ be a real linear space, and $Y$ and $W$ be two real locally convex spaces that their zero element is denoted by $0_Y$ and $0_W$, respectively. The cone generated by a subset $B$ of $Y$ is defined as $coneB := \{\lambda b : b \in B, \lambda \geq 0\}$. The closure of $coneB$ is denoted by $clconeB$. We say that a cone $U \subseteq Y$ is pointed if $U \cap (-U) = \{0\}$. We assume that $U \subseteq Y$ and $V \subseteq W$ are two nontrivial pointed closed convex cones that their interior is nonempty.

## 2. Benson properly efficient solution

**Definition 2.1** ([1], Definition 2.1)**.** Let $U$ be a pointed closed convex cone in $Y$. A nonempty subset $S$ of $Y$ is said to be an improvement set with respect to $U$ if $0 \notin S$ and $S + U = S$. The family of all improvement sets in $Y$ with respect to $U$ is denoted by $\mathfrak{L}_U$.

**Definition 2.2.** [[3], Definition 3.1] Let $S \in \mathfrak{L}_U$ and $D$ be a nonempty subset of $Y$.

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- (a) An element $d \in D$ is called an $S$-efficient point of $D$ if $(d - S) \cap D = \emptyset$. The set of all $S$-efficient points of $D$ is denoted by $O^S(D)$.
- (b) An element $d \in D$ is called an $S$-quasi efficient point of $D$ if $(d - S - (U \setminus \{0\})) \cap D = \emptyset$. The set of all $S$-quasi efficient points of $D$ is denoted by $O_q^S(D)$.
- (c) An element $d \in D$ is called an $S$-weakly efficient point of $D$ if $(d - intS) \cap D = \emptyset$. The set of all $S$-weakly efficient points of $D$ is denoted by $O^{intS}(D)$.

**Remark 2.3** ([3], Remark 3.2)**.** When $S = U \setminus \{0\}$, (a) and (c) of Definition 2.2 reduce to the definitions [3, Definition 2.4.1] and [3, Definition 2.4.2] in vector optimization, respectively.

**Remark 2.4** ([3], Remark 3.3)**.** It is easy to see that the following inclusion relations hold:

$$O^S(D) \subseteq O_q^S(D) \subseteq O^{intS}(D).$$

However, $O^{intS}(D) \nsubseteq O_q^S(D)$ and $O_q^S(D) \nsubseteq O^S(D)$.

**Definition 2.5** ([2], Definition 4.2)**.** Let $S \in \mathfrak{L}_U$ and $D$ be a nonempty subset of $Y$. An element $d \in D$ is called a type $I - S$-Benson properly efficient point of $D$ if $clcone(D + S - d) \cap (-U) = \{0\}$. The set of all type $I - S$-Benson properly efficient points of $D$ is denoted by $O_{(BS,I)}^S(D)$.

**Remark 2.6** ([3], Remark 3.6)**.** When $S = U \setminus \{0\} \in \mathfrak{L}_U$, we have

$$clcone(D + S - d) = clcone(D + U \setminus \{0\} - d) + clcone(D + U - d).$$

**Remark 2.7** ([3], Remark 3.7)**.** It follows from [[2], Remark 4.1 and Example 4.1] that $O_{(BS,I)}^S(D) \subseteq O_q^S(D)$ and $O_q^S(D) \nsubseteq O_{(BS,I)}^S(D)$.

**Definition 2.8** ([4], Definition 3.2)**.** Let $S \in \mathfrak{L}_U$ and $D$ be a nonempty subset of $Y$. An element $d \in D$ is called a type $II - S$-Benson properly efficient point of $D$ if $clcone(D + U - d) \cap (-S) = \emptyset$. The set of all type $II - S$-Benson properly efficient points of $D$ is denoted by $O_{(BS,II)}^S(D)$..

**Remark 2.9** ([3], Remark 3.11)**.** It is easy to see that $O_{(BS,I)}^S(D) \subseteq O^S(D)$.

**Definition 2.10** ([2], Definition 4.2)**.** Let $S \in \mathfrak{L}_U$ and $D$ be a nonempty subset of $Y$.

- A sequence $(d_n) \subseteq D$ is called a type $I - S$-Benson approximate solution sequence of $D$ if $clcone(D + S - d_n) \cap (-U) = \{0\}$.
- A sequence $(d_n) \subseteq D$ is called a type $II - S$-Benson approximate solution sequence of $D$ if $clcone(D + U - d_n) \cap (-S) = \emptyset$.

Let $F : B \longrightarrow 2^Y$ and $G : B \longrightarrow 2^W$ be a set-valued map. We investigate the following set-valued optimization problem (for short, $SVOP$):

$$\min_{x \in B} F(x), \qquad (SVOP)$$

where $A := \{x \in B : G(x) \cap (-V) \neq \emptyset\}$.

**Definition 2.11** ([3], Definition 4.1)**.** Let $\bar{x} \in A$ and $\bar{y} \in F(\bar{x})$.

- The element $\bar{x}$ is called a type $I - S$-Benson properly efficient solution of $(SVOP)$ if

$$clcone(F(B) + S - \bar{y}) \cap (-U) = \emptyset.$$

- The element $\bar{x}$ is called a type $II - S$-Benson properly efficient solution of $(SVOP)$ if

$$clcone(F(B) + U - \bar{y}) \cap (-S) = \emptyset.$$

The pair $(\bar{x}, \bar{y})$ is called a type $II - S$-Benson proper minimizer of $(SVOP)$.

**Definition 2.12.** Let sequence $\{x_n\} \subseteq B$ and $\{y_n\} \subseteq F(\{x_n\})$.

- The sequence $\{x_n\}$ is called an approximate type $I - S$-Benson properly efficient solution sequence of $(SVOP)$, if $clcone(F(B) + S - y_n) \cap (-U + \varepsilon_n) = \{0\}$.
- The sequence $\{x_n\}$ is called an approximate type $II - S$-Benson properly efficient solution sequence of $(SVOP)$ if
$$clcone(F(B) + U - y_n) \cap (-S + \varepsilon_n) = \emptyset.$$

**Definition 2.13.** Let $\bar{S}$ denote the solution set of the $(SVOP)$. Then we say that the $(SVOP)$ is

- Tykhonov well-posed if $\bar{S}$ is a singelton set and every approximate type $I - S$-Benson properly efficient solution sequence for the $(SVOP)$ converges to the unique solution.
- generalized well-posed if $\bar{S}$ is a nonempty set and every approximate type $I - S$-Benson properly efficient solution sequence for the $(SVOP)$ has a subsequence which converges to some $\bar{x} \in \bar{S}$.
- Tykhonov well-posed if $\bar{S}$ is a singelton set and every approximate type $II - S$-Benson properly efficient solution sequence for the $(SVOP)$ converges to the unique solution.
- generalized well-posed if $\bar{S}$ is a nonempty set and every approximate type $II - S$-Benson properly efficient solution sequence for the $(SVOP)$ has a subsequence which converges to some $\bar{x} \in \bar{S}$.

In the sequal, we specify approximate solution set to $(SVOP)$, then we recognize metric characterization of the well-posed. Let $\varepsilon \in \mathbb{R}^+$, we costract approximate solution set to $(SVOP)$, as

$$S(\varepsilon_I) = \{(\bar{x}, \bar{z}) : \bar{z} \in F(\bar{x}), clcone(F(B) + U - \bar{z}) \cap (-S + \varepsilon) = \emptyset.$$

$$S(\varepsilon_I I) = \{(\bar{x}, \bar{z}) : \bar{z} \in F(\bar{x}), clcone(F(B) + U - \bar{z}) \cap (-S + \varepsilon) = \emptyset.$$

Note that $\cup_{\varepsilon>0} S(\varepsilon) = S$ and if $\varepsilon < \varepsilon_0$, then $S(\varepsilon) < S(\varepsilon_0)$.

**Theorem 2.14.** *Necessary and sufficient conditions for generalized well-posed of problem* $(SEP)$ *are*

- *solution set* $\bar{S}$ *is nonempty*
- $diam(S(\varepsilon)) \longrightarrow 0$ *as* $\varepsilon \longrightarrow 0$.

**Theorem 2.15.** *Necessary and sufficient conditions for generalized well-posed of Problem* $(SEP)$ *are*

- *solution set* $\bar{S}$ *is nonempty and compact,*
- $H(S(\varepsilon), S) \longrightarrow 0$ *as* $\varepsilon \longrightarrow 0$.

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# The Generalized Laguerre fractional differential equation and a Heuristic Method for its Solution

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**Abstract**

In this paper, an analytical method for solving the fractional Laguerre differential equation using the fractional Laplace transform defined by Abdeljawad [1] is presented. The aim of this paper is to convert the fractional Laguerre differential equation into ordinary differential equations of first order, this is done by using the fractional Laplace transform of $(\alpha+\beta)$ order.

## 1. First Section

The aim of this paper is to present a novel analytical method for solving the fractional Laguerre differential equation using the conformable fractional derivative and the fractional Laplace transform. The conformable fractional derivative is a relatively new concept in fractional calculus that requires further investigation. The conformable fractional derivative of order $0 < \alpha \leq 1$ starting from $a \in \mathbb{R}$ the function
$f : [a, +\infty) \to \mathbb{R}$, is defined by

$$({}_tT_\alpha^a f)(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon(t-a)^{1-\alpha}\right) - f(t)}{\epsilon}. \tag{1}$$

When $a = 0$, we have:

$$({}_tT_\alpha^0 f)(t) = {}_tT_\alpha f(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon} = .$$

By this definition we have

$$\left(T_\alpha^a f\right)(t) = (t-a)^{1-\alpha} f'(t) \tag{2}$$

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The conformable fractional Laplace transform (CFLT) of function $f:[0,\infty)\to\mathbb{R}$ for $t>0$, of order $0<\alpha\leq 1$, starting from a of u is defined by

$$L_\alpha^a\{f(t)\}=\int_a^\infty e^{-s\frac{(t-a)^\alpha}{\alpha}}f(t)(t-a)^{\alpha-1}dt=F_\alpha^a(s). \tag{3}$$

If a=0, we have

$$L_\alpha^0\{f(t)\}=\int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}f(t)t^{\alpha-1}dt=F_\alpha^0(s)=F_\alpha(s). \tag{4}$$

In particular, if $\alpha=1$, then Eq, (4) is reduced to the definition of the Laplace transform.

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=F(s). \tag{5}$$

**Theorem 1.1.** *Let* $f:[a,\infty)\to\mathbb{R}$ *be twice differentiable on* $(a,\infty)$, $\alpha,\ \beta>0$ *and* $\alpha+\beta\leq 1$. *Then,*

$$ {}_tT_\beta[{}_tT_\alpha f(t)]=(1-\alpha)t^{1-(\alpha+\beta)}f'(t)+t^{2-(\alpha+\beta)}f''(t). \tag{6}$$

*Proof.* See [5]. □

**Theorem 1.2.** *Let* $f:[a,\infty)\to\mathbb{R}$ *be twice differentiable on* $(a,\infty)$, $\alpha,\ \beta>0$ *and* $\alpha+\beta\leq 1$. *Then,*

$$ {}_tT_\beta[{}_tT_\alpha f(t)]=(1-\alpha)t^{1-(\alpha+\beta)}f'(t)+t^{2-(\alpha+\beta)}f''(t). \tag{7}$$

*Proof.* See [5]. □

**Theorem 1.3.** *If* $f:[a,\infty)\to\mathbb{R}$ *is twice differentiable on* $(a,\infty)$, $\alpha,\ \beta>0$ *and* $\alpha+\beta\leq 1$, *then the following equalities hold.*

1) $L_{(\alpha+\beta)}^0\left\{{}_tT_\beta\left({}_tT_\alpha f(t)\right)\right\}=\alpha f(0)-(2\alpha+\beta)sF_{(\alpha+\beta)}(s)-(\alpha+\beta)s^2F'_{(\alpha+\beta)}(s).$

2) $L_{\alpha+\beta}^0\left\{t^{1-(\alpha+\beta)}f'(t)\right\}=L_{\alpha+\beta}^0\left\{T_{\alpha+\beta}f(t)\right\}=-f(0)+sF_{\alpha+\beta}(s).$

3) $L_{\alpha+\beta}^0\left\{t^{2-(\alpha+\beta)}f''(t)\right\}=\big(-1-(\alpha+\beta)\big)sF_{\alpha+\beta}(s)+f(0)-\big(\alpha+\beta\big)s^2F'_{\alpha+\beta}(s).$

4) $L_{\alpha+\beta}^0\left\{(m+1)t^{1-(\alpha+\beta)}f'(t)+t^{2-(\alpha+\beta)}f''(t)\right\}=-mf(0)-\big(\alpha+\beta-m\big)sF_{\alpha+\beta}(s)-\big(\alpha+\beta\big)s^2F'_{\alpha+\beta}(s).$

*Proof.* [6] □

## 2. solving the Laguerre fractional differential equation

In the following, a new analytical method for solving the fractional Laguerre differential equation is presented. This method is developed based on the special properties of Laguerre functions and the generalization of fractional derivatives, and it can serve as an effective tool for a more precise analysis of such equations.

**Remark 2.1.** (The Laguerre fractional differential equation) Suppose that $f(t)$ is twice differentiable onIn the following, a new analytical method for solving the fractional Laguerre differential equation is presented. This method is developed based on the special properties of Laguerre functions and the generalization of fractional derivatives, and it can serve as an effective tool for a more precise analysis of such equations. $(0,\infty)$, $\alpha,\beta>0$, $\alpha+\beta\leq 1$, $k,m,n\in\mathbb{R}$ and $k\neq 0$. Then, the solution of the CFDE

$$ {}_tT_\beta\Big({}_tT_\alpha f(t)\Big)+\big(\alpha+m\big)\Big({}_tT_{\alpha+\beta}f(t)\Big)+ktf'(t)+\big(n-m\big)f(t)=q(t) \tag{8}$$

is given by

$$f(t)=\left(L_{\alpha+\beta}^0\right)^{-1}\left\{\left((s+k)^{\frac{(k+1)m-n}{k(\alpha+\beta)}}s^{\frac{n-m}{k(\alpha+\beta)}-1}\right)\left(\int\frac{-mf(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{n-m}{k(\alpha+\beta)}}}(s+k)^{\frac{n-m(k+1)}{k(\alpha+\beta)}-1}ds+C\right)\right\}.$$

**Proposition 2.2.** *From, Eq. 7 and Theorem 1.3, the Eq. 8 can be rewritten as*

$$t^{2-(\alpha+\beta)}f''(t)+(m+1)t^{1-(\alpha+\beta)}f'(t)+ktf'(t)+(n-m)f(t)=q(t). \tag{9}$$

*Now, by applying $L^0_{\alpha+\beta}$ to the both sides of (9), and using Theorem 1.3 we obtain*

$$-mf(0)-(\alpha+\beta-m)sF_{\alpha+\beta}(s)-(\alpha+\beta)s^2F'_{\alpha+\beta}(s)+k\Big\{-(\alpha+\beta)F_{\alpha+\beta}(s)-(\alpha+\beta)sF'_{\alpha+\beta}(s)\Big\}$$
$$+(n-m)F_{\alpha+\beta}(s)=Q_{\alpha+\beta}(s).$$

*Therefore,*

$$F'_{\alpha+\beta}(s)+\Big(\frac{(\alpha+\beta-m)s+(\alpha+\beta)k+m-n}{(\alpha+\beta)s(s+k)}\Big)F_{\alpha+\beta}(s)=\frac{-mf(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)(s)(s+k)}.$$

*By solving this first-order ODE we obtain*

$$F_{\alpha+\beta}(s)=\Big((s+k)^{\frac{(k+1)m-n}{k(\alpha+\beta)}}s^{\frac{n-m}{k(\alpha+\beta)}-1}\Big)\Big(\int\frac{-mf(0)-Q_{\alpha+\beta}(s)}{(\alpha+\beta)s^{\frac{n-m}{k(\alpha+\beta)}}}(s+k)^{\frac{n-m(k+1)}{k(\alpha+\beta)}-1}ds+C\Big).$$

*Now, the solution $f(t)$ can be found by applying the CF inverse transform.*
*In particular, if $\alpha+\beta=1$, $k=-1$ and $m, n\in\mathbb{N}\cup\{0\}$ in (9), then we obtain the solution of the differential equation*

$$tf''(t)+(m+1-t)f'(t)+(n-m)f(t)=0 \tag{10}$$

*as*

$$f(t)=L^{-1}\Big\{(s-1)^n s^{m-n-1}\Big(\int\frac{-mf(0)}{s^{m-n}}(s-1)^{-n-1}ds+C\Big)\Big\}.$$

*On the other hand, we can find the Laguerre dependent function using equation (10):*

$$\psi_n^m(t)=\frac{d^m}{dt^m}\psi_n(t)=\frac{d^m}{dt^m}\Big(e^t\frac{d^n}{dt^n}(t^ne^{-t})\Big)=L^{-1}\Big\{(s-1)^n s^{m-n-1}\Big(\int\frac{-mf(0)}{s^{m-n}}(s-1)^{-n-1}ds+C\Big)\Big\}.$$

*For example, letting m=0 in (10) we find that the solution of the Laguerre differential equation*

$$tf''(t)+(1-t)f'(t)+nf(t)=0$$

*is given by*

$$f(t)=L^{-1}\Big\{C\frac{(s-1)^n}{s^{n+1}}\Big\}.$$

*So, we obtain the Laguerre polynomial of degree n, that is,*

$$\psi_n(t)=L^{-1}\Big\{C\frac{(s-1)^n}{s^{n+1}}\Big\}.$$

## 3. Conclusion

Fractional Laplace transform is an effective tool for analyzing fractional differential equations. Compared to the classical Laplace transform, this method offers greater capability and enables the analytical solution of a class of problems that other existing methods have proven inadequate or incapable of solving.

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# New Chebyshev type inequality for fuzzy integrals in Sugeno sense

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**Abstract**

This work is devoted to establishing a weighted Chebyshev-type inequality in the context of fuzzy integration. We prove that the validity of this inequality hinges critically on an alignment condition between the integrated functions, a necessity rigorously demonstrated via illustrative counterexamples. The findings highlight the pivotal role of functional interdependence and fundamental measure-theoretic properties in determining the monotonic behavior of Sugeno integrals.

## 1. Introduction

The concept of fuzziness was first introduced by Zadeh [14] in 1965, and it soon found applications across various branches of science. Nearly a decade later, in 1974, Sugeno [12] extended this idea by defining the fuzzy integral, now widely known as the Sugeno integral. This integral bears a resemblance to the classical Lebesgue integral, except that the operations of addition and multiplication are replaced by the supremum and infimum, respectively.
Ralesco and Adams [5] provided several equivalent formulations of the fuzzy integral, while Wang and Kiler [6] proposed a general framework for fuzzy measurability and integration. Later, Agahi et al. [1] established Chebyshev-type inequalities for fuzzy integrals. Building on this, Ouyang and Zhang [7] extended the main results for the Sugeno integral of monotone functions under the Lebesgue measure. Subsequently, Ouyang [8] generalized the fuzzy Chebyshev-type inequalities to arbitrary fuzzy measures. More recently, Daraby and his collaborators have derived generalized versions of Fritz–Carlson-type inequalities and other important results concerning the Sugeno integral [2–4].
The Chebyshev-type integral inequality stands out as one of the most fundamental and widely applicable inequalities in mathematics. It finds use in contexts such as the normal (and other arbitrary) distributions, especially when parameters like the mean and variance are unknown, and it also plays a role in proving the weak law of large numbers. In [13], Tong stated and proved the Chebyshev-type integral inequality as follows:

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**Theorem 1.1** ([13]). *Let $f, h \in L[0,1]$ be two real-valued, continuous and comonotone functions and let $\mu$ be the Lebesgue measure on $\mathbb{R}$. Then*

$$\int_0^1 f(x)h(x)\,d\mu \geq \left(\int_0^1 f(x)\,d\mu\right)\left(\int_0^1 h(x)\,d\mu\right),$$

*holds.*

## 2. Preliminaries

In this section, we recall some fundamental notions concerning fuzzy measures and fuzzy integrals.
Let $X$ be a non-empty set, and let $\Sigma$ be a $\sigma$-algebra of subsets of $X$.

**Definition 2.1** ([5]). A fuzzy measure is a set function $\mu : \Sigma \to [0, +\infty]$ satisfying the following conditions:

1. $\mu(\emptyset) = 0$,
2. if $A, B \in \Sigma$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$,
3. if $(A_n)$ is an increasing sequence in $\Sigma$, i.e., $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$,
4. if $(A_n)$ is a decreasing sequence in $\Sigma$, i.e., $A_n \supseteq A_{n+1}$ for each $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then $\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

Throughout this paper, we assume that $(X, \Sigma, \mu)$ is a fuzzy measure space, where $\mu : \Sigma \to [0, +\infty]$ is a fuzzy measure, and $F_+^{\mu}(X)$ denotes the class of all non-negative and $\mu$-measurable functions with respect to $\Sigma$.
For any $f \in F_+^{\mu}(X)$, we define

$$F_\alpha = \{x \in X \mid f(x) \geq \alpha\}, \quad \alpha \in [0, \infty],$$

where $F_\alpha$ is called the $\alpha$-cut of $f$.

**Definition 2.2** ([6]). Let $A \in \Sigma$ and $f \in F_+^{\mu}(X)$. The fuzzy integral of $f$ over $A$ with respect to $\mu$, denoted by $(S)\int_A f\,d\mu$, is defined as

$$(S)\int_A f\,d\mu = \sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)],$$

where the symbol $\wedge$ denotes the inf operation on $[0, \infty]$.

**Proposition 2.3.** *Let $(X, \Sigma, \mu)$ be a fuzzy measure space, $A, B \in \Sigma$ and $f \in F_+^{\mu}(X)$. Then:*

*(i)* $(S)\int_A f\,d\mu \leq \mu(A)$;
*(ii) if $A \subseteq B$, then* $(S)\int_A f\,d\mu \leq (S)\int_B f\,d\mu$.

## 3. Main results

Let $m$ be a fuzzy measure on $X = [0,1]$ and $g, h$ be nondecreasing functions and let $f$ be a function with $f(x) > x$ for every $x \in X$. The fundamental problem we consider is the feasibility of the two inequalities presented below:

$$(S)\int_0^1 \frac{x}{f(x)} h(x)\,g(x)\,dm \;\leq\; \left[(S)\int_0^1 \frac{x}{f(x)} g(x)\,dm\right]\left[(S)\int_0^1 \frac{x}{f(x)} h(x)\,dm\right], \tag{1}$$

or the reversed inequality

$$(S)\int_0^1 \frac{x}{f(x)} h(x)\,g(x)\,dm \;\geq\; \left[(S)\int_0^1 \frac{x}{f(x)} g(x)\,dm\right]\left[(S)\int_0^1 \frac{x}{f(x)} h(x)\,dm\right]. \tag{2}$$

We will show that there is no universal inequality of either kind without extra assumptions. Concretely, we give a simple finite-space counterexample that show the first inequality does not hold and the reverse inequality can hold.

**Example 3.1.** Take the two–point space $X = \{1, 2\}$. Define a simple (additive) fuzzy measure by

$$m(\{1\}) = 0.6, \quad m(\{2\}) = 0.4, \quad m(\{1, 2\}) = 1.$$

Choose

$$f(1) = 3, \quad f(2) = 5 \quad (\text{so } f(x) > x),$$

and take $g$ and $h$ nondecreasing with

$$g(1) = h(1) = 0.2, \quad g(2) = h(2) = 0.9.$$

Compute the three pointwise functions appearing in the integrals:

$$a(x) := f(x)\, g(x)\, h(x), \qquad b(x) := f(x)\, g(x), \qquad c(x) := f(x)\, h(x).$$

Numerical values:

| $x$ | $a(x)$ | $b(x)$ | $c(x)$ |
|---|---|---|---|
| 1 | $\frac{1}{3} \times 0.2 \times 0.2 = 0.013333\ldots$ | $\frac{1}{3} \times 0.2 = 0.066666\ldots$ | $\frac{1}{3} \times 0.2 = 0.066666\ldots$ |
| 2 | $\frac{2}{5} \times 0.9 \times 0.9 = 0.324$ | $\frac{2}{5} \times 0.9 = 0.36$ | $\frac{2}{5} \times 0.9 = 0.36$ |

Now compute Sugeno integrals (using the discrete definition):

$$(S)\int a\, dm = \max_{t \geq 0} \min\big(t,\, m(\{x : a(x) \geq t\})\big).$$

For calculating $a$: the top value is $a(2) = 0.324$ with

$$m(\{x : a(x) \geq 0.324\}) = m(\{2\}) = 0.4.$$

Thus one candidate is $\min(0.324,\, 0.4) = 0.324$. The other candidate, from the smaller value, is $\min(0.01333,\, 1) = 0.01333$. Hence

$$(S)\int a\, dm = 0.324.$$

For calculating $b$ (and likewise $c$): the top value is $0.36$ at $x = 2$ with measure $0.4$, so

$$(S)\int b\, dm = 0.36, \qquad (S)\int c\, dm = 0.36.$$

Take the product on the right-hand side:

$$(S)\int b\, dm \cdot (S)\int c\, dm = 0.36 \times 0.36 = 0.1296.$$

Comparing:

$$(S)\int a\, dm = 0.324 \quad \text{while} \quad (S)\int b\, dm \cdot (S)\int c\, dm = 0.1296,$$

We obtain,

$$(S)\int f(x)\, g(x)\, h(x)\, dm = 0.324 \;>\; 0.1296 = (S)\int f(x)\, g(x)\, dm \cdot (S)\int f(x)\, h(x)\, dm.$$

Hence the inequality with "$\leq$" is false in general. The reversed inequality (with "$\geq$") happens to hold in this particular example, but this do not imply that it holds universally.

**Theorem 3.2.** *Let $m$ be a fuzzy measure on $X = [0,1]$, $g, h$ be nondecreasing functions, $f$ be a function with $f(x) > x$ for every $x \in X$ and*

$$u(x) := \frac{x}{f(x)}\, g(x), \qquad v(x) := \frac{x}{f(x)}\, h(x),$$

$$\alpha := (S)\int u\, dm, \qquad \beta := (S)\int v\, dm.$$

*If there exist $s, t \in [0,1]$ such that*

$$s \le \alpha, \qquad t \le \beta,$$

*and*

$$m\big(\{x : u(x) \ge s\} \cap \{x : v(x) \ge t\}\big) \ \ge \ s\, t, \tag{$\star$}$$

*then*

$$(S)\int (uv)\, dm \ \ge \ \alpha\beta.$$

*Proof.* By definition of the Sugeno integral (level–set form),

$$(S)\int (uv)\, dm = \sup_{r \in [0,1]} \min\big(r,\, m(\{uv \ge r\})\big).$$

Take $r = st$. Note the elementary inclusion

$$\{x : u(x) \ge s\} \cap \{x : v(x) \ge t\} \subseteq \{x : u(x)v(x) \ge st\}.$$

Therefore, using ($\star$),

$$m(\{uv \ge st\}) \ \ge \ m(\{u \ge s\} \cap \{v \ge t\}) \ \ge \ st.$$

Hence

$$\min\big(st,\, m(\{uv \ge st\})\big) = st.$$

Taking the supremum over $r$ gives

$$(S)\int (uv)\, dm \ \ge \ st.$$

Since $s \le \alpha$ and $t \le \beta$, we obtain $st \le \alpha\beta$. But we can choose $s$ arbitrarily close to $\alpha$ and $t$ arbitrarily close to $\beta$ (provided ($\star$) holds in the limit); in the usual cases one takes $s = \alpha$ and $t = \beta$ when the Sugeno integrals are realized as the maximizing levels. consequently,

$$(S)\int (uv)\, dm \ \ge \ \alpha\beta,$$

as claimed. This completes the proof that condition ($\star$) is a sufficient condition for the reversed inequality. □

*Verification of condition ($\star$) for $s = \alpha,\ t = \beta$..* For $s = \alpha$,

$$\{u \ge \alpha\} = \{x \in [0.6, 1] : 0.45x \ge \alpha\} = \{x \in [0.6, 1] : x \ge \tfrac{\alpha}{0.45}\}.$$

Substituting $\alpha \approx 0.310345$, we get

$$\frac{\alpha}{0.45} \approx 0.689655, \qquad m(\{u \ge \alpha\}) = 1 - 0.689655 \approx 0.310345 = \alpha.$$

Since $u = v$, it follows that

$$m(\{u \ge \alpha\} \cap \{v \ge \beta\}) = m(\{u \ge \alpha\}) = \alpha \approx 0.310345.$$

On the other hand,

$$s\, t = \alpha\beta = \alpha^2 \approx 0.0963.$$

Therefore, condition ($\star$) is clearly satisfied because

$$m(\{u \geq \alpha\} \cap \{v \geq \beta\}) \approx 0.310345 \geq 0.0963 = \alpha^2.$$

Since the condition is satisfied, we obtain

$$(S)\int uv\, dm \geq \alpha\beta \approx 0.0963.$$

A direct computation gives $(S)\int uv\, dm = 0.288136$, which is indeed greater than $\alpha\beta$. Hence, the reverse inequality

$$(S)\int \frac{x}{f(x)} g(x)h(x)\, dm \geq \left((S)\int \frac{x}{f(x)} g(x)\, dm\right) \cdot \left((S)\int \frac{x}{f(x)} h(x)\, dm\right)$$

holds true in this example, with the validity grounded on the fulfillment of condition ($\star$).

## 4. Conclusion

We establish and prove a Chebyshev weighted inequality for the Sugeno (fuzzy) integral. The indispensability of a monotonic alignment condition for this inequality is rigorously demonstrated via constructed examples. These results underscore the fundamental influence of both functional correlation and the properties of the fuzzy measure on the monotonic behavior of the integral.

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# Study on the certain inequalities for analytic functions subordinated to the exponential function

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**Abstract**

In this paper, we introduce new subclasses of analytic functions, the subclasses defined by quantum derivatives. Upper bounds of initial coefficient of fuctions belong to the these classes are given, Also the best possible for upper bound of Fekte-Szego and Kurushkal inequalities are obtained.

## 1. Intoduction

The class of analytic functions in $U = \{z : z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ was denoted by $\mathcal{A}$. Functions belonging to this class can be displayed in the form of the following power series

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n. \tag{1}$$

The class of univalent functions in $\mathcal{A}$ which normalized with the conditions $f(0) = f'(0) - 1 = 0$ was represented by $\mathcal{S}$.
The class of functions which are analytic in $U$ and satisfying the following conditions $Re\, p(z) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \quad z \in U,$$

denoted by $\mathcal{P}$.
Let $\mathcal{R}$ denote the class of bounded turning functions in $\mathcal{A}$, which are univalent in $U$. Also, the class $\mathcal{R}$ is analytically defined as,

$$\mathcal{R} = \{f \in \mathcal{S} : f'(z) \in \mathcal{P}\}, \quad z \in U. \tag{2}$$

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The $(p,q)-$derivative operator is is denoted by $D_{p,q}$ for a function $f \in \mathcal{A}$ and $z \neq 0$ is defined by

$$D_{p,q}f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \in U, 0 < q < p \leq 1. \tag{3}$$

For more details, see [5]. Using equations (1) and (3), for $n \in \mathbb{N}$ and $z \in U$ we obtained

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}, \quad 0 < q < p \leq 1, \tag{4}$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}; \ [0]_{p,q} = 0. \tag{5}$$

It is clear that $[1]_{p,q} = 1$.

we say, $f$ is subordinate to $g$ and is written as $f(z) \prec g(z)$, if there exists a function $w(z)$ belonging to class of Schwartz functions which are satisfying $w(0) = 0$ and $|w(z)| < 1$, such that

$$f(z) = g(w(z))$$

where, $f$ and $g$ are analytic in $U$
Let $g$ be univalent function, $f(z) \prec g(z)$ if and only if $f(0) = 0$ and $f(U) \subset g(U)$.

In the sequel, for obtaining our main results, we used the following lemma [2].

**Lemma 1.1.** *If $p \in \mathcal{P}$, then $|p_k| \leq 2$ for each $k$.*

**Lemma 1.2.** *If $p \in \mathcal{P}$, then $2p_2 = p_1^2 + (4 - p_1^2)x$ for some $x$ with $|x| < 1$.*

**Lemma 1.3.** *If $p \in \mathcal{P}$, then, for complex number $\mu$, we have*

$$|p_2 - \mu p_1^2| \leq 2 \max\{1, |2\mu - 1|\}, \tag{6}$$

where the inequality 6 is obtained in [6].

**Lemma 1.4.** *If $p \in \mathcal{P}$, then, we have*

$$|p_{i+j} - \mu p_i p_j| \leq 2, \quad 0 \leq \mu \leq 1. \tag{7}$$

where the inequality 7 is obtained in [3].

## 2. Main Results

In this section, we introduce new class of bonded turning functions associated with $(p,q)-$derivative operator, in first subsection, estimation of initial coefficient are obtained, also in second subsection the best possible for upper bound of Fekte-Szego and Kurushkal inequalities are given.

**Definition 2.1.** A function $f(z)$ given by 1 is said to be in the class $\mathcal{R}_e^{p,q}$, if the following condition is satisfied

$$D_{p,q}f(z) \prec e^z \tag{8}$$

We note that $D_{p,q}f(z) \to f'(z)$ as $p = 1$ and $q \to 1^{-1}$ where $f'(z)$ denotes the ordinary derivative of the function $f(z)$.
In particular, a function $f(z)$ given by 1 is said to be in the class $\mathcal{R}_e$, if the following condition is satisfied

$$f'(z) \prec e^z. \tag{9}$$

[1].

*2.1. Coefficient bounds for the functions class* $\mathcal{R}_e^{p,q}$

In this section, we find estimates on the coefficients $|a_2|, |a_3|$ and $|a_4|$, for functions in the class $\mathcal{R}_e^{p,q}$.

**Theorem 2.2.** *Let* $f(z)$ *be in the class* $\mathcal{R}_e^{p,q}$ *where* $0 < q < p \leq 1$, *then*

$$|a_2| \leq \frac{1}{[2]_{p,q}}, \quad |a_3| \leq \frac{1}{[3]_{p,q}},$$
$$|a_4| \leq \frac{7}{6[4]_{p,q}},$$

**Corollary 2.3.** *Let* $f(z)$ *be in the class* $\mathcal{R}_e$, *then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{3},$$
$$|a_4| \leq \frac{7}{24}.$$

*2.2. Fekete-Szego inequality*

In this section, we will fined the best possible bounds of Fekete-Szego functional $|a_3 - \eta a_2^2|, \eta \in \mathbb{C}$ for $f(z)$ be in the class $\mathcal{R}_e^{p,q}$.

**Theorem 2.4.** *If* $f(z)$ *of the form 1 belong to* $\mathcal{R}_e^{p,q}$, *then for any complex number* $\eta$

$$|a_3 - \eta a_2^2| \leq \frac{1}{[3]_{p,q}} \max\{1, |\phi(\eta)|\},$$

*where,*

$$\phi(\eta) = \frac{4[3]_{p,q}.\eta + 2[2]_{p,q}^2}{4[2]_{p,q}^2}.$$

*2.3. Kurushkal inequality*

In this section, we will show that the best possible bounds of kurushkal functional $|a_4 - a_2^3|$, for $f(z)$ be in the class $\mathcal{R}_e^{p,q}$.

**Theorem 2.5.** *If* $f(z)$ *of the form 1 belong to* $\mathcal{R}_e^{p,q}$ *then,*

$$|a_4 - a_2^3| \leq |\frac{[2]_{p,q}^3 - 6[4]_{p,q}}{6[4]_{p,q}.[2]_{p,q}^3}| + \frac{11}{12[4]_{p,q}}.$$

## 3. Conclusion

In this work, we've explored how subordination method can be used to estimates coefficient bounds and the sharp bounds of Fekete-Szego and Kurushkal functional. Future studies might extend these techniques to classes of multi-valent functions, or investigate on bounds of the Hankel and Teoplitz determinant with various order.

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# Numerical solution of the epidemic model by the weak Galerkin finite element method

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**Abstract**

Understanding infectious disease epidemics is critical for predicting outbreak dynamics and guiding effective public health interventions. Epidemic models, such as the classical SIR (Susceptible-Infected-Removed) framework, enable researchers and decision-makers to analyze disease transmission, recovery rates, and the projected impacts of control strategies. Spatial extensions of these models account for local interactions, population movement, and geographic heterogeneity—features essential for capturing complex real-world contagion scenarios. This study investigates the numerical solution of a spatial SIR epidemic model governed by nonlinear reaction-diffusion equations, employing the weak Galerkin finite element method (WGFEM). The WGFEM allows for flexible domain geometries and efficient discretization of weak derivatives, making it a promising approach for simulating spatially heterogeneous epidemic processes. The article details both semi-discrete and fully-discrete formulations of WGFEM and applies the scheme to analyze spatiotemporal pattern formation in disease spread, including evolution toward steady states and emergence of network-like infection topologies. Numerical experiments validate the method's capability to resolve dynamics characteristic of Turing instability and local herd immunity effects in structured populations. While the proposed approach demonstrates significant utility in addressing spatial epidemic models, systematic comparisons with alternative numerical techniques and further extensions to model-data integration and adaptive interventions remain promising avenues for future research. The findings underscore the importance of advanced numerical methods for the study of epidemiological phenomena, offering insights that are vital for both theoretical understanding and practical disease management.

## 1. Introduction

Infectious diseases, caused by transmissible agents, have had profound impacts on human populations throughout history, influencing demographics, economy, culture, and daily life. For example, devastating epidemics such as the European bubonic plague (Black Death) killed large proportions of populations, reshaped urbanization, and slowed economic growth while increasing demand for labor among survivors. Over centuries, numerous epidemics, including

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smallpox, cholera, yellow fever, influenza, and plague repeatedly caused widespread mortality and social disruption. According to [1], mathematical modeling has become a central theme in understanding epidemics [2, 3]. Such models use systems of differential equations to represent disease dynamics and the life history of disease agents, helping predict outbreaks, study transmission mechanisms, and optimize control strategies. These models serve epidemiology and systems biology by forecasting outbreaks, designing containment measures, and studying complex biological processes of pathogens at multiple scales. The SIR (Susceptible-Infected-Removed) model is typically formulated as a set of ordinary differential equations (ODEs), but when spatial dynamics are introduced (for example, considering how the disease spreads across different geographical locations), the model can be extended to partial differential equations (PDEs). This leads to the development of spatially explicit models, such as the reaction-diffusion SIR model. The numerical methods for PDE SIR models have evolved from classical ODE frameworks to sophisticated reaction-diffusion systems with various numerical schemes such as Finite Difference Methods (FDM), Finite Element Methods (FEM), Method of Characteristics, Particle Methods, Cellular Automata, Collocation methods, Spectral Methods, and Hybrid models. These methods are designed for accurate and efficient computation of disease dynamics and epidemiological thresholds in spatially heterogeneous populations [4–7]. The choice of numerical method for solving partial differential equations in the context of SIR models depends on various factors, including the specific characteristics of the problem, desired accuracy, computational resources, and ease of implementation. Each method has its strengths and weaknesses, making it crucial to select the appropriate approach based on the specific requirements of the study at hand. A novel finite element method, the weak Galerkin finite element method (WGFEM), was first developed by Wang and Ye [8, 9] for solving second-order elliptic equations based on a discrete weak gradient arising from local RT or BDM elements. In the WGFEM procedure, differential operators (e.g., gradient, divergence, curl) in the variational form are approximated by their weak forms as distributions [10–12]. Weak functions and their weak derivatives are approximated by piecewise polynomials with various degrees. Different combinations of polynomial spaces lead to different WGFEM methods, which makes this method highly flexible and efficient in practical computation [11]. In this paper, we are concerned with the WGFEM for solving the epidemic model. This paper is structured as follows. Section 2 offers a brief review of the definition of weak functions, weak gradients, discrete weak functions, and discrete weak gradients. In Section 3, we introduce a weak Galerkin finite element scheme. Section 4 relates to introducing the SIR model. In Section 5, the numerical results related to solving the SIR model using WGFEM are provided by one example. Section 6 completes this paper with a brief conclusion.

## 2. Weak Gradient and Discrete Weak Gradient

Let $T$ be a polygonal domain with boundary $\partial T$ and interior $T^0$, a weak function on the element $T$ is a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(T)$ and $v_b \in H^{\frac{1}{2}}(\partial T)$. $v_0$ is the value of $v$ in the interior of element $T$ and $v_b$ is the value of $v$ on the boundary of element $T$. The space of weak functions on the element $T$ is represented by $W(T)$:

$$W(T) = \{v = \{v_0, v_b\} | v_0 \in L^2(T)\ ,\ v_b \in H^{\frac{1}{2}}(\partial T)\},$$

A discrete weak function is a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P^j(T^0)$ and $v_b \in P^l(\partial T), (j, l \geq 0)$ are polynomials with a degree no more than $j$ and $l$ in the interior of element $T$ and on boundary of element $T$, respectively. The space of discrete weak function is denoted by:

$$W(T, j, l) = \{v = \{v_0, v_b\} | v_0 \in P^j(T^0)\ ,\ v_b \in P^l(\partial T)\}.$$

We define the weak gradient and discrete weak gradient of a weak Galerkin finite element function $v$ as follows [12].

**Definition 2.1.** For any weak function $v \in W(T)$, its weak gradient $\nabla_w v$ is defined as a linear functional on $H(div, T)$ which is given by:

$$(\nabla_w v, w) = -\int_{T^0} v_0(\nabla \,.\, w) + \int_{\partial T} v_b(w.\mathbf{n}), \qquad \forall w \in H(div, T),$$

where $H(div; \Omega) = \{\mathbf{v} : \mathbf{v} \in [L^2(\Omega)]^2\ , \nabla.\mathbf{v} \in L^2(\Omega)\}$ and $\mathbf{n}$ is the outward normal direction to $\partial T$.

**Definition 2.2.** A discrete weak gradiant of $v = \{v_0, v_b\}$ is defined to be a function $\nabla_w v \in [P_k(T)]^2$ such that satisfying the following equality:

$$\int_T (\nabla_{w,d} v).w = \int_{\partial T} v_b(w.n) - \int_{T^0} v_0(\nabla . w), \qquad \forall w \in [P_k(T)]^2 .$$

## 3. Weak Galerkin Scheme

Let $\mathcal{T}_h$ be a partition of domain $\Omega$ which is shape regular so that the inverse inequality holds. Assume that for each $T \in \mathcal{T}_h$, $P^j(T^0)$ be the set of polynomials on $T^0$ with degree less than or equal to $j$ on $T^0$ and $P^l(\partial T)$ the set of polynomials on $\partial T$ with degree less than or equal to $l$ on $\partial T$. If each of $W(T, j, l)$ patching over all element $T \in \mathcal{T}_h$, the weak galerkin finite element space introduce as:

$$V_h = \{v = \{v_0, v_b\}, v|_T \in W(T, j, l), \forall T \in \mathcal{T}_h\}.$$

## 4. The SIR model

The SIR (Susceptible-Infected-Removed) model is an epidemic model [13, 14], this model is a system of two non-linear Partial Differential Equations (PDEs) for S and I, both S and I are functions of time and denote the susceptible and infected compartments of a given population in the presence of an infectious disease. The SIR and its variants have played a crucial role in several recent studies of the COVID–19 epidemic. The PDE formulation of the SIR model that provides a powerful framework for understanding the spatial dynamics of infectious disease outbreaks and can be adapted to incorporate various complexities relevant to specific diseases or populations. By solving these PDEs, researchers and public health officials can gain valuable insights into epidemic dynamics, aiding in effective disease management and control strategies.
We concern with the following PDE

$$\begin{cases} \frac{\partial S}{\partial t} = A - dS - \beta S^p I^q + D_1 \nabla^2 S \,, \mathbf{x} \in \Omega, \; t > 0 \\ \frac{\partial I}{\partial t} = \beta S^p I^q - (d + \mu) I + D_2 \nabla^2 I, \mathbf{x} \in \Omega, \; t > 0 \end{cases} \tag{1}$$

where $\Omega$ denotes a bounded and open domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. From the biological point of view, we assume all the parameters are positive throughout the paper. $\frac{\partial S}{\partial t}$ and $\frac{\partial I}{\partial t}$ represent the rate of change of the susceptible and infected populations over time. $A$ is input or birth rate of susceptible, $d$ is the natural death rate. $\beta$ is the transmission rate. Exponents $p$ and $q$ generalize the incidence term $\beta S^p I^q$, here we set $p = 1$ and $q = 2$. $\mu$ is the additional death rate due to infection. $D_1$, $D_2$ are, respectively, the susceptible and infected individuals diffusion coefficients. $\nabla^2$ is the Laplacian operator, indicating spatial diffusion.This type of model is used to study the spread of infectious diseases in space and time. This models needs to be analyzed with the initial populations

$$S(\mathbf{x}, 0) = S_0, \quad I(\mathbf{x}, 0) = I_0, \tag{2}$$

and Neumman boundary conditions

$$\frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, \;\; t > 0, \tag{3}$$

$\frac{\partial}{\partial \mathbf{n}}$ means the outward normal derivative on $\partial\Omega$.

## 5. Numerical Results

In this work, we consider the SIR model (1) in the unit square domain $\Omega = [0,10] \times [0,10]$ with homogeneous Neumann boundary conditions, and initial conditions are taken as small random perturbations around the steady-state. The system (1) has three steady-state points, we can find the steady-state points by solving the following equations

$$f(S,I) = A - dS - \beta S^p I^q = 0,$$

$$g(S,I) = \beta S^p I^q - (d+\mu)I = 0,$$

1. $E_1 = (\frac{A}{d}, 0)$,

   which is corresponding to extinction of the epidemic.

2. $$E_2 = \Big(\frac{A\beta + \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d+\mu)}{A\beta + \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}\Big),$$

   which is corresponding to the coexistence of the S and I.

3. $$E^\star = \Big(\frac{A\beta - \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}{2d\beta}, \frac{2d(d+\mu)}{A\beta - \sqrt{A^2\beta^2 - 4d^3\beta - 8d^2\beta\mu - 4d\beta\mu^2}}\Big),$$

   which is corresponding to the coexistence of the S and I.

By direct calculations, we can show that $E_2$ is unstable, indicating it is a saddle point. Thus, the stability behavior of the interior equilibrium point $E^\star$ is considered.
The parameters are $A = 1, d = 1, \beta = 16, D_1 = 6, D_2 = 1, \mu = 1$,where the parameter values are in the domain of Turing space. We assume the population grows with constant rate and set $p = 1$ and $q = 2$. We have shown the distribution of susceptible and infected by an analysis of pattern formation. Figure 1 shows the evolution of the spatial pattern of the infected population at different iterations. We can see, once the irregular pattern develops, it expands slightly, and ultimately, the network-like patterns dominate the entire domain, with no further changes occurring in the system's dynamics.
Figure 2 shows the graph of approximate solutions of the spatial pattern of susceptible and infected population at steady-state, which corresponds to the coexistence of the $S$ and $I$, respectively.

## 6. Conclusion

In this paper, we constructed the WGFEM method for the solution of the epidemic model. We showed that this method is suitable for finding the numerical solution of these problems. Numerical results presented the applicability and efficiency of the proposed scheme.

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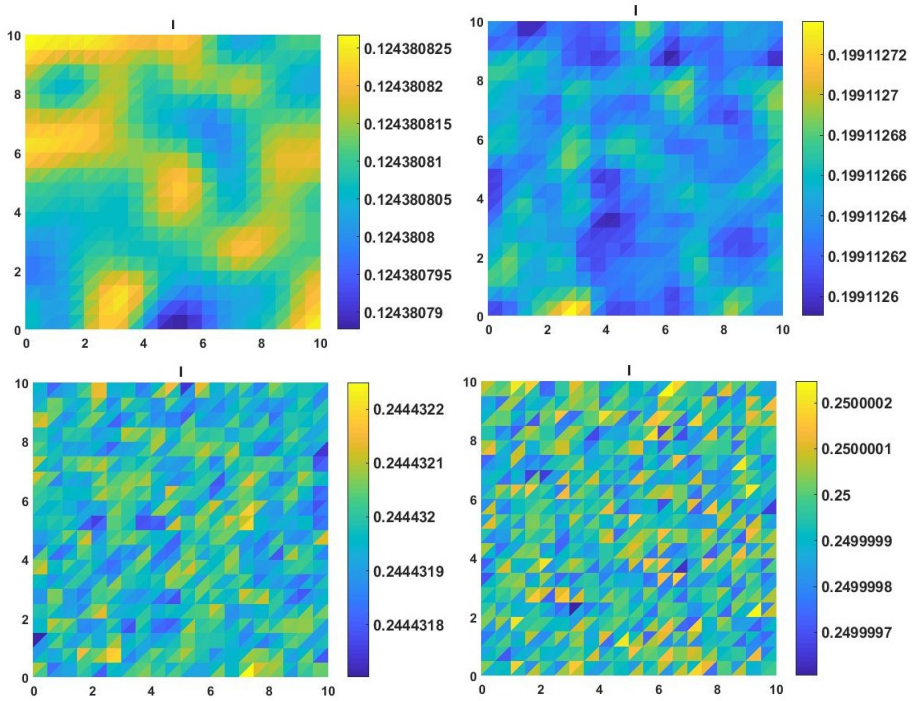


Fig. 1. Snapshots of contour pictures of the time evolution of the infected (I) at different iteration.

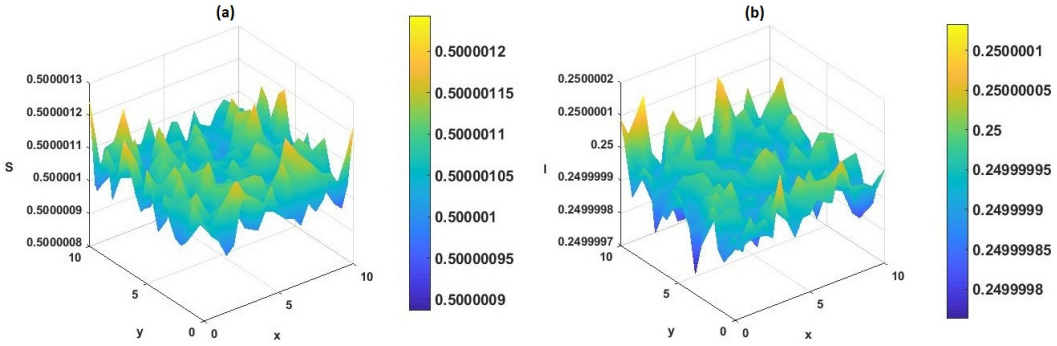


Fig. 2. Graph of approximate solutions for (a) suspected (b) infected at steady-state.

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# A Systematic Review of the Applications of Large Language Models Focusing on Teaching and Learning Mathematics

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**Abstract**

Large Language Models (LLMs) are built on mathematical foundations such as vector operations and probability, using neural networks to understand language patterns. In mathematics education, they transform abstract theories into interactive lessons and step-by-step problem-solving. This article provides an overview of the evolution of LLMs and their growing role in education, including personalized learning, content generation, teacher support, and student feedback. It also highlights key challenges such as accuracy, bias, privacy, and academic integrity, concluding that effective and safe use of LLMs requires thoughtful design, human oversight, and clear ethical guidelines.

## 1. Introduction

Language is essential for human communication, but machines cannot understand it naturally. To enable this, AI and Natural Language Processing (NLP) build systems that read, write, and speak like humans. At the core of NLP are language models, which predict or generate text based on context. These models have evolved from simple statistical rules and neural networks to pre-trained models and today's large language models (LLMs), which use massive data and advanced algorithms to become highly capable and human-like [1]. Large language models (LLMs) like GPT, ChatGPT, and Gemini can generate human-like text and perform language tasks accurately [2]. However, many people do not fully understand them. This study explains how LLMs work and their applications in a clear, accessible way to help more people understand and use their full potential.

Recent studies on large language models (LLMs) across fields like software, medicine, finance, law, and education have identified key trends, gaps, and developments. This provides a clear overview of progress, challenges, and future directions for research and industry [1], [3]. LLMs transform math education by acting as intelligent tutors that interpret problems, explain proofs, and provide personalized, interactive feedback to deepen conceptual understanding.

AI and large language models can automate educational tasks like question generation, feedback, and grading, reducing teachers' workloads and improving learning efficiency. However, challenges remain, including bias, privacy, and transparency. Two key factors for success in education are practicality (feasibility and real-world benefit) and ethicality

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(fairness, responsibility, and societal impact). Both are essential for developing trustworthy and effective AI tools in education [4]–[6].

LLMs are used in education for question generation, automated feedback, chatbots, and essay scoring. For instance, Kurdi et al. [7] reviewed methods for automatic question generation, noting that LLMs support semantic-based approaches. The rise of ChatGPT has increased discussions on practical and ethical challenges, as highlighted by Kasneci et al. [8], who reviewed opportunities and challenges for students and teachers.

In mathematics, LLMs can generate LaTeX expressions, automate deductive reasoning, and translate natural language into formal proofs or code. In education, they create dynamic problem sets that adjust to student performance. However, the hallucination problem—generating plausible but incorrect formulas—remains a major challenge, requiring human expertise to ensure mathematical accuracy. AI in education must be applied fairly and responsibly due to risks such as errors, bias (e.g., in grading or access), student cheating, and privacy violations [9], [10]. LLMs make learning more engaging, personalized, and interactive; provide quick feedback; generate content; support collaboration and accessibility; and assist teachers with planning and evaluation [11]–[13].

In this article, Section 2 outlines the development of language models, explaining their progression from early statistical approaches to neural models, pretrained systems, and finally large language models. Section 3 discusses the educational applications of large language models, including their role in personalized learning, content generation, automated feedback, and support for both teachers and students. Section 4 addresses the main challenges and limitations of using these models in educational settings, such as concerns about accuracy, bias, privacy, and academic misuse. Finally, Section 5 provides a conclusion that summarizes the key points and highlights the importance of responsible and well-managed integration of large language models into teaching and learning environments.

## 2. EVOLUTION OF LARGE LANGUAGE MODELS

To understand the development of large language models (LLMs), it is helpful to first look at the evolution of language models (LMs). LMs have progressed from statistical language models (SLMs) to neural language models (NLMs), pre-trained language models (PLMs), and finally to large language models (LLMs) [1], [14]-[15]. Statistical language models (SLMs) emerged in the 1990s [1], [14]. They estimate the likelihood of words and sentences appearing in a text using probabilities. For instance, consider the sentence:

If we label each word as: $\omega_1 =$ “The”, $\omega_2 =$ “cat”, $\omega_3 =$ “sat”, $\omega_4 =$ “on”, and $\omega_5 =$ “the mat” the probability of this sentence can be expressed as: $P(S) = P(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ Using conditional probabilities, this becomes: $P(S) = P(\omega_1) \cdot P(\omega_2 \mid \omega_1) \cdot P(\omega_3 \mid \omega_1, \omega_2) \cdots P(\omega_5 \mid \omega_1, \dots, \omega_4)$. Here, each term represents the probability of a word given the previous words. For example, $P$(“cat” | “The”) is the probability that ”cat” follows ”The”. To estimate these probabilities from a text corpus, Maximum Likelihood Estimation (MLE) is commonly used: $P(w_i \mid w_1 w_2 \dots w_{i-1}) = \frac{C(w_1 \dots w_i)}{C(w_1 \dots w_{i-1})}$.

SLMs often assume that each word depends only on the previous $n-1$ words, which is known as the n-gram model. For example, a bigram model ($n = 2$) considers only the previous word, while a trigram model ($n = 3$) looks at the two preceding words. This makes computation feasible but limits the model’s ability to capture long-range dependencies in text. A major challenge of SLMs is that the number of possible sentences grows exponentially with vocabulary size. For instance, with a vocabulary of 5000 words, the number of possible sequences of length n is . Therefore, n is usually kept small (2 or 3), which simplifies storage and computation but reduces accuracy for modeling longer context. Neural Language Models (NLMs) [1], [16] use neural networks to predict what word comes next in a sentence. They are better than older models because they can understand longer sentences and don’t have the limits of small “n-grams” in statistical models. Before we talk about neural networks, we need to understand word vectors. Humans easily understand word meanings. For example, “cat” and “dog” are more similar to each other than “cat” and “tree” because they are both animals. Computers, however, only understand 0s and 1s, so we need a way to turn words into numbers. Word vectors do this: each word becomes a list of numbers (a vector). Words that are similar in meaning end up closer together in this number space. For example, the vectors for “cat” and “dog” are closer together than the vectors for “cat” and “tree”. Tools like Word2Vec [17] create these vectors, helping computers understand the meaning of words and the relationships between them. Pre-trained language models (PLMs) [1], [18] first learn from a large amount of text to understand the basics of language, like vocabulary, grammar, meaning, and logic. This first step is called pre-training. After that, the model can be used for different language tasks, such as translating text, summarizing

articles, or answering questions. To make the model perform better for a specific task, it is trained a second time on a smaller, task-specific dataset. This step is called fine-tuning. Together, this approach is called pre-training and fine-tuning. This pre-training and fine-tuning method is the foundation for many modern language models, like GPT and BERT. Large language models (LLMs) are trained on enormous text datasets and contain tens or even hundreds of billions of parameters. Examples include GPT-3 [19], GPT-4 [20], PaLM [21], and LLaMA [22]. The main objective of LLMs is to enable machines to understand human language and follow human-aligned values. Unlike pre-trained language models (PLMs), which are often specialized for specific tasks, LLMs combine extensive general pre-training with alignment to human values, making them highly adaptable for a wide range of applications [1]. The dramatic increase in model size, training data, and computational resources has led to significant improvements in performance and new capabilities that smaller models cannot achieve. For instance, GPT-3 can leverage contextual information from task examples to improve its responses, a feature not present in GPT-2. LLMs are typically much larger than PLMs; while the largest GPT-2 model has 1.5 billion parameters and was trained on 40 GB of text, the largest GPT-3 model has 175 billion parameters and uses 570 GB of training data. This highlights the substantial difference between LLMs and earlier PLMs in terms of both scale and capability.

## 3. APPLICATIONS OF LLMS IN EDUCATION

LLMs serve as logical engines in math education, breaking down problems step by step and creating adaptive problem sets that act like personal tutors. They enable real-time personalized learning beyond traditional or basic digital tools, though ethical and equity concerns persist [23]–[26].

In recent years, on-device and locally deployed LLMs—such as Llama 3, Phi-3, and Mistral 7B—have introduced a new wave of innovation by improving privacy, speed, and offline functionality without relying on cloud systems [27]. These technologies help reduce educational inequality, support content localization, and enable culturally responsive smart classrooms [27]. Statistics show that intelligent tutoring systems improve student performance beyond 75% of peers in traditional settings [28], and the global AI-in-education market is projected to reach 20.2 billion by 2027 [29], with 47% of learning tools expected to integrate AI capabilities within the next three years. This trajectory signals the rise of highly personalized AI-driven learning environments and advanced systems—such as ITS, ILP, AGS, and DCG—that function as 24/7 adaptive tutors for every learner.

Second, LLMs are powerful content generators: they can produce practice questions, worked examples, explanations at multiple difficulty levels, lesson summaries, and scaffolding prompts quickly and at scale. This capability reduces teachers' content creation burden and supports rapid iteration of instructional materials, although quality control and alignment with curriculum goals remain necessary.

Third, LLMs can support automated assessment and feedback. Recent studies show that, with carefully designed prompting or rubric-aligned prompting, LLMs can approach or complement traditional automated essay scoring systems and provide timely, formative feedback that helps learners revise and improve. However, validity, fairness across student groups, and robustness to adversarial or off-domain writing are active concerns.

Fourth, LLMs function as assistants for both teachers and students: they can suggest lesson plans, generate exemplars, help draft formative assessments, and act as on-demand tutors that answer questions or explain concepts. These assistant roles can increase teacher productivity and provide learners with immediate, scaffolded help, but they require careful integration into pedagogy and oversight to avoid overreliance.

Finally, LLMs facilitate domain-specific learning — for example, language learning, programming practice, and STEM problem solving — by generating targeted exercises, simulating conversational practice, or providing step-by-step code explanations. Empirical work finds promising gains when LLMs are used to create interactive, context-aware practice, though domain alignment and evaluation remain important to ensure pedagogical effectiveness. Across all these uses, two key points are important. First, LLMs only work well if the system is designed carefully, the prompts or fine-tuning are good, and humans monitor the results. Second, ethical and technical issues — like bias, privacy concerns, cheating, and mistakes in automated decisions — must be solved before LLMs can be used responsibly in real classrooms. Overall, the research shows that LLMs have a lot of potential, but schools still need strong evaluation, teacher-focused design, and clear policies to make sure these tools lead to real and reliable improvements in education.

## 4. CHALLENGES AND LIMITATIONS OF USING LLMS IN EDUCATIONS

Large language models face three main difficulties in mathematics. First, they sometimes make simple calculation errors because they predict patterns rather than performing precise arithmetic. Second, logical gaps may appear in step-by-step proofs, especially in geometry or calculus, where an invalid connection between steps can mislead students. Third, students may become over-reliant on LLMs and skip the productive struggle that is essential for building deep understanding and persistence. Therefore, LLMs should be used as a support tool to explain concepts, not as a shortcut to bypass hard thinking.

The integration of large language models (LLMs) in education offers many advantages, but it also brings important challenges and limitations that must be carefully considered. One major concern is the possibility of incorrect or fabricated responses, often referred to as hallucinations. Since LLMs generate text based on patterns in data rather than verified facts, they may provide answers that sound confident but are inaccurate. This can mislead students, especially younger learners or those unfamiliar with the subject.

Another limitation involves biases within the models. Because LLMs are trained on large datasets that contain social, cultural, or linguistic biases, these biases can unintentionally appear in their outputs. This may result in unfair or inappropriate content, prioritizing certain groups over others, or representing information in a skewed way. Ethical concerns also arise when AI tools are used widely in learning environments—for example, questions about transparency, accountability, and the role of AI in decision-making processes.

In addition, LLMs raise challenges related to academic integrity. Students can use AI tools to generate essays, solve assignments, or complete tasks without genuinely learning the material, leading to plagiarism or dependence on AI. This increases the need for digital literacy, meaning that both teachers and students must learn how to use AI responsibly and critically. Finally, privacy concerns are significant, as interacting with AI systems may require sharing personal or sensitive information. Ensuring that student data is protected and that AI tools comply with privacy regulations is essential for safe and responsible use in education.

## 5. CONCLUSION

In conclusion, the integration of Large Language Models into mathematics education marks a shift from traditional calculation to a more interactive form of logical reasoning. These models have shown that they can assist students in visualizing the steps of a proof and understanding the structure of complex equations. By providing a bridge between abstract mathematical language and clear, human-like explanations, LLMs help demystify mathematics and make it more accessible to a broader range of learners.

However, as this article has explored, the role of LLMs in math is best seen as a partnership between technology and human judgment. While these tools can generate immediate solutions and practice problems, the core of mathematical learning—critical thinking and precision—still requires human guidance. Ensuring that these models are used to support, rather than replace, the student's own problem-solving process is the key to their success in the classroom.

Large language models have become influential tools that are reshaping how learners and educators interact with information. Their ability to generate text, provide feedback, and support personalized learning demonstrates significant potential for improving educational experiences. However, the effective use of these systems depends on recognizing their limitations, including issues related to accuracy, bias, and responsible use. As educational institutions adopt these technologies, careful planning, clear guidelines, and ongoing human oversight are essential to ensure that the benefits outweigh the risks. Overall, the findings of this article emphasize that large language models, when implemented thoughtfully, can serve as valuable assistants in teaching and learning, offering new opportunities for innovation while requiring continuous evaluation to maintain trust, fairness, and educational integrity.

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# On the Hosoya Index of Trees

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**Abstract**

For a graph $G$, the Hosoya index is defined as the sum of the number of all matchings of $G$. In this paper, we investigate several well-known families of trees, including the comet and double comet trees, $T_p^t$, and $E_p^t$, and compute the Hosoya index for each of these graph classes.

## 1. Introduction

Let $G = (V, E)$ be a simple, connected graph with $n$ vertices, and $m$ edges. Among various topological indices of graphs, the Hosoya index, introduced by Haruo Hosoya in 1971 [6], plays a prominent role. A subset of edges in $G$ is called a matching if no two edges in the subset are incident to the same vertex. For a given graph $G$, the Hosoya index, denoted by $Z(G)$, is defined as follows:

$$Z(G) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} M_G(k),$$

Here $M_G(k)$ denotes the number of $k$-matchings in $G$, that is, set of k independent edges. Clearly, for any graph $G$, we have $M_G(0) = 1$, and $M_G(1) = m$. Moreover, for $k > \left\lfloor \frac{n}{2} \right\rfloor$, $M_G(k) = 0$. The Hosoya index has been extensively studied in the context of mathematical chemistry, and numerous researchers have examined methods for its computation across various graph classes. Due to its crucial role in the study of chemical isomers and in the characterization of molecular topological features, this index has become an important and widely used tool in chemistry [6].
In recent years, a number of techniques have been introduced to compute this index for specific families of graphs. In 2012, Cheng and Yang conducted a study on the Hosoya index of the corona product of two graphs [2]. Liu et al. investigated the extremal values of the Hosoya index in non-cyclic graphs whose diameter does not exceed four [7]. In this paper, we determine the Hosoya index for several well-known families of trees, including comet trees, double comet tree, $T_p^t$, and $E_p^t$. By establishing the corresponding recurrence relations for each class, we provide a more comprehensive understanding of the behavior of this index within these graph structures.

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## 2. Preliminaries

In this section, we present several well-established results concerning the Hosoya index of graphs, which will be utilized in the subsequent section.

**Theorem 2.1** ([5]). *Let $G$ be a graph and let $u, v$ in $V(G)$, wuth $e = uv$ in $E(G)$. Then*

1. $Z(G) = Z(G - \{uv\}) + Z(G - \{u, v\})$,
2. $Z(G) = Z(G - \{v\}) + \sum_{u \in N_G(v)} Z(G - \{u, v\})$.

**Theorem 2.2** ([5]). *Let $F_n$ denote the Fibonacci sequence defined by the recursive formula $F_{n+1} = F_n + F_{n-1}$ and conditions $F_0 = 0$, $F_1 = 1$. Then*

1. *The Hosoya index of a path of order $n > 0$ is equal to $F_{n+1}$,*
2. *The Hosoya index of a cycle of order $n \geq 3$ is equal to $F_{n+1} + F_{n-1}$.*

**Theorem 2.3** ([5]). *Let $G$ be a graph and $G_1, G_2, \ldots, G_k$ be the connected components of it. Then*

$$Z(G) = \prod_{i=1}^{k} Z(G_i).$$

**Proposition 2.4.** *Let $K_{1,n}$ denote the star graph with one central vertex and $n$ leaves. Then*

$$Z(K_{1,n}) = n + 1.$$

*Proof.* By convention, the number of 0-edge matchings is 1, and the number of 1-edge matchings is equal to the number of edges, which is $n$. Since the star graph does not contain any 2-edge matchings, the result follows. □

**Proposition 2.5.** *For the complete graph $K_n$, we have*

$$Z(\overline{K_n}) = 1.$$

*Proof.* The result follows directly from the definition of the Hosoya index and Theorem 2.3. □

## 3. Main Results

In this section, we compute the Hosoya index for several well-known families of trees, including comet, $T_p^t$, and $E_p^t$. We start with the following definition.

**Definition 3.1** ([1]). A comet tree, $C(k, s)$, is a tree formed by a path $P_k$ whose terminal vertex is the central vertex of the star $K_{1,s}$.

Figure 1 presents a comet tree containing $k + s = n$ vertices.

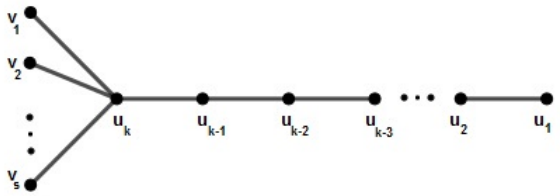


Fig. 1. The comet tree $C(k, s)$.

**Theorem 3.2.** *Let $G = C(k,s)$ be a comet tree. Then*

$$Z(G) = (s+1)F_k + F_{k-1}, \quad F_0 = 0,\ F_1 = 1.$$

*Where $F_k$ denotes the Fibonacci number.*

*Proof.* By applying Theorem 2.1,

$$\begin{aligned} Z(G) =& Z(G - u_k u_{k-1}) + Z(G - \{u_k, u_{k-1}\}) \\ =& Z(K_{1,s})Z(P_{k-1}) + Z(\overline{K_s})Z(P_{k-2}) \\ =& (s+1)F_k + F_{k-1} \end{aligned}$$

Here, $K_s$ is the complete graph with $s$ vertices, and $K_{1,s}$ is the star graph. □

**Definition 3.3** ([4])**.** Let $x, y \geq 1$, and $n \geq x + y + 2$. A double comet tree, denoted by $DC(n,x,y)$, is constructed by attaching $x$, and $y$ vertices to the two ends of a path with $n - x - y$ vertices. Consequently, a double comet tree, $DC(n,x,y)$, contains $n$ vertices.

Figure 2 illustrates a double comet tree with $n$ vertices.

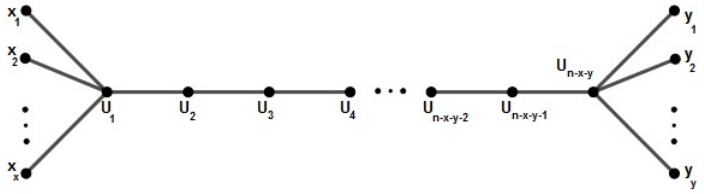


Fig. 2. The double comet tree $DC(n,x,y)$.

**Theorem 3.4.** *Let $G = DC(n,x,y)$ be a double comet tree. Then*

$$\begin{aligned} Z(G) =& (xy + x + y + 1)F_(n - x - y - 1) + (xy + x + 2y + 2)F_{n-x-y-2} + F_{n-x-y-3}, \\ & F_0 = 0, F_1 = 1. \end{aligned}$$

*Where $F_t$ denotes the Fibonacci number.*

*Proof.* We have,

$$Z(G) = Z(G - \{u_1u_2\}) + Z(G - \{u_1, u_2\})$$

That,

$$\begin{aligned} Z(G - \{u_1u_2\}) =& Z(K_{1,x}) \cdot Z(C(n - x - y - 1, y)) \\ =& (x+1) \cdot [yF_{n-x-y-1} + F_{n-x-y-1} + yF_{n-x-y-2} + F_{n-x-y-2}] \\ =& xyF_{n-x-y-1} + xF_{n-x-y-1} + xyF_{n-x-y-2} + xF_{n-x-y-2} + yF_{n-x-y-1} \\ & + F_{n-x-y-1} + yF_{n-x-y-2} + F_{n-x-y-2} \\ =& (xy + x + y - 1)(F_{n-x-y-1} + F_{n-x-y-2}), \end{aligned}$$

Also,

$$\begin{aligned} Z(G - \{u_1, u_2\}) =& Z(\overline{K_x}) \cdot Z(C(n - x - y - 2, y)) \\ =& (y+1)F_{n-x-y-2} + F_{n-x-y-3}. \end{aligned}$$

By summing the two relations above, we obtain

$$Z(G) = (xy + x + y + 1)F_{n-x-y-1} + (xy + x + 2y + 2)F_{n-x-y-2} + F_{n-x-y-3}.$$

□

**Definition 3.5** ([3]). The graph $T_p^t$, (a star of paths), is a tree with $t$ legs and a central vertex $u$. Each leg consists of $p$ vertices, and therefore $T_p^t$ has $tp + 1$ vertices.

Figure 3 illustrates the tree $T_p^t$.

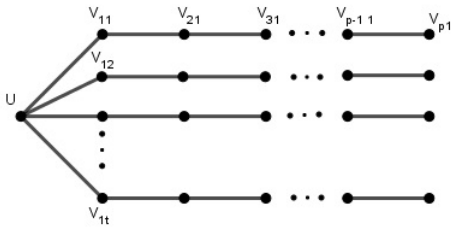


Fig. 3. The $T_p^t$ tree.

**Definition 3.6** ([3]). The graph $E_p^t$ is a tree with t legs and a central edge. Each leg consists of $p$ vertices, and therefore $E_p^t$ has $tp + 2$ vertices.

The $E_p^t$, tree is shown in Figure 4.

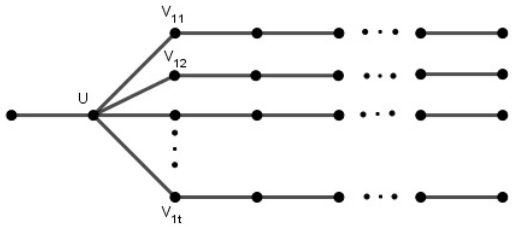


Fig. 4. The $E_p^t$ tree.

**Theorem 3.7.** *Let $G = E_p^t$ tree. Then*

$$Z(G) = (F_{p+1})^{t-1}(tF_p + 2F_{p+1}), \quad F_0 = 0, F_1 = 1,$$

*Where $F_t$ denotes the Fibonacci number.*

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# Symmetry analysis and exact solutions of fractional Zeldovich-Frank-Kamenetskii equation

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**Abstract**

The article focuses on investigating Lie symmetry analysis of time-fractional Zeldovich-Frank-Kamenetskii equation with Riemann-Liouville derivative. The fractional reaction-diffusion equation describes how planar laminar premixed flames spread in combustion theory. The use of Lie method is also illustrated to obtain Lie symmetry generators, symmetry reduction solutions, invariant properties. Furthermore, we convert time-fractional Zeldovich-Frank-Kamenetskii equation to a nonlinear fractional ordinary differential equation (ODE) with Erdélyi-Kober derivative using its Lie point symmetries. This decreased fractional ODE is investigated by explicit power series. In addition, some figures for obtained explicit solution are presented.

## 1. Introduction

In recent years, fractional differential equations (FDEs), as a generalization of partial differential equations (PDEs), have been applied in physics, engineering, chemistry, biology, blood flow phenomena, quantum mechanics, etc. In fact, many physical and natural phenomena may depend on their current and historical situations that can be formulated by the theory of fractional derivatives and integrals. The most popular models of this calculus are differential equations with fractional-order derivatives. Also, there are numerous methods available in the literature for solving FDEs. Lie symmetry analysis, as an analytic method which is introduced by Lie, provides an effective way to derive exact solutions for PDEs and FDEs.

The Lie symmetry group of a system of differential equations has many numerous practical applications. One of the most important efforts is its ability to generate new solutions from known ones. Additionally, a Lie symmetry group provides a method for categorizing different classes of symmetrical solutions. Another valuable application of Lie group theory is to identify the conservation laws of equations, playing a pivotal role in examining solution properties, including their existence, uniqueness and stability [1].

In the present article, we show that Lie symmetry techniques can also be applied to discover solutions of a reaction-diffusion equation, named Zeldovich-Frank-Kamenetskii (ZFK). Further, we concentrate on one of the easiest variants

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of ZFK equation [2]. The time-fractional differential equation is defined by

$$\partial_t^{\eta}\theta = \theta_{xx} + \frac{\beta^2}{2}\theta(1-\theta)e^{-\beta(1-\theta)}, \tag{1}$$

in which $t \geq 0$, $x \in R$, $\beta$ is Zeldovich number, and $\theta$ is a non-dimensional variable that quantifies the ratio of burnt to unburnt gas in a easy reaction including two modes and is called reduced temperature. The term $\omega(\theta,\beta) = \frac{\beta^2}{2}\theta(1-\theta)e^{-\beta(1-\theta)}$ is called reaction term which depends on Zeldovich number $\beta \gg 1$ [2]. The focus of the article is on investigating Lie point symmetries, similarity reduction for time-fractional ZFK equation.
The continuation of this paper is 6 sections including a conclusion. Section 2 recalls some primary definitions and features of fractional calculus. Section 3 explains how to determine Lie point symmetries of FDEs. In section 4, Lie point symmetries of time-fractional ZFK equation are found. Next section is devoted to applying Lie symmetries to construct novel invariant solutions for this equation. In Section 6, we obtain the power series solutions of (1). Some conclusions are also presented in last section.

## 2. Preliminaries

First, some preliminary notations and definitions of fractional calculus are presented. We should mention that there are various definitions for fractional derivative like Caputo, Riemann-Liouville, Riesz, Grünwald-Letnikov, Miller-Ross, Hadamard and Erdélyi-Kober fractional derivatives. Here, we employ Riemann-Liouville and Erdélyi-Kober fractional derivative to obtain symmetries and exact solutions of system (1) (see [4**?** ]).

**Definition 2.1.** [3, 7] On $[a,b]$,

$$\begin{aligned} {}_aD_t^{\eta}f(t) &= D_t^n({}_aI_t^{n-\eta}f(t)) = \frac{1}{\Gamma(n-\eta)}\left(\frac{d}{dt}\right)^n\int_a^t \frac{f(r)}{(t-r)^{\eta-n+1}}dr, \\ {}_tD_b^{\eta}f(t) &= (-1)^n D_t^n({}_tI_b^{n-\eta}f(t)) = \frac{(-1)^n}{\Gamma(n-\eta)}\left(\frac{d}{d}\right)^n\int_t^b \frac{f(r)}{(r-t)^{\eta-n+1}}dr, \end{aligned} \tag{2}$$

are left and right Riemann-Liouville fractional partial derivatives of order $\eta > 0$ for $f(x)$, where $n = [\eta] + 1$ and $\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}$ is Gamma function [3].

Some applicable features of Riemann-Liouville partial derivative are as follows [7]:

- $D_t^{\eta}C = \frac{Ct^{-\eta}}{\Gamma(1-\eta)}$, $C$ is a constant;
- $D_t^{\eta}t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\eta)}t^{\beta-\eta}$;
- $D_t^{\eta}[h(t)g(t)] = \sum_{m=0}^{\infty}\binom{\eta}{m}D_t^{\eta-m}h(t)D_t^m g(t), \eta > 0$ where $\binom{\eta}{m} = \frac{(-1)^{m-1}\eta\Gamma(m-\eta)}{\Gamma(1-\eta)\Gamma(m+1)}$.

Further, we need the Erdélyi-Kober fractional integral operator

$$(\mathcal{K}_{\delta}^{\tau,\eta}h)(z) = \begin{cases} \frac{1}{\Gamma(\eta)}\int_1^{\infty}(s-1)^{\eta-1}s^{-(\tau+\eta)}h(zs^{\frac{1}{\delta}})ds, & \eta > 0, \\ h(z), & \eta = 0. \end{cases} \tag{3}$$

Using this notion, we can define the extended left-hand side of this operator with

$$\left(\mathcal{P}_{\delta}^{\tau,\eta}h\right)(z) = \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\delta}z\frac{\partial}{\partial z}\right)(\mathcal{K}_{\delta}^{\tau+\eta,n-\eta}h)(z), \tag{4}$$

where $z > 0$, $\delta > 0$, $\eta > 0$ and $n = \begin{cases} [\eta]+1, & \eta \notin N, \\ \eta, & \eta \in N. \end{cases}$

## 3. Finding lie symmetries of time-fractional PDEs regarding Riemann-Liouville derivative

Lie symmetries for FDEs is first proposed by Gazizov et al. [5] and then many authors considered Lie group theory for analyzing FDEs. This section presents several short points of Lie symmetry analysis for time-fractional PDEs with over independent variables.Consider a scaler time-fractional PDE as follows:

$$\partial_t^{\eta}\theta(t,x) = F(t,x,\theta,\theta_x,\theta_{xx},\cdots), \tag{5}$$

for the order $0 < \eta < 1$, where $\partial_t^{\eta}$ indicates Riemann-Liouville fractional derivative and indexes are partial derivatives. According to Lie group theory, if (5) is invariant under a one-parameter Lie group of point transformations, it remains unchanged under invertible transformations of variables $t, x, \theta$, and the derivatives of $\theta$ for independent variables are

$$\begin{aligned} \bar{t} &= t + \epsilon\tau(t,x,\theta) + \mathcal{O}(\epsilon^2), \quad \bar{x} = x + \epsilon\xi(t,x,\theta) + \mathcal{O}(\epsilon^2), \\ \bar{\theta} &= \theta + \epsilon\Phi(t,x,\theta) + \mathcal{O}(\epsilon^2), \quad \frac{\partial\bar{\theta}}{\partial\bar{x}} = \frac{\partial\theta}{\partial x} + \epsilon\Phi^{x}(t,x,\theta) + \mathcal{O}(\epsilon^2), \\ \frac{\partial^2\bar{\theta}}{\partial\bar{x}^2} &= \frac{\partial^2\theta}{\partial x^2} + \epsilon\Phi^{xx}(t,x,\theta) + \mathcal{O}(\epsilon^2), \quad \frac{\partial^{\eta}\bar{\theta}}{\partial\bar{t}^{\eta}} = \frac{\partial^{\eta}\theta}{\partial t^{\eta}} + \epsilon\Phi^{(\eta,t)}(t,x,\theta) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{6}$$

in which $\tau, \xi$ and $\Phi$ are called infinitesimal and $\Phi^{x}$, $\Phi^{xx}$ and $\Phi^{(\eta,t)}$ are called generalized infinitesimals of order 1, 2 and $\eta$, respectively. The associated infinitesimal generator for these group transformations is defined as:

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \Phi\frac{\partial}{\partial\theta}, \tag{7}$$

where $\frac{d\bar{t}}{d\epsilon}\Big|_{\epsilon=0} = \tau$, $\frac{d\bar{x}}{d\epsilon}\Big|_{\epsilon=0} = \xi$, $\frac{d\bar{\theta}}{d\epsilon}\Big|_{\epsilon=0} = \Phi$. Based the invariance criterion of Lie theory, (5) accepts Lie group transformation (6) and Lie symmetry vector field (7) as generator iff prolonged vector field $Pr^{(\eta,t)}X$ annihilates (5) on its solution; that is, $Pr^{(\eta,t)}X(\Delta)\Big|_{\delta=0} = 0, \qquad \Delta = \partial_t^{\eta}\theta(t,x) - F(t,x,\theta,\theta_x,\theta_{xx},\cdots)$.
This equation is named determining equation. By keeping essential terms, the prolongation of operator $Pr^{(\eta,t)}X$ is

$$Pr^{(\eta,t)}X = X + \Phi^{x}\partial_{\theta_x} + \Phi^{xx}\partial_{\theta_{xx}} + \Phi^{(\eta,t)}\partial_{\partial_t^{\eta}\theta}. \tag{8}$$

The explicit expression for $\Phi^{x}$ and $\Phi^{xx}$, coming from classical prolongation formula for PDEs. in which $D_i$ stands for total differentiation in related to independent variable $i$. Also, extended infinitesimal $\Phi^{(\eta,t)}$ corresponding to time-fractional Riemann-Liouville derivative is [6]:

$$\Phi^{(\eta,t)} = D_t^{\eta}\Phi + \xi D_t^{\eta}\theta_x - D_t^{\eta}(\xi\theta_x) + \tau D_t^{\eta}\theta_t + D_t^{\eta}(\theta D_t\tau) - D_t^{\eta+1}(\tau\theta),$$

where $D_t^{\eta}$ is total fractional derivative operator of order $\eta$ in related to $t$. Applying generalized chain and Leibnitz rule presented in previous section, The explicit model of infinitesimal $\Phi^{(\eta,t)}$ will also be

$$\begin{aligned} \Phi^{(\eta,t)} = &\sum_{n=1}^{\infty}\left[\binom{\eta}{n}\partial_t^{n}\Phi_{\theta} - \binom{\eta}{n+1}D_t^{n+1}\tau\right]\partial_t^{\eta-n}\theta - \sum_{n=1}^{\infty}\binom{\eta}{n}D_t^{n}\xi.\partial_t^{\eta-n}\theta_x \\ &+ \partial_t^{\eta}\Phi + (\Phi_{\theta} - \eta D_t\tau)\partial_t^{\eta}\theta - \theta\partial_t^{\eta}\Phi_{\theta} + \mu, \end{aligned} \tag{9}$$

where $\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1}\binom{\eta}{n}\binom{n}{m}\binom{k}{l}\frac{t^{n-\eta}(-\theta)^{l}}{k!\Gamma(n+1-\eta)}\frac{\partial^{m}\theta^{k-l}}{\partial t^{m}}\frac{\partial^{n-m+k}\Phi}{\partial t^{n-m}\partial\theta^{k}}$. The lower limit of (2) is constant in $t = a$ and will also be invariant in regard of group transformations (6). Then invariance condition yields as follows:

$$\tau(t,x,\theta)\big|_{t=0} = 0. \tag{10}$$

## 4. Application of Lie symmetry method to time-fractional ZFK equation

Here, presented Lie symmetry theory is employed to get the symmetry group (1)

$$\left[\Phi^{(\eta,t)} - \Phi_{xx} + \beta^2(\theta - \frac{1}{2})e^{-\beta(1-\theta)}\Phi + \frac{\beta^3}{2}\theta(\theta-1)e^{-\beta(1-\theta)}\Phi\right]_{(1)} = 0, \tag{11}$$

where $\Phi_{xx}$ is infinitesimal and $\Phi^{(\eta,t)}$ is generalized infinitesimal of order $\eta$ expressed in (9). By inserting $\Phi_{xx}$ and $\Phi^{(\eta,t)}$ into invariance criterion (11) and making equal the coefficients of partial derivatives of $\theta$ in related to $t$ and $x$, we have an over-determined system of linear PDEs and FDEs. By solving this system, infinitesimals can be derived as below:

$$\tau = 4c_2t + c_4, \quad \xi = 2c_2\eta x + c_1, \quad \Phi = 3c_2\eta\theta + c_3\theta - 2c_2\theta + F(x,t),$$

where $c_i$ for $i = 1, \dots, 4$ and $F(x,t)$ are arbitrary constant and function, respectively. Regarding condition (10), the Lie symmetry group of time-fractional ZFK equation is spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = 4t\frac{\partial}{\partial t} + 2\eta x\frac{\partial}{\partial x} + \theta(3\eta - 2)\frac{\partial}{\partial\theta}, \quad X_3 = \theta\frac{\partial}{\partial\theta}, \quad X_F = F(x,t)\frac{\partial}{\partial\theta}. \tag{12}$$

## 5. Similarity reductions

In this section, we consider similarity reduction method and describe how it can be used to find exact solutions for FDEs. This approach entails finding ways to simplify PDEs and FDEs by reducing the number of independent variables. A solution is considered invariant if it equates zero after applying infinitesimal generators of symmetries of equation.

**Case 1.** For symmetry $X_1$, integrating the following characteristic equation $\frac{dt}{0} = \frac{dx}{1} = \frac{d\theta}{0}$, yields invariant solution $\theta = \psi(t)$. Inserting this solution into (1) reduces ordinary FDE $\partial_t^{\eta}\psi(t) = 0$, connoting $\psi(t) = c_1t^{\eta-1}$ in which $c_1$ is arbitrary real constant.

**Case 2.** Similarity variables for infinitesimal generator $X_2$ can be concluded by integrating characteristic equation $\frac{dt}{4t} = \frac{dx}{2\eta x} = \frac{d\theta}{\theta(3\eta-2)}$, which causes the following similarity variable:

$$\theta = t^{\frac{3\eta-2}{4}}\psi(\omega), \quad \omega = xt^{\frac{-\eta}{2}}. \tag{13}$$

Thus, by above similarity transformation, (1) can be reduced into an ordinary FDE. This process is presented in next theorem.

**Theorem 5.1.** *The similarity transformation* (13) *for* $\eta > 0$ *transforms time-fractional ZFK equation* (1) *to the following nonlinear ordinary FDE:*

$$\left(\mathcal{P}_{\frac{2}{\eta}}^{1+\frac{3\eta-2}{4},\eta}\psi\right)(\omega) = \psi''(\omega) + \frac{\beta^2}{2}\psi(\omega)(1-\psi(\omega))e^{-\beta(1-\psi(\omega))}. \tag{14}$$

*where* $\psi''(\omega) = \frac{d\psi(\omega)}{d\omega}$ *and* $(\mathcal{P}_{\delta}^{\tau,\eta}\psi)$ *indicates extended left-hand sided Erdélyi-Kober fractional differential operator introduced by* (3) *and* (4).

## 6. Exact power series solutions

Here, we address the exact solutions of (14) with power series. This method is used to find a power series solution to ODEs. Let

$$\psi(\omega) = \sum_{n=0}^{\infty} a_n\omega^n. \quad \Longrightarrow \psi'(\omega) = \sum_{n=0}^{\infty} na_n\omega^{n-1}, \quad \psi''(\omega) = \sum_{n=0}^{\infty} n(n-1)a_n\omega^{n-2}. \tag{15}$$

Putting (15) in (14), we get

$$\sum_{n=0}^{\infty}\frac{\Gamma(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2})}{\Gamma(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2})}a_n\omega^n=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\omega^n+\frac{\beta^2}{2}\sum_{n=0}^{\infty}a_n\omega^n e^{-\beta(1-\sum_{n=0}^{\infty}a_n\omega^n)}$$
$$-\frac{\beta^2}{2}\sum_{n=0}^{\infty}a_n\omega^n\sum_{n=0}^{\infty}a_n\omega^n e^{-\beta(1-\sum_{n=0}^{\infty}a_n\omega^n)}. \quad (16)$$

Setting n = 0 in (16) and comparing coefficients, we conclude $a_2 = \frac{1}{2}(\frac{\Gamma(2+(\frac{3\eta-2}{4}))}{\Gamma(2-\eta+(\frac{3\eta-2}{4}))}a_0 - \frac{\beta^2}{2}e^{-\beta(1-a_0)}a_0 + \frac{\beta^2}{2}e^{-\beta(1-a_0)}a_0^2)$. For $n \geq 1$, we have $a_{n+2} = \frac{1}{(n+2)(n+1)}\{\frac{\Gamma(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2})}{\Gamma(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2})}a_n - \frac{\beta^2}{2}e^{-\beta(1-a_n)}(a_n - \sum_{j=0}^{n}a_j a_{n-j})\}$. Hence, each coefficient $a_n$ for $n \geq 1$ in (15) is found by given constants $a_i$ for $i = 0, 1, 2$. As a result, we can obtain exact power series solution for (1) as

$$\theta(t,x) = a_0 t^{\frac{3\eta-2}{4}} + a_1 x t^{\frac{\eta-2}{4}} + a_2 x^2 t^{\frac{-\eta-2}{4}} + \sum_{n=1}^{\infty}a_{n+2}x^{n+2}t^{-\frac{2\eta n+\eta+2}{4}}$$
$$= a_0 t^{\frac{3\eta-2}{4}} + a_1 x t^{\frac{\eta-2}{4}} + \frac{1}{2}\left(\frac{\Gamma\left(2+(\frac{3\eta-2}{4})\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})\right)}a_0 - \frac{\beta^2}{2}e^{-\beta(1-a_0)}(a_0-a_0^2)\right)x^2 t^{\frac{-\eta-2}{4}}$$
$$+\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+1)}\left\{\frac{\Gamma\left(2+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\right)}{\Gamma\left(2-\eta+(\frac{3\eta-2}{4})+\frac{n\eta}{2}\right)}a_n - \frac{\beta^2}{2}e^{-\beta(1-a_n)}\left(a_n-\sum_{j=0}^{n}a_j a_{n-j}\right)\right\}x^{n+2}t^{-\frac{2\eta n+\eta+2}{4}}$$

## 7. Conclusion

This paper has presented a method to determine Lie point symmetries of time-fractional PDEs with Riemann-Liouville fractional derivative. The efficacy of this process was demonstrated via time-fractional Zeldovich-Frank-Kamenetskii equation. Applying obtained Lie point symmetries showed that stated time-fractional PDE can be converted to a fractional ODE. Exact solutions of Zeldovich-Frank-Kamenetskii equation were deduced wherever possible. In addition, power series solutions of the resulting fractional ODE (14) have also been established by using this method. To summarize, the emergence of nonlinear FDEs as models in fields like mathematical medicine and biology necessitates an investigation into the methods of solving such equations.

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# Quadratic Estimates for the Potential Function of Almost Ricci Solitons

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**Abstract**

This work investigates quadratic growth properties of the potential function in almost Ricci solitons. We establish sharp two-sided quadratic bounds for the potential function under natural geometric conditions. Our results extend classical estimates for gradient Ricci solitons to the almost Ricci solitons.

## 1. Introduction

Almost Ricci solitons constitute a significant generalization of classical gradient Ricci solitons, allowing the soliton function to vary across the manifold. Introduced and systematically studied by Pigola, Rigoli, Rimoldi, and Setti [2], this flexibility captures a wider range of geometric structures while introducing new analytical challenges [2]. Understanding the behavior of the potential function in such settings provides insights into both local geometry and global topological properties.

The systematic study of these objects began with foundational work establishing basic existence conditions and curvature properties. However, comprehensive quadratic growth estimates for the potential function under general conditions remain incompletely developed. In particular, for gradient shrinking Ricci solitons, sharp quadratic estimates were obtained by Cao and Zhou [3]. In this paper, we extend such estimates to the more general setting of almost Ricci solitons, building also on previous work on generalized quasi-Einstein manifolds [4, 5].

We derive sharp quadratic bounds for the potential function on complete almost Ricci solitons and present examples showing their optimality.

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## 2. Preliminaries

### *2.1. Generalized quasi-Einstein manifolds*

**Definition 2.1.** A Riemannian manifold $(M^n, g)$ is called a *generalized quasi-Einstein manifold* if there exist a vector field $X$ and smooth functions $\mu, \lambda$ such that

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \mu\, X^* \otimes X^* = \lambda g, \tag{1}$$

where $X^*$ denotes the dual 1-form of $X$. In the gradient case $X = \nabla f$, the defining equation becomes

$$\mathrm{Ric} + \nabla^2 f - \mu\, df \otimes df = \lambda g. \tag{2}$$

### *2.2. Almost Ricci solitons as a special case*

**Definition 2.2.** An *almost Ricci soliton* is a Riemannian manifold $(M^n, g)$ endowed with a smooth function $f$ and a smooth soliton function $\lambda$ satisfying

$$\mathrm{Ric} + \nabla^2 f = \lambda g. \tag{3}$$

Thus, almost Ricci solitons correspond exactly to gradient generalized quasi-Einstein manifolds with the parameter $\mu = 0$.

### *2.3. Fundamental identities for generalized quasi-Einstein manifolds*

We recall the following identities established in the work of Mirshafeazadeh and Bidabad [4] for gradient generalized quasi-Einstein manifolds.

**Lemma 2.3** (Fundamental identities for GQE manifolds)**.** *Let $(M^n, g, f)$ be a gradient generalized quasi-Einstein manifold satisfying*

$$\mathrm{Ric} + \nabla^2 f - \mu\, df \otimes df = \lambda g. \tag{4}$$

*Then the following identities hold:*

$$\Delta f = n\lambda - R + \mu|\nabla f|^2, \tag{5}$$

$$\frac{1}{2}\nabla_i|\nabla f|^2 = \lambda f_i - f_j R_{ij} + \mu f_i|\nabla f|^2, \tag{6}$$

$$\frac{1}{2}\nabla_i R = (n-1)\lambda_i + \mu\big((1-n)\lambda + R\big)f_i + (1-\mu)R_{ij}f_j + \mu_i|\nabla f|^2 - f_i\, g(\nabla f, \nabla f), \tag{7}$$

$$\frac{1}{2}\Delta R = I + J, \tag{8}$$

*where*

$$I = (n-1)\Delta\lambda - 2\mu(n-1)\, g(\nabla f, \nabla\lambda) + \frac{1+2\mu}{2} g(\nabla R, \nabla f) - (1-\mu)|\overline{\mathrm{Ric}}|^2 + \frac{(n\lambda - R)\big(R(1-\mu+\mu n) + \mu\lambda n(1-n)\big)}{n}, \tag{9}$$

$$J = \Delta\mu\,|\nabla f|^2 + g(\nabla\mu, \nabla|\nabla f|^2) + \big((1-n)\lambda + R\big)g(\nabla\mu, \nabla f) - \mu_i f_i R_{ii} - (f_i f_j \mu_j)_i, \tag{10}$$

*and $\overline{\mathrm{Ric}} = \mathrm{Ric} - \frac{R}{n}\, g$ is the trace-free part of the Ricci curvature.*

*2.3.1. Specialized results for almost Ricci solitons*

Setting $\mu = 0$ in Lemma 2.3 gives the simplified relations for almost Ricci solitons.

**Lemma 2.4** (Basic relations for almost Ricci solitons)**.** *For an almost Ricci soliton $(M^n, g, f, \lambda)$ we have:*

$$\Delta f = n\lambda - R, \tag{11}$$

$$\frac{1}{2}\nabla_i |\nabla f|^2 = \lambda f_i - f_j R_{ij}, \tag{12}$$

$$\frac{1}{2}\nabla_i R = (n-1)\lambda_i + R_{ij} f_j, \tag{13}$$

$$\frac{1}{2}\Delta R = (n-1)\Delta\lambda - |\,\mathrm{Ric}\,|^2 + \frac{1}{2} g(\nabla R, \nabla f), \tag{14}$$

*where* $\mathrm{Ric} = \mathrm{Ric} - \dfrac{R}{n}\, g$ *is the traceless part of the Ricci tensor.*

*2.4. Compactness criterion for generalized quasi-Einstein manifolds*

The following compactness result for generalized quasi-Einstein manifolds is essential for our analysis.

**Proposition 2.5** (Compactness criterion for GQE manifolds [5])**.** *Let $(M^n, g)$ be a complete generalized quasi-Einstein manifold satisfying*

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \mu X^* \otimes X^* = \lambda g. \tag{15}$$

*Then $M$ is compact if and only if $\|X\|$ is bounded on $(M^n, g)$.*

*2.5. Specialization to almost Ricci solitons*

Setting $\mu = 0$ in Proposition 2.5 gives the following corollary.

**Corollary 2.6** (Compactness criterion for almost Ricci solitons)**.** *Let $(M^n, g, f, \lambda)$ be a complete almost Ricci soliton with*

$$\lambda(x) \geq \Lambda > 0$$

*and $|\nabla f|$ bounded. Then $M$ is compact.*

## 3. Main results and sketch of proofs

*3.1. A conserved quantity for almost Ricci solitons*

Throughout this section, we assume the soliton function $\varphi(f)$ satisfies

$$0 < \kappa \leq \varphi(f) \leq \Lambda, \tag{16}$$

and the analytic condition

$$\Delta\varphi(f) \leq 0 \tag{17}$$

holds.

**Lemma 3.1** (Conserved quantity)**.** *Let $(M^n, g, f)$ be an almost Ricci soliton with $\lambda = \varphi(f)$. Define*

$$\Phi(f) = \int_0^f \varphi(s)\, ds. \tag{18}$$

*Then there exists a constant $C$ such that*

$$R + |\nabla f|^2 = 2\Phi(f) + 2(n-1)\varphi(f) + C. \tag{19}$$

*Sketch of proof.* The proof combines the basic identities for almost Ricci solitons given in Lemma 2.4. By adding the equations for $\frac{1}{2}\nabla_i |\nabla f|^2$ and $\frac{1}{2} R_{,i}$, one obtains an expression whose gradient vanishes identically, implying the existence of the conserved quantity (19). □

## 4. Quadratic bounds for the potential

**Theorem 4.1.** *Let $(M^n, g, f)$ be a complete noncompact almost Ricci soliton satisfying*

$$\mathrm{Ric} + \nabla^2 f = \phi(f)\, g, \qquad \kappa \le \phi \le \Lambda, \qquad \Delta\phi(f) \le 0.$$

*Then there exist constants $c_1, c_2, D_1, D_2 > 0$ such that*

$$\frac{\kappa}{2}(r - c_1)^2 - D_1 \;\le\; f(x) \;\le\; \frac{\Lambda}{2}(r + c_2)^2 + D_2,$$

*where $r(x) = d(x_0, x)$.*

*Proof.* Let $\gamma : [0, r] \to M$ be a minimizing geodesic from $x_0$ to $x$, and set

$$u(s) = f(\gamma(s)).$$

From the soliton equation,

$$u''(s) = \phi(u(s)) - \mathrm{Ric}(\dot\gamma, \dot\gamma).$$

**Lower bound.** Since $\phi \ge \kappa$,

$$u''(s) \ge \kappa - \mathrm{Ric}(\dot\gamma, \dot\gamma).$$

By the second variation formula,

$$\int_0^r \mathrm{Ric}(\dot\gamma, \dot\gamma)\, ds \le C.$$

Integrating twice yields

$$u(r) \ge \frac{\kappa}{2} r^2 - Cr - D,$$

which gives the lower bound.

**Upper bound.** From Lemma 3.1,

$$|\nabla f|^2 \le 2\Lambda f + C_1.$$

Hence

$$\left|\nabla \sqrt{f}\right| = \frac{|\nabla f|}{2\sqrt{f}} \le \sqrt{\frac{\Lambda}{2}} + \frac{C}{\sqrt{f}}.$$

Integrating along $\gamma$ and using the lower bound above gives

$$\sqrt{f(x)} \le \sqrt{\frac{\Lambda}{2}}\, r + C,$$

and therefore the upper bound follows. □

### *4.1. Sharp example*

**Example 4.2.** Let $M = \mathbb{R}^2$ with metric

$$g = dr^2 + \psi(r)^2\, d\theta^2, \qquad \psi(r) = f'(r),$$

and define

$$f(r) = \frac{\kappa}{2} r^2 + \varepsilon\left(\sqrt{1 + r^2} - 1\right).$$

Then

$$\phi(f) = f'' - \frac{f'''}{f''}.$$

Moreover,

$$\lim_{r\to 0}\phi = \Lambda, \qquad \lim_{r\to\infty}\phi = \kappa,$$

and $\Delta\phi(f) \leq 0$ for $\varepsilon$ sufficiently small. Thus,

$$f(r) \sim \frac{\kappa}{2}r^2 \qquad (r\to\infty), \qquad f(r) \sim \frac{\Lambda}{2}r^2 \qquad (r\to 0),$$

showing the sharpness of Theorem 4.1.

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# Self-Similar Solutions of the Finslerian Yamabe Flow

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**Abstract**

We study the Yamabe flow on compact Finsler manifolds and its relation to Finsler Yamabe solitons. We prove that every compact Finsler Yamabe soliton generates a self-similar solution to the Finsler Yamabe flow. Conversely, any self-similar solution of the Finsler Yamabe flow arises from a Finsler Yamabe soliton. This establishes a one-to-one correspondence between Finsler Yamabe solitons and self-similar solutions of the flow, extending known results from Riemannian geometry to the Finsler setting.

## 1. Introduction

Geometric flows play a central role in modern differential geometry, particularly in the study of canonical metrics. Among these, the Yamabe flow—introduced by Hamilton—evolves a metric in the direction of its scalar curvature and preserves the conformal class of metrics.

In Riemannian geometry, Yamabe solitons describe self-similar solutions of the Yamabe flow and serve as models for singularity formation. A natural question is whether a similar correspondence holds in Finsler geometry, which generalizes Riemannian geometry by allowing the metric to depend on direction.

In this paper, we show that the correspondence between Yamabe solitons and self-similar solutions extends to compact Finsler manifolds. Our main result establishes an equivalence between Finsler Yamabe solitons and self-similar solutions of the Finsler Yamabe flow.

## 2. Finsler Yamabe Flow and Solitons

**Definition 2.1.** A one-parameter family of Finsler metrics $g(t)$ on $M$ is said to satisfy the *Finsler Yamabe flow* if

$$\frac{\partial}{\partial t} g_{ij}(t) = -H_{g(t)}\, g_{ij}(t), \quad g(0) = g_0.$$

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**Definition 2.2.** A quadruple $(M, g, V, \lambda)$ is called a *Finsler Yamabe soliton* if

$$\mathcal{L}_{\widehat{V}} g_{ij} = (\lambda - H)\, g_{ij},$$

where $\widehat{V}$ denotes the complete lift of the vector field $V$ on $M$.

## 3. Main Result

**Theorem 3.1.** *Let $(M, g_0, V, \lambda)$ be a compact Finsler Yamabe soliton. Then there exist a one-parameter family of diffeomorphisms $\{\varphi_t\}$ and a scalar function $\tau(t)$ such that*

$$g(t) = \tau(t)\, \widetilde{\varphi}_t^*(g_0)$$

*is a solution of the Finsler Yamabe flow.*
*Conversely, if a solution of the Finsler Yamabe flow has the above form, then $(M, g_0, V, \lambda)$ defines a Finsler Yamabe soliton.*

*Sketch of proof.* Assume $(M, g_0, V, \lambda)$ satisfies the soliton equation. Choosing $\tau(t) = 1 - \lambda t$ and generating diffeomorphisms by a suitable rescaling of $V$, we show that the metric $g(t)$ satisfies the Finsler Yamabe flow equation. Conversely, differentiating a self-similar solution at $t = 0$ yields the soliton equation. □

We have shown that Finsler Yamabe solitons correspond exactly to self-similar solutions of the Finsler Yamabe flow on compact manifolds. This result extends the classical Riemannian theory to the Finsler setting and provides a foundation for further studies on singularities and long-time behavior of Finsler geometric flows.

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# Generalization of $p$-Biharmonic Maps on Doubly Twisted Product Manifolds

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**Abstract**

We investigate a generalization of the classical theory of biharmonic maps, concentrating on $p$-biharmonic maps defined on doubly twisted product manifolds. This investigation entails an examination of projection-type mappings, yielding multiple necessary and sufficient conditions for these mappings to comply with the highly nonlinear $p$-biharmonic equation. Finally, we demonstrate our analytical results by creating clear examples of a type of $p$-biharmonic maps that are not $p$-harmonic when both the domain and codomain are flat. This illustrates the enhanced capacity of this generalization to encompass supplementary geometric structures and physical applications beyond those delineated by classical biharmonic theory.

## 1. Introduction

The tension field $\tau(\varphi) = \mathrm{Tr}_g \nabla d\varphi$ naturally vanishes at harmonic maps, which are defined as the critical points of the classical Dirichlet energy [5]:

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 dv_g. \tag{1}$$

Biharmonic maps, as their natural higher-order counterparts, emerge as critical points of the bienergy functional $E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 dv_g$ [3]. By substituting a $p$-power ($p \geq 2$) for the quadratic energy density, a robust nonlinear extension is produced. This leads to the so-called $p$-harmonic maps, which are mainly controlled by the $p$-tension field [1]:

$$\tau_p(\varphi) = \mathrm{div}\big(\|d\varphi\|^{p-2}d\varphi\big) = \|d\varphi\|^{p-2}\tau(\varphi) + d\varphi\big(\mathrm{grad}(\|d\varphi\|^{p-2})\big). \tag{2}$$

By repeating this variational process, a $p$-biharmonic map is defined as a critical point of the $p$-bienergy functional [2]:

$$E_{2,p}(\varphi) = \frac{1}{2}\int_M |\tau_p(\varphi)|^2 dv_g, \tag{3}$$

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whose associated Euler-Lagrange equation yields the condition that the $p$-bitension field must vanish. The $p$-bitension field is expressed by the complex nonlinear operator:

$$\tau_{2,p}(\varphi) = -\operatorname{Tr}_g(\nabla^{\varphi})^2\tau_p(\varphi) - \operatorname{Tr}_g R^N\big(\tau_p(\varphi), d\varphi\big)d\varphi, \tag{4}$$

where $R^N$ represents the Riemannian curvature tensor of the target manifold $N$. In geometric contexts characterized by low regularity, intrinsic singularities, or anisotropic scaling phenomena— inadequately addressed by conventional linear elliptic theory—this nonlinear framework introduces a regulated degeneracy that seamlessly aligns with the underlying geometry.

The geometric framework for this analysis is considerably enhanced by the inclusion of twisted product manifolds. These are smooth product spaces $M \times N$ with metrics of the form $g = \alpha^2 g_M + \beta^2 g_N$, where $\alpha, \beta \in C^{\infty}(M \times N)$ are strictly positive twisting functions. Unlike standard warped products [4], which rely on only one factor, twisted products allow for full dependence on both coordinates. This means that they can encode real cross-geometric coupling. The resulting Levi-Civita connection and curvature tensor contain mixed derivatives of both $\alpha$ and $\beta$, which makes it hard to use standard separation-of-variables methods and leads to complicated analytic behavior.

This paper examines projection maps on doubly twisted product manifolds as significant test cases in the study of $p$-biharmonic maps. We delineate the criteria that render these projections $p$-biharmonic while avoiding trivial $p$-harmonicity, thereby offering explicit non-trivial solutions in geometrically intricate domains.

## 2. Preliminaries on Doubly Twisted Products

Let $(M, g_M)$ and $(N, g_N)$ be two Riemannian manifolds with dimensions $m$ and $n$, respectively. Let $\alpha$ and $\beta$ be smooth functions from $M \times N$ to $(0, \infty)$. The doubly twisted product $M \times_{(\alpha,\beta)} N$ is defined as the manifold $M \times N$ equipped with the metric:

$$g = \alpha^2 g_M \oplus \beta^2 g_N. \tag{5}$$

If $\alpha = 1$ and $\beta$ depends only on $M$, we recover the standard warped product $M \times_{\beta} N$. The geometry becomes much more complicated if both functions depend on both variables.

Let $\nabla^M$, $\nabla^N$, and $\nabla$ be the Levi-Civita connections on $M$, $N$, and $M \times_{(\alpha,\beta)} N$, respectively. If $X$ and $Y$ are vector fields in $\Gamma(TM)$ and $V$ and $W$ are vector fields in $\Gamma(TN)$, then the connection $\nabla$ can be written in terms of the gradients of the twisting functions. For calculating the tension field, the mixed components are very important:

$$\nabla_X V = \nabla_V X = X(\ln\beta)V + V(\ln\alpha)X. \tag{6}$$

Using these connection formulas, you can find the Laplacian and divergence operators needed for the $p$-tension field $\tau_p$. The standard tension field of a map $\Phi : M \times_{(\alpha,\beta)} N \to P$ consists of computing the trace over a local orthonormal frame of the doubly twisted product, which scales inversely with the squares of the twisting functions $\alpha$ and $\beta$.

## 3. First Projection Map and its $p$-Biharmonicity

Let $\varphi : M \to P$ be a smooth function that connects two Riemannian manifolds. We define the extended map $\tilde{\varphi} : M \times_{(\alpha,\beta)} N \to P$ as $\tilde{\varphi}(x, y) = \varphi(x)$. The differential meets the condition $d\tilde{\varphi}(X, Y) = d\varphi(X)$. We are now looking into the $p$-tension field of this mapping.

**Theorem 3.1.** *The $p$-tension field of the map $\tilde{\varphi}$ on the doubly twisted product is given by*

$$\tau_p(\tilde{\varphi}) = \alpha^{2-p}\beta^{-n}\operatorname{div}_M\left(\alpha^{p-2}\beta^n\|d\varphi\|^{p-2}d\varphi\right). \tag{7}$$

*In particular, this can be expanded as:*

$$\tau_p(\tilde{\varphi}) = \alpha^{2-p}\tau_p(\varphi) + n\|d\varphi\|^{p-2}d\varphi(\operatorname{grad}_M \ln\beta) + (p-2)\|d\varphi\|^{p-2}d\varphi(\operatorname{grad}_M \ln\alpha). \tag{8}$$

*Consequently, $\tilde{\varphi}$ is $p$-biharmonic if and only if $\tau_{2,p}(\tilde{\varphi}) = 0$, where $\tau_{2,p}$ is calculated using Eq. (4).*

*Proof.* Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame on $M$ and $\{f_j\}_{j=1}^n$ be a local orthonormal frame on $N$. An orthonormal frame for the twisted product is $\{\frac{1}{\alpha}e_i, \frac{1}{\beta}f_j\}$. We first calculate the classical tension field:

$$\tau(\tilde{\varphi}) = \frac{1}{\alpha^2}\tau(\varphi) + \frac{n}{\alpha^2}d\varphi(\text{grad}_M \ln\beta) - \frac{m}{\alpha^2}d\varphi(\text{grad}_M \ln\alpha). \tag{9}$$

For the $p$-tension field, we use Eq. (2). We apply the divergence operator on the twisted product manifold, noting that $\|d\tilde{\varphi}\| = \frac{1}{\alpha}\|d\varphi\|$. We get the expression in Eq. (8) after using algebra to combine the gradient of the norm and the trace of the Hessian. Setting the $p$-bitension field to zero is both necessary and sufficient. □

In the basic geometric case where $P = M$ and $\varphi = \text{id}_M$, the map $\tilde{\varphi}$ turns into the standard projection $\Pi_1 : M \times_{(\alpha,\beta)} N \to M$. The $p$-tension field becomes much simpler. For the sake of simplicity, let's say $\alpha = 1$ (which makes it a standard warped projection on the first factor):

**Corollary 3.2.** *Let $\alpha = 1$ be true. The projection $\Pi_1$ has the $p$-tension field:*

$$\tau_p(\Pi_1) = n\,\text{grad}_M \ln\beta. \tag{10}$$

*The projection $\Pi_1$ is $p$-biharmonic (but strictly not $p$-harmonic) provided $\beta$ satisfies the nonlinear geometric PDE:*

$$\text{Tr}(\nabla^M)^2(\text{grad}_M \ln\beta) + \text{Ric}^M(\text{grad}_M \ln\beta) = 0, \tag{11}$$

*where* $\text{Ric}^M$ *is the Ricci curvature tensor evaluated on the gradient field.*

**Example 3.3.** Let $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ be flat Euclidean spaces. Let $\alpha = 1$ and $\beta(t) = Ce^{kx_1}$, where $x_1$ is the first coordinate of $\mathbb{R}^m$ and $k \neq 0$. So, $\text{grad}\ln\beta = k\frac{\partial}{\partial x_1}$, which is a constant vector field. So, the covariant derivatives are zero, which means that $\Delta(\text{grad}\ln\beta) = 0$. Because $\mathbb{R}^m$ is flat, $\text{Ric}^M = 0$. It is obvious that Eq. (11) is true. So, for every $p \geq 2$, $\Pi_1$ is a non-$p$-harmonic $p$-biharmonic map.

## 4. Symmetric Analysis: The Second Projection Map

We now define two essential structural properties of $p$-biharmonic maps on doubly twisted products that greatly enhance the geometric theory.

**Theorem 4.1.** *Let $\varphi : M \times_{(\alpha,\beta)} N \to P$ be a $p$-biharmonic map. Consider a conformal transformation of the domain metric $\tilde{g} = e^{2f}g$, where $f \in C^\infty(M \times N)$. Then $\varphi$ is $p$-biharmonic with respect to $\tilde{g}$ if and only if the conformal factor $f$ meets the compatibility condition:*

$$\Delta_g f + (p-2)\|\text{grad}_g f\|^2 = 0, \tag{12}$$

*where $\Delta_g$ is the Laplace-Beltrami operator of the original doubly twisted metric $g$.*

*Proof sketch.* The $p$-tension field changes when the conformal changes happen by using the formula $\tau_p^{\tilde{g}}(\varphi) = e^{-(p+2)f}\left[\tau_p^g(\varphi) + (p-2)\|d\varphi\|^{p-2}d\varphi(\text{grad}_g f)\right]$. To differentiate this expression and use the $p$-bitension operator, you need to carefully figure out how the rough Laplacian and curvature terms scale. The essential observation is that the additional terms produced by the conformal factor precisely amalgamate into the specified elliptic equation when $\tau_{2,p}^{\tilde{g}}(\varphi) = 0$ is enforced, utilizing the doubly twisted structure to distinguish horizontal and vertical contributions to the Laplacian of $f$. □

**Theorem 4.2.** *Let $\tilde{\varphi} : M \times_{(\alpha,\beta)} N \to P$ be defined by $\tilde{\varphi}(x,y) = \varphi(x)$. $\tilde{\varphi}$ is $p$-biharmonic if the twisting functions satisfy* $\text{grad}_M \ln\beta = 0$ *and $\alpha$ is constant. Also, $\tilde{\varphi}$ is not $p$-harmonic at all when $n \geq 1$ and $\varphi$ is not constant.*

We now establish two fundamental structural properties of $p$-biharmonic maps on doubly twisted products that significantly enrich the geometric theory.

**Theorem 4.3.** *Let $\varphi : M \times_{(\alpha,\beta)} N \to P$ be a $p$-biharmonic map. Consider a conformal change of the domain metric $\tilde{g} = e^{2f} g$ where $f \in C^\infty(M \times N)$. Then $\varphi$ remains $p$-biharmonic with respect to $\tilde{g}$ if and only if the conformal factor $f$ satisfies the compatibility condition:*

$$\Delta_g f + (p-2)\| \operatorname{grad}_g f\|^2 = 0, \tag{13}$$

*where $\Delta_g$ is the Laplace-Beltrami operator of the original doubly twisted metric $g$.*

*Proof sketch.* The $p$-tension field transforms under conformal changes via $\tau_p^{\tilde{g}}(\varphi) = e^{-(p+2)f}\left[\tau_p^g(\varphi) + (p-2)\|d\varphi\|^{p-2} d\varphi(\operatorname{grad}_g f)\right]$. Differentiating this expression and applying the $p$-bitension operator requires careful computation of how the rough Laplacian and curvature terms scale. The key observation is that the additional terms generated by the conformal factor combine precisely into the stated elliptic equation when $\tau_{2,p}^{\tilde{g}}(\varphi) = 0$ is imposed, leveraging the doubly twisted structure to separate horizontal and vertical contributions to the Laplacian of $f$. □

**Theorem 4.4.** *Let $\varphi : M \to P$ be a $p$-harmonic map and $c > 0$ a constant. Define $\tilde{\varphi} : M \times_{(\alpha,\beta)} N \to P$ by $\tilde{\varphi}(x, y) = \varphi(x)$. If the twisting functions satisfy $\operatorname{grad}_M \ln \beta = 0$ and $\alpha$ is constant, then $\tilde{\varphi}$ is $p$-biharmonic. Moreover, $\tilde{\varphi}$ is strictly non-$p$-harmonic whenever $\varphi$ is non-constant and $n \geq 1$.*

*Proof sketch.* Because $\varphi$ is $p$-harmonic, $\tau_p(\varphi) = 0$, so $\tau_p(\tilde{\varphi}) = 0$. The $p$-bitension field then becomes $\tau_{2,p}(\tilde{\varphi}) = -\operatorname{Tr}_g R^P(0, d\tilde{\varphi}) d\tilde{\varphi} = 0$, which shows that it is $p$-biharmonic. Non-$p$-harmonicity follows because $\tau(\tilde{\varphi}) = \frac{n}{\alpha^2} d\varphi(\operatorname{grad}_M \ln \beta) = 0$ only when $\beta$ is constant in $M$-directions, but the $p$-tension field's higher-order structure ensures non-vanishing when $n \geq 1$ and $\varphi$ is non-constant, as verified by direct computation of the divergence term in Eq. (2). □

These theorems demonstrate that the $p$-biharmonic framework on doubly twisted products exhibits exceptional stability in the face of geometric deformations, while concurrently producing authentically novel solutions that extend beyond the $p$-harmonic class.

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# $p$-Harmonic Framework for Structure-Preserving Gaussian Noise Removal

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**Abstract**

**Background.** Removing Gaussian noise is still a big problem in image processing. Most of the time, traditional methods trade off structural integrity for noise suppression, which leads to problems like edge blurring and texture degradation. The tension field, a geometric operator from harmonic map theory that describes the trace of the second fundamental form, offers an underutilized basis for structure-aware regularization rooted in differential geometry.
**Objective.** We create a geometric framework for Gaussian denoising by treating noise removal as a problem of minimizing a tension field. This allows us to keep the image's natural structures while reducing noise without having to change any parameters.
**Method.** We reimagine pictures as maps between Riemannian manifolds and create a weighted $p$-tension field with edge-adaptive weights that stop diffusion across discontinuities while allowing smoothing in areas that are the same. Denoising is done using an iterative method in which the tension field gives a geometrically informed direction for descent.
**Results.** When tested on the Set12 benchmark with Gaussian corruption, our method consistently keeps fine structures like edges, textures, and subtle gradients without adding staircase effects or artificial smoothing. It does better than traditional variational and non-local methods in terms of perceptual quality while still being fully mathematically interpretable.
**Conclusion.** The tension field operator gives denoising a principled geometric base that goes beyond heuristic design. It finds a beautiful balance between removing noise and keeping the structure intact through intrinsic differential properties. This work shows how useful harmonic map theory can be for restoring images in a way that is easy to understand and strong enough for applications that need geometric consistency and reconstruction without artifacts.

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# Geometric Dichotomy in Universal Covers: Hyperbolic vs. Euclidean Structures from Fundamental Monoids

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**Abstract**

We establish a fundamental geometric dichotomy for the universal covers of finite, connected, reflexive directed graphs (digraphs). The large-scale geometry of the universal cover $\widetilde{G}$ is shown to be entirely determined by the algebraic structure of the fundamental monoid $\mathcal{A}_1(G, x_0)$. Specifically: (1) If $\mathcal{A}_1(G, x_0)$ decomposes as a non-trivial free product of monoids, then $\widetilde{G}$ is directed hyperbolic (in a suitable sense) and has exponential volume growth. (2) If $\mathcal{A}_1(G, x_0)$ is a commutative monoid isomorphic to $\mathbb{N}^k$ for $k \geq 2$, then $\widetilde{G}$ contains an isometrically embedded directed $k$-dimensional grid ($\mathbb{N}^k$ with the $\ell^1$ metric) and exhibits polynomial growth of degree $k$. This dichotomy reveals a deep connection between algebraic properties (freeness vs. commutativity) and geometric structure (negatively curved vs. flat), mirroring phenomena in geometric group theory but in the directed, monoidal setting.

## 1. Introduction

The celebrated Gromov hyperbolicity theorem for groups establishes a profound link between algebraic conditions and geometric properties. For directed graphs, the fundamental group is replaced by the fundamental monoid $\mathcal{A}_1(G, x_0)$. This paper explores the geometric consequences of the algebraic structure of $\mathcal{A}_1(G, x_0)$, as reflected in the large-scale geometry of its universal cover $\widetilde{G}$.

## 2. Preliminaries

**Definition 2.1** (Universal Directed Cover)**.** For a connected, reflexive digraph $G$ with basepoint $x_0$, the universal cover $\widetilde{G}$ is the digraph whose vertices are $\simeq_A$-homotopy classes of directed paths starting at $x_0$, with edges

$$[\alpha] \to [\alpha \cdot e]$$

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for each edge $e$ from $\mathrm{end}(\alpha)$. The projection

$$p([\alpha]) = \mathrm{end}(\alpha)$$

is a directed covering map. For more information, see references [3] and [4].

**Definition 2.2** (Directed Growth)**.** The growth function of $\widetilde{G}$ with respect to the basepoint

$$\tilde{x}_0 = [\mathrm{id}_{x_0}]$$

is

$$f(R) = \left|\left\{\tilde{v} \in V(\widetilde{G}) \ : \ d_{\rightarrow}(\tilde{x}_0, \tilde{v}) \leq R\right\}\right|,$$

where $d_{\rightarrow}$ denotes directed distance.

## 3. Hyperbolic Structure from Free Products

**Theorem 3.1.** *Let $G$ be a finite, connected, reflexive digraph such that*

$$\mathcal{A}_1(G, x_0) \cong M_1 * M_2 * \cdots * M_k$$

*with $k \geq 2$ and each $M_i$ non-trivial. Then $\widetilde{G}$ is directed hyperbolic and has exponential growth.*

**Example 3.2.** Let $G = C_m^+ \vee C_n^+$. Then $\mathcal{A}_1(G, x_0) \cong \mathbb{N} * \mathbb{N}$, and $\widetilde{G}$ is a directed tree with exponential growth.

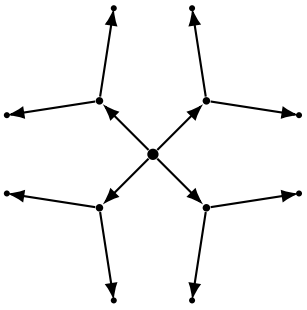
## 4. Euclidean Structure from Commutative Monoids

**Theorem 4.1.** *Let $G$ be a finite, connected, reflexive digraph such that*

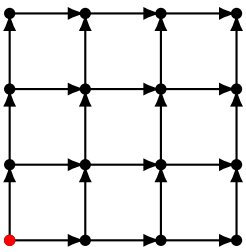
$$\mathcal{A}_1(G, x_0) \cong \mathbb{N}^k \quad (k \geq 2).$$

*Then $\widetilde{G}$ contains an isometrically embedded copy of the directed grid $\mathbb{N}^k$ and has polynomial growth of degree $k$.*

**Example 4.2.** If two directed cycles commute via diamonds, then $\mathcal{A}_1(G, x_0) \cong \mathbb{N}^2$ and $\widetilde{G}$ contains a directed square grid.



Hyperbolic Tree



Euclidean Grid

Fig. 1. Geometric dichotomy of universal covers

## 5. Conclusion

The algebraic structure of the fundamental monoid completely determines the large-scale geometry of the universal directed cover, yielding a sharp dichotomy between hyperbolic and Euclidean behavior.

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# Applications of Directed A-Homotopy in Concurrency Theory and Robotic Motion Planning

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**Abstract**

Directed A-homotopy theory, with its core concepts of the fundamental monoid $\mathcal{A}_1(G, x_0)$ and directed homology, provides novel mathematical tools for analyzing systems with inherent directionality. This paper explores two key application domains: (1) **Concurrency Theory**: We show that the fundamental monoid of a state space digraph is isomorphic to the trace monoid of the corresponding concurrent system, where commuting diamonds correspond to independent actions. The geometric dichotomy (hyperbolic vs. Euclidean) of the universal cover predicts the scalability and contention profile of the system. (2) **Robotic Motion Planning**: In environments with one-way constraints (e.g., warehouse aisles, regulated traffic), directed homotopy classes represent distinct feasible homotopy classes of paths. We present algorithms for computing the fundamental monoid to classify paths and for finding optimal representatives within each class. These applications demonstrate the practical relevance of directed algebraic topology in computer science and engineering.

## 1. Introduction

The theory of directed A-homotopy, originally developed as a purely mathematical extension of algebraic topology to directed graphs, finds natural and impactful applications in areas where irreversibility is fundamental. This paper bridges the gap between this theory and two applied fields: concurrency in computer science and motion planning in robotics.

In concurrency theory, the execution sequences of a parallel program can be modeled as paths in a state space digraph. Independence between actions (leading to potentially different but equivalent interleavings) corresponds to commuting diamonds in the digraph. The fundamental monoid $\mathcal{A}_1(G, x_0)$ then classifies traces (equivalence classes of execution sequences) [1, 2, 4]. In motion planning, especially for autonomous vehicles or robots operating in environments with directional restrictions (e.g., one-way streets, conveyor belts), the space of feasible paths is a directed graph. Directed homotopy provides a notion of deformation among feasible paths, crucial for path optimization and congestion analysis [3].

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## 2. Directed Homotopy in Concurrency Theory

**Definition 2.1** (Concurrent System as a Digraph)**.** A concurrent system with a finite set of actions $Act$ can be modeled by a state space digraph $G$:

- Vertices: Global states of the system.
- Edges: Transitions $s \xrightarrow{a} s'$ for some action $a \in Act$.

Two actions $a$ and $b$ are *independent* if they commute (i.e., executing $a$ then $b$ leads to the same state as $b$ then $a$). This induces commuting diamonds in $G$.

**Proposition 2.2** (Fundamental Monoid as Trace Monoid)**.** *Let $G$ be the state space digraph of a concurrent system. The fundamental monoid $\mathcal{A}_1(G, x_0)$ is isomorphic to the* trace monoid *(free partially commutative monoid) generated by $Act$ with independence relation given by the diamond commutations.*

*Proof.* The isomorphism sends each action $a$ to the homotopy class of the loop that executes $a$ and its inverse (if reversible) or a suitable return path. Diamond commutations correspond exactly to the independence relations in the trace monoid. □

**Example 2.3** (Dining Philosophers)**.** Consider a simple system with two processes competing for two resources. The state space contains diamonds when processes act on independent resources. $\mathcal{A}_1(G, x_0)$ is a non-commutative monoid (free product), reflecting contention. Its universal cover is hyperbolic, indicating exponential state space growth and potential for deadlock.

**Remark 2.4** (Geometric Dichotomy and Scalability)**.** If $\mathcal{A}_1(G, x_0)$ is commutative ($\cong \mathbb{N}^k$), the system exhibits high concurrency; the universal cover is a Euclidean grid, implying polynomial state space growth and good scalability. If $\mathcal{A}_1(G, x_0)$ is a free product, the system has high contention; the universal cover is hyperbolic with exponential growth, indicating poor scalability and potential state explosion.

## 3. Directed Homotopy in Motion Planning

**Definition 3.1** (Directed Configuration Space)**.** For a robot moving in an environment with one-way constraints (e.g., prescribed directions in certain corridors), the *directed configuration space* is a digraph $G$:

- Vertices: Distinct locations or configurations.
- Edges: Feasible directed moves from one configuration to the next.

**Definition 3.2** (Feasible Homotopy Classes)**.** Two directed paths $\alpha$ and $\beta$ in $G$ are *feasibly homotopic* if they are related by directed A-homotopy (identity and diamond moves). The set of equivalence classes is $\mathcal{A}_1(G, x_0)$ for loops, or more generally, the set of homotopy classes of paths between two points.

**Theorem 3.3** (Path Classification for Planning)**.** *Computing the fundamental monoid $\mathcal{A}_1(G, x_0)$ (or the fundamental category) provides a complete classification of all possible directed paths up to feasible homotopy. This classification can be used to:*

1. *Identify distinct homotopy classes between start and goal.*
2. *Select an optimal (e.g., shortest time) path within each class.*
3. *Detect unavoidable congestion points (bottlenecks) corresponding to non-trivial generators of $\mathcal{A}_1(G, x_0)$.*

**Example 3.4** (Warehouse Robot)**.** Consider a warehouse grid with one-way aisles. The configuration space digraph is a directed grid with possible obstacles. Directed homotopy classes correspond to different ways to navigate around obstacles while respecting one-way constraints. Computing $\mathcal{A}_1(G, x_0)$ helps in generating distinct routing plans.

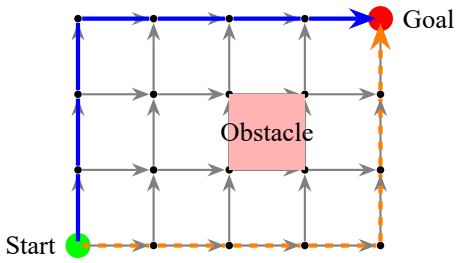


Fig. 1. Motion planning in a directed grid with an obstacle. The blue and orange paths are in distinct directed homotopy classes because they cannot be deformed into each other while respecting one-way edges and avoiding the obstacle.

## 4. Algorithms and Future Directions

- **Algorithm for $\mathcal{A}_1(G, x_0)$**: Use the directed Seifert–van Kampen theorem to decompose the digraph and compute the monoid via amalgamated free products.
- **Persistent Directed Homology**: For time-varying networks (e.g., changing traffic rules), apply directed homology in a persistent framework to detect evolving cycles and bottlenecks.
- **Integration with Control Theory**: Use the fundamental category (a generalization of the fundamental monoid) for optimal control in directed environments.

## 5. Conclusion

Directed A-homotopy theory is not merely an abstract generalization but a practical framework with significant applications in computer science and robotics. By modeling concurrency and motion constraints through directed graphs and analyzing them via fundamental monoids and homology, we gain powerful tools for system verification, performance prediction, and path planning. The cross-fertilization between algebraic topology and these applied fields promises further fruitful developments.

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# Bayesian optimal designs for Gumbel regression model with uniformly decreasing weights

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**Abstract**

This paper investigates Bayesian optimal design strategies for the Gumbel regression model under a framework of uniformly decreasing weights. The Gumbel distribution, widely applied in extreme value analysis and reliability studies, provides a flexible regression structure for modeling skewed data. We develop Bayesian optimal designs for the Gumbel regression model by employing a Dirichlet process prior with the truncated normal distribution as a base measure. To approximate the Dirichlet process, we adopt the zarepour representation, which provides a constructive and computationally tractable framework for generating prior distributions. This work contributes to the literature on Bayesian nonparametric optimal design and provides practical guidelines for implementing efficient experimental designs when prior uncertainty is modeled by a Dirichlet process.

## 1. Introduction

As discussed in our previous work [1, 2, 29], in experimental design, the term ”optimal design” refers to a specific class of designs categorized according to particular statistical criteria. Optimal experimental design plays a central role in statistical inference, providing a systematic framework for selecting experimental settings that maximize the efficiency of parameter estimation. In regression models with complex distributions, such as the Gumbel distribution, the design problem becomes more challenging due to the presence of skewness and heavy tails. The Gumbel regression model, rooted in extreme value theory, is widely applied in reliability engineering, hydrology, survival analysis, and risk management, where modeling of extreme responses is crucial.

Optimal designs are sought using optimality criteria, typically based on the information matrix. In nonlinear models, the presence of unknown parameters introduced complexities in the design problem, as the optimality criteria depends on these unknown parameters [4, 7]. To address this challenge, researchers proposed various solutions, including local optimal designs [3, 9, 13, 20], sequential optimal designs, minimax optimal designs, Bayesian optimal designs [5, 13, 30, 32, 23-27], and pseudo-Bayesian designs [26]. Chernoff [9] proposed to use a suitable guess of the parameter value

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which leads to locally optimal design. This approach aimed to overcome the difficulties associated with the dependence of the design problem on unknown parameters in nonlinear models. It's important to note that local designs for nonlinear models are derived subsequent to an initial linearization of the model, using the parameter set as a reference point.

The Bayesian approach is a statistical framework that combines prior knowledge about model parameters with observed data through Bayes' theorem. In the context of optimal design, this method incorporates prior distributions for the unknown parameters into the design criterion, ensuring that the resulting design is not only efficient for a specific set of parameter values but also robust across a range of plausible values. The choice of the prior distribution is subjective, relying on the researcher's beliefs, and it significantly influences the final outcome. Unfortunately, the Bayesian approach lacks a definitive method for selecting the prior distribution. Subjective specification of a prior distribution which captures all existing information about the uncertainty of the parameter values was employed by many authors such as Chaloner and Larentz [12], Chaloner and Duncan [10], Burghaus and Dette [6], Chaloner and Vardinelli [11], Pronzato and Walter [31], Mukhopadhyay and Haines [28], Dette and Ngobauer [14,15], Fedorov [16,17], and Firth and Hinde [19] that have contributed extensively to this field. Chapter 18 of Atkinson et al.'s book [4] provides further reading on this topic. Often the choice of the prior is a compromise of a representation of uncertainty of the parameter values that may not be correct and can lead to misleading decision. Moreover, using the custom priors is a restrictive choice of priors, and it is possible to distort the actual prior distribution of the parameter. Indeed, no single probability distribution can model ignorance satisfactory, therefore large classes of distributions are needed. In this regards, we show that such a class is attained by considering infinite dimensional families of prior distributions; We consider the prior distribution function as an unknown distribution, which belongs to a class of distribution functions. We apply a nonparametric Bayesian approach that puts a prior on such families. In nonparametric Bayesian analysis, the prior process in terms of a probability measure P instead of the corresponding distribution function is discussed. The Dirichlet process, which is defined by Ferguson [18], plays a central role in nonparametric Bayesian methods. Dirichlet process (DP) involved two desirable properties of a prior; First, DP is defined on an arbitrary probability space. Second, it belonged to a conjugate family of priors. The aim of this paper is to apply Dirichlet process priors to obtain the Bayesian D-optimal design.

This research paper presents the optimal design for nonlinear models in section 2. In Section3, the nonparametric Bayesian D-optimal design for Gumbel regresion model is presented. Finally, Section 4 concludes the paper with some closing remarks.

## 2. Introduction to optimal designs in nonlinear models

In the realm of nonlinear experimental design, a common scenario arises where in the relationship between the response variable $y$ and the independent variable $x$ is given by the equation $y = \eta(x, \boldsymbol{\theta}) + \epsilon$ where $x \in \chi \subseteq \mathrm{R}$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is the unknown parameter vector and $\epsilon$ is a normally distributed residual value with mean 0 and known variance $\sigma^2 > 0$. For simplicity, we assume $\sigma^2 = 1$ in this problem. If $\eta(x, \boldsymbol{\theta})$ is differentiable with respect to $\boldsymbol{\theta}$ then, the information matrix $I(x, \boldsymbol{\theta})$ at a given point $x$ can be represented as follows [4]:

$$I(x, \boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \eta(x, \boldsymbol{\theta}) \frac{\partial}{\partial \boldsymbol{\theta}^T} \eta(x, \boldsymbol{\theta}). \tag{1}$$

There exist several optimality criteria used to obtain the optimal design [4]:

$$\Psi_D(\xi, \boldsymbol{\theta}) = -\log(\det(M(\xi, \boldsymbol{\theta}))) \ , \ \Psi_A(\xi, \boldsymbol{\theta}) = tr(M^{-1}(\xi, \boldsymbol{\theta})),$$

where $\xi$ denotes a design with two components; the first component represents specific values from the design space $\chi$ while the second component corresponds to the weights assigned to these values, so that design $\xi$ can be defined as follows:

$$\boldsymbol{\xi} = \begin{Bmatrix} x_1 & x_2 & \dots & x_\ell \\ w_1 & w_2 & \dots & w_\ell \end{Bmatrix} \in \boldsymbol{\Xi}, \tag{2}$$

where $\boldsymbol{\Xi} = \{\xi \mid 0 \le w_j \le 1; \sum_{j=1}^{\ell} w_j = 1, x \in \boldsymbol{\chi}\}, p \le \ell \le \frac{p(p+1)}{2}$ [27].

For a discrete probability measure $\xi$ with finite support, the information function of $\xi$ can be expressed as follows [4]:

$$M(\xi, \boldsymbol{\theta}) = \sum_{j=1}^{\ell} w_j I(x_j, \boldsymbol{\theta}). \tag{3}$$

In generalized linear and non-linear models, due to the dependence of the information matrix $M(\xi, \boldsymbol{\theta})$ to the unknown parameter $\boldsymbol{\theta}$, one approach to address this issue is to employ the Bayesian method and incorporate a prior distribution of the parameter vector instead of the local method to achieve a better result. The Bayesian D-optimality criterion can be formulated as follows [4]:

$$\Psi_{\Pi}(\xi) = E(\psi(\xi; \boldsymbol{\theta})) = \int_{\Theta} \psi(\xi; \boldsymbol{\theta}) d\Pi(\boldsymbol{\theta}) = \int_{\Theta} -\log(\det(M(\xi, \boldsymbol{\theta}))) d\Pi(\boldsymbol{\theta}), \tag{4}$$

where $\Pi$ represents the prior distribution for $\boldsymbol{\theta}$ and the Bayesian D-optimal design is attained by minimizing (2.4).

## 3. Nonparametric Bayesian D-optimal design for Gumbel regresion model with Respect to Prior Processes

Now in the following section, we will discuss Nonparametric Bayesian optimal designs for Gumbel regression model. Now suppose we have the following regression model:

$$E(y|x) = \eta(x, \theta) = \theta[\exp(-\exp(-\theta x))]\exp(-\theta x);\ x \in \mathbb{R}, \theta > 0. \tag{5}$$

let's consider the regression model (3.1), therefore, the Bayesian D-optimality criterion, denoted as $\Psi_{\Pi}(\xi)$ can be expressed as follows [22]:

$$\begin{aligned}\Psi_{\Pi}(\xi) =& E(\psi(\xi; \boldsymbol{\theta})) \\ &= \int_{\Theta} \psi(\xi; \theta) d\Pi(\boldsymbol{\theta}) = \int_{\Theta} -\log(\sum_{j=1}^{\ell} w_j [\exp(-x_j)((\exp(-\theta \exp(-x_j))) \\ &+\theta(-\exp(-x_j))(\exp(-\theta \exp(-x_j))))]^2) d\Pi(\theta)\end{aligned} \tag{6}$$

where $\Pi$ is the prior distribution for $\theta$. The Bayesian D-optimal design is attained by minimizing equation (3.2).

In the nonparametric Bayesian framework, we consider $P \sim \mathrm{DP}(\alpha, P_0)$ and its collective representation as $P(.) = \sum_{i=1}^{\infty} p_i\, \delta_{\theta_i}(.)$. In this context, the optimality criterion can be expressed as follows:

$$\Psi_{\Pi}(\xi) = \sum_{i=1}^{\infty} p_i(-\log(\sum_{j=1}^{\ell} w_j [\exp(-x_j)((\exp(-\theta_i \exp(-x_j))) + \theta_i(-\exp(-x_j))(\exp(-\theta_i \exp(-x_j))))]^2)). \tag{7}$$

According to Ferguson's definition of the DP, the calculation (3.2) is not easily possible, so to solve this problem in obtaining the optimal nonparametric Bayesian criterion, methods such as the stick breaking process are used to approximate this criterion. Another method has been presented by Zarepour and Ellabadi [35] whose simulation speed and accuracy is much higher than the stick breaking process [33]. We have used this method in this article.

Now, in this section we consider truncated normal distribution as the base measure in DP. We get the results by using a nonlinear optimization programing with R package Rsolnp [21]. To better understanding of the effect of the $\alpha$ parameter, we tabulate the results for four different values of $\alpha$=1, 5, 10, 50, in Tables 1-4. Without loss of generality, we consider a bounded design space $\chi$=[0, 1]. Tables 1-4 represent the results when the concentration parameter and uncertainty in the base measure increase. According to the results, when the value of $\alpha$ increases, the support points in two points design do not significantly change. The weight of maximum point increases rapidly and the largest point will have the most weight that this weight almost increases or remains fixed by increasing the concentration parameter. Also for three points design, maximum support point has the largest weight. In addition, in the range under

Table 1. Nonparametric Bayesian D-optimal designs for gumbel regression model with $\alpha$=1.

| *Parameters* | *Design* | *Two points* | | *Three points* | | |
|---|---|---|---|---|---|---|
| $\mu = 5, \sigma = 2$ | $x$ | 0.178063 | 0.790214 | – | – | – |
| | $w$ | 0.000008 | 0.999992 | – | – | – |
| $\mu = 50, \sigma = 30$ | $x$ | 0.103135 | 0.989394 | 0.099031 | 0.300948 | 1.000000 |
| | $w$ | 0.000001 | 0.999999 | 0.000001 | 0.000001 | 0.999998 |
| $\mu = 150, \sigma = 90$ | $x$ | 0.099999 | 1.00000 | 0.100000 | 0.302812 | 1.00000 |
| | $w$ | 0.000005 | 0.999995 | 0.000003 | 0.000003 | 0.999994 |

Table 2. Nonparametric Bayesian D-optimal designs for gumbel regression model with $\alpha$=5.

| *Parameters* | *Design* | *Two points* | | *Three points* | | |
|---|---|---|---|---|---|---|
| $\mu = 5, \sigma = 2$ | $x$ | 0.170275 | 0.787736 | – | – | – |
| | $w$ | 0.00000001 | 0.99999999 | – | – | – |
| $\mu = 50, \sigma = 30$ | $x$ | 0.102144 | 0.995086 | 0.099031 | 0.300948 | 1.00000 |
| | $w$ | 0.000001 | 0.99999 | 0.000001 | 0.000001 | 0.999998 |
| $\mu = 150, \sigma = 90$ | $x$ | 0.998362 | 1.000000 | 0.100000 | 0.302812 | 1.00000 |
| | $w$ | 0.000005 | 0.999995 | 0.000003 | 0.000003 | 0.999994 |

Table 3. Nonparametric Bayesian D-optimal designs for gumbel regression model with $\alpha$=10.

| *Parameters* | *Design* | *Two points* | | *Three points* | | |
|---|---|---|---|---|---|---|
| $\mu = 5, \sigma = 2$ | $x$ | 0.1647445 | 0.7984882 | – | – | – |
| | $w$ | 0.00000002 | 0.99999998 | – | – | – |
| $\mu = 50, \sigma = 30$ | $x$ | 0.1005972 | 0.9972532 | 0.1000427 | 0.3052599 | 1.000000 |
| | $w$ | 0.0000012 | 0.9999998 | 0.0000008 | 0.0000009 | 0.9999983 |
| $\mu = 150, \sigma = 90$ | $x$ | 0.09999915 | 1.0000000 | 0.100000 | 0.300255 | 1.00000 |
| | $w$ | 0.0000060 | 0.9999940 | 0.000003 | 0.000003 | 0.999994 |

Table 4. Nonparametric Bayesian D-optimal designs for gumbel regression model with $\alpha$=50.

| *Parameters* | *Design* | *Two points* | | *Three points* | | |
|---|---|---|---|---|---|---|
| $\mu = 5, \sigma = 2$ | $x$ | 0.191323 | 0.755625 | – | – | – |
| | $w$ | 0.00000007 | 0.99999993 | – | – | – |
| $\mu = 50, \sigma = 30$ | $x$ | 0.099720 | 1.000000 | 0.100020 | 0.302281 | 1.000000 |
| | $w$ | 0.000001 | 0.999999 | 0.000001 | 0.000001 | 0.999998 |
| $\mu = 150, \sigma = 90$ | $x$ | 0.099950 | 1.00000 | 0.100000 | 0.300117 | 0.301432 |
| | $w$ | 0.000005 | 0.999995 | 0.000003 | 0.000003 | 0.999994 |

investigation, the results show that we don't have a three - point design for $\mu = 5, \sigma = 2$, and in fact, it converts to the design by less points. This observation is more clear for larger concentration parameter. But, by increasing the parameter space, optimal two and three - point designs are obtained.

Now, if we assume the mean of the base distribution to be constant and increase the variance, it can be seen that in the two-point designs, the largest point has the most weight. The results related to this case has been presented in the table 5. As is clear the table, as the variance increases, the weight of some points goes towards zero, and in this case, a design with fewer points is obtained.

Table 5. Nonparametric Bayesian D-optimal designs with constant mean when $\alpha$=1.

| *Parameters* | *Design* | *Two points* | | *Three points* | | |
|---|---|---|---|---|---|---|
| $\mu = 50, \sigma = 30$ | $x$ | 0.102018 | 0.989789 | – | – | – |
| | $w$ | 0.000001 | 0.999999 | – | – | – |
| $\mu = 50, \sigma = 90$ | $x$ | 0.100540 | 0.996792 | 0.100023 | 0.305002 | 1.000000 |
| | $w$ | 0.000003 | 0.999997 | 0.000002 | 0.000002 | 0.999996 |
| $\mu = 50, \sigma = 500$ | $x$ | 0.0999992 | 1.00000 | 0.100003 | 0.302462 | 1.00000 |
| | $w$ | 0.000003 | 0.999997 | 0.000003 | 0.000003 | 0.999994 |

## 4. Concluding Remarks And Future Works

In this study, we investigated the problem of constructing optimal designs for the Gumbel regression model when the prior distribution is represented by a truncated normal distribution as the base measure under a Bayesian nonparametric framework. By employing the zarepour method for the Dirichlet process, we achieved a flexible representation of the prior, which allows capturing uncertainty in a more realistic manner compared to parametric approaches.
In the investigated range, the results reveal interesting findings. For small parameter values, there are no three-point designs observed. However, By increasing uncertainty in the base measure, another optimal point is obtained with a very small weight, resulting in a design where the largest point has the highest weight.
It is important to note that this approach can be applied to other optimality criteria and various models with two or more parameters. For example, nonparametric Bayesian optimal designs using the A- or E-optimality criterion for the nonlinear model discussed in this paper, along with a Dirichlet process prior, hold potential for further research. We hope to report new results in this area in the near future.

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# Generalized Unit Exponential Distribution and its applications

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**Abstract**

The Generalized Unit Exponential (GUE) distribution is a flexible extension of the unit exponential distribution defined on the unit interval $(0, 1)$. It introduces with two parameters: $\theta > 0$ controlling the rate and $\alpha > 0$ controlling the shape that enhance its ability to model data with different levels of skewness and tail behavior. The GUE distribution includes several well-known models as special cases, such as the unit exponential and the exponentiated unit exponential. Due to its flexibility, the GUE distribution has found applications in survival analysis, reliability engineering, actuarial science, Bayesian inference, and environmental studies.

## 1. Introduction

Modeling lifetime data is very important in many applied sciences like engineering, medicine, and finance. Recently, several lifetime distributions have been introduced by [9-11, 3], that have some advantages and disadvantages in modeling lifetime data.

Probability distributions defined on the unit interval $(0, 1)$ play a central role in modeling proportions, rates, and bounded lifetime data [4-7]. Among these, the unit exponential distribution has been widely employed due to its simplicity and tractable mathematical properties [1, 2]. However, its lack of flexibility in capturing diverse data shapes and hazard rate behaviors has limited its usefulness in more complex applications.

To overcome these limitations, generalized families of the unit exponential distribution have been proposed. One of the most notable is the Generalized Unit Exponential (GUE) distribution, which introduces two additional parameters, $\alpha$ and $\theta$, providing greater adaptability for modeling skewed data and varying hazard structures method. This family encompasses several important sub-models as special cases, such as the unit exponential and the exponentiated unit exponential distributions.

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The GUE distribution has been successfully applied in diverse areas such as survival analysis, reliability engineering, Bayesian inference, actuarial science, and environmental statistics. Its flexibility and interpretability make it a promising candidate for analyzing real-world bounded datasets.
Let $Y \sim \text{Exp}(\theta)$ be an exponential random variable with rate parameter $\theta > 0$, i.e.,

$$f_Y(y) = \theta e^{-\theta y}, \quad y \geq 0.$$

Define the transformation:

$$X = \frac{Y}{1+Y}, \quad 0 < X < 1.$$

Then the Cumulative Distribution Function (CDF) for unit exponential as follow as:

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) = \Pr\left(\frac{Y}{1+Y} \leq x\right) \\ &= \Pr\left(Y \leq \frac{x}{1-x}\right) = F_Y\left(\frac{x}{1-x}\right) \\ &= 1 - \exp\Big(-\theta\frac{x}{1-x}\Big). \end{aligned}$$

The probability density function (PDF) of this distribution is obtained by taking the derivative of the Cumulative Distribution Function as follows:

$$f_X(x) = \frac{\theta}{(1-x)^2}\exp\Big(-\theta\frac{x}{1-x}\Big), \quad 0 < x < 1.$$

Now, we present a generalization of the unit exponential distribution.

## 2. Generalized Unit Exponential Distribution

In this section we create a generalized exponential distribution by adding a new parameter to the distribution:

$$F_X^{(\alpha)}(x) = [F_X(x)]^\alpha = \left(1 - \exp\Big(-\theta\frac{x}{1-x}\Big)\right)^\alpha, \quad 0 < x < 1.$$

Differentiating the CDF with respect to $x$ gives:

$$f_X^{(\alpha)}(x) = \frac{d}{dx}F_X^{(\alpha)}(x) = \alpha\left(1 - \exp\Big(-\theta\frac{x}{1-x}\Big)\right)^{\alpha-1} \cdot \frac{\theta}{(1-x)^2}\exp\Big(-\theta\frac{x}{1-x}\Big), \quad 0 < x < 1.$$

The hazard rate function of this distribution is also easily obtained from the following equation:

$$h(x) = \frac{f(x)}{1-F(x)}, \quad 0 < x < 1. \tag{1}$$

Plots of the PDF and Hazard Function of the Generalized Unit Exponential (GUE) distribution for different values of the parameters $\theta$ and $\alpha$ are presented in Figure 1.
As is clear, the probability density function of this distribution is decreasing and unimodal, and the hazard rate function of this distribution is also decreasing with respect to the values of its parameters.
This research paper presents the statistical and mathematical properties of the new distribution in section 3. In Section 4 and 5, respectively, the Maximum Likelihood Estimation (MLE) and a simulation study for generalized unit exponential distribution are presented. Finally, Section 6 and 7 presents applications of the new distribution and results in this area, as well as some ideas for further work.

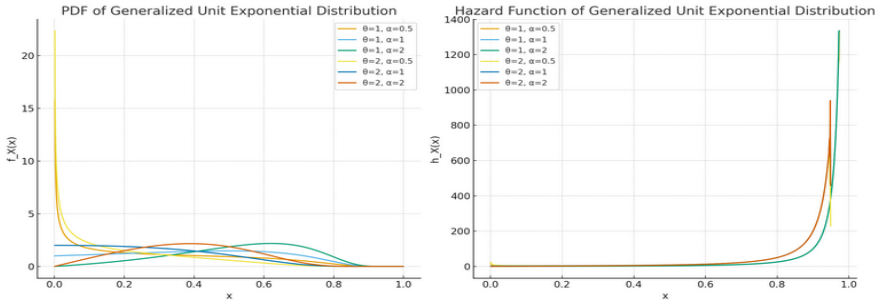


Fig. 1. Plot of (a) pdf and (b) hrf of new distribution for different values of the parameters.

## 3. Statistical and mathematical properties of the new distribution

In this section, we present the statistical and mathematical properties of the new distribution. Suppose that the random variable X has a Generalized Unit Exponential Distribution then the survival function as follow as:

$$S_X(x) = 1 - F_X^{(\alpha)}(x) = 1 - \left(1 - e^{-\theta \frac{x}{1-x}}\right)^{\alpha}$$

*Quantile Function*

and the Quantile Function (Inverse CDF) as follow as:

$$F_X^{(\alpha)}(x) = u \quad \Rightarrow \quad X = \frac{-\ln\left(1 - u^{1/\alpha}\right)}{\theta - \ln\left(1 - u^{1/\alpha}\right)}$$

*Moment representation*

The $r$-th raw moment is

$$m_r = \mathbb{E}[X^r] = \alpha\theta \int_0^1 \frac{x^r}{(1-x)^2} e^{-\theta \frac{x}{1-x}} \left(1 - e^{-\theta \frac{x}{1-x}}\right)^{\alpha-1} dx.$$

Using the change of variable

$$t = \frac{\theta x}{1-x} \quad \Leftrightarrow \quad x = \frac{t}{\theta + t}, \qquad dx = \frac{\theta}{(\theta + t)^2}\, dt,$$

one obtains the simplified integral form

$$\boxed{m_r = \alpha \int_0^{\infty} \frac{t^r}{(\theta + t)^r} e^{-t} \left(1 - e^{-t}\right)^{\alpha-1} dt}.$$

*Skewness and kurtosis*

Based on the first four moments, we have

$$\begin{aligned}
\mu &= m_1, \\
\sigma^2 &= m_2 - m_1^2, \\
\gamma_1 &= \frac{m_3 - 3m_1 m_2 + 2m_1^3}{\sigma^3}, \\
\gamma_2 &= \frac{m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4}{\sigma^4} - 3,
\end{aligned}$$

where $\gamma_1$ is skewness and $\gamma_2$ excess kurtosis.

*Remark.*. In general, closed-form solutions of $m_r$ are not available for arbitrary $\alpha, \theta$. For integer $\alpha - 1$, a binomial expansion $(1-e^{-t})^{\alpha-1} = \sum_{k=0}^{\alpha-1}(-1)^k\binom{\alpha-1}{k}e^{-kt}$ may be applied, leading to expressions involving special functions, but numerical integration is the most practical approach.
The following table summarizes the statistical and mathematical properties of the new distribution.

Table 1. Statistical measures for GUE distribution for various $\alpha$ and $\theta$.

| $\alpha$ | $\theta$ | Mean | Variance | Std. Dev. | Skewness | Kurtosis | CV |
|---|---|---|---|---|---|---|---|
| 0.5 | 1.0 | 0.273400 | 0.053398 | 0.231052 | 0.671291 | 0.266460 | 0.845658 |
| 0.5 | 2.0 | 0.181674 | 0.030103 | 0.173477 | 0.880955 | 0.861031 | 0.955045 |
| 0.5 | 4.0 | 0.111560 | 0.010011 | 0.100055 | 1.047381 | 1.360241 | 0.896515 |
| 0.5 | 8.0 | 0.058998 | 0.003147 | 0.056107 | 1.267326 | 1.930713 | 0.950091 |
| 1.0 | 1.0 | 0.333333 | 0.055556 | 0.235702 | 0.707107 | 0.250000 | 0.707107 |
| 1.0 | 2.0 | 0.200000 | 0.026667 | 0.163299 | 0.912871 | 0.866667 | 0.816496 |
| 1.0 | 4.0 | 0.111111 | 0.012346 | 0.111111 | 1.224745 | 1.800000 | 1.000000 |
| 1.0 | 8.0 | 0.057143 | 0.004082 | 0.063893 | 1.499360 | 2.922872 | 1.118624 |
| 2.0 | 1.0 | 0.375000 | 0.046875 | 0.216506 | 0.707107 | 0.250000 | 0.577350 |
| 2.0 | 2.0 | 0.250000 | 0.020833 | 0.144338 | 0.912871 | 0.866667 | 0.577350 |
| 2.0 | 4.0 | 0.166667 | 0.013889 | 0.117851 | 1.224745 | 1.800000 | 0.707107 |
| 2.0 | 8.0 | 0.090909 | 0.006173 | 0.078576 | 1.499360 | 2.922872 | 0.864935 |
| 3.0 | 1.0 | 0.406250 | 0.042969 | 0.207278 | 0.671291 | 0.266460 | 0.510166 |
| 3.0 | 2.0 | 0.312500 | 0.024531 | 0.156611 | 0.880955 | 0.861031 | 0.500355 |
| 3.0 | 4.0 | 0.290650 | 0.010893 | 0.104361 | 1.047381 | 1.360241 | 0.359035 |
| 3.0 | 8.0 | 0.176246 | 0.008455 | 0.091927 | 1.267326 | 1.930713 | 0.521233 |

## 4. Maximum Likelihood Estimation (MLE)

Let $x_1, \cdots, x_n$ be an observed random sample of size $n$ from the new distribution. Then, the log-likelihood function for the parameters $\alpha$ and $\theta$ is:

$$\ell(\alpha, \theta) = \sum_{i=1}^{n}\left[\log\alpha + \log\theta - 2\log(1-x_i) - \theta\frac{x_i}{1-x_i} + (\alpha-1)\log\left(1 - e^{-\theta\frac{x_i}{1-x_i}}\right)\right] \tag{2}$$

The MLE estimates $(\hat{\alpha}, \hat{\theta})$ can be obtained numerically by solving:

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=1}^{n}\left[\frac{1}{\alpha} + \log\left(1 - e^{-\theta\frac{x_i}{1-x_i}}\right)\right] = 0, \tag{3}$$

$$\frac{\partial\ell}{\partial\theta} = \sum_{i=1}^{n}\left[\frac{1}{\theta} - \frac{x_i}{1-x_i} + (\alpha-1)\frac{\frac{x_i}{1-x_i}e^{-\theta\frac{x_i}{1-x_i}}}{1 - e^{-\theta\frac{x_i}{1-x_i}}}\right] = 0. \tag{4}$$

We may use numerical methods to find the solutions of the above equations.

## 5. A simulation study

In this section, we evaluate the performance of the ML estimators of the parameters of the new model by means of a simulation study. The inverse transform algorithm is used to generate random data from the new distribution. The precision of the ML estimators is discussed by means of the bias, mean squared error (MSE). We obtained the ML estimates of the parameters for each generated sample and the standard errors of the ML estimators were obtained by

inverting the observed information matrix. Let $\hat{\alpha}$ be the ML estimator of $\alpha$ and $\hat{\alpha}_i$ be the ML estimator of $\alpha$ that is obtained in the $i$-th iteration, then the estimated bias, MSE of $\hat{\alpha}$ can be obtained using the following equations

$$\widehat{Bias}_{\alpha}(N) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha), \quad \widehat{MSE}_{\alpha}(N) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2,$$

respectively. We can obtain the estimated bias, MSE of $\hat{\theta}$ (the ML estimator of $\theta$) similarly.The simulation results reported in Table 2 provide the following insights: The MSE values generally decrease as the sample size $n$ increases, which confirms the consistency and asymptotic efficiency of the MLEs.
Overall, the simulation study confirms that the proposed maximum likelihood estimators behave well in moderate to large samples, but small $\alpha$ and large $\theta$ combinations may lead to instability in estimating $\theta$.

Table 2. Simulation study (n=50, reps=300): MLE performance across $\alpha$ and $\theta$.

| $\alpha$ | $\theta$ | Conv.% | Mean $\hat{\alpha}$ | Bias$_\alpha$ | RMSE$_\alpha$ | SD$_\alpha$ | Mean $\hat{\theta}$ | Bias$_\theta$ | RMSE$_\theta$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 1.0 | 96.7 | 0.6768 | 0.1768 | 0.4129 | 0.3716 | 1.2363 | 0.2363 | 0.7416 |
| 0.5 | 2.0 | 97.0 | 0.7868 | 0.2868 | 0.5051 | 0.4153 | 2.2526 | 0.2526 | 0.8668 |
| 0.5 | 4.0 | 97.7 | 0.8732 | 0.3732 | 0.5763 | 0.4109 | 4.3664 | 0.3664 | 1.0894 |
| 0.5 | 8.0 | 95.0 | 1.0478 | 0.5478 | 0.8713 | 0.6911 | 8.6744 | 0.6744 | 1.7203 |
| 1.0 | 1.0 | 97.0 | 1.1619 | 0.1619 | 0.3675 | 0.3194 | 1.3828 | 0.3828 | 0.7689 |
| 1.0 | 2.0 | 96.7 | 1.2270 | 0.2270 | 0.4440 | 0.3734 | 2.4071 | 0.4071 | 0.9996 |
| 1.0 | 4.0 | 97.7 | 1.2901 | 0.2901 | 0.4867 | 0.3780 | 4.4751 | 0.4751 | 1.1323 |
| 1.0 | 8.0 | 96.0 | 1.5332 | 0.5332 | 0.8630 | 0.7112 | 8.8686 | 0.8686 | 1.9350 |
| 2.0 | 1.0 | 98.3 | 2.2671 | 0.2671 | 0.5091 | 0.4369 | 1.4358 | 0.4358 | 0.9432 |
| 2.0 | 2.0 | 98.0 | 2.3968 | 0.3968 | 0.5917 | 0.4115 | 2.4570 | 0.4570 | 1.0799 |
| 2.0 | 4.0 | 98.7 | 2.5865 | 0.5865 | 0.7188 | 0.4313 | 4.2309 | 0.2309 | 0.7956 |
| 2.0 | 8.0 | 97.7 | 3.0436 | 1.0436 | 1.2740 | 0.9486 | 8.3547 | 0.3547 | 1.1065 |

## 6. Application of new distribution

In this section, we present a real data applications in order to showthe flexibility of the generalized unit exponential distribution. We compare the fit of the new distribution with those of some other lifetime distributions which are Exponential, Gamma and Weibull distributions. The data set focuses on the monthly water capacity data sourced from the Shasta reservoir in California, USA, spanning from February 1991 to 2010, as retrieved from http://cdec.water.ca.gov/reservoirmap.html, in line with Mukhopadhyay (1995) [8].

Table 3. Fitted models, information criteria, and Kolmogorov–Smirnov test on the real dataset ($n = 20$).

| Model | Parameters | LogLik | AIC | BIC | KS stat | $p$-value |
|---|---|---|---|---|---|---|
| GUE | $\hat{\alpha} = 2.9725,\ \hat{\theta} = 0.5329$ | 15.6025 | -27.2051 | -25.2136 | 0.187 | 0.434 |
| Exponential | $\hat{\lambda} = 1.3864$ | -13.4657 | 28.9314 | 29.9271 | 0.419 | 0.001 |
| Gamma | $\hat{k} = 17.7949,\ \widehat{scale} = 0.0405$ | 7.3246 | -10.6492 | -8.6578 | 0.291 | 0.054 |
| Weibull | $\hat{k} = 7.2987,\ \widehat{scale} = 0.7748$ | 12.2673 | -20.5347 | -18.5432 | 0.222 | 0.239 |

We use the Kolmogorov-Smirnov (K-S) test statistic, the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) in order to compare the distributions. The computed ML estimates, K-S test statistics and the values of AIC and BIC for data sets is given in Table 3. It is well known that the smaller values of AIC, BIC, and K-S test statistic mean a better fit to data set. Here, it is observed from Table 3 that the new model outperforms all the other considered models in the sense of the considered criteria.

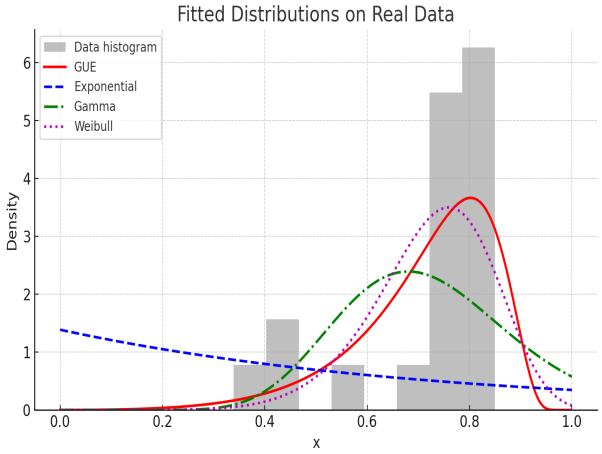


Fig. 2. Plot of (a) pdf and (b) hrf of new distribution for different values of the parameters.

## 7. Concluding remarks and future works

In this paper, we introduced a new distribution and worked on some of its mathematical properties. The new distribution has only two parameters and its cdf and hrf have simple forms. The hrf of the new model can be increasing depending on the values of the parameters. To sum up, we can claim that the proposed distribution provides a quite flexible model for fitting many positive data sets that may appear in the areas like engineering, survival analysis, hydrology, economics, and so on, in comparison with some other distributions. We note that the figure plotting and computations of this paper have been performed using R [12].

There exist many other properties of the new distribution, like the reliability parameter, Kullback-Leibler divergence and so on, that have not been discussed in this paper. Moreover, some interesting inferential topics related to the new distribution are the Bayesian estimation of the parameters, prediction of future observations and estimation based on censored samples. We will probably report our findings regarding the mentioned subjects in the future.

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# Prediction Based onType-II Censored Coherent System Lifetime Data under the Rayleigh distribution with the scale paramete

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**Abstract**

In this paper, we consider a $k$-component coherent system while the system lifetimes are observed, the system structure is known and the component lifetime distribution is the Rayleigh distribution with the scale parameter. Different point and interval predictors based on classical and Bayesian approaches are derived. A numerical example is presented to illustrate the prediction methods used in this paper.

## 1. Introduction

In a Type-II censored experiment, $n$ independent $k$-components systems with the same system structure are placed on a life-testing experiment and the experiment is terminated when the $m$-th (where $m \leq n$ is pre-fixed) system fails. In other words, only the first $m$ failures out of the $n$ systems in the life-test will be observed. The ordered system data obtained from such a life-test, denote as $T_{1:n} < T_{2:n} < \ldots < T_{m:n}$, is referred to as a Type-II censored system lifetime data. The likelihood function of $\sigma$ based on the Type-II censored system lifetime data is given by

$$L(\sigma \mid \mathbf{t}) = \frac{n!}{(n-m)!} \prod_{j=1}^{m} f(t_j;\sigma)\,[1 - F(t_m;\sigma)]^{n-m}, \tag{1}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ is the vector of observations. [7] showed that the probability density function and survival function of the system lifetime $T$ can be written as $f_T(t) = \sum_{i=1}^{k} p_i \binom{k}{i} i f_X(t)[F_X(t)]^{i-1}[\overline{F}_X(t)]^{k-i}$, and $\overline{F}_T(t) = \sum_{i=1}^{k} p_i \sum_{j=0}^{i-1} \binom{k}{j} [F_X(t)]^j [\overline{F}_X(t)]^{k-j}$, respectively, where $p_i$ is the $i$-th component in the system signature vector for the $k$-component system such that $p_i = \Pr(T = X_{i:k})$, and $\sum_{i=1}^{k} p_i = 1$. [4] showed that the survival function of a coherent system can be expressed as

$$\overline{F}_T(t) = \sum_{i=1}^{k} a_i \overline{F}_{1:i}(t) = \sum_{i=1}^{k} a_i [\overline{F}_X(t)]^i. \tag{2}$$

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where $\overline{F}_{1:i}(\cdot)$ is the survival function of the series system lifetime with $i$ components. The vector $\mathbf{a} = (a_1, a_2, \dots, a_k)$ is called the minimal signature of the system and $a_1, a_2, \dots, a_k$ are some negative and nonnegative integers that do not depend on $F_X$, and satisfy $\sum_{i=1}^{k} a_i = 1$. In this paper, it is supposed that component lifetime distribution in a coherent system is the Rayleigh distribution with the scale parameter $\sigma$. The probability density function and survival function of Rayleigh distribution are given by

$$f_X(x) = \frac{x}{\sigma^2}\exp(\frac{-x^2}{2\sigma^2}),\ x > 0,\ \sigma > 0 \quad \text{and} \quad \overline{F}_X(x) = \exp(\frac{-x^2}{2\sigma^2}),\ x > 0,\ \sigma > 0 \tag{3}$$

where $\sigma > 0$ is the scale parameter. Based on the Eqs. (2) and (3), the probability density function and survival function of the coherent system lifetime are given by

$$f_T(t,\sigma) = \frac{t}{\sigma^2}\sum_{i=1}^{k} i\, a_i e^{\frac{-it^2}{2\sigma^2}}\ t > 0,\ \sigma > 0, \quad \text{and} \quad \overline{F}_T(t,\sigma) = \sum_{i=1}^{k} a_i e^{\frac{-it^2}{2\sigma^2}}\ t > 0,\ \sigma > 0. \tag{4}$$

Based on the observed Type-II censored coherent system lifetime data, we derive the maximum likelihood predictor, the best unbiased predictor, the conditional median predictor, and the Bayesian point predictor for future system failures $T' = T_{s+m:n}(s = 1, 2, \dots, n-m)$. Furthermore, we present the prediction intervals (PIs) for future failures $T' = T_{s+m:n}(s = 1, 2, \dots, n-m)$. In recent years, many authors studied the statistical inference and prediction of the component lifetime distribution based on system lifetime data when the system signature is known; see, for example, [1, 2, 5].

## 2. Point Predictors

The conditional probability density function of $T'$ given $T_{m:n} = t_m$ for $t' \geq t_m$ is:

$$f(t' \mid t_m;\sigma) = s\binom{n-m}{s} f(t';\sigma)\,[F(t';\sigma) - F(t_m;\sigma)]^{s-1}\,[1 - F(t';\sigma)]^{n-m-s}\,[1 - F(t_m;\sigma)]^{-(n-m)} \tag{5}$$

### *2.1. Maximum likelihood predictor*

The predictive likelihood function (PLF) of $T'$ and $\sigma$, can be written as

$$L(T', \sigma \mid \mathbf{t}) = f(t' \mid \mathbf{t};\sigma)L(\sigma \mid \mathbf{t}) = f(t' \mid t_m;\sigma)L(\sigma \mid \mathbf{t}). \tag{6}$$

Generally, if $\widehat{T'} = u(\mathbf{T})$ and $\widehat{\theta} = v(\mathbf{T})$ are statistics for which, $L(u(\mathbf{t}), v(\mathbf{t}) \mid \mathbf{t}) = \sup_{(t',\sigma)} L(t', \sigma \mid \mathbf{t})$, then, $u(\mathbf{T})$ is the maximum likelihood predictor (MLP) of $T'$ and $v(\mathbf{T})$ is the predictive maximum likelihood estimator (PMLE) of $\sigma$, respectively. By maximizing the logarithm predictive likelihood function, $\ln L(T', \sigma \mid \mathbf{t})$, the predictive maximum likelihood estimator (PMLE) is obtained for the parameter $\sigma$ and the maximum likelihood estimator (MLP) for $T'$.

### *2.2. Best Unbiased Predictor*

The BUP of $T'$ is the mean of conditional distribution of $T'$ given $\mathbf{T} = \mathbf{t}$.

$$\widehat{T'}_{BUP} = E_\sigma(T' \mid \mathbf{t}) = E_\sigma(T' \mid t_m) = \int_{t_m}^{\infty} t' f(t' \mid t_m;\sigma)\, dt'.$$

if the parameter $\sigma$ is unknown, the best unbiased predictor of $T'$ can be approximated by replacing $\sigma$ with its corresponding MLE.

*2.3. Conditional median predictor*

A predictor $\widehat{T'}$ is called the CMP of $T'$, if $\widehat{T'}$ is the median of the conditional distribution of $T'$ given $T_{m:n} = t_m$ observation, i.e., $\Pr_\sigma(T' \le \widehat{T'} \mid T_{m:n} = t_m) = \Pr_\sigma(T' \ge \widehat{T'} \mid T_{m:n} = t_m)$. Now observe that the distribution of random variable, $Z = 1 - \frac{\sum_{i=1}^{k} a_i e^{\frac{-it'^2}{2\sigma^2}}}{\sum_{i=1}^{k} a_i e^{\frac{-it_m^2}{2\sigma^2}}}$, given $T_{m:n} = t_m$ is a beta distribution with parameters $s$ and $n - m - s + 1$.

Therefore, the CMP of $T'$ can be obtained by solving the equation, $1 - \frac{\sum_{i=1}^{k} a_i e^{\frac{-i\widehat{T'}^2}{2\sigma^2}}}{\sum_{i=1}^{k} a_i e^{\frac{-it_m^2}{2\sigma^2}}} = Med(B)$, where $Med(B)$ is stands for median of beta distribution. When $\sigma$ is unknown, we can substitute $\sigma$ with its maximum likelihood estimate and obtain an approximation of the CMP of $T'$.

*2.4. Bayesian predictor*

In the Bayesian approach, $\sigma$ is considered as a random variable with a prior distribution $\pi(\sigma)$. Here, we take the natural conjugate prior,

$$\pi(\sigma) = \frac{\alpha^\beta}{\Gamma(\beta)2^{\beta-1}}\sigma^{-2\beta-1}\exp(\frac{-\alpha}{2\sigma^2}), \quad \beta, \alpha > 0, \tag{7}$$

where $\beta$ and $\alpha$ are positive hyperparameters [3].The squared error loss (SEL) function for the parameter $\sigma$ can be expressed as, $L(\hat{\sigma}, \sigma) = (\hat{\sigma} - \sigma)^2$, where $\hat{\sigma}$ is an estimator of $\sigma$. By combining (1) and (7), we obtain the posterior of $\sigma$ as

$$\pi(\sigma \mid \mathbf{t}) = K(\mathbf{t})\sigma^{-2(m+\beta)-1}e^{\frac{-\alpha}{2\sigma^2}}\prod_{j=1}^{m}\left\{\sum_{i=1}^{k} i\, a_i e^{\frac{-it_j^2}{2\sigma^2}}\right\}\left[\sum_{i=1}^{k} a_i e^{\frac{-it_m^2}{2\sigma^2}}\right]^{n-m}, \tag{8}$$

where

$$K(\mathbf{t}) = \left[\int_0^\infty \sigma^{-2(m+\beta)-1}e^{\frac{-\alpha}{2\sigma^2}}\prod_{j=1}^{m}\left\{\sum_{i=1}^{k} i\, a_i e^{\frac{-it_j^2}{2\sigma^2}}\right\}\left[\sum_{i=1}^{k} a_i e^{\frac{-it_m^2}{2\sigma^2}}\right]^{n-m} d\sigma\right]^{-1}$$

The posterior predictive density for $t' > t_m$ is given by

$$f^*(t' \mid t_m) = \int_0^\infty f(t' \mid t_m; \sigma)\pi(\sigma \mid \mathbf{t})d\sigma. \tag{9}$$

Therefore, the Bayesian point predictor of $T' = T_{s+m:n}$ $(s = 1, 2, \dots, n-m)$, under the squared error loss, can be obtained as

$$T'_{Bayes} = \int_{t_m}^\infty t'\, f^*(t' \mid t_m)dt'. \tag{10}$$

Due to the complicated form of $f^*(t' \mid t_m)$, the Bayesian point predictor in Eq. (10) cannot be computed explicitly. Consequently, we use the Markov Chain Monte Carlo (MCMC) samples to generate samples from the predictive distributions. The Metropolis-Hastings algorithm steps are given below:

1. Set the initial value $\sigma^{(0)}$; Set $q = 1$;
2. Using the Metropolis-Hastings algorithm, generate $\sigma$ from $\pi(\sigma^{(q-1)} \mid \mathbf{t})$ with the normal distribution with mean $\sigma^{(q-1)}$ and variance $S_\sigma^2$ (denoted as $N(\sigma^{(q-1)}, S_\sigma^2)$) as proposal distribution, where $S_\sigma^2$ is inverse of the Fisher information;
3. Set $q = q + 1$;
4. Repeat Steps 2 and 3, $N$ times to compute the MCMC samples $\sigma_1, \sigma_2, \dots, \sigma_N$.

Based on MCMC samples $\{\sigma_q : q = 1, 2, \dots, N - M\}$, the simulation consistent estimator of $f^*(t' \mid t_m)$ and $\widehat{T'}_{Bayes}$ can be obtained as, (M is the burn-in period)

$$\widehat{f}^*(t' \mid t_m) = \frac{1}{N-M}\sum_{q=1}^{N-M} f(t' \mid t_m; \sigma_q), \quad \widehat{T'}_{Bayes} = \frac{1}{N-M}\sum_{q=1}^{N-M}\int_{t_m}^\infty t'\, f(t' \mid t_m; \sigma_q)dt' \tag{11}$$

## 3. Prediction interval

### 3.1. Pivotal method

We have already discussed that that the distribution of random variable $Z$, given $T_{m:n} = t_m$ is a beta distribution with parameters $s$ and $n - m - s + 1$. Therefore, we can consider $Z$ as a pivotal quantity to obtain the prediction interval for $T'$. Taking $1 - \gamma$ $(0 < \gamma < 1)$ as prediction coefficient, we have $\Pr(B_{\frac{\gamma}{2}} < Z < B_{1-\frac{\gamma}{2}} \mid T_{m:n} = t_m) = 1 - \gamma$, where $B_\gamma$ is the $100\gamma$-th upper percentile of $Beta(s, n - m - s + 1)$ distribution.

### 3.2. Highest conditional density method

The distribution of $Z$ given $T_{m:n} = t_m$ is a $Beta(s, n - m - s + 1)$ distribution with the probability density function, $f(z \mid t_m) = \frac{(n-m)!}{(s-1)!(n-m-s)!} z^{s-1}(1-z)^{n-m-s}$, $(0 < z < 1)$, which is unimodal function of $z$ for $(s = 1, 2, \dots, n-m)$. The $100(1 - \gamma)\%$ HCD prediction interval for $T'$ is $(L_1(\mathbf{T}), U_1(\mathbf{T}))$, where $L_1(\mathbf{T})$ and $U_1(\mathbf{T})$ are the solutions of

$$\left(\sum_{i=1}^{k} a_i e^{\frac{-it_m^2}{2\sigma^2}}\right)(1 - w_1) - \left(\sum_{i=1}^{k} a_i e^{\frac{-iL_1(\mathbf{T})^2}{2\sigma^2}}\right) = 0, \left(\sum_{i=1}^{k} a_i e^{\frac{-it_m^2}{2\sigma^2}}\right)(1 - w_2) - \left(\sum_{i=1}^{k} a_i e^{\frac{-iU_1(\mathbf{T})^2}{2\sigma^2}}\right) = 0.$$

where $w_1$ and $w_1$ are the simultaneous solution of the following equations: (see, [6])

$$B_{w_2}(s, n - m - s + 1) - B_{w_1}(s, n - m - s + 1) = 1 - \gamma, \quad \left(\frac{1 - w_2}{1 - w_1}\right)^{n-m-s} = \left(\frac{w_1}{w_2}\right)^{s-1}$$

Notice that $B_t(a, b)$, $B_t(a, b) = \frac{1}{B(a,b)} \int_0^t x^{a-1}(1 - x)^{b-1}\, dx$ is the incomplete beta function. When parameter $\sigma$ is unknown, the prediction interval $(L_1(\mathbf{T}), U_1(\mathbf{T}))$ can be approximated by replacing $\sigma$ with its corresponding MLE. It should be mentioned here that for the case that $s = 1$ or $s = n - m$, the function $f(z \mid t_m)$ is not unimodal and the HCD prediction interval cannot be obtained in these cases.

### 3.3. Bayesian prediction interval

A $100(1 - \gamma)\%$ Bayesian prediction interval for the $(s + m)$-th censored system lifetime data $T' = T_{s+m:n}(s = 1, 2, \dots, n - m)$ is given by $(L(t_m), U(t_m))$, where $L(t_m)$ and $U(t_m)$ can be obtained by solving the following two nonlinear equations simultaneously

$$\Pr(T' > L(t_m) \mid t_m) = \int_{L(t_m)}^{\infty} f^*(t' \mid t_m; \sigma) dt' = \frac{1}{N - M} \sum_{q=1}^{N-M} \int_{L(t_m)}^{\infty} f(t' \mid t_m; \sigma_q) dt' = 1 - \frac{\gamma}{2},$$

$$\Pr(T' > U(t_m) \mid t_m) = \int_{U(t_m)}^{\infty} f^*(t' \mid t_m; \sigma) dt' == \frac{1}{N - M} \sum_{q=1}^{N-M} \int_{L(t_m)}^{\infty} f(t' \mid t_m; \sigma_q) dt' = \frac{\gamma}{2}.$$

## 4. Numerical Example

Algorithm to generate system lifetimes steps are given below, [1]:

1. Generate $u, u_1, u_2, \dots, u_k$ independently from uniform distribution in $[0, 1]$;
2. Set Generate $x_i$, $i = 1, 2, \dots, k$ of the Rayleigh distribution with the scale parameter;
3. Sort $x_1, x_2, \dots, x_k$ in ascending order to obtain $x_{1:k} < x_{2:k} < \dots < x_{k:k}$;
4. Take $t = x_{j:k}$ if $\sum_{i=1}^{j-1} p_i < u < \sum_{i=1}^{j} p_i, (j = 1, 2, \dots, k)$,.
5. Repeat Steps 1–4, $n$ times to obtain the system lifetimes $\mathbf{t} = (t_1, t_2, \dots, t_n)$.

To obtain a Type-II censored sample based on the simulated system lifetimes $\mathbf{t} = (t_1, t_2, \dots, t_n)$, we sort $(t_1, t_2, \dots, t_n)$ in ascending order to obtain $t_{1:n} < t_{2:n} < \dots < t_{n:n}$ and take the first $m$ order statistics $t_{1:n} < t_{2:n} < \dots < t_{m:n}$ as the Type-II censored sample.

**Example 4.1.** Using the algorithm to generate system lifetimes, we generate a sample of $n = 30$ from a 4- component system with system signature $p = (1/4, 1/4, 1/2, 0)$ and the corresponding minimal signature and structure function are $a = (0, 3, -3, 1)$ and $\min\{X_1, \max\{X_2, X_3, X_4\}\}$, respectively. We assume that the component lifetimes follow the Rayleigh distribution with the scale parameter $\sigma = 1$. The simulated system lifetimes are given in Table 1. We consider

Table 1. Simulated 5-component system lifetimes with system signature $\mathbf{p} = (1/4, 1/4, 1/2, 0)$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.039 | 0.275 | 0.373 | 0.472 | 0.569 | 0.599 | 0.676 | 0.746 | 0.796 | 0.801 |
| 0.805 | 0.885 | 0.887 | 0.908 | 0.995 | 1.123 | 1.268 | 1.275 | 1.303 | 1.329 |
| 1.388 | 1.389 | 1.414 | 1.444 | 1.481 | 1.628 | 1.697 | 1.711 | 1.776 | 1.961 |

the case when we observe the first 20 observations and the rest are censored, i.e., we observe a Type-II censored sample with $n = 30$ and $m = 25$. With this Type-II censored sample, we compute the point and interval prediction for future system failures $T' = T_{s+m:n}$ $(s = 1, 2, \dots, n - m)$ as described in Sections 2 and 3. For the Bayesian prediction, we use the Metropolis-Hastings algorithm to compute the Bayesian point prediction of $T' = T_{s+20:30} (s = 1, 2, \dots, 10)$. In the Metropolis-Hastings algorithm, the MLE $\hat{\sigma}$ is considered as the initial value of the chain and the inverse of Fisher information is used as the variance of the proposal distribution, $S_\sigma^2$. We have generated $N = 50000$ values by Metropolis-Hastings algorithm with $S_\sigma^2 = 0.0098$ and the acceptance rate is about 69%. We discarded the initial $M = 5000$ burn-in samples and computed the Bayes estimates based on averaging the remaining $N - M = 45000$ values. For the Bayesian prediction, we consider the case that the prior of $\sigma$ is almost improper, i.e, $\alpha = \beta = 0.0001$. The results are presented in Table 2.

Table 2. Point predictors and 95% prediction intervals for $T'$ based on the data set in example.

| | Exact | | Point predictors | | | | Prediction intervals | | |
|---|---|---|---|---|---|---|---|---|---|
| $s$ | value | PMLE | MLP | BUP | CMP | Bayes | Pivotal | HCD | HPD |
| 1 | 1.388 | 0.841 | 1.329 | 1.376 | 1.362 | 1.379 | (1.330,1.494) | – | (1.330,1.514) |
| 2 | 1.389 | 0.842 | 1.363 | 1.425 | 1.412 | 1.431 | (1.342,1.580) | (1.333,1.548) | (1.341,1.613) |
| 3 | 1.414 | 0.843 | 1.400 | 1.477 | 1.465 | 1.487 | (1.362,1.660) | (1.352,1.635) | (1.361,1.707) |
| 4 | 1.444 | 0.843 | 1.440 | 1.533 | 1.522 | 1.546 | (1.390,1.743) | (1.383,1.725) | (1.388,1.805) |
| 5 | 1.481 | 0.844 | 1.484 | 1.596 | 1.584 | 1.611 | (1.425,1.831) | (1.422,1.824) | (1.421,1.911) |
| 6 | 1.628 | 0.845 | 1.535 | 1.666 | 1.654 | 1.684 | (1.467,1.930) | (1.470,1.938) | (1.459,2.023) |
| 7 | 1.697 | 0.846 | 1.594 | 1.748 | 1.736 | 1.772 | (1.517,2.046) | (1.529,2.080) | (1.505,2.151) |
| 8 | 1.711 | 0.847 | 1.666 | 1.849 | 1.835 | 1.876 | (1.579,2.196) | (1.602,2.280) | (1.562,2.325) |
| 9 | 1.776 | 0.848 | 1.763 | 1.987 | 1.969 | 2.017 | (1.659,2.415) | (1.698,2.664) | (1.639,2.559) |
| 10 | 1.961 | 0.849 | 1.918 | 2.227 | 2.195 | 2.265 | (1.781,2.858) | – | (1.755,3.028) |

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# Statistical inference from coherent system lifetime data under the two-parameter Weibull distribution

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Abstract

In this paper, we discuss the statistical inference of the lifetime distribution of components in a $n$-component coherent system when the system structure is known and the component lifetime follows the two-parameter Weibull distribution. Different estimation methods, the maximum likelihood estimator and Bayes estimator for the component lifetime parameters are discussed. As it seems that the integrals of the Bayes estimates do not possess closed forms, the Metropolis-Hastings method is applied to approximating these integrals. confidence intervals based on the asymptotic distribution of the MLE and the HPD credible interval are computed. Finally, A numerical example is used to illustrate the methodologies developed in this paper

## 1. Introduction

Let $T$ be the lifetime of a coherent system with independent and identically distributed (i.i.d.) component lifetimes $X_1, X_2, \dots, X_n$ with common absolutely continuous cumulative distribution function $F_X(\cdot)$, probability density function $f_X(\cdot)$, and survival function $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$. Suppose $m$ independent $n$-component systems with the same structure are placed on a life-test with the corresponding lifetimes $T_1, T_2, \dots, T_m$ being identically distributed as $T$ with cumulative distribution function $F_T(\cdot)$, probability density function $f_T(\cdot)$ and survival function $\bar{F}_T(\cdot) = 1 - F_T(\cdot)$. [9] showed that the probability density function and survival function of the system lifetime $T$ can be written as $f_T(t) = \sum_{i=1}^{n} p_i \binom{n}{i} i f_X(t)[F_X(t)]^{i-1}[\overline{F}_X(t)]^{n-i}$, and $\overline{F}_T(t) = \sum_{i=1}^{n} p_i \sum_{j=0}^{i-1} \binom{n}{j} [F_X(t)]^j [\overline{F}_X(t)]^{n-j}$, respectively, where $p_i$ is the $i$-th component in the system signature vector for the $n$-component system such that $p_i = \Pr(T = X_{i:n})$, and $\sum_{i=1}^{n} p_i = 1$. [4] showed that the survival function of a coherent system can be expressed as

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t) = \sum_{i=1}^{n} a_i [\overline{F}_X(t)]^i. \tag{1}$$

where $\overline{F}_{1:i}(\cdot)$ is the survival function of the series system lifetime with $i$ components. The vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ is called the minimal signature of the system and $a_1, a_2, \dots, a_n$ are some negative and nonnegative integers that do not

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depend on $F_X$, and satisfy $\sum_{i=1}^{n} a_i = 1$. In this paper, it is supposed that component lifetime distribution in a coherent system is the two-parameter Weibull distribution. The probability density function and survival function of Weibull distribution are given by

$$f_X(x;\theta,\lambda) = \theta\lambda^{\theta}x^{\theta-1}\exp[-(\lambda x)^{\theta}], \quad \text{and} \quad \overline{F}_X(x;\theta,\lambda) = \exp[-(\lambda x)^{\theta}], \tag{2}$$

for $x > 0$, $\theta > 0$ and $\lambda > 0$, where $\theta$ and $\lambda$ are the shape and scale parameters. Based on the Eqs. (1) and (2), the probability density function and survival function of the coherent system lifetime are given by

$$\bar{F}_T(t;\theta,\lambda) = \sum_{i=1}^{n} a_i e^{-i(\lambda t)^{\theta}}, \quad and \quad f_T(t;\theta,\lambda) = \theta\lambda^{\theta}t^{\theta-1}\sum_{i=1}^{n} i\, a_i e^{-i(\lambda t)^{\theta}} \;\; t > 0, \;\; \theta,\lambda > 0. \tag{3}$$

In recent years, many authors studied the statistical inference of the component lifetime distribution based on system lifetime data when the system signature is known; see, for example, [3, 5, 6].

## 2. Estimation

### *2.1. Maximum likelihood estimation*

Suppose $m$ independent $n$-component systems with the same distribution as $T$ are placed on a life-test and that the corresponding lifetimes $T_1, T_2, \ldots, T_m$ are observed. The likelihood function of $\theta$ and $\lambda$ based on a complete sample is given by

$$L(t;\theta,\lambda) = \prod_{k=1}^{m} f_T(t_k;\theta,\lambda) = \theta^m\lambda^{\theta m}\prod_{k=1}^{m} t_k^{\theta-1}\prod_{k=1}^{m}\left(\sum_{i=1}^{n} i\, a_i e^{-i(\lambda t_k)^{\theta}}\right), \tag{4}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_m)$ is the vector of observations. The maximum likelihood estimates of the parameters $\theta$, $\widehat{\theta}_{MLE}$ and $\lambda$, $\hat{\lambda}_{MLE}$ are obtained by solving the following two non-linear equations simultaneously.

$$\frac{d\ln L(t;\theta,\lambda)}{d\theta} = \frac{m}{\theta} + m\ln\lambda + \sum_{k=1}^{m}\ln t_k - \lambda^{\theta}\sum_{k=1}^{m} t_k^{\theta}[\ln\lambda + \ln t_k]\left(\frac{\sum_{i=1}^{n} i^2 a_i e^{-i(\lambda t_k)^{\theta}}}{\sum_{i=1}^{n} i a_i e^{-i(\lambda t_k)^{\theta}}}\right) = 0, \tag{5}$$

$$\frac{d\ln L(t;\theta,\lambda)}{d\lambda} = \frac{m\theta}{\lambda} - \theta\lambda^{\theta-1}\sum_{k=1}^{m} t_k^{\theta}\left[\frac{\sum_{i=1}^{n} i^2 a_i e^{-i(\lambda t_k)^{\theta}}}{\sum_{i=1}^{n} i a_i e^{-i(\lambda t_k)^{\theta}}}\right] = 0. \tag{6}$$

Under some order conditions, the asymptotic joint distribution of the estimators is as follows:
$\begin{pmatrix} \widehat{\theta}_{MLE} - \theta \\ \hat{\lambda}_{MLE} - \lambda \end{pmatrix} \xrightarrow{d} N_2(0, I^{-1}(\theta,\lambda))$, where,

$$I(\theta,\lambda) = \begin{pmatrix} -\frac{d^2\ln L(t;\theta,\lambda)}{d\theta^2} & -\frac{d^2\ln L(t;\theta,\lambda)}{d\theta d\lambda} \\ -\frac{d^2\ln L(t;\theta,\lambda)}{d\lambda d\theta} & -\frac{d^2\ln L(t;\theta,\lambda)}{d\lambda^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}.$$

By placing the corresponding the maximum likelihood estimation of each parameter in the rows of the matrix $I(\theta,\lambda)$, it is possible to estimate the Fisher information matrix. If the matrix $I^{-1}(\theta,\lambda)$ represents the inverse of the matrix $I(\theta,\lambda)$, then an estimator for the asymptotic variance-covariance matrix of the vector $(\widehat{\theta}_{MLE}, \hat{\lambda}_{MLE})^T$ is as follows:

$$I^{-1}(\theta,\lambda)|_{(\theta,\lambda)=(\widehat{\theta}_{MLE},\hat{\lambda}_{MLE})} = \begin{pmatrix} \widehat{Var}(\widehat{\theta}_{MLE}) & \widehat{Cov}(\hat{\lambda}_{MLE},\widehat{\theta}_{MLE}) \\ \widehat{Cov}(\widehat{\theta}_{MLE},\hat{\lambda}_{MLE}) & \widehat{Var}(\hat{\lambda}_{MLE}) \end{pmatrix}.$$

Hence, an asymptotic $100(1-\tau)\%$ confidence intervals for $\theta$ and $\lambda$ can be constructed as

$$[\widehat{\theta}_l, \widehat{\theta}_u] = \hat{\theta}_{MLE} \pm z_{1-\frac{\tau}{2}}\sqrt{\widehat{Var}(\widehat{\theta}_{MLE})}, \qquad [\hat{\lambda}_l, \hat{\lambda}_u] = \hat{\lambda}_{MLE} \pm z_{1-\frac{\tau}{2}}\sqrt{\widehat{Var}(\hat{\lambda}_{MLE})},$$

where $z_q$ is the $q$-th percentile of the standard normal distribution.

### 2.2. Bayes estimation

In this section, we discuss the Bayesian inference of the unknown parameters $\theta$ and $\lambda$. For parameters estimation we have considered squared error loss function. Under the Bayesian paradigm, $\theta$ and $\lambda$ are independent random variables. It is assumed that $\theta$ and $\lambda$ have independent gamma priors with the pdfs

$$\pi(\theta) = \frac{\beta_1^{\alpha_1}\theta^{\alpha_1-1}\, e^{-\beta_1\theta}}{\Gamma(\alpha_1)}, \quad \pi(\lambda) = \frac{\beta_2^{\alpha_2}\lambda^{\alpha_2-1}\, e^{-\beta_2\lambda}}{\Gamma(\alpha_2)} \qquad \theta > 0,\ \alpha,\beta > 0. \tag{7}$$

Here the hyper-parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$ are chosen to reflect the prior knowledge about the unknown parameters. By combining Eqs. (4) and (7), we obtain the posterior joint probability density function of $\theta$ and $\lambda$ as

$$\pi(\theta,\lambda|\mathbf{t}) = m(\mathbf{t})\ \theta^{m+\alpha_1-1}\ e^{-\theta\left[\beta_1-\sum_{k=1}^{m}\ln t_k\right]}\ \lambda^{m\theta+\alpha_2-1}\ e^{-\beta_2\lambda}\ \prod_{k=1}^{m}\left(\sum_{i=1}^{n} i\ a_i e^{-i(\lambda t_k)^{\theta}}\right), \tag{8}$$

where

$$m(\mathbf{t}) = \left[\int_0^{\infty}\int_0^{\infty} \theta^{m+\alpha_1-1}\ e^{-\theta\left[\beta_1-\sum_{k=1}^{m}\ln t_k\right]}\ \lambda^{m\theta+\alpha_2-1}\ e^{-\beta_2\lambda}\ \prod_{k=1}^{m}\left(\sum_{i=1}^{n} i\ a_i e^{-i(\lambda t_k)^{\theta}}\right) d\theta d\lambda\right]^{-1}.$$

Subsequently, the Bayes estimator of any function, say $g(\theta,\lambda)$ under the squared error loss function, is given by

$$\widehat{g(\theta,\lambda)}_{BS} = E(g(\theta,\lambda)|\mathbf{t}) = \int_0^{\infty}\int_0^{\infty} g(\theta,\lambda)\pi(\theta,\lambda|\mathbf{t})d\theta d\lambda = \frac{\int_0^{\infty}\int_0^{\infty} g(\theta,\lambda)\pi(\theta,\lambda|\mathbf{t})d\theta d\lambda}{\int_0^{\infty}\int_0^{\infty} \pi(\theta,\lambda|\mathbf{t})d\theta d\lambda}. \tag{9}$$

Due to the complexity of the posterior joint probability density function, $\pi(\theta,\lambda|\mathbf{t})$, the Bayes estimator of any function, $g(\theta,\lambda)$ in Eq. (9), cannot be obtained analytically. We will use Metropolis-Hastings algorithm with a bivariate normal proposal distribution to generate random variates from posterior distribution [8]. Based on the Markov Chain Monte Carlo (MCMC) samples generated from the Metropolis-Hastings algorithm, we compute the Bayes estimate of $\theta$ and $\lambda$ and the corresponding credible intervals. The steps for the Metropolis-Hastings algorithm are described as follows:

1. Set the initial value $\zeta_0 = (\theta_0, \lambda_0)$;
2. Set $q = 1$;
3. Using the Metropolis-Hastings algorithm, generate $\zeta_q = (\theta_q, \lambda_q)$ from $\pi(\theta,\lambda|\mathbf{t})$ with the bivariate normal distribution with mean $\zeta_{q-1}$ and variance $\Sigma$ (denoted as $N_2(\zeta_{q-1}, \Sigma)$) as proposal distribution, where $\Sigma$ is the inverse of the estimated information matrix around the parameters $\theta$ and $\lambda$. In other words, we put (in each repetition);

$$I(\theta,\lambda) = \begin{pmatrix} -\frac{d^2\ln L(t;\theta,\lambda)}{d\theta^2} & -\frac{d^2\ln L(t;\theta,\lambda)}{d\theta d\lambda} \\ -\frac{d^2\ln L(t;\theta,\lambda)}{d\lambda d\theta} & -\frac{d^2\ln L(t;\theta,\lambda)}{d\lambda^2} \end{pmatrix}^{-1} \Big|_{(\theta,\lambda)=(\widehat{\theta}_{MLE},\widehat{\lambda})_{MLE}}$$

4. Set $q = q + 1$;
5. Repeat Steps 3 and 4, $N$ times to obtain the Markov chain Monte Carlo (MCMC) sample of size $N$: $(\theta_1, \lambda_1) \ldots, (\theta_N, \lambda_N)$.

Now we can get the approximate value of (9) using the MCMC technique as follows

$$\widehat{g(\theta,\lambda)}_{BS} = \frac{1}{N-M}\sum_{q=M+1}^{N} g(\theta_q,\lambda_q),$$

where M is the burn-in period. Now the approximate Bayes estimation under squared error loss function become

$$\widehat{\theta}_{BS} = \frac{1}{N-M}\sum_{q=M+1}^{N}\theta_q, \quad \text{and} \quad \widehat{\lambda}_{BS} = \frac{1}{N-M}\sum_{q=M+1}^{N}\lambda_q.$$

Based on the above simulated values of $\theta$ and $\lambda$, $\{(\theta_q, \lambda_q); q = M+1, \dots, N\}$, and using the method proposed by [1], we obtain the HPD credible interval for $\theta$ and $\lambda$. We assume that $\theta_{[1]} < \dots < \theta_{[N-M]}$ is the ordered MCMC sample of $\{\theta_q; q = M+1, \dots, N\}$. A $100(1-\tau)\%$ the HPD credible interval for $\theta$ is $\left(\theta_{[(\frac{\tau}{2})(N-M)]}, \theta_{[(1-\frac{\tau}{2})(N-M)]}\right)$, where $\theta_{[\frac{\tau}{2}(N-M)]}$ and $\theta_{[(1-\frac{\tau}{2})(N-M)]}$ are the $[\frac{\tau}{2}(N-M)]$-th smallest integer and the $[(1-\frac{\tau}{2})(N-M)]$-th smallest integer of $\{\theta_q \; q = 1, 2, \cdots, N-M\}$, respectively. Similarly, the HPD credible interval for parameter $\lambda$ can be obtained.

## 3. Numerical Example

In this section, We first discuss the algorithm to generate a random sample of $m$ i.i.d. system lifetimes $T_1, T_2, \dots, T_m$ for systems with Weibull distributed components and system signature $\mathbf{p} = (p_1, p_2, \dots, p_n)$ $(0 < p_i < 1, \sum_{i=1}^{n} p_i = 1)$. Algorithm to generate system lifetimes steps are given below, [3]:

1. Generate $u, u_1, u_2, \dots, u_n$ independently from uniform distribution in $[0, 1]$;
2. Set Generate $x_i$, $i = 1, 2, \dots, n$ of the Rayleigh distribution with the scale parameter;
3. Sort $x_1, x_2, \dots, x_n$ in ascending order to obtain $x_{1:n} < x_{2:n} < \dots < x_{n:n}$;
4. Take $t = x_{j:n}$ if $\sum_{i=1}^{j-1} p_i < u < \sum_{i=1}^{j} p_i, (j = 1, 2, \dots, n)$, i.e.,

$$T = \begin{cases} x_{1:n} & 0 < u < p_1 \\ x_{2:n} & p_1 < u < p_1 + p_2 \\ x_{3:n} & p_1 + p_2 < u < p_1 + p_2 + p_3 \\ \vdots & \vdots \\ x_{n:n} & \sum_{i=1}^{n-1} p_i < u < \sum_{i=1}^{n} p_i \end{cases}$$

5. Repeat Steps 1–4, $m$ times to obtain the system lifetimes $\mathbf{t} = (t_1, t_2, \dots, t_m)$.

**Example 3.1.** In this section, to explain the obtained results, consider the coherent system of 4 components according to Figure 1. A sample of $m = 20$ is generated from a 4-component system with lifetime

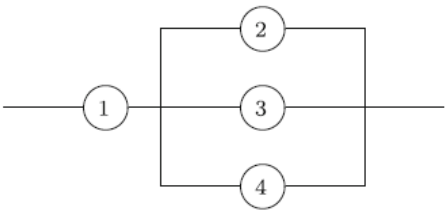


Fig. 1. The coherent system of 4-components.

$$T = \min\{X_1, \max\{X_2, X_3, X_4\}\}.$$

The corresponding system signature is $\mathbf{P} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and the corresponding minimal signature is $\mathbf{a} = (0, 3, -3, 1)$. The component lifetimes follow the Weibull distribution with parameters $\theta = 2$ and $\lambda = 1$. The simulated system lifetimes are presented in Table 1. Based on the system lifetime data in Table (1), we obtain the corresponding maximum likelihood estimates of $\theta$ and $\lambda$, and their associated 95% asymptotic confidence interval estimates. We also compute Bayes estimates and HPD interval estimates using the MH algorithm under the non-informative prior. For computing Bayes estimates and highest posterior density (HPD) intervals, we assumed that the priors on $\theta$ and $\lambda$ are the non-informative priors, i.e., $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0.0001$. It is to be noted that, while generating samples from the posterior distribution using the Metropolis-Hastings algorithm, initial values of $\zeta = (\theta, \lambda)$ are considered as $(\theta_0, \lambda_0) = (\widehat{\theta}_{MLE}, \widehat{\lambda}_{MLE})$. Furthermore, we considered the inverse of the estimated information matrix $\Sigma$, which can be easily obtained. We sampled $N = 50000$ values by Metropolis-Hastings algorithm with $\Sigma = \begin{pmatrix} 0.1380 & -0.0068 \\ -0.0068 & 0.0078 \end{pmatrix}$

Table 1. Simulated 4-component system lifetimes with system signature $\mathbf{P} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$.

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t_{m:20}$ | 0.2119 | 0.3059 | 0.4542 | 0.5788 | 0.6008 | 0.6279 | 0.6322 | 0.6759 | 0.6762 | 0.7487 |
| $m$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $t_{m:20}$ | 0.8005 | 0.8204 | 0.8882 | 0.9332 | 0.9682 | 1.0123 | 1.0714 | 1.3703 | 1.4702 | 1.6246 |

and the acceptance rate is about 69%. We discarded the initial $M = 5000$ burn-in samples and computed Bayes estimates and HPD interval estimates using the method of [1] based on averaging the remaining $N - M = 45000$ values. The results are also presented in Table (2). The computations are performed in R [7] with the MHadaptive

Table 2. Maximum likelihood, Bayesian and associated interval estimates for simulate data set.

| | Maximum likelihood estimationions | | Bayesian estimation | |
|---|---|---|---|---|
| Parameter | MLE | Asymptotic confidence interval. | MH algorithm | HPD interval |
| $\theta$ | 2.0871 | (1.3588, 2.8154) | 2.0101 | (1.3417, 2.7841) |
| $\lambda$ | 0.9678 | (0.7942, 1.1414) | 0.9642 | (0.7823, 1.1655) |

package [2].

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# Convolutional Neural Network Pruning Based on Topological Data Analysis

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**Abstract**

Pruning neural networks requires identifying which components are insignificant and can be removed. However, the high complexity of neural networks makes it difficult to fully understand the nature and function of their individual neurons. To address this, tools are employed to gain insights; one such tool is Topological Data Analysis (TDA). Convolutional Neural Networks, particularly the ResNet family, have numerous industrial applications. However, even the smallest member, ResNet-18, with its 11 million parameters and size of nearly 45 MB, is too large for resource-constrained environments like the automotive industry's SPC5 microcontrollers, which often have less than 2 MB of memory. Reducing ResNet-18's parameters by 90 could make it deployable on such devices, provided the reduction does not significantly impact performance. This research aims to prune ResNet networks effectively, minimizing the impact on accuracy to enable their cost-efficient use in industries like automotive. We propose a novel pruning method that utilizes TDA, specifically the Mapper algorithm, to identify and remove redundant kernels based on their topological density within the network's feature space.

## 1. Introduction

With the advancement of computer technology, Artificial Intelligence (AI) and its most prominent branch, neural networks, are being integrated into various industries. A significant barrier to this industrialization is the high cost, both in infrastructure (powerful hardware) and development time. Large neural networks like those in the ResNet family, while highly accurate for tasks such as image recognition, require substantial computational resources. Reducing the size of these networks without compromising their accuracy can lower hardware costs and accelerate inference, making AI more accessible. Pruning, the process of removing unimportant components from a trained network, is a key technique for model compression. However, accurately identifying which parts to prune remains challenging due to the "black-box" nature of neural networks. In 2021, Vadera et al. [5] presented a review paper in which they categorized and introduced 150 pruning methods. Figure1 is also adapted from the paper by Vadera. Later, other authors presented the other methods[2]
This paper introduces a pruning method grounded in Topological Data Analysis (TDA). Some articles used the tools of TDA to overcome issues which come from neural nets context [1],[3]. TDA provides a mathematical framework

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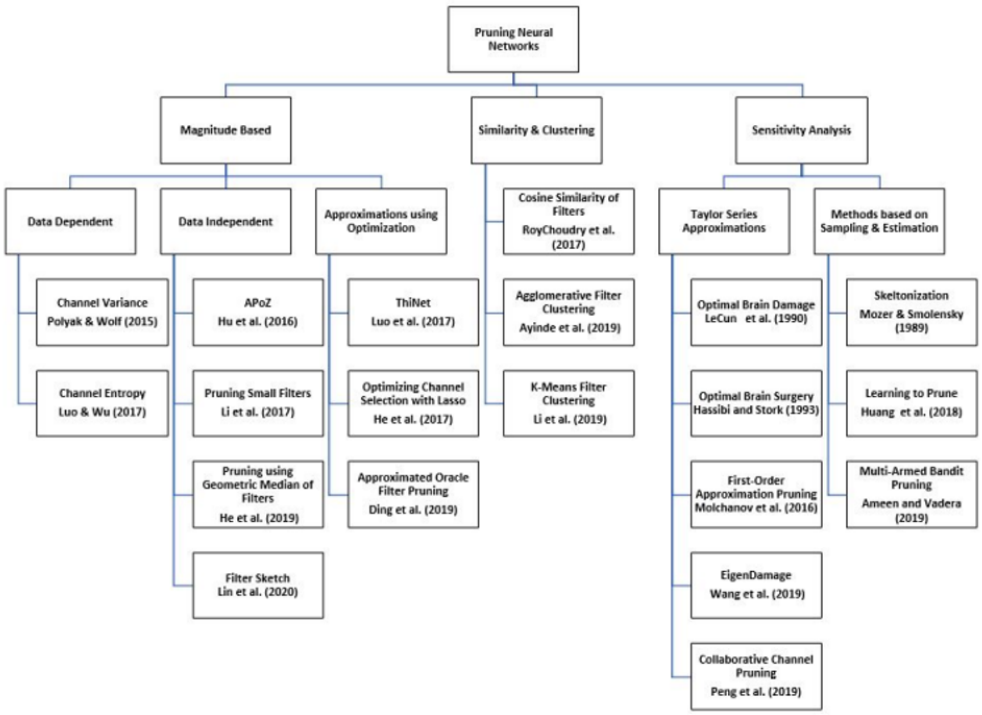


Fig. 1. Categorization of Neural Network Pruning Methods

for analyzing the shape and structure of high-dimensional data, which in our context corresponds to the parameters and activations within a neural network. By applying TDA, we hypothesize that we can more intelligently identify redundant regions (densely connected, similar components) for removal. Our primary contributions are:

1. A novel pruning framework that uses TDA, specifically the Mapper algorithm, to guide the pruning process in CNNs.
2. The application of this method to the ResNet family, demonstrating effective pruning on the CIFAR-10 dataset.
3. Evidence that our method performs genuine pruning (refining an existing network) rather than inducing training from scratch, and a comparative analysis showing its competitive performance against a established baseline (APoZ).

## 2. Background and Related Work

### *2.1. Neural Network Pruning*

Pruning is an iterative process typically involving three stages:

- Identifying a target component (weight, filter, channel);
- Removing or deactivating it;
- Fine-tuning the network to recover accuracy.

Pruning methods can be categorized along several axes:

- Structured vs. Unstructured: Unstructured pruning removes individual weights, offering high flexibility but often requiring specialized hardware for speedup. Structured pruning removes entire units (filters, channels), leading to direct model size reduction and faster inference on general hardware.

- Criteria: Methods vary in how they identify targets, using metrics like weight magnitude, activation sparsity (e.g., APoZ), or heuristic scores. Recent literature reviews, such as that by Vadera et al. (2021) which categorized over 150 methods, indicate a trend towards unstructured pruning due to its ability to escape local minima, though structured pruning remains crucial for practical deployment. A common practice in pruning research, especially for large networks like ResNet, is to use smaller datasets like CIFAR-10 for feasibility, followed by limited-epoch fine-tuning with a reduced learning rate (e.g., 10-20 epochs at 0.01x the original learning rate).

### *2.2. Topological Data Analysis (TDA)*

TDA is a set of techniques for finding robust topological features in complex datasets. Unlike traditional statistical methods, TDA can capture global shape properties like connectedness and holes. Key tools include:

- Persistent Homology: Tracks the birth and death of topological features (like connected components, loops) across scales.
- Mapper Algorithm: A practical tool for creating a combinatorial graph (a ”nerve”) that summarizes the data’s shape

Its stages are:

1. Filter Function: A real-valued function (e.g., $L^2$ norm of a kernel’s weights) is applied to the data.
2. Covering and Clustering: The range of the filter is partitioned into overlapping intervals. Data points within each interval are clustered (e.g., using K-means).
3. Nerve Construction: A graph is built where nodes represent clusters, and edges connect nodes whose corresponding clusters share data points due to the overlap.

## 3. Proposed TDA-Based Pruning Method

Our core assumption is that within a trained neural network, regions of higher importance for the task are associated with greater density and connectivity in the parameter or activation space. We operationalize this using the Mapper algorithm to find dense regions for pruning.

### *3.1. Methodology Overview*

The pruning process is iterative. In each iteration:

1. Dense Region Identification: For each convolutional layer, we treat its kernels as a point cloud in a high-dimensional weight space. We apply the Mapper algorithm with the following components:
   - Filter Function: The $L^2$ norm of each kernel’s weight vector.
   - Resolution: The filter range is partitioned into 20 intervals.
   - Overlap: An overlap of 40 percent between intervals is used (empirically chosen for balanced graph connectivity).
   - Clustering: K-means is applied within each interval. This produces a nerve graph for the layer.
2. Target Selection: The node in the nerve graph with the highest degree (most connections) is identified as the densest region. Across all layers, the node with the absolute highest degree in the entire network is selected for pruning.
3. Kernel Removal: Instead of removing a single kernel, we randomly select and completely remove 25 percent of the kernels within the chosen dense node. This batch removal addresses the inefficiency of single-kernel pruning and helps reduce the region’s density more decisively.
4. Fine-tuning: The pruned network undergoes short retraining for 10-20 epochs with a significantly reduced learning rate (0.001, 1 percent of the initial rate) to recover accuracy.

## 4. Experimental Evaluation

### *4.1. Setup*

- Models: ResNet-18 and ResNet-30.
- Dataset: CIFAR-10 (60,000 images, 10 classes).
- Baseline: APoZ (Average Percentage of Zeros) pruning method.

### *4.2. Validation of Pruning (vs. Training from Scratch)*

To prove our method performs genuine pruning and does not merely instigate training a new network from scratch, we provide two pieces of evidence:

1. Retraining Speed: As shown in Figure 2, accuracy recovers to 90 percent within the first epoch of fine-tuning. Training the same architecture from scratch with the low learning rate (0.001) used for fine-tuning would require orders of magnitude more epochs to reach similar accuracy, indicating the pre-learned features are largely preserved.

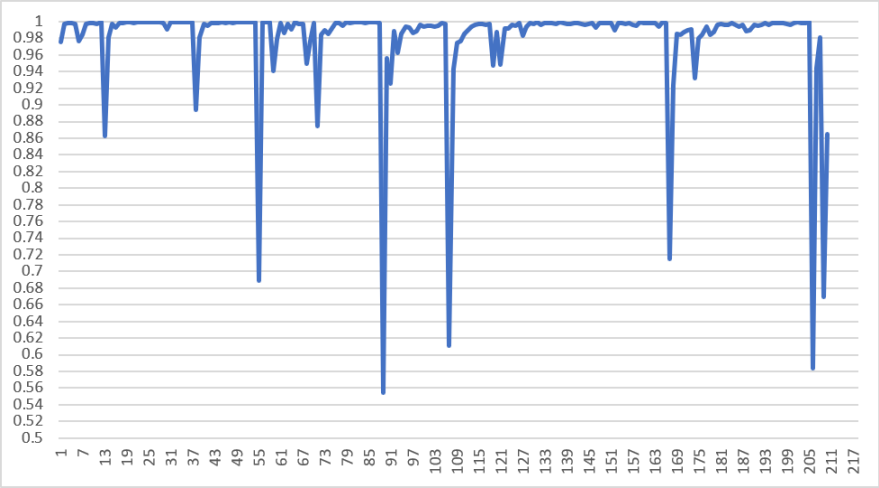


Fig. 2. Network accuracy after one epoch of retraining following a pruning step.

2. Topological Consistency: We compare the nerve graphs of a layer before pruning and after fine-tuning (20 epochs). As shown in Figure 3, the graphs retain significant structural similarity, suggesting the underlying data manifold (the set of effective kernels) remains largely unchanged despite the removal of some points.

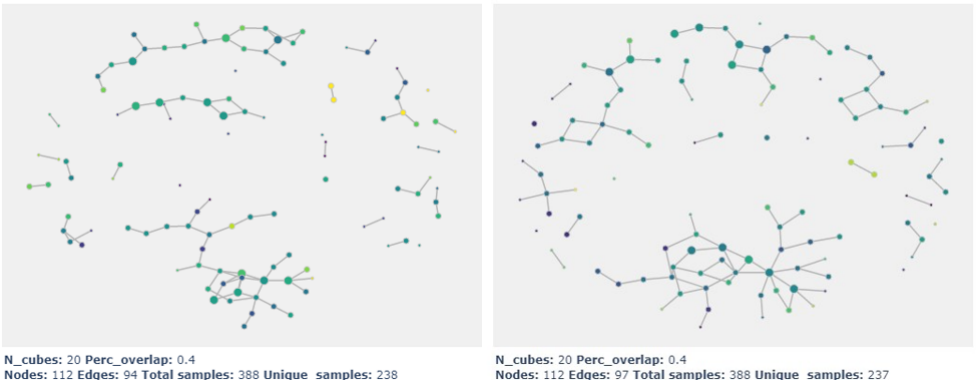


Fig. 3. Nerve graph of a convolutional layer (a) before pruning and (b) after 20 epochs of retraining.

### 4.3. *Performance Comparison*

We compare our TDA-based method against APoZ pruning which has brought in [4]. Figure 4 plots network accuracy against the percentage of parameters pruned.Two important points can be extracted from the diagram.

- Result: Our method maintains higher accuracy than APoZ across most of the pruning range. This superiority is attributed to TDA's ability to identify structurally redundant groups rather than relying solely on local activation statistics.
- Limitation: As pruning becomes extreme and the nerve graphs become sparse (due to very few kernels remaining), the density heuristic becomes less reliable, and accuracy eventually declines. This defines a practical limit for our method's applicability.

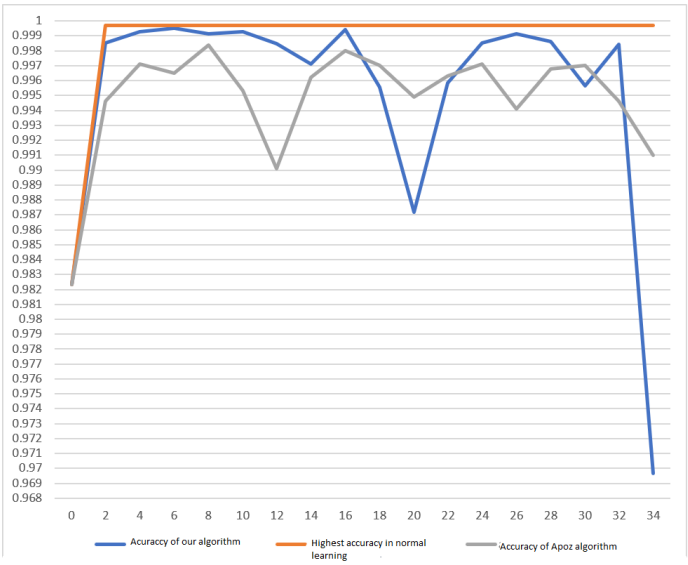


Fig. 4. Comparison of accuracy vs. pruning percentage between the proposed TDA-based method and the APoZ baseline on CIFAR-10 training data

## 5. Conclusion and Future Work

This paper presented a novel neural network pruning method that leverages Topological Data Analysis. By using the Mapper algorithm to identify dense, redundant regions in a network's parameter space, we can effectively guide structured pruning. Applied to ResNet models on CIFAR-10, the method demonstrates competitive performance against the APoZ baseline and provides evidence that it performs genuine network refinement rather than inducing retraining from scratch. The method is particularly effective in the early and middle stages of pruning where topological density is a reliable indicator of redundancy. Future Work will focus on:

1. Hybrid Strategies: Combining our TDA-based method with other pruning criteria in later stages to push the compressibility limit further.
2. Advanced Clustering: Replacing K-means with density-based algorithms (e.g., DBSCAN) within the Mapper pipeline for potentially more accurate density estimation.
3. Enhanced Filter Functions: Exploring filter functions beyond the $L^2$ norm, such as those based on angular similarity in polar coordinates, which may align better with the properties of normalized weights (e.g., after Batch Normalization layers).

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# Machine Learning-–Based Solution to Estimating Equation in Causal Structural Logistic Mean Models

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**Abstract**

This paper explores machine learning-–based solution to estimating equation in causal structural logistic mean model. We show how flexible, data-adaptive machine learning (ML) algorithm can be incorporated into estimating equation frameworks to estimate nuisance functions and construct valid estimating equations. Bias correction is achieved through orthogonalization of the estimating equation, reducing sensitivity to nuisance model misspecification. Simulation study demonstrates that the proposed ML substantially reduce bias under model misspecification while maintaining valid statistical inference.

## 1. Introduction

Causal structural mean models (SMMs) are widely used for estimating treatment effects in observational studies with a binary outcome. The structural logistic mean model, a special case of SMMs, models the log-odds of the conditional mean outcome under treatment as a linear function of parameters representing causal contrasts [1]. Traditional estimation solves unbiased estimating equations derived from assumptions such as no unmeasured confounding, consistency, and correct model specification [1, 2]. However, parametric model misspecification remains a major limitation. ML offers an alternative by allowing flexible, data-adaptive estimation of nuisance functions (e.g., propensity scores, outcome regressions). This paper shows how to integrate ML into estimating equation frameworks for causal SMMs while preserving asymptotic properties [3, 4]. Our simulation study demonstrates that the proposed ML-based estimating equation method reduces bias under model misspecification and maintains valid inference.

## 2. Causal Structural Logistic Mean Models

Let $A$ be a binary treatment, $Y$ the observed outcome, and $X$ a vector of pre-treatment covariates. Under potential outcomes framework, let $Y_a$ denote the potential outcome that would be observed if treatment $A = a$ were assigned. The logistic structural mean model (LSMM) assumes [1];

$$\operatorname{logit} E[Y^a|X, A = a] - \operatorname{logit} E[Y^0|X, A = a] = a \cdot (X^\top \psi_0) \tag{1}$$

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where $\text{logit}(p) = \log(\frac{p}{1-p})$, $\psi_0$ is the causal log-odds ratio parameter of interest, and $Y_0$ is the potential outcome. Under no unmeasured confounding $Y^a \perp\!\!\!\perp A \mid X$ and consistency, this implies the observed data model:

$$E[Y \mid X, A] = \text{expit}\big(m(X) + \psi_0 A\big) \tag{2}$$

where $m(X)$ is an unknown function of $X$ and $\text{expit}(z) = \frac{e^z}{1+e^z}$ is the inverse logit function.

The parameter $\psi_0$ is identified through the estimating equation:

$$U(\psi) = \sum_{i=1}^{n} D(X_i, A_i)\left\{Y_i - \text{expit}\big(m(X_i) + \psi A_i\big)\right\} = 0 \tag{3}$$

where $D(X, A)$ is a chosen weighting matrix. Naive estimation requires correct specification of $m(X)$.

## 3. Machine Learning-Based Solution

Suppose we have the data described in Section 2. The goal is to estimate causal parameter $\psi_0$ from the model definition and consistency ($Y = Y^A$):

$$E[Y^0|X, A] = \text{expit}\left(\text{logit}(E[Y|X, A]) - A \cdot (X^\top \psi_0)\right)$$

where $\text{expit}(u) = 1/(1 + e^{-u})$.

### Step 1: ML Nuisance Functions with K-Fold Cross-Fitting

Split indices $\{1, \ldots, n\}$ into $K$ folds $I_1, \ldots, I_K$. For each fold $k = 1, \ldots, K$:

*1.1 Propensity Score Model [1, 2]*

Train on $I_{-k} = \bigcup_{j \neq k} I_j$:

$$\hat{e}^{(-k)}(x) = \hat{P}(A = 1|X = x) \quad \text{using ML on } \{(X_i, A_i) : i \in I_{-k}\}$$

We consider logistic regression with regularization, random forests, and gradient boosting — all common ML choices:

$$\hat{e}^{(-k)}(x) = \frac{1}{B}\sum_{b=1}^{B} \text{expit}(f_b(x)) \quad \text{(random forest)}$$

or

$$\hat{e}^{(-k)}(x) = \text{expit}\left(\sum_{m=1}^{M} \beta_m h_m(x)\right) \quad \text{(gradient boosting)}$$

*1.2 Outcome Regression Model*

Train on $I_{-k}$:

$$\hat{\mu}^{(-k)}(x, a) = \hat{E}[Y|X = x, A = a] \quad \text{using ML on } \{(X_i, A_i, Y_i) : i \in I_{-k}\}$$

For binary $Y$, this is a classification problem [3]:

$$\hat{\mu}^{(-k)}(x, a) = \sigma\left(\sum_{l=1}^{L} w_l \phi_l(x, a)\right) \quad \text{(e.g., neural network with sigmoid output)}$$

*1.3 Cross-Fitted Predictions*

For each $i \in I_k$:

$$\hat{e}_i = \hat{e}^{(-k(i))}(X_i), \quad \hat{\mu}_i = \hat{\mu}^{(-k(i))}(X_i, A_i)$$

where $k(i)$ is the fold containing observation $i$. For a candidate $\psi$, define the imputed potential outcome under no treatment:

$$g_i(\psi) = \text{expit}\left(\text{logit}(\hat{\mu}_i) - A_i \cdot (X_i^\top \psi)\right)$$

The efficient score function (influence function) is:

$$U_i(\psi) = (A_i - \hat{e}_i) \cdot [Y_i - g_i(\psi)] \cdot X_i$$

This is a $p$-dimensional vector (same dimension as $\psi$).

**Step 2: Solving the Estimating Equation**

We need to find $\hat{\psi}$ such that:

$$\frac{1}{n}\sum_{i=1}^{n} U_i(\hat{\psi}) = 0$$

We used Newton-Raphson Algorithm with initial value of $\psi^{(0)} = 0$ or from simple logistic regression. For iteration $t = 0, 1, 2, \ldots$:

label=1. Compute function value:

$$U_n(\psi^{(t)}) = \frac{1}{n}\sum_{i=1}^{n} U_i(\psi^{(t)})$$

lbbel=2. Compute Jacobian (negative Hessian approximation):

$$J_n(\psi^{(t)}) = -\frac{1}{n}\sum_{i=1}^{n} (A_i - \hat{e}_i) \cdot g_i(\psi^{(t)})[1 - g_i(\psi^{(t)})] \cdot A_i \cdot X_i X_i^\top$$

lcbel=3. Update:

$$\psi^{(t+1)} = \psi^{(t)} + [J_n(\psi^{(t)})]^{-1} U_n(\psi^{(t)})$$

ldbel=4. Check convergence: Stop when $\|\psi^{(t+1)} - \psi^{(t)}\|_2 < \epsilon$ (e.g., $\epsilon = 10^{-6}$).

**Step 3: Variance Estimation and Confidence Interval**

*3.1 Sandwich Variance Estimator*

Let:

$$\hat{\Sigma} = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial U_i(\hat{\psi})}{\partial \psi^\top} = J_n(\hat{\psi})$$

$$\hat{\Omega} = \frac{1}{n}\sum_{i=1}^{n} U_i(\hat{\psi}) U_i(\hat{\psi})^\top$$

Then:

$$\widehat{\text{Var}}(\hat{\psi}) = \frac{1}{n}\hat{\Sigma}^{-1}\hat{\Omega}(\hat{\Sigma}^{-1})^\top$$

For $j = 1, \ldots, p$:

$$\text{SE}(\hat{\psi}_j) = \sqrt{[\widehat{\text{Var}}(\hat{\psi})]_{jj}}$$

*4.1 Wald Confidence Interval*

For parameter $\psi_{0j}$:

$$\hat{\psi}_j \pm z_{1-\alpha/2} \cdot \mathrm{SE}(\hat{\psi}_j)$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of standard normal.

*4.2 Hypothesis Test*

For $H_0 : \psi_{0j} = 0$:

$$Z_j = \frac{\hat{\psi}_j}{\mathrm{SE}(\hat{\psi}_j)} \sim \text{approx } N(0,1)$$

**Step 4: Mathematical Justification**

*4.1 Double Robustness*

The estimator satisfies:

$$E[U(\psi_0)] = 0$$

if either:

1. Propensity score correct: $e(X) = P(A = 1|X)$
2. Outcome model correct: $\mu(X, A) = E[Y|X, A]$

Proof sketch:

$$E[U(\psi_0)|X] = E\left[(A - e(X)) \cdot (Y^0 - E[Y^0|X]) \cdot X \mid X\right]$$
$$= E[A - e(X) \mid X] \cdot E[Y^0 - E[Y^0|X] \mid X] \cdot X = 0$$

*4.2 Orthogonality (Neyman Orthogonality)*

The derivative with respect to nuisance parameters is zero at truth:

$$\frac{\partial}{\partial \eta} E[U(\psi_0, \eta)]\Big|_{\eta=\eta_0} = 0$$

where $\eta = (e(\cdot), \mu(\cdot,\cdot))$.

*4.3 Asymptotic Distribution*

Under regularity conditions and using cross-fitting:

$$\sqrt{n}(\hat{\psi} - \psi_0) \xrightarrow{d} N(0, V)$$

where $V = E\left[\frac{\partial U}{\partial \psi}\right]^{-1} E[UU^\top] \left(E\left[\frac{\partial U}{\partial \psi}\right]^{-1}\right)^\top$.

This algorithm provides $\sqrt{n}$-consistent, asymptotically normal estimates of $\psi_0$ even with high-dimensional or non-parametric nuisance functions.

## 4. Simulation Study

### *4.1. Simulation Design*

We conducted a simulation study to evaluate the performance of our proposed ML-based estimating equation method and compare it with traditional parametric approaches. We generated data according to the causal structural logistic mean model defined in Section 2. The data generating process includes both linear and nonlinear components to create realistic scenarios where parametric models may be misspecified:

$$\begin{aligned} X &\sim N_4(0,\Sigma), \quad \Sigma_{ij} = 0.5^{|i-j|}, \quad X \in \mathbb{R}^4 \\ e(X) &= P(A=1|X) = \text{expit}\left(0.3X_1 - 0.2X_2^2 + 0.1X_3 + 0.15|X_4|\right) \\ Y|X,A &\sim \text{Bernoulli}\left(\text{expit}\left(e(X) + A(X^\top \psi_0)\right)\right) \end{aligned}$$

where $\psi_0 = (0.8, -0.5, 0.3)^\top$ are the true causal parameters of interest. The parameters are non-zero, representing relevant treatment effect modifiers. We compared three estimation approaches:

1. SLMM: Traditional parametric estimating equations with linear logistic regression models
2. ML-Based (Random Forest): ML method using random forests for nuisance function estimation
3. ML-Based (Gradient Boosting): ML method using gradient boosting machines

Two ML methods used 5-fold cross-fitting as described in Section 3.

### *4.2. Performance Metrics*

We evaluated three methods across $B = 500$ simulation runs using consistent performance metrics:

$$\text{Bias}_j = \frac{1}{B}\sum_{b=1}^{B}(\hat{\psi}_j^{(b)} - \psi_{0j})$$

$$\text{RMSE}_j = \sqrt{\frac{1}{B}\sum_{b=1}^{B}(\hat{\psi}_j^{(b)} - \psi_{0j})^2}$$

$$\text{Empirical SE}_j = \sqrt{\frac{1}{B-1}\sum_{b=1}^{B}(\hat{\psi}_j^{(b)} - \bar{\psi}_j)^2}$$

$$\text{Coverage}_j = \frac{1}{B}\sum_{b=1}^{B}\mathbb{I}\left(\psi_{0j} \in [\hat{\psi}_j^{(b)} \pm 1.96 \times \widehat{\text{SE}}_j^{(b)}]\right)$$

The bias and RMSE provide direct comparisons of point estimation accuracy across methods. The empirical standard error quantifies the variability of estimates across simulation runs. The methods differ in how they estimate standard errors ($\widehat{\text{SE}}_j^{(b)}$) for confidence intervals. LSMM uses robust sandwich estimators derived from the parametric model specification. ML Methods (Random Forest and Gradient Boosting) use bootstrap standard errors with 200 resamples. This allows us to compare both point estimation performance (bias, RMSE) and inference quality (coverage) across fundamentally different approaches.

### *4.3. Implementation Details*

For ML methods, we used the following implementations and hyperparameters:

- Parametric: `glm` function with main effects only
- Random Forest: `ranger` package with 500 trees, minimum node size of 10
- Gradient Boosting: `xgboost` package with learning rate 0.01, maximum depth 3, 1000 rounds with early stopping

The Newton-Raphson algorithm for solving estimating equations used convergence tolerance $\epsilon = 10^{-6}$ and maximum iterations 100.

### *4.4. Simulation Results*

Table 1 presents the Bias and Root Mean Square Error (RMSE) for estimating $\psi_1$, $\psi_2$, and $\psi_3$. The ML-based methods show substantially lower Bias and RMSE compared to the parametric approach.

Table 1. Bias and RMSE Comparison Across Methods

| | $\psi_1 = 0.8$ | | $\psi_2 = -0.5$ | | $\psi_3 = 0.3$ | |
|---|---|---|---|---|---|---|
| Method | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| LSMM | 0.152 | 0.467 | -0.121 | 0.392 | 0.089 | 0.285 |
| ML (Random Forest) | 0.032 | 0.198 | -0.028 | 0.165 | 0.021 | 0.132 |
| ML (Gradient Boosting) | 0.029 | 0.187 | -0.025 | 0.158 | 0.019 | 0.126 |

Table 2 shows the empirical coverage rates of 95% confidence intervals. The ML-based methods maintain nominal coverage close to 95%, while the parametric method undercovers due to model misspecification.

Table 2. Empirical Coverage Rates of 95% Confidence Intervals

| Method | $\psi_1$ | $\psi_2$ | $\psi_3$ |
|---|---|---|---|
| Parametric LSMM | 0.890 | 0.910 | 0.885 |
| ML (Random Forest) | 0.940 | 0.950 | 0.935 |
| ML (Gradient Boosting) | 0.945 | 0.955 | 0.940 |

## 5. Discussion

The simulation results demonstrate that our proposed ML-based estimating equation method substantially reduces bias when parametric models are misspecified. Maintains valid statistical inference with proper coverage rates. These findings support the use of ML-based estimating equations in practical applications where the functional forms of nuisance parameters are unknown or complex.

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# Advancements in Spatial Scan Statistics: From Circular Windows to Flexible Modeling of Irregularly Shaped Clusters

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**Abstract**

This paper reviews the Spatial Scan Statistic (SSS) and compares traditional circular scanning methods with advanced techniques for detecting irregularly shaped spatial clusters. The main limitation of circular methods is shape bias, which leads to inaccurate boundary estimates and reduced statistical power, especially for non-compact clusters like those along rivers, transport networks, or elongated regions. By synthesizing flexible, elliptical, and graph-based scan statistics, the study shows that irregular scanning methods provide more accurate spatial inference. Although these advanced methods require more computation, they better capture the true shape of spatial phenomena, which is essential for improving precision in public health and social science research. The review concludes that shifting toward irregular cluster detection is necessary for more reliable spatial analysis.

## 1. Introduction

Spatial cluster detection constitutes a central component of geographic data analysis, with wide-ranging applications in epidemiology [8], criminology [11], image analysis [17], and others. Identifying statistically significant spatial concentrations of events enables researchers and policymakers to uncover underlying risk factors, allocate resources efficiently, and design targeted interventions. Among the available methodologies, the Spatial Scan Statistic (SSS), first popularized through Kulldorff's framework [10], has emerged as one of the most widely used and operationally accessible tools due to its solid statistical foundation and adaptability to diverse probability models (e.g., Poisson, Bernoulli, and space–time formulations).

Despite its popularity, the classical implementation of SSS relies predominantly on regular geometric scanning windows—most commonly circular, and in some extensions elliptical [9]. While computationally efficient and mathematically tractable, these regular windows impose strong geometric assumptions on the underlying cluster structure. In real-world settings, however, many spatial phenomena exhibit highly irregular and non-compact morphologies. Disease outbreaks may spread along river basins, pollution follows transportation or industrial corridors, and socio-economic

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deprivation often aligns with administratively or geographically elongated regions. Under such circumstances, circular windows introduce what is often termed **shape bias**, potentially leading to boundary misclassification, inclusion of low-risk areas, exclusion of true high-risk zones, and an overall dilution of statistical power.

To fix the flaws of circular scanning methods, researchers have developed irregular and flexible scan statistics that better capture the true shape of spatial clusters. However, these advanced methods bring challenges like higher computational cost and complexity. Given the growing availability of high-resolution spatial data, this paper reviews and compares traditional circular approaches with modern irregular techniques to clarify their strengths, weaknesses, and practical use in spatial analysis.

Continuation of the paper is as follows. In Section 2, the Spatial Scan Statistic (SSS) for detecting spatial clusters is introduced. Section 3 focuses on the detection of irregularly shaped spatial clusters, and Section 4 presents the conclusions.

## 2. Circular Scan statistics

The most well-known method for detecting spatial clusters was introduced by [10]. They proposed the **Spatial Scan Statistic (SSS)**, a method based on the Likelihood Ratio Test (LRT). In engineering and applied sciences, a similar idea is sometimes referred to as moving window analysis. The core intuition of this method is simple and visual. Events, such as disease cases, are represented as points on a map, and the goal is to determine whether these points are more concentrated in some areas than would be expected by chance alone.

To formalize this idea, a candidate subregion is denoted by $z$, while the rest of the study area is denoted by $z^c$. It is assumed that the probability of observing an event inside $z$ is $p$, whereas outside $z$ the probability is $q$. Detecting a cluster is therefore equivalent to testing whether the risk inside a region is higher than outside it. This leads to the following hypothesis test

$$H_0 : p = q, \quad \forall z \in \mathcal{Z} \qquad \text{vs.} \qquad H_1 : p > q, \quad \exists z \in \mathcal{Z}. \tag{1}$$

Under the null hypothesis, there is no spatial cluster and the risk is homogeneous across the entire region. Under the alternative hypothesis, there exists at least one subregion where the risk is significantly higher, indicating the presence of a high-risk cluster.

In the original framework proposed by [10], circular scanning windows are used to search for potential clusters. A large number of circles with different centers and radii are imposed on the map. Each circle represents a possible cluster $z$. For every candidate circle, the likelihood ratio is computed as $\lambda(z) = L(z)/L_0$, where $L(z)$ is the likelihood under the alternative hypothesis and $L_0$ is the likelihood under the null hypothesis. The circle that maximizes this likelihood ratio is identified as the **Most Likely Cluster (MLC)**, meaning it is the region with the strongest statistical evidence of elevated risk.

A practical challenge is that the exact sampling distribution of the scan statistic is analytically intractable. For this reason, statistical significance is evaluated using Monte Carlo simulation. In this procedure, a large number of random datasets are generated under the null hypothesis, typically 999 replications. For each simulated dataset, the scan statistic is recalculated. The observed statistic from the real data, denoted by $\lambda_R$, is then compared with the distribution obtained from the simulated datasets. If the observed value exceeds most of the simulated values, the detected cluster is considered statistically significant and unlikely to have occurred by random variation alone.

Although the circular scan statistic is widely used due to its simplicity and interpretability, it has several important limitations. One key issue is related to shape restriction. Because the scanning window is circular, the method is naturally suited to detecting circular clusters. However, real-world clusters are often irregular or elongated, for example along rivers, roads, or environmental exposure paths. In such situations, circular windows may fail to accurately capture the true spatial pattern.

Another limitation concerns size estimation. Since the imposed circular window may not align well with the actual cluster boundary, the detected cluster can be either overestimated or underestimated in size. This mismatch between the scanning shape and the true cluster geometry can lead to biased delineation of risk areas.

A further practical limitation is computational cost. The need to evaluate a large number of scanning windows, combined with hundreds or thousands of Monte Carlo simulations, can make the method computationally intensive, particularly when applied to large datasets or when computational resources are limited.

To fix these issues, researchers developed better tools. [9] create an elliptic scan statistic to find non-circular shapes. However, even ellipses cannot find completely irregular shapes. Other researchers expanded the method to handle different types of data, such as ordinal data [7] or multivariate data [3].
In the next section, we focus on spatial clustering methods designed to detect clusters with irregular shapes. Unlike circular scan statistics, these approaches are capable of identifying complex and non-compact spatial patterns that more closely reflect real-world risk structures. We review the main methodological developments in this area and discuss how they improve cluster detection when the true cluster boundaries are not circular.

## 3. Irregular Scan Statistics

To detect spatial clusters with irregular shapes, [1] propose the dynamic minimum spanning tree (dMST) method. The main motivation behind this method is that real-world clusters are rarely circular. Instead, they often follow complex geographic or environmental patterns. Therefore, methods that are not restricted to circular scanning windows are needed.
The dMST approach is based on graph theory. In this framework, the study region is represented as a connected graph in which each spatial cell corresponds to a vertex. A spanning tree is defined as a subgraph that connects all vertices without forming cycles. Among all possible spanning trees, the minimum spanning tree (MST) is the one with the smallest total edge weight.
In the dMST method, centroids of neighboring cells are connected by edges. In this way, the spatial map is transformed into an undirected graph. A weight $w(i,j)$ is assigned to each edge $(v_i, v_j)$ using the Kullback–Leibler distance. This weight reflects the dissimilarity between the densities of events (or cases) in the two connected cells. Larger weights indicate greater differences in density. Once the weighted graph is constructed, the MST is obtained using the algorithm of [14].
After constructing the MST, the method proceeds by removing one edge at a time. Removing an edge divides the tree into two subtrees. The smaller subtree is considered as a candidate cluster and evaluated using the likelihood ratio test (LRT) of [10]. The removed edge is then restored, and another edge is deleted. This process is repeated for all edges, generating multiple candidate clusters. Among all candidates, the one with the maximum likelihood ratio is identified as the Most Likely Cluster (MLC). Monte Carlo simulation is then used to evaluate the statistical significance of the detected cluster.
Although the dMST method improves spatial scan statistics in two important aspects, namely flexibility in cluster shape and reduction in the number of candidate regions (from many circular windows to only $I$ candidates, where $I$ is the number of spatial cells), it still has several limitations. First, it is designed to detect only one irregular cluster. Second, it often identifies clusters that are larger than the true underlying cluster, a phenomenon known as overestimation or the octopus effect. Third, the reliance on Monte Carlo simulation makes the method computationally time-consuming.
To address the overestimation problem of the dMST, [2] proposed three constrained spatial scan statistics for detecting irregular clusters. These approaches impose ad-hoc restrictions on cluster expansion using three criteria: edMST (early stopping dMST), double connection, and mlink (maximum linkage).
The key difference between the edMST and the unrestricted dMST is that the edMST procedure stops expanding a candidate cluster when adding neighboring units no longer increases the likelihood function. In the double connection approach, a neighboring unit is eligible to join the candidate cluster only if it is connected to at least two units already included in the cluster and if its inclusion increases the LRT. An exception occurs in the first step, when the candidate consists of only one unit. If no neighboring unit satisfies the connection rule, the algorithm stops and restarts from another unit. This strategy generally produces more compact clusters than the edMST method. Finally, the mlink spatial scan statistic searches for candidates only among neighbors that have the maximum number of connections to the current cluster and evaluates whether they should be included. For all these constrained methods, Monte Carlo simulation is used to assess statistical significance.
Recently, [18] introduced another extension called the Adaptive Minimum Spanning Tree (AMST), which is described in detail in the next subsection.

### *3.0.1. Adaptive Minimum Spanning Trees in Detection of Irregularly Shaped Spatial Clusters*

The adaptive minimum spanning tree (AMST) method was introduced by [4]. The central idea of this method is to determine the best way to partition the MST into subgraphs that represent potential clusters. To achieve this, a validity

index is defined to evaluate the quality of each partition:

$$val_{index} = \frac{Intra_{cluster}}{Inter_{cluster}}.$$

The numerator measures the within-partition variation, while the denominator measures the separation between partitions. According to [6], this index reflects how well the graph is separated into distinct groups. Formally,

$$Intra_{dist} = \sum_{i=1}^{K} \sum_{j \in C} |\lambda_{C^i j} - \lambda_{C^i}|^2 / K,$$

$$Inter_{dist} = \max |\lambda_{C^i} - \lambda_{C^j}|^2.$$

Here, $\lambda_{C^i}$ denotes the event rate in sub-partition $C^i$, and $\lambda_{C^i j}$ represents the rate for cell $j$ within that sub-partition. These parameters are typically estimated using maximum likelihood methods. The symbol $K$ denotes the number of subgraphs in a given partition of the MST. According to [18], the optimal partition is the one that minimizes the validity index, since this corresponds to partitions that are internally homogeneous and well separated from each other.
Algorithm 1 outlines the main steps of the AMST procedure.

**Algorithm 1** Adaptive Minimum Spanning Tree Algorithm

1: find MST using the Prim algorithm,
2: order weights of the edges of the MST,
3: the second lightest weight is considered a threshold and heavier edges are removed,
4: compute the validity index for the obtained partition,
5: add the next lightest edge and return to Step 4 until the original MST is reconstructed.

A partition with the minimum validity index is considered the best representation of spatial clusters because it simultaneously achieves high compactness within subgraphs and strong separation between them [18].
This algorithm was designed to address the overestimation problem observed in [1]. In addition, by incorporating the linear time subset scan (LTSS) property [12], the number of candidate regions $\mathcal{Z}$ is drastically reduced.
It was shown in [12] that scan statistics and several of their extensions satisfy the LTSS condition, which enables exact and efficient optimization over subsets. Instead of evaluating all possible subsets, LTSS requires only ordered subsets to be examined. Suppose $K$ is the number of sub-partitions in the optimal MST partition and $PF(.)$ is a priority function such as incidence rate. Let $o_{(j)}$ denote the index of the sub-partition with the $j$-th largest priority value, where $o_{(j)} \in \{1, \dots, K\}$. Define ordered subsets $o_i = \{o_{(1)}, \dots, o_{(i)}\}$ for $i = 1, \dots, K$. LTSS shows that, instead of evaluating all $2^K$ subsets, only the $K$ ordered subsets $\{o_1, \dots, o_K\}$ need to be considered to identify the MLC.
As in other spatial scan methods, Monte Carlo simulation is finally used to assess statistical significance. Compared with the original MST approach, AMST is computationally faster and effectively controls the overestimation problem. Therefore, [18] improved irregular cluster detection in two ways: by mitigating the overestimation effect through adaptive partitioning and by reducing the candidate search space using LTSS. Furthermore, [16] proposed a hybrid approach that combines MST with the restricted likelihood ratio method of [15] to accelerate irregular cluster detection. It is also important to mention the adaptive multiresolution procedure proposed by [17], known as spatial multiresolution cluster detection (MCD). This method is particularly effective for identifying irregular clusters without heavy computational burden and is well suited for large spatial datasets such as images and fMRI data. Another related development is the particle swarm optimization approach introduced by [5], which optimizes scanning windows for detecting irregular spatial clusters. Finally, for hierarchically structured data represented as trees, [13] proposed a flexible scan statistic based on minimum description length penalization, which helps prevent the detection of excessively complex or unrealistic cluster shapes.

## 4. Conclusion

In this review, we have highlighted the critical role of the Spatial Scan Statistic in identifying meaningful spatial clusters across various applications, from public health to social science. While traditional circular and elliptical

scanning windows offer computational simplicity, they are limited by shape bias and may fail to accurately capture non-compact clusters. Irregular and flexible scan methods, including adaptive, graph-based, and network-constrained approaches, provide higher fidelity in detecting the true morphology of clusters. Although these advanced methods involve increased computational complexity, their ability to more accurately delineate spatial patterns is essential for informed decision-making and targeted interventions. Overall, adopting irregular cluster detection frameworks enhances the precision of spatial analysis and strengthens its utility in contemporary geographic research.

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# Statistical inference of the weighted compounded inverse Weibull distribution

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**Abstract**

The weighted distributions offer extending distributions for further flexibility to fit a data set. The inverse Weibull geometric distribution can be applied to a model lifetime of the series system. In this paper, we present a weighted inverse Weibull geometric distribution. The survival, hazard rate functions, Rényi entropy, estimation the parameters using the maximum likelihood estimation, expectation-maximization algorithm and the method moment estimation are derived. Moreover, simulation studies are done.

## 1. Introduction

The inverse Weibull (IW) distribution is competent for statistical analysis of lifetime in reliability engineering investigation. This distribution has been utilized to an extent range of sites including utilizations in medication, reliability, and ecology. Shafiei et al. (2016)[6] showed that IW distribution is a suitable model for hazard function unimodal and indicated it's as one of the common distributions in supplementary risk matters. In recent years, numerous investigators have suggested compounding of favorable lifetime distributions to model lifetime data. In 2008, Tahmasbi and Rezaei[7] recommended exponential logarithmic distribution with decreasing failure rate. Later Chung and Kang (2014)[2] introduced the exponential Weibull geometric distribution. Chung and (2017)[1] proposed a new exponential inverse Weibull geometric (IWG) distribution. Assume $\{X_i\}_{i=1}^{n}$ n are independent and identically distributed (iid) random variables from the IW distribution with scale parameter$\lambda > 0$ and shape parameter $\alpha > 0$ that denote failure times of the components of a system and n is an integer. Therefore the corresponding pdf and cdf are exposed through, respectively

$$f(x;\lambda,\alpha) = \lambda\alpha x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}}, \quad F(x;\lambda,\alpha) = e^{-\lambda x-\alpha} \;\; ; \;\; x,\lambda,\alpha > 0.$$

Suppose Z is a random variable characterizing number of components and Y (independent of Z) is observable random variable representing minimum component lifetime. So the random variable Y illustrated by $y = min(x_1 \dots x_z)$. Consequently, the conditional cdf and pdf Y given Z=z is yield by respectively

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$$F_{Y|Z=z}(y;\lambda,\alpha) = 1-(1-e^{-\lambda y^{-\alpha}})^{z} \;\; ; \;\; y,\lambda,\alpha>0,$$

and

$$f_{Y|Z=z}(y;\lambda,\alpha) = z\alpha\lambda y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}}(1-e^{-\lambda y^{-\alpha}})^{z-1} \;\; ; \;\; y,\lambda,\alpha>0.$$

Assuming random variable Z has geometric distribution with parameter p and its probability function as

$$P(z;p) = p(1-p)^{z-1}; z=1,2,\ldots,0<p<1.$$

The unconditional marginal pdf, cdf and rth raw moment of Y can be gotten as, respectively

$$f(y;\lambda,\alpha,p) = p\alpha\lambda y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}})^{-2} \;\; ; \;\; y,\lambda,\alpha>0, \tag{1}$$

$$F(y;\lambda,\alpha,p) = e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}})^{-1}; y,\lambda,\alpha>0, 0<p<1, \tag{2}$$

$$E_r = E(Y^r) = p\alpha\lambda\int_0^{\infty} y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}}))^{-2}dy = p\alpha\lambda B_r,$$

where

$$B_r = \int_0^{\infty} y^{-(\alpha+1)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}}))^{-2}dy.$$

Weighted distribution theory denotes a consolidated path to dealing with model stipulation and data interpretation problems. The examination of weighted distribution can be applied for better comprehension of standard distributions and gives techniques of extending distributions for further flexibility to fit a data set. The weighted distributions apply in studies linked to the reliability, medicine, ecology, survival analysis, and branching processes can be seen in Gupta and Kirmani (1990)[3] and Patil (2006)[5] and and the references therein. Patil and Rao (1978)[4] introduced weighted probability density functions (pdf). Suppose X is the nonnegative random variable that has f(x) as the pdf, then the pdf of the weighted random variable Xw can be attained through

$$f_w(x) = \frac{(w(x)f(x))}{(E(w(X)))}; \;\; x\geq 0, \;\; where \;\; 0<E(w(X)) = \int_0^{\infty} w(x)f(x)dx<\infty. \tag{3}$$

The random variable $X_w$ is weight version of X, and its distribution is named weighted distribution with weight function w(x). The weight function w(x) obtains a different practical. In this paper, we contemplate$w(x)=x^c$, and the model is attained called as size biased distribution. For c=1, it is named length biased distribution, and when c=2, is referred to as area biased distribution.

## 2. The Weighted Inverse Weibull Geometric (WIWG) Distribution

The pdf, cdf and kth raw moment for WIWG distribution using (1), (2), (3) and weight of $w(y)=y^r$ can be derived as, respectively

$$f_w(y;\lambda,\alpha,p,r) = B_r^{-1}y^{-(\alpha+1-r)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}})^{-2}; y,\lambda,\alpha>0, 0<p<1, \tag{4}$$

$$F_w(y;\lambda,\alpha,p,r) = B_r^{-1}\int_0^{y} x^{-(\alpha+1-r)}e^{-\lambda x^{-\alpha}}(p+(1-p)e^{-\lambda x^{-\alpha}})^{-2}dx, \tag{5}$$

$$E_k = E(Y^k) = B_r^{-1}B_(r+k).$$

The possible relations of the WIWG distribution by using the following values of the parameters are as follows: a) for r=0, WIWG distribution reduces to IWG distribution, for r=1, WIWG distribution is called length biased IWG distribution, and for r=2, WIWG distribution is called area biased IWG distribution, b) for r=0,$\alpha=1$, inverse exponential geometric distribution is achieved, c) for $r=0,\lambda=1$, Fréchet geometric distribution is achieved, d) for r=0,$\alpha=2$,

inverse Rayleigh geometric distribution is derived, e) if p approaches 1, IW distribution is obtained. The survival (or reliability) and hazard (or failure) rate functions are expressed by, respectively

$$S_w(y;\lambda,\alpha,p,r) = 1 - B_r^{-1}\int_0^y x^{-(\alpha+1-r)}e^{-\lambda x^{-\alpha}}(p+(1-p)e^{-\lambda x^{-\alpha}})^{-2}dx,$$

$$h_w(y;\lambda,\alpha,p,r) = \frac{B_r^{-1}y^{-(\alpha+1-r)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}})^{-2}}{1-B_r^{-1}\int_0^y x^{-(\alpha+1-r)}e^{-\lambda x^{-\alpha}}(p+(1-p)e^{-\lambda x^{-\alpha}})^{-2}dx} \;;\; y,\lambda,\alpha>0, 0<p<1.$$

## 3. Rényi entropy

Entropy is a measure of uncertainty or randomness and is applied in various statuses in engineering, physics, statistical mechanics and other applied sciences. The Rényi entropy of the WIWG distribution is yielded to be

$$I_R(\gamma) = \frac{1}{1-\gamma}log(B_r^{-\gamma}\int_0^\infty y^{-\gamma(\alpha+1-r)}e^{-\gamma\lambda y^{-\alpha}}(p+(1-p)e^{-\gamma\lambda y^{-\alpha}})^{-\gamma}dy) \;;\; \gamma\neq 1,\gamma>0).$$

## 4. Estimation of parameters

### *4.1. The maximum likelihood estimation (MLE)*

Suppose $Y_1, Y_2, ..., Y_n$ be a random sample of size n from WIWG distribution (4) and $\phi = (\lambda,\alpha,p,r)^T$ the vector of unknown parameters. The log-likelihood function for $\phi$ would be

$$l(\phi) = -n\log B_r - \sum_{i=1}^n[(\alpha+1-r)\log y_i + \lambda y_i^{-\alpha} + 2log(p=+(1-p)e^{-\lambda y^{-\alpha}}))].$$

Let $U(\phi) = (\frac{\partial l}{\partial\lambda},\frac{\partial l}{\partial\alpha},\frac{\partial l}{\partial p},\frac{\partial l}{\partial r})^T$, the MLEs of the parameters are calculated with solving the nonlinear system equations $U(\phi)=0$, that is yielded by

$$\frac{\partial l}{\partial\lambda} = -\frac{nB_r'^{(\lambda)}}{B_r} - \sum_{i=1}^n\frac{y_i^{-\alpha}(p-(1-p)e^{-\lambda y^{-\alpha}})}{p+(1-p)e^{-\lambda y^{-\alpha}}} = 0,$$

$$\frac{\partial l}{\partial\alpha} = -\frac{nB_r'^{(\alpha)}}{B_r} - \sum_{i=1}^n[\log(y_i) - \frac{\lambda y_i^{-\alpha}\log(y_i)(p-(1-p)e^{-\lambda y^{-\alpha}})}{p+(1-p)e^{-\lambda y^{-\alpha}}}] = 0,$$

$$\frac{\partial l}{\partial p} = -\frac{nB_r'^{(p)}}{B_r} - 2\sum_{i=1}^n\frac{1-e^{-\lambda y^{-\alpha}}}{p+(1-p)e^{-\lambda y^{-\alpha}}} = 0,\; fracdldr = -\frac{nB_r'^{(r)}}{B_r} + \sum_{i=1}^n\log(y_i) = 0,$$

where

$$B_r'^{(\lambda)} = \frac{\partial B_r}{\partial\lambda} = \int_0^\infty y^{-(2\alpha+1-r)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}})^{-3}((1-p)(2-e^{-\lambda y^{-\alpha}})-p)dy,$$

$$B_r'^{(\alpha)} = \frac{\partial B_r}{\partial\alpha} = \int_0^\infty y^{-(\alpha+1-r)}e^{-\lambda y^{-\alpha}}\log(y_i)(p+(1-p)e^{-\lambda y^{-\alpha}})^{-3}(p(\lambda y^{-\alpha}-1)-(1-p)e^{-\lambda y^{-\alpha}}(\lambda y^{-\alpha}+1))dy,$$

$$B_r'^{(p)} = \frac{\partial B_r}{\partial p} = -2\int_0^\infty y^{-(\alpha+1-r)}e^{-\lambda y^{-\alpha}}(p+(1-p)e^{-\lambda y^{-\alpha}})^{-3}(1-e^{-\lambda y^{-\alpha}})dy,$$

$$B_r^{\prime(r)} = \frac{\partial B_r}{\partial r} = \int_0^{\infty} y^{-(\alpha+1-r)} e^{-\lambda y^{-\alpha}} \log(y_i)(p + (1-p)e^{-\lambda y^{-\alpha}})^{-2}.$$

These equations can be solved numerically via iterative approaches like Newton-Raphson method using statistical software.

*4.1.1. EM Algorithm*

In order to use EM algorithm, we define the complete data distribution as follows

$$f(y,z) = \frac{z}{B_r} y^{-(\alpha+1-r)} e^{-\lambda y^{-\alpha}} ((1-p)(1-e^{-\lambda y^{-\alpha}}))^{(}(z-1)) \;\; ; \;\; y,\alpha,\lambda > 0, 0 < p < 1, z \in \mathbb{N}.$$

Thus, it's the log-likelihood function is arrived by

$$l_c(\phi) = -n \log B_r - \sum_{i=1}^{n} [(\alpha + 1 - r) \log y_i - \log z_i + \lambda y^{-\alpha} - (z_i - 1)(\log(1-p) + \log(1 - e^{-\lambda y^{-\alpha}}))]. \quad (6)$$

Here, for E-step of EM algorithm, we need the conditional expectation of$(Z|Y; \phi^{(k)})$, where $\phi^{(k)} = (\lambda^{(k)}, \alpha^{(k)}, p^{(k)}, r^{(k)})^T$. The conditional density function of $(Z|Y)$ is expressed by

$$f_(Z|Y = y)(z; \lambda, \alpha, p) = z((1-p)(1-e^{-\lambda y^{-\alpha}}))^{(}(z-1))(p + (1-p)e^{-\lambda y^{-\alpha}})^2 \;\; ; \;\; y,\alpha,\lambda > 0, 0 < p < 1, z \in \mathbb{N}.$$

Therefore, the conditional expectation is arrived as

$E(Z|Y = y) = \frac{1+(1-p)(1-e^{-\lambda y^{-\alpha}})}{p+(1-p)e^{-\lambda y^{-\alpha}}}$. $Let \;\; v_i^{(k)} = \frac{1+(1-p^{(k)})(1-e^{-\lambda^{(k)} y_i^{-\alpha^{(k)}}})}{1-(1-p^{(k)})(1-e^{-\lambda^{(k)} y_i^{-\alpha^{(k)}}})}$,

now, the missing Z being replaced by $v_i^{(k)}$ in (6) and afterward, for M-step of EM algorithm, we obtain MLE over using log-likelihood the complete data distribution. Therefore, we can obtain the expressions for $\phi^{(k+1)} = (\lambda^{(k+1)}, \alpha^{(k+1)}, p^{(k+1)}, r^{(k+1)})^T$

*4.2. Method of Moment*

The method of moment estimators (MME) of the parameters are obtained with equating the first four raw moments of the WIWG distribution (4) to the corresponding raw moments of the sample. We require to solve the system of nonlinear equations that is

$$E_k = E(Y^k) = \frac{1}{n} \sum_{i=1}^{n} y_i^k \;\; ; \;\; k = 1, 2, 3, 4.$$

## 5. Simulation study

We supply the results of simulation study in two cases known and unknown $(p, r)$, for $r = 1, 2$ and $(\lambda, \alpha, p) = \{(1.3, 3.2, 0.1)\}$, with using 1000 iterations per sample size, $n = 50, 150, 600$. The conclusions have been curtailed after four decimal places. It is descried that MSEs often decrease while sample size increases and r decreases in case of both methods of estimations and also the rate convergence is faster in the EM algorithm than in the method of MME. Tables A.1. and A.2. in Appendix A exhibit the simulation results of WIWG distribution.

## 6. Conclusion

In this paper, we introduced WIWG distribution. It's reliability properties such as survival, hazard rate functions, and Rényi entropy are derived. The parameters of the distribution are estimated through MLE, EM-algorithm, and MME. The simulation results indicated that the MSEs of the estimations decrease when sample size increases and r decreases in case of both method of EM-algorithm and MME, and also the MSEs of the method of EM-algorithm is smaller than the method of MME.

# Appendix A

Table 1. Table A.1. The average bias and MSEs for $(\hat{\lambda}, \hat{\alpha})$using EM-algorithm and MME in case of known (p,r).

| EM-Algorithm | | | | | | | MME | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda, \alpha, p)$ | r | n | Bias($\hat{\lambda}$ ) | MSE($\hat{\lambda}$) | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | Bias($\hat{\lambda}$) | MSE($\hat{\lambda}$) | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) |
| (1.3,3.2,0.1) | 1 | 50 | -0.0757 | 0.0324 | -0.06139 | 0.4171 | 0.1897 | 0.1863 | 0.8339 | 0.839 |
| | | 150 | -0.0737 | 0.0133 | -0.6484 | 0.434 | 0.1563 | 0.0676 | 0.6789 | 0.5252 |
| | | 600 | -0.0715 | 0.0072 | -0.6608 | 0.44 | 0.1389 | 0.0303 | 0.5916 | 0.3787 |
| | 2 | 50 | -0.4458 | 0.2152 | 0.3458 | 0.1574 | 0.1077 | 0.2051 | 1.9189 | 3.8276 |
| | | 150 | -0.4488 | 0.2064 | 0.3137 | 0.1098 | 0.0431 | 0.0558 | 1.7676 | 3.1983 |
| | | 600 | -0.4488 | 0.2029 | 0.3023 | 0.0939 | 0.0087 | 0.0123 | 1.6713 | 2.8271 |

Table 2. Table A.2. The average bias and MSEs for $(\hat{p}, \hat{r})$using EM-algorithm and MME in case of known $(\lambda, \alpha)$.

| EM-Algorithm | | | | | | | MME | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $(\lambda, \alpha, p)$ | r | n | Bias($\hat{p}$ ) | MSE($\hat{p}$) | Bias($\hat{r}$) | MSE($\hat{r}$) | Bias($\hat{p}$) | MSE($\hat{p}$) | Bias($\hat{r}$) | MSE($\hat{r}$) |
| (1.3,3.2,0.1) | 1 | 50 | -0.0163 | 0.0012 | -0.1204 | 0.1584 | -0.0086 | 0.0016 | -0.1167 | 0.1071 |
| | | 150 | -0.0194 | 0.0006 | -0.0866 | 0.5174 | -0.0108 | 0.0006 | -0.0755 | 0.0345 |
| | | 600 | -0.0203 | 0.0004 | -0.0732 | 0.0205 | -0.0115 | 0.0002 | -0.0618 | 0.0142 |
| | 2 | 50 | -0.0449 | 0.0023 | -0.4774 | 0.5362 | -0 | 0.0082 | -0.5432 | 0.6459 |
| | | 150 | -0.0465 | 0.0022 | -0.4537 | 0.4359 | -0.0818 | 0.0081 | -0.5112 | 0.5398 |
| | | 600 | -0.047 | 0.0022 | -0.4434 | 0.3986 | -0.0904 | 0.0081 | 0.4986 | 0.5014 |

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# A Comparative Study of EM and Metaheuristic Algorithms for Student's t-Distribution Parameter Estimation

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**Abstract**

This study presents a comparative analysis of parameter estimation methods for the Student's t-distribution. We evaluate the performance of the classical Expectation-Maximization (EM) algorithm against three metaheuristic optimization algorithms: Particle Swarm Optimization (PSO), Differential Evolution (DE), and Moth-Flame Optimization (MFO). A comprehensive Monte Carlo simulation is conducted, generating samples with known parameters for two sample sizes ($n = 30$ and $n = 100$). Performance is rigorously assessed using the Root Mean Square Error (RMSE) metric across multiple replications. The results demonstrate a significant weakness in the EM algorithm, which exhibits high instability and error specifically in estimating the degrees of freedom parameter. In contrast, all three metaheuristic algorithms provide robust and accurate estimates for all parameters (location, scale, and degrees of freedom). The findings strongly suggest that metaheuristic global optimization techniques offer a more reliable and effective framework for t-distribution parameter estimation compared to the standard EM approach, especially when accurate tail inference is required.

## 1. Introduction

The Student's t-distribution is a fundamental tool in robust statistical analysis, prized for its heavier tails which effectively model data with outliers or high kurtosis, common in fields like finance and engineering[13]. However, its practical use is constrained by the complexity of estimating its parameters ($\mu$, $\sigma^2$, $\nu$) via Maximum Likelihood Estimation (MLE), which suffers from sensitivity to initial values, local optima, and numerical instability[2, 5]. The Expectation-Maximization (EM) algorithm has been widely adopted as a reliable iterative method for MLE, simplifying optimization via a scale mixture of normals representation. Nevertheless, it remains sensitive to initialization and can be slow[1, 8]. In contrast, metaheuristic optimization algorithms like Particle Swarm Optimization (PSO), Differential Evolution (DE), and Moth Flame Optimization (MFO) are effective for global search in complex, multimodal parameter spaces without requiring gradients[10, 11]. This study conducts a comprehensive comparative analysis of

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the EM algorithm and the three metaheuristics (PSO, DE, MFO) for estimating t-distribution parameters. Performance is evaluated via an extensive Monte Carlo simulation study with sample sizes $n = 30$ and $n = 100$, using Root Mean Square Error (RMSE) based on 30 replications per configuration. The remainder of the paper is organized as follows. Section 2 presents the theoretical background. Section 3 details the experimental design. Section 4 discusses the numerical results. Finally, Section 5 provides conclusions and future research directions.

## 2. Theoretical Framework and Preliminaries

*Student's t-Distribution: Core Concepts*

The Student's t-distribution is a fundamental probability model in robust statistics, widely employed to model data exhibiting heavier tails or outliers compared to the Gaussian distribution. For a random variable $X$, its probability density function (PDF) is characterized by three parameters: the location $\mu \in \mathbb{R}$, the scale $\sigma^2 > 0$, and the degrees of freedom $\nu > 0$. The PDF is given by:

$$f(x \mid \mu, \sigma^2, \nu) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi\sigma^2}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{1}{\nu}\frac{(x-\mu)^2}{\sigma^2}\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R}, \tag{1}$$

where $\Gamma(\cdot)$ denotes the gamma function. The parameter $\nu$ governs the tail weight; lower values produce heavier tails, and the distribution converges to $\mathcal{N}(\mu, \sigma^2)$ as $\nu \to \infty$. A representation pivotal for estimation is its formulation as a scale mixture of normals. Specifically, the distribution can be expressed hierarchically by introducing a latent variable $w > 0$:

$$w \sim \text{Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right), \tag{2}$$

$$X \mid w \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{w}\right). \tag{3}$$

Marginalizing over $w$ yields the t-density in (1). From this representation, the conditional expectation of the latent weight, given an observation and the parameters, is:

$$\mathbb{E}[w \mid x, \mu, \sigma^2, \nu] = \frac{\nu + 1}{\frac{(x-\mu)^2}{\sigma^2} + \nu}, \tag{4}$$

a key quantity utilized in the Expectation-Maximization algorithm [6, 7].

### *2.1. Maximum Likelihood Estimation and the EM Algorithm*

Given an independent and identically distributed sample $\mathbf{x} = (x_1, \ldots, x_n)$, the goal of maximum likelihood estimation (MLE) is to maximize the observed-data log-likelihood $\ell(\theta \mid \mathbf{x}) = \sum_{i=1}^{n} \log f(x_i \mid \mu, \sigma^2, \nu)$. Direct optimization is challenging due to the complex likelihood surface. The Expectation-Maximization (EM) algorithm provides an efficient iterative framework by leveraging the latent variable representation in equations (2) and (3).

The algorithm treats the latent weights $\mathbf{w} = (w_1, \ldots, w_n)$ as missing data. Starting from an initial estimate $\theta^{(t)} = (\mu^{(t)}, \sigma^{2(t)}, \nu^{(t)})$, iteration $t$ consists of two steps. In the E-step, the conditional expectation of each latent weight is computed using the current parameters:

$$\hat{w}_i^{(t)} = \mathbb{E}[w_i \mid x_i, \theta^{(t)}] = \frac{\nu^{(t)} + 1}{\frac{(x_i - \mu^{(t)})^2}{\sigma^{2(t)}} + \nu^{(t)}}. \tag{5}$$

Subsequently, the M-step updates the parameters by maximizing the expected complete-data log-likelihood, yielding the following closed-form updates for $\mu$ and $\sigma^2$:

$$\mu^{(t+1)} = \frac{\sum_{i=1}^{n} \hat{w}_i^{(t)} x_i}{\sum_{i=1}^{n} \hat{w}_i^{(t)}}, \tag{6}$$

$$\sigma^{2(t+1)} = \frac{1}{n} \sum_{i=1}^{n} \hat{w}_i^{(t)} \left(x_i - \mu^{(t+1)}\right)^2. \tag{7}$$

The degrees of freedom $\nu$ are updated by numerically solving the equation:

$$\log\left(\frac{\nu}{2}\right) + 1 - \psi\left(\frac{\nu}{2}\right) + \frac{1}{n} \sum_{i=1}^{n} \left(\hat{\eta}_i^{(t)} - \hat{w}_i^{(t)}\right) = 0, \tag{8}$$

where $\psi(\cdot)$ is the digamma function and $\hat{\eta}_i^{(t)}$ is the conditional expectation of $\log w_i$. The algorithm guarantees monotonic convergence to a local maximum of the likelihood function, though its performance and final estimates are sensitive to the choice of initial parameters $\theta^{(0)}$ [1, 3, 12]

### *2.2. Metaheuristic Optimization Algorithms*

To address the limitations of the EM algorithm, particularly its sensitivity to initialization and potential convergence to local optima, we employ three population-based metaheuristic algorithms for global optimization: Particle Swarm Optimization (PSO), Differential Evolution (DE), and Moth-Flame Optimization (MFO). These algorithms perform a stochastic search within the parameter space without requiring gradient information, making them suitable for maximizing the complex, non-convex log-likelihood function. In adapting these algorithms to our maximum likelihood estimation problem, the objective function for all three is the negative observed-data log-likelihood, $-\ell(\theta \mid \mathbf{x})$, derived from Eq. (1), which is to be minimized. The parameter vector is defined as $\theta = (\mu, \sigma^2, \nu)$, with the positivity constraints for $\sigma^2$ and $\nu$ handled internally within each algorithm's search mechanism. Particle Swarm Optimization (PSO) is inspired by the social behavior of bird flocking, where candidate solutions (particles) navigate the search space by adjusting their positions based on individual and collective best-found solutions. Differential Evolution (DE) is an evolutionary strategy that generates new candidate solutions by combining scaled differences between population members, promoting robust exploration. Moth-Flame Optimization (MFO) mimics the navigation of moths relative to light sources, with candidates moving in a spiral trajectory around the current best solutions to balance exploration and exploitation. Each algorithm offers a distinct mechanism for global search, providing a comparative basis for estimating the parameters of the Student's t-distribution beyond the local convergence characteristics of the EM algorithm [4, 9, 14].

## 3. Methodology

A comprehensive Monte Carlo simulation study is conducted to evaluate the performance of the four estimation algorithms. Data are generated from a Student's t-distribution with true parameters $\mu_0 = 0$, $\sigma_0^2 = 1$, and $\nu_0 = 5$, representing a case with moderately heavy tails. Two sample sizes are considered: $n = 30$ and $n = 100$. For each sample size, $R = 30$ independent datasets are generated, each denoted as $\mathbf{x}^{(r)} = (x_1^{(r)}, \ldots, x_n^{(r)})$ for $r = 1, \ldots, 30$. Each algorithm EM, PSO, DE, and MFO is applied to every generated dataset. To ensure a fair comparison, all metaheuristic algorithms are run for a fixed budget of $T = 500$ iterations. The EM algorithm is also run until convergence or until it reaches the same iteration limit. For a given dataset $r$ and algorithm $a$, the procedure yields a parameter estimate vector $\hat{\theta}^{(a,r)} = (\hat{\mu}^{(a,r)}, \hat{\sigma^2}^{(a,r)}, \hat{\nu}^{(a,r)})$. All algorithms are initialized using the same procedure for a given dataset: the metaheuristics are initialized randomly within a feasible range, while the EM algorithm is initialized with method-of-moments estimates derived from the sample. The estimation accuracy for each algorithm, sample size, and

parameter is evaluated using the Root Mean Square Error (RMSE). For a specific algorithm $a$, sample size $n$, and parameter $\theta$ (where $\theta$ is $\mu$, $\sigma^2$, or $\nu$), the RMSE is computed as:

$$\mathrm{RMSE}_{\theta}^{(a,n)} = \sqrt{\frac{1}{R}\sum_{r=1}^{R}\left(\hat{\theta}^{(a,r)} - \theta_0\right)^2}, \tag{9}$$

where $\hat{\theta}^{(a,r)}$ is the estimate from the $r$-th replication and $\theta_0$ is the true parameter value. A lower RMSE indicates higher estimation accuracy. The RMSE is calculated separately for each parameter, providing a detailed assessment of performance across location, scale, and degrees of freedom. In summary, the experimental design involves 4 algorithms, 2 sample sizes, and 30 replications, resulting in 240 individual algorithm executions. The RMSE values derived from these executions form the basis for the comparative analysis presented in the following section.

## 4. Results and Discussion

This section presents the numerical outcomes and analysis of the Monte Carlo simulation study designed to compare the estimation performance of the Expectation-Maximization (EM), Particle Swarm Optimization (PSO), Differential Evolution (DE), and Moth-Flame Optimization (MFO) algorithms for the parameters of the Student's t-distribution. The evaluation is based on the Root Mean Square Error (RMSE) metric, computed over $R = 30$ independent replications for two distinct sample sizes: $n = 30$ and $n = 100$. The data were generated from a t-distribution with true parameters $\mu_0 = 0$, $\sigma_0^2 = 1$, and $\nu_0 = 5$.

### *4.1. RMSE Analysis*

The RMSE values for each parameter, algorithm, and sample size are presented in Table 1. A primary observation from the table is the substantial error associated with the EM algorithm in estimating the degrees of freedom parameter $\nu$. Its RMSE for $\nu$ is 21.52 for $n = 30$ and increases to 24.99 for $n = 100$, indicating a severe and persistent estimation issue. In contrast, the three metaheuristic algorithms (PSO, DE, MFO) maintain RMSE values for $\nu$ below 4.2, demonstrating significantly greater stability. For the location parameter $\mu$, all algorithms achieve relatively low error levels. The EM and MFO algorithms show comparable accuracy for $n = 30$, while DE yields the lowest RMSE (0.0612). With the larger sample size ($n = 100$), PSO and DE produce the most accurate estimates of $\mu$. Regarding the scale parameter $\sigma^2$, all methods perform adequately, with RMSE values ranging from 0.25 to 0.45. The EM algorithm shows the lowest error for $\sigma^2$ at both sample sizes, while the metaheuristics exhibit slightly higher but consistent RMSE values.

Table 1. Root Mean Square Error (RMSE) for parameters $\mu$, $\sigma^2$, and $\nu$ by algorithm and sample size.

| | **n=30** | | | **n=100** | | |
|---|---|---|---|---|---|---|
| **Algorithm** | **RMSE□** | **RMSE□²** | **RMSE□** | **RMSE□** | **RMSE□²** | **RMSE□** |
| EM | 0.1061 | 0.2539 | 21.5243 | 0.1309 | 0.2490 | 24.9929 |
| PSO | 0.1859 | 0.3623 | 4.1902 | 0.0911 | 0.2445 | 3.4597 |
| DE | 0.0612 | 0.4480 | 4.0872 | 0.0969 | 0.2578 | 2.5970 |
| MFO | 0.1052 | 0.3055 | 3.6769 | 0.1001 | 0.2186 | 3.4070 |

### *4.2. Distribution of Estimation Errors*

The variability and robustness of the absolute estimation errors across the 30 replications are analyzed to assess the stability of each algorithm. For the location parameter $\mu$, all algorithms exhibit low median errors and tight interquartile ranges, indicating consistent estimation accuracy. The distribution of errors for the scale parameter $\sigma$ is similarly compact for all methods, with no algorithm showing a propensity for extreme outliers. The most revealing behavior appears for the degrees of freedom $\nu$. The estimation errors for the EM algorithm frequently exceed 20, which reflects substantial instability and aligns with the high RMSE values reported in Table 1. Conversely, the estimation errors

for PSO, DE, and MFO remain much smaller, typically below 5, and display significantly fewer extreme deviations. The error variability for these algorithms also decreases for the larger sample size ($n = 100$), indicating improved precision with more data.

### *4.3. Multi-Criteria Performance Visualization*

For a holistic comparison that considers all three parameters simultaneously, a combined multi-criteria assessment is performed. This assessment jointly evaluates the RMSE values of $\mu$, $\sigma^2$, and $\nu$, providing an overall perspective on the error profile of each algorithm. Across all three parameters, the EM algorithm exhibits substantially larger combined error levels compared to the metaheuristics. Among the metaheuristics, the overall error profile for the larger sample size ($n = 100$) is generally more compact and balanced, indicating improved and more consistent performance with increased data. The relative performance patterns of PSO, DE, and MFO differ, suggesting variations in their error trade-offs across the three parameters. For instance, one algorithm may yield slightly larger errors in $\sigma^2$ while producing smaller errors in $\nu$. The collective evidence from the numerical RMSE results and the multi-criteria error assessment consistently indicates a pronounced weakness in the EM algorithm's ability to estimate the tail heaviness parameter $\nu$ under the studied conditions. The metaheuristic algorithms, in contrast, demonstrate stable and accurate estimation across all parameters, with their performance generally improving as the sample size increases.

## 5. Conclusion

This study conducted a comprehensive comparative analysis of four algorithms for estimating the parameters of the Student's t-distribution: the classical Expectation-Maximization (EM) algorithm and three metaheuristic approaches Particle Swarm Optimization (PSO), Differential Evolution (DE), and Moth-Flame Optimization (MFO). The evaluation, based on an extensive Monte Carlo simulation, revealed a fundamental limitation of the EM algorithm: its pronounced instability and high error in estimating the degrees of freedom parameter $\nu$, which governs the tail heaviness. In contrast, the metaheuristic algorithms demonstrated robust and accurate estimation across all three parameters $\mu$, $\sigma^2$, and $\nu$ for both small and moderate sample sizes. Their global search capability effectively navigates the complex likelihood surface, avoiding the local optima that often trap the EM algorithm. The findings robustly indicate that metaheuristic optimization algorithms are a superior and more reliable alternative to the EM algorithm for maximum likelihood estimation of t-distribution parameters, especially when accurate inference about tail behavior is critical. For practitioners, this suggests adopting methods such as DE or MFO, which showed the best balanced performance in this study, can lead to more stable and trustworthy models in the presence of heavy-tailed data. Future research could extend this work in several directions. Investigating the performance of these algorithms with real world, high dimensional datasets would provide valuable insights into their practical scalability. Furthermore, adapting advanced variants of these metaheuristics, such as hybrid or adaptive parameter control strategies, may yield even greater estimation efficiency. Finally, exploring the application of this comparative framework to other complex, non-Gaussian statistical models presents a promising avenue for methodological advancement.

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# An Adjoint Theory for Affine Lipschitz Mappings in $CAT(0)$ Spaces

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**Abstract**

In this paper, we develop an adjoint theory for a class of affine Lipschitz mappings in $CAT(0)$ spaces. Using the fact that real-valued geodesically convex Lipschitz functions which are additive along geodesic concatenations admit representations in terms of elements of the linear dual space, we are motivated to introduce an adjoint notion for such mappings. In addition, we establish a comparison between the proposed adjoint notion and the classical adjoint operator in the setting of Hilbert spaces.

## 1. Introduction and Preliminaries

A *geodesic path* in the metric space $(X,d)$ from $a \in X$ to $b \in X$ is a continuous mapping $\gamma_{a,b} : [0,1] \to X$ such that $\gamma_{a,b}(0) = a, \gamma_{a,b}(1) = b$ and $d(\gamma_{a,b}(t), \gamma_{a,b}(s)) = |t-s|d(a,b)$, for all $t,s \in [0,1]$. A *geodesic space* is a metric space $(X,d)$ such that every two points $a,b \in X$ are connected by a geodesic path. We use $\mathcal{G}eo(X)$ to denote the set of all geodesic path contained in $X$. The geodesic space $(X,d)$ is called a $CAT(0)$ *space* if for all $a,b,c \in X$ and $t \in [0,1]$ the following inequality, so-called *CN-inequality*,

$$d(c,\gamma_{a,b}(t))^2 \leq (1-t)d(c,a)^2 + td(c,b)^2 - t(1-t)d(a,b)^2 \text{ for all } a,b,c \in X, t \in [0,1]. \tag{1}$$

In a $CAT(0)$ space $X$, any geodesic path from $a \in X$ to $b \in X$ is denoted by $\gamma_{a,b}(t) := (1-t)a \oplus tb$, whenever $t \in [0,1]$. The *geodesic segment* joining $a \in X$ to $b \in X$ is defined by $[\![a,b]\!] := \{\gamma_{a,b}(t) : t \in [0,1]\}$. An *Hadamard space* is a complete $CAT(0)$ space. Important examples of Hadamard spaces include Hilbert spaces, Hadamard manifolds, $\mathbb{R}$-trees, nonlinear Lebesgue spaces, Hilbert balls. We denote by $X^2 := \{\overrightarrow{xy} : x,y \in X\}$ is the set of all *bound vectors*. The *dual space* of $X$, denoted by $X^*$, was introduced in [1] via the notion of quasilinearization in abstract metric spaces. Briefly, $X^*$ consists of all equivalence classes $x^* = [t\overrightarrow{uv}]$, where $(t,u,v) \in \mathbb{R} \times X^2$, such that $\langle x^*, \overrightarrow{ab}\rangle := \frac{t}{2}\big(d(u,b)^2 + d(v,a)^2 - d(u,a)^2 - d(v,b)^2\big)$ ,whenever $\overrightarrow{ab} \in X^2$. Moreover, the element $\mathbf{0} := [\overrightarrow{uu}]$ represents the zero of $X^*$. The *linear dual space* of $X$ is defined by $X^\diamond := \operatorname{lin} X^*$. Indeed,

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$X^{\diamond} = \{\sum_{i=1}^{n} \alpha_i x_i^* : \alpha_i \in \mathbb{R}, x_i^* \in X^*, n \in \mathbb{N}\}$. Moreover, $X^{\diamond}$ acts on $X^2$; $\langle \sum_{i=1}^{n} \alpha_i x_i^*, \overrightarrow{xy} \rangle := \sum_{i=1}^{n} \alpha_i \langle x_i^*, \overrightarrow{xy} \rangle$ where $x, y \in X$. It is easy to verify that the following mapping is a norm on $X^{\diamond}$;

$$\|x^{\diamond}\|_{\diamond} := \sup \left\{ \frac{|\langle x^{\diamond}, \overrightarrow{ab} \rangle - \langle x^{\diamond}, \overrightarrow{cd} \rangle|}{d(a,b) + d(c,d)} : a, b, c, d \in X, (a,c) \neq (b,d) \right\}. \tag{2}$$

Also, It is known [6, Remark 2.2] that the mapping $\| \cdot \|_{\blacklozenge} : X \to \mathbb{R}$ is given by

$$\|x^{\diamond}\|_{\blacklozenge} = \sup \left\{ \frac{|\langle x^{\diamond}, \overrightarrow{ab} \rangle|}{d(a,b)} : a, b \in X, a \neq b \right\}. \tag{3}$$

defines a norm on $X^{\diamond}$ wich is equivalent to original norm (2);
One can see that for each $a, b, w \in X$ and each $x^{\diamond} \in X^{\diamond}$

$$\langle x^{\diamond}, \overrightarrow{ab} \rangle = \langle x^{\diamond}, \overrightarrow{aw} \rangle + \langle x^{\diamond}, \overrightarrow{wb} \rangle. \tag{4}$$

We refer to [1–3] for the details of the background of $CAT(0)$ spaces.

**Definition 1.1.** Let $(X, d)$ and $(Y, \rho)$ are two metric spaces.

(i) [2, Section 2.2] Let $X$ be a $CAT(0)$ space. A function $f : X \to ]-\infty, \infty]$ is called *convex* (or *geodesically convex*) if

$$f((1-\lambda)x \oplus \lambda y) \leq (1-\lambda) f(x) + \lambda f(y), \quad x, y \in X, \lambda \in [0,1]. \tag{5}$$

Morover, $f$ is *affine* if equality holds in (5).
In addition, let $X$ and $Y$ are $CAT(0)$ spaces. A mapping $f : X \to Y$ is said to be a *geodesically affine* (or *affine*) if the following holds:

$$f((1-t)a \oplus_X tb) = (1-t) f(a) \oplus_Y t f(b), \quad \forall a, b \in X, \ t \in [0,1].$$

The set of all affine mapping from $X$ to $Y$ is denoted by $\mathrm{Aff}(X, Y)$.

(ii) Consider the mapping $f : X \to Y$. We say that $f : X \to Y$ is called *Lipschitz* (or *Lipschitzian*) if there is $L > 0$ such that $\rho(f(x), f(y)) \leq L d(x, y)$, for all $x, y \in X$. The set of all Lipschitz functions from $X$ to $Y$ is denoted by $\mathrm{Lip}(X, Y)$.
Recall that a base point (or distinguished point) in a metric space $(X, d)$ is a fixed element $p \in X$. A metric space with a specified base point $p$ is called a *pointed metric space* which that is denoted by $(X, d, p)$. If $X, Y$ are pointed metric spaces with base points $p$ and $q$, respectively, then one denotes by

$$\mathrm{Lip}_0(X, Y) := \{ f : X \to Y : \ f \text{ is Lipschitz on } X, \ f(p) = q \}.$$

When more precision is required, one writes $\mathrm{Lip}_p(X, Y)$. Also, we put $\mathrm{Lip}_p(X) := \mathrm{Lip}_p(X, X)$. In special case $Y := \mathbb{R}$, we put $\mathrm{Lip}_0(X) := \mathrm{Lip}_0(X, \mathbb{R})$. One can see that $(\mathrm{Lip}_0(X), \| \cdot \|_{\mathrm{Lip}})$ is a normed space, where $\|f\|_{\mathrm{Lip}} := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x,y)}$.

In addition, we use the following notations;

$$\mathrm{AL}(X, Y) := \mathrm{Aff}(X, Y) \cap \mathrm{Lip}(X, Y), \text{ and } \mathrm{AL}(X) := \mathrm{AL}(X, X).$$

## 2. Main Results

**Definition 2.1.** Let $X$ be a $CAT(0)$ space and let $\delta : \mathcal{G}eo(X) \to \mathbb{R}$, $\gamma_{x,z} \mapsto \delta(\gamma_{x,z})$ be a real-valued function. if there exists $f \in \mathrm{Lip}_0(X, \mathbb{R})$ such that $\delta(\gamma_{x,z}) = f(z) - f(x)$, in this case $\delta$ is called *increment of $f$ along the geodesic path from $x$ to $z$*. For notational convenience, we may also write $\delta_f(\gamma_{x,z})$ as $f(x \to z)$, i.e. $f(x \to z) := \delta_f(\gamma_{x,z})$.
It is immediate that

$$f(x \to z) = f(x \to y) + f(y \to z), \ x, z \in X, \ y \in [\![x, z]\!].$$

In this case, the mapping $f$ is called *additive along geodesic concatenation*.

**Example 2.2.** (i) Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a real Hilbert space and let $v \in \mathcal{H}$ be fixed. Consider the mapping $\delta_v : \mathcal{G}eo(\mathcal{H}) \to \mathbb{R}$, $\gamma_{x,z} \mapsto \langle v, z-x\rangle_{\mathcal{H}}$ and for each $x \in \mathcal{H}$, set $f_v(x) := \langle v, x\rangle_{\mathcal{H}}$. It follows from $f_v(0_{\mathcal{H}}) = \langle v, 0_{\mathcal{H}}\rangle_{\mathcal{H}} = 0$ and Cauchy-Schwarz inequality that $f_v \in \mathrm{Lip}_0(X,\mathbb{R})$. Moreover,

$$\delta_v(\gamma_{x,z}) = \langle v, z-x\rangle_{\mathcal{H}} = \langle v, z\rangle_{\mathcal{H}} - \langle v, x\rangle_{\mathcal{H}} = f_v(z) - f_v(x).$$

Thus $\delta_v$ is increment of $f$ along the geodesic path from $x$ to $z$.

(ii) It should also be noted that any mapping $\delta : \mathcal{G}eo(X) \to \mathbb{R}$ cannot induce by a $f \in \mathrm{Lip}_0(X,\mathbb{R})$. To see this, consider the mapping $\delta : \mathcal{G}eo(X) \to \mathbb{R}$ by $\delta(\gamma_{x,z}) = d(x,z)$ and suppose that there exists $f \in \mathrm{Lip}_0(X,\mathbb{R})$ such that $\delta(\gamma_{x,z}) = f(z) - f(x)$. Now assume that $x \neq z$. It follows from $d(x,z) = f(z) - f(x)$ and $d(z,x) = f(x) - f(z)$ that $f(x) = f(z)$. Consequently, $d(x,z) = 0$ and so $x = z$, which is a contradiction.

**Remark 2.3.** Let $X$ be a $CAT(0)$ space, $x^\diamond \in X^\diamond$ and $p \in X$ be fixed. Define the mapping $\tau_{p,x^\diamond} : X \to \mathbb{R}$; $x \mapsto \langle x^\diamond, \overrightarrow{px}\rangle$. One can easily see that for each $p, q \in X$, the mapping $\tau_{p,x^\diamond}$ is geodesically convex if and only if $\tau_{q,x^\diamond}$ so is. To see this, it suffices to show that the convexity of $\tau_{p,x^\diamond}$ implies that $\tau_{q,x^\diamond}$ is convex. Let $a, b \in X$, $\lambda \in [0,1]$ and set $\gamma_{a,b}(\lambda) := (1-\lambda)a \oplus \lambda b$. Then

$$\begin{aligned}\tau_{q,x^\diamond}(\gamma_{a,b}(\lambda)) &= \left\langle x^\diamond, \overrightarrow{q(\gamma_{a,b}(\lambda))}\right\rangle = \left\langle x^\diamond, \overrightarrow{p(\gamma_{a,b}(\lambda))}\right\rangle - \langle x^\diamond, \overrightarrow{pq}\rangle = \tau_{p,x^\diamond}(\gamma_{a,b}(\lambda)) - \langle x^\diamond, \overrightarrow{pq}\rangle \\ &= \tau_{p,x^\diamond}(\gamma_{a,b}(\lambda)) - (\langle x^\diamond, \overrightarrow{pa}\rangle - \langle x^\diamond, \overrightarrow{qa}\rangle) \\ &\leq (1-\lambda)\tau_{p,x^\diamond}(a) + \lambda\tau_{p,x^\diamond}(b) - (\langle x^\diamond, \overrightarrow{pa}\rangle - \langle x^\diamond, \overrightarrow{qa}\rangle) \\ &= (1-\lambda)\tau_{p,x^\diamond}(a) + \lambda\tau_{p,x^\diamond}(b),\end{aligned}$$

which shows that $\tau_{q,x^\diamond}$ is convex. Therefore, the convexity of $\tau_{p,x^\diamond}$ is independent of the choice of $p \in X$. Moreover, straightforward calculations show that for each $x, y, z \in X$ be such that $y \in [\![x,z]\!]$;

$$\tau_{p,x^\diamond}(x \to z) = \tau_{p,x^\diamond}(x \to y) + \tau_{p,x^\diamond}(y \to z),$$

thus $\tau_{p,x^\diamond}$ is additive along geodesic concatenation. Now, suppose that $u, v \in X$. By using Remark 4 and (3) we get

$$\left|\tau_{p,x^\diamond}(v) - \tau_{p,x^\diamond}(u)\right| = |\langle x^\diamond, \overrightarrow{pv}\rangle - \langle x^\diamond, \overrightarrow{pu}\rangle| = |\langle x^\diamond, \overrightarrow{uv}\rangle| \leq \|x^\diamond\|_\bullet d(u,v),$$

and since $\tau_{p,x^\diamond}(p) = \langle x^\diamond, \overrightarrow{pp}\rangle = 0$ we conclude that $\tau_{p,x^\diamond} \in \mathrm{Lip}_0(X,\mathbb{R})$.
Now, suppose that $A$ is a nonempty subset of a $X$. We define

$$\mathcal{C}_\tau(A) := \left\{x^\diamond \in X^\diamond : \tau_{x^\diamond} \text{ is geodesically convex in } A\right\}. \tag{6}$$

One can check that $\mathcal{C}_\tau(A)$ is a positive cone in $X^\diamond$. It will be shown that $\mathcal{C}_\tau(X)$ is a vector space only when $X$ is flat.

**Theorem 2.4.** [4, Theorem 2.9] *Let $(X,d)$ be an Hadamard space with the linear dual space $X^\diamond$ and $p \in X$ be arrbitararay and fixed element. For a function $f : X \to \mathbb{R}$, the following assertions are equivalent:*

*(a) There exists an element $x^\diamond \in X^\diamond$ such that $f(\cdot) = \langle x^\diamond, \overrightarrow{p\cdot}\rangle$ and is geodesically convex.*
*(b) (i) The function $f$ is geodesically convex and $f \in \mathrm{Lip}_0(X,\mathbb{R})$.*
*(ii) $f$ is additive along geodesic concatenation on $X$.*

*Moreover, if such an $x^\diamond$ exists, it is unique.*

**Definition 2.5.** Given $F \in \mathrm{AL}(X,Y)$, in which $(X,d)$ and $(Y,\rho)$ are $CAT(0)$ spaces with $p \in Y$ be arbitrary and fixed, such that $F^{-1}(\{p\}) \neq \emptyset$, we define a mapping $F^{p,\sharp} : \mathcal{C}_\tau(Y) \to X^\diamond$ as follows:
For each $y^\diamond \in \mathcal{C}_\tau(Y)$,

$$F^{p,\sharp} := \tau_{p,y^\diamond} \circ F, \quad y^\diamond \mapsto F^{p,\sharp}(y^\diamond). \tag{7}$$

We refer to $F^{p,\sharp}$ as the *p-adjoint* of $F$. Indeed, $F^{p,\sharp}(y^\diamond)(x) := \left\langle y^\diamond, \overrightarrow{p(F(x))}\right\rangle$, $x \in X$.

**Remark 2.6.** One can see that $F^{p,\sharp}(y^\diamond)$ is really in $X^\diamond$. To see this, it follows from $F \in \mathrm{AL}(X,Y)$ and $\tau_{p,y^\diamond} \in \mathcal{C}_\tau(Y)$ that for each for each $a, b \in X$ and each $\lambda \in [0,1]$,

$$\begin{aligned} F^{p,\sharp}(y^\diamond)((1-\lambda)a \oplus_X \lambda b) &= \langle y^\diamond, \overrightarrow{p(F((1-\lambda)a \oplus_X \lambda b))}\rangle = \langle y^\diamond, \overrightarrow{p((1-\lambda)F(a) \oplus_Y \lambda F(b))}\rangle \\ &\le (1-\lambda)\langle y^\diamond, \overrightarrow{p(F(a))}\rangle + \lambda\langle y^\diamond, \overrightarrow{p(F(b))}\rangle \\ &\le (1-\lambda)F^{p,\sharp}(y^\diamond)(a) + \lambda F^{p,\sharp}(y^\diamond)(b), \end{aligned}$$

which shows that the mapping $F^{p,\sharp}(y^\diamond) : X \to \mathbb{R}$ is geodesically convex. On the other hand, $F$ is a Lipschitz mapping and so there exists $L > 0$ such that for each $u, v \in X$,

$$\rho(F(v), F(u)) \le L d(u,v). \tag{8}$$

Moreover, by using (3) and (8) we obtain that

$$|F^{p,\sharp}(y^\diamond)(u) - F^{p,\sharp}(y^\diamond)(v)| = |\langle y^\diamond, \overrightarrow{pF(u)}\rangle - \langle y^\diamond, \overrightarrow{pF(v)}\rangle| = |\langle y^\diamond, \overrightarrow{F(v)F(u)}\rangle| \le \|y^\diamond\|_\blacklozenge \rho(F(v),F(u)) \le \|y^\diamond\|_\blacklozenge L d(u,v).$$

Now, taking the supremum over all $u \neq v$ gives

$$\|F^{p,\sharp}(y^\diamond)\|_{\mathrm{Lip}} \le L\|y^\diamond\|_\blacklozenge < \infty, \tag{9}$$

therefore, $F^{p,\sharp}(y^\diamond)$ is a Lipschitz mapping. In addition, it follows from $F^{-1}(\{p\}) \neq \emptyset$ that there exists $a \in X$ such that $F(a) = p$. Hence, $F^{p,\sharp}(y^\diamond)(a) = \langle y^\diamond, \overrightarrow{p(F(a))}\rangle = \langle y^\diamond, \overrightarrow{pp}\rangle = 0$. Consequently, $F^{p,\sharp}(y^\diamond) \in \mathrm{Lip}_0(X,\mathbb{R})$. Now, the Theorem 2.4 guarantees that $F^{p,\sharp}(y^\diamond) \in X^\diamond$ and so the mapping $F^{p,\sharp}$ is well-defined.

**Remark 2.7.** [5, Definition 2.2 & Corollary 2.4] Let $\Omega$ be a closed and convex subset of an Hadamard space $(X,d)$ with a non-empty interior. Let $p \in \mathrm{Int}(\Omega)$, for some $\epsilon > 0$, $B_\epsilon[p] := \{x \in X : d(x,p) \le \epsilon\} \subseteq \Omega$. Set $\lambda := \frac{d(p,x)}{\epsilon}$ and define the mapping $T_{p,\epsilon} : X \longrightarrow \Omega^*$ by

$$T_{p,\epsilon}(x) := \begin{cases} \left[\frac{1}{\lambda}\overrightarrow{p\,((1-\lambda)p \oplus \lambda x)}\right] & 0 < \lambda \le 1, \\ \mathbf{0} & \lambda = 0, \\ \left[\lambda\overrightarrow{p\,((1-\frac{1}{\lambda})p \oplus \frac{1}{\lambda}x)}\right] & \lambda \ge 1. \end{cases} \tag{10}$$

Then, $X$ is flat if and only if $T_{p,\epsilon}(x) = [\overrightarrow{px}]$, for each $p, x \in X$.

**Theorem 2.8.** *Let $X$ be a flat Hadamard space, $p \in X$ fixed and $F \in \mathrm{AL}(X)$. Then the following hold:*

*(i) $\mathcal{C}_\tau(X) = X^\diamond$. In special case, $\mathcal{C}_\tau(\mathcal{H}) = \mathcal{H}$, where $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ is the real Hilbert space.*

*(ii) The mapping $F^{p,\sharp} : X^\diamond \to X^\diamond$ is a linear operator.*

*(iii) If $L > 0$ is the Lipschitz constant of $F$. Then, for each $x^\diamond \in X^\diamond$;*

$$\|F^{p,\sharp}(x^\diamond)\|_{\mathrm{Lip}} \le L\|x^\diamond\|_\blacklozenge. \tag{11}$$

*Proof.* (i) Let $X$ be a flat Hadamard space. It suffices to show that $X^\diamond \subseteq \mathcal{C}_\tau(X)$. Suppose that $x^\diamond \in X^\diamond$, $a, b \in X$ and $\lambda \in [0,1]$. By using Remark 2.7 and (4) we obtain that

$$\begin{aligned} \tau_{p,x^\diamond}((1-\lambda)a \oplus \lambda b)) &= \langle x^\diamond, \overrightarrow{p((1-\lambda)a \oplus \lambda b)}\rangle = \langle x^\diamond, \overrightarrow{pa} + \overrightarrow{a((1-\lambda)a \oplus \lambda b)}\rangle = \langle x^\diamond, \overrightarrow{pa}\rangle + \langle x^\diamond, \overrightarrow{a((1-\lambda)a \oplus \lambda b)}\rangle \\ &= \langle x^\diamond, \overrightarrow{pa}\rangle + \lambda\langle x^\diamond, T_{a,\epsilon}(b)\rangle = \langle x^\diamond, \overrightarrow{pa}\rangle + \lambda\langle x^\diamond, \overrightarrow{ab}\rangle = \langle x^\diamond, \overrightarrow{pa}\rangle + \lambda\langle x^\diamond, \overrightarrow{pb}\rangle - \lambda\langle x^\diamond, \overrightarrow{pa}\rangle \\ &= (1-\lambda)\langle x^\diamond, \overrightarrow{pa}\rangle + \lambda\langle x^\diamond, \overrightarrow{pb}\rangle = (1-\lambda)\tau_{p,x^\diamond}(a) + \lambda\tau_{p,x^\diamond}(b), \end{aligned}$$

which shows that $\tau_{p,x^\diamond} \in \mathrm{Aff}(X,\mathbb{R})$. So, $\tau_{p,x^\diamond}$ is a convex mapping. Thus, $x^\diamond \in \mathcal{C}_\tau(X)$ as required.
(ii) Let $a^\diamond, b^\diamond \in X^\diamond$ and $\alpha,\beta \in \mathbb{R}$. we can see that, for each $x \in X$;

$$F^{p,\sharp}(\alpha a^\diamond + \beta b^\diamond)(x) = \langle \alpha a^\diamond + \beta b^\diamond, \overrightarrow{p(F(x))}\rangle = \alpha\langle a^\diamond, \overrightarrow{p(F(x))}\rangle + \beta\langle b^\diamond, \overrightarrow{p(F(x))}\rangle = \alpha F^{p,\sharp}(a^\diamond)(x) + \beta F^{p,\sharp}(b^\diamond)(x).$$

So, $F^{p,\sharp}(\alpha a^\diamond + \beta b^\diamond) = \alpha F^{p,\sharp}(a^\diamond) + \beta F^{p,\sharp}(b^\diamond)$. Consequently, $F^{p,\sharp}$ is a linear operator
(iii) This is an immediate consequence of Remark 2.6. □

**Remark 2.9.** Assume that $A$ is a closed convex subset of the real Hilbert space $(\mathcal{H}, \langle\cdot\,|\,\cdot\rangle_{\mathcal{H}})$. If $\mathrm{Int}(A) \neq \emptyset$, then $A^* = \mathcal{H}$. More precisely, if $z \in \mathrm{Int}(A)$, then there is $\epsilon > 0$ such that

$$\{u \in \mathcal{H} : \|u - z\| \leq \epsilon\} \subseteq A.$$

Define $j_z : \mathcal{H} \to A^*$ by

$$j_z(a) := \begin{cases} \frac{\|a\|}{\epsilon}\left[\overrightarrow{zz_a}\right] & a \neq 0, \\ [\mathbf{0}_{\overrightarrow{aa}}] & a = 0, \end{cases} \qquad (a \in \mathcal{H}) \tag{12}$$

where $z_a := z + \frac{\epsilon a}{\|a\|}$. By [7, Remark 2.4], we get $j_z$ is a surjective isomorphism. Note that, if $A = \mathcal{H}$ then by taking $z = 0$ we get $j_0(a) = \left[\overrightarrow{0a}\right]$ for any $a \in \mathcal{H}$. So, in the sense of isomorphism, we have $\mathcal{H}^* = \mathcal{H}^\diamond = \mathcal{H}$.

**Example 2.10.** Let $(\mathcal{H}, \langle\cdot\,|\,\cdot\rangle_{\mathcal{H}})$ be a real Hilbert space and $p \in \mathcal{H}$ be fixed and arbitrary. Consider the affine map $F : \mathcal{H} \to \mathcal{H}$, $x \mapsto Ax + b$ in which $A : \mathcal{H} \to \mathcal{H}$ is a linear operator and $b \in \mathcal{H}$ be fixed. In view of Remark 2.9, surjectivity of $j_{0_{\mathcal{H}}}$ implies that there exists $u \in \mathcal{H}$ for which $j_{0_{\mathcal{H}}}(u) = x^\diamond$. Then

$$F^{p,\sharp}(x^\diamond)(x) = \langle x^\diamond, \overrightarrow{p(F(x))}\rangle = \langle j_{0_{\mathcal{H}}}(u), \overrightarrow{p(F(x))}\rangle = \langle u, F(x) - p\rangle_{\mathcal{H}} = \langle u, Ax + b - p\rangle_{\mathcal{H}} = \langle u, Ax\rangle_{\mathcal{H}} + \langle u, b - p\rangle_{\mathcal{H}}. \tag{13}$$

Therefore,

$$\left\langle j_{0_{\mathcal{H}}}(u), \overrightarrow{p(F(x))}\right\rangle = \langle u, Ax\rangle_{\mathcal{H}} + \langle u, b - p\rangle_{\mathcal{H}} \tag{14}$$

As a special case of (14), let $p := 0_{\mathcal{H}}$, $b := 0_{\mathcal{H}}$ and let $A \in \mathfrak{B}(\mathcal{H})$, where $\mathfrak{B}(\mathcal{H})$ denotes the space of bounded linear operators on $\mathcal{H}$, we obtain that

$$\left\langle j_{0_{\mathcal{H}}}(u), \overrightarrow{0_{\mathcal{H}}(Ax)}\right\rangle = \langle u, Ax\rangle_{\mathcal{H}}.$$

Now apply $\langle u, Ax\rangle_{\mathcal{H}} = \langle A^*u, x\rangle_{\mathcal{H}}$, where $A^*$ denotes the adjoint of $A$, in the sense of adjoint in $\mathfrak{B}(\mathcal{H})$, which yields $\left\langle j_{0_{\mathcal{H}}}(u), \overrightarrow{0_{\mathcal{H}}(Ax)}\right\rangle = \langle A^*u, x\rangle_{\mathcal{H}}$ we conclude that $\left\langle j_{0_{\mathcal{H}}}(u), \overrightarrow{0_{\mathcal{H}}(Ax)}\right\rangle = \langle x, A^*u\rangle_{\mathcal{H}}$. Finally, (13) implies that

$$A^{0_{\mathcal{H}},\sharp}(j_{0_{\mathcal{H}}}(u))(x) = \langle A^*u, x\rangle_{\mathcal{H}}.$$

## 3. Conclusion

In this work, we developed a notion of adjoint for affine Lipschitz mappings between $CAT(0)$ spaces based on the dual structure induced by quasilinearization. The central tool was a representation theorem characterizing geodesically convex Lipschitz functions that are additive along geodesic concatenations via unique elements of the linear dual space. This representation enabled the introduction of the $p$-adjoint as a natural dual operator associated with affine Lipschitz mappings and ensured its well-definedness under mild assumptions.
Furthermore, we demonstrated that the geometric properties of the underlying space are essential: the cone of dual elements inducing geodesically convex functions coincides with the entire linear dual space if and only if the Hadamard space is flat. In this case, the $p$-adjoint reduces to a bounded linear operator satisfying a Lipschitz-type norm estimate. Finally, the Hilbert space framework shows that the proposed adjoint recovers the classical adjoint operator, thereby confirming the coherence and generality of the developed theory.

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# Module Left Biflatness for Banach Algebras Related to Characters

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**Abstract**

Amenability of Banach algebras is a fundamental concept in harmonic analysis. Over time, the notions of biflatness, $\varphi$-biflatness, and left $\varphi$-biflatness (where $\varphi$ is a character on the Banach algebra) were introduced and studied, such that each concept is weaker than the previous one, and various results clarifying the relationships among these concepts were obtained. In this paper, we extend the notion of left $\varphi$-biflatness to the module setting, which is a new and important concept, for the first time. Several new results for this generalized notion are established, and its connections with the earlier concepts are investigated.

## 1. Introduction

Amenability of Banach algebras [2] has long been recognized as a central concept in harmonic analysis. A Banach algebra $A$ is called *amenable* whenever for every Banach $A$-bimodule $X$, every continuous derivation from $A$ into $X^*$ is inner. Building upon this foundational notion, the concept of *biflatness* [1] was introduced to provide a deeper understanding of the homological structure of Banach algebras and their modules. Further refinements, such as *$\varphi$-biflatness* [5] and *left $\varphi$-biflatness* [4], represent weaker but often more flexible properties, allowing the study of modules and algebras under less restrictive conditions.

In this paper, we take a further step in this direction by focusing on left $\varphi$-biflatness in the module setting, which is a new and important generalization. We first establish results linking the biflatness properties of a Banach $\mathfrak{A}$-bimodule $A$ with those of its second dual $A^{**}$, providing insight into how these properties are preserved under duality. Next, we explore the inheritance of module left $(\phi, \varphi)$-biflatness by closed ideals, illustrating how structural properties can be transferred to submodules. Finally, we investigate how biflatness behaves under direct sums of modules, demonstrating the stability of this property under standard algebraic operations.

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## 2. Preliminaries

### 2.1. Classical Notions Biflatness

Let $A \widehat{\otimes} A$, be the projective tensor product, and let $\omega_A : A \widehat{\otimes} A \to A$ be the canonical tensor map defined by $\omega_A(a \otimes b) = ab$. The space $A \widehat{\otimes} A$ becomes a Banach $A$-bimodule via

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

It is known that the Banach algebra $A$ is amenable if and only if $A$ has a *virtual diagonal*; that is, there exists an element $m \in (A \widehat{\otimes} A)^{**}$ such that

$$a \cdot m = m \cdot a, \qquad \omega_A^{**}(m)a = a \quad (a \in A).$$

In homological concepts, important notions such as *biflatness* arise, which are directly related to amenability. The notion of biflatness of Banach algebras was introduced by Helemskii in [1]. Later, he investigated its details and proved that a Banach algebra $A$ is amenable if and only if $A$ is biflat and has a bounded approximate identity.
Let $X$ and $Y$ be Banach $A$-bimodules. A bounded linear map $T : X \to Y$ is called an *$A$-bimodule homomorphism* if

$$T(a \cdot x) = a \cdot T(x), \qquad T(x \cdot a) = T(x) \cdot a \quad (a \in A,\ x \in X).$$

Let $A$ be a Banach algebra and $\varphi \in \Delta(A)$, where $\Delta(A)$ denotes the set of all nonzero multiplicative linear functionals on $A$. Throughout this paper, for each $\varphi \in \Delta(A)$, the extension $\widehat{\varphi} \in \Delta(A^{**})$ is defined by $\widehat{\varphi}(F) = F(\varphi)$ for all $F \in A^{**}$. The following definition contains concepts, each of which is an extension of the previous one, and they are presented in order based on references [3], [5], and [4].

**Definition 2.1.** Let $A$ be a Banach algebra and $\varphi \in \Delta(A) \cup \{0\}$. $A$ is called

- *biflat* if there exists a bounded $A$-bimodule homomorphism $\rho : A \to (A \widehat{\otimes} A)^{**}$ such that

$$\omega_A^{**} \circ \rho = \mathrm{id}_A.$$

- *$\varphi$-biflat* if there exists a bounded $A$-bimodule homomorphism $\rho : A \to (A \widehat{\otimes} A)^{**}$ such that

$$\widehat{\varphi} \circ \omega_A^{**} \circ \rho = \varphi.$$

- *left $\varphi$-biflat* if there exists a bounded linear map $\rho : A \to (A \widehat{\otimes} A)^{**}$ such that

$$\rho(ab) = \varphi(b)\rho(a) = a \cdot \rho(b), \qquad \widehat{\varphi} \circ \omega_A^{**} \circ \rho(a) = \varphi(a).$$

### 2.2. (Module) Projective Tensor Product

Let $A$ and $\mathfrak{A}$ be Banach algebras such that $A$ is a Banach $\mathfrak{A}$-bimodule with compatible actions:

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \qquad (ab) \cdot \alpha = a(b \cdot \alpha), \quad (a, b \in A,\ \alpha \in \mathfrak{A}).$$

Let $X$ be both a Banach $A$-bimodule and Banach $\mathfrak{A}$-bimodule with compatible actions:

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \qquad a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \qquad (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a),$$

and similarly on the right actions. Then $X$ is called an $A$-$\mathfrak{A}$-module. It is called a commutative $A$-$\mathfrak{A}$-module if $\alpha \cdot x = x \cdot \alpha \quad (\alpha \in \mathfrak{A},\ x \in X)$. If $X$ is a (commutative) $A$-$\mathfrak{A}$-module, then its dual $X^*$ also becomes a (commutative) $A$-$\mathfrak{A}$-module via

$$\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \qquad \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle.$$

A bounded operator $h : X \to Y$ is an *$A$-$\mathfrak{A}$-module homomorphism* if for all $a \in A, \alpha \in \mathfrak{A}, x, y \in X$:

$$h(x \pm y) = h(x) \pm h(y), \qquad h(\alpha \cdot x) = \alpha \cdot h(x), \quad h(a \cdot x) = a \cdot h(x), \quad h(x \cdot a) = h(x) \cdot a.$$

The projective tensor product $A\widehat{\otimes}A$ becomes an $\mathfrak{A}$-module via

$$\alpha \cdot (a \otimes b) = (\alpha \cdot a) \otimes b, \qquad (a \otimes b) \cdot \alpha = a \otimes (b \cdot \alpha).$$

Let $I_{\mathfrak{A}}$ be the closed ideal generated by

$$\{ a \cdot \alpha \otimes b - a \otimes \alpha \cdot b : a, b \in A,\ \alpha \in \mathfrak{A} \},$$

and let $J_A$ be the closed ideal of $A$ generated by

$$\{ \omega_A(a \cdot \alpha \otimes b - a \otimes \alpha \cdot b) = (a \cdot \alpha)b - a(\alpha \cdot b) : a, b \in A,\ \alpha \in \mathfrak{A} \}.$$

Define the *module projective tensor product* by $A\widehat{\otimes}_{\mathfrak{A}}A := \frac{A\widehat{\otimes}A}{I_{\mathfrak{A}}}$. Then $A/J_A$ becomes an $A$-$\mathfrak{A}$-module naturally. The map

$$\widetilde{\omega}_A : A\widehat{\otimes}_{\mathfrak{A}}A \to A/J_A, \qquad \widetilde{\omega}_A(a \otimes b + I_{\mathfrak{A}}) = ab + J_A,$$

is an $A$-$\mathfrak{A}$-module homomorphism, and so is its adjoint $\widetilde{\omega}_A^*$.

## 3. Module Left $(\phi, \varphi)$-Biflat Banach Algebras and Main Results

In this section, we present the main definition, which forms the core of the paper, followed by the key results. Before proceeding, we first introduce the following two notations that will be used throughout the discussion:
Let $A$ and $\mathfrak{A}$ be Banach algebras such that $A$ is a Banach $\mathfrak{A}$-bimodule:

- $\Omega_A$ denotes the set of linear maps $\phi : A \to \mathfrak{A}$ satisfying

$$\phi(ab) = \phi(a)\phi(b), \qquad \phi(\alpha \cdot a) = \varphi(\alpha)\phi(a).$$

- For $\phi \in \Omega_A$, define

$$\widetilde{\phi} : A/J_A \to \mathfrak{A}, \qquad \widetilde{\phi}(a + J_A) = \phi(a).$$

  This is well-defined since $\phi\big((a \cdot \alpha)b - a(\alpha \cdot b)\big) = 0$.

With these notations in place, we are ready to state the main definition and the subsequent theorems.

**Definition 3.1.** Let $A$ be a Banach $\mathfrak{A}$-bimodule, $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$, and $\phi \in \Omega_A$. We say that $A$ is $\mathfrak{A}$-*module left* $(\phi, \varphi)$-*biflat* if there exists a bounded linear map

$$\widetilde{\rho} : A/J_A \longrightarrow \Big((A\widehat{\otimes}_{\mathfrak{A}}A)/I_{\mathfrak{A}}\Big)^{**},$$

such that for all $a, b \in A$:

$$\widetilde{\rho}\big((ab) + J_A\big) = \phi(b)\,\widetilde{\rho}(a + J_A), \tag{D1}$$

$$\big(\widehat{\varphi \circ \widetilde{\phi}}\big)\big(\widetilde{\omega}_A^{**}(\widetilde{\rho}(a + J_A))\big) = (\varphi \circ \widetilde{\phi})(a + J_A). \tag{D2}$$

Here, $\widetilde{\rho}$ is only required to be bounded linear; it is not assumed to be a module homomorphism.

**Remark 3.2.** Condition (D1) encodes the "left $\phi$-behaviour" of the map $\widetilde{\rho}$, while (D2) ensures compatibility with the character $\varphi \circ \widetilde{\phi}$ after applying the product map and the canonical extension of the character. Note that $\widetilde{\rho}$ is only required to be bounded linear; it is not assumed to be a module homomorphism.

The result below establishes a clear connection between the module biflatness properties of $A^{**}$ and $A$, providing a powerful tool for understanding how biflatness behaves under duality in the module setting. This theorem sets the stage for several subsequent results and illustrates a key structural property that the reader can expect to see throughout the paper.

**Theorem 3.3.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule, $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$, and $\phi \in \Omega_A$. If $A^{**}$ is $\mathfrak{A}$-module left $(\widehat{\phi}, \varphi)$-biflat, then $A$ is $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat.*

*Proof.* Let $\widetilde{\rho}^{**} : A^{**}/J_{A^{**}} \to \left((A^{**}\widehat{\otimes}_{\mathfrak{A}}A^{**})/I_{\mathfrak{A},A^{**}}\right)^{**}$ be the bounded linear map witnessing the $\mathfrak{A}$-module left $(\widehat{\phi}, \varphi)$-biflatness of $A^{**}$. It satisfies:

$$\widetilde{\rho}^{**}((F * G) + J_{A^{**}}) = \widehat{\phi}(G)\,\widetilde{\rho}^{**}(F) \quad \text{and} \quad \widehat{(\varphi \circ \widetilde{\phi})}\left(\widetilde{\omega}^{**}_{A^{**}}(\widetilde{\rho}^{**}(F))\right) = (\varphi \circ \widetilde{\phi})(F)$$

for all $F, G \in A^{**}$.
Consider the natural embedding $\iota : A \hookrightarrow A^{**}$ and define

$$\widetilde{\rho}(a + J_A) := \widetilde{\rho}^{**}(a + J_{A^{**}}), \quad a \in A.$$

If $a + J_A = b + J_A$, then $a - b \in J_A \subseteq J_{A^{**}}$, so $\widetilde{\rho}(a + J_A) = \widetilde{\rho}(b + J_A)$. Hence $\widetilde{\rho}$ is well-defined. $\widetilde{\rho}$ is the restriction of a bounded linear map $\widetilde{\rho}^{**}$, so it is bounded linear.
For $a, b \in A$,

$$\widetilde{\rho}((ab) + J_A) = \widetilde{\rho}^{**}((ab) + J_{A^{**}}) = \widehat{\phi}(b)\,\widetilde{\rho}^{**}(a) = \phi(b)\,\widetilde{\rho}(a + J_A),$$

where we use the identification $\widehat{\phi}|_A = \phi$.
Now, for $a \in A$ we have

$$\widehat{(\varphi \circ \widetilde{\phi})}\left(\widetilde{\omega}^{**}_{A}(\widetilde{\rho}(a + J_A))\right) = \widehat{(\varphi \circ \widetilde{\phi})}\left(\widetilde{\omega}^{**}_{A^{**}}(\widetilde{\rho}^{**}(a + J_{A^{**}}))\right) = (\varphi \circ \widetilde{\phi})(a + J_A).$$

Therefore, $\widetilde{\rho}$ witnesses that $A$ is $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat. □

Before presenting Theorem 3.4, we emphasize the importance of understanding how $\mathfrak{A}$-module left $(\phi, \varphi)$-biflatness behaves with respect to closed ideals. This result demonstrates that the biflatness property of a Banach $\mathfrak{A}$-bimodule $A$ can be inherited by its closed ideals under natural non-vanishing conditions on the character $\phi$. It provides a crucial insight into the internal structure of Banach modules and prepares the reader for applications where examining submodules or ideals is essential for extending biflatness properties.

**Theorem 3.4.** *Let $A$ be a Banach $\mathfrak{A}$-bimodule and $\phi \in \Omega_A$, $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$. Suppose $I$ is a closed ideal of $A$ such that $\phi|_I \neq 0$, and $A$ is $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat. Then $I$ is $\mathfrak{A}$-module left $(\phi|_I, \varphi)$-biflat.*

*Proof.* Since $A$ is $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat, there exists a bounded linear map

$$\widetilde{\rho}_A : A/J_A \longrightarrow \left((A\widehat{\otimes}_{\mathfrak{A}}A)/I_{\mathfrak{A}}\right)^{**}$$

satisfying the $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat conditions (D1) and (D2).
Let $J_I$ be the ideal of $I$ generated analogously to $J_A$, and let $\iota : I \hookrightarrow A$ be the canonical inclusion. This induces a map $\tilde{\iota} : I/J_I \to A/J_A$. Define

$$\widetilde{\rho}_I := \widetilde{\rho}_A \circ \tilde{\iota} : I/J_I \longrightarrow \left((A\widehat{\otimes}_{\mathfrak{A}}A)/I_{\mathfrak{A}}\right)^{**}.$$

Now we verify the $\mathfrak{A}$-module left $(\phi|_I, \varphi)$-biflat conditions:

1. For $x, y \in I$,

$$\begin{aligned}\widetilde{\rho}_I((xy) + J_I) &= \widetilde{\rho}_A((xy) + J_A)\\ &= \phi(y)\widetilde{\rho}_A(x + J_A)\\ &= (\phi|_I)(y)\,\widetilde{\rho}_I(x + J_I),\end{aligned}$$

so condition (D1) holds.

2. For $x \in I$,

$$\begin{aligned}(\widehat{\varphi \circ \widetilde{\phi}})(\widetilde{\omega}_A^{**}(\widetilde{\rho}_I(x + J_I))) &= (\widehat{\varphi \circ \widetilde{\phi}})(\widetilde{\omega}_A^{**}(\widetilde{\rho}_A(x + J_A))) \\ &= (\varphi \circ \widetilde{\phi})(x + J_A) \\ &= (\varphi \circ \widetilde{\phi}|_I)(x + J_I).\end{aligned}$$

Therefore, (D2) also holds. Finally, $\widetilde{\rho}_I$ is a bounded $(I/J_I)$-$\mathfrak{A}$-linear map, being the restriction of the $(A/J_A)$-$\mathfrak{A}$-module map $\widetilde{\rho}_A$. Hence, $I$ is $\mathfrak{A}$-module left $(\phi|_I, \varphi)$-biflat. □

In the next theorem, we explore how $\mathfrak{A}$-module left $(\phi, \varphi)$-biflatness behaves under standard algebraic constructions and shows that if both $A$ and $B$ are $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat, then their direct sum $A \oplus B$ (with componentwise operations) also possesses this property. This result illustrates the structural stability of module biflatness and provides a practical tool for constructing more complex modules from simpler ones.

**Theorem 3.5.** *Let A and B be Banach $\mathfrak{A}$-bimodules, $\varphi \in \Delta(\mathfrak{A}) \cup \{0\}$, and $\phi \in \Omega_A \cap \Omega_B$. If both A and B are $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat, then the direct sum $A \oplus B$ (with componentwise operations) is $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat.*

*Proof.* Let

$$\widetilde{\rho}_A : A/J_A \to \left((A\widehat{\otimes}_{\mathfrak{A}}A)/I_{\mathfrak{A},A}\right)^{**}, \quad \widetilde{\rho}_B : B/J_B \to \left((B\widehat{\otimes}_{\mathfrak{A}}B)/I_{\mathfrak{A},B}\right)^{**}$$

be the bounded linear maps witnessing $\mathfrak{A}$-module left $(\phi, \varphi)$-biflatness of $A$ and $B$, respectively.
Define

$$\widetilde{\rho}_{A\oplus B} : (A \oplus B)/(J_A \oplus J_B) \to \left((A \oplus B)\widehat{\otimes}_{\mathfrak{A}}(A \oplus B)/I_{\mathfrak{A},A\oplus B}\right)^{**},$$

by

$$\widetilde{\rho}_{A\oplus B}((a, b) + J_A \oplus J_B) := (\widetilde{\rho}_A(a + J_A), \widetilde{\rho}_B(b + J_B)).$$

For $(a_1, b_1), (a_2, b_2) \in A \oplus B$,

$$\begin{aligned}\widetilde{\rho}_{A\oplus B}((a_1, b_1) * (a_2, b_2) + J_{A\oplus B}) &= (\widetilde{\rho}_A(a_1a_2 + J_A), \widetilde{\rho}_B(b_1b_2 + J_B)) \\ &= (\phi(a_2)\widetilde{\rho}_A(a_1), \phi(b_2)\widetilde{\rho}_B(b_1)) \\ &= \phi((a_2, b_2))\, \widetilde{\rho}_{A\oplus B}((a_1, b_1) + J_{A\oplus B}).\end{aligned}$$

For $(a, b) \in A \oplus B$,

$$\begin{aligned}(\widehat{\varphi \circ \widetilde{\phi}})\left(\widetilde{\omega}_{A\oplus B}^{**}(\widetilde{\rho}_{A\oplus B}((a, b) + J_{A\oplus B}))\right) &= (\widehat{\varphi \circ \widetilde{\phi}})(\widetilde{\omega}_A^{**}(\widetilde{\rho}_A(a + J_A)), \widetilde{\omega}_B^{**}(\widetilde{\rho}_B(b + J_B))) \\ &= (\varphi \circ \widetilde{\phi})(a + J_A) + (\varphi \circ \widetilde{\phi})(b + J_B),\end{aligned}$$

which satisfies the $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat condition componentwise. Thus, $A \oplus B$ is $\mathfrak{A}$-module left $(\phi, \varphi)$-biflat. □

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# On the Duality Structure of Tangent Spaces in Hadamard Spaces

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**Abstract**

This paper investigates the structure of the tangent space of a Hadamard space and its associated linear dual. We introduce a family of mappings corresponding to directions at a fixed point, and show that the linear dual space is spanned by these mappings. A key intermediate result is a Cauchy–Schwarz type inequality in the tangent space. Our results provide an explicit representation of dual elements and clarify their action on bound vectors.

## 1. Introduction and Preliminaries

A *geodesic path* in the metric space $(X,d)$ from $a \in X$ to $b \in X$ is a continuous mapping $\gamma : [0,1] \to X$ such that $\gamma(0) = a, \gamma(1) = b$ and $d(\gamma(t),\gamma(s)) = |t-s|d(a,b)$, for all $t,s \in [0,1]$. A *geodesic space* is a metric space $(X,d)$ such that every two points $a,b \in X$ are connected by a geodesic path. We use $\mathcal{G}eo(X)$ to denote the set of all geodesic path contained in $X$. The geodesic space $(X,d)$ is called a $CAT(0)$ *space* if for all $a,b,c \in X$ and $t \in [0,1]$ the following inequality, so-called *CN-inequality*,

$$d(c,\gamma(t))^2 \leq (1-t)d(c,a)^2 + td(c,b)^2 - t(1-t)d(a,b)^2 \text{ for all } a,b,c \in X, t \in [0,1]. \tag{1}$$

In a $CAT(0)$ space $X$, any geodesic path from $a \in X$ to $b \in X$ is denoted by $\gamma(t) := (1-t)a \oplus tb$, whenever $t \in [0,1]$. The *geodesic segment* joining $a \in X$ to $b \in X$ is defined by $[\![a,b]\!] := \gamma_{a,b}([0,1])$. A geodesic path $\gamma : [0,1] \to X$ is called constant at $p \in X$ if $\gamma([0,1]) = \{p\}$. The *Alexandrov angle* between two nonconstant geodesics $\gamma,\eta \in \mathcal{G}eo(X)$ issuing at the same point $p = \gamma(0) = \gamma'(0)$ is well-defined and can be determined as $\angle_p(\gamma,\eta) = \lim_{t,s\searrow 0} \angle_{\overline{p}}\left(\overline{\gamma(t)},\overline{\eta(s)}\right)$, where $\Delta(\overline{\gamma(t)},\overline{p},\overline{\eta(s)})$ is a comparison triangle in $\mathbb{R}^2$. We also denote it by $\angle_p(\gamma,\eta)$. An *Hadamard space* is a complete $CAT(0)$ space. Important examples of Hadamard spaces include Hilbert spaces, Hadamard manifolds, $\mathbb{R}$-trees, nonlinear Lebesgue spaces, Hilbert balls.
Fix $p \in X$. Denote by $\Theta_p X$ the set of all nonconstant geodesics issuing at $x$, i.e.,

$$\Theta_p X := \{\gamma \in \mathcal{G}eo(X) : \gamma(0) = p, \gamma \text{ is a nonconstant geodesic.}\}$$

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The Alexandrov angle induces a metric on the set $\Sigma_p X$ of equivalence classes of geodesics in $\Theta_p X$, where two geodesics $\gamma, \eta \in \Theta_p X$, are considered equivalent if $\angle(\gamma, \eta) = 0$.
For a geodesic $\gamma \in \Theta_p X$, we denote by $[\gamma]$ its equivalence class. At the same time, when working with an equivalence class in $\Sigma_p X$, we assume that it is represented by a geodesic in $\Theta_p X$. Under our assumptions, the metric space $(\Sigma_p X, \angle)$ is complete.
We will refer to the elements in $\Sigma_p X$ as directions at $p$ and to $(\Sigma_p X, \angle)$ as the space of directions at $p$. The tangent space $\mathsf{T}_p X$ of $X$ at $p$ is the Euclidean cone over the metric space $(\Sigma_p X, \angle)$ (see [15, Chapter I, Definition 5.6]). More precisely, $\mathsf{T}_p X := (\Sigma_p X \times [0, \infty))/\sim$, where for $([\gamma], r), ([\eta], s) \in \Sigma_p X \times [0, \infty)$,

$$([\gamma], r) \sim ([\eta], s) :\Leftrightarrow (r = s = 0) \text{ or } (r = s > 0 \text{ and } [\gamma] = [\eta]).$$

For brevity and where no confusion is likely, we denote the tangent space of the Hadamard space $X$ at $p$ by $\mathfrak{C}$, indeed $\mathfrak{C} := \mathsf{T}_p X$. The equivalence class of $([\gamma], 0) \in \mathfrak{C}$ is called the origin of $\mathfrak{C}$ and will be denoted by $\mathbf{o}_p$. (When dealing with the origin $\mathbf{o}_p$, the direction at $p$ bears no relevance and sometimes, to simplify the writing, we consider the geodesic constantly equal to $p$.) The metric $d_{\mathfrak{C}}$ on $\mathfrak{C}$ is defined for $v = ([\gamma], r), w = ([\eta], s) \in \mathfrak{C}$ by

$$\rho_{\mathfrak{C}}(v, w)^2 = r^2 + s^2 - 2rs \cos \angle_p(\gamma, \eta),$$

while the scalar product of $v$ and $w$ is given by $\langle v, w \rangle_{\mathfrak{C}} = rs \cos \angle_p(\gamma, \eta)$. The multiplication with a nonnegative scalar $\lambda \geq 0$ is defined for $v = ([\gamma], r)$ by $\lambda v = ([\gamma], \lambda r)$. We say that $v = ([\gamma], r), w = ([\eta], s) \in \mathfrak{C}$ are opposite to each other if $(r = s = 0)$ or $(r = s > 0$ and $\angle(\gamma, \eta) = \pi)$. Observe that, under our assumptions on $X$, $(\mathfrak{C}, \rho_{\mathfrak{C}})$ is an Hadamard space.
Let $X$ be an Hadamard space and $x, y \in X$. The ordered pair $(x, y) \in X^2$ is denoted by $\overrightarrow{xy}$, which is called a *bound vector*. The *zero bound vector* at $x \in X$ is denoted by $\mathbf{0}_x := \overrightarrow{xx}$. We identify two bound vectors $-\overrightarrow{xy}$ and $\overrightarrow{yx}$. The bound vectors $\overrightarrow{xy}$ and $\overrightarrow{uz}$ are called *compatible* if $y = u$. Moreover, the *addition* of two compatible bound vectors $\overrightarrow{xy}$ and $\overrightarrow{yz}$ is defined by $\overrightarrow{xy} + \overrightarrow{yz} = \overrightarrow{xz}$. The *quasilinearization mapping* is $\langle \overrightarrow{xy}, \overrightarrow{uv} \rangle := \frac{1}{2}\big(d(x,v)^2 + d(y,u)^2 - d(x,u)^2 - d(y,v)^2\big)$, where $x, y, u, v, \in X$. Given an Hadamard space $(X, d)$, define the map $\Theta : \mathbb{R} \times X^2 \to C(X, \mathbb{R})$

$$(t, a, b) \mapsto \Theta(t, a, b)x =: t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (a, b, x \in X, t \in \mathbb{R}). \tag{2}$$

is a Lipschitz function with the Lipschitz semi-norm $L(\Theta(t, a, b)) = |t| d(a, b)$, $(a, b \in X,\ t \in \mathbb{R})$. Also, $(\mathbb{R} \times X^2, D)$ is a *pseudometric* space, where $D\big((t, a, b), (s, c, d)\big) = L\big(\Theta(t, a, b) - \Theta(s, c, d)\big)$, for each $a, b, c, d \in X$, $t, s \in \mathbb{R}$. It is easy to check that the following relation on $\mathbb{R} \times X^2$ is an equivalence relation $(t, a, b) \sim (s, c, d) :\Leftrightarrow D\big((t, a, b), (s, c, d)\big) = 0$. Now, the set $X^* := \big\{[t\overrightarrow{ab}] : a, b \in X, t \in \mathbb{R}\big\}$ is called the *dual space* of $X$. It is easily seen that $(X^*, D)$ is a metric space. Moreover, the element $\mathbf{0} := [\overrightarrow{uu}]$ represents the zero of $X^*$. The *linear dual space* of $X$ is defined by $X^\diamond := \operatorname{lin} X^*$. Indeed,

$$X^\diamond = \Big\{ \sum_{i=1}^{n} \alpha_i x_i^* : \alpha_i \in \mathbb{R},\ x_i^* \in X^*,\ n \in \mathbb{N} \Big\}. \tag{3}$$

Moreover, $X^\diamond$ acts on $X^2$; $\langle \sum_{i=1}^n \alpha_i x_i^*, \overrightarrow{xy} \rangle := \sum_{i=1}^n \alpha_i \langle x_i^*, \overrightarrow{xy} \rangle$ where $x, y \in X$. It follows from [3, Corollary 3] that a geodesic space $(X, d)$ is a $CAT(0)$ space if and only if it satisfies the *Cauchy-Schwarz inequality*, that is

$$|\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle| \leq d(a, b) d(c, d), \quad a, b, c, d \in X. \tag{4}$$

one can see that for each $a, b, w \in X$ and each $x^\diamond \in X^\diamond$

$$\langle x^\diamond, \overrightarrow{ab} \rangle = \langle x^\diamond, \overrightarrow{aw} \rangle + \langle x^\diamond, \overrightarrow{wb} \rangle. \tag{5}$$

We refer to [1, 2, 4, 5] for the details of the background of $CAT(0)$ spaces and tangent spaces.

## 2. Main Results

**Lemma 2.1.** *Let $X$ be an Hadamard space and let $\mathfrak{C}$ be the tangent cone of $X$ at $p \in X$. Then*

$$\langle \overrightarrow{o_p u}, \overrightarrow{o_p v} \rangle = \langle u, v \rangle_{\mathfrak{C}}, \quad u, v \in \mathfrak{C} \tag{6}$$

*Proof.* Let $u := ([\gamma], r) \in \mathfrak{C}$, $v := ([\eta], s) \in \mathfrak{C}$ and let $\mathbf{o}_p := ([\alpha], 0)$. By using definition of quasilinearization map we obtain that

$$\begin{aligned}\langle \overrightarrow{\mathbf{o}_p u}, \overrightarrow{\mathbf{o}_p v} \rangle &= \left\langle \overrightarrow{([\alpha],0)([\gamma],r)}, \overrightarrow{([\alpha],0)([\eta],s)} \right\rangle = \frac{1}{2}\Big(\rho_{\mathfrak{C}}\big(([\alpha],0),([\eta],s)\big)^2 + \rho_{\mathfrak{C}}\big([\gamma],r),([\alpha],0)\big)^2 \\ &\quad - \rho_{\mathfrak{C}}\big(([\alpha],0),([\alpha],0)\big)^2 - \rho_{\mathfrak{C}}\big(([\gamma],r),([\eta],s)\big)^2\Big) \\ &= rs\cos\angle_p(\gamma,\eta) = \langle u, v \rangle_{\mathfrak{C}},\end{aligned}$$

as required. □

**Remark 2.2.** As a direct consequence of Lemma 6, for each $u_i := ([\gamma_i], r_i)$, $v_i = ([\eta_i], s_i) \in \mathfrak{C}$ where $i = 1, 2$

$$\langle \overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2} \rangle = \langle u_1, u_2 \rangle_{\mathfrak{C}} - \langle u_1, v_2 \rangle_{\mathfrak{C}} - \langle v_1, u_2 \rangle_{\mathfrak{C}} + \langle v_1, v_2 \rangle_{\mathfrak{C}}.$$

To see this, by (5) we get

$$\begin{aligned}\langle \overrightarrow{u_1 v_1}, \overrightarrow{u_2 v_2} \rangle &= \langle \overrightarrow{\mathbf{o}_p v_1} - \overrightarrow{\mathbf{o}_p u_1}, \overrightarrow{\mathbf{o}_p v_2} - \overrightarrow{\mathbf{o}_p u_2} \rangle = \langle \overrightarrow{\mathbf{o}_p v_1}, \overrightarrow{\mathbf{o}_p v_2} \rangle - \langle \overrightarrow{\mathbf{o}_p v_1}, \overrightarrow{\mathbf{o}_p u_2} \rangle - \langle \overrightarrow{\mathbf{o}_p u_1}, \overrightarrow{\mathbf{o}_p v_2} \rangle + \langle \overrightarrow{\mathbf{o}_p u_1}, \overrightarrow{\mathbf{o}_p u_2} \rangle \\ &= \langle u_1, u_2 \rangle_{\mathfrak{C}} - \langle u_1, v_2 \rangle_{\mathfrak{C}} - \langle v_1, u_2 \rangle_{\mathfrak{C}} + \langle v_1, v_2 \rangle_{\mathfrak{C}}.\end{aligned}$$

**Lemma 2.3.** *Let $X$ be an Hadamard space and $\mathfrak{C}$ be the tangent cone of $X$ at $p \in X$. Then*

$$|\langle u, v \rangle_{\mathfrak{C}}| \le \rho_{\mathfrak{C}}(u, \mathbf{o}_p)\rho_{\mathfrak{C}}(v, \mathbf{o}_p), \quad u, v \in \mathfrak{C} \tag{7}$$

*Proof.* Suppose that $u := ([\gamma], r) \in \mathfrak{C}$, $v := ([\eta], s) \in \mathfrak{C}$ and let $\mathbf{o}_p := ([\alpha], 0)$. It follows from Cauchy-Schwarz inequality (4) and Remark 2.2 that

$$|\langle u, v \rangle_{\mathfrak{C}}| = |\langle \overrightarrow{\mathbf{o}_p u}, \overrightarrow{\mathbf{o}_p v} \rangle| \le \rho_{\mathfrak{C}}(\mathbf{o}_p, u)\rho_{\mathfrak{C}}(\mathbf{o}_p, v),$$

and the proof is complete. □

**Lemma 2.4.** *Let $\mathfrak{C}$ be the tangent cone of Hadamard space $X$ at $p \in X$. Let $\xi \in \Sigma_p X$ be arbitrary and fixed. Define the mapping*

$$\psi_\xi : \mathfrak{C} \to \mathbb{R}, \qquad \psi_\xi\big(([\eta], s)\big) := s\cos\angle_p(\xi, \eta), \quad s \ge 0,\ \eta \in \Sigma_p X.$$

*Then the following hold.*

*(i) For every $x^* = [t\overrightarrow{uv}] \in \mathfrak{C}^*$, $u = ([\xi], b)$ and $v = ([\zeta], a) \in \mathfrak{C}$*

$$\langle [t\overrightarrow{uv}], \overrightarrow{o_p([\eta], s)} \rangle = t\Big(r_2\psi_{\xi_2}\big(([\eta], s)\big) - r_1\psi_{\xi_1}\big(([\eta], s)\big)\Big) \qquad \forall\, ([\eta], s) \in \mathfrak{C}.$$

*(ii) Let $u := ([\xi], b), v := ([\zeta], a) \in \mathfrak{C}$. Then $x^* := [\overrightarrow{uv}] \in \mathfrak{C}^*$ satisfies*

$$\langle x^*, \overrightarrow{o_p([\eta], s)} \rangle = a\,\psi_\zeta\big(([\eta], s)\big) - b\,\psi_\xi\big(([\eta], s)\big) \qquad \forall\, ([\eta], s) \in \mathfrak{C}.$$

*Consequently, every function of the form $a\psi_\zeta - b\psi_\xi$ arises as the restriction to radial bound vectors of some element of $\mathfrak{C}^*$.*

*Proof.* $u = ([\xi], b)$ and $v = ([\zeta], a)$ (i) Let $x^* = [t\overrightarrow{uv}] \in \mathfrak{C}^*$ and let $u = ([\xi], b), v = ([\zeta], a)$. Fix an arbitrary $w := ([\eta], s) \in \mathfrak{C}$. It follows from Remark 2.2 that

$$\langle x^*, \overrightarrow{\mathbf{o}_p w}\rangle = \langle [t\overrightarrow{uv}], \overrightarrow{\mathbf{o}_p w}\rangle = t\langle \overrightarrow{uv}, \overrightarrow{\mathbf{o}_p w}\rangle = t\langle \overrightarrow{\mathbf{o}_p v}, \overrightarrow{\mathbf{o}_p w}\rangle - t\langle \overrightarrow{\mathbf{o}_p u}, \overrightarrow{\mathbf{o}_p w}\rangle = t\big(\langle v, w\rangle_{\mathfrak{C}} - \langle u, w\rangle_{\mathfrak{C}}\big)$$
$$= t\big(as\cos\angle_p(\zeta,\eta) - bs\cos\angle_p(\xi,\eta)\big) = t\Big(a\,\psi_\zeta\big(([\eta], s)\big) - b\,\psi_\xi\big(([\eta], s)\big)\Big).$$

This proves (i).
(ii) This is an immediate consequence of (i). □

**Theorem 2.5.** *Let $X$ be a Hadamard space and let $\mathfrak{C}$ be the tangent cone of $X$ at $p \in X$. Then*

$$\mathfrak{C}^\diamond = \operatorname{lin}\{\psi_{\xi_i} : \xi_i \in \Sigma_p X,\ 1 \le i \le n, n \in \mathbb{N}\}.$$

*Proof.* Suppose that $x^\diamond = \sum_{i=1}^n \lambda_i x_i^* \in \mathfrak{C}^\diamond$. By using Lemma 2.4(i) there exist $a_i, b_i \ge 0$ and $\xi_i, \zeta_i \in \Sigma_p X$ where $(1 \le i \le n)$ such that for every $w \in \mathfrak{C}$, we obtain that $\langle x_i^*, \overrightarrow{\mathbf{o}_p w}\rangle = a_i\psi_{\zeta_i}(w) - b_i\psi_{\xi_i}(w)$. Hence for every $w \in \mathfrak{C}$,

$$\langle x^\diamond, \overrightarrow{\mathbf{o}_p w}\rangle = \Big\langle \sum_{i=1}^n \lambda_i x_i^*, \overrightarrow{\mathbf{o}_p w}\Big\rangle = \sum_{i=1}^n \lambda_i\langle x_i^*, \overrightarrow{\mathbf{o}_p w}\rangle = \sum_{i=1}^n \lambda_i\big(a_i\psi_{\zeta_i}(w) - b_i\psi_{\xi_i}(w)\big), \tag{8}$$

which (8) shows that $x^\diamond \in \operatorname{lin}\{\psi_{\xi_i} : \xi_i \in \Sigma_p X,\ n \in \mathbb{N},\ 1 \le i \le n\}$.
Conversely, let $\xi \in \Sigma_p X$ be fixed and choose any $\zeta \in \Sigma_p X$. Put $u := ([\xi], 0)$ and $v := ([\zeta], 1)$ in $\mathfrak{C}$. Then, using Proposition 2.4(ii), we obtain that $x^* := [\overrightarrow{uv}] \in \mathfrak{C}^*$ satisfies

$$\langle x^*, \overrightarrow{\mathbf{0}\, w}\rangle = \psi_\zeta(w) \qquad w \in \mathfrak{C}.$$

In particular, $\psi_\zeta$ is the restriction of an element of $\mathfrak{C}^*$. Since $\zeta$ was arbitrary, it follows that each $\psi_\xi$ belongs to $\operatorname{span}(\mathfrak{C}^*) = \mathfrak{C}^\diamond$. This completes the proof. □

**Corollary 2.6.** *Let $X$ be an Hadamard space and $x^\diamond = \sum_{i=1}^n \alpha_i\psi_{\xi_i} \in \mathfrak{C}^\diamond$ whenever $\alpha_i \in \mathbb{R}$, $\xi_i \in \Sigma_p X$ with $1 \le i \le n$, $n \in \mathbb{N}$. Then for every $u, v \in \mathfrak{C}$*

$$\langle x^\diamond, \overrightarrow{uv}\rangle = \sum_{i=1}^n \alpha_i\big(\psi_{\xi_i}(v) - \psi_{\xi_i}(u)\big). \tag{9}$$

*In particular, if $u = ([\eta], r)$ and $v = ([\zeta], s)$ with $r, s \ge 0$ and $\eta, \zeta \in \Sigma_p X$, then*

$$\langle x^\diamond, \overrightarrow{uv}\rangle = \sum_{i=1}^n \alpha_i\Big(s\cos\angle_p(\xi_i,\zeta) - r\cos\angle_p(\xi_i,\eta)\Big). \tag{10}$$

*Proof.* Let $u, v \in \mathfrak{C}$. Then Proposition 2.5 and (5) imply that

$$\langle x^\diamond, \overrightarrow{uv}\rangle = \langle x^\diamond, \overrightarrow{\mathbf{o}_p v}\rangle - \langle x^\diamond, \overrightarrow{\mathbf{o}_p u}\rangle = \sum_{i=1}^n \alpha_i\psi_{\xi_i}(v) - \sum_{i=1}^n \alpha_i\psi_{\xi_i}(u) = \sum_{i=1}^n \alpha_i\big(\psi_{\xi_i}(v) - \psi_{\xi_i}(u)\big).$$

Finally, (10) is a direct consequence of (9). □

## 3. Conclusion

In this work, we have provided a detailed investigation of the tangent cone $\mathfrak{C}$ of a Hadamard space $X$ and its linear dual space $\mathfrak{C}^\diamond$. By introducing the family of functions $\psi_\xi$ corresponding to directions at $p$, we have shown that $\mathfrak{C}^\diamond$ can be fully represented as the linear span of these functions. This representation allows an explicit computation of the action of linear dual elements on bound vectors, including radial vectors, and demonstrates the interplay between the metric, angular, and dual structures of the tangent cone.
Our results extend the classical notions of duality from linear Hilbert spaces to the non-linear framework of Hadamard spaces, providing a concrete tool for studying $CAT(0)$ spaces.

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# $r$-Homotopically null cycles in topological graph theory

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**Abstract**

In this talk, after introducing $r$-homotopy in topological graph theory, we study various types of cycles that are $r$-homotopically null, including cycles of length at most $2r$ and lollipops whose heads are cycles with length at most $2r$.

## 1. Introduction and Preliminaries

T. Matsushita [9] introduced the notion of $r$-homotopy provides a natural framework for detecting and classifying homotopically trivial cycles of bounded length. Matsushita's work laid the foundation for connecting combinatorial structures in graphs with ideas inspired by algebraic topology.
Here, after introducing preliminaries in topological graph theory and $r$-homotopy, we study various types of cycles that are $r$-homotopically null, including cycles of length at most $2r$ and lollipops whose heads are cycles with length at most $2r$.

## 2. Preliminaries

In this section, we define notation and terminologies in this paper and mainly follow [4] and [5]. We work in the category Gph of simple graphs: undirected graphs without multiple edges. Moreover, throughout this paper, we will assume that graphs are connected and finite.
A **simple graph** $G$ is defined by a finite set $V_G$ of vertices and a set $E_G$ of edges connecting vertices. Each edge is given by an unordered pair of vertices. Any pair of vertices has at most one edge connecting them. Loops and isolated vertices are not allowed: each vertex must be connected to at least one other. A connecting edge will be notated by $v_1 \sim v_2$ and we say $v_1$ is adjacent with $v_2$.
A **graph homomorphism** $f : G \longrightarrow H$ is a map $f : V_G \longrightarrow V_H$ such that for $v_1, v_2 \in V_G$, if $v_1 \sim v_2 \in E_G$ then either $f(v_1) = f(v_2)$ or $f(v_1) \sim f(v_2) \in E_H$ (we shall remove subscripts whenever there is no confusion). In other words, a graph homomorphism maps vertices to vertices and edges to edges in a way that preserves the incidence relationship.

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A graph with $n+1$ vertices $\{0,1,2,...,n\}$ such that $i \sim i+1$ for $i = 0,1,...,n-1$, is called **path graph** and denoted by $P_n$. The **cycle graph** $C_n$ has $n$ vertices $\{0,1,2,...,n-1\}$ such that $i$ adjacent to $j$ if and only if $i-j \equiv \pm 1 \ mod\ n$, for $i = 0,1,...,n-1$.
A **walk** in $G$ of length $n$ from $x \in V_G$ to $y \in V_G$ is a morphism $\alpha : P_n \longrightarrow G$ with $s(\alpha) = \alpha(0) = x$ and $t(\alpha) = \alpha(n) = y$. We shall usually describe a walk by a list of image vertices $(v_0v_1v_2...v_n)$ such that $v_i \sim v_{i+1}$. The symbol $\overline{\alpha}$ denote the walk $(v_nv_{n-1}...v_1v_0)$ and we call it the inverse of $\alpha$. A graph $G$ is called **connected** if for every choice of $x$ and $y$ there is a walk from $x$ to $y$.
For $u, v \in V_G$, $d(u,v)$ denotes the distance between $u$ and $v$, i.e., the length of a shortest walk connecting $u$ and $v$ and $k$-neighborhood of $u \in V_G$ is

$$N_k(u) = \{v \in V_G | d(u,v) \leq k\}.$$

The diameter of a given graph $G$ is the maximal distance between any two vertices in $G$.
A walk is called a **path** if $v_{i+2} \neq v_i$. A closed path is called a **cycle** if $v_0 = v_n$. By $u$-based $n$-cycle $\alpha : P_n \longrightarrow G$, we mean a closed path $\alpha$ of length $n$ in which $\alpha(0) = \alpha(n) = u$. The $(u)$ is considered as a closed walk of length 0 and also as a $u$-based cycle. A **tree** is a connected graph that contains no cycles.
The **concatenation** $\alpha * \beta$ of walks $\alpha$ and $\beta$ is defined to be the walk $\alpha$ followed by the walk $\beta$ when $t(\alpha) = s(\beta)$, i.e. if $\alpha = (v_0,...,v_m)$, $\beta = (u_0,...,u_n)$ and $v_m = u_0$, then $\alpha * \beta = (v_0,...,v_m,u_1,...,u_n)$. The walk of the form $\alpha * \overline{\alpha}$ is called a backtrack.
Two walks $\alpha$ and $\beta$ are called equivalent and denoted by $\alpha \sim \beta$ if $s(\alpha) = t(\alpha) = s(\beta) = t(\beta)$ and there exists a sequence

$$\alpha = \alpha_0,\ \alpha_1,\ ...,\ \alpha_n = \beta,$$

such that $\alpha_i$ and $\alpha_{i+1}$ differ form either the insertion or the removal of backtracks. The equivalence class of $\alpha$ is usually denoted by $[\alpha]$.
For a given $u \in V_G$, the **fundamental group** of $G$ at $u$ denoted by $\pi_1(G,u)$, is the set of all the equivalence classes of $u$-based cycles with the product operation: $[\alpha] * [\beta] = [\alpha * \beta]$. The trivial element of this group is $[(u)]$ and the inverse element of $[\alpha]$ is $[\overline{\alpha}]$. For connected graph $G$, this group is independent of the choice of the base vertex $u$, up to isomorphism.
In the sequel, we recall the basic definitions and properties of the $r$-homotopy, and establish several results that will be used throughout the paper.
Let $r$ be a fixed positive integer.

**Definition 2.1.** [9] Let $G$ be a graph and $v, w \in V_G$. Two paths $\varphi$ and $\psi$ in $G$ from $v$ to $w$ are said to be *$r$-homotopic* and denoted by $\varphi \simeq_r \psi$ if there exists a finite sequence $\varphi = \varphi_0, \varphi_1, ..., \varphi_k = \psi$ such that for each $i$ one of the following holds:

(A) $l(\varphi_{i+1}) = l(\varphi_i) \pm 2$, and $\varphi_{i+1}$ is obtained from $\varphi_i$ by inserting or deleting a backtrack.

(B)$_r$ $l(\varphi_{i+1}) = l(\varphi_i)$, and $\varphi_i$ and $\varphi_{i+1}$ differ in fewer than $r$ vertices.

In the $(B)_r$ we say that $\varphi_{i+1}$ is obtained by replacing an $r$-path inside the $\varphi_i$.
The $r$-homotopy relation $\simeq_r$ on the set of closed paths in $G$ is easily seen to be an equivalence relation. Moreover, it is compatible with path concatenation and inversion, which allows us to define the group operation in $\pi_1^r(G,u)$.

**Definition 2.2.** [9] For a based graph $(G,u)$, the *$r$-fundamental group* $\pi_1^r(G,u)$ is defined as the set of $r$-homotopy classes of cycles in $(G,u)$ with concatenation as the group operation.

**Proposition 2.3.** *[9] Let based graph homomorphism $f : (G,u) \longrightarrow (H,v)$. Then $f$ induces a group homomorphism $f_* : \pi_1^r(G,u) \longrightarrow \pi_1^r(H,v)$.*

## 3. Main Results

After recalling the definitions of $r$-homotopy and $r$-fundamental group of a graph, we list here several simple properties that follow directly from the definition. These statements are elementary consequences and will be used frequently in the subsequent sections.

**Proposition 3.1.** *If two paths $\alpha$ and $\beta$ in $G$ from $v$ to $w$ differ only by the insertion or removal of a subpath of length at most $r$ that starts and ends at the same vertex, then $\alpha \simeq_r \beta$.*

*Proof.* Easily comes from the condition $B_r$ in Definition 2.1. □

**Proposition 3.2.** *Let $G$ be a graph and $v, w \in V_G$. Then*

*(A) 2-homotopy is the usual homotopy on graph $G$;*

*(B) If $\varphi \simeq_r \psi$ for $\varphi, \psi \in P(G, v, w)$, then $\varphi \simeq_k \psi$, for every $k \geq r$;*

*(C) If $\varphi \simeq \psi$ for $\varphi, \psi \in P(G, v, w)$, then $\varphi \simeq_r \psi$, for every non-negative integer $r$;*

*(D) If $\varphi \simeq_r \psi$, then $\overleftarrow{\varphi} \simeq_r \overleftarrow{\psi}$;*

*(E) If $\varphi_1 \simeq_r \psi_1$ and $\varphi_2 \simeq_r \psi_2$, then $\varphi_1 * \varphi_2 \simeq_r \psi_1 * \psi_2$ whenever the concatenations are defined.*

*(F) If $\varphi \in P(G, v, w)$, then $\varphi * \overleftarrow{\varphi} \simeq_r C_v$.*

*Proof.* (A) This is streight because for $r = 2$, condition $B_r$ is equivalent to condition $A$.

(B) When $\varphi$ and $\psi$ differ in at most $r$ vertices, for every $k > r$, they differ in at most $k$ vertices, as desired.

(C) When $\varphi \simeq \psi$, they are equivalent by condition (A) of definition 2.1 and so $\varphi \simeq_r \psi$.

(D) This is because when two path differ only by the insertion or removal of a subpath of length at most $r$, so are their inversions.

(E) Since $\varphi_1 \simeq_r \psi_1$, there exists a finite sequence $\varphi_1 = \alpha_0, \alpha_1, \dots, \alpha_k = \psi_1$ such that for each $i$, $\alpha_i$ and $\alpha_{i+1}$ satisfy condition (A) or $(B_r)$ and also since $\varphi_2 \simeq_r \psi_2$, there exists a finite sequence $\varphi_2 = \beta_0, \beta_1, \dots, \beta_m = \psi_2$ such that for each $i$, $\beta_i$ and $\beta_{i+1}$ satisfy condition (A) or $(B_r)$. Without losing the generality, assume that $k > m$. for every $m < j < k$, let $\beta_j = \psi_2$. Now the finite sequence $\alpha_i * \beta_i$ makes $\varphi_1 * \varphi_2 \simeq_r \psi_1 * \psi_2$.

(F) It is obviouse that by finitely many removing of backtracks, we have constant path.

□

The next statement is an immediate consequence of the definition of $r$-homotopy and will be used repeatedly in what follows. Its proof is omitted, as it can be verified directly. A stronger version, valid when the initial cycle has length at most $2r$, will be established in Proposition 3.10.

**Proposition 3.3.** *Let $\alpha$ be a closed path based at $v$. If $\alpha$ can be written as a concatenation $\alpha = \lambda * \beta$ where $\lambda$ is a subpath of length $\leq r$ forming a small cycle at $v$, then $\alpha \simeq_r \beta$.*

In preparation for the main results, we examine the behavior of short cycles with respect to the $r$-homotopy relation. It turns out that every cycle of length at most $2r$ is $r$-homotopic to the constant cycle at its base vertex, and the converse also holds by some conditions. These properties provide a fundamental characterization of local cycles in the context of $r$-homotopy and form the basis for the proofs of the subsequent theorems.

**Proposition 3.4.** *Let $(G, u)$ be a based graph and $\alpha$ be a $u$-based $n$-cycle in $G$. If $n \leq 2r$, then $\alpha$ is $r$-homotopic to $C_u$.*

*Proof.* In the strictest sense, let $\alpha = (u = u_0, u_1, \dots, u_{2r} = u)$. If we set $\alpha_1 = (u = u_0, u_1, \dots, u_r)$ and $\alpha_2 = (u = u_{2r}, u_{2r-1}, \dots, u_{r+1}, u_r)$, then $\alpha_1$ and $\alpha_2$ differ in at most $(r-1)$ vertices, and thus $\alpha_1 \simeq_r \alpha_2$. Therefore, $\alpha = \alpha_1 * \overleftarrow{\alpha_2} \simeq_r C_u$, as desired. □

In the following example, we show that the converse of Proposition 3.4 is not necessarily true.

**Example 3.5.** Let the 8-cycle

$$\alpha = (v_0, v_1, v_2, v_3, v_0, v_4, v_5, v_6, v_0)$$

which can be written as the concatenation of two 4-cycles:

$$(v_0, v_1, v_2, v_3, v_0) * (v_0, v_4, v_5, v_6, v_0).$$

Although $(v_0, v_1, v_2, v_3, v_0) \simeq_2 C_{v_0}$ by Proposition 3.4 and also $(v_0, v_4, v_5, v_6, v_0) \simeq_2 C_{v_0}$, which implies that $\alpha \simeq_2 C_{v_0}$, we have $8 \nleq 2r$.

Intuitively, what makes the converse of Proposition 3.4 fail is the existence of smaller cycles in the structure of the cycle $\alpha$.
In order to find a partial reverse of Proposition 3.4, we introduce the following definition.

**Definition 3.6.** Let $\alpha = (v_0, \dots, v_m)$ be an $m$-cycle ($v_m = v_0$). For $n < m$, we say that the $n$-cycle $\beta = (u_0, u_1, \dots, u_n)$ is a *proper subcycle* of $\alpha$ if there exists $j \in \{0, 1, \dots, m-1\}$ such that $u_i = v_{j+i} \bmod m$ for all $0 \leq i \leq n$.

Indeed, a subcycle is a segment of a larger cycle which itself is a cycle. By definition, cycles contain no backtrack, and consequently, neither do their subcycles.

**Proposition 3.7.** *Let $(G, u)$ be a based graph, $\alpha$ be a $u$-based $n$-cycle with no proper subcycle, and $\alpha \simeq_r C_u$. Then $n \leq 2r$.*

*Proof.* By contradiction, assume that $n > 2r$ and without loss of generality, $n = 2r + 1$. Let

$$\alpha = (u = u_0, u_1, \dots, u_{2r}, u_{2r+1} = u).$$

If

$$\alpha_1 = (u = u_0, u_1, \dots, u_{r+1}) \quad \text{and} \quad \alpha_2 = (u_{2r+1}, u_{2r}, \dots, u_{r+2}, u_{r+1}),$$

then $\alpha_1 * \overleftarrow{\alpha_2} = \alpha \simeq_r C_u$, which implies that $\alpha_1 \simeq_r \alpha_2$. By the definition of $r$-homotopy, $\alpha_1$ and $\alpha_2$ differ in at most $r - 1$ vertices. So there is at least one vertex $u_i$ in $\alpha_1$ such that it is either repetitious in $\alpha_1$ or equal to a vertex in $\alpha_2$. Both cases cause a subcycle in $\alpha$, which is a contradiction. □

**Corollary 3.8.** *Let $(G, u)$ be a based graph, $\gamma$ be a $u$-based closed walk with no proper subcycle and $\gamma \simeq_r C_u$. Then there exists a $u$-based $n$-cycle $\alpha$ such that $n \leq 2r$ and $\gamma \simeq \alpha$.*

*Proof.* It is enough to consider a representation of the $\gamma$ that has the shortest possible length and use Proposition 3.7 . In fact, such a representation is obtained by removing all the backtracks from $\gamma$. □

**Remark 3.9.** It should be noted that if $\varphi$ and $\psi$ are $r$-homotopic and the related sequence of walks $\varphi_i$ in Definition 2.1 all satisfy condition $A$, then $\varphi$ and $\psi$ differ by a backtrack. In other words, a sequence of equivalences using condition $A$ results in a backtrack.

In Proposition 3.3, we showed that a small cycle of length at most $r$ at the beginning of a cycle can be removed by an $r$-homotopy. In fact, the same conclusion remains true when the loop has length up to $2r$. This stronger statement is recorded below.

**Proposition 3.10.** *Let $\alpha$ be a $u$-based cycle in a graph $G$. Suppose that $\alpha$ can be written as a concatenation $\alpha = \lambda * \beta$, where $\lambda$ is a $u$-based subcycle of length at most $2r$. Then $\alpha \simeq_r \beta$.*

*Proof.* By Proposition 3.4, $\lambda \simeq_r C_u$ which implies

$$\alpha = \lambda * \beta \simeq_r C_u * \beta \simeq_r \beta.$$

□

Lollipops play a key role in analyzing how small cycles can be transferred along paths while preserving $r$-homotopy equivalence. In particular, we show that when the head of a lollipop has length at most $2r$, the entire lollipop turns out to be $r$-homotopic to the constant loop, and conversely by some conditions. These results form an important step toward understanding the generation of $\pi_1^r(G,u)$ and the reduction of cycles under $r$-homotopy.

**Definition 3.11.** Let $(G,u)$ be a based graph. A $u$-based lollipop is a cycle of the form $\alpha * \beta * \overleftarrow{\alpha}$, where $\alpha$ is a path from $u$ to a vertex $v$ and $\beta$ is a cycle at $v$. We call $\beta$ the head of lollipop and if $\beta$ is an $n$-cycle, we call $\alpha * \beta * \overleftarrow{\alpha}$ an $n$-lollipop.

**Proposition 3.12.** *Let $(G,u)$ be a based graph, $n \leq 2r$ and $\alpha * \beta * \overleftarrow{\alpha}$ be a $u$-based $n$-lollipop. Then $\alpha * \beta * \overleftarrow{\alpha} \simeq_r C_u$.*

*Proof.* Let $v = \alpha(1) = \beta(0) = \beta(1)$. Since $\beta$ is an $n$-cycle and $n \leq 2r$, so Proposition 3.4 implies that $\beta \simeq_r C_v$. Then, by Proposition 3.2 we have

$$\alpha * \beta * \overleftarrow{\alpha} \simeq_r \alpha * C_v * \overleftarrow{\alpha} \simeq_r \alpha * \overleftarrow{\alpha} \simeq_r C_u.$$

□

**Proposition 3.13.** *Let $(G,u)$ be a based graph and $\alpha*\beta*\overleftarrow{\alpha}$ be a $u$-based $n$-lollipop in which $\beta$ has no proper subcycle. If $\alpha * \beta * \overleftarrow{\alpha} \simeq_r C_u$, then $n \leq 2r$.*

*Proof.* Let $v = \alpha(1)$. Since $\alpha * \beta * \overleftarrow{\alpha} \simeq_r C_u$, $\beta \simeq_r \overleftarrow{\alpha} * C_u * \alpha \simeq_r C_v$. Hence $\beta$ is a $v$-based $n$-cycle with no proper subcycle and $\beta \simeq_r C_v$. Proposition 3.7 gives that $n \leq 2r$. □

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# A generalized Wiener Index Incorporating Bond Multiplicity with Applications to Molecular Graphs

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**Abstract**

In this talk, we introduce a generalized version of the Wiener index that explicitly incorporates bond multiplicity into the distance metric of molecular graphs. Each bond is assigned a weight equal to the reciprocal of its multiplicity, leading to generalized shortest-path distances and a corresponding generalized Wiener index. We show that the proposed index is a genuine extension of the classical Wiener index and coincides with it for molecular graphs containing only single bonds.

## 1. Introduction and Preliminaries

Topological indices are numerical descriptors associated with the structure of molecular graphs and have been widely used in quantitative structure–property and structure–activity relationship studies (QSPR/QSAR) to correlate structural features of molecules with their physical, chemical, and biological properties. Among these descriptors, the *Wiener index* is one of the most classical and extensively studied graph invariants in chemical graph theory. The Wiener index, introduced by Harry Wiener in 1947, was originally proposed to correlate structural features of organic compounds, such as paraffins, with their boiling points and other physicochemical properties [8]. It is defined as the sum of the shortest-path distances between all pairs of vertices in a graph representing a molecule, where vertices correspond to non-hydrogen atoms and edges correspond to chemical bonds [4].
Extensions and variations, including the *Hyper-Wiener index* and degree-generalized versions, have been proposed to capture additional structural nuances of graphs [2, 7].
In chemical graph theory, efforts have been made to introduce generalized or generalized indices that distinguish between different types of edges or account for additional structural information. generalized Wiener indices have been studied in the context of alkenes and alkadienes, showing improved correlation with properties such as boiling points [5]. More recent mathematical generalizations, such as degree-generalized Wiener indices, further demonstrate the utility of incorporating supplementary structural data into distance-based descriptors [7].

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Nevertheless, a systematic approach to explicitly integrate bond multiplicity into the Wiener index framework remains underdeveloped. Motivated by this gap, in this work we propose an enhanced version of the Wiener index that incorporates bond multiplicity directly into the distance metric. The proposed *generalized Wiener index*, defined via reciprocal bond multiplicities, is expected to better reflect structural and electronic differences among molecular graphs with multiple bonds. In the following sections, we present the formal definition of the enhanced index, computational results for selected molecules, and a comparative analysis demonstrating its improved discriminatory power over the classical Wiener index.

## 2. Preliminaries

In this section, we briefly recall the classical Wiener index and establish the notation that will be used throughout the paper. We also compute the Wiener index for several simple molecular graphs that will serve as reference examples in subsequent sections.

**Molecular Graphs:**

A molecular structure is represented by a simple connected graph

$$G = (V(G), E(G)),$$

called a *molecular graph*, where vertices correspond to non-hydrogen atoms and edges correspond to chemical bonds between them. Throughout this work, hydrogen atoms are suppressed, as is standard in chemical graph theory. Unless otherwise stated, molecular graphs are assumed to be simple, undirected, and connected.

**Graph Distance:**

For two vertices $u, v \in V(G)$, the *graph distance* $d_G(u, v)$ is defined as the length (number of edges) of the shortest path connecting $u$ and $v$ in $G$.

**The Classical Wiener Index:**

The *Wiener index* of a connected graph $G$, denoted by $W(G)$, is defined as

$$W(G) = \sum_{\{u,v\} \subset V(G)} d_G(u, v),$$

where the summation runs over all unordered pairs of distinct vertices of $G$.

## 3. A generalized Wiener Index Incorporating Bond Multiplicity

In this section, we introduce a modified version of the Wiener index that explicitly accounts for bond multiplicity in molecular graphs. We then compute the proposed index for several representative molecules and compare the results with the classical Wiener index.

**Edge-generalized Molecular Graphs:**

Let $G = (V, E)$ be a connected molecular graph. For each edge $e \in E$, let $m(e) \in \{1, 2, 3\}$ denote the bond multiplicity associated with $e$, corresponding to single, double, or triple bonds, respectively.

We define a generalized molecular graph by assigning to each edge $e$ a positive weight

$$w(e) = \frac{1}{m(e)}.$$

This choice reflects the increased bond strength and reduced effective distance associated with multiple bonds.

**generalized Graph Distance:**

For vertices $u, v \in V(G)$, the generalized distance $d_w(u, v)$ is defined as the minimum total weight among all paths connecting $u$ and $v$:

$$d_w(u, v) = \min_{\gamma: u \to v} \sum_{e \in \gamma} w(e).$$

When all bonds are single, $w(e) = 1$ for all $e$, and the generalized distance coincides with the classical graph distance.

**Definition of the generalized Wiener Index:**
The *generalized Wiener index* of $G$ is defined as

$$W_w(G) = \sum_{\{u,v\}\subset V(G)} d_w(u,v).$$

It is immediate that $W_w(G) = W(G)$ for graphs with only single bonds.
We first show that the proposed generalized Wiener index is a genuine extension of the classical Wiener index.

**Proposition 3.1.** *Let $G = (V, E)$ be a connected molecular graph in which all edges correspond to single bonds. Then the generalized Wiener index coincides with the classical Wiener index, i.e.,*

$$W_w(G) = W(G).$$

*Proof.* Easily comes from the definition. □

We next establish several fundamental properties of the generalized Wiener index and compare it with the classical Wiener index.

**Proposition 3.2.** *Let $G = (V, E)$ be a connected molecular graph. Then*

$$W_w(G) \leq W(G),$$

*with equality if and only if all bonds in $G$ are single.*

*Proof.* Easily comes from the definition. □

**Corollary 3.3.** *If a molecular graph $G$ contains at least one multiple bond, then*

$$W_w(G) < W(G).$$

The following proposition shows the sensitivity of the generalized Wiener index to local changes in bond multiplicity.

**Proposition 3.4.** *Let $G$ be a molecular graph and let $e \in E(G)$ be an edge whose bond multiplicity is increased from $m$ to $m + 1$. Let $G'$ denote the modified graph. Then*

$$W_w(G') < W_w(G).$$

*Proof.* Increasing the bond multiplicity of $e$ decreases its weight from $\frac{1}{m}$ to $\frac{1}{m+1}$. Therefore, any shortest path that includes $e$ becomes strictly shorter in generalized length. Since at least one pair of vertices has a shortest path containing $e$, the total sum of generalized distances decreases. Hence,

$$W_w(G') < W_w(G).$$

□

**Remark 3.5.** The magnitude of the decrease depends on the number of shortest paths that include the modified edge. Consequently, bonds located in structurally central positions have a stronger impact on $W_w(G)$.

Let $P_n$ be a path graph on $n$ vertices representing a linear molecular chain. Suppose that the edges of $P_n$ have bond multiplicities

$$(m_1, m_2, \dots, m_{n-1}),$$

where $m_i$ denotes the multiplicity of the bond between vertices $v_i$ and $v_{i+1}$.

**Proposition 3.6.** *The generalized Wiener index of $P_n$ is given by*

$$W_w(P_n) = \sum_{1\leq i<j\leq n} \sum_{k=i}^{j-1} \frac{1}{m_k}.$$

*Proof.* In a path graph, there exists a unique path between any two vertices $v_i$ and $v_j$. The generalized distance between them is therefore equal to the sum of the weights of the edges along that path:

$$d_w(v_i, v_j) = \sum_{k=i}^{j-1} \frac{1}{m_k}.$$

Summing over all unordered pairs of vertices yields the stated formula. □

**Example 3.7.** Consider a path graph $P_4$ corresponding to a four-carbon chain.

- If all bonds are single, then
$$W_w(P_4) = W(P_4) = 10.$$
- If the bond multiplicities are $(2, 1, 2)$, corresponding to 1,3-butadiene, then
$$W_w(P_4) = \frac{1}{2} + 1 + \frac{1}{2} + \left(\frac{1}{2} + 1\right) + \left(1 + \frac{1}{2}\right) = \frac{19}{2}.$$

*3.1. Computation for Selected Molecular Graphs*

We now compute $W_w(G)$ for the molecular graphs introduced in the previous section.
**Ethane ($C_2H_6$):** The molecular graph consists of a single bond with $m(e) = 1$. Thus,

$$W_w(\text{Ethane}) = 1.$$

**Ethene ($C_2H_4$):** The two carbon atoms are connected by a double bond, so $m(e) = 2$ and $w(e) = \frac{1}{2}$. Hence,

$$W_w(\text{Ethene}) = \frac{1}{2}.$$

**Butane ($C_4H_{10}$):** All bonds are single. Therefore,

$$W_w(\text{Butane}) = W(\text{Butane}) = 10.$$

**1,3-Butadiene ($C_4H_6$):** The carbon chain consists of alternating single and double bonds with multiplicities $(2, 1, 2)$. Using generalized shortest-path distances, we obtain

$$W_w(\text{1,3-Butadiene}) = \frac{19}{2}.$$

**Cyclohexane ($C_6H_{12}$):** All bonds are single, hence

$$W_w(\text{Cyclohexane}) = 27.$$

**Benzene ($C_6H_6$):** Benzene is represented by the cycle graph $C_6$ with alternating single and double bonds in a Kekulé structure. Hence, single bonds have weight 1, while double bonds have weight $\frac{1}{2}$.
Along the cycle, the edge weights alternate as

$$\frac{1}{2},\ 1,\ \frac{1}{2},\ 1,\ \frac{1}{2},\ 1.$$

We compute the generalized Wiener index by summing generalized distances over all unordered vertex pairs.

*Distance 1 (adjacent vertices)..* There are three single bonds and three double bonds. Their total contribution is

$$3 \times 1 + 3 \times \frac{1}{2} = 4.5.$$

*Distance 2..* Each pair of vertices at graph distance two is connected by a path containing one single bond and one double bond, yielding generalized distance

$$1 + \frac{1}{2} = 1.5.$$

There are six such pairs, contributing

$$6 \times 1.5 = 9.$$

*Distance 3 (opposite vertices)..* For each opposite pair, there are two possible paths of length three. The generalized lengths of these paths are

$$\frac{1}{2} + 1 + \frac{1}{2} = 2, \qquad 1 + \frac{1}{2} + 1 = 2.5.$$

The shortest generalized distance is therefore 2. There are three such pairs, contributing

$$3 \times 2 = 6.$$

Summing all contributions, we obtain

$$W_w(\text{Benzene}) = 4.5 + 9 + 6 = 19.5.$$

**Remark 3.8.** Although benzene contains three formal double bonds, the presence of multiple shortest paths in the cyclic structure limits the influence of bond multiplicity, resulting in a moderate reduction of the generalized Wiener index compared to its classical value $W(C_6) = 27$.

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